MATH104: Topology **Fall 2023**

Homework 6

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Chapter 9 (The Fundamental Group) Problems.

Section 54 (The Fundamental Group of the Circle), 54.7

Extend the proof of Theorem 54.5 (the fundamental group of S^1 is isomorphic to the additive group of integers) to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Solution. Let $p \colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ be the covering map of Theorem 53.1 extended to two dimensions, so that $p(x, y) = ((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y))$, let $e_0 = (0, 0)$, and let $p(e_0) = b_0$ for $b_0 \in S^1 \times S^1$. Then $p^{-1}(b_0)$ is the set $\mathbb{Z} \times \mathbb{Z}$. By Theorem 54.4, since $\mathbb{R} \times \mathbb{R}$ is simply connected, the lifting correspondence

$$\varphi \colon \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}$$

is bijective. To show that $\pi_1(S^1 \times S^1, b_0)$ is isomoporhic to $\mathbb{Z} \times \mathbb{Z}$, then, it remains to show that φ is a homomorphism.

Let $[f]$ and $[g]$ be two elements of $\pi_1(S^1 \times S^1, b_0)$, and let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths on $\mathbb{R} \times \mathbb{R}$ beginning at $e_0 = (0, 0)$. Let $\tilde{f}(1) = (a, b)$ and $\tilde{g}(1) = (c, d)$. This tells us that $\varphi([f]) = (a, b)$ and $\varphi([g]) = (c, d)$. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = (a, b) + \tilde{g}(s)$$

on $\mathbb{R} \times \mathbb{R}$. Since $p((a, b) + z) = p(z)$ for all $z \in \mathbb{R} \times \mathbb{R}$, the path $\tilde{\tilde{g}}$ is a lifting of g, beginning at (a, b). It follows that the product $\tilde{f} * \tilde{\tilde{g}}$ is defined – it is the lifting on $f \times g$ beginning at $(0, 0)$. The endpoint of this path is $\tilde{\tilde{g}}(1) = (a + c, b + d)$. Thus, we see that

$$\varphi([f] * [g]) = (a + c, b + d) = (a, b) + (c, d) = \varphi([f]) + \varphi([g])$$

and so φ is a homomorphism. We conclude that φ is an isomorphism between $\pi_1(S^1 \times S^1, b_0)$ and $\mathbb{Z} \times \mathbb{Z}$. Thus, the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$. ■

Section 55 (Retractions and Fixed Points), 55.1

Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Solution. Since A is a retract of B^2, by definition, there exists a continuous map $r\colon B^2 \to A$ such that $r \mid A$ is the identity map of A. Let f be an arbitrary continuous map from A to A, and let $j\colon A \to B^2$ be the inclusion map, which is continuous.

Since j, f and r are continuous maps and the composition of continuous maps is continuous, it follows that $j \circ f \circ r$ is a continuous map from B^2 to B^2.

By Brouwer's Fixed-Point Theorem for the Disc, it follows that $j \circ f \circ r$ has a fixed point $x \in B^2$. Furthermore, since the image of $j \circ f \circ r$ is the subspace A of B^2 (as j is the inclusion map from A to B^2), it must be that $x \in A$. Consequently, for that fixed point $x \in A$, we have that

$$x = (j \circ f \circ r)(x) = j(f(r(x))) = j(f(x)) = f(x)$$

where the third equality follows from the fact that r is the retraction map and the fourth equality from the fact that j is the inclusion map. Thus, we must have that $f(x) = x$ for some $x \in A$, so every continuous map $f\colon A \to A$ must have a fixed point, as desired. ■

Section 57 (The Borsuk-Ulam Theorem), 57.2

Show that if $g\colon S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective. [*Hint:* If $p \in S^2$, then $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$.]

Solution. Suppose for the sake of contradiction that g is not surjective. Then there exists some $p \in S^2$ such that $g(x) \neq p$ for all $x \in S^2$. Note that per the hint, $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$; let f be the homeomorphism from $S^2 - \{p\}$ to $\mathbb{R}^2$.

Since f and g are both continuous and the composition of continuous functions is continuous, it follows that $f \circ g$ is a continuous map from S^2 to $\mathbb{R}^2$. By the Borsuk-Ulam Theorem for S^2, we know that there is a point $y \in S^2$ such that $(f \circ g)(y) = (f \circ g)(-y)$, or equivalently,

$$f(g(y)) = f(g(-y)).$$

Since f is a homeomorphism, it must be injective. Consequently, if $f(g(y)) = f(g(-y))$, then $g(y) = g(-y)$ for that $y \in S^2$. However, this contradicts the fact that $g(x) \neq g(-x)$ for all $x \in S^2$.

Thus, we conclude that if $g\colon S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all $x \in S^2$, then g is surjective, as desired. ■

Section 58 (Deformation Retracts and Homotopy Type), 58.5

Recall that a space X is said to be *contractible* if the identity map of X to itself is nulhomotopic. Show that X is contractible if and only if X has the homotopy type of a one-point space.

Solution. We will prove both directions of the implication.

Suppose that X is contractible. Then the identity map of X is homotopic to a constant map f, where $f(x) = x_0$ for all $x \in X$ and some $x_0 \in X$. Let Y be the one-point space consisting of the point x_0. We will show that X and Y are homotopy equivalent. Consider the map $g\colon Y \to X$ by inclusion. It follows that $g \circ f = f$. Furthermore, X is contractible, so f is homotopic to the identity map of X. Thus, we conclude that $g \circ f$ is homotopic to the identity map of X. Furthermore, note that $f \circ g$ maps x_0 to itself; consequently, since x_0 is the only element in Y, it follows that $f \circ g$ is homotopic to the identity map of Y. Since $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y, we know that X and Y have the same homotopy type, and so X has the homotopy type of a one-point space.

On the other hand, suppose that X has the homotopy type of a one-point space; equivalently, X is homotopy equivalent to the space $Y = \{y\}$. By definition, this means that there exists continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ where $f \circ g$ is homotopic to the identity map of Y and $g \circ f$ is homotopic to the identity map of X. Note that since Y is a one-point space consisting of the single element y, g is defined by where it sends y; let $g(y) = x_0$ for some $x_0 \in X$. It follows that for all $x \in X$,

$$(g \circ f)(x) = g(f(x)) = g(y) = x_0$$

meaning that $g \circ f$ is the constant map sending every element of X to x_0. Equivalently, $g \circ f$ is homotopic to the constant map sending everything to x_0, and so X is by definition contractible.

We conclude that X is contractible if and only if X has the homotopy type of a one-point space, as desired. ∎