

# Math 104: Topology

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## Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor [Allison N. Miller](#), at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at [dyang5@swarthmore.edu](mailto:dyang5@swarthmore.edu).

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## 1 Chapter 1: Set Theory and Logic

### 1.2 Functions

#### Definition 1.1 (Injective, Surjective, Bijection)

A function  $f: A \rightarrow B$  is said to be **injective** (or **one-to-one**) if for each pair of distinct points of  $A$ , their images under  $f$  are distinct.

It is said to be **surjective** if every element of  $B$  is the image of some element of  $A$  under  $f$ .

If  $f$  is both **injective** and **surjective**, it said to be **bijective**.

### 1.3 Relations

#### Definition 1.2 (Relation)

A **relation** on a set  $A$  is a subset  $C$  of the Cartesian product  $A \times A$ .

#### Definition 1.3 (Equivalence Relation)

An **equivalence relation**  $\sim$  on a set  $A$  is a relation  $C$  on  $A$  having the following three properties:

- (Reflexivity)  $x \sim x$  for every  $x$  in  $A$ .
- (Symmetry) If  $x \sim y$ , then  $y \sim x$ .
- (Transitivity) If  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .

#### Definition 1.4 (Order Relation)

A relation  $C$  on a set  $A$  is an **order relation** (also simple order, or linear order) if it has the following properties:

- (Comparability) For every  $x$  and  $y$  in  $A$  for which  $x \neq y$ , either  $xCy$  or  $yCx$ .
- (Nonreflexivity) For no  $x$  in  $A$  does the relation  $xCx$  hold.
- (Transitivity) If  $xCy$  and  $yCz$ , then  $xCz$ .

*Note: the relation  $C$  is often replaced as  $<$ , just as how it is synonymous with  $\sim$  in the case of an equivalence relation.*

**Remark.** It follows that  $xCy$  and  $yCx$  cannot both be true. If so, then transitivity implies  $xCx$ , contradicting nonreflexivity.

**Example.** Suppose that  $A$  and  $B$  are two sets with order relations  $<_A$  and  $<_B$  respectively. The order relation  $<$  on  $A \times B$  defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if  $a_1 <_A a_2$  or if  $a_1 = a_2$  and  $b_1 <_B b_2$  is known as the **dictionary order relation** on  $A \times B$ .

#### Definition 1.5 (Immediate Predecessors and Successors)

If  $X$  is a set and  $<$  is an order relation on  $X$ , and if  $a < b$ , the **open interval**  $(a, b)$  on  $X$  is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty,  $a$  is the **immediate predecessor** of  $b$  and  $b$  is the **immediate successor** of  $a$ .

### Definition 1.6 (Order Type)

Suppose that  $A$  and  $B$  are two sets with order relations  $<_A$  and  $<_B$ , respectively.  $A$  and  $B$  have the same **order type** if there is a bijective correspondence between them that preserves order.

That is, if there exists a bijective function  $f: A \rightarrow B$  such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

**Example.** The interval  $(-1, 1)$  of real numbers has the same order type as  $\mathbb{R}$ . The function  $f: (-1, 1) \rightarrow \mathbb{R}$  such that

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijective correspondence.

### Definition 1.7 (Supremum and Infimum)

Let  $A$  be an ordered set. The subset  $A_0$  of  $A$  is **bounded above** if there is an element  $b$  of  $A$  such that  $x \leq b$  for every  $x \in A_0$ :  $b$  is an **upper bound** for  $A_0$ . If the set of all upper bounds for  $A_0$  has a smallest element, that element is the **supremum** of  $A_0$  (also the least upper bound).

The subset  $A_0$  of  $A$  is **bounded below** if there is an element  $b$  of  $A$  such that  $b \leq x$  for every  $x \in A_0$ :  $b$  is a **lower bound** for  $A_0$ . If the set of all lower bounds for  $A_0$  has a largest element, that element is the **infimum** of  $A_0$  (also the greatest lower bound).

### Definition 1.8 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set  $A$  is said to have the **least upper bound property** if every nonempty subset  $A_0$  of  $A$  that is bounded above has a least upper bound.

An ordered set  $A$  is said to have the **greatest lower bound property** if every nonempty subset  $A_0$  of  $A$  that is bounded below has a greatest lower bound.

## 1.4 The Integers and the Real Numbers

**Theorem 1.1 (Well-Ordering Principle).** Every nonempty subset of  $\mathbb{Z}_+$  has a smallest element.

## 1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g.  $\{1, \dots, n\}$ ,  $\mathbb{Z}_+$ ), tuples, sequences, and Cartesian products.

