

## Connected Subspaces of the Real Line

David Yang and Adin Aberbach

## 1 Introduction and Relevant Definitions

**Definition (Linear Continuum).** A simply ordered set  $L$  having more than one element is called a **linear continuum** if the following hold:

1.  $L$  has the least upper bound property.
2. If  $x < y$ , there exists  $z$  such that  $x < z < y$ .

**Example (1, page 155).** The ordered square (under the dictionary topology) is a linear continuum. (See example in textbook for more details.)

**Definition (Path and Path Connectedness).** Given points  $x$  and  $y$  of the space  $X$ , a **path** in  $X$  from  $x$  to  $y$  is a continuous map  $f: [a, b] \rightarrow X$  of some closed interval in the real line into  $X$ , such that  $f(a) = x$  and  $f(b) = y$ .

A space  $X$  is said to be **path connected** if every pair of points of  $X$  can be joined by a path in  $X$ .

**Theorem (21.3, page 130).** Let  $f: X \rightarrow Y$ . If the function  $f$  is continuous, then for every convergent sequence  $x_n \rightarrow x$  in  $X$ , the sequence  $f(x_n)$  converges to  $f(x)$ . The converse holds if  $X$  is metrizable.

**Theorem (23.4, page 150).** Let  $A$  be a connected subspace of  $X$ . If  $A \subset B \subset \overline{A}$ , then  $B$  is also connected.

**Theorem (23.5, page 150).** The image of a connected space under a continuous map is connected.

**Theorem (24.1, page 153).** If  $L$  is a linear continuum in the order topology, then  $L$  is connected, and so are intervals and rays in  $L$ .

**Theorem (24.3, page 154 – Intermediate Value Theorem).** Let  $f: X \rightarrow Y$  be a continuous map, where  $X$  is a connected space and  $Y$  is an ordered set in the order topology. If  $a$  and  $b$  are

two points of  $X$  and if  $r$  is a point of  $Y$  lying between  $f(a)$  and  $f(b)$ , then there exists a point  $c$  of  $X$  such that  $f(c) = r$ .

## 2 Main Examples

**Example** (Example 6, page 156). The ordered square  $I_o^2$  is connected but not path connected.

**Proof.** By 24.1, since  $I_o^2$  is a linear continuum under the order topology, it is connected. We will show that it is not path connected by showing that there is no path between points  $p = 0 \times 0$  and  $q = 1 \times 1$  in  $I_o^2$ .

Suppose for the sake of contradiction that there is a path  $f: [a, b] \rightarrow I_o^2$ .  $f$  is a continuous map from the connected interval  $[a, b]$  to  $I_o^2$ . By the Intermediate Value, the image set  $f([a, b])$  (which contains  $p$  and  $q$ , the smallest and largest elements in  $I_o^2$ ) must contain every point  $x \times y$  of  $I_o^2$ .

Consider the subsets

$$U_x = f^{-1}(x \times (0, 1))$$

for each  $x \in I$ . Note that since  $f$  is continuous, each  $U_x$  is open. Furthermore, by construction, each  $U_x$  is disjoint, as  $f^{-1}(x \times (0, 1)) \cap f^{-1}(y \times (0, 1))$  for  $x \neq y$ .

For each  $x \in I$ , pick a rational number  $q_x \in \mathbb{Q} \cap U_x$ . Consider the map  $g: I \rightarrow \mathbb{Q}$

$$g(x) = q_x.$$

Since each  $Q_x$  is disjoint, this is an injective mapping from  $I$  into  $\mathbb{Q}$ . Consequently, we find that  $|\mathbb{Q}| \geq |I|$ . But  $\mathbb{Q}$  is countable whereas  $I$  is uncountable, so we have a contradiction.

We conclude that there is no path between points  $p$  and  $q$  in  $I_o^2$ , so  $I_o^2$  is not path connected.  $\square$

**Example.** Let  $S$  denote the following subset of the plane:

$$S = \{x \times \sin\left(\frac{1}{x}\right) \mid 0 < x < 1\}.$$

$\overline{S}$  is known as the **topologist's sine curve**, and is not path-connected.

Before we begin, note that by 23.5,  $S$  is connected.

Furthermore, by 23.4, it follows that  $\overline{S} = S \cup \{0 \times [-1, 1]\}$  is connected.

**Proof.** Suppose for the sake of contradiction that there is a path  $f: [a, c] \rightarrow \overline{S}$  beginning at  $0 \times 0$  and ending at some point of  $S$ . Define

$$L = \{t \mid f(t) \in 0 \times [-1, 1]\}.$$

Since  $f$  is continuous,  $L$  is closed, so it has a largest element, which we can denote as  $b$ . By construction,  $f|_{[b, c]}$  ( $f$  restricted to the interval  $[b, c]$ ) is a path with  $f(b) \in 0 \times [-1, 1]$  and  $f((b, c]) \subseteq S$ . For the remainder of the proof, we will focus on the restricted map  $f|_{[b, c]}$ .

Let  $f(t) = (x(t), y(t))$ . Note that by definition\* of  $b$ ,  $x(b) = 0$  and  $x(t) > 0$  for any  $t > b$ . Furthermore,  $y(t) = \sin\left(\frac{1}{x(t)}\right)$  for  $t > b$ .

To show that  $f$  is in fact not continuous, we show that there is a sequence of points  $(t_n) \subseteq [b, c]$  such that  $t_n \rightarrow b$  and  $y(t_n) = (-1)^n$ , contradicting the result from 21.3.

Let  $n \in \mathbb{N}$ . Choose  $u$  such that

$$x(b) < u < x\left(b + \frac{1}{n}\right)$$

satisfying  $\sin\left(\frac{1}{u}\right) = (-1)^n$ ; such a value  $u$  exists as there are infinitely many oscillations between  $x(b) = 0$  and  $x\left(b + \frac{1}{n}\right)$ .

Since  $x$  is a continuous function from a connected set  $[b, c]$  to the ordered set  $[0, 1]$  in  $\overline{S}$ , it follows from the Intermediate Value that there exists some  $t_n \in (b, b + \frac{1}{n})$  satisfying  $x(t_n) = u$ .

Thus, we've constructed a sequence of points  $(t_n) \subseteq [b, c]$  such that  $t_n \rightarrow b$  and  $y(t_n) = (-1)^n$ ; since the sequence  $f(t_n) = (x(t_n), y(t_n))$  does not converge to  $f(b)$ , we know by 21.3 that  $f$  is not continuous. We conclude that  $\overline{S}$  is not path-connected.  $\square$