

## Homework 1

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*Chapter 1 (Set Theory and Logic) Problems.*Section 2 (Functions), 2.5

In general, let us denote the *identity function* for a set  $C$  by  $i_C$ . That is, define  $i_C: C \rightarrow C$  to be the function given by the rule  $i_C(x) = x$  for all  $x \in C$ . Given  $f: A \rightarrow B$ , we say that a function  $g: B \rightarrow A$  is a *left inverse* for  $f$  if  $g \circ f = i_A$ ; and we say that  $h: B \rightarrow A$  is a *right inverse* for  $f$  if  $f \circ h = i_B$ .

- a) Show that if  $f$  has a left inverse,  $f$  is injective; and if  $f$  has a right inverse,  $f$  is surjective.

*Solution.* Suppose that  $f$  has a left inverse. Then there exists a function  $g: B \rightarrow A$  such that

$$g \circ f = i_A.$$

Suppose that  $f(a) = f(a')$  for two (not necessarily distinct) elements  $a$  and  $a'$  in  $A$ . Note that since  $f(a) = f(a')$ , it follows that

$$g(f(a)) = g(f(a'))$$

However, since  $g \circ f = i_A$ , we know that  $g(f(a)) = (g \circ f)(a) = a$  and  $g(f(a')) = (g \circ f)(a') = a'$ . Thus, it follows that  $a = a'$  and so  $f$  is injective. We conclude that if  $f$  has a left inverse,  $f$  is injective.

Similarly, suppose that  $f$  has a right inverse. Then there exists a function  $h: B \rightarrow A$  such that

$$f \circ h = i_B.$$

Consider any  $b \in B$ . Note that  $(f \circ h)(b) = f(h(b)) = b$ . Consequently, for any  $b \in B$ , there exists an element  $a = h(b)$  in  $A$  such that  $f(a) = b$ , and equivalently,  $f$  is surjective. We conclude that if  $f$  has a right inverse,  $f$  is surjective. ■

- b) Give an example of a function that has a left inverse but no right inverse.

*Solution.* From part (a), we know that a function that has a left inverse but no right inverse is injective but not surjective. One such function  $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is one such function; its left inverse is the square root function and it has no right inverse as it is not surjective – for example,  $f(-3) = f(3) = 9$ . ■

- c) Give an example of a function that has a right inverse but no left inverse.

*Solution.* From part (a), we know that a function that has a right inverse but no left inverse is surjective but not injective.  $f: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  defined by  $f(x) = x^2$  is one such function; its right inverse is the square root function and it has no left inverse as it is not injective – for example,  $f(-3) = f(3) = 9$ . ■

d) **Can a function have more than one left inverse? More than one right inverse?**

*Solution.* A function can have more than one left inverse. Consider the sets  $A = \{1, 2\}$  and  $B = \{1, 2, 3\}$ . Define the function  $f: A \rightarrow B$  as  $f(a) = a$  for any element  $a \in A$ . Define  $g: B \rightarrow A$  such that  $g(1) = 1$ ,  $g(2) = 2$ , and  $g(3) = 1$ , and  $g'$  similarly, such that  $g'(1) = 1$ ,  $g'(2) = 2$ , and  $g'(3) = 2$ .  $g$  and  $g'$  are two distinct left inverses of  $f$ .

Similarly, a function can have more than one right inverse. Consider the sets  $A = \{1, 2\}$  and  $B = \{1\}$ . Define the function  $f: A \rightarrow B$  as  $f(1) = f(2) = 1$ . Furthermore, define  $h: B \rightarrow A$  such that  $h(1) = 1$  and  $h'$  similarly, such that  $h'(1) = 2$ .  $h$  and  $h'$  are two distinct right inverses of  $f$ . ■

e) **Show that if a function  $f$  has both a left inverse  $g$  and right inverse  $h$ , then  $f$  is bijective and  $g = h = f^{-1}$ .**

