MATH104: Topology **Fall 2023**

Homework 11

David Yang

Chapter 13 (Classification of Covering Spaces) Problems.

Section 79 (Equivalence of Covering Spaces), 79.5(b)

Let $T = S^1 \times S^1$ be the torus; let $x_0 = b_0 \times b_0$. Prove the following:

Theorem. If E is a covering space of T, then E is homeomorphic either to $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T.

(Hint: You may use the following result from algebra: if F is a free abelian group of rank 2 and N is a nontrivial subgroup, then there is a basis a_1, a_2 for F such that either (1) ma_1 is a basis for N, for some positive integer m, or (2) ma_1, na_2 is a basis for N, where m and n are positive integers.)

Solution. Note that the fundamental group of $T = S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. For any covering map from a covering space E to T, we know that the induced homomorphism of the covering map gives a subgroup of the fundamental group of T, namely $\mathbb{Z} \times \mathbb{Z}$. Consequently, to consider the covering spaces of T, it suffices to consider the subgroups of $\mathbb{Z} \times \mathbb{Z}$, a free abelian group of rank 2. The subgroups of $\mathbb{Z} \times \mathbb{Z}$ include the trivial subgroup, the subgroup generated by ma_1, or the subgroup generated by ma_1 and na_2, where m and n are positive integers and a_1 and a_2 represent some basis for $\mathbb{Z} \times \mathbb{Z}$.[1]

Without loss of generality, let this basis of $\mathbb{Z} \times \mathbb{Z}$ be the canonical basis $(1, 0)$ and $(0, 1)$.[2] We will show that there are covering spaces corresponding to each of the three flavors of subgroups of $\mathbb{Z} \times \mathbb{Z}$.

Recall from Example 1 that the covering spaces of S^1 are $\mathbb{R}$ and S^1; the respective covering maps include $p_m : S^1 \to S^1$ with $p_m(z) = z^m$ for any positive integer m, and $q : \mathbb{R} \to S^1$ with $q(t) = (\cos(2\pi t), \sin(2\pi t))$. The subgroups of $\mathbb{Z}$, the fundamental group of S^1, induced by the covering maps are $m\mathbb{Z}$ and the trivial subgroup, respectively. By Theorem 53.3, the product of covering maps is a covering map.

Consequently, $p_m \times p_n : S^1 \times S^1 \to S^1 \times S^1$ is a covering map from T to T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times 0$ and $0 \times n$ for positive integers m and n. Similarly, $p_m \times q : S^1 \times \mathbb{R} \to S^1 \times S^1$ is a covering map from $S^1 \times \mathbb{R}$ to T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $m \times 0$ for positive integer m. Finally, $q \times q : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map from $\mathbb{R}^2$ to T corresponding to the trivial subgroup of $\mathbb{Z} \times \mathbb{Z}$.

These give us three distinct covering spaces for T. By Theorem 79.2, we know there is an equivalence between covering spaces when the induced subgroups are equal. Since the respective induced

[1] the form for the latter two nontrivial subgroups follows directly from the hint.

[2] this step requires further justification. The idea presented in office hours is to use 79.5(a) – that every isomorphism of $\pi_1(T, x_0)$ with itself is induced by a homeomorphism of T with itself that maps x_0 to x_0, and to somehow use a homomorphism between the canonical basis and ma_1 and na_2.

subgroups represent all subgroups of $\mathbb{Z}\times\mathbb{Z}$, the fundamental group of T, we know that any covering space E of T is homeomorphic to one of the covering spaces represented above: $\mathbb{R}^2$, $S^1 \times \mathbb{R}$, or to T, as desired. $\blacksquare$

<u>Section 80 (The Universal Covering Space), 80.1(a)</u>

Let $q\colon X \to Y$ and $r\colon Y \to Z$ be maps; let $p = r \circ q$.

a) **Let q and r be covering maps. Show that if Z has a universal covering space, then p is a covering map. (Compare Exercise 4 of Section 53.)**[3]

Solution. Let E be the universal covering space of Z; E is simply connected. By definition, there is a covering map $s\colon E \to Z$. By Theorem 80.3, since $s\colon E \to Z$ and $r\colon Y \to Z$ are covering maps, there exists a covering map $t\colon E \to Y$ such that $s = r \circ t$. Furthermore, by Theorem 80.3, since $t\colon E \to Y$ and $q\colon X \to Y$ are covering maps, there exists a covering map $u\colon E \to X$ such that $t = q \circ u$.

