Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor Allison N. Miller, at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

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1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)

A function $f\colon A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f.

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)

A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)

An **equivalence relation** $\sim$ on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x \sim x$ for every x in A.
2. (Symmetry) If $x \sim y$, then $y \sim x$.
3. (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)

A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz, then xCz.

Note: the relation C is often replaced as $<$, just as how it is synonyous with $\sim$ in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx, contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Immediate Predecessors and Successors)

If X is a set and $<$ is an order relation on X, and if $a < b$, the **open interval** (a, b) on X is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty, a is the **immediate predecessor** of b and b is the **immediate successor** of a.

Definition 1.6 (Order Type)

Suppose that A and b are two sets with order relations $<_A$ and $<_B$, respectively. A and B have the same **order type** if there is a bijective correspondence between them that preserves order.

That is, if there exists a bijective function $f\colon A \to B$ such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

Example. The interval $(-1, 1)$ of real numbers has the same order type as $\mathbb{R}$. The function $f\colon (-1, 1) \to B$ such that

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijective correspondence.

Definition 1.7 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that elements is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that elements is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.8 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \ldots, n\}$, $\mathbb{Z}_+$), tuples, sequences, and Cartesian products.

Definition 1.9 (ω-tuple / Sequence)
Given a set X, a ω-tuple of elements of X is a function

$$\mathbf{x} \colon \mathbb{Z}_+ \to X$$

also known as a **sequence** (or infinite sequence) of elements of X.

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

1.7 Countable and Uncountable Sets

Definition 1.10 (Countably Infinite)
A set A is infinite if it is not finite. It is **countably infinite** of there is a bijective correspondence

$$f \colon A \to \mathbb{Z}_+.$$

Example. The set $\mathbb{Z}$ of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f \colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ and $g \colon A \to \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. $\square$

Definition 1.11 (Countable and Uncountable Sets)
A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0, 1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and $\mathbb{R}$ are examples of uncountable sets.

Theorem 1.2. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.

2. There is a surjective function $f\colon \mathbb{Z}_+ \to B$.

3. There is an injective function $g\colon B \to \mathbb{Z}_+$.

Theorem 1.3 (Countable Union of Countable Sets)**.** A countable union of countable sets is countable.

1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

1.9 Infinite Sets and the Axiom of Choice

Theorem 1.4. Let A be a set. The following statements about A are equivalent:

1. A is infinite.

2. There exists an injective function $f\colon \mathbb{Z}_+ \to A$

3. There exists a bijection of A with a proper subset of itself.

Theorem 1.5 (Axiom of Choice)**.** Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

1.10 Well-Ordered Sets

Definition 1.12 (Well-Ordered Sets)

A set A with an order relation $<$ is **well-ordered** if every nonempty subset of A has a smallest element.

Example. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set $(\mathbb{Z}_+)^\omega$.

Theorem 1.6 (Well-Ordering Theorem; Zermelo, 1904)**.** If A is a set, there exists an order relation on A that is a well-ordering.

Corollary 1.1. There exists an uncountable well-ordered set.

Definition 1.13 (Section of a Set)

Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}.$$

S_α is the **section** of X by α.

Lemma 1.1. There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem 1.7. If A has a countable subset of S_Ω, then A has an upper bound in S_Ω.

1.11 The Maximum Principle

Definition 1.14 (Partial Order)

Given a set A, a relation $\prec$ on A is a **strict partial order** on A if it has the following properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

If the relation $\prec$ is instead $\preceq$, where $a \preceq b$ implies $a = b$ or $a \prec b$, $\preceq$ is a **partial order** on A.

Remark. These are the second and third properties of a simple order, defined in Definition 1.4. Consequently, a strict partial order behaves like a simple order except that it need not be true that for every pair of distinct x and y in the set, either $x \prec y$ or $y \prec x$.

Theorem 1.8 (The Maximum Principle)**.** Let A be a set and let $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered subset B of A.

Example. If $\mathcal{A}$ is the collection of all circular regions in the plane under the "proper subset of" relation, a maximal simply ordered subcollection of $\mathcal{A}$ consists of all circular regions with centers at the origin.

Definition 1.15

Let A be a set and let $\prec$ be a strict partial order on A. If B is a subset of A, an **upper bound** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

A **maximal element** of A is an element m on A such that for no element a of A does the relation $m \prec a$ hold.

