MATH104: Topology **Fall 2023**

Homework 9

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Chapter 11 (The Seifert-van Kampen Theorem) Problems.

<u>Section 70 (The Seifert-van Kampen Theorem), 70.2</u>

Suppose that i_2 is surjective.

a) **Show that j_1 induces an epimorphism**

$$h\colon \pi_1(U, x_0) \to \pi_1(X, x_0),$$

where M is the least normal subgroup of $\pi_1(U, x_0)$ containing $i_1(\ker i_2)$. [*Hint:* Show j_1 is surjective.]

b) **Show that h is an isomorphism. [*Hint:* Use Theorem 70.1 to define a left inverse for h.]**

Hatcher Problem

Let X be the union of n lines through the origin in $\mathbb{R}^3$. Compute the fundamental group of $\mathbb{R}^3 - X$.

Solution. We claim that there is a deformation retract of $\mathbb{R}^3 - X$ to S^2 with $2n$ points removed. To conceptualize this, we describe the deformation retract. For any point of $\mathbb{R}^3 - X$ that is on S^2, then stay as is. Otherwise, for any point y of $R^3 - X$ that is not on S^2, let ℓ be the line passing through the origin and y. Deformation retract y to its closest intersection of ℓ with S^2. Each of the n lines removed passes through S^2 twice, so this description characterizes a deformation retract from $\mathbb{R}^3 - X$ to S^2 with $2n$ points removed.

Through stereographic projection, we know that there is a homeomorphism from S^2 – pt to $\mathbb{R}^2$. By defining one of our $2n$ points removed from S^2 to be the "north pole" and applying the stereographic projection map, we get a homeomorphism from S^2 with $2n$ points removed to $\mathbb{R}^2$ with $2n - 1$ points removed.

The fundamental group of $\mathbb{R}^2$ with $2n - 1$ points removed is simply the fundamental group of the wedge of $2n - 1$ circles, which is the free group with $2n - 1$ generators. ■