

Math 104: Topology

DAVID YANG

Spring 2024

Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor [Allison N. Miller](#), at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

Contents

1 Chapter 1: Set Theory and Logic	1
1.2 Functions	1
1.3 Relations	1
1.4 The Integers and the Real Numbers	2
1.5 Cartesian Products	3
1.6 Finite Sets	3

1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)

A function $f: A \rightarrow B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A , their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f .

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)

A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)

An **equivalence relation** \sim on a set A is a relation C on A having the following three properties:

- (Reflexivity) $x \sim x$ for every x in A .
- (Symmetry) If $x \sim y$, then $y \sim x$.
- (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)

A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

- (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx .
- (Nonreflexivity) For no x in A does the relation xCx hold.
- (Transitivity) If xCy and yCz , then xCz .

Note: the relation C is often replaced as $<$, just as how it is synonymous with \sim in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx , contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0 . If the set of all upper bounds for A_0 has a smallest element, that element is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0 . If the set of all lower bounds for A_0 has a largest element, that element is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.6 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle). Every nonempty subset of \mathbb{Z}_+ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \dots, n\}$, \mathbb{Z}_+), tuples, sequences, and Cartesian products.

Definition 1.7 (ω -tuple / Sequence)

Given a set X , a ω -tuple of elements of X is a function

$$\mathbf{x}: \mathbb{Z}_+ \rightarrow X$$

also known as a **sequence** (or infinite sequence) of elements of X .

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.