MATH104: Topology **Fall 2023**

Homework 7

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Chapter 9 (The Fundamental Group) Problems.

Section 60 (Fundamental Groups of Some Surfaces), 60.2

Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to the projective plane P^2.

Solution. [1] Let π represent the quotient map from B^2 to X defined by identifying each point x of S^1 with its antipode. Consider the projection of the upper hemisphere of S^2 onto B^2, a homemomorphism. Let h represent the inverse of this projection; h is a homeomorphism from B^2 to S^2 mapping the unit disk to the upper hemisphere. Finally, let p represent the canonical quotient map from S^2 to P^2, defined by identifying each point of S^2 with its antipode.

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[1] constructed with Hillary's help.

Section 63 (The Jordan Curve Theorem), 63.1

Let C_1 and C_2 be disjoint simple closed curves in S^2.

a) **Show that $S^2 - C_1 - C_2$ has precisely three components. [*Hint:* If W_1 is the component of $S^2 - C_1$ disjoint from C_2, and if W_2 is the component of $S^2 - C_2$ disjoint from C_1, show that $\overline{W}_1 \cup \overline{W}_2$ does not separate S^2.]**

Solution. [2] By the Jordan Curve Theorem, C_1 separates S^2 into two components, which we will denote as W_1 and V_1. Similarly, by the Jordan Curve Theorem, C_2 separates S^2 into two components, which we will denote as W_2 and V_2. Furthermore, define W_1 and W_2 to be the components matching the hint: W_1 is the component of $S^2 - C_1$ disjoint from C_2 and W_2 is the component of $S^2 - C_2$ disjoint from C_1.

Note that since C_1 and C_2 are disjoint, it follows that $W_1 \subset V_2$ and $W_2 \subset V_1$, with $W_1 \cap W_2 = \varnothing$. Consequently, W_1 and W_2 are two components of $S^2 - C_1 - C_2$. Note that by construction, $V_1 = S^2 - \overline{W}_1$, and $V_2 = S^2 - \overline{W}_2$. Furthermore, C_1 and C_2 are disjoint, so $\overline{W}_1 \cap \overline{W}_2 = \varnothing$, and $S^2 - (\overline{W}_1 \cap \overline{W}_2) = S^2$ is simply connected. Finally, neither $\overline{W}_1$ nor $\overline{W}_2$ separate S^2, so by Theorem 63.3 (the general nonseparation theorem), $\overline{W}_1 \cup \overline{W}_2$ does not separate S^2. Thus, the third component of $S^2 - C_1 - C_2$ is precisely $\overline{W}_1 \cup \overline{W}_2 = V_1 \cap V_2$.

We conclude that $S^2 - C_1 - C_2$ has three components: W_1, W_2, and $V_1 \cap V_2$. ■

b) **Show that these three components have boundaries C_1 and C_2 and $C_1 \cup C_2$, respectively.**

Solution. [3] By construction, the boundaries of W_1 and W_2, two of the components of $S^2 - C_1 - C_2$, are C_1 and C_2, respectively. It remains to show that the boundary of $V_1 \cap V_2$ is $C_1 \cup C_2$. Since $V_1 = S^2 - \overline{W}_1$, and $V_2 = S^2 - \overline{W}_2$, we have that

$$\begin{aligned}
\overline{V}_1 \cap \overline{V}_2 - V_1 \cap V_2 &= ((S^2 - W_1) \cap (S^2 - W_2)) - ((S^2 - \overline{W}_1) \cap (S^2 - \overline{W}_2)) \\
&= (\overline{W}_1 \cup \overline{W}_2) - (W_1 \cup W_2) \\
&= (\overline{W}_1 - W_1) \cup (\overline{W}_2 - W_2) \\
&= C_1 \cup C_2.
\end{aligned}$$

Thus, the boundary of $V_1 \cap V_2$ is by definition $C_1 \cup C_2$, as desired. ■

[2]constructed with Hillary's help.
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