MATH104: Topology **Fall 2023**

Homework 11

David Yang

Chapter 13 (Classification of Covering Spaces) Problems.

Section 79 (Equivalence of Covering Spaces), 79.5(b)

Let $T = S^1 \times S^1$ be the torus; let $x_0 = b_0 \times b_0$. Prove the following:

Theorem. If E is a covering space of T, then E is homeomorphic either to $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T.

(Hint: You may use the following result from algebra: if F is a free abelian group of rank 2 and N is a nontrivial subgroup, then there is a basis a_1, a_2 for F such that either (1) ma_1 is a basis for N, for some positive integer m, or (2) ma_1, na_2 is a basis for N, where m and n are positive integers.)

Solution. Let E be a covering space of T, with p representing the covering map from E to T. Pick some $e_0 \in p^{-1}(x_0)$. Since $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$, and $p_*(\pi_1(E, e_0))$ must be a subgroup of $\mathbb{Z} \times \mathbb{Z}$, a free abelian group of rank 2, we know that $p_*(\pi_1(E, e_0))$ is either the trivial subgroup $\{(0, 0)\}$, the subgroup generated by ma_1, or the subgroup generated by ma_1 and na_2, where m and n are positive integers and a_1 and a_2 represent some basis for $\mathbb{Z} \times \mathbb{Z}$. [1]

Since there is an isomorphism of $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ mapping the respective basis elements a_1 and a_2 to the respective canonical basis elements $(1, 0)$ and $(0, 1)$, we know by Exercise 79.5(a), that this isomorphism is induced by a homeomorphism $f : T \to T$ mapping x_0 to x_0. Consider now the map $q : E \to T$ defined by $q = f \circ p$. Since f is the composition of a covering map and a homeomorphism, it is itself a covering map.[2]

Consider now $q_*(\pi_1(E, e_0)) = (f_* \circ p_*)(\pi_1(E, e_0))$. Note that p_* is a map from $\pi_1(E, e_0)$ onto one of the three classified subgroups $\{0, 0\}$, $\langle ma_1 \rangle$, or $\langle ma_1, na_2 \rangle$ of $\pi_1(T, x_0)$. Then f_* maps these subgroups to the respective subgroups in the canonical basis of $\pi_1(T, x_0)$, represented as the trivial subgroup $\{0, 0\}$, $\langle (m, 0) \rangle$, or $\langle (m, 0), (0, n) \rangle$ of $\pi_1(T, x_0)$. These represent our three possible images of $q_*(\pi_1(E, e_0))$.

Recall from Example 1 that the covering spaces of S^1 are $\mathbb{R}$ and S^1; the respective covering maps include $p_m : S^1 \to S^1$ with $p_m(z) = z^m$ for any positive integer m, and $q : \mathbb{R} \to S^1$ with $q(t) = (\cos(2\pi t), \sin(2\pi t))$. By Theorem 53.3, since the product of covering maps is a covering map, we can form three distinct covering maps p'_1, p'_2, and p'_3 each of which map from some space E' to T. Let $e'_0 = p'^{-1}(x_0)$.

We see that

[1]the form for the latter two nontrivial subgroups follows directly from the hint.

[2]alternatively, f is a 1-fold covering map, and by Exercise 54.4, since both f and p are covering maps and $f^{-1}(x)$ is finite for each $x \in T$, q is also a covering map.

1. $p'_1\colon S^1 \times S^1 \to S^1 \times S^1$ defined by $p'_1 = p_m \times p_n$, with $p'_1(E', e'_0) = \langle (m, 0), (0, n) \rangle$
2. $p'_2\colon S^1 \times \mathbb{R} \to S^1 \times S^1$ defined by $p'_2 = p_m \times q$, with $p'_2(E', e'_0) = \langle (m, 0) \rangle$
3. $p'_3\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ defined by $p'_3 = q \times q$, with $p'_3(E', e'_0) = \{(0, 0)\}$

are three distinct covering maps of $T = S^1 \times S^1$. Note that these covering maps $p'_1, p'_2, p'_3\colon E' \to T$ induce the same images in $\pi_1(T, x_0)$ as the three distinct possibilities for the image of the induced homomorphism $q_*(\pi_1(E, e_0))$ in $\pi_1(T, x_0)$.

