Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor Allison N. Miller, at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

Contents

1 Chapter 1: Set Theory and Logic

2 Functions

Definition 2.1 (Injective, Surjective, Bijection)
A function $f\colon A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f.

If f is both **injective** and **surjective**, it said to be **bijective**.

3 Relations

Definition 3.1 (Relation)
A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 3.2 (Equivalence Relation)
An **equivalence relation** $\sim$ on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x \sim x$ for every x in A.
2. (Symmetry) If $x \sim y$, then $y \sim x$.
3. (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 3.3 (Order Relation)
A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz, then xCz.

Note: the relation C is often replaced as $<$, just as how it is synonyous with $\sim$ in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx, contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 3.4 (Immediate Predecessors and Successors)
If X is a set and $<$ is an order relation on X, and if $a < b$, the **open interval** (a, b) on X is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty, a is the **immediate predecessor** of b and b is the **immediate successor** of a.

Definition 3.5 (Order Type)
Suppose that A and b are two sets with order relations $<_A$ and $<_B$, respectively. A and B have the same **order type** if there is a bijective correspondence between them that preserves

order.

That is, if there exists a bijective function $f\colon A \to B$ such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

Example. The interval $(-1, 1)$ of real numbers has the same order type as $\mathbb{R}$. The function $f\colon (-1,1) \to B$ such that

$$f(x) = \frac{x}{1-x^2}$$

is an order-preserving bijective correspondence.

Definition 3.6 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that elements is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that elements is the **infimum** of A_0 (also the greatest lower bound).

Definition 3.7 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

4 The Integers and the Real Numbers

Theorem 4.1 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \ldots, n\}$, $\mathbb{Z}_+$), tuples, sequences, and Cartesian products.

Definition 5.1 (ω-tuple / Sequence)

Given a set X, a ω-tuple of elements of X is a function

$$\mathbf{x}\colon \mathbb{Z}_+ \to X$$

also known as a **sequence** (or infinite sequence) of elements of X.

6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

7 Countable and Uncountable Sets

Definition 7.1 (Countably Infinite)

A set A is infinite if it is not finite. It is **countably infinite** of there is a bijective correspondence

$$f\colon A \to \mathbb{Z}_+.$$

Example. The set $\mathbb{Z}$ of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ and $g\colon A \to \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. □

Definition 7.2 (Countable and Uncountable Sets)

A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0, 1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and $\mathbb{R}$ are examples of uncountable sets.

Theorem 7.1. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.
2. There is a surjective function $f\colon \mathbb{Z}_+ \to B$.
3. There is an injective function $g\colon B \to \mathbb{Z}_+$.

Theorem 7.2 (Countable Union of Countable Sets)**.** A countable union of countable sets is countable.

8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

9 Infinite Sets and the Axiom of Choice

Theorem 9.1. Let A be a set. The following statements about A are equivalent:

1. A is infinite.
2. There exists an injective function $f\colon \mathbb{Z}_+ \to A$
3. There exists a bijection of A with a proper subset of itself.

Theorem 9.2 (Axiom of Choice)**.** Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

10 Well-Ordered Sets

Definition 10.1 (Well-Ordered Sets)

A set A with an order relation $<$ is **well-ordered** if every nonempty subset of A has a smallest element.

Example. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set $(\mathbb{Z}_+)^\omega$.

Theorem 10.1 (Well-Ordering Theorem; Zermelo, 1904)**.** If A is a set, there exists an order relation on A that is a well-ordering.

Corollary 1.1. There exists an uncountable well-ordered set.

Definition 10.2 (Section of a Set)

Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}.$$

S_α is the **section** of X by α.

Lemma 10.1 (First Uncountable Ordinal)**.** There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem 10.2. If A has a countable subset of S_Ω, then A has an upper bound in S_Ω.

11 The Maximum Principle

Definition 11.1 (Partial Order)

Given a set A, a relation $\prec$ on A is a **strict partial order** on A if it has the following properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

If the relation $\prec$ is instead $\preceq$, where $a \preceq b$ implies $a = b$ or $a \prec b$, $\preceq$ is a **partial order** on A.

Remark. These are the second and third properties of a simple order, defined in Definition 3.3. Consequently, a strict partial order behaves like a simple order except that it need not be true that for every pair of distinct x and y in the set, either $x \prec y$ or $y \prec x$.

Theorem 11.1 (The Maximum Principle)**.** Let A be a set and let $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered subset B of A.

Example. If $\mathcal{A}$ is the collection of all circular regions in the plane under the "proper subset of" relation, a maximal simply ordered subcollection of $\mathcal{A}$ consists of all circular regions with centers at the origin.

Definition 11.2

Let A be a set and let $\prec$ be a strict partial order on A. If B is a subset of A, an **upper bound** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

A **maximal element** of A is an element m on A such that for no element a of A does the relation $m \prec a$ hold.

Theorem 11.2 (Zorn's Lemma)**.** Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

2 Chapter 2: Topological Spaces and Continuous Functions

12 Topological Spaces

Definition 12.1 (Topology)
A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is a **topological space**.

Definition 12.2 (Open Set)
If X is a topological space with topology $\mathcal{T}$, a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Remark. Using the open set terminology, a topological space can be thought of as a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Example. The topologies on $X = \{a, b, c\}$ can be specified by permuting a, b, and c and specifying the open sets defined by subsets of X.