*Solution.* Consider  $(g \circ f) \circ h$  and  $g \circ (f \circ h)$ , which are equivalent due to the associativity of functions. Note that since  $g$  is a left inverse of  $f$ ,  $g \circ f = id_A$ ; similarly, since  $h$  is a right inverse of  $f$ ,  $f \circ h = id_B$ . It follows that

$$(g \circ f) \circ h = id_A \circ h = h \text{ and } g \circ (f \circ h) = g \circ id_B = g.$$

Thus,  $g = h$  and they are both equal to  $f^{-1}$ , as  $f^{-1}$  is by definition the function that is both the right and left inverse of  $f$ . ■

**Prove the following Theorem: *If an ordered set  $A$  has the least upper bound property, then it has the greatest lower bound property.***

*Solution.* Let  $A$  be an ordered set with the least upper bound property and let  $B$  be a nonempty subset of  $A$  that is bounded below. Consider the set of lower bounds of  $B$ , which we denote as  $L(B)$ . Since  $B$  is bounded below,  $L(B)$  is nonempty. Fix some element  $b \in B$ , and consider any element  $l \in L(B)$ . Since  $l$  is a lower bound in  $B$ , by definition,  $l \leq b$ .

Consequently, the element  $b$  is an upper bound of  $L(B)$ .  $L(B)$  is a subset of  $A$  that is bounded above by the element  $b$  in  $A$ . Since  $A$  has the least upper bound property, we know that  $L(B)$  must have a least upper bound, which we will denote  $l'$ . We claim that  $l'$  is the greatest lower bound of  $B$ ; to prove this, we will show that  $l'$  is both a lower bound of  $B$  and is the greatest such lower bound.

First, we claim that  $l'$  is a lower bound of  $B$ . Consider any  $b \in B$ . By construction,  $b$  is an upper bound for  $L(B)$ . Furthermore,  $l'$  is the least upper bound of  $L(B)$ ; consequently,  $l' \leq b$  for any  $b \in B$ , and so it is a lower bound for  $B$ .

It remains to show that  $l'$  is in fact the greatest lower bound of  $B$ . Note that since  $l'$  is a lower bound of  $B$ ,  $l'$  is in  $L(B)$ . Consider any other lower bound  $l$  of  $B$ , where  $l$  is in  $L(B)$  by definition. Since by construction,  $l'$  is the least upper bound of  $L(B)$ , it follows that  $l \leq l'$ . Thus,  $l'$  is the greatest lower bound of  $B$ .

We conclude that  $B$ , an arbitrary nonempty subset of  $A$  that is bounded below, has a greatest lower bound. Thus, every nonempty subset of  $A$  that is bounded below has a greatest lower bound and so by definition,  $A$  has the greatest lower bound property. ■

Let  $J$  be a well-ordered set. A subset  $J_0$  of  $J$  is said to be *inductive* if for every  $\alpha \in J$ ,

$$(S_\alpha \subset J_0) \implies \alpha \in J_0.$$

**Theorem (The Principle of Transfinite Induction).** *If  $J$  is a well-ordered set and  $J_0$  is an inductive subset of  $J$ , then  $J_0 = J$ .*

*Solution.* Let  $J_0$  be an inductive subset of  $J$ , and assume for the sake of contradiction that  $J_0 \neq J$ . Then the set  $J \setminus J_0$  is nonempty, and since it is a subset of the well-ordered set  $J$ , it by definition has a minimal element, which we will denote as  $m$ .

Since  $m$  is by definition the minimal element in  $J$  that is not in  $J_0$ , all elements smaller than  $J$  under the well-order relation are in  $J_0$ . Equivalently, the section  $S_m$  is a subset of  $J_0$ . Since  $J_0$  is an inductive subset of  $J$ , this implies that  $m$  is itself in  $J_0$ , which contradicts the fact that  $m$  is an element in  $J \setminus J_0$ .

Thus,  $J_0$  must be equal to  $J$ . We conclude that if  $J$  is a well-ordered set and  $J_0$  is an inductive subset of  $J$ , then  $J_0 = J$ . ■