**MATH104: Topology** **Spring 2024**

# The Order Topology and the First Uncountable Ordinal

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## 1 Introduction

**Definition 1.1** (Order Topology)

Let $X$ be a set with a simple order relation and assume that $X$ has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals $(a, b)$ in $X$.
2. All intervals of the form $[a_0, b)$ where $a_0$ is the smallest element (if any) of $X$.
3. All intervals of the form $(a, b_0]$ where $b_0$ is the largest element (if any) of $X$.

The collection $\mathcal{B}$ is a basis for a topology on $X$, known as the **order topology**.

*If $X$ has no smallest element, there are no sets of type 2, and if $X$ has no largest element, there are no sets of type 3.*

**Lemma 1.1** (Lemma 13.1, page 80)**.** Let $X$ be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on $X$. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

**Lemma 1.2** (Lemma 10.2, page 66)**.** There exists a well-ordered set $A$ having a largest element $\Omega$, such that $S_\Omega$ of $A$ by $\Omega$ is uncountable but every other section of $A$ is uncountable.

We can actually construct such a well-ordered set! Assuming the existence of an uncountable well-ordered set $B$ (a weaker result following from the axiom of choice), define $C$ to be the well-ordered set $\{1, 2\} \times B$ in the dictionary order; a section of $C$ is uncountable (for example, the section of $C$ by any element of the form $2 \times b$). Let $\Omega$ be the smallest element of $C$ for which the section of $C$ by $\Omega$ is uncountable, and let $A$ be this section along with $\Omega$.

**Definition 1.2** (Minimal Uncountable Well-Ordered Set)

$S_\Omega$ is an uncountable well-ordered set, every section of which is countable; it is known as the **minimal uncountable well-ordered set**.

Alternatively, $S_\Omega$ is known as the **first uncountable ordinal**, and the closure of $\overline{S}_\Omega$ is $\overline{S}_\Omega = S_\Omega \cup \{\Omega\}$.

## 2 The Order Topology of the First Uncountable Ordinal

- Sets of type 2 in the order topology can be thought of as sections of $S_\Omega$, and are countable.
- By problem 10.6(a) on page 67, $S_\Omega$ has no largest element, so no set of type 3 exists in $S_\Omega$.
- Since $S_\Omega$ is well-ordered, any nonempty subset has to have a minimal element; consequently, all basis elements in the order topology can be written in the form $[a, b)$.
- We can characterize all open sets of $S_\Omega$: they must be unions of our basis elements. These look like unions of disjoint intervals of open sets, or are of the form of our basis elements.