

Homework 11

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Chapter 13 (Classification of Covering Spaces) Problems.

Section 79 (Equivalence of Covering Spaces), 79.5(b)

Let $T = S^1 \times S^1$ be the torus; let $x_0 = b_0 \times b_0$. Prove the following:

***Theorem.* If E is a covering space of T , then E is homeomorphic either to \mathbb{R}^2 , or to $S^1 \times \mathbb{R}$, or to T .**

(Hint: You may use the following result from algebra: if F is a free abelian group of rank 2 and N is a nontrivial subgroup, then there is a basis a_1, a_2 for F such that either (1) ma_1 is a basis for N , for some positive integer m , or (2) ma_1, na_2 is a basis for N , where m and n are positive integers.)

Solution. Let E be a covering space of T , with p representing the covering map from E to T . Pick some $e_0 \in p^{-1}(x_0)$. Since $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$, and $p_*(\pi_1(E, e_0))$ must be a subgroup of $\mathbb{Z} \times \mathbb{Z}$, a free abelian group of rank 2, we know that $p_*(\pi_1(E, e_0))$ is either the trivial subgroup, the subgroup generated by ma_1 , or the subgroup generated by ma_1 and na_2 , where m and n are positive integers and a_1 and a_2 represent some basis for $\mathbb{Z} \times \mathbb{Z}$.¹

Since there is an isomorphism of $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ mapping the respective basis elements a_1 and a_2 to the respective canonical basis elements $(1, 0)$ and $(0, 1)$, we know by Exercise 79.5(a), that this isomorphism is induced by a homeomorphism $f: T \rightarrow T$ mapping x_0 to x_0 . Consider now the map $q: E \rightarrow T$ defined by $q = f \circ p$. Since f is the composition of a covering map and a homeomorphism, it is itself a covering map.²

Consider now $q_*(\pi_1(E, e_0)) = (f_* \circ p_*)(\pi_1(E, e_0))$.

■

¹the form for the latter two nontrivial subgroups follows directly from the hint.

²alternatively, f is a 1-fold covering map, and by Exercise 54.4, since both f and p are covering maps and $f^{-1}(x)$ is finite for each $x \in T$, q is also a covering map.

Let $q: X \rightarrow Y$ and $r: Y \rightarrow Z$ be maps; let $p = r \circ q$.

- a) Let q and r be covering maps. Show that if Z has a universal covering space, then p is a covering map. (Compare Exercise 4 of Section 53.)³

Solution. Let E be the universal covering space of Z ; E is simply connected. By definition, there is a covering map $s: E \rightarrow Z$. By Theorem 80.3, since $s: E \rightarrow Z$ and $r: Y \rightarrow Z$ are covering maps, there exists a covering map $t: E \rightarrow Y$ such that $s = r \circ t$. Furthermore, by Theorem 80.3, since $t: E \rightarrow Y$ and $q: X \rightarrow Y$ are covering maps, there exists a covering map $u: E \rightarrow X$ such that $t = q \circ u$.

Note that $u: E \rightarrow X$ and $s: E \rightarrow Z$ are covering maps. Furthermore, we have that

$$p \circ u = (r \circ q) \circ u = r \circ (q \circ u) = r \circ t = s$$

by construction. By Lemma 80.2(b), since $s = p \circ u$, and u and s are covering maps, then so is p . Thus, if q and r are covering maps, and if Z has a universal covering space, then p is a covering map. ■

³We replace the requirement that $r^{-1}(z)$ is finite for each $z \in Z$ with the condition that Z has a universal covering space to get that p is a covering map.

Let $p: X \rightarrow B$ be a covering map (not necessarily regular); let G be its group of covering transformations.

a) Show that the action of G on X is properly discontinuous.

Solution. Let $x \in X$, so that $p(x) \in B$. Since p is a covering map, there exists a neighborhood V of $p(x)$ in B that is evenly covered by p . Equivalently, the inverse image $p^{-1}(V)$ is a disjoint union of neighborhoods in U_α in X such that the restriction of p to U_α is a homeomorphism of U_α onto V , for each α . Let U be the neighborhood of $p^{-1}(V)$ in X containing x . We will show that $g(U)$ and U are disjoint, for any $g \neq e$.

Suppose for the sake of contradiction that there exists some element $y \in g(U) \cap U$. Then $y \in U$ and $y = g(z)$ for some $z \neq y$ in U (as a non-identity covering transformation has no fixed points). Since g is a covering transformation, it follows that $p \circ g = p$, so $p(g(z)) = p(z)$. Simplifying, we get that $p(y) = p(z)$, where both y and z are in U and $y \neq z$. The restriction of p to U cannot be a homeomorphism of U onto V , as it is not injective; thus, we arrive at a contradiction, and conclude that $g(U)$ and U are disjoint for every $g \neq e$ in G .

By definition, it follows that the action of G on X is properly discontinuous. ■

Let G be a group of homeomorphisms of X . The action of G on X is said to be fixed-point free if no element of G other than the identity e has a fixed point. Show that if X is Hausdorff, and if G is a finite group of homeomorphisms of X whose action is fixed-point free, then the action of G is properly discontinuous.

Solution. Let $x \in X$. Let $\{g_1, \dots, g_n\}$ be the finite group of homeomorphisms in G of X that are not equal to the identity. Since the action of G on X is fixed-point free, it follows that $g_i(x) \neq x$ for all $i \in \{1, \dots, n\}$. Since x and $g_i(x)$ are distinct points in X and X is Hausdorff, it follows that there are disjoint open sets V_i and W_i about x and $g_i(x)$ that are disjoint in X for each $i \in \{1, \dots, n\}$.

Let $\tilde{V} = \bigcap_{i \in \{1, \dots, n\}} V_i$; by construction, \tilde{V} is the intersection of finitely many open sets, so it is itself open. Consider

$$U = \bigcap_{i \in \{1, \dots, n\}} g_i^{-1}(W_i) \cap \tilde{V}$$

Since each g_i is a homeomorphism, they are each continuous, and so each $g_i^{-1}(W_i)$ is open, and by construction contains x . Consequently, each $g_i^{-1}(W_i) \cap \tilde{V}$ is open, and so U , the intersection of finitely many such open sets, is also open. Furthermore, U is nonempty, as it contains x by construction. Consider $U = g^{-1}(W) \cap V$.

Note that for each g_i , $U \subseteq V_i$, and $U \subseteq g_i^{-1}(W_i)$, so $g_i(U) \subseteq W_i$. Since each V_i and W_i are disjoint open sets by construction, U and $g_i(U)$ must be disjoint open sets, for all $i \in \{1, \dots, n\}$.

Since for every $x \in X$, there is a neighborhood U of x such that for all g_i in the finite group of homeomorphisms of X that are not the identity, $g_i(U)$ is disjoint from U , the action of G is properly discontinuous, as desired. ■