Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor Allison N. Miller, at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

Contents

1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)
A function $f: A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f.

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)
A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)
An **equivalence relation** $\sim$ on a set A is a relation C on A having the following three properties:

1. (Reflexivity) $x \sim x$ for every x in A.
2. (Symmetry) If $x \sim y$, then $y \sim x$.
3. (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)
A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz, then xCz.

Note: the relation C is often replaced as $<$, just as how it is synonyous with $\sim$ in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx, contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively.

The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Immediate Predecessors and Successors)

If X is a set and $<$ is an order relation on X, and if $a < b$, the **open interval** (a, b) on X is the set

$$(a, b) = \{x \mid a < x < b\}.$$

If this set is empty, a is the **immediate predecessor** of b and b is the **immediate successor** of a.

Definition 1.6 (Order Type)

Suppose that A and b are two sets with order relations $<_A$ and $<_B$, respectively. A and B have the same **order type** if there is a bijective correspondence between them that preserves order.

That is, if there exists a bijective function $f \colon A \to B$ such that

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2).$$

Example. The interval $(-1, 1)$ of real numbers has the same order type as $\mathbb{R}$. The function $f \colon (-1, 1) \to B$ such that

$$f(x) = \frac{x}{1 - x^2}$$

is an order-preserving bijective correspondence.

Definition 1.7 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that elements is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that elements is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.8 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \ldots, n\}$, $\mathbb{Z}_+$), tuples, sequences, and Cartesian products.

Definition 1.9 (ω-tuple / Sequence)

Given a set X, a ω-tuple of elements of X is a function

$$\mathbf{x} \colon \mathbb{Z}_+ \to X$$

also known as a **sequence** (or infinite sequence) of elements of X.

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

1.7 Countable and Uncountable Sets

Definition 1.10 (Countably Infinite)

A set A is infinite if it is not finite. It is **countably infinite** of there is a bijective correspondence

$$f \colon A \to \mathbb{Z}_+.$$

Example. The set $\mathbb{Z}$ of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f \colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ and $g \colon A \to \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. $\square$

Definition 1.11 (Countable and Uncountable Sets)

A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0,1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and $\mathbb{R}$ are examples of uncountable sets.

Theorem 1.2. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.
2. There is a surjective function $f\colon \mathbb{Z}_+ \to B$.
3. There is an injective function $g\colon B \to \mathbb{Z}_+$.

Theorem 1.3 (Countable Union of Countable Sets)**.** A countable union of countable sets is countable.

1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

1.9 Infinite Sets and the Axiom of Choice

Theorem 1.4. Let A be a set. The following statements about A are equivalent:

1. A is infinite.
2. There exists an injective function $f\colon \mathbb{Z}_+ \to A$
3. There exists a bijection of A with a proper subset of itself.

Theorem 1.5 (Axiom of Choice)**.** Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

1.10 Well-Ordered Sets

Definition 1.12 (Well-Ordered Sets)

A set A with an order relation $<$ is **well-ordered** if every nonempty subset of A has a smallest element.

Example. The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a well-ordered set in the dictionary order.

However, the dictionary order does not give a well-ordering of the set $(\mathbb{Z}_+)^\omega$.

Theorem 1.6 (Well-Ordering Theorem; Zermelo, 1904)**.** If A is a set, there exists an order relation on A that is a well-ordering.

Corollary 1.1. There exists an uncountable well-ordered set.

Definition 1.13 (Section of a Set)

Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set

$$S_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}.$$

S_α is the **section** of X by α.

Lemma 1.1 (First Uncountable Ordinal)**.** There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Theorem 1.7. If A has a countable subset of S_Ω, then A has an upper bound in S_Ω.

1.11 The Maximum Principle

Definition 1.14 (Partial Order)

Given a set A, a relation $\prec$ on A is a **strict partial order** on A if it has the following properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

If the relation $\prec$ is instead $\preceq$, where $a \preceq b$ implies $a = b$ or $a \prec b$, $\preceq$ is a **partial order** on A.

Remark. These are the second and third properties of a simple order, defined in Definition 1.4. Consequently, a strict partial order behaves like a simple order except that it need not be true that for every pair of distinct x and y in the set, either $x \prec y$ or $y \prec x$.

Theorem 1.8 (The Maximum Principle)**.** Let A be a set and let $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered subset B of A.

Example. If $\mathcal{A}$ is the collection of all circular regions in the plane under the "proper subset of" relation, a maximal simply ordered subcollection of $\mathcal{A}$ consists of all circular regions with centers at the origin.

