MATH104: Topology **Spring 2024**

The Order Topology and the First Uncountable Ordinal

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1 Introduction

Definition (Order Topology)**.** Let X be a set with a simple order relation and assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, known as the **order topology**.

If X has no smallest element, there are no sets of type 2, and if X has no largest element, there are no sets of type 3.

Lemma (13.1, page 80)**.** Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Lemma (10.2, page 66)**.** There exists a well-ordered set A having a largest element Ω, such that S_Ω of A by Ω is uncountable but every other section of A is uncountable.

We can actually construct such a well-ordered set! Assuming the existence of an uncountable well-ordered set B (a weaker result following from the axiom of choice), define C to be the well-ordered set $\{1, 2\} \times B$ in the dictionary order; a section of C is uncountable (for example, the section of C by any element of the form $2 \times b$). Let Ω be the smallest element of C for which the section of C by Ω is uncountable, and let A be this section along with Ω.

Definition (Minimal Uncountable Well-Ordered Set)**.** S_Ω is an uncountable well-ordered set, every section of which is countable; it is known as the **minimal uncountable well-ordered set**.

Alternatively, S_Ω is known as the **first uncountable ordinal**, and the closure of $\overline{S}_\Omega$ is $\overline{S}_\Omega = S_\Omega \cup \{\Omega\}$.

2 The Order Topology of the First Uncountable Ordinal

- Sets of type 2 in the order topology can be thought of as sections of S_Ω, and are countable.

- By problem 10.6(a) on page 67, S_Ω has no largest element, so no set of type 3 exists in S_Ω.
- Since S_Ω is well-ordered, any nonempty subset has to have a minimal element; consequently, all basis elements in the order topology can be written in the form $[a, b)$.
- We can characterize all open sets of S_Ω: they must be unions of our basis elements. These look like unions of disjoint intervals of open sets, or are of the form of our basis elements.