Note that $u\colon E \to X$ and $s\colon E \to Z$ are covering maps. Furthermore, we have that

$$p \circ u = (r \circ q) \circ u = r \circ (q \circ u) = r \circ t = s$$

by construction. By Lemma 80.2(b), since $s = p \circ u$, and u and s are covering maps, then so is p. Thus, if q and r are covering maps, and if Z has a universal covering space, then p is a covering map. ■

[3] We replace the requirement that $r^{-1}(z)$ is finite for each $z \in Z$ with the condition that Z has a universal covering space to get that p is a covering map.

Section 81 (Covering Transformations), 81.3

Let $p\colon X \to B$ be a covering map (not necessarily regular); let G be its group of covering transformations.

a) **Show that the action of G on X is properly discontinuous.**

Solution. Let $x \in X$, so that $p(x) \in B$. Since p is a covering map, there exists a neighborhood V of $p(x)$ in B that is evenly covered by p. Equivalently, the inverse image $p^{-1}(V)$ is a disjoint union of neighborhoods in U_α in X such that the restriction of p to U_α is a homeomorphism of U_α onto V, for each α. Let U be the neighborhood of $p^{-1}(V)$ in X containing x. We will show that $g(U)$ and U are disjoint, for any $g \neq e$.

Suppose for the sake of contradiction that there exists some element $y \in g(U) \cap U$. Then $y \in U$ and $y = g(z)$ for some $z \neq y$ in U (as a non-identity covering transformation has no fixed points). Since g is a covering transformation, it follows that $p \circ g = p$, so $p(g(z)) = p(z)$. Simplifying, we get that $p(y) = p(z)$, where both y and z are in U and $y \neq z$. The restriction of p to U cannot be a homeomorphism of U onto V, as it is not injective; thus, we arrive at a contradiction, and conclude that $g(U)$ and U are disjoint for every $g \neq e$ in G.

By definition, it follows that the action of G on X is properly discontinuous. ■

Let G be a group of homeomorphisms of X. The action of G on X is said to be fixed-point free if no element of G other than the identity e has a fixed point. Show that if X is Haussdorf, and if G is a finite group of homeomorphisms of X whose action is fixed-point free, then the action of G is properly discontinuous.

Solution. Let $x \in X$. Let $\{g_1, \ldots, g_n\}$ be the finite group of homeomorphisms in G of X that are not equal to the identity. Since the action of G on X is fixed-point free, it follows that $g_i(x) \neq x$ for all $i \in \{1, \ldots, n\}$. Since x and $g_i(x)$ are distinct points in X and X is Hausdorff, it follows that there are disjoint open sets V_i and W_i about x and $g(x)$ that are disjoint in X for each $i \in \{1, \ldots, n\}$.

Let $\tilde{V} = \bigcap_{i \in \{1,\ldots,n\}} V_i$; by construction, $\tilde{V}$ is the intersection of finitely many open sets, so it is itself open. Consider

$$U = \bigcap_{i \in \{1,\ldots,n\}} g_i^{-1}(W_i) \cap \tilde{V}$$

Since each g_i is a homeomorphism, they are each continuous, and so each $g_i^{-1}(W_i)$ is open, and by construction contains x. Consequently, each $g_i^{-1}(W_i) \cap \tilde{V}$ is open, and so U, the intersection of finitely many such open sets, is also open. Furthermore, U is nonempty, as it contains x by construction. Consider $U = g^{-1}(W) \cap V$.

Note that for each g_i, $U \subseteq V_i$, and $U \subseteq g_i^{-1}(W_i)$, so $g_i(U) \subseteq W_i$. Since each V_i and W_i are disjoint open sets by construction, U and $g_i(U)$ must be disjoint open sets, for all $i \in \{1, \ldots, n\}$.

Since for every $x \in X$, there is a neighborhood U of x such that for all g_i in the finite group of homeomorphisms of X that are not the identity, $g_i(U)$ is disjoint from U, the action of G is properly discontinuous, as desired. ∎