MATH104: Topology **Fall 2023**

Homework 4

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Chapter 9 (The Fundamental Group) Problems.

Section 53 (Covering Spaces), 53.3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{b}$.

Solution. We begin with the forward implication. Let $[f] \in \pi_1(X, x_0)$ and let α and β be two paths from x_0 to x_1. Since $[\bar{\alpha}] * [\alpha] = e_{x_1}$ and $[\beta] * e_{x_1} * [\bar{\beta}] = e_{x_0}$, it follows that

$$[f] = [\beta] * [\bar{\alpha}] * [\alpha] * [\bar{\beta}] * [f].$$

Equivalently,

$$[f] = [\beta * \bar{\alpha}] * [\alpha * \bar{\beta}] * [f].$$

Furthermore, note that $[\alpha * \bar{\beta}]$ is a loop based at x_0, so it is in $\pi_1(X, x_0)$, which is abelian. Consequently, $[\alpha * \bar{\beta}]$ commutes with $[f]$, so $[\alpha * \bar{\beta}] * [f] = [f] * [\alpha * \bar{\beta}]$. This gives us

$$\begin{aligned}
[f] &= [\beta * \bar{\alpha}] * [\alpha * \bar{\beta}] * [f] \\
&= [\beta * \bar{\alpha}] * [f] * [\alpha * \bar{\beta}] \\
&= [\beta] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\beta}]
\end{aligned}$$

Finally, multiplying both sides by $[\bar{\beta}]$ on the left and by $[\beta]$ on the right and simplifying, we get that

$$[\bar{\beta}] * [f] * [\beta] = [\bar{\alpha}] * [f] * [\alpha]$$

and this is equivalent to

$$\hat{\alpha}([f]) = \hat{\beta}([f]).$$

We conclude that for any two paths α and β from x_0 to x_1, $\hat{\alpha} = \hat{\beta}$, as desired.

It remains to show the reverse implication. Suppose that for any two paths α and β of paths from x_0 to x_1, $\hat{\alpha} = \hat{b}$. Let $[f_1]$ and $[f_2]$ be distinct path homotopy classes in $\pi_1(X, x_0)$. To show that $\pi_1(X, x_0)$ is abelian, we will show that $[f_1] * [f_2] = [f_2] * [f_1]$.

Note that $f_1 * \alpha$ and $f_2 * \alpha$ are two paths from x_0 to x_1, so we know that $\widehat{f_1 * \alpha} = \widehat{f_2 * \alpha}$. It follows that $\widehat{f_1 * \alpha}([f_1]) = \widehat{f_2 * \alpha}([f_1])$, so

$$[\overline{f_1 * \alpha}] * [f_1] * [f_1 * \alpha] = [\overline{f_2 * \alpha}] * [f_1] * [f_2 * \alpha].$$

Simplifying, we have that

$$\begin{aligned}
& [\overline{f_1 * \alpha}] * [f_1] * [f_1 * \alpha] = [\overline{f_2 * \alpha}] * [f_1] * [f_2 * \alpha] \\
\implies & [\bar{\alpha} * \bar{f_1}] * [f_1] * [f_1 * \alpha] = [\bar{\alpha} * \bar{f_2}] * [f_1] * [f_2 * \alpha] \\
\implies & [\bar{\alpha}] * [\bar{f_1}] * [f_1] * [f_1] * [\alpha] = [\bar{\alpha}] * [\bar{f_2}] * [f_1] * [f_2] * [\alpha].
\end{aligned}$$

Multiplying both sides on the left by $[\alpha]$ and on the right by $[\bar{\alpha}]$ and simplifying further, we have that

$$\begin{aligned} &[\bar{\alpha}] * [\bar{f}_1] * [f_1] * [f_1] * [\alpha] = [\bar{\alpha}] * [\bar{f}_2] * [f_1] * [f_2] * [\alpha]. \\ \implies &[f_1] = [\bar{f}_2] * [f_1] * [f_2]. \end{aligned}$$

Finally, multiply both sides on the left by $[f_2]$, we get that

$$[f_2] * [f_1] = [f_1] * [f_2]$$

as desired. Thus, we conclude that for any two elements $[f_1]$ and $[f_2]$ in $\pi_1(X, x_0)$, $[f_1] * [f_2] = [f_2] * [f_1]$, so $\pi_1(X, x_0)$ is abelian, as desired. ■

Section 53 (Covering Spaces), 53.6(b)

Let $p\colon E \to B$ be a covering map.

b) **If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.**

Solution. ■