MATH104: Topology **Fall 2023**

Homework 4

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Chapter 4 (Countability and Separation Axioms) Problems.[1]

Section 31 (The Separation Axioms), 31.1

Show that if X is regular, every pair of points[2] of X have neighborhoods whose closures are disjoint.

Solution. Let X be a regular space. Since regular spaces are Hausdorff, it follows that X is a Hausdorff space. Since X is Hausdorff, for any pair of distinct points x_1, x_2 in X, there exist disjoint neighborhoods U_{x_1} and U_{x_2} in X of x_1 and x_2, respectively. By Lemma 31.1(a), since X is regular, there must exist neighborhoods V_{x_1} and V_{x_2} of x_1 and x_2 respectively, such that $\overline{V}_{x_1} \subset U_{x_1}$ and $\overline{V}_{x_2} \subset U_{x_2}$.

By construction, since U_{x_1} and U_{x_2} are themselves disjoint, V_{x_1} and V_{x_2} are two neighborhoods of x_1 and x_2 whose closures are disjoint. Thus, if X is regular, every pair of points of X have neighborhoods whose closures are disjoint ■

[1] James Wang helped me "make-up" these problems after missing class.

[2] assuming distinct (not explicitly mentioned in textbook problem)

Section 31 (The Separation Axioms), 31.7(a)

Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$ (such a map is called a *perfect map*).

a) **Show that if X is Hausdorff, then so is Y.**

Solution. Let X be Hausdorff, and let y_1 and y_2 be two distinct points in Y. To show that Y is Hausdorff, we will show that there are two disjoint neighborhoods of y_1 and y_2 in Y.

Note that since p is continuous, the preimages $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are closed. Since p is perfect, each preimage is compact. Furthermore, due to the surjectivity of p and the fact that there is only one input for each output, the sets of preimages $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are nonempty and disjoint. In summary, we know that $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are each nonempty, closed and compact, and are themselves disjoint.

It follows from Exercise 26.5 that since $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are disjoint compact subspaces of X, there exist nonempty disjoint open sets U_1 and U_2 of X containing $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$, respectively.

We now claim that $Y \setminus p(X \setminus U_1)$ and $Y \setminus p(X \setminus U_2)$ are two disjoint open neighborhoods of y_1 and y_2, respectively. Equivalently, we will first show that $Y \setminus p(X \setminus U_1)$ is open and a neighborhood of y_1 (the argument for $Y \setminus p(X \setminus U_2)$ follows by symmetry), and then show that $Y \setminus p(X \setminus U_1)$ is disjoint from $Y \setminus p(X \setminus U_2)$.

We first show that $Y \setminus p(X \setminus U_1)$ is open. By construction, U_1 is open in X. Consequently, $X \setminus U_1$ is closed. Since p is a closed map, it follows that $p(X \setminus U_1)$ is closed. Thus, $Y \setminus p(X \setminus U_1)$ is open, as desired. Next, we show that $Y \setminus p(X \setminus U_1)$ is a neighborhood of y_1. Suppose for the sake of contradiction that $y_1 \notin Y \setminus p(X \setminus U_1)$. Then $y_1 \in p(X \setminus U_1)$, in which case $p^{-1}(y_1) \subset X \setminus U_1$. It follows that $p^{-1}(y_1) \not\subset U_1$, which contradicts the construction of U_1, which was defined to contain $p^{-1}(y_1)$. Thus, $Y \setminus p(X \setminus U_1)$ is open. By symmetry, it follows that $Y \setminus p(X \setminus U_2)$ is open and a neighborhood of y_2.

Finally, we show that $Y \setminus p(X \setminus U_1)$ is disjoint from $Y \setminus p(X \setminus U_2)$. Suppose once again for the sake of contradiction that there exists some $\tilde{y} \in (Y \setminus p(X \setminus U_1)) \cap (Y \setminus p(X \setminus U_2))$. It follows that $\tilde{y} \in Y \setminus p(X \setminus U_1)$ and $\tilde{y} \in Y \setminus p(X \setminus U_2)$, or equivalently, $\tilde{y} \notin p(X \setminus U_1)$ and $\tilde{y} \notin p(X \setminus U_2)$. Consequently, $p^{-1}(\{\tilde{y}\}) \not\subset X \setminus U_1$ and $p^{-1}(\{\tilde{y}\}) \not\subset X \setminus U_2$, or equivalently, $p^{-1}(\{y\}) \subset U_1$ and $p^{-1}(\{y\}) \subset U_2$; $p^{-1}(\{y\})$ is in both U_1 and U_2. However, this contradicts the construction of U_1 and U_2, which were defined to be disjoint open sets in X. Thus, we conclude that $Y \setminus p(X \setminus U_1)$ and $Y \setminus p(X \setminus U_2)$ are disjoint.

From an arbitrary choice of distinct y_1 and y_2 in Y, we constructed two nonempty, disjoint open sets $Y \setminus p(X \setminus U_1)$ and $Y \setminus p(X \setminus U_2)$ in Y containing y_1 and y_2, respectively. Equivalently, Y is Hausdorff, as desired. ■

Section 32 (Normal Spaces), 32.3

Show that every locally compact Hausdorff space is regular.

