MATH104: Topology **Fall 2023**

Homework 10

David Yang

Chapter 12 (Classification of Surfaces) Problems.

Section 74 (Classification of Surfaces), 74.7

If $m > 1$, show the fundamental group of the m-fold projective plane is not abelian. [*Hint*: There is a homomorphism mapping this group onto the group $\mathbb{Z}/2 * \mathbb{Z}/2$.]

Solution. Let $m > 1$. G be the free group on the set $\{\alpha_1, \dots, \alpha_m\}$ and let H be the group $\mathbb{Z}/2 * \mathbb{Z}/2$, which we define to have generators γ and δ satisfying $\gamma^2 = \delta^2 = 1$. Consider the homomorphism φ of G onto H defined by

$$\varphi(\alpha_i) = \begin{cases} \gamma & \text{if } i = 1 \\ \delta & \text{otherwise} \end{cases}.$$

(Defining where the generators of G are mapped defines the entire homomorphism).

Note that under the homomorphism φ, we have that

$$\varphi(\alpha_1^2 \dots \alpha_m^2) = \varphi(\alpha_1)^2 \cdots \varphi(\alpha_m)^2 = \gamma^2 \cdots \delta^2 = 1,$$

as both γ and δ are defined to have order 2.

Consider the least normal subgroup N of G generated by $\alpha_1^2 \dots \alpha_m^2$. By the Very Useful Lemma, since φ is a homomorphism from G to H vanishing on the normal subgroup N, we have a well-defined homomorphism $\psi \colon G/N \to H$ mapping gN to $\varphi(g)$.

Note that the domain G/N of ψ is isomorphic to the fundamental group of the m-fold projective plane. Thus, ψ is a homomorphism from the fundamental group of the m-fold projective plane onto the group $H \cong \mathbb{Z}/2 * \mathbb{Z}/2$, which is not abelian.

Take two elements h_1 and h_2 in H where $h_1 h_2 \neq h_2 h_1$, and let g_1 and g_2 be elements in G/N such that $\psi(g_1) = h_1$ and $\psi(g_2) = h_2$ (g_1 and g_2 exist as ψ is surjective). If the fundamental group of the m-fold projective plane were abelian, then $g_1 g_2 = g_2 g_1$, so we would have

$$h_1 h_2 = \psi(g_1)\psi(g_2) = \psi(g_1 g_2) = \psi(g_2 g_1) = \psi(g_2)\psi(g_1) = h_2 h_1,$$

contradicting the fact that h_1 does not commute with h_2. Thus, for $m > 1$, the fundamental group of the m-fold projective plane is not abelian. ∎

Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labelling scheme $acadbcb^{-1}d$.

a) **Check that all vertices of P are mapped to the same point of the quotient space X by the pasting map.**

Solution. Consider the vertex corresponding that is the base of the edge a. This vertex is directly identified to the end of edges c and d. The end of edge c is identified to the end of edge b, and the end of edge d is identified to the base of b. Finally, the base of edge b is identified to the base of edge d, which is in turn identified to the end of edge a, which is identified to the base of edge c.

Thus, all vertices of P are mapped to the same point of the quotient space X by the pasting map. ■

b) **Calculate $H_1(X)$.**

Solution. By Theorem 74.2, since the pasting map identifies all vertices of P to the same point of the quotient space, $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the four generators a, b, c, d (which we will denote F) by the least normal subgroup containing the element $acadbcb^{-1}d$. We will use x to denote the element $acadbcb^{-1}d$ and N to denote the least normal subgroup containing x.

Since $\pi_1(X, x_0) = F/N$, by Corollary 75.2, we have that

$$H_1(X) \cong \frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]} \cong \frac{(F/[F, F])}{\langle p(N) \rangle}$$

where p is the projection map from F to $F/[F, F]$. Note that $F/[F, F]$ is simply the free abelian group on four generators a, b, c, d, i.e. $\mathbb{Z}^4$. On the other hand, $p(N) = a^2c^2d^2$, the "abelianized" version of the element x.

