

Homework 2

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Chapter 2 (Topological Spaces and Continuous Functions) Problems.

Section 16 (The Subspace Topology), 16.5 (reformulated)

Let X and Y be two sets, each with two topologies. So we have 4 topological spaces: (X, T) , (X, T') , (Y, U) , (Y, U') . Let S be the product topology on $X \times Y$ induced by T and U , and let S' be the product topology on $X \times Y$ induced by T' and U' . (So a *basis* for S consists of sets of the form $V \times W$ for V in T and W in U , but S itself does not just equal $T \times U$).

- a) Suppose that (X, T) is coarser than (X, T') and (Y, U) is coarser than (Y, U') . Prove that $(X \times Y, S)$ is coarser than $(X \times Y, S')$.

Solution. Let $A \times B$ be an open set in $(X \times Y, S)$. Since (X, T) is coarser than (X, T') , it follows that $A \subset A'$, where A' is an open set in (X, T') . Similarly, since (Y, U) is coarser than (Y, U') , it follows that $B \subset B'$ where B' is an open set in (Y, U') . Thus, $A' \times B'$ is an open set in $(X \times Y, S')$. Since every open set in $(X \times Y, S)$ is an open set in $(X' \times Y', S')$, it follows that $(X \times Y, S)$ is coarser than $(X \times Y, S')$, as desired. ■

- b) Suppose that $(X \times Y, S)$ is coarser than $(X \times Y, S')$. Does it necessarily follow that (X, T) is coarser than (X, T') and that (Y, U) is coarser than (Y, U') ?

Solution. Let $(X \times Y, S)$ be coarser than $(X \times Y, S')$. To show that (X, T) is coarser than (X, T') , we will show that any arbitrary open set in (X, T) is also an open set in (X, T') . Let A be a nonempty open set in (X, T) . We will show that A is open in (X, T') .

By definition, since A is open in (X, T) and Y is open in U , it follows that $A \times Y$ is open in the product topology S . Furthermore, since $(X \times Y, S)$ is coarser than $(X \times Y, S')$, this means that $A \times Y$ is also open in the product topology S' . By definition, then, it can be written as the union of nonempty open sets in S' :

$$A \times Y = \bigcup_{\alpha \in J} V_{\alpha} \times W_{\alpha}$$

where V_{α} is nonempty and open in (X, T') and W_{α} is nonempty open in (Y, U') .¹

We claim that $A = \bigcup_{\alpha \in J} V_{\alpha}$, and we will show containment in both directions. First, we will show that $A \subset \bigcup_{\alpha \in J} V_{\alpha}$. Let $x \in A$ and let y be an arbitrary element in Y . Since

¹Note that we can assume that V_{α} and W_{α} are nonempty for each α as if either of them were empty for a given α , then the product $V_{\alpha} \times W_{\alpha}$ would be the empty set.

$A \times Y = \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$ and $x \in A$, it follows that $x \times y \in \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$ and so $x \times y \in V_\alpha \times W_\alpha$ for a specific value of α . Consequently, $x \in V_\alpha$ for that value of α , and so $x \in \bigcup_{\alpha \in J} V_\alpha$. Thus, $A \subseteq \bigcup_{\alpha \in J} V_\alpha$.

For the other direction of containment, let $x \in \bigcup_{\alpha \in J} V_\alpha$. Then $x \in V_\alpha$ for some value of α . We can pick any arbitrary $y \in W_\alpha$ for that same value of α , as W_α is nonempty. Then $x \times y \in \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$. Since $A \times Y = \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$, it follows that $x \times y \in A \times Y$ and so $x \in A$ as desired.

Thus, by proving containment in both directions, we know $A = \bigcup_{\alpha \in J} V_\alpha$. Since A is a union of open sets V_α in (X, T') , it follows that A is also open in (X, T') .

Finally, since every open set in (X, T) is also open in (X, T') , we know that (X, T) is coarser than (X, T') . The same argument can be made to show that (Y, U) is coarser than (Y, U') . We conclude that if $(X \times Y, S)$ is coarser than $(X \times Y, S')$, it follows that (X, T) is coarser than (X, T') and that (Y, U) is coarser than (Y, U') . ■

Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Let A , B , and A_α denote subsets of a space X . Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions \subset or \supset holds.

a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Solution. We claim that

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Let $x \in \overline{A \cap B}$. By definition, every open set U of X containing x must intersect $A \cap B$. Consequently, U must intersect both A and B . Since every open set U of X containing x intersects A , and every open set U of X containing x intersects B , it follows that $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \in \overline{A} \cap \overline{B}$.

On the other hand, let A be the set of all positive real numbers, and let B be the set of all negative real numbers; A and B are subsets of \mathbb{R} . Note that $\overline{A} \cap \overline{B} = \{0\}$ whereas $\overline{A \cap B} = \overline{\emptyset} = \emptyset$. Thus, $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$. ■

b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.

Solution. ■

c) $\overline{A - B} = \overline{A} - \overline{B}$.