However, note that not every collection of subsets of X is a topology on X: consider

$$\{\{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

This is not closed under intersection.

Example. Let X be any set. The collection of all subsets of X is a topology on X known as the **discrete topology**.

The collection consisting of only $\varnothing$ and X is also a topology on X known as the **indiscrete topology** (or **trivial topology**).

Example. Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or all of X. Then $\mathcal{T}_f$ is a topology on X known as the **finite complement topology**.

Similarly, let X be a set and let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

Definition 12.3 (Finer and Coarser Topologies)

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, then $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$.

In the resepctive situations, we say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$; and $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$.

$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

Intuition. Think of a topological space as something like a truckload full of gravel. Then the pebbles and all unions of collections of pebbles are the open sets.

Furthermore, if the pebbles are smashed into smaller pieces, the collection of open sets has been enlarged, adn the topology, like the gravel, is made finer by the operation.

Remark. The finer and coaser terminology can also be replaced by **larger** and **smaller** as well as **stronger** and **weaker**, respectively.

13 Basis for a Topology

Instead of specifying a topology by describing the entire collection $\mathcal{T}$ of open sets, it is often easier to specify a smaller collection of subsets of X and to define the topology from that collection.

Definition 13.1 (Basis for Topology)

If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (**basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 = B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these conditions, then the **topology $\mathcal{T}$ generated by** $\mathcal{B}$ is defined as follows: a subset U of X is said to be open in X (or an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$ (each basis element is itself an element of $\mathcal{T}$).

Remark. The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ satisfies the requirements for a topology on X:

1. The empty set and X are both in $\mathcal{T}$
2. $U = \bigcup_{\alpha \in J} U_\alpha$ for an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}$ belongs to $\mathcal{T}$
3. The intersection of two elements U_1 and U_2 in $\mathcal{T}$ is also in $\mathcal{T}$; the finite intersection case

follows from induction.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane; $\mathcal{B}$ is a basis. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every $x \in U$ lies in some circular region contained in U.

Example. If X is any set, the collection of all one-point subsets in X is a basis for the discrete topology on X.

Lemma 13.1. Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Remark. The above lemma tells us that every open set U in X can be expressed as a union of basis elements. However, unlike the notion of a basis in linear algebra, this expression may not be unique.

Lemma 13.2 (Obtaining a Basis from a Topology)**.** Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3 (Comparing Fineness of Topologies)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Intuition. Recall the analogy between a topological space and a truckload of gravel. Here, the pebbles are the basis elements of the topology; after the pebbles are smashed into dust, the dust particles are the basis elements for the new topology. The new topology is finer than the old one, and the dust particle was previously contained in a pebble.

Remark. The collection of all circular regions in the plane generates the same topology as the collection of all rectangular regions in the plane: each rectangle in the rectangular basis has a circle enclosed in it, and each circle in the circular basis has a rectangle enclosed in it. This tells us that each topology is finer than the other, and so they generate the same topology.

Definition 13.2 (Topologies on $\mathbb{R}$)

Let $\mathcal{B}$ be the collection of open intervals on the real line; the **standard topology** on the real line is the topology generated by $\mathcal{B}$.

Let $\mathcal{B}'$ be the collection of all half-open intervals $[a, b)$ on the real line; the **lower limit topology** on the real line is the topology generated by $\mathcal{B}'$ and is denoted by $\mathbb{R}_\ell$.

Finally, let K be the set of all numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$. Let $\mathcal{B}''$ be the collection of all open intervals along with sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ is the **K-topology** on $\mathbb{R}$, denoted by $\mathbb{R}_k$.

Definition 13.3 (Subbasis for a Topology)
A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

The **topology generated by the subbasis** $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

14 The Order Topology

Definition 14.1 (Order Topology)
Let X be a set with a simple order relation and assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, known as the **order topology**.

If X has no smallest element, there are no sets of type 2, and if X has no largest element, there are no sets of type 3.

Example. The standard topology on $\mathbb{R}$ is the order topology derived from the usual order on $\mathbb{R}$.

The order topology on $\mathbb{R} \times \mathbb{R}$ in the dictionary order has as basis the collection of all intervals of the form $(a \times b, c \times d)$ for $a < c$ and for $a = c$ and $b < d$. The subcollection consisting of only intervals of the second type is also an order topology on $\mathbb{R} \times \mathbb{R}$.

Example. Consider the set $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order, and denote $1 \times n$ as a_n and $2 \times n$ by b_n. Then $X = \{a_1, a_2, \dots; b_1, b_2, \dots\}$.

Note that the order topology is not the discrete topology (of which the collection of one-point subsets is a basis), as the one-point set $\{b_1\}$ is not open. Any open set containing b_1 must contain a basis element about b_1, and all basis elements of b_1 contain points in a_i.

Definition 14.2 (Rays of X)

If X is an ordered set and a is an element of X, there are four subsets of X known as the **rays** determined by a: two open rays and two closed rays.

Remark. The open rays in X are open sets in the order topology. Furthermore, the open rays form a subbasis for the order topology on X.

15 The Product Topology on $X \times Y$

Definition 15.1

Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 15.1. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Example. The standard topology on $\mathbb{R}$ is the order topology. The product of this topology with itself is the **standard topology** on $\mathbb{R} \times \mathbb{R}$.

One such basis, by the above theorem, for the standard topology is the collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$.

16 The Subspace Topology

Definition 16.1

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, known as the **subspace topology**.

