

Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor [Allison N. Miller](#), at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

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1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)

A function $f: A \rightarrow B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A , their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f .

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)

A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)

An **equivalence relation** \sim on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x \sim x$ for every x in A .
2. (Symmetry) If $x \sim y$, then $y \sim x$.
3. (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)

A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx .
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz , then xCz .

Note: the relation C is often replaced as $<$, just as how it is synonymous with \sim in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx , contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Immediate Predecessors and Successors)

If X is a set and $<$ is an order relation on X , and if $a < b$, the **open interval** (a, b) on X is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty, a is the **immediate predecessor** of b and b is the **immediate successor** of a .

Definition 1.6 (Order Type)

Suppose that A and B are two sets with order relations $<_A$ and $<_B$, respectively. A and B have the same **order type** if there is a bijective correspondence between them that preserves order.

That is, if there exists a bijective function $f: A \rightarrow B$ such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

Example. The interval $(-1, 1)$ of real numbers has the same order type as \mathbb{R} . The function $f: (-1, 1) \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijective correspondence.

Definition 1.7 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0 . If the set of all upper bounds for A_0 has a smallest element, that element is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0 . If the set of all lower bounds for A_0 has a largest element, that element is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.8 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle). Every nonempty subset of \mathbb{Z}_+ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \dots, n\}$, \mathbb{Z}_+), tuples, sequences, and Cartesian products.

Definition 1.9 (ω -tuple / Sequence)

Given a set X , a ω -tuple of elements of X is a function

$$\mathbf{x}: \mathbb{Z}_+ \rightarrow X$$

also known as a **sequence** (or infinite sequence) of elements of X .

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

1.7 Countable and Uncountable Sets

Definition 1.10 (Countably Infinite)

A set A is infinite if it is not finite. It is **countably infinite** if there is a bijective correspondence

$$f: A \rightarrow \mathbb{Z}_+.$$

Example. The set \mathbb{Z} of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow A$ and $g: A \rightarrow \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to \mathbb{Z} , so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to \mathbb{Z} . □

Definition 1.11 (Countable and Uncountable Sets)

A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0, 1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and \mathbb{R} are examples of uncountable sets.

Theorem 1.2. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.
2. There is a surjective function $f: \mathbb{Z}_+ \rightarrow B$.
3. There is an injective function $g: B \rightarrow \mathbb{Z}_+$.

Theorem 1.3 (Countable Union of Countable Sets). A countable union of countable sets is countable.

1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

1.9 Infinite Sets and the Axiom of Choice

Theorem 1.4. Let A be a set. The following statements about A are equivalent:

1. A is infinite.
2. There exists an injective function $f: \mathbb{Z}_+ \rightarrow A$
3. There exists a bijection of A with a proper subset of itself.

Theorem 1.5 (Axiom of Choice). Given a collection \mathcal{A} of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of \mathcal{A} ; that is, a set C such that C is contained in the union of the elements of \mathcal{A} , and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

1.10 Well-Ordered Sets

Definition 1.12 (Well-Ordered Sets)

A set A with an order relation $<$ is **well-ordered** if every nonempty subset of A has a smallest element.

Example. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set $(\mathbb{Z}_+)^{\omega}$.

Theorem 1.6 (Well-Ordering Theorem; Zermelo, 1904). If A is a set, there exists an order relation on A that is a well-ordering.

Corollary 1.1. There exists an uncountable well-ordered set.

Definition 1.13 (Section of a Set)

Let X be a well-ordered set. Given $\alpha \in X$, let S_{α} denote the set

$$S_{\alpha} = \{x \mid x \in X \text{ and } x < \alpha\}.$$

S_α is the **section** of X by α .

Lemma 1.1. There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem 1.7. If A has a countable subset of S_Ω , then A has an upper bound in S_Ω .

1.11 The Maximum Principle

Definition 1.14 (Strict Partial Order)

Given a set A , a relation \prec on A is a **strict partial order** on A if it has the following properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

Remark. These are the second and third properties of a simple order, defined in [Definition 1.4](#). Consequently, a strict partial order behaves like a simple order except that it need not be true that for every pair of distinct x and y in the set, either $x \prec y$ or $y \prec x$.

Theorem 1.8 (The Maximum Principle). Let A be a set and let \prec be a strict partial order on A . Then there exists a maximal simply ordered subset B of A .

2 Chapter 2: Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.1 (Topology)

A **topology** on a set X is a collection \mathcal{T} of subsets of X having the following properties:

1. \emptyset and X are in \mathcal{T} .
2. The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .
3. The intersection of the elements of any finite subcollection of \mathcal{T} is in \mathcal{T} .

A set X for which a topology \mathcal{T} has been specified is a **topological space**.