Homework 12

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Prove that if G and G' are homeomorphic finite linear graphs, then they have the same Euler characteristic.

Solution. Let G and G' be homeomorphic finite linear graphs. We will first consider the case where G and G' are both connected. Since G and G' are homeomorphic, they have isomorphic fundamental groups. By Theorem 85.2, since G and G' are both finite, connected linear graphs, the cardinality of a system of free generators for the fundamental groups for G and G' are $1 - \chi(G)$ and $1 - \chi(G')$, respectively.

We know that that the fundamental groups of G and G', two finite linear graphs, must be free groups (if G and G' are trees, their fundamental groups are trivial, and we can realize these as a free group with 0 generators). Let the fundamental groups of G and G' be F_m and F_n, the free groups with m and n generators, respectively. If $F_m \cong F_n$, then their abelianizations $\mathbb{Z}^m$ and $\mathbb{Z}^n$ must be isomorphic. By Theorem 67.8, it follows that $m = n$. Thus, the cardinalities of the system of free generators for the fundamental groups of G and G' must be equal, so

$$1 - \chi(G) = 1 - \chi(G'),$$

meaning $\chi(G) = \chi(G')$ as desired.

We now consider the case where G and G' are not connected. Since G and G' are homeomorphic, there is a homeomorphism between their connected components; consequently, G and G' must also have the same number of connected components. We know by the above logic that the Euler characteristics are the same for each pair of homeomorphic connected components G_α and G'_α. Thus, summing over all connected components G_α and G'_α using the property $\chi(G_\alpha) = \chi(G'_\alpha)$, we find that

$$\chi(G) = \sum_{\alpha \in J} \chi(G_\alpha) = \sum_{\alpha \in J} \chi(G'_\alpha) = \chi(G').$$

Thus, if G and G' are homeomorphic finite linear graphs, then they have the same Euler characteristic, as desired. ■

Let $F = \langle a, b\rangle$ be the free group on two generators, and let $F' = [F, F]$. We now know that F', as a subgroup of a free group, is free. Find a set of free generators for F' by using covering space theory.

Solution. Let B represent the wedge of two circles; the fundamental group of B at its intersection point is precisely F. Consider the integer lattice grid covering space E of the wedge of B, where the loops a and b of B lift to one unit horizontal and vertical shifts in E, and let p represent the covering map from E to B. Let $e_0 \in p^{-1}(b_0)$. We claim that $p_*(\pi_1(E, e_0)) = F'$.

We will show containment in both directions. First, let x be generated by the commutators of F, so that $x \in F'$. Since the powers of a and b on every commutator in F sum to 0, so do the powers of a and b in x, which is generated by the commutators. Consequently, when we lift x to E, the net horizontal and vertical shift from e_0 is 0, and so we get a loop in E based at e_0. Thus, $x \in p_*(\pi_1(E, e_0))$, and we conclude that $F' \subseteq p_*(\pi_1(E, e_0))$.

It remains to show that $p_*(\pi_1(E, e_0)) \subseteq F'$. We will use results from Exercise 85.3, which followed from Theorem 84.7. As we did in Exercise 85.3, consider some maximal tree T in E; one such maximal tree consists of all vertical grid lines and one horizontal grid line passing through e_0. From the result of Exercise 85.3 and Theorem 84.7, we know that the system of free generators for the subgroup $p_*(\pi_1(E, e_0))$ is in bijective correspondence with each edge of E not in T, and can be expressed as the set of all $a^m b^n a b^{-n} a^{-(m+1)}$, for integer m and n. Note that each of these generators is itself a commutator:

$$a^m b^n a b^{-n} a^{-(m+1)} = (a^m b^n) a (b^{-n} a^{-m}) a^{-1} = (a^m b^n) a (a^m b^n)^{-1} a^{-1}.$$

Consider some loop ℓ based at e_0 in E, and decompose it into $p_1 q_1 p_2 \dots q_r p_r$ where the p terms represent paths along the maximal tree T and the q terms represent edges of E not in T. Note that the free generator of $p_*(\pi_1(E, e_0))$ corresponding to the edge q_1 is a commutator, and traverses p_1. Similarly, for any $i \in \{1, \dots, r\}$, the free generator of $p_*(\pi_1(E, e_0))$ corresponding to the edge q_i is a commutator, and traverses p_i. Thus, the image of the induced homomorphism of the loop ℓ based at e_0 can be represented as a product of commutators, and so if $x \in p_*(\pi_1(E, e_0))$, then $x \in F'$.

We have shown that $p_*(\pi_1(E, e_0)) = F'$. From Exercise 85.3, we know the system of free generators for $p_*(\pi_1(E, e_0))$ – the set of elements of the form $a^m b^n a b^{-n} a^{-(m+1)}$ for integer m, n. Thus, since $F' = p_*(\pi_1(E, e_0))$, we know that a set of free generators for F' is simply the same set of free generators for $p_*(\pi_1(E, e_0))$, namely

$$\boxed{\{a^m b^n a b^{-n} a^{-(m+1)} \mid m, n \in \mathbb{Z}\}}.$$

■

<u>Section 85 (Subgroups of Free Groups), 85.2</u>

Let F be a free group on two generators α and β. Let H be the subgroup generated by α. Show that H has infinite index in F.

Solution. Suppose for the sake of contradiction that H has finite index $k > 0$ in F. By Theorem 85.3, since F has $1 + 1 = 2$ free generators, H must have $k + 1$ free generators. However, H has one free generator, meaning $k = 0$, which contradicts the assumption of finite index. Thus, H has infinite index in F. ■

Solution. Note that any element β^i for $i \in \mathbb{Z}^+$ gives a distinct coset $\beta^i H$ of F. Thus, since there are infinitely many such elements (which are not necessarily even representative of all cosets of H in F), H must have infinite index in F. ■