MATH104: Topology **Fall 2023**

Homework 9

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Chapter 11 (The Seifert-van Kampen Theorem) Problems.

Section 70 (The Seifert-van Kampen Theorem), 70.2

Suppose that i_2 is surjective.

a) **Show that j_1 induces an epimorphism**

$$h\colon \pi_1(U, x_0)/M \to \pi_1(X, x_0),$$

where M is the least normal subgroup of $\pi_1(U, x_0)$ containing $i_1(\ker i_2)$. [*Hint:* Show j_1 is surjective.]

Solution. We begin by following the hint and showing that j_1 is surjective. Since i_2 is surjective, it follows that for all $v \in \pi_1(V, x_0)$, there exists some $y \in \pi_1(U \cap V, x_0)$ such that $i_2(y) = v$. It follows then that

$$j_2(v) = j_2(i_2(y)) = i_*(y) = j_1(i_1(y)).$$

Since $j_2(v) = j_1(i_1(y))$ for all $v \in \pi_1(V, x_0)$ (the domain of j_2), we know that $\operatorname{im}(j_2) \subseteq \operatorname{im}(j_1)$. By Theorem 59.1, the images of j_1 and j_2 generate $\pi_1(X, x_0)$. Since $\operatorname{im}(j_2) \subseteq \operatorname{im}(j_1)$, it follows that $\operatorname{im}(j_1)$ generates $\pi_1(X, x_0)$ and so j_1 is surjective.

Let M be the least normal subgroup of $\pi_1(U, x_0)$ containing $i_1(\ker\ i_2)$. We will show that $i_1(\ker i_2) \subseteq \ker j_1$, from which it would follow by the construction of M that M is a normal subgroup of $\ker j_1$. Let $x \in \ker i_2$ where $x \in \pi_1(U \cap V, x_0)$. Then

$$(j_2 \circ i_2)(x) = j_2(i_2(x)) = j_2(e) = e.$$

Furthermore, $(j_2 \circ i_2)(x) = i_*(x) = (j_1 \circ i_1)(x)$, so $(j_1 \circ i_1)(x) = e$. Equivalently, we have that $j_1(i_1(x)) = e$, and so $i_1(x) \in \ker j_1$. Since this holds true for any $x \in \ker i_2$, we have that $i_1(\ker i_2) \subset \ker j_1$.

Since M is by definition the least normal subgroup of $\pi_1(u, x_0)$ containing $i_1(\ker i_2)$, we can conclude that M is a normal subgroup of $\ker j_1$. By the Very Useful Lemma[1], j_1 induces a homomorphism

$$h\colon \pi_1(U, x_0)/M \to \pi_1(X, x_0).$$

Since j_1 is surjective, so is h; this gives us the epimorphism $h\colon \pi_1(U, x_0)/M \to \pi_1(X, x_0)$, as desired. ■

[1] Let $\psi\colon G \to G'$ be a group homomorphism. Suppose N is a normal subgroup of G, and that $N \subseteq \ker(\psi)$. Then there exists a group homomorphism $\overline{\psi}\colon G/N \to G'$ given by $\overline{\psi}(gN) = \psi(g)$.

b) **Show that h is an isomorphism. [*Hint:* Let $H = \pi_1(U, x_0)/M$. Let $\varphi_1 \colon \pi_1(U, x_0) \to H$ be the projection. Use the fact that $\pi_1(U \cap V, x_0)/\ker i_2$ is isomorphic to $\pi_1(V, x_0)$ to define a homomorphism $\varphi_2 \colon \pi_1(V, x_0) \to H$. Use Theorem 70.1 to define a left inverse for h.]**

Solution. We follow the hint. Let $H = \pi_1(U, x_0)/M$, and let $\varphi_1 \colon \pi_1(U, x_0) \to H$ be the projection. By the First Isomorphism Theorem, since i_2 is surjective, we know that $\pi_1(U \cap V, x_0)/\ker i_2$ is isomorphic to $\pi_1(V, x_0)$. Let $f \colon \pi_1(V, x_0) \to \pi_1(U \cap V, x_0)/\ker i_2$ be such an isomorphism.

Note that by the construction of M, we have that $\ker i_2 \subseteq \ker(\varphi_1 \circ i_1)$. Consequently, by the Very Useful Lemma, we have a homomorphism $\varphi_1 \circ i_1 \colon \pi_1(U \cap V, x_0) \to H$. Since f and $\varphi_1 \circ i_1$ are both well-defined homomorphisms, it follows that their composition $\varphi_1 \circ i_1 \circ f$ is also a well-defined homomorphism from $\pi_1(V, x_0)$ to H. Let us use φ_2 to denote this homomorphism.