Thus, $H_1(X) \cong \mathbb{Z}^4/\langle a^2c^2d^2 \rangle$. For convenience, let us think of $a = (1, 0, 0, 0), b = (0, 1, 0, 0), c = (0, 0, 1, 0)$, and $d = (0, 0, 0, 1)$, so $a^2c^2d^2 = (2, 0, 2, 2)$. By considering $\mathbb{Z}^4$ as a four-tuple of integers, one such generating set for $\mathbb{Z}^4$ is $(1, 0, 0, 0)$, $(0, 1, 0, 0), (0, 0, 1, 0)$, and $(1, 0, 1, 1)$.

It follows that

$$\begin{aligned} H_1(X) &\cong \frac{[\langle (1,0,0,0) \rangle \times \langle (0,1,0,0) \rangle \times \langle (0,0,1,0) \rangle \times \langle (1,0,1,1) \rangle]}{\langle 2,0,2,2 \rangle} \\ &\cong \langle (1,0,0,0) \rangle \times \langle (0,1,0,0) \rangle \times \langle (0,0,1,0) \rangle \times \mathbb{Z}/2\mathbb{Z} \\ &\cong \boxed{\mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}}. \blacksquare \end{aligned}$$

c) **Assuming X is homeomorphic to one of the surfaces given in Theorem 75.5 (which it is), which surface is it?**

Solution. The first homology group of the 4-fold projective plane is precisely $\mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}$, so X is homeomorphic to $\boxed{P_4}$, the 4-fold projective plane. ■

Section 78 (Constructing Compact Surfaces), 78.2(a)(b)

Let H^2 be the subspace of $\mathbb{R}^2$ consisting of all points (x_1, x_2) with $x_2 \geq 0$. A 2-*manifold with boundary (or surface with boundary)* is a Hausdorff space X with a countable basis such that each point x of X has a neighborhood homeomorphic with an open set of $\mathbb{R}^2$ or H^2. The *boundary* of X (denoted ∂X) consists of those points x such that x has no neighborhood homeomorphic with an open set of $\mathbb{R}^2$.

a) **Show that no point of H^2 of the form $(x_1, 0)$ has a neighborhood (in H^2) that is homeomorphic to an open set of $\mathbb{R}^2$.**

Solution. Suppose for the sake of contradiction that there is a point y of H^2 of the form $(x_1, 0)$ that has a neighborhood U in H^2 that is homeomorphic to an open set U' of $\mathbb{R}^2$. Let h be the homeomorphism between U and U'. By construction, $h(y) \in U'$.

Since U and U' are homeomorphic by assumption, removing a point from both U and U' should preserve their homeomorphic nature. However, this is not the case. Removing the point y from U preserves the simply-connectedness nature of U, and so the fundamental group of $U \setminus \{y\}$ is trivial. On the other hand, removing a point from an open set U' of $\mathbb{R}^2$ gives us a space which has fundamental group $\mathbb{Z}$. Thus, $U \setminus \{y\}$ and $U' \setminus \{\text{pt}\}$ are not homeomorphic, and so U and U' cannot be homeomorphic.

We conclude that there is no point of H^2 of the form $(x_1, 0)$ that has a neighborhood U in H^2 that is homeomorphic to an open set U' of $\mathbb{R}^2$. ■

b) **Show that $x \in \partial X$ if and only if there is a homeomorphism h mapping a neighborhood of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times 0$.**

Solution. Suppose that $x \in \partial X$. Let h be a homeomorphism from U a neighborhood of X to U', an open set of H^2. Suppose for the sake of contradiction that $h(x) \in \mathbb{R} \times c$ for some $c > 0$. Then $U' \cap B_{\frac{c}{2}}(h(x))$ is an open set of $\mathbb{R}^2$, and $h^{-1}(U \cap B_{\frac{c}{2}}(h(x)))$ is a neighborhood of X homeomorphic to an open set of $\mathbb{R}^2$. By definition, since x has a neighborhood homeomorphic with an open set of $\mathbb{R}^2$, then $x \notin \partial X$, contradicting our initial assumption. It follows that $h(x) \in \mathbb{R} \times 0$. Thus, if $x \in \partial X$, then there is a homeomorphism h mapping a neighborhood of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times 0$.

On the other hand, suppose that there is a homeomorphism h mapping a neighborhood of x onto an open set of H^2 such that $h(x) \in \mathbb{R} \times 0$. From part (a), since $h(x) = (x_1, 0)$, $h(x)$ cannot have a neighborhood in H^2 that is homeomorphic to an open set of $\mathbb{R}^2$. By definition, then, $x \in \partial X$. ■