MATH104: Topology **Fall 2023**

Homework 5

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Chapter 9 (The Fundamental Group) Problems.

Section 53 (Covering Spaces), 53.3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{b}$.

Solution. We begin with the forward implication. Let $[f] \in \pi_1(X, x_0)$ and let α and β be two paths from x_0 to x_1. Since $[\bar{\alpha}] * [\alpha] = e_{x_1}$ and $[\beta] * e_{x_1} * [\bar{\beta}] = e_{x_0}$, it follows that

$$[f] = [\beta] * [\bar{\alpha}] * [\alpha] * [\bar{\beta}] * [f].$$

Equivalently,

$$[f] = [\beta * \bar{\alpha}] * [\alpha * \bar{\beta}] * [f].$$

Furthermore, note that $[\alpha * \bar{\beta}]$ is a loop based at x_0, so it is in $\pi_1(X, x_0)$, which is abelian. Consequently, $[\alpha * \bar{\beta}]$ commutes with $[f]$, so $[\alpha * \bar{\beta}] * [f] = [f] * [\alpha * \bar{\beta}]$. This gives us

$$\begin{aligned} [f] &= [\beta * \bar{\alpha}] * [\alpha * \bar{\beta}] * [f] \\ &= [\beta * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] \\ &= [\beta] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\beta}] \end{aligned}$$

Finally, multiplying both sides by $[\bar{\beta}]$ on the left and by $[\beta]$ on the right and simplifying, we get that

$$[\bar{\beta}] * [f] * [\beta] = [\bar{\alpha}] * [f] * [\alpha]$$

and this is equivalent to

$$\hat{\alpha}([f]) = \hat{\beta}([f]).$$

We conclude that for any two paths α and β from x_0 to x_1, $\hat{\alpha} = \hat{\beta}$, as desired.

It remains to show the reverse implication. Suppose that for any two paths α and β of paths from x_0 to x_1, $\hat{\alpha} = \hat{b}$. Let $[f_1]$ and $[f_2]$ be distinct path homotopy classes in $\pi_1(X, x_0)$. To show that $\pi_1(X, x_0)$ is abelian, we will show that $[f_1] * [f_2] = [f_2] * [f_1]$.

Note that $f_1 * \alpha$ and $f_2 * \alpha$ are two paths from x_0 to x_1, so we know that $\widehat{f_1 * \alpha} = \widehat{f_2 * \alpha}$. It follows that $\widehat{f_1 * \alpha}([f_1]) = \widehat{f_2 * \alpha}([f_1])$, so

$$[\overline{f_1 * \alpha}] * [f_1] * [f_1 * \alpha] = [\overline{f_2 * \alpha}] * [f_1] * [f_2 * \alpha].$$

Simplifying, we have that

$$\begin{aligned} & [\overline{f_1 * \alpha}] * [f_1] * [f_1 * \alpha] = [\overline{f_2 * \alpha}] * [f_1] * [f_2 * \alpha] \\ \implies & [\bar{\alpha} * \bar{f}_1] * [f_1] * [f_1 * \alpha] = [\bar{\alpha} * \bar{f}_2] * [f_1] * [f_2 * \alpha] \\ \implies & [\bar{\alpha}] * [\bar{f}_1] * [f_1] * [f_1] * [\alpha] = [\bar{\alpha}] * [\bar{f}_2] * [f_1] * [f_2] * [\alpha]. \end{aligned}$$

Multiplying both sides on the left by $[\alpha]$ and on the right by $[\bar{\alpha}]$ and simplifying further, we have that

$$[\bar{\alpha}] * [\bar{f_1}] * [f_1] * [f_1] * [\alpha] = [\bar{\alpha}] * [\bar{f_2}] * [f_1] * [f_2] * [\alpha].$$
$$\implies [f_1] = [\bar{f_2}] * [f_1] * [f_2].$$

Finally, multiplying both sides on the left by $[f_2]$, we get that

$$[f_2] * [f_1] = [f_1] * [f_2]$$

as desired. Thus, we conclude that for any two elements $[f_1]$ and $[f_2]$ in $\pi_1(X, x_0)$, $[f_1] * [f_2] = [f_2] * [f_1]$, so $\pi_1(X, x_0)$ is abelian, as desired. ■

Section 53 (Covering Spaces), 53.6(b)

Let $p\colon E \to B$ be a covering map.

