Homework 7

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Chapter 9 (The Fundamental Group) and Chapter 10 (Separation Theorems in the Plane) Problems.

Section 60 (Fundamental Groups of Some Surfaces), 60.2

Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to the projective plane P^2.

Solution. [1] Let π represent the quotient map from B^2 to X defined by identifying each point x of S^1 with its antipode. Consider the projection of the upper hemisphere of S^2 onto B^2, a homemomorphism. Let h represent the inverse of this projection; h is a homeomorphism from B^2 to S^2 mapping the unit disk to the upper hemisphere. Finally, let p represent the canonical quotient map from S^2 to P^2, defined by identifying each point of S^2 with its antipode.

Note that by construction, $p \circ h \colon B^2 \to P^2$ is a map that is constant on each set $\pi^{-1}(\{x\})$ for each $x \in X$: if $x \in S^1$, then $\pi^{-1}(\{x\}) = \{x, -x\}$ which is mapped to one point by $p \circ h$. On the other hand, if $x \notin S^1$, then $\pi^{-1}(x)$ is a one-point set, and is consequently mapped to a constant by $p \circ h$. Since π is a quotient map from B^2 to X, we know by Theorem 22.2 that $p \circ h$ induces a map $f \colon X \to P^2$ such that $f \circ \pi = p \circ h$. It remains to show that f is a homeomorphism.

Note that $X = \{(p \circ h)^{-1}(\{z\}) \mid z \in P^2\}$. We show containment in both directions. Suppose that $x \in X$. Then $x \in \{(p \circ h)^{-1}(\{z\}) \mid z \in P^2\}$ as $(p \circ h)^{-1}((p \circ h)(x)) = x$, where $(p \circ h)(x)$ is a singular point in P^2. On the other hand, suppose that $y \in \{(p \circ h)^{-1}(\{z\}) \mid z \in P^2\}$. Then $y = \pi(y) \in X$. It follows that $X = \{(p \circ h)^{-1}(\{z\}) \mid z \in P^2\}$.

We also claim that $p \circ h$ is a quotient map. Note that p and h are both continuous maps, and so their composition is continuous. p is a quotient map from S^2 to P^2, so it is surjective. Also, h is a homeomorphism from the unit disk to the upper hemisphere, so it is by definition a bijection, and in turn, is surjective. The composition of two surjective maps p and h must be surjective, so $p \circ h$ is also surjective. Finally, p is a quotient map so both p and p^{-1} map open sets to open sets. Furthermore, h is a homeomorphism so both h and h^{-1} are continuous, and map open sets to open sets. Thus, it follows that a subset U of P^2 is open if and only if $(p \circ h)^{-1}(U)$ is open in B^2. Since $p \circ h$ is also both continuous and surjective, it follows by definition that $p \circ h$ is a quotient map from B^2 to P^2.

Thus, by Corollary 22.3, since $p \circ h$ is a quotient map (and so it is both surjective and continuous), the induced continuous map $f \colon X \to P^2$ is a homeomorphism. Thus, X is homeomorphic to P^2, as desired. ■

[1] constructed with Hillary's help.

Section 63 (The Jordan Curve Theorem), 63.1

Let C_1 and C_2 be disjoint simple closed curves in S^2.

a) **Show that $S^2 - C_1 - C_2$ has precisely three components. [*Hint:* If W_1 is the component of $S^2 - C_1$ disjoint from C_2, and if W_2 is the component of $S^2 - C_2$ disjoint from C_1, show that $\overline{W}_1 \cup \overline{W}_2$ does not separate S^2.]**

Solution. [2] By the Jordan Curve Theorem, C_1 separates S^2 into two components, which we will denote as W_1 and V_1. Similarly, by the Jordan Curve Theorem, C_2 separates S^2 into two components, which we will denote as W_2 and V_2. Furthermore, define W_1 and W_2 to be the components matching the hint: W_1 is the component of $S^2 - C_1$ disjoint from C_2 and W_2 is the component of $S^2 - C_2$ disjoint from C_1.

Note that since C_1 and C_2 are disjoint, it follows that $W_1 \subset V_2$ and $W_2 \subset V_1$, with $W_1 \cap W_2 = \varnothing$. Consequently, W_1 and W_2 are two components of $S^2 - C_1 - C_2$. Note that by construction, $V_1 = S^2 - \overline{W}_1$, and $V_2 = S^2 - \overline{W}_2$. Furthermore, C_1 and C_2 are disjoint, so $\overline{W}_1 \cap \overline{W}_2 = \varnothing$, and $S^2 - (\overline{W}_1 \cap \overline{W}_2) = S^2$ is simply connected. Finally, neither $\overline{W}_1$ nor $\overline{W}_2$ separate S^2, so by Theorem 63.3 (the general nonseparation theorem), $\overline{W}_1 \cup \overline{W}_2$ does not separate S^2. Thus, the third component of $S^2 - C_1 - C_2$ is precisely $\overline{W}_1 \cup \overline{W}_2 = V_1 \cap V_2$.

We conclude that $S^2 - C_1 - C_2$ has three components: W_1, W_2, and $V_1 \cap V_2$. ■

b) **Show that these three components have boundaries C_1 and C_2 and $C_1 \cup C_2$, respectively.**

Solution. [3] By construction, the boundaries of W_1 and W_2, two of the components of $S^2 - C_1 - C_2$, are C_1 and C_2, respectively. It remains to show that the boundary of $V_1 \cap V_2$ is $C_1 \cup C_2$. Since $V_1 = S^2 - \overline{W}_1$, and $V_2 = S^2 - \overline{W}_2$, we have that

$$\begin{aligned}
\overline{V}_1 \cap \overline{V}_2 - V_1 \cap V_2 &= ((S^2 - W_1) \cap (S^2 - W_2)) - ((S^2 - \overline{W}_1) \cap (S^2 - \overline{W}_2)) \\
&= (\overline{W}_1 \cup \overline{W}_2) - (W_1 \cup W_2) \\
&= (\overline{W}_1 - W_1) \cup (\overline{W}_2 - W_2) \\
&= C_1 \cup C_2.
\end{aligned}$$

Thus, the boundary of $V_1 \cap V_2$ is by definition $C_1 \cup C_2$, as desired. ■

[2]constructed with Hillary's help.
[3]constructed with Hillary's help.