MATH104: Topology **Fall 2023**

Homework 1

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Chapter 1 (Set Theory and Logic) Problems.

Section 2 (Functions), 2.5

In general, let us denote the *identity function* for a set C by i_C. That is, define $i_C\colon C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f\colon A \to B$, we say that a function $g\colon B \to A$ is a *left inverse* for f if $g \circ f = i_A$; and we say that $h\colon B \to A$ is a *right inverse* for f if $f \circ h = i_B$.

a) **Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.**

Solution. Suppose that f has a left inverse. Then there exists a function $g\colon B \to A$ such that

$$g \circ f = i_A.$$

Suppose that $f(a) = f(a')$ for two (not necessarily distinct) elements a and a' in A. Note that since $f(a) = f(a')$, it follows that

$$g(f(a)) = g(f(a'))$$

However, since $g \circ f = i_A$, we know that $g(f(a)) = (g \circ f)(a) = a$ and $g(f(a')) = (g \circ f)(a') = a'$. Thus, it follows that $a = a'$ and so f is injective. We conclude that if f has a left inverse, f is injective.

Similarly, suppose that f has a right inverse. Then there exists a function $h\colon B \to A$ such that

$$f \circ h = i_B.$$

Consider any $b \in B$. Note that $(f \circ h)(b) = f(h(b)) = b$. Consequently, for any $b \in B$, there exists an element $a = h(b)$ in A such that $f(a) = b$, and equivalently, f is surjective. We conclude that if f has a right inverse, f is surjective. ■

b) **Give an example of a function that has a left inverse but no right inverse.**

Solution. From part (a), we know that a function that has a left inverse but no right inverse is injective but not surjective. One such function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $f(x) = x^2$ is one such function; its left inverse is the square root function and it has no right inverse as it is not surjective – for example, $f(-3) = f(3) = 9$. ■

c) **Give an example of a function that has a right inverse but no left inverse.**

Solution. From part (a), we know that a function that has a right inverse but no left inverse is surjective but not injective. $f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^2$ is one such function; its right inverse is the square root function and it has no left inverse as it is not injective – for example, $f(-3) = f(3) = 9$. ■

d) **Can a function have more than one left inverse? More than one right inverse?**

Solution. A function can have more than one left inverse. Consider the sets $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Define the function $f\colon A \to B$ as $f(a) = a$ for any element $a \in A$. Define $g\colon B \to A$ such that $g(1) = 1$, $g(2) = 2$, and $g(3) = 1$, and g' similarly, such that $g'(1) = 1$, $g'(2) = 2$, and $g'(3) = 2$. g and g' are two distinct left inverses of f.

Similarly, a function can have more than one right inverse. Consider the sets $A = \{1, 2\}$ and $B = \{1\}$. Define the function $f\colon A \to B$ as $f(1) = f(2) = 1$. Furthermore, define $h\colon B \to A$ such that $h(1) = 1$ and h' similarly, such that $h'(1) = 2$. h and h' are two distinct right inverses of f. ■

e) **Show that if a function f has both a left inverse g and right inverse h, then f is bijective and $g = h = f^{-1}$.**

Solution. Consider $(g \circ f) \circ h$ and $g \circ (f \circ h)$, which are equivalent due to the associativity of functions. Note that since g is a left inverse of f, $g \circ f = id_A$; similarly, since h is a right inverse of f, $f \circ h = id_B$. It follows that

$$(g \circ f) \circ h = id_A \circ h = h \text{ and } g \circ (f \circ h) = g \circ id_B = g.$$

Thus, $g = h$ and they are both equal to f^{-1}, as f^{-1} is by definition the function that is both the right and left inverse of f. ■

Section 3 (Relations), 3.13

Prove the following Theorem: ***If an ordered set A has the least upper bound property, then it has the greatest lower bound property.***

Solution. Let A be an ordered set with the least upper bound property and let B be a nonempty subset of A that is bounded below. Consider the set of lower bounds of B, which we denote as $L(B)$. Since B is bounded below, $L(B)$ is nonempty. Fix some element $b \in B$, and consider any element $l \in L(B)$. Since l is a lower bound in B, by definition, $l \leq b$.

Consequently, the element b is an upper bound of $L(B)$. $L(B)$ is a nonempty subset of A that is bounded above by the element b in A. Since A has the least upper bound property, we know that $L(B)$ must have a least upper bound, which we will denote l'. We claim that l' is the greatest lower bound of B; to prove this, we will show that l' is both a lower bound of B and that it is the greatest such lower bound.

First, we claim that l' is a lower bound of B. Consider any $b \in B$. By construction, b is an upper bound for $L(B)$. Furthermore, l' is the least upper bound of $L(B)$; consequently, $l' \leq b$ for any $b \in B$, and so l' is a lower bound for B.

It remains to show that l' is in fact the greatest lower bound of B. Note that since l' is a lower bound of B, l' is in $L(B)$. Consider any other lower bound l of B, where l is in $L(B)$ by definition. Since by construction, l' is the least upper bound of $L(B)$, it follows that $l \leq l'$ for any $l \in L(B)$. Thus, l' is the greatest lower bound of B.

We conclude that B, an arbitrary nonempty subset of A that is bounded below, has a greatest lower bound. Thus, every nonempty subset of A that is bounded below has a greatest lower bound and so by definition, A has the greatest lower bound property. ∎

Section 10 (Well-Ordered Sets), 10.7

Let J be a well-ordered set. A subset J_0 of J is said to be *inductive* if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \implies \alpha \in J_0.$$

Theorem (The Principle of Transinfinite Induction). If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.

Solution. Let J_0 be an inductive subset of J, and assume for the sake of contradiction that $J_0 \neq J$. Then the set $J \setminus J_0$ is nonempty, and since it is a subset of the well-ordered set J, it by definition has a minimal element, which we will denote as m.

Since m is by definition the minimal element in J that is not in J_0, all elements smaller than J under the well-order relation are in J_0. Equivalently, the section S_m is a subset of J_0. Since J_0 is an inductive subset of J, this implies that m is itself in J_0, which contradicts the fact that m was defined to be the minimal element in $J \setminus J_0$.

Thus, J_0 must be equal to J. We conclude that if J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$. ■