MATH104: Topology **Fall 2023**

Homework 13

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Assorted Topological Groups Problems.

Section 26 (Compact Spaces), 26.13(a)(b)

Let G be a topological group.

a) **Let A and B be subspaces of G. If A is closed and B is compact, show $A \cdot B$ is closed. [*Hint:* If c is not in $A \cdot B$, find a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.]**

Solution. To show that $A \cdot B$ is closed, we will show that $(A \cdot B)^C$ is open. Let $c \in (A \cdot B)^C$. As the hint says, we will show that there is a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.

Since G is a topological group, it follows from Exercise 145.1 that the map $f\colon G \times G \to G$ sending (x, y) to xy^{-1} is continuous. A is closed, so A^C is open. Since f is continuous, the preimage of an open set is open, and so $f^{-1}(A^C)$ is open. The one-point set $\{c\}$ and B are each compact. We claim that $\{c\} \times B \subseteq f^{-1}(A^C)$. Equivalently, we will show that $f(\{c\} \times B) \subseteq A^C$. Since $c \in (A \cdot B)^C$, or equivalently, $c \notin A \cdot B$, it follows that $c \neq ab$ for any $a \in A$, $b \in B$. Equivalently, $cb^{-1} \neq a$ for any $a \in A$ and $b \in B$. Since $f(\{c\} \times B)$ is defined as $\{cb^{-1} \mid b \in B\}$, it follows that $f(\{c\} \times B)$ and A are disjoint, or equivalently, $f(\{c\} \times B) \subseteq A^C$.

To summarize, we have that $f^{-1}(A^C)$ is an open set in $G \times G$, and $\{c\}$ and B are each compact in G, with $\{c\} \times B \subseteq f^{-1}(A^C)$. It follows by Exercise 26.9 that there exist open sets W and V in G such that

$$\{c\} \times B \subseteq W \times V \subseteq f^{-1}(A^C).$$

Note that $c \in W$ by construction, and $W \times B \subseteq W \times V$. Consequently, since $W \times B \subseteq W \times V \subseteq f^{-1}(A^C)$, it follows that $f(W \times B) \subseteq A^C$, or equivalently

$$W \cdot B^{-1} \subseteq A^C.$$

It follows that $(W \cdot B^{-1}) \cap A = \varnothing$. Consequently, $wb^{-1} \neq a$, or $w \neq ab$ for any $w \in W, a \in A$ and $b \in B$. Thus, $W \cap (A \cdot B) = \varnothing$.

For any arbitrary $c \in (A \cdot B)^C$, we have found a neighborhood W of c such that $W \cap (A \cdot B) = \varnothing$, or equivalently, $W \subseteq (A \cdot B)^C$. It follows by definition that $(A \cdot B)^C$ is open in G, so $A \cdot B$ is closed, as desired. ■

b) **Let H be a subgroup of G; let $p\colon G \to G/H$ be the quotient map. If H is compact, show that p is a closed map.**

Solution. Let A be a closed set in G. Note that $p(A)$ consists of all the cosets of G/H containing elements of A. Then $p^{-1}(p(A))$ is

$$p^{-1}(p(A)) = \{ah \mid a \in A, h \in H\} = A \cdot H.$$

Since H is compact and A is closed, we know from part (a) that $A \cdot H$ is closed. Since p is a quotient map, we know by definition that $p^{-1}(p(A))$ is closed if and only if $p(A)$ is closed. Consequently, since $p^{-1}(p(A))$ is closed, $p(A)$ must also be closed, so p is a closed map, as desired. ■

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

c) **Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]**

Solution. Let $[f]$ and $[g]$ be any two arbitrary elements in $\pi_1(G, x_0)$. We will show that $[f] \otimes [g] = [f] * [g]$. Consider $[f] \otimes [g]$. Since $[f] = [f] * [e_{x_0}]$ and $[g] = [e_{x_0}] * [g]$, it follows that

$$\begin{aligned}
[f] \otimes [g] &= ([f] * [e_{x_0}]) \otimes ([e_{x_0}] * [g]) \\
&= ([f] * [e_{x_0}]) \cdot ([e_{x_0}] * [g]) \\
&= ([f] \cdot [e_{x_0}]) * ([e_{x_0}] \cdot [g])
\end{aligned}$$

where the final step follows by the construction of products of paths:

$$\begin{aligned}
([f] * [e_{x_0}]) \cdot ([e_{x_0}] * [g]) &= \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ e_{x_0} & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \cdot \begin{cases} e_{x_0} & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= \begin{cases} f(2s) \cdot e_{x_0} & \text{for } s \in \left[0, \frac{1}{2}\right] \\ e_{x_0} \cdot g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= ([f] \cdot [e_{x_0}]) * ([e_{x_0}] \cdot [g]).
\end{aligned}$$

Simplifying further by noting that $[f] \cdot [e_{x_0}] = [f]$ and $[e_{x_0}] \cdot [g] = [g]$, we get that

$$\begin{aligned}
[f] \otimes [g] &= ([f] \cdot [e_{x_0}]) * ([e_{x_0}] \cdot [g]) \\
&= [f] * [g]
\end{aligned}$$

and so the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. ■

c) **Show that $\pi_1(G, x_0)$ is abelian.**

Solution. Let $[f]$ and $[g]$ be two arbitary elements in $\pi_1(G, x_0)$. so we will show that $\pi_1(G, x_0)$ is abelian by showing $[f] * [g] = [g] * [f]$.

Note that $[f] = [e_{x_0}] * [f] = [e_{x_0} * f]$ and $[g] = [g] * [e_{x_0}] = [g * e_{x_0}]$, so

$$\begin{aligned}
[f] \otimes [g] &= [e_{x_0} * f] \otimes [g * e_{x_0}] \\
&= [e_{x_0} * f] \cdot [g * e_{x_0}] \\
&= [e_{x_0} \cdot g] * [f \cdot e_{x_0}]^1 \\
&= [g] * [f].
\end{aligned}$$

[1] this follows by the "distributive property" proved in the previous part.

From part (c), we know that the group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same, so $[f] \otimes [g] = [f] * [g]$. It follows that

$$[f] * [g] = [g] * [f],$$

so $\pi_1(G, x_0)$ is abelian, as desired. ■