With this topology, Y is a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Theorem 16.1. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Remark. Let X be an ordered set in the order topology and let Y be a subset of X. The order relation on X when restricted to Y makes Y into an ordered set. However, the *resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X.*

Example. Consider the subset $Y = [0, 1]$ on $\mathbb{R}$ in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$ for $(a, b) \in \mathbb{R}$, which is

$$(a,b) \cap Y = \begin{cases} (a,b), & \text{if } a \text{ and } b \text{ are in } Y; \\ [0,b), & \text{if only } b \text{ is in } Y; \\ (a,1], & \text{if only } a \text{ is in } Y; \\ Y \text{ or } \varnothing, & \text{if neither } a \text{ nor } b \text{ is in } Y; \end{cases}$$

Each of these sets is open in Y, but sets of the second and third types are not open in $\mathbb{R}$. Furthermore, note that these sets form a basis for the order topology on Y, so the order topology and subspace topology of Y as a subspace of $\mathbb{R}$ are the same.

Definition 16.2 (Convex Subsets)

Given an ordered set X, the subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y.

Note: intervals and rays in X are convex in X.

Theorem 16.2. Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

17 Closed Sets and Limit Points

Definition 17.1 (Closed Set)

A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Example. The subsets $[a, b]$ and $[a, \infty)$ of $\mathbb{R}$ are closed, while $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example. In the discrete topology on the set X, every set is both open and closed.

Definition 17.2 (Interior and Closure)

Given a subset A of a topological space X, the **interior** of A, denoted $\operatorname{Int} A$, is the union of all open sets contained in A.

The **closure** of A, denoted $\operatorname{Cl} A$ or $\overline{A}$, is the intersection of all closed sets containing A.

It follows by definition that $\operatorname{Int} A \subset A \subset \overline{A}$.

Theorem 17.1. Let Y be a subspace of X, let A be a subset of Y, and let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Theorem 17.2. Let A be a subset of the topological space X.

a) Then $x \in \overline{A}$ if and only if every open set U containing x (i.e. neighborhood of x) intersects A.

b) Supposing that the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Example. If $A = (0, 1]$, then $\overline{A} = [0, 1]$. If $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then $\overline{B} = \{0\} \cup B$.

Example. Consider $Y = (0, 1]$, a subspace of $\mathbb{R}$, and the set $A = (0, \frac{1}{2})$, a subset of Y.

$\overline{A} = [0, \frac{1}{2}]$ in $\mathbb{R}$ while the closure of A in Y is $\overline{A} \cap Y = (0, \frac{1}{2}]$.

Definition 17.3 (Limit Points)

Let A be a subset of the topological space X and let x be a point of X.

x is a **limit point** (or “cluster point”, or “point of accumulation”) of A if every neighborhood of x intersects A in some point *other than x itself*.

Equivalently, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Theorem 17.3. Let A be a subset of the topological space X and let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'.$$

Definition 17.4 (Hausdorff Spaces)

A topological space X is a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 17.4. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X, known as the **limit** of that sequence of points.

18 Continuous Functions

Definition 18.1 (Continuous Functions)

Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Remark. To prove continuity of f, it suffices to show that the inverse image of every basis element is open; it can even suffice to show that the inverse of each subbasis is open.

Theorem 18.1 (Continuity Equivalences)**.** Let X and Y be topological spaces; let $f\colon X \to Y$. Then the following are equivalent:

1. f is continuous.
2. For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
4. For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

If condition 4 holds for the point x of X, we say f is **continuous at the point** x.

Definition 18.2 (Homeomorphism)

Let X and Y be topological spaces; let $f\colon X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1}\colon Y \to X$$

are continuous, then f is a **homeomorphism**.

Equivalently, f is a homeomorphism if there is a bijective correspondence $f\colon X \to Y$ such that $f(U)$ is open if and only if U is open.

Remark. Just as isomorphisms preserve algebraic properties of groups/rings, homeomorphisms preserve topological structures (e.g. connectedness).

Remark. Two spaces are homemorphic if and only if the two resulting spaces, after removing a point, are also homeomorphic.

Definition 18.3 (Imbedding)

Let $f\colon X \to Y$ be an injective continuous map between topological spaces X and Y. Let Z be the image set $f(X)$ considered as a subspace of Y; then the function $f'\colon X \to Z$ obtained by restricting the range is bijective.

If f' is a homemorphism of X with Z, then the map f is a **topological imbedding** (or **imbedding**) of X in Y.

Example. The function $F : (-1,1) \to \mathbb{R}$ defined by $F(x) = \frac{x}{1-x^2}$ is a homemomorphism.

Remark. A bijective function $f\colon X \to Y$ can be continuous without being a homomorphism: consider $F : [0,1) \to S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$; f^{-1} is not continuous, as $f^{-1}([0, \frac{1}{4}))$ does not lie in an open set V of $\mathbb{R}^2$ such that $V \cap S^1 \subset f([0, \frac{1}{4}))$.

Theorem 18.2 (Rules for Constructing Continuous Functions)**.** Let X, Y, and Z be topological spaces.

a) (Constant function) If $f\colon X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

b) (Inclusion) If A is a subspace of X, the inclusion function $j\colon A \to X$ is continuous.

c) (Composites) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

d) (Restricting the domain) If $f\colon X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f \mid A\colon A \to Y$ is continuous.

e) (Restricting or expanding the range) Let $f\colon X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g\colon X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h\colon X \to Z$ obtained by by expanding the range of f is continuous.

f) (Local formulation of continuinity) The map $f\colon X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f \mid U_\alpha$ is continuous for each α.

