MATH104: Topology **Fall 2023**

Homework 3

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Chapter 3 (Connectedness and Compactness) Problems.

Section 22 (Connected Spaces), 22.4(a)

Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_0 \times y_1 \text{ if } x_0 + y_0^2 = x_1 + y_1^2.$$

Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it? [*Hint:* Set $g(x \times y) = x + y^2$.]

Solution. We will show that X^* is homeomomorphic to $\mathbb{R}$. We define, as per the hint, the continuous map $g \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x \times y) = x + y^2$. Let us define the continuous map $f \colon \mathbb{R} \to \mathbb{R}^2$ by $f(x) = x \times 0$. Note that

$$(g \circ f)(x) = g(f(x)) = g(x \times 0) = x,$$

for any $x \in \mathbb{R}$, so $g \circ f$ is the identity map of $\mathbb{R}$. By Exercise 22.2(a), it follows that g is a quotient map from $\mathbb{R}^2$ to $\mathbb{R}$, or equivalently, from X to R.

Define $p \colon X \to X^*$ to be the surjective map mapping each point in the plane X to its equivalence class. Equivalently, since g is a map from X to $\mathbb{R}$, we can write the quotient space as

$$X^* = \{g^{-1}(\{r\}) \mid r \in \mathbb{R}\},$$

matching the definition in Corollary 22.3. By Corollary 22.3(a), since g is a quotient map, it induces a homeomorphism from the quotient space X^* to $\mathbb{R}$.

Thus, we conclude that X^* is homeomorphic to $\mathbb{R}$. ∎

Section 26 (Compact Spaces), 26.5

Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

Solution. Let a be an element of A. Since A and B are disjoint, it follows that a is not in B. By Lemma 26.4, since B is a compact subspace of the Hausdorff space X and $a \notin B$, we know that there exist disjoint open sets U_a and V_a containing a and B, respectively.

Consider the union of each of these respective open sets U_a across all elements of a in A:

$$\tilde{U} = \bigcup_{a \in A} U_a.$$

By construction, $\tilde{U}$ is a union of open sets that contains every element in A, so it is an open covering of A. Since A is compact, $\tilde{U}$ must have a finite subcover consisting of open sets $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$ where each $a_1, \ldots, a_n$ is a distinct element of A. Similarly, consider the corresponding finite collection of open sets $V_{a_1}, V_{a_2}, \ldots, V_{a_n}$ in B defined for those same elements $a_1, \ldots, a_n$.

We define

$$U = \bigcup_{i \in \{1,\ldots,n\}} U_{a_i}$$

and

$$V = \bigcap_{i \in \{1,\ldots,n\}} V_{a_i}.$$

Note that U is by construction a union of open sets in A covering A – a finite subcover of A – so it is open and contains A. Similarly, V is an intersection of finitely many open sets in B, so it is also open. Furthermore, since each open set V_{a_i} contains B by construction, the intersection V also contains B.

It remains to show that U and V are disjoint. Indeed, suppose that $x \in U \cap V$. It follows that $x \in U$ so $x \in U_{a_j}$ for some j. Since $x \in V$, x is in every V_{a_i} for i from 1 to n. Consequently, $x \in V_{a_j}$. It follows that $x \in U_{a_j} \cap V_{a_j}$ for the same j, contradicting the disjoint property each pair of corresponding open sets given by Lemma 26.4.

Thus, $U = \bigcup_{i \in \{1,\ldots,n\}} U_{a_i}$ and $V = \bigcap_{i \in \{1,\ldots,n\}} V_{a_i}$ are disjoint open sets of X containing A and B, as desired. ■

Section 24 (Connected Subspaces of the Real Line), 24.2

Let $f\colon S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Solution. Let us define $g\colon S^1 \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$. Since $f(x)$ and $f(-x)$ are continuous, so is $g(x)$, as it is a difference of two continuous functions. Furthermore, note that

$$g(x) = f(x) - f(-x) \text{ and } g(-x) = f(-x) - f(x) = -g(x)$$

for any $x \in S^1$. Consider some y in S^1. We can assume that $f(y) \neq f(-y)$, otherwise we would be done. Equivalently, $g(y) \neq 0$. Suppose without loss of generality that $g(y) > 0$. It follows that $g(-y) = -g(y) < 0$.

Since g is a continuous map from the connected space S^1 to the ordered set $\mathbb{R}$ in the order topology, and 0 is a point of $\mathbb{R}$ between $g(-y)$ and $g(y)$, we know by the Intermediate Value Theorem that there must be some point x of S^1 such that $g(x) = 0$. It follows that for that point x, $f(x) = f(-x)$, as desired. ■

Section 24 (Connected Subspaces of the Real Line), 24.3

Let $f\colon X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f. What happens if X equals $[0, 1)$ or $(0, 1)$?]

Solution. Let $X = [0, 1]$. Let us define $g\colon X \to [-1, 1]$ by $g(x) = f(x) - x$. To show that f has a fixed point, it suffices to show that g has a zero in X. Note that g is continuous, as it is a difference of continuous functions $f(x)$ and x.

Assume without loss of generality that $f(0) \neq 0$ and $f(1) \neq 1$; otherwise, 0 or 1 would be a fixed point for f. Since $f\colon [0, 1] \to [0, 1]$ It follows that

$$g(0) = f(0) - 0 > 0 \text{ and } g(1) = f(1) - 1 < 0.$$

Since g is a continuous map from a connected space $[0, 1]$ to the ordered set $[-1, 1]$ and $g(0) < 0 < g(1)$, it follows by the Intermediate Value Theorem that there must exist some point $x \in (0, 1)$ such that $g(x) = 0$.

By construction, this point x in X satisfies $f(x) - x = 0$, and so it is a fixed point of f.

Note that f does not necessarily have a fixed point if X equals $[0, 1)$ or $(0, 1)$. In the former case, consider the function

$$f\colon [0, 1) \to [0, 1) \text{ defined by } f(x) = \frac{x}{2} + \frac{1}{2}.$$

This has no fixed point in $[0, 1)$: $\frac{x}{2} + \frac{1}{2} = x$ if and only if $x = 1 \notin [0, 1)$. Similarly, in the latter case, consider

$$f\colon (0, 1) \to (0, 1) \text{ defined by } f(x) = \frac{x}{2}.$$

This has no fixed point in $(0, 1)$: $\frac{x}{2} = x$ if and only if $x = 0 \notin (0, 1)$. ■