Thus, by Theorem 79.2, there must be an equivalence between E and the spaces of E', which were homeomorphic to $S^1 \times S^1$, $S^1 \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{R}$. It follows that if E is a covering space of T, then E is homeomorphic either to $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T, as desired. ■

Section 80 (The Universal Covering Space), 80.1(a)

Let $q\colon X \to Y$ and $r\colon Y \to Z$ be maps; let $p = r \circ q$.

a) **Let q and r be covering maps. Show that if Z has a universal covering space, then p is a covering map. (Compare Exercise 4 of Section 53.)**[3]

Solution. Let E be the universal covering space of Z; E is simply connected. By definition, there is a covering map $s\colon E \to Z$. By Theorem 80.3, since $s\colon E \to Z$ and $r\colon Y \to Z$ are covering maps, there exists a covering map $t\colon E \to Y$ such that $s = r \circ t$. Furthermore, by Theorem 80.3, since $t\colon E \to Y$ and $q\colon X \to Y$ are covering maps, there exists a covering map $u\colon E \to X$ such that $t = q \circ u$.

Note that $u\colon E \to X$ and $s\colon E \to Z$ are covering maps. Furthermore, we have that

$$p \circ u = (r \circ q) \circ u = r \circ (q \circ u) = r \circ t = s$$

by construction. By Lemma 80.2(b), since $s = p \circ u$, and u and s are covering maps, then so is p. Thus, if q and r are covering maps, and if Z has a universal covering space, then p is a covering map. ∎

[3]We replace the requirement that $r^{-1}(z)$ is finite for each $z \in Z$ with the condition that Z has a universal covering space to get that p is a covering map.

Let $p\colon X \to B$ be a covering map (not necessarily regular); let G be its group of covering transformations.

a) **Show that the action of G on X is properly discontinuous.**

Solution. Let $x \in X$, so that $p(x) \in B$. Since p is a covering map, there exists a neighborhood V of $p(x)$ in B that is evenly covered by p. Equivalently, the inverse image $p^{-1}(V)$ is a disjoint union of neighborhoods in U_α in X such that the restriction of p to U_α is a homeomorphism of U_α onto V, for each α. Let U be the neighborhood of $p^{-1}(V)$ in X containing x. We will show that $g(U)$ and U are disjoint, for any $g \neq e$.

Suppose for the sake of contradiction that there exists some element $y \in g(U) \cap U$. Then $y \in U$ and $y = g(z)$ for some $z \neq y$ in U (as a non-identity covering transformation has no fixed points). Since g is a covering transformation, it follows that $p \circ g = p$, so $p(g(z)) = p(z)$. Simplifying, we get that $p(y) = p(z)$, where both y and z are in U and $y \neq z$. The restriction of p to U cannot be a homeomorphism of U onto V, as it is not injective; thus, we arrive at a contradiction, and conclude that $g(U)$ and U are disjoint for every $g \neq e$ in G.

By definition, it follows that the action of G on X is properly discontinuous. ∎

Section 81 (Covering Transformations), 81.4

Let G be a group of homeomorphisms of X. The action of G on X is said to be fixed-point free if no element of G other than the identity e has a fixed point. Show that if X is Haussdorf, and if G is a finite group of homeomorphisms of X whose action is fixed-point free, then the action of G is properly discontinuous.

Solution. Let $x \in X$. Let $\{g_1, \dots, g_n\}$ be the finite group of homeomorphisms in G of X that are not equal to the identity. Since the action of G on X is fixed-point free, it follows that $g_i(x) \neq x$ for all $i \in \{1, \dots, n\}$. Since x and $g_i(x)$ are distinct points in X and X is Hausdorff, it follows that there are disjoint open sets V_i and W_i about x and $g(x)$ that are disjoint in X for each $i \in \{1, \dots, n\}$.

Let $\tilde{V} = \bigcap_{i \in \{1,\dots,n\}} V_i$; by construction, $\tilde{V}$ is the intersection of finitely many open sets, so it is itself open. Consider

$$U = \bigcap_{i \in \{1,\dots,n\}} g_i^{-1}(W_i) \cap \tilde{V}$$

Since each g_i is a homeomorphism, they are each continuous, and so each $g_i^{-1}(W_i)$ is open, and by construction contains x. Consequently, each $g_i^{-1}(W_i) \cap \tilde{V}$ is open, and so U, the intersection of finitely many such open sets, is also open. Furthermore, U is nonempty, as it contains x by construction. Consider $U = g^{-1}(W) \cap V$.

Note that for each g_i, $U \subseteq V_i$, and $U \subseteq g_i^{-1}(W_i)$, so $g_i(U) \subseteq W_i$. Since each V_i and W_i are disjoint open sets by construction, U and $g_i(U)$ must be disjoint open sets, for all $i \in \{1, \dots, n\}$.

Since for every $x \in X$, there is a neighborhood U of x such that for all g_i in the finite group of homeomorphisms of X that are not the identity, $g_i(U)$ is disjoint from U, the action of G is properly discontinuous, as desired. ■