Theorem 18.3 (The Pasting Lemma)**.** Let $X = A \cup B$ where A and B are closed in X, and let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h\colon X \to Y$, defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Theorem 18.4 (Maps into Products)**.** Let $f\colon A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if the functions $f_1\colon A \to X$ and $f_2\colon A \to Y$ are continuous.

19 The Product Topology

Definition 19.1 (Box Topology)

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is the **box topology**.

Definition 19.2 (Projection Mapping)

Let $\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its βth coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta.$$

π_β is the **projection mapping** associated with index β.

Definition 19.3 (Product Topology)

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let S denote the union of these collections, $S = \bigcup_{\beta \in J} \mathcal{S}_\beta$.

The topology generated by the subbasis $\mathcal{S}$ is the **product topology** and the topology $\prod_{\alpha \in J} X_\alpha$ is a **product space**.

Theorem 19.1 (Comparison of Box and Product Topologies)**.** The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α.

The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Remark. For finite products, these topologies are identical. Furthermore, the box topology is in general finer than the product topology.

20 The Metric Topology

Definition 20.1 (Metric)

A **metric** on a set X is a function $d \colon X \times X \to R$ having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds iff $x = y$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

There is also the idea of a **ε-ball centered at x**: $B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ which we have grown to love from our Analysis classes.

Definition 20.2 (The Metric Topology)

If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

Remark. Equivalently, we can think of the metric topology in terms of its open sets: U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Definition 20.3

If X is a topological space, X is **metrizable** if there exists a metric d on a set X that induces the topology of X.

A **metric space** is a metrizable space X together with a specific metric d that gives the topology on X.

Theorem 20.1 (Standard Bounded Metric)**.** Let X be a metric space with metric d. Define $\overline{d}\colon X \times X \to \mathbb{R}$ by the equation

$$\overline{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\overline{d}$ is known as the **standard bounded metric** corresponding to d, and it is a metric that induces the same topology as d.

Lemma 20.1. Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B'_d(x, \delta) \subset B_d(x, \varepsilon).$$

Definition 20.4 (Uniform Metric and Topology)

Given an index set J and points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, the metric $\overline{\rho}$ on $\mathbb{R}^J$ defined by

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$ is known as the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**.

21 The Metric Topology (Continued)

Theorem 21.1. Let $f: X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

Lemma 21.1 (The Sequence Lemma). Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

Theorem 21.2. Let $f: X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Definition 21.1 (Countable Basis, First Countability Axiom)

A space X is said to have a **countable basis at the point** x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains *at least one* of the sets U_n.

A space X that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

Theorem 21.3 (Uniform Limit Theorem). Let $f_n: X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Example. $\mathbb{R}^\omega$ in the box topology is not metrizable.

An uncountable product of $\mathbb{R}$ with itself is not metrizable.

22 The Quotient Topology

Definition 22.1 (Quotient Map)

Let X and Y be topological spaces; let $p: X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Equivalently, p is a quotient map if every subset A of Y is closed in Y if and only if $p^{-1}(A)$ is closed in X.

Definition 22.2

A subset C of X is **saturated** (with respect to the surjective map $p\colon X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the converse inverse image of a subset of Y.

The statement "p is a quotient map" is equivalent to the statement that "p is continuous and maps saturated open sets of X to open sets of Y" (or saturated closed sets of X to closed sets of Y).

Definition 22.3 (Open and Closed Maps)

A map $f\colon X \to Y$ is an **open map** if for each open set U of X, the set $f(U)$ is open in Y.

A map $f\colon X \to Y$ is a **closed map** if for each closed set A of X, the set $f(A)$ is closed in Y.

Remark. Any surjective continuous map $p\colon X \to Y$ that is either open or closed is a quotient map.

However, there are quotient maps that are neither open nor closed.

Definition 22.4 (Quotient Topology)

If X is a space and A is a set and if $p\colon X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map, and this is the **quotient topology** induced by p.

The topology itself consists of subsets U of A such that $p^{-1}(U)$ is open in X.

Definition 22.5 (Quotient Space)

Let X be a topological space, and let X^* be a partition of X into distinct subsets whose union is X. Let $p\colon X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

Theorem 22.1. Let $p\colon X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q\colon A \to p(A)$ be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map.

2. If p is either an open map or a closed map, then q is a quotient map.

Theorem 22.2. Let $p\colon X \to Y$ be a quotient map. Let Z be a space and let $g\colon X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f\colon Y \to Z$ such that $f \circ p = g$.

> The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

3 Chapter 3: Connectedness and Compactness

23 Connected Spaces

Definition 23.1 (Separation and Connected Spaces)
Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

The space is said to be **connected** if there does not exist a separation of X.

Remark. An equivalent formulation of a connected space is that X is a connected space if and only if the subsets of X that are both open and closed in X are the empty set and X itself.

Lemma 23.1. If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of each other. The space Y is connected if there exists no separation of Y.

Example. The rationals are not connected: if Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number $a \in (p, q)$ and write Y as the union of the open sets

$$Y \cap (-\infty, a) \text{ and } Y \cap (a, \infty),$$

thus showing that there exists a separation of Y and proving that Y is not connected.