Definition 1.15

Let A be a set and let $\prec$ be a strict partial order on A. If B is a subset of A, an **upper bound** on B is an element c of A such that for every b in B, either $b = c$ or $b \prec c$.

A **maximal element** of A is an element m on A such that for no element a of A does the relation $m \prec a$ hold.

Theorem 1.9 (Zorn's Lemma)**.** Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

2 Chapter 2: Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.1 (Topology)
A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is a **topological space**.

Definition 2.2 (Open Set)
If X is a topological space with topology $\mathcal{T}$, a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Remark. Using the open set terminology, a topological space can be thought of as a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

Example. The topologies on $X = \{a, b, c\}$ can be specified by permuting a, b, and c and specifying the open sets defined by subsets of X.

However, note that not every collection of subsets of X is a topology on X: consider

$$\{\{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

This is not closed under intersection.

Example. Let X be any set. The collection of all subsets of X is a topology on X known as the **discrete topology**.

The collection consisting of only $\varnothing$ and X is also a topology on X known as the **indiscrete topology** (or **trivial topology**).

Example. Let X be a set and let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or all of X. Then $\mathcal{T}_f$ is a topology on X known as the **finite complement topology**.

Similarly, let X be a set and let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X.

Definition 2.3 (Finer and Coarser Topologies)

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, then $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$.

In the resepctive situations, we say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$; and $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$.

$\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

Intuition. Think of a topological space as something like a truckload full of gravel. Then the pebbles and all unions of collections of pebbles are the open sets.

Furthermore, if the pebbles are smashed into smaller pieces, the collection of open sets has been enlarged, adn the topology, like the gravel, is made finer by the operation.

Remark. The finer and coaser terminology can also be replaced by **larger** and **smaller** as well as **stronger** and **weaker**, respectively.

2.13 Basis for a Topology

Instead of specifying a topology by describing the entire collection $\mathcal{T}$ of open sets, it is often easier to specify a smaller collection of subsets of X and to define the topology from that collection.

Definition 2.4 (Basis for Topology)

If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (**basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 = B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these conditions, then the **topology $\mathcal{T}$ generated by** $\mathcal{B}$ is defined as follows: a subset U of X is said to be open in X (or an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$ (each basis element is itself an element of $\mathcal{T}$).

Remark. The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ satisfies the requirements for a topology on X:

1. The empty set and X are both in $\mathcal{T}$
2. $U = \bigcup_{\alpha \in J} U_\alpha$ for an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}$ belongs to $\mathcal{T}$
3. The intersection of two elements U_1 and U_2 in $\mathcal{T}$ is also in $\mathcal{T}$; the finite intersection case

follows from induction.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane; $\mathcal{B}$ is a basis. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every $x \in U$ lies in some circular region contained in U.

Example. If X is any set, the collection of all one-point subsets in X is a basis for the discrete topology on X.

Lemma 2.1. Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Remark. The above lemma tells us that every open set U in X can be expressed as a union of basis elements. However, unlike the notion of a basis in linear algebra, this expression may not be unique.

Lemma 2.2 (Obtaining a Basis from a Topology)**.** Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 2.3 (Comparing Fineness of Topologies)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.

2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Intuition. Recall the analogy between a topological space and a truckload of gravel. Here, the pebbles are the basis elements of the topology; after the pebbles are smashed into dust, the dust particles are the basis elements for the new topology. The new topology is finer than the old one, and the dust particle was previously contained in a pebble.

Remark. The collection of all circular regions in the plane generates the same topology as the collection of all rectangular regions in the plane: each rectangle in the rectangular basis has a circle enclosed in it, and each circle in the circular basis has a rectangle enclosed in it. This tells us that each topology is finer than the other, and so they generate the same topology.

Definition 2.5 (Topologies on $\mathbb{R}$)

Let $\mathcal{B}$ be the collection of open intervals on the real line; the **standard topology** on the real line is the topology generated by $\mathcal{B}$.

Let $\mathcal{B}'$ be the collection of all half-open intervals $[a, b)$ on the real line; the **lower limit topology** on the real line is the topology generated by $\mathcal{B}'$ and is denoted by $\mathbb{R}_\ell$.

Finally, let K be the set of all numbers of the form $\frac{1}{n}$ for $n \in \mathbb{Z}_+$. Let $\mathcal{B}''$ be the collection of all open intervals along with sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ is the **K-topology** on $\mathbb{R}$, denoted by $\mathbb{R}_k$.

Definition 2.6 (Subbasis for a Topology)
A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X.

The **topology generated by the subbasis** $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

2.14 The Order Topology

Definition 2.7 (Order Topology)
Let X be a set with a simple order relation and assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, known as the **order topology**.

