MATH104: Topology **Fall 2023**

Homework 2

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Chapter 2 (Topological Spaces and Continuous Functions) Problems.

Section 16 (The Subspace Topology), 16.5 (reformulated)

Let X and Y be two sets, each with two topologies. So we have 4 topological spaces: (X, T), (X, T'), (Y, U), (Y, U'). Let S be the product topology on $X \times Y$ induced by T and U, and let S' be the product topology on $X \times Y$ induced by T' and U'. (So a *basis* for S consists of sets of the form $V \times W$ for V in T and W in U, but S itself does not just equal $T \times U$).

a) **Suppose that (X, T) is coarser than (X, T') and (Y, U) is coarser than (Y, U'). Prove that $(X \times Y, S)$ is coarser than $(X \times Y, S')$.**

Solution. Let $A \times B$ be an open set in $(X \times Y, S)$. Since (X, T) is coarser than (X, T'), it follows that $A \subset A'$, where A' is an open set in (X, S'). Similarly, since (Y, U) is coarser than (Y, U'), it follows that $B \subset B'$ where B' is an open set in (Y, U'). Thus, $A' \times B'$ is an open set in $(X \times Y, S')$. Since every open set in $(X \times Y, S)$ is an open set in $(X \times Y, S')$, it follows that $(X \times Y, S)$ is coarser than $(X \times Y, S')$, as desired. ■

b) **Suppose that $(X \times Y, S)$ is coarser than $(X \times Y, S')$. Does it necessarily follow that (X, T) is coarser than (X, T') and that (Y, U) is coarser than (Y, U')?**

Solution. Let $(X \times Y, S)$ be coarser than $(X \times Y, S')$. To show that (X, T) is coarser than (X, T'), we will show that any arbitrary open set in (X, T) is also an open set in (X, T'). Let A be a nonempty open set in (X, T). We will show that A is open in (X, T').

By definition, since A is open in (X, T) and Y is open in U, it follows that $A \times Y$ is open in the product topology S. Furthermore, since $(X \times Y, S)$ is coarser than $(X \times Y, S')$, this means that $A \times Y$ is also open in the product topology S'. Consequently, $A \times Y$ can be written as the union of nonempty open sets in S':

$$A \times Y = \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$$

where V_α is nonempty and open in (X, T') and W_α is nonempty open in (Y, U').[1]

We claim that $A = \bigcup_{\alpha \in J} V_\alpha$, and we will show containment in both directions. First, we will show that $A \in \bigcup_{\alpha \in J} V_\alpha$. Let $x \in A$ and let y be an arbitrary element in Y. Since

[1] Note that we can assume that V_α and W_α are nonempty for each α as if either of them were empty for a given α, then the product $V_\alpha \times W_\alpha$ would be the empty set.

$A \times Y = \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$ and $x \in A$, it follows that $x \times y \in \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$ and so $x \times y \in V_\alpha \times W_\alpha$ for at least one value of α. Consequently, $x \in V_\alpha$ for that value of α, and so $x \in \bigcup_{\alpha \in J} V_\alpha$. Thus, $A \subseteq \bigcup_{\alpha \in J} V_\alpha$.

For the other direction of containment, let $x \in \bigcup_{\alpha \in J} V_\alpha$. Then $x \in V_\alpha$ for some value of α. We can pick any arbitrary $y \in W_\alpha$ for that same value of α, as W_α is nonempty. Then $x \times y \in \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$. Since $A \times Y = \bigcup_{\alpha \in J} V_\alpha \times W_\alpha$, it follows that $x \times y \in A \times Y$ and so $x \in A$ as desired.

Thus, by proving containment in both directions, we know $A = \bigcup_{\alpha \in J} V_\alpha$. Since A is a union of open sets V_α in (X, T'), it follows that A is also open in (X, T').

