MATH104: Topology **Fall 2023**

Homework 10

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Chapter 13 (Classification of Covering Spaces) Problems.

<ins>Section 79 (Equivalence of Covering Spaces), 79.5(b)</ins>

Let $T = S^1 \times S^1$ be the torus; let $x_0 = b_0 \times b_0$. Prove the following:

Theorem. If E is a covering space of T, then E is homeomorphic either to $\mathbb{R}^2$, or to $S^1 \times \mathbb{R}$, or to T.

(Hint: You may use the following result from algebra: if F is a free abelian group of rank 2 and N is a nontrivial subgroup, then there is a basis a_1, a_2 for F such that either (1) ma_1 is a basis for N, for some positive integer m, or (2) ma_1, na_2 is a basis for N, where m and n are positive integers.)

Solution. ∎

Section 80 (The Universal Covering Space), 80.1(a)

Let $q\colon X \to Y$ and $r\colon Y \to Z$ be maps; let $p = r \circ q$.

a) **Let q and r be covering maps. Show that if Z has a universal covering space, then p is a covering map. (Compare Exercise 4 of Section 53.)**

Solution. Let E be the universal covering space of Z; E is simply connected. By definition, there is a covering map $s\colon E \to Z$. By Theorem 80.3, since $s\colon E \to Z$ and $r\colon Y \to Z$ are covering maps, there exists a covering map $t\colon E \to Y$ such that $s = r \circ t$. Furthermore, by Theorem 80.3, since $t\colon E \to Y$ and $q\colon X \to Y$ are covering maps, there exists a covering map $u\colon E \to X$ such that $t = q \circ u$.

Note that $u\colon E \to X$ and $s\colon E \to Z$ are covering maps. Furthermore, we have that

$$p \circ u = (r \circ q) \circ u = r \circ (q \circ u) = r \circ t = s$$

by construction. By Lemma 80.2(b), since $s = p \circ u$, and u and s are covering maps, then so is p. Thus, if q and r are covering maps, and if Z has a universal covering space, then p is a covering map. ∎

Section 81 (Covering Transformations), 81.4

Let G be a group of homeomorphisms of X. The action of G on X is said to be fixed-point free if no element of G other than the identity e has a fixed point. Show that if X is Haussdorf, and if G is a finite group of homeomorphisms of X whose action is fixed-point free, then the action of G is properly discontinuous.

Solution. Let $x \in X$. Let g be a homeomorphism in G not equal to the identity. Consider $g(x)$. Since the action of G on X is fixed-point free, it follows that $g(x) \neq x$. Since x and $g(x)$ are distinct points in X and X is Hausdorff, it follows that there are disjoint open sets V and W about x and $g(x)$ that are disjoint in X.

Since g is a homeomorphism, it is continuous, and so $g^{-1}(W)$ is also open, and contains x. Consider $U = g^{-1}(W) \cap V$. Note that by construction, $U \subseteq V$ and $g(U) \subseteq W$. Since $V \cap W$ is empty, it follows that $g(U)$ and U are disjoint.

Thus, since for every $x \in X$, there is a neighborhood U of x such that $g(U)$ is disjoint from U whenever $g \neq e$, the action of G is properly discontinuous, as desired. ∎