If X has no smallest element, there are no sets of type 2, and if X has no largest element, there are no sets of type 3.

Example. The standard topology on $\mathbb{R}$ is the order topology derived from the usual order on $\mathbb{R}$.

The order topology on $\mathbb{R} \times \mathbb{R}$ in the dictionary order has as basis the collection of all intervals of the form $(a \times b, c \times d)$ for $a < c$ and for $a = c$ and $b < d$. The subcollection consisting of only intervals of the second type is also an order topology on $\mathbb{R} \times \mathbb{R}$.

Example. Consider the set $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order, and denote $1 \times n$ as a_n and $2 \times n$ by b_n. Then $X = \{a_1, a_2, \ldots; b_1, b_2, \ldots\}$.

Note that the order topology is not the discrete topology (of which the collection of one-point subsets is a basis), as the one-point set $\{b_1\}$ is not open. Any open set containing b_1 must contain a basis element about b_1, and all basis elements of b_1 contain points in a_i.

Definition 2.8 (Rays of X)

If X is an ordered set and a is an element of X, there are four subsets of X known as the **rays** determined by a: two open rays and two closed rays.

Remark. The open rays in X are open sets in the order topology. Furthermore, the open rays form a subbasis for the order topology on X.

2.15 The Product Topology on $X \times Y$

Definition 2.9

Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 2.1. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y then the collection

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Example. The standard topology on $\mathbb{R}$ is the order topology. The product of this topology with itself is the **standard topology** on $\mathbb{R} \times \mathbb{R}$.

One such basis, by the above theorem, for the standard topology is the collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$.

2.16 The Subspace Topology

Definition 2.10

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, known as the **subspace topology**.

With this topology, Y is a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Theorem 2.2. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Remark. Let X be an ordered set in the order topology and let Y be a subset of X. The order relation on X when restricted to Y makes Y into an ordered set. However, the *resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X.*

Example. Consider the subset $Y = [0, 1]$ on $\mathbb{R}$ in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap Y$ for $(a, b) \in \mathbb{R}$, which is

$$(a,b) \cap Y = \begin{cases} (a,b), & \text{if } a \text{ and } b \text{ are in } Y; \\ [0,b), & \text{if only } b \text{ is in } Y; \\ (a,1], & \text{if only } a \text{ is in } Y; \\ Y \text{ or } \varnothing, & \text{if neither } a \text{ nor } b \text{ is in } Y; \end{cases}$$

Each of these sets is open in Y, but sets of the second and third types are not open in $\mathbb{R}$. Furthermore, note that these sets form a basis for the order topology on Y, so the order topology and subspace topology of Y as a subspace of $\mathbb{R}$ are the same.

Definition 2.11 (Convex Subsets)
Given an ordered set X, the subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y.

Note: intervals and rays in X are convex in X.

Theorem 2.3. Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

2.17 Closed Sets and Limit Points

Definition 2.12 (Closed Set)
A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Example. The subsets $[a, b]$ and $[a, \infty)$ of $\mathbb{R}$ are closed, while $[a, b)$ of $\mathbb{R}$ is neither open nor closed.

Example. In the discrete topology on the set X, every set is both open and closed.

Definition 2.13 (Interior and Closure)
Given a subset A of a topological space X, the **interior** of A, denoted $\operatorname{Int} A$, is the union of all open sets contained in A.

The **closure** of A, denoted $\operatorname{Cl} A$ or $\overline{A}$, is the intersection of all closed sets containing A.

It follows by definition that $\operatorname{Int} A \subset A \subset \overline{A}$.

Theorem 2.4. Let Y be a subspace of X, let A be a subset of Y, and let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Theorem 2.5. Let A be a subset of the topological space X.

a) Then $x \in \overline{A}$ if and only if every open set U containing x (i.e. neighborhood of x) intersects A.

b) Supposing that the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A.

Example. If $A = (0, 1]$, then $\overline{A} = [0, 1]$. If $B = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then $\overline{B} = \{0\} \cup B$.

Example. Consider $Y = (0, 1]$, a subspace of $\mathbb{R}$, and the set $A = (0, \frac{1}{2})$, a subset of Y.

$\overline{A} = [0, \frac{1}{2}]$ in $\mathbb{R}$ while the closure of A in Y is $\overline{A} \cap Y = (0, \frac{1}{2}]$.

Definition 2.14 (Limit Points)

Let A be a subset of the topological space X and let x be a point of X.

x is a **limit point** (or "cluster point", or "point of accumulation") of A if every neighborhood of x intersects A in some point *other than x itself*.

Equivalently, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Theorem 2.6. Let A be a subset of the topological space X and let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'.$$

Definition 2.15 (Hausdorff Spaces)

A topological space X is a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 2.7. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X, known as the **limit** of that sequence of points.