Finally, since every open set in (X, T) is also open in (X, T'), we know that (X, T) is coarser than (X, T'). The same argument can be made to show that (Y, U) is coarser than (Y, U'). We conclude that if $(X \times Y, S)$ is coarser than $(X \times Y, S')$, it follows that (X, T) is coarser than (X, T') and that (Y, U) is coarser than (Y, U'). $\blacksquare$

Section 17 (Closed Sets and Limit Points), 17.13

Show that X is Hausdorff if and only if the diagonal $\triangle = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution. Suppose that X is Hausdorff. Consider two distinct elements x and y in X. Since X is Hausdorff, there exists disjoint open sets A and B in X such that $x \in A$, $y \in B$. Since A and B are disjoint, they have no elements of X in common. Consequently, it follows that $(A \times B) \cap \triangle = \varnothing$. Since $A \times B$ is an open set U in $X \times X$ containing (x, y) that does not intersect $\triangle$, it follows that any point (x, y) cannot be in the closure of $\triangle$, and so $\triangle = \overline{\triangle}$. Equivalently, $\triangle$ is closed.

The reverse implication follows similarly. Suppose that $\triangle$ is closed, so it is its own closure. Consequently, every point $(x, y) \in X \times X$ with $x \neq y$ is not in $\overline{\triangle}$. It follows that there exists an open set $A \times B$ in $X \times X$ containing (x, y) such that $(A \times B) \cap \triangle$ is empty. Equivalently, A and B must be disjoint open sets in X where $x \in A$ and $y \in B$. By definition, this tells us that X is Hausdorff.

Thus, we conclude that X is Hausdorff if and only if the diagonal $\triangle = \{x \times x \mid x \in X\}$ is closed in $X \times X$. ∎

Section 17 (Closed Sets and Limit Points), 17.8

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subset$ or $\supset$ holds.

a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

Solution. We claim that
$$\boxed{\overline{A \cap B} \subset \overline{A} \cap \overline{B}}.$$
Let $x \in \overline{A \cap B}$. By definition, every open set U of X containing x must intersect $A \cap B$. Consequently, U must intersect both A and B. Since every open set U of X containing x intersects A, and every open set U of X containing x intersects B, it follows that $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \in \overline{A} \cap \overline{B}$. We conclude that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

As a counterexample for the other direction of containment, let A be the set of all positive real numbers, and let B be the set of all negative real numbers; A and B are subsets of $\mathbb{R}$. Note that $\overline{A} \cap \overline{B} = \{0\}$ whereas $\overline{A \cap B} = \overline{\varnothing} = \varnothing$. Thus, $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$. ∎

b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.

Solution. We claim that
$$\boxed{\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}}.$$
Let $x \in \overline{\bigcap A_\alpha}$. By definition, every open set U of X containing x must intersect $\bigcap A_\alpha$. Since U intersects the intersection of the A_α, it must also intersect each A_α. Thus, for each α, $x \in \overline{A_\alpha}$; equivalently, $x \in \bigcap \overline{A_\alpha}$. We conclude that $\overline{\bigcap A_\alpha} \subset \bigcap \overline{A_\alpha}$.

As a counterexample for the other direction of containment, we can refer to the example in part (a); let A_1 be the set of all positive real numbers, and let A_2 be the set of all negative real numbers; A_1 and A_2 are subsets of $\mathbb{R}$. Since $\overline{A_1} \cap \overline{A_2} = \{0\}$ whereas $\overline{A_1 \cap A_2} = \overline{\varnothing} = \varnothing$, we conclude that $\bigcap \overline{A_\alpha} \not\subset \overline{\bigcap A_\alpha}$. ∎

c) $\overline{A - B} = \overline{A} - \overline{B}$.

Solution. We claim that
$$\boxed{\overline{A} - \overline{B} \subset \overline{A - B}}.$$
Let $x \in \overline{A} - \overline{B}$. By definition, every open set U of X containing x must intersect A and some open set U' containing x will not intersect B. However, every open set of X containing x will intersect $A - B$. Thus, $x \in \overline{A - B}$ and we conclude that $\overline{A} - \overline{B} \subset \overline{A - B}$.

As a counterexample for the other direction of containment, let A be the set of real numbers and let B be the set of rationals; A and B are both subsets of $\mathbb{R}$. Note that $\overline{A} = \mathbb{R}$, $\overline{B} = \mathbb{R}$ (as the rationals are dense in the reals), and $\overline{A - B} = \mathbb{R}$ (as the irrationals are also dense in the reals). Consequently, $\overline{A - B} = \mathbb{R}$ is not contained in $\overline{A} - \overline{B} = \varnothing$, and we conclude that $\overline{A - B} \not\subset \overline{A} - \overline{B}$. ∎