Homework 7

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Chapter 11 (The Seifert-van Kampen Theorem) Problems.

Section 67 (Direct Sums of Abelian Groups), 67.4(b), (c)

The *order* of an element a of an abelian group G is the smallest positive integer m such that $ma = 0$, if such exists; otherwise, the order of a is said to be infinite. The order of a thus equals the order of the subgroup generated by a.

b) **Show that if G is free abelian, it has no elements of finite order.**

   Solution. Since G is free abelian, it has the elements $\{a_\alpha\}$ as a basis, where each a_α generates an infinite cyclic subgroup G_α. Let a be a general element of G, and let its order be denoted by m, so that $ma = 0$. Since G is free abelian, it is by definition also the direct sum of the groups $\{G_\alpha\}$, so

   $$a = \sum_{\alpha_i} m_{\alpha_i} a_{\alpha_i}$$

   where each a_{α_i} is a basis element of G and so $m_{\alpha_i} a_{\alpha_i}$ is an element of the group G_{α_i}. Multiplying both sides of the above equation by m and using the fact that the order of a is defined to be m, we know that

   $$0 = ma = m \sum m_{\alpha_i} a_{\alpha_i} = \sum (m m_{\alpha_i}) a_{\alpha_i}.$$

   By the uniqueness caused by G being a direct sum of the groups G_α, we know that each $(m m_{\alpha_i}) a_{\alpha_i} = 0$; however, each a_{α_i} is of infinite order as they are the generator of an infinite cyclic group G_{α_i}. Consequently, we must have that $m m_{\alpha_i} = 0$ for each α_i.

   By definition, since $m > 0$, it follows that $m_{\alpha_i} = 0$ for each α_i. Thus, $a = 0$, and so the only element of finite order in G is 0. We conclude that if G is free abelian, it has no elements of finite order. ∎

b) **Show the additive group of rationals has no elements of finite order, but is not free abelian. [*Hint*: If $\{a_\alpha\}$ is a basis, express $\frac{1}{2}a_\alpha$ in terms of this basis.]**

Section 68 (Free Product of Groups), 68.4

Prove Theorem 68.4 (Uniqueness of Free Products): Let $\{G_\alpha\}$ be a family of groups. Suppose G and G' are groups and $i_\alpha \colon G_\alpha \to G$ and $i'_\alpha \colon G_\alpha \to G'$ are families of monomorphisms, such that the families $\{i_\alpha(G_\alpha)\}$ and $\{i'_\alpha(G_\alpha)\}$ generate G and G', respectively. If both G and G' have the extension property stated in the preceding lemma, then there is a unique isomorphism $\varphi \colon G \to G'$ such that $\varphi \cdot i_\alpha = i'_\alpha$ for all α.

Solution. ∎