2.18 Continuous Functions

Definition 2.16 (Continuous Functions)

Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Remark. To prove continuity of f, it suffices to show that the inverse image of every basis element is open; it can even suffice to show that the inverse of each subbasis is open.

Theorem 2.8 (Continuity Equivalences). Let X and Y be topological spaces; let $f\colon X \to Y$. Then the following are equivalent:

1. f is continuous.
2. For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.
3. For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
4. For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

If condition 4 holds for the point x of X, we say f is **continuous at the point** x.

Definition 2.17 (Homeomorphism)

Let X and Y be topological spaces; let $f\colon X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1}\colon Y \to X$$

are continuous, then f is a **homeomorphism**.

Equivalently, f is a homeomorphism if there is a bijective correspondence $f\colon X \to Y$ such that $f(U)$ is open if and only if U is open.

Remark. Just as isomorphisms preserve algebraic properties of groups/rings, homeomorphisms preserve topological structures.

Definition 2.18 (Imbedding)

Let $f\colon X \to Y$ be an injective continuous map between topological spaces X and Y. Let Z be the image set $f(X)$ considered as a subspace of Y; then the function $f'\colon X \to Z$ obtained by restricting the range is bijective.

If f' is a homemorphism of X with Z, then the map f is a **topological imbedding** (or **imbedding**) of X in Y.

Example. The function $F : (-1, 1) \to \mathbb{R}$ defined by $F(x) = \frac{x}{1-x^2}$ is a homemomorphism.

Remark. A bijective function $f\colon X \to Y$ can be continuous without being a homomorphism: consider $F : [0, 1) \to S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$; f^{-1} is not continuous, as $f^{-1}([0, \frac{1}{4}))$ does not lie in an open set V of $\mathbb{R}^2$ such that $V \cap S^1 \subset f([0, \frac{1}{4}))$.

Theorem 2.9 (Rules for Constructing Continuous Functions)**.** Let X, Y, and Z be topological spaces.

a) (Constant function) If $f\colon X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

b) (Inclusion) If A is a subspace of X, the inclusion function $j\colon A \to X$ is continuous.

c) (Composites) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

d) (Restricting the domain) If $f\colon X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f \mid A\colon A \to Y$ is continuous.

e) (Restricting or expanding the range) Let $f\colon X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g\colon X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h\colon X \to Z$ obtained by by expanding the range of f is continuous.

f) (Local formulation of continuinity) The map $f\colon X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f \mid U_\alpha$ is continuous for each α.

Theorem 2.10 (The Pasting Lemma)**.** Let $X = A \cup B$ where A and B are closed in X, and let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h\colon X \to Y$, defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Theorem 2.11 (Maps into Products)**.** Let $f\colon A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if the functions $f_1\colon A \to X$ and $f_2\colon A \to Y$ are continuous.

2.19 The Product Topology

Definition 2.19 (Box Topology)

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is the **box topology**.

Definition 2.20 (Projection Mapping)
Let $\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its βth coordinate,

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta.$$

π_β is the **projection mapping** associated with index β.

Definition 2.21 (Product Topology)
Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let S denote the union of these collections, $S = \bigcup_{\beta \in J} \mathcal{S}_\beta$.

The topology generated by the subbasis $\mathcal{S}$ is the **product topology** and the topology $\prod_{\alpha \in J} X_\alpha$ is a **product space**.

Theorem 2.12 (Comparison of Box and Product Topologies)**.** The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α.

The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Remark. For finite products, these topologies are identical. Furthermore, the box topology is in general finer than the product topology.

2.20 The Metric Topology

Definition 2.22 (Metric)
A **metric** on a set X is a function $d \colon X \times X \to R$ having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

There is also the idea of a ε**-ball centered at** x: $B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ which we have grown to love from our Analysis classes.

Definition 2.23 (The Metric Topology)

If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

Remark. Equivalently, we can think of the metric topology in terms of its open sets: U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Definition 2.24

If X is a topological space, X is **metrizable** if there exists a metric d on a set X that induces the topology of X.

A **metric space** is a metrizable space X together with a specific metric d that gives the topology on X.

Theorem 2.13 (Standard Bounded Metric)**.** Let X be a metric space with metric d. Define $\overline{d} \colon X \times X \to \mathbb{R}$ by the equation

$$\overline{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\overline{d}$ is known as the **standard bounded metric** corresponding to d, and it is a metric that induces the same topology as d.