The assumptions of the Seifert-van Kampen are satisfied. By construction, we have that $\varphi_2 \circ i_2 = \varphi_1 \circ i_1$, for homomorphisms

$$\varphi_1 \colon \pi_1(U, x_0) \to H \text{ and } \varphi_2 \colon \pi_1(V, x_0) \to H.$$

By Theorem 70.1 (Seifert-van Kampen), it follows that there is a unique homomorphism $\Phi \colon \pi_1(X, x_0) \to H$ satisfying $\Phi \circ j_1 = \varphi_1$ and $\Phi \circ j_2 = \varphi_2$.

We claim that Φ is a left inverse of h: note that

$$\Phi \circ h \circ \varphi_1 = \Phi(h(\varphi_1)) = \Phi \circ j_1 = \varphi_1$$

Thus, $\Phi \circ h$ is the identity map of H, and so Φ is the left inverse of h. Thus, h is an isomorphism, as desired. ∎

Section 71 (The Fundamental Group of a Wedge of Circles), 71.5(a)

Let S_n be the circle of radius n in $\mathbb{R}^2$ whose center is at the point $(n, 0)$. Let Y be the subspace of $\mathbb{R}^2$ that is the union of these circles; let p be their common point.

a) **Show that Y is not homeomorphic to a countably infinite wedge of circles, nor to the space of Example 1 (in Section 71).**

Solution. Let X be a countably infinite wedge of the circles T_α. By definition, the topology X is coherent with the subspaces T_α. To show that Y is not homeomorphic to X, we will show that Y is not coherent with its subspaces S_i for $i \in \mathbb{N}$. Let C be the set of Y defined as the set of all points of intersection (with positive y-coordinate) of S_i with the circle of radius $\frac{1}{i}$ centered at the origin, for all $i \in \mathbb{N}$. Each $C \cap S_i$ for $i \in \mathbb{N}$ is a singleton element, and thus, is closed. However, the set C is not closed in Y, as it does not contain its limit point at the origin. Thus, the topology of Y is not coherent with its subspaces S_i for $i \in \mathbb{N}$. Since X is coherent with its subspaces and Y is not, we know that Y cannot be homeomorphic to X.

To show that Y and the infinite earring space in Example 1 are not homeomorphic, we consider a neighborhood around the respective shared points at the origin. A neighborhood around the origin in Y contains at most finitely many loops of the respective circular subspaces, whereas a neighborhood around the origin in the infinite earring contains infinitely many loops of its respective circular subspaces. Thus, since the neighborhoods of the common points in the infinite earring and Y are not homeomorphic, we conclude that Y is also not homeomorphic to the infinite earring space. ■

Hatcher Problem

Let X be the union of n lines through the origin in $\mathbb{R}^3$. Compute the fundamental group of $\mathbb{R}^3 - X$.

Solution. We claim that there is a deformation retract of $\mathbb{R}^3 - X$ to S^2 with $2n$ points removed. To conceptualize this, we describe the deformation retract. For any point of $\mathbb{R}^3 - X$ that is on S^2, the point stays as is. Otherwise, for any point y of $R^3 - X$ that is not on S^2, let ℓ be the line passing through the origin and y. Deformation retract y to its closest intersection of ℓ with S^2. Each of the n lines removed passes through S^2 twice, so this description characterizes a deformation retract from $\mathbb{R}^3 - X$ to S^2 with $2n$ points removed.

Through stereographic projection, we know that there is a homeomorphism from S^2 with a point removed (at ∞ i.e. the north pole) to $\mathbb{R}^2$. By defining one of our $2n$ points removed from S^2 to be the "north pole" and applying the stereographic projection map, we get a homeomorphism from S^2 with $2n$ points removed to $\mathbb{R}^2$ with $2n - 1$ points removed.

Extending the procedure in Section 58 Example 2, which highlights a deformation retract from the doubly punctured plane to the figure eight space (which is the wedge of two circles), we can get a deformation retract from $\mathbb{R}^2$ with $2n - 1$ points to the wedge of $2n - 1$ circles. Thus, the fundamental group of $\mathbb{R}^2$ with $2n-1$ points removed is simply the fundamental group of the wedge of $2n - 1$ circles. By Theorem 71.1, this is the $\boxed{\text{free group with } 2n - 1 \text{ generators}}$. ∎