The Fundamental Group of Some Surfaces

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Section 60 (The Fundamental Group of Some Surfaces)

Theorem (60.1)**.** *$\pi_1(X \times Y, x_0, y_0)$ is isomorphic to $\pi_1(X, x_0) \times \pi_2(Y, y_0)$.*

Proof. We can consider their projective mappings and thus the induced homomorphisms $p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0)$ and $q_x : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0)$. This allows us to define a homomorphism $\phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ with $\phi([f]) = p_*([f]) \times q_*([f]) = [p \circ f] \times [q \circ f]$. The homomorphism is surjective since $f = g \times h$ implies that $\phi([f]) = [g] \times [h]$ for any arbitrary loops g and h starting at x_0 and y_0, respectively. Now let $\phi([f])$ be the identity element, where f is a loop based at $x_0 \times y_0$, then we know that $[p \circ f]$ and $[q \circ f]$ are path homotopic to the constant loops at x_0 and y_0 under G and H, respectively. Then, $F = G \times H$ is a path homotopy between f and the constant loop based at $x_0 \times y_0$. □

Corollary (60.2)**.** *The fundamental group of the torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

Definition (Projective Plane)**.** *The* ***projective plane*** *P^2 is the quotient space obtained from S^2 by identifying each point x of S^2 with its antipodal point $-x$.*

Remark. *The above definition can be generalized to define the* ***projective n-space****, obtained by identifying points with their antipodes in S^n.*

An equivalent definition is that the projective plane is the quotient of $\mathbb{R}^3 - 0$ under the equivalence relation $a \sim ta$ for $t \neq 0$. (so the projective plane can be thought of as "the set of lines in $\mathbb{R}^3 - 0$ passing through the origin.")

Definition. *A* ***surface*** *is a Hausdorff space with a countable basis, each point of which has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^2$.*

Theorem (60.3)**.** *The projective plane P^2 is a compact surface, and the quotient map $p\colon S^2 \to P^2$ is a covering map.*

Proof Outline. p is an open map: let U be an open set in S^2. Then the antipodal map $a\colon S^2 \to S^2$ given by $a(x) = -x$ is a homeomorphism of S^2, so $p^{-1}(p(U)) = U \cup a(U)$ is open, as it is a union of open sets. By definition, this tells us that the quotient map p is open.

p is a covering map: given $y \in P^2$, choose $x \in p^{-1}(y)$, and let U be an ϵ-neighborhood of x in S^2 for $\epsilon < 1$. Then U has no pair of antipodal points, so $p\colon U \to p(U)$ and $p\colon a(U) \to p(a(U)) = p(U)$ are homeomorphisms. Then $p^{-1}(p(U)) = U \cup a(U)$, which are disjoint open sets that are mapped homeomorphically to $p(U)$ under p, from which it follows that $p(U)$ is a neighborhood of P^2 that is evenly covered by p.

P^2 is Hausdorff: let y_1 and y_2 be distinct points in P^2. Then $p^{-1}(y_1) \cup p^{-1}(y_2)$ consists of four points; let 2ϵ be the minimum distance between them, and let U_1 be the ϵ-neighborhood of one point of $p^{-1}(y_1)$ and let U_2 be the ϵ-neighborhood of one point of $p^{-1}(y_2)$. Then $U_1 \cup a(U_1)$ and

$U_2 \cup a(U_2)$ are disjoint, and so $p(U_1)$ and $p(U_2)$ are disjoint neighborhoods of y_1 and y_2 in P^2.

P^2 has a countable basis: S^2 has a countable basis $\{U_n\}$ so P^2 has the countable basis $\{p(U_n)\}$.

Since S^2 is a compact surface, and every point of P^2 is homeomorphic to an open subset of S^2, P^2 is also a compact surface. $\square$

Remark. *We can also use Section 31 Exercise 6. We would need to show that p is closed (which can be done similar to how we've shown p is open). Since p is a covering map, though, it is also continuous and subjective. Furthermore, S^2 is normal, so by the exercise, P^2 is normal and in turn Hausdorff.*

Corollary. $\pi_1(P^2, y)$ *is a group of order* 2.

Proof. Since S^2 is simply connected, Theorem 54.4 tells us that there is a bijective correspondence between $\pi_1(P^2, y)$ and $p^{-1}(y)$, a two element set. Thus, $\pi_1(P^2, y)$ is a group of order 2, i.e. isomorphic to $\mathbb{Z}/2\mathbb{Z}$. $\square$