

Connected Subspaces of the Real Line

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1 Introduction and Relevant Definitions

Definition (Linear Continuum). A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Example (1, page 155). The ordered square (under the dictionary topology) is a linear continuum. (See example in textbook for more details.)

Definition (Path and Path Connectedness). Given points x and y of the space X , a **path** in X from x to y is a continuous map $f: [a, b] \rightarrow X$ of some closed interval in the real line into X , such that $f(a) = x$ and $f(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X .

Theorem (21.3, page 130). Let $f: X \rightarrow Y$. If the function f is continuous, then for every convergent sequence $x_n \rightarrow x$ in X , the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem (23.4, page 150). Let A be a connected subspace of X . If $A \subset B \subset \overline{A}$, then B is also connected.

Theorem (23.5, page 150). The image of a connected space under a continuous map is connected.

Theorem (24.1, page 153). If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L .

Theorem (24.3, page 154 – Intermediate Value Theorem). Let $f: X \rightarrow Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are

two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

2 Main Examples

Example (Example 6, page 156). The ordered square I_o^2 is connected but not path connected.

Proof. By 24.1, since I_o^2 is a linear continuum under the order topology, it is connected. We will show that it is not path connected by showing that there is no path between points $p = 0 \times 0$ and $q = 1 \times 1$ in I_o^2 .

Suppose for the sake of contradiction that there is a path $f: [a, b] \rightarrow I_o^2$. f is a continuous map from the connected interval $[a, b]$ to I_o^2 . By the Intermediate Value, the image set $f([a, b])$ (which contains p and q , the smallest and largest elements in I_o^2) must contain every point $x \times y$ of I_o^2 .

Consider the subsets

$$U_x = f^{-1}(x \times (0, 1))$$

for each $x \in I$. Note that since f is continuous, each U_x is open. Furthermore, by construction, each U_x is disjoint, as $f^{-1}(x \times (0, 1)) \cap f^{-1}(y \times (0, 1))$ for $x \neq y$.

For each $x \in I$, pick a rational number $q_x \in \mathbb{Q} \cap U_x$. Consider the map $g: I \rightarrow \mathbb{Q}$

$$g(x) = q_x.$$

Since each U_x is disjoint, this is an injective mapping from I into \mathbb{Q} . Consequently, we find that $|\mathbb{Q}| \geq |I|$. But \mathbb{Q} is countable whereas I is uncountable, so we have a contradiction.

We conclude that there is no path between points p and q in I_o^2 , so I_o^2 is not path connected. \square

Example. Let S denote the following subset of the plane:

$$S = \{x \times \sin\left(\frac{1}{x}\right) \mid 0 < x < 1\}.$$

\overline{S} is known as the **topologist's sine curve**, and is not path-connected.

Before we begin, note that by 23.5, S is connected.

Furthermore, by 23.4, it follows that $\overline{S} = S \cup \{0 \times [-1, 1]\}$ is connected.

Proof. Suppose for the sake of contradiction that there is a path $f: [a, c] \rightarrow \overline{S}$ beginning at 0×0 and ending at some point of S . Define

$$L = \{t \mid f(t) \in 0 \times [-1, 1]\}.$$

Since f is continuous, L is closed, so it has a largest element, which we can denote as b . By construction, $f|_{[b, c]}$ (f restricted to the interval $[b, c]$) is a path with $f(b) \in 0 \times [-1, 1]$ and $f((b, c]) \subseteq S$. For the remainder of the proof, we will focus on the restricted map $f|_{[b, c]}$.

Let $f(t) = (x(t), y(t))$. Note that by definition* of b , $x(b) = 0$ and $x(t) > 0$ for any $t > b$. Furthermore, $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t > b$.

To show that f is in fact not continuous, we show that there is a sequence of points $(t_n) \subseteq [b, c]$ such that $t_n \rightarrow b$ and $y(t_n) = (-1)^n$, contradicting the result from 21.3.

Let $n \in \mathbb{N}$. Choose u such that

$$x(b) < u < x\left(b + \frac{1}{n}\right)$$

satisfying $\sin\left(\frac{1}{u}\right) = (-1)^n$; such a value u exists as there are infinitely many oscillations between $x(b) = 0$ and $x\left(b + \frac{1}{n}\right)$.

Since x is a continuous function from a connected set $[b, c]$ to the ordered set $[0, 1]$ in \overline{S} , it follows from the Intermediate Value that there exists some $t_n \in (b, b + \frac{1}{n})$ satisfying $x(t_n) = u$.

Thus, we've constructed a sequence of points $(t_n) \subseteq [b, c]$ such that $t_n \rightarrow b$ and $y(t_n) = (-1)^n$; since the sequence $f(t_n) = (x(t_n), y(t_n))$ does not converge to $f(b)$, we know by 21.3 that f is not continuous. We conclude that \overline{S} is not path-connected. \square