Connected Subspaces of the Real Line

David Yang and Adin Aberbach

1 Introduction and Relevant s

Definition (Linear Continuum)**.** A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Example (1, page 155)**.** The ordered square (under the dictionary topology) is a linear continuum. (*See example in textbook for more details.*)

Definition (Path and Path Connectedness)**.** Given points x and y of the space X, a **path** in X from x to y is a continuous map $f\colon [a,b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Theorem (21.3, page 130)**.** Let $f\colon X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem (23.4, page 150)**.** Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Theorem (23.5, page 150)**.** The image of a connected space under a continuous map is connected.

Theorem (24.1, page 153)**.** If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.

Theorem (24.3, page 154 – Intermediate Value Theorem)**.** Let $f\colon X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

2 Main Examples

Example (Example 6, page 156)**.** The ordered square I_o^2 is connected but not path connected.

Proof. By 24.1, since I_o^2 is a linear continuum under the order topology, it is connected. We will show that it is not path connected by showing that there is no path between points $p = 0 \times 0$ and $q = 1 \times 1$ in I_o^2.

Suppose for the sake of contradiction that there is a path $f\colon [a, b] \to I_o^2$. f is a continuous map from the connected interval $[a, b]$ to I_o^2. By the Intermediate Value , the image set $f([a, b])$ (which contains p and q, the smallest and largest elements in I_o^2) must contain every point $x \times y$ of I_o^2.

Consider the subsets

$$U_x = f^{-1}(x \times (0, 1))$$

for each $x \in I$. Note that since f is continuous, each U_x is open. Furthermore, by construction, each U_x is disjoint, as $f^{-1}(x \times (0, 1)) \cap f^{-1}(y \times (0, 1))$ for $x \neq y$.

For each $x \in I$, pick a rational number $q_x \in \mathbb{Q} \cap U_x$. Consider the map $g\colon I \to \mathbb{Q}$

$$g(x) = q_x.$$

Since each Q_x is disjoint, this is an injective mapping from I into $\mathbb{Q}$. Consequently, we find that $|\mathbb{Q}| \geq |I|$. But $\mathbb{Q}$ is countable whereas I is uncountable, so we have a contradiction.

We conclude that there is no path between points p and q in I_o^2, so I_o^2 is not path connected. □

Example. Let S denote the following subset of the plane:

$$S = \{x \times \sin\left(\frac{1}{x}\right) \mid 0 < x < 1\}.$$

$\overline{S}$ is known as the **topologist's sine curve**, and is not path-connected.

Before we begin, note that by 23.5, S is connected.

Furthermore, by 23.4, it follows that $\overline{S} = S \cup \{0 \times [-1, 1]\}$ is connected.

Proof. Suppose for the sake of contradiction that there is a path $f\colon [a, c] \to \overline{S}$ beginning at 0×0 and ending at some point of S. Define

$$L = \{t \mid f(t) \in 0 \times [-1, 1]\}.$$

Since f is c continuous, L is closed, so it has a larger element, which we can denote as b. By construction, $f \mid [b, c]$ (f restricted to the interval $[b, c]$) is a path with $f(b) \in 0 \times [-1, 1]$ and $f((b, c]) \subseteq S$. For the remainder of the proof, we will focus on the restricted map $f \mid [b, c]$.

Let $f(t) = (x(t), y(t))$. Note that by definition* of b, $x(b) = 0$ and $x(t) > 0$ for any $t > b$. Furthermore, $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t > b$.

To show that f is in fact not continuous, we show that there is a sequence of points $(t_n) \subseteq [b, c]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$, contradicting the result from 21.3.

Let $n \in \mathbb{N}$. Choose u such that

$$x(b) < u < x\left(b + \frac{1}{n}\right)$$

satisfying $\sin\left(\frac{1}{u}\right) = (-1)^n$; such a value u exists as there are infinitely many oscillations between $x(b) = 0$ and $x\left(b + \frac{1}{n}\right)$.

Since x is a continuous function from a connected set $[b, c]$ to the ordered set $[0, 1]$ in $\overline{S}$, it follows from the Intermediate Value that there exists some $t_n \in \left(b, b + \frac{1}{n}\right)$ satisfying $x(t_n) = u$.

Thus, we've constructed a sequence of points $(t_n) \subseteq [b, c]$ such that $t_n \to b$ and $y(t_n) = (-1)^n$; since the sequence $f(t_n) = (x(t_n), y(t_n))$ does not converge to $f(b)$, we know by 21.3 that f is not continuous. We conclude that $\overline{S}$ is not path-connected. $\square$