### Definition 1.9 ( $\omega$ -tuple / Sequence)

Given a set  $X$ , a  $\omega$ -tuple of elements of  $X$  is a function

$$\mathbf{x}: \mathbb{Z}_+ \rightarrow X$$

also known as a **sequence** (or infinite sequence) of elements of  $X$ .

## 1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

## 1.7 Countable and Uncountable Sets

### Definition 1.10 (Countably Infinite)

A set  $A$  is infinite if it is not finite. It is **countably infinite** if there is a bijective correspondence

$$f: A \rightarrow \mathbb{Z}_+.$$

**Example.** The set  $\mathbb{Z}$  of all integers is countably infinite. Similarly,  $\mathbb{Z} \times \mathbb{Z}$  is countably infinite.

**Proof (Countability of  $\mathbb{Z} \times \mathbb{Z}$ ).** *Proof 1.* Consider the bijections  $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow A$  and  $g: A \rightarrow \mathbb{Z}_+$  defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition  $g \circ f$  is also a bijection from  $\mathbb{Z} \times \mathbb{Z}$  to  $\mathbb{Z}$ , so  $\mathbb{Z} \times \mathbb{Z}$  is countably infinite.

*Proof 2.* Consider  $f(n, m) = 2^n 3^m$ , an injective map from  $\mathbb{Z} \times \mathbb{Z}$  to  $\mathbb{Z}$ . □

### Definition 1.11 (Countable and Uncountable Sets)

A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

**Example.**  $\{0, 1\}^\omega$ ,  $\mathcal{P}(\mathbb{Z}_+)$ , and  $\mathbb{R}$  are examples of uncountable sets.

**Theorem 1.2.** Let  $B$  be a nonempty set. Then the following are equivalent:

1.  $B$  is countable.
2. There is a surjective function  $f: \mathbb{Z}_+ \rightarrow B$ .
3. There is an injective function  $g: B \rightarrow \mathbb{Z}_+$ .

**Theorem 1.3** (Countable Union of Countable Sets). A countable union of countable sets is countable.

## 1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

## 1.9 Infinite Sets and the Axiom of Choice

**Theorem 1.4.** Let  $A$  be a set. The following statements about  $A$  are equivalent:

1.  $A$  is infinite.
2. There exists an injective function  $f: \mathbb{Z}_+ \rightarrow A$
3. There exists a bijection of  $A$  with a proper subset of itself.

**Theorem 1.5** (Axiom of Choice). Given a collection  $\mathcal{A}$  of disjoint nonempty sets, there exists a set  $C$  consisting of exactly one element from each element of  $\mathcal{A}$ ; that is, a set  $C$  such that  $C$  is contained in the union of the elements of  $\mathcal{A}$ , and for each  $A \in \mathcal{A}$ , the set  $C \cap A$  contains a single element.

## 1.10 Well-Ordered Sets

**Definition 1.12** (Well-Ordered Sets)

A set  $A$  with an order relation  $<$  is **well-ordered** if every nonempty subset of  $A$  has a smallest element.

**Example.** The set  $\mathbb{Z}_+ \times \mathbb{Z}_+$  is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set  $(\mathbb{Z}_+)^{\omega}$ .

**Theorem 1.6** (Well-Ordering Theorem; Zermelo, 1904). If  $A$  is a set, there exists an order relation on  $A$  that is a well-ordering.

**Corollary 1.1.** There exists an uncountable well-ordered set.

**Definition 1.13** (Section of a Set)

Let  $X$  be a well-ordered set. Given  $\alpha \in X$ , let  $S_{\alpha}$  denote the set

$$S_{\alpha} = \{x \mid x \in X \text{ and } x < \alpha\}.$$

$S_\alpha$  is the **section** of  $X$  by  $\alpha$ .

**Lemma 1.1.** There exists a well-ordered set  $A$  having a largest element  $\Omega$  such that the section  $S_\Omega$  of  $A$  by  $\Omega$  is uncountable but every other section of  $A$  is countable.

**Theorem 1.7.** If  $A$  has a countable subset of  $S_\Omega$ , then  $A$  has an upper bound in  $S_\Omega$ .

## 2 Chapter 2: Topological Spaces and Continuous Functions

### 2.12 Topological Spaces

**Definition 2.1** (Topology)

A **topology** on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  having the following properties:

1.  $\emptyset$  and  $X$  are in  $\mathcal{T}$ .
2. The union of the elements of any subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .
3. The intersection of the elements of any finite subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .

A set  $X$  for which a topology  $\mathcal{T}$  has been specified is a **topological space**.