Consequently, the only connected subspaces of $\mathbb{Q}$ are the one-point sets.

Example. Consider the following subset of the plane $\mathbb{R}^2$:

$$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = \frac{1}{x}\}.$$

Then X is not connected, as neither subset contains a limit point of the other.

Lemma 23.2. If the sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Theorem 23.1. The union of a collection of connected subspaces of X that have a point in common is connected.

Theorem 23.2. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Theorem 23.3. The image of a connected space under a continuous map is connected.

Theorem 23.4. A finite cartesian product of connected spaces is connected.

24 Connected Subspaces of the Real Line

Definition 24.1 (Linear Continuum)

A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 24.1. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Theorem 24.2 (Intermediate Value Theorem)**.** Let $f: X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Definition 24.2

Given points x and y of the space X, a **path** in X from x to y is a continuous map $f: [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Remark. A path-connected space X is connected; suppose that $X = A \cup B$ is a separation of X. Let $f: [a, b] \to X$ be any path in X. The set $f([a, b])$ is the image of a connected set, so it is connected, and must lie entirely in A or B. Thus, there is no path X joining a point of A to a point of B, contradicting the assumption of path connectedness of X.

On the other hand, the converse does not hold.

Example. The ordered square I_o^2 is connected but not path connected.

Example. Let S denote the following subset of the plane:

$$S = \{x \times \sin\left(\frac{1}{x}\right) \mid 0 < x < 1\}.$$

$\overline{S}$ is known as the **topologist's sine curve**, and is not path-connected.

25 Components and Local Connectedness

Definition 25.1 (Components)
Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y.

The equivalence classes are the **components** (or connected components) of X.

Theorem 25.1. The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition 25.2 (Path Components)
Given X, define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y.

The equivalence classes form the **path components**.

Theorem 25.2. The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.

Remark. Each component of X is closed in X, since its closure is also connected. If X has a finite number of components, each component is also open.

On the other hand, the path components of X do not need not be open nor closed in X.

Example. Each component of the subspace $\mathbb{Q}$ of $\mathbb{R}$ is a single point – none of these components are open in $\mathbb{Q}$.

$\overline{S}$ is a space with a single component and two path components (S and $V = 0 \times [-1, 1]$). Note that S in open in $\overline{S}$ but not closed, while V is closed but not open.

Definition 25.3 (Local Connectedness)
A space X is said to be **locally connected at** x if for every neighborhood U of x, there is a

connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be **locally connected**.

Similarly, a space X is said to be **locally path connected at** x if for every neighborhood U of x, there is a path connected neighborhood V of x contained in U. If X is locally path connected at each of its points, it is said simply to be **locally path connected**.

Example. The subspace $[-1,0) \subset (0,1]$ of $\mathbb{R}$ is not connected, but is locally connected.

The topologist's sine curve is connected but not locally connected. The rationals are neither connected nor locally connected.

Theorem 25.3. A space X is locally (path) connected if and only if for every open set U of X, each (path) component of U is open in X.

Theorem 25.4. If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

26 Compact Spaces

Definition 26.1 (Cover and Open Cover)

A collection $\mathcal{A}$ of subsets of a space X **covers** X if the union of the elements of $\mathcal{A}$ is equal to X.

It is an **open cover** if the subsets in $\mathcal{A}$ are open sets.

Definition 26.2 (Compact)

A space X is **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that covers X.

Example. $\mathbb{R}$ is not compact but the subspace

$$X = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}$$

of $\mathbb{R}$ is compact.

Lemma 26.1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Theorem 26.1. Every closed subspace of a compact space is compact.

Theorem 26.2. Every compact subspace of a Hausdorff space is closed.

Lemma 26.2. If Y is a compact subspace of the Hausdorff space X and x_0 in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

Theorem 26.3. The image of a compact space under a continuous map is compact.

Theorem 26.4. Let $f\colon X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Theorem 26.5. The product of finitely many compact spaces is compact.

Proof. Apply tube lemma (below), and take the union of finitely many tubes to be a finite subcollection covering the entire space. □

Lemma 26.3 (Tube Lemma)**.** Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.

Theorem 26.6 (Tychnoff Theorem)**.** The product of infinitely many compact spaces is compact.

Definition 26.3 (Finite Intersection Property)

A collection $\mathcal{C}$ of subsets of X has the **finite intersection property** if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.7. Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets of X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

27 Compact Subspaces of the Real Line

Theorem 27.1 (Extreme Value Theorem)**.** Let $f\colon X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Lemma 27.1 (Lebesgue Number Lemma)**.** Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.

Remark. δ is known as the **Lebsegue number** for the covering $\mathcal{A}$.

Definition 27.1 (Uniform Continuity)

A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is **uniformly continuous** if given $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 27.2 (Uniform Continuity Theorem)**.** Let $f \colon X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Theorem 27.3. Let X be a nonempty compact Hausdorff space. If X has no isolated points (one-point open sets) then X is uncountable.

Remark. It follows that every closed interval in $\mathbb{R}$ is uncountable.

28 Limit Point Compactness

Section skipped.

29 Local Compactness

Definition 29.1 (Locally Compact)

A space X is **locally compact at** x if there is some compact subspace C of X that contains a neighborhood of x.

If X is locally compact at each of its points, X is **locally compact**.

Example. $\mathbb{R}$ is locally compact – any point $x \in (a, b)$ which is contained in the compact subspace $[a, b]$.