Solution. Let X be a locally compact Hausdorff space. By Theorem 29.1, since X is locally compact Hausdorff, there exists a space Y such that Y is compact Hausdorff, $Y - X$ consists of a single point, and X is a subspace of Y. By Theorem 32.3, since Y is a compact Hausdorff space, it is normal. Since Y is a normal space, it is also regular. Finally, by Theorem 31.2, since X is a subspace of a regular space Y, X must also be regular. We conclude that X, a locally compact Hausdorff space, must also be regular. Thus, every locally compact Hausdorff space is regular, as desired. ■

Section 35 (The Tietze Extension Theorem), 35.4(a), (b)

Let Z be a topological space. If Y is a subspace of Z, we say that Y is a *retract* of Z if there is a continuous map $r\colon Z \to Y$ such that $r(y) = y$ for each $y \in Y$.

a) **Show that if Z is Hausdorff and Y is a retract of Z, then Y is closed in Z.**

Solution. Let Z be Hausdorff, and let Y be a retract of Z. To show that Y is closed in Z, it is equivalent to show that $Z \setminus Y$ is open in Z. We will do so by showing that any arbitrary $z \in Z \setminus Y$ has a neighborhood around it in $Z \setminus Y$.

Let z be an arbitrary element in $Z \setminus Y$. Since Y is a retract of Z, there exists some continuous map $r\colon Z \to Y$ such that $r(y) = y$ for each $y \in Y$. Let $r(z) = \tilde{y}$, where $\tilde{y} \in Y$. Since $z \in Z \setminus Y$, it follows that z and Y are disjoint, so z and $\tilde{y}$ are two distinct points in Z. Since Z is Hausdorff, it follows that there must be two disjoint neighborhoods U_z and $U_{\tilde{y}}$ of Z containing z and $\tilde{y}$, respectively.

We claim that $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is a neighborhood of z in $Z \setminus Y$. We will show three properties separately: $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is open, $z \in (r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$, and that $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is disjoint from Y.

We first show that $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is open in Z. Note that by construction, $U_{\tilde{y}}$ is open in Z. Furthermore, Y is a subspace of Z. Thus, under the subspace topology, $U_{\tilde{y}} \cap Y$ is open in Z. Furthermore, r is by definition continuous, so the preimage $(r^{-1}(U_{\tilde{y}} \cap Y))$ is open in Z. Finally, by construction, U_z is open in Z. Since the intersection of open sets is open, it follows that $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is open in Z, as desired.

Next, we show that z is contained in $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$. By construction, $z \in U_z$. Furthermore, since by construction, $r(z) = \tilde{y}$ and $\tilde{y} \in Y \cap U_y$, we have that $z \in r^{-1}(U_{\tilde{y}} \cap Y)$. Thus, since $z \in U_z$ and $z \in r^{-1}(U_{\tilde{y}}) \cap Y$, it follows that $z \in (r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$.

Finally, we show that Y and $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ are disjoint, to show that $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ is actually a neighborhood of z in $Z \setminus Y$. Suppose for the sake of contradiction that there exists some $y \in Y$ such that $y \in (r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$. Then $y \in U_z$ and $y \in (r^{-1}(U_{\tilde{y}} \cap Y))$. The latter condition, coupled with the continuity of r, tells us that $r(y) \in U_{\tilde{y}} \cap Y$, so $r(y) \in U_{\tilde{y}}$. Note however that since Y is a retraction of Z, $r(y) = y$ by definition. Consequently, we have that $y = r(y) \in U_{\tilde{y}}$ and $y \in U_z$, so y is in both $U_{\tilde{y}}$ and U_z. However, this contradicts the construction of $U_{\tilde{y}}$ and U_z, which were defined to be disjoint neighborhoods in Z. Thus, we know that Y and $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ are disjoint.

To summarize, for an arbitrary element $z \in Z \setminus Y$, we constructed a neighborhood $(r^{-1}(U_{\tilde{y}} \cap Y)) \cap U_z$ which is open in $Z \setminus Y$. Thus, $Z \setminus Y$ is open and so equivalently, Y is closed in Z. ■

b) **Let A be a two-point set in $\mathbb{R}^2$. Show that A is not a retract of $\mathbb{R}^2$.**

Solution. Let $A = \{a_1, a_2\}$ for distinct points a_1 and a_2 in $\mathbb{R}^2$. We will show that A is not a retract of $\mathbb{R}^2$. Suppose for the sake of contradiction that A is a retract of $\mathbb{R}^2$. By definition, this means that there exists a continuous function $r\colon \mathbb{R}^2 \to A$ such that $r(a_1) = a_1$ and $r(a_2) = a_2$.

By Theorem 23.5, the image of a connected space under a continuous map is connected. Since $\mathbb{R}^2$ is connected and r is a continuous map from $\mathbb{R}^2$ to A, it follows that A must be connected. However, $A = \{a_1, a_2\}$ is certainly not connected; let $r = \frac{d(a_1,a_2)}{2}$, and take $B_r(a_1) \cap \{a_1, a_2\}$ and $B_r(a_2) \cap \{a_1, a_2\}$. By construction, these are two disjoint open sets in the subspace topology on A, and thus, they form a separation of A. Since A has a separation, it is not connected, giving a contradiction. Thus, A is not a retract of $\mathbb{R}^2$. $\blacksquare$