Lemma 2.4. Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each x in X and each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B'_d(x, \delta) \subset B_d(x, \varepsilon).$$

Definition 2.25 (Uniform Metric and Topology)

Given an index set J and points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, the metric $\overline{\rho}$ on $\mathbb{R}^J$ defined by

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$ is known as the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**.

2.21 The Metric Topology (Continued)

Theorem 2.14. Let $f: X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

Lemma 2.5 (The Sequence Lemma)**.** Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

Definition 2.26 (Countable Basis, First Countability Axiom)

A space X is said to have a **countable basis at the point** x if there is a countable collection $\{U_n\}_{n\in\mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains *at least one* of the sets U_n.

A space X that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

Theorem 2.15 (Uniform Limit Theorem)**.** Let $f_n: X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Example. $\mathbb{R}^\omega$ in the box topology is not metrizable.

An uncountable product of $\mathbb{R}$ with itself is not metrizable.

2.22 The Quotient Topology

Definition 2.27 (Quotient Map)

Let X and Y be topological spaces; let $p: X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Equivalently, p is a quotient map if every subset A of Y is closed in Y if and only if $p^{-1}(A)$ is closed in X.

Definition 2.28

A subset C of X is **saturated** (with respect to the surjective map $p: X \to Y$) if C contains every set $p^{-1}(\{y\})$ that it intersects. Thus C is saturated if it equals the converse inverse image of a subset of Y.

The statement "p is a quotient map" is equivalent to the statement that "p is continuous and maps saturated open sets of X to open sets of Y" (or saturated closed sets of X to closed sets of Y).

Definition 2.29 (Open and Closed Maps)

A map $f\colon X \to Y$ is an **open map** if for each open set U of X, the set $f(U)$ is open in Y.

A map $f\colon X \to Y$ is a **closed map** if for each closed set A of X, the set $f(A)$ is closed in Y.

Remark. Any surjective continuous map $p\colon X \to Y$ that is either open or closed is a quotient map.

However, there are quotient maps that are neither open nor closed.

Definition 2.30 (Quotient Topology)

If X is a space and A is a set and if $p\colon X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map, and this is the **quotient topology** induced by p.

The topology itself consists of subsets U of A such that $p^{-1}(U)$ is open in X.

Definition 2.31 (Quotient Space)

Let X be a topological space, and let X^* be a partition of X into distinct subsets whose union is X. Let $p\colon X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

Theorem 2.16. Let $p\colon X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q\colon A \to p(A)$ be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map.
2. If p is either an open map or a closed map, then q is a quotient map.

3 Chapter 3: Connectedness and Compactness

3.23 Connected Spaces

Definition 3.1 (Separation and Connected Spaces)
Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

The space is said to be **connected** if there does not exist a separation of X.

Remark. An equivalent formulation of a connected space is that X is a connected space if and only if the subsets of X that are both open and closed in X are the empty set and X itself.

Lemma 3.1. If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of each other. The space Y is connected if there exists no separation of Y.

Example. The rationals are not connected: if Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an irrational number $a \in (p, q)$ and write Y as the union of the open sets

$$Y \cap (-\infty, a) \text{ and } Y \cap (a, \infty),$$

thus showing that there exists a separation of Y and proving that Y is not connected.

Consequently, the only connected subspaces of $\mathbb{Q}$ are the one-point sets.

Example. Consider the following subset of the plane $\mathbb{R}^2$:

$$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = \frac{1}{x}\}.$$

Then X is not connected, as neither subset contains a limit point of the other.

Lemma 3.2. If the sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Theorem 3.1. The union of a collection of connected subspaces of X that have a point in common is connected.

Theorem 3.2. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Theorem 3.3. The image of a connected space under a continuous map is connected.

Theorem 3.4. A finite cartesian product of connected spaces is connected.

3.24 Connected Subspaces of the Real Line

Definition 3.2 (Linear Continuum)

A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 3.5. If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Theorem 3.6 (Intermediate Value Theorem)**.** Let $f\colon X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Definition 3.3

Given points x and y of the space X, a **path** in X from x to y is a continuous map $f\colon [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Remark. A path-connected space X is connected; suppose that $X = A \cup B$ is a separation of X. Let $f\colon [a, b] \to X$ be any path in X. The set $f([a, b])$ is the image of a connected set, so it is connected, and must lie entirely in A or B. Thus, there is no path X joining a point of A to a point of B, contradicting the assumption of path connectedness of X.

On the other hand, the converse does not hold.

Example. The ordered square I_o^2 is connected but not path connected.

Example. Let S denote the following subset of the plane:

$$S = \{x \times \sin\left(\frac{1}{x}\right) \mid 0 < x < 1\}.$$

$\overline{S}$ is known as the **topologist's sine curve**, and is not path-connected.