Similarly, $\mathbb{R}^n$ is locally compact, but $\mathbb{R}^\omega$ is not.

Theorem 29.1. Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

1. X is a subspace of Y.
2. The set $Y - X$ consists of a single point.
3. Y is a compact Hausdorff space.

Definition 29.2 ((One-point) Compactification)
If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is a **compactification** of X.

If $Y - X$ equals a single point, then Y is the **one-point-compactification** of X (Y is unique up to homeomorphism).

Remark. We have shown that X has a one-point compactification Y if and only if X is a locally compact Hausdorff space that is not itself compact.

Example. The one-point compactification of the real line $\mathbb{R}$ is homeomorphic to the circle; similarly, the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2.

Connection to Complex Analysis: if $\mathbb{R}^2$ is thought of as the space $\mathbb{C}$, then $C \cup \{\infty\}$ is the Riemann sphere, or extended complex plane.

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact if and only if given x in X and a neighborhood U of x, there exist a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

4 Countability and Separation Axioms

30 The Countability Axioms

Definition 30.1 (First Countability Axiom)
A space X is said to have a **countable basis at** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Theorem 30.1. Let X be a topological space.

a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

b) let $f\colon X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the function $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition 30.2 (Second Countability Axiom)
If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Remark. The second axiom implies the first; if $\mathcal{B}$ is a countable basis for X, then the elements of $\mathcal{B}$ containing x form a countable basis at x.

Example. $\mathbb{R}$ and $\mathbb{R}^n$ have countable bases, constructed using the collection of (products) of intervals having rational end points.

Similarly, $\mathbb{R}^\omega$ has a countable basis consisting of all products

$$\prod_{n \in \mathbb{Z}_+} U_n$$

where U_n is an open interval with rational end points for finitely many values of n and $U_n = \mathbb{R}$ for all other values of n.

Theorem 30.2. A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.

The same properties hold for second-countable spaces.

Definition 30.3 (Dense)
A subset A of a space is **dense** in X if $\overline{A} = X$.

Theorem 30.3. Suppose that X has a countable basis. Then

a) Every open covering of X contains a countable subcollection covering X.

b) There exists a countable subset of X that is dense in X.

Definition 30.4
A space for which every open covering contains a countable subcovering is a **Lindelof space**. A space having a countable dense subset is often said to be **separable**.

Each of these properties is equivalent to the second countability axiom when the space is metrizable.

Example. The product of two Lindelof spaces need not be Lindelof.

$\mathbb{R}_l$ is Lindelof, whereas $\mathbb{R}_l^2$, the **Sorgenfrey plane**, is not.

31 The Separation Axioms

As a reminder, a space is *Hausdorff* if for each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.

Definition 31.1 (Regular and Normal Spaces)
Suppose that one-point sets are closed in X. Then X is **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

The space X is **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Remark. A regular space is Hausdorff and a normal space is regular.

These relations constitute the **separation axioms**.

Lemma 31.1. Let X be a topological space. Let one-point sets in X be closed.

a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.

b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

32 Normal Spaces

Theorem 32.1. Every regular space with a countable basis is normal.

Theorem 32.2. Every metrizable space is normal.

Theorem 32.3. Every compact Hausdorff space is normal.

Theorem 32.4. Every well-ordered set X is normal in the order topology.

Remark. A stronger result is that *every* order topology is in fact normal.

33 Urysohn Lemma

Theorem 33.1 (Urysohn Lemma)**.** Let X be a normal space. let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f \colon X \to [a, b]$$

such that $f(x) = a$ for every x in A and $f(x) = b$ for every x in B.

Definition 33.1

If A and B are two subsets of the topological space X and if there is a continuous function $f \colon X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B **can be separated by a continuous function**.

Remark. The Urysohn lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is trivial ($f^{-1}[0, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1]$ are disjoint open sets containing A and B, respectively.

Definition 33.2

A space X is **completely regular** if one-point sets are closed in X and for each point x_0 and each closed set A not containing x_0, there is a continuous function $f \colon X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Remark. A normal space is completely regular by the Urysohn lemma. A completely regular space is regular.

Theorem 33.2. A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

34 The Urysohn Metrization Theorem

Theorem 34.1 (Urysohn Metrization Theorem)**.** Every regular space X with a countable basis is metrizable.

Theorem 34.2 (Imbedding Theorem)**.** Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha \colon X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then the function $F \colon X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$ for each α, then F imbeds X in $[0,1]^J$.

Theorem 34.3. A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.

35 The Tietze Extension Theorem

Theorem 35.1 (The Tietze Extension Theorem)**.** Let X be a normal space; let A be a closed subspace of X.

a) Any continuous map of A into the closed interval $[a,b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a,b]$.

b) Any continuous map A of $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

9 The Fundamental Group

51 Homotopy of Paths

Definition 51.1 (Homotopy)

If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F\colon X \times I \to Y$ such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x)$$

for each x. (Here $I = [0, 1]$.) The map F is called a **homotopy** between f and f'.

If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.

Remark. Think of a homotopy as a continuous one-parameter family of maps from X to Y; the homotopy F represents a continuous "deforming" of the map f to f', as t goes from 0 to 1.

