MATH104: Topology **Fall 2023**

Homework 13

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Assorted Topological Groups Problems.

Section 26 (Compact Spaces), 26.13(a)(b)

Let G be a topological group.

1. **Let A and B be subspaces of G. If A is closed and B is compact, show $A \cdot B$ is closed. [*Hint:* If c is not in $A \cdot B$, find a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.]**

   Solution. To show that $A \cdot B$ is closed, we will show that $(A \cdot B)^C$ is open. Let $c \in (A \cdot B)^C$. As the hint says, we will show that there is a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.

   Since G is a topological group, it follows from Exercise 145.1 that the map $f \colon G \times G \to G$ sending (x, y) to xy^{-1} is continuous. A is closed, so A^C is open. Since f is continuous, the preimage of an open set is open, and so $f^{-1}(A^C)$ is open. The one-point set $\{c\}$ and B are each compact. We claim that $\{c\} \times B \subseteq f^{-1}(A^C)$. Equivalently, we will show that $f(\{c\} \times B) \subseteq A^C$. Since $c \in (A \cdot B)^C$, or equivalently, $c \notin A \cdot B$, it follows that $c \neq ab$ for any $a \in A$, $b \in B$. Equivalently, $cb^{-1} \neq a$ for any $a \in A$ and $b \in B$. Since $f(\{c\} \times B)$ is defined as $\{cb^{-1} \mid b \in B\}$, it follows that $f(\{c\} \times B)$ and A are disjoint, or equivalently, $f(\{c\} \times B) \subseteq A^C$.

   To summarize, we have that $f^{-1}(A^C)$ is an open set in $G \times G$, and $\{c\}$ and B are each compact in G, with $\{c\} \times B \subseteq f^{-1}(A^C)$. It follows by Exercise 26.9 that there exist open sets W and V in G such that
   $$\{c\} \times B \subseteq W \times V \subseteq f^{-1}(A^C).$$
   Note that $c \in W$ by construction, and $W \times B \subseteq W \times V$. Consequently, since $W \times B \subseteq W \times V \subseteq f^{-1}(A^C)$, it follows that $f(W \times B) \subseteq A^C$, or equivalently
   $$W \cdot B^{-1} \subseteq A^C.$$
   It follows that $(W \cdot B^{-1}) \cap A = \varnothing$. Consequently, $wb^{-1} \neq a$ for any $w \in W, a \in A$ and $b \in B$. Equivalently, $w \neq ab$, so $W \cap (A \cdot B) = \varnothing$.

   For any arbitrary $c \in (A \cdot B)^C$, we have found a neighborhood W of c such that $W \cap (A \cdot B) = \varnothing$, or equivalently, $W \subseteq (A \cdot B)^C$. It follows by definition that $(A \cdot B)^C$ is open in G, so $A \cdot B$ is closed, as desired. ■

2. **Let H be a subgroup of G; let $p \colon G \to G/H$ be the quotient map. If H is compact, show that p is a closed map.**

   Solution. ■

<u>Section 52 (The Fundamental Group), 52.7(c)(d)</u>

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

a) **Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]**

b) **Show that $\pi_1(G, x_0)$ is abelian.**

Solution. ■