Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor Allison N. Miller, at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

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1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)

A function $f\colon A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f.

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)

A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)

An **equivalence relation** $\sim$ on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x \sim x$ for every x in A.
2. (Symmetry) If $x \sim y$, then $y \sim x$.
3. (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)

A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz, then xCz.

Note: the relation C is often replaced as $<$, just as how it is synonyous with $\sim$ in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx, contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Immediate Predecessors and Successors)

If X is a set and $<$ is an order relation on X, and if $a < b$, the **open interval** (a, b) on X is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty, a is the **immediate predecessor** of b and b is the **immediate successor** of a.

Definition 1.6 (Order Type)

Suppose that A and b are two sets with order relations $<_A$ and $<_B$, respectively. A and B have the same **order type** if there is a bijective correspondence between them that preserves order.

That is, if there exists a bijective function $f\colon A \to B$ such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

Example. The interval $(-1, 1)$ of real numbers has the same order type as $\mathbb{R}$. The function $f\colon (-1, 1) \to B$ such that

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijective correspondence.

Definition 1.7 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that elements is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that elements is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.8 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \ldots, n\}$, $\mathbb{Z}_+$), tuples, sequences, and Cartesian products.

Definition 1.9 (ω-tuple / Sequence)
Given a set X, a ω-tuple of elements of X is a function

$$\mathbf{x}\colon \mathbb{Z}_+ \to X$$

also known as a **sequence** (or infinite sequence) of elements of X.

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

1.7 Countable and Uncountable Sets

Definition 1.10 (Countably Infinite)
A set A is infinite if it is not finite. It is **countably infinite** of there is a bijective correspondence

$$f\colon A \to \mathbb{Z}_+.$$

Example. The set $\mathbb{Z}$ of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ and $g\colon A \to \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. □

Definition 1.11 (Countable and Uncountable Sets)
A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0, 1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and $\mathbb{R}$ are examples of uncountable sets.

Theorem 1.2. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.
2. There is a surjective function $f\colon \mathbb{Z}_+ \to B$.
3. There is an injective function $g\colon B \to \mathbb{Z}_+$.

Theorem 1.3 (Countable Union of Countable Sets)**.** A countable union of countable sets is countable.

1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

1.9 Infinite Sets and the Axiom of Choice

Theorem 1.4. Let A be a set. The following statements about A are equivalent:

1. A is infinite.
2. There exists an injective function $f\colon \mathbb{Z}_+ \to A$
3. There exists a bijection of A with a proper subset of itself.

Theorem 1.5 (Axiom of Choice)**.** Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

1.10 Well-Ordered Sets

Definition 1.12 (Well-Ordered Sets)

A set A with an order relation $<$ is **well-ordered** if every nonempty subset of A has a smallest element.

Example. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set $(\mathbb{Z}_+)^\omega$.

Theorem 1.6 (Well-Ordering Theorem; Zermelo, 1904)**.** If A is a set, there exists an order relation on A that is a well-ordering.

Corollary 1.1. There exists an uncountable well-ordered set.

Definition 1.13 (Section of a Set)

Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}.$$

S_α is the **section** of X by α.

Lemma 1.1. There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem 1.7. If A has a countable subset of S_Ω, then A has an upper bound in S_Ω.

1.11 The Maximum Principle

Definition 1.14 (Partial Order)

Given a set A, a relation $\prec$ on A is a **strict partial order** on A if it has the following properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

If the relation $\prec$ is instead $\preceq$, where $a \preceq b$ implies $a = b$ or $a \prec b$, $\preceq$ is a **partial order** on A.

Remark. These are the second and third properties of a simple order, defined in Definition 1.4. Consequently, a strict partial order behaves like a simple order except that it need not be true that for every pair of distinct x and y in the set, either $x \prec y$ or $y \prec x$.

Theorem 1.8 (The Maximum Principle)**.** Let A be a set and let $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered subset B of A.

Example. If $\mathcal{A}$ is the collection of all circular regions in the plane under the "proper subset of" relation, a maximal simply ordered subcollection of $\mathcal{A}$ consists of all circular regions with centers at the origin.

Definition 1.15

Let A be a set and let $\prec$ be a strict partial order on A. If B is a subset of A, an **upper bound** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

A **maximal element** of A is an element m on A such that for no element a of A does the relation $m \prec a$ hold.

Theorem 1.9 (Zorn's Lemma)**.** Let A be a set that is strictly partially ordered. If every simply ordered set of A has an upper bound in A, then A has a maximal element.

2 Chapter 2: Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.1 (Topology)

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is a **topological space**.