The Countability Axioms

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1 Introduction and Relevant Theorems

Definition (First Countability Axiom). A space X is said to have a **countable basis at** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Theorem. Let X be a topological space.

a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable.

b) let $f\colon X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the function $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition (Second Countability Axiom). If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, or to be **second-countable**.

Motivation: A topology on a space can have multiple bases of various sizes. We want to settle the size of a basis.

2 Examples

Example. $\mathbb{R}^\omega$ is first-countable but not second-countable.

Note: this should illustrate the difference between having a countable basis (second-countable) and having a countable basis at each of its points (first-countable).

Lemma. If X is a space having a countable basis B, then any discrete subspace A of X must be countable.

Proof. Choose, for each $a \in A$, a basis element B_a that intersects A in the point a alone.

Then the map $a \mapsto B_a$ is injective; (as if $a \neq b$, the sets B_a and B_b are disjoint). It follows that A must be countable. $\square$

Proof of Example. First, note that $\mathbb{R}^\omega$ satisfies first countability axiom, as it is metrizable.

We will show that it is not second-countable. Consider the subspace A of $\mathbb{R}^\omega$ consisting of all sequences of 0's and 1's; this subspace is uncountable.

Furthermore, this space has the discrete topology as for any distinct $x, y \in A$, $\bar{\rho}(x, y) = 1$. By the above lemma, since A is uncountable, it follows that $\mathbb{R}^\omega$ cannot have a countable basis, so it is not second-countable. $\square$

Example. $\mathbb{R}^n$ is second-countable.

Proof. We use the fact $\mathbb{Q}$ is dense in $\mathbb{R}$ (i.e. $\overline{\mathbb{Q}} = \mathbb{R}$).

$$\mathbb{B}_1 = \{B_r(x) \mid x \in \mathbb{R}^n, r \in \mathbb{R}^+\}$$
$$\mathbb{B}_2 = \{B_q(x) \mid x \in \mathbb{Q}^n, q \in \mathbb{Q}^+\}$$

We want to show $\mathbb{B}_2$ is also a basis for $\mathbb{R}^n$. Let U be an open set of $\mathbb{R}^n$, then for all $u \in U$, there exists some $r \in \mathbb{R}^+$ such that $u \in B_r(u) \subseteq U$. By a version of the Archimedean Property, there exists some $q \in \mathbb{Q}$ such that $q \leq r$.

Recall that $u \in \overline{\mathbb{Q}^n}$ if and only if every open set U containing u intersects $\mathbb{Q}^n$ (theorem* 17.5a), so there exists $p \in \mathbb{Q}^n$ such that $d(u, p) < q/2$. We claim

$$u \in B_{q/2}(p) \subseteq B_r(u) \subseteq U, \text{ and}$$
$$\bigcup B_2 = U.$$

$\square$