Theorem 1.9 (Zorn's Lemma). Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

2 Chapter 2: Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.1 (Topology)
A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is a **topological space**.

Definition 2.2 (Open Set)
If X is a topological space with topology $\mathcal{T}$, a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Remark. Using the open set terminology, a topological space can be thought of as a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Example. The topologies on $X = \{a, b, c\}$ can be specified by permuting a, b, and c and specifying the open sets defined by subsets of X.

However, note that not every collection of subsets of X is a topology on X: consider

$$\{\{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

This is not closed under intersection.

Example. Let X be any set. The collection of all subsets of X is a topology on X known as the **discrete topology**.

The collection consisting of only $\varnothing$ and X is also a topology on X known as the **indiscrete topology** (or **trivial topology**).

Example. Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - -U$ either is finite or all of X. Then $\mathcal{T}_f$ is a topology on X known as the **finite complement topology**.

Similarly, let X be a set and let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - -U$ either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

Definition 2.3 (Finer and Coarser Topologies)

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, then $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$.

In the resepctive situations, we say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$; and $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$.

$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

Intuition. Think of a topological space as something like a truckload full of gravel. Then the pebbles and all unions of collections of pebbles are the open sets.

Furthermore, if the pebbles are smashed into smaller pieces, the collection of open sets has been enlarged, adn the topology, like the gravel, is made finer by the operation.

Remark. The finer and coaser terminology can also be replaced by **larger** and **smaller** as well as **stronger** and **weaker**, respectively.

2.13 Basis for a Topology

Instead of specifying a topology by describing the entire collection $\mathcal{T}$ of open sets, it is often easier to specify a smaller collection of subsets of X and to define the topology from that collection.

Definition 2.4 (Basis for Topology)

If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (**basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 = B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these conditions, then the **topology $\mathcal{T}$ generated by** $\mathcal{B}$ is defined as follows: a subset U of X is said to be open in X (or an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$ (each basis element is itself an element of $\mathcal{T}$).

Remark. The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ satisfies the requirements for a topology on X:

1. The empty set and X are both in $\mathcal{T}$
2. $U = \bigcup_{\alpha \in J} U_\alpha$ for an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}$ belongs to $\mathcal{T}$
3. The intersection of two elements U_1 and U_2 in $\mathcal{T}$ is also in $\mathcal{T}$; the finite intersection case

follows from induction.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane; $\mathcal{B}$ is a basis. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every $x \in U$ lies in some circular region contained in U.

Example. If X is any set, the collection of all one-point subsets in X is a basis for the discrete topology on X.

Lemma 2.1. Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Remark. The above lemma tells us that every open set U in X can be expressed as a union of basis elements. However, unlike the notion of a basis in linear algebra, this expression may not be unique.

Lemma 2.2 (Obtaining a Basis from a Topology)**.** Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 2.3 (Comparing Fineness of Topologies)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.

2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Intuition. Recall the analogy between a topological space and a truckload of gravel. Here, the pebbles are the basis elements of the topology; after the pebbles are smashed into dust, the dust particles are the basis elements for the new topology. The new topology is finer than the old one, and the dust particle was previously contained in a pebble.

Remark. The collection of all circular regions in the plane generates the same topology as the collection of all rectangular regions in the plane: each rectangle in the rectangular basis has a circle enclosed in it, and each circle in the circular basis has a rectangle enclosed in it. This tells us that each topology is finer than the other, and so they generate the same topology.

Definition 2.5 (Topologies on $\mathbb{R}$)

Let $\mathcal{B}$ be the collection of open intervals on the real line; the **standard topology** on the real line is the topology generated by $\mathcal{B}$.

Let $\mathcal{B}'$ be the collection of all half-open intervals $[a, b)$ on the real line; the **lower limit topology** on the real line is the topology generated by $\mathcal{B}'$ and is denoted by $\mathbb{R}_\ell$.

Finally, let K be the set of all numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$. Let $\mathcal{B}''$ be the collection of all open intervals along with sets of the form $(a, b) - -K$. The topology generated by $\mathcal{B}''$ is the **K-topology** on $\mathbb{R}$, denoted by $\mathbb{R}_k$.

Definition 2.6 (Subbasis for a Topology)

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

The **topology generated by the subbasis** $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.