b) **If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.**

Solution. [1] Let $U = \bigcup_{\alpha \in J} U_\alpha$ be an open cover of E. Since p is a covering map, it is both an open map, so that $p(U_\alpha)$ is open, and surjective onto B. Consequently, $\bigcup_{\alpha \in J} p(U_\alpha)$ is a union of open sets which covers E; equivalently, $\bigcup_{\alpha \in J} p(U_\alpha)$ is an open cover of B. Let $b \in B$. Since p is a covering map, b has a neighborhood V_b that is openly covered by p. For each $b \in B$, pick an arbitrary $p(U_\alpha^b)$ containing it; we denote this as $\tilde{V}_\alpha^b$. It follows that the collection of

$$\{V_b \cap \tilde{V}_\alpha^b \mid b \in B\}$$

is an open covering of B – each element is the intersection of open sets which is open, and this set contains every point $b \in B$. Since B is compact, this open covering has a finite subcovering, which we will describe as

$$\{V_{b_i} \cap \tilde{V}_\alpha^{b_i} \mid i \in \{1, \ldots, n\}\}$$

for each $b_i \in B$.

Consider now

$$\begin{aligned} p^{-1}\left(\bigcup_{i \in \{1,\ldots,n\}} V_{b_1} \cap \tilde{V}_\alpha^{b_1}\right) &= \bigcup_{i \in \{1,\ldots,n\}} p^{-1}(V_{b_1}) \cap p^{-1}(\tilde{V}_\alpha^{b_1}) \\ &= \bigcup_{i \in \{1,\ldots,n\}} p^{-1}(V_{b_1}) \cap U_\alpha^{b_1} \end{aligned}$$

which is an open cover of E – each element is an intersection of open sets in E and it covers E. Furthermore, it has a finite number of open sets, as $p^{-1}(b)$ is finite for any $b \in B$.[2]

It follows that we can pick the set $U_\alpha^{b_i}$ for $i \in \{1, \ldots, n\}$ to be an finite subcovering of the open cover U.[3] Thus, any open cover of E has a finite subcovering, and so E is compact. ■

[1] constructed with Zoe's help. Also, I think there are a few steps that are not quite correct in the proof but I have tried for quite a while to fix them to no avail. Will be at Office Hours to talk about it.

[2] I don't think is quite true as I'm working with an open set and using the fact that the inverse image of a point in E is finite.

[3] Also not sure about this, as I feel that I am neglecting the $p^{-1}(V_{b_1})$ part of the construction

Suppose X is contractible. Prove that there is a bijection between $[X, Y]$ and the set of path components of Y.

Solution. [4] Since X is contractible, the identity map i_X is homotopic to a constant map. Let F be this homotopy, the continuous map $F\colon X \times I \to X$ satisfying

$$F(x, 0) = i_X(x) = x \text{ and } F(x, 1) = x_0$$

for all $x \in X$ and some fixed $x_0 \in X$.

Let f be a continuous map from X into Y where $f(x_0) = y_0$ for some $y_0 \in Y$. It follows that f is homotopic to the constant map from X to Y sending any element in X to y_0. The homotopy between these two maps, which we will denote as $G\colon X \times I \to Y$, is defined by $G(x, t) = f(F(x, t))$ and satisfies

$$G(x, 0) = f(F(x, 0)) = f(x) \text{ and } G(x, 1) = f(F(x, 1)) = f(x_0) = y_0.$$

Furthermore, let $y_1 \in Y$ be in the same path component as y_1. We claim that f is homotopic to the constant map from X to Y sending any element of X to y_0. Since y_0 and y_1 are in the same path component of Y, there is some path p connecting them. Let us parametrize p so that $p\colon I \to Y$ satisfies $p(0) = y_0$ and $p(1) = y_1$. The homotopy between f and the constant map from X to Y sending any element of X to y_1 is $H\colon X \times I \to Y$ defined by

$$H(x, t) = \begin{cases} G(x, 2t) & \text{if } t \leq \frac{1}{2} \\ p(2t - 1) & \text{if } t > \frac{1}{2} \end{cases}.$$

H is continuous, as $H\left(x, \frac{1}{2}\right) = G(x, 1) = p(0) = y_0$; furthermore,

$$H(x, 0) = G(x, 0) = f(x) \text{ and } H(x, 1) = p(1) = y_1.$$

Thus, the maps f of X into Y correspond directly to the constant maps from X to Y. Furthermore, if two constant maps yield values in the same path component of Y, they are in the same homotopy class. Thus, we conclude that there is a bijection between $[X, Y]$, the set of homotopy classes of maps of X into Y and the path components of Y, as desired. ■

[4] constructed with Phil's help.