Definition 51.2 (Path Homotopy)

Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be **path homotopic** if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F\colon I \times I \to X$ such that

$$\begin{gathered} F(s, 0) = f(s) \text{ and } F(s, 1) = f'(s), \\ F(0, t) = x_0 \text{ and } F(1, t) = x_1, \end{gathered}$$

for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

Remark. The first condition says that F is a homotopy between f and f', or equivalently, that F representsa continuous way of deforming the path f to the path f'.

The second condition says that for each t, the path f_t definde by $f_t(s) = F(s, t)$ is a path from x_0 to x_1, or equivalently, the endpoints of path remain fixed during the deformation.

Lemma 51.1. The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Example. Let f and g be any two maps of a space X into $\mathbb{R}^2$. The map

$$F(x, t) = (1 - t)f(x) + tg(x)$$

is known as the **straight-line homotopy** between them.

Remark. If f and g are paths from x_0 to x_1, then F is a path homotopy.

More generally, let A be a convex subspace of $\mathbb{R}^n$. Then any two paths f, g in A from x_0 to x_1 are path homotopic in A, as the straight-line homotopy F between them has image set in A.

Definition 51.3 (Product)

If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, the **product** $f * g$ of f and g is the path h given by

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Remark. The function h is well-defined and continuous, by the pasting lemma. It is a path in X from x_0 to x_2.

The product operation on paths induce a well-defined operation on path-homotopy classes: $[f] * [g] = [f * g]$.

Theorem 51.1. The operation $*$ has the following properties:

1. (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.
2. (Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x \colon I \to X$ carrying all of I to the point x. If f is a path from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f].$$

3. (Inverse) Given the path f in X from x_0 to x_1, let $\overline{f}$ be the path defined by $\overline{f}(s) = f(1-s)$. It is the **reverse** of f. Then

$$[f] * [\overline{f}] = [e_{x_0}] \text{ and } [\overline{f}] * [f] = [e_{x_1}].$$

Theorem 51.2. Let f be a path in X and let $a_0, \ldots, a_n$ be numbers such that $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $f_i \colon I \to X$ be the path that equals the positive linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * \cdots * [f_n].$$

52 The Fundamental Group

Definition 52.1 (Algebra Review)

A **homomorphism** $f \colon G \to G'$ is a map such that $f(x \cdot y) = f(x) \cdot f(y)$. (It follows that

$f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$.)

The **kernel** of f is the set $f^{-1}(e')$ and is a subgroup of G. The image is also a subgroup of G.

The homomorphism f is called a **monomorphism** if it is injective (or equivalently, the kernel consists of e alone), an **epimorphism** if it is surjective, and an **isomorphism** if it is bijective.

Definition 52.2 (Quotient Group)

If H is a normal subgroup of G, the operation $(xH)\cdot(yH) = (x\cdot y)H$ is a well-defined operation on G/H that makes it a group. This is known as the **quotient** of G by H.

Remark. The map $f\colon G \to G/H$ mapping x to xH is an epimorphism with kernel H.

Conversely, if $f\colon G \to G'$ is an epimorphism, then its kernel N is a normal subgroup of G and f induces an isomorphism $G/N \to G'$ that carries xN to $f(x)$ for each $x \in G$.

Definition 52.3 (The Fundamental Group)

Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a **loop** based at x_0.

The set of path homotopy classes of loops based at x_0, with the operation $*$, is the **fundamental group** of X relative to the **base point** x_0, and is denoted by $\pi_1(X, x_0)$. This group is also sometimes called the **first homotopy group** of X.

Example. $\pi_1(\mathbb{R}^n, x_0)$ is the trivial group – if f is a loop in $\mathbb{R}^n$ based at x_0, the straight-line homotopy is a path homotopy between f and the constant path at x_0.

If X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial group.

Definition 52.4

Let α be a path in X from x_0 to x_1. Define the map

$$\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha].$$

Theorem 52.1. $\hat{\alpha}$ is a group isomorphism.

Corollary 9.1. If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Definition 52.5 (Simply Connected)

A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial one-element group for some $x_0 \in X$, and hence for every $x_0 \in X$. (The trivial fundamental group is often expressed as $\pi_1(X, x_0) = 0$.)

Lemma 52.1. In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Definition 52.6

Let $h\colon (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

by the equation $h_*([f]) = [h \circ f]$. The map h_* is called the **homomorphism induced by** h, relative to the base point x_0.

Theorem 52.2. If $h\colon (X, x_0) \to (Y, y_0)$ and $k\colon (Y, y_0) \to (Z, z_0)$ is continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i\colon (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Corollary 9.2. If $h\colon (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

53 Covering Spaces

Definition 53.1

Let $p\colon E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into **slices**.

Definition 53.2 (Covering Spaces)

Let $p\colon E \to B$ be a continuous surjective map. If every point b of B has a neighborhood U that is evenly covered by p, then p is a **covering map**, and E is said to be a **covering space** of B.

Example. Let X be any space; let $i\colon X \to X$ be the identity map. Then i is a covering map.

More generally, let E be the space $X \times \{1, 2, \ldots, n\}$ consisting of n disjoint copies of X. The map $p\colon E \to X$ given by $p(x, i) = x$ for all i is a covering map.

Theorem 53.1. The map $p\colon \mathbb{R} \to S^1$ given by

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.

Definition 53.3 (Local homeomorphism)

If $p\colon E \to B$ is a covering map, then p is a **local homeomorphism** of E with B.

That is, each point e of E has a neighborhood that is mapped homeomorphically by p onto an open subset of B.

