# The Countability Axioms

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## 1 Introduction and Relevant Theorems

**Definition** (First Countability Axiom)**.** *A space $X$ is said to have a **countable basis at** $x$ if there is a countable collection $\mathscr{B}$ of neighborhoods of $x$ such that each neighborhood of $x$ contains at least one of the elements of $\mathscr{B}$.*

*A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.*

**Theorem.** *Let $X$ be a topological space.*

*a) Let $A$ be a subset of $X$. If there is a sequence of points of $A$ converging to $x$, then $x \in \overline{A}$; the converse holds if $X$ is first-countable.*

*b) let $f: X \to Y$. If $f$ is continuous, then for every convergent sequence $x_n \to x$ in $X$, the function $f(x_n)$ converges to $f(x)$. The converse holds if $X$ is first-countable.*

**Definition** (Second Countability Axiom)**.** *If a space $X$ has a countable basis for its topology, then $X$ is said to satisfy the **second countability axiom**, or to be **second-countable**.*

**Motivation:** A topology on a space can have multiple bases of various sizes. We want to settle the size of a basis.

## 2 Examples

**Example.** *$\mathbb{R}^\omega$ is first-countable but not second-countable.*

*Note: this should illustrate the difference between having a countable basis (second-countable) and having a countable basis at each of its points (first-countable).*

**Lemma.** *If $X$ is a space having a countable basis $B$, then any discrete subspace $A$ of $X$ must be countable.*

*Proof.* Choose, for each $a \in A$, a basis element $B_a$ that intersects $A$ in the point $a$ alone.

Then the map $a \mapsto B_a$ is injective; (as if $a \neq b$, the sets $B_a$ and $B_b$ are disjoint). It follows that $A$ must be countable. □

*Proof of Example.* First, note that $\mathbb{R}^\omega$ satisfies first countability axiom, as it is metrizable.

We will show that it is not second-countable. Consider the subspace $A$ of $\mathbb{R}^\omega$ consisting of all sequences of 0's and 1's; this subspace is uncountable.

Furthermore, this space has the discrete topology as for any distinct $x, y \in A$, $\bar{\rho}(x, y) = 1$. By the above lemma, since $A$ is uncountable, it follows that $\mathbb{R}^\omega$ cannot have a countable basis, so it is not second-countable. $\square$

**Example.** $\mathbb{R}^n$ *is second-countable.*

*Proof.* We use the fact $\mathbb{Q}$ is dense in $\mathbb{R}$ (i.e. $\overline{\mathbb{Q}} = \mathbb{R}$).

$$\mathbb{B}_1 = \{B_r(x) \mid x \in \mathbb{R}^n, r \in \mathbb{R}^+\}$$
$$\mathbb{B}_2 = \{B_q(x) \mid x \in \mathbb{Q}^n, q \in \mathbb{Q}^+\}$$

We want to show $\mathbb{B}_2$ is also a basis for $\mathbb{R}^n$. Let $U$ be an open set of $\mathbb{R}^n$, then for all $u \in U$, there exists some $r \in \mathbb{R}^+$ such that $u \in B_r(u) \subseteq U$. By a version of the Archimedean Property, there exists some $q \in \mathbb{Q}$ such that $q \leq r$.

Recall that $u \in \overline{\mathbb{Q}^n}$ if and only if every open set $U$ containing $u$ intersects $\mathbb{Q}^n$ (Theorem 17.5$a$), so there exists $p \in \mathbb{Q}^n$ such that $d(u, p) < q/2$. We claim

$$u \in B_{q/2}(p) \subseteq B_r(u) \subseteq U, \text{ and}$$
$$\bigcup B_2 = U.$$

$\square$