Theorem 53.2. Let $p\colon E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0\colon E_0 \to B_0$ obtained by restricting p is a coveirng map.

Theorem 53.3. If $p\colon E \to B$ and $p'\colon E' \to B'$ are covering maps, then

$$p \times p'\colon E \times E' \to B \times B'$$

is a covering map.

54 The Fundamental Group of the Circle

Definition 54.1 (Lifting)

Let $p\colon E \to B$ be a map. If f is a continuous mapping of some space X into B, a **lifting** of f is a map $\tilde{f}\colon X \to E$ such that $p \circ \tilde{f} = f$.

Example. Consider the covering $p\colon \mathbb{R} \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$.

The path $f\colon [0,1] \to S^1$ beginning at $b_0 = (1,0)$ given by $f(s) = (\cos(\pi s), \sin(\pi s))$ lifts to the path $\tilde{f}(s) = \frac{s}{2}$ beginning at 0 and ending at $\frac{1}{2}$.

Lemma 54.1. Let $p\colon E \to B$ be a covering map, let $p(e_0) = b_0$. Any path $f\colon [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

Lemma 54.2. Let $p\colon E \to B$ be a covering map, let $p(e_0) = b_0$. Let the map $F\colon I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F}\colon I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Theorem 54.1. Let $p\colon E \to B$ be a covering map, let $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1 and let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

Definition 54.2 (Lifting Correspondence)

Let $p\colon E \to B$ be a covering map and let $b_0 \in B$. Choose e_0 so that $p(e_0) = b_0$. Given an element $[f] \in \pi_1(X, x_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the endpoint of $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi\colon \pi_1(B, b_0) \to p^{-1}(b_0).$$

ϕ is the **lifting correspondence** derived from the covering map p, and depends on the choice of e_0.

Theorem 54.2. Let $p\colon E \to B$ be a covering map, let $p(e_0) = b_0$. If E is path connected then the lifting correspondence

$$\phi\colon \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Theorem 54.3. The fundamental group of S^1 is isomorphic to the additive group of integers.

Theorem 54.4. Let $p\colon E \to B$ be a covering map, let $p(e_0) = b_0$.

a) The homomorphism $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism (injective).

b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi\colon \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

55 Retractions and Fixed Points

Definition 55.1

If $A \subset X$, a **retraction** of X onto A is a continuous map $r\colon X \to A$ such that $r \mid A$ is the identity map of A.

If such a map exists, then A is known as a **retract** of X.

Lemma 55.1. If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j\colon A \to X$ is injective.

Theorem 55.1. There is no retraction of B^2 onto S^1.

Proof. The homomorphism induced by inclusion cannot be injective, as the fundamental group of S^1 is nontrivial and the fundamental group of B^2 is trivial. □

Lemma 55.2. Let $h\colon S^1 \to X$ be a continuous map. Then the following conditions are equivalent:

1) h is nulhomotopic.

2) h extends to a continuous map $k\colon B^2 \to X$.

3) h_* is the trivial homomorphism of fundamental groups.

Theorem 55.2 (Brouwer Fixed-Point Theorem for the Disc)**.** If $f\colon B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.

56 The Fundamental Theorem of Algebra

Theorem 56.1 (Fundamental Theorem of Algebra)**.** A polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

57 The Borsuk-Ulam Theorem

Theorem 57.1 (Borsok-Ulam Theorem for S^2)**.** Given a continuous map $f\colon S^2 \to \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.

Theorem 57.2 (The Bisection Theorem)**.** Given two bounded polygonal regions in $\mathbb{R}^2$, there exists a line in $\mathbb{R}^2$ that bisects both of them.

58 Deformation Retracts and Homotopy Type

Definition 58.1 (Deformation Retracts)

Let A be a subspace of X. A is a **deformation retract** of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy.

This means that there is a continuous map $H: X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$.

The homotopy H is a **deformation retraction** of X onto A. The map $r: X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$ where $j: A \to X$ is inclusion.

Theorem 58.1 (Isomorphism of Fundamental Groups under Deformation Retracts)**.** Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion map

$$j: (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

Example. Let B denote the z-axis in $\mathbb{R}^3$ and consider $\mathbb{R}^3 - B$. It has the punctured xy-plane $(\mathbb{R}^2 - \mathbf{0}) \times 0$ as a deformation retract.

Example. $\mathbb{R}^2 - p - q$, the doubly punctured plane, has the figure eight space as a deformation retract.

Example. Another deformation retraction of the doubly punctured plane $\mathbb{R}^2 - p - q$ is the theta space

$$\theta = S^1 \cup (0 \times [-1, 1]).$$

Definition 58.2 (Homotopy Equivalence)

Let $f: X \to Y$ and $g: Y \to X$ be continuous maps. Suppose that $g \circ f: X \to X$ is homotopic to the identity map of X, and the map $f \circ g: Y \to Y$ is homotopic to the identity map of Y.

Then the maps f and g are **homotopy equivalences** and each is said to be a **homotopy inverse** of the other.

Two spaces that are homotopy equivalent have the same **homotopy type**.

Theorem 58.2. Let $f: X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Remark. The relation of homotopy equivalence is more general than the notion of deformation retraction.

Both the theta space and the figure eight spaces are deformation retracts of the doubly punctured plane, so they are homotopy equivalent to the doubly punctured plane and thus to each other. But neither is homeomorphic to a deformation retract of the other; in fact, neither of them can even be imbedded in each other.