MATH104: Topology **Fall 2023**

Homework 14

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Swarthmore Honors Exams Problems.

2022, Problem 11

Let X be the one point union of a torus and a 2-sphere.

a) **Compute $\pi_1(X)$.**

Solution. Let x_0 be the shared point between the torus T and 2-sphere S in X. Consider

$$U = T \vee W_1,\ V = S \vee W_2$$

where W_1 and W_2 are neighborhoods of x_0 in S and T that deformation retract to x_0, respectively. Note that U and V are each open in X. Furthermore, the intersection $U \cap V = W_1 \vee W_2$ is simply connected (it deformation retracts to x_0 which has trivial fundamental group and is path-connected – for any two points in the intersection, there is either a path fully in W_1 or W_2 or there is a path in W_1 to x_0 to W_2 or vice versa). By Seifert-Van Kampen, since $U \cap V$ is simply connected, the fundamental group of X is isomorphic to the free product of the fundamental groups of U and V. Thus, since $\pi_1(U, x_0) \cong \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(V, x_0)$ is trivial, we get that $\boxed{\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}}$. ■

b) **Describe the universal cover of X.**

Solution. The universal cover of X is $\mathbb{R} \times \mathbb{R}$ with a 2-sphere attached at each point in $\mathbb{Z} \times \mathbb{Z}$.

It remains to show that this universal cover is simply connected. Note that it is path-connected; $\mathbb{R}^2$ is path-connected, and the union of a 2-sphere at each point in $\mathbb{Z} \times \mathbb{Z}$ is a union of path-connected spaces at a common point, which is path-connected. On the other hand, the fundamental group of the universal cover is also trivial. Consider any loop in the universal covering space. It is composed of a number of components, which are either paths in $\mathbb{R}^2$ or loops around a 2-sphere at an integer grid point. Since the 2-sphere is simply connected, each of the loops around a 2-sphere can be continuously deformed to a constant map at the point of intersection. Consequently, the loop in the universal covering space can deformation retract to a loop in $\mathbb{R}^2$ (by applying the above deformation to each part of the loop around a 2-sphere). This loop in $\mathbb{R}^2$ similarly deformation retracts to the trivial loop at a given base point. Thus, the fundamental group of our universal cover is trivial, and we have our universal cover. ■

2019, Problem 2

Let X be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. For each $t \in \mathbb{R}$, let X_t be the subset of all $f \in X$ such that

$$\sup_{x\in\mathbb{R}} f(x) < t.$$

Let τ be the coarsest topology on X that contains X_t for each $t \in \mathbb{R}$.

a) **Show that (X,τ) is not Hausdorff.**

Solution. First, note that open sets in (X,τ) are simply the sets X_t; any union or intersection of such sets remain of the same form.

Let $f(x) = x$ and let $g(x) = 1$. Note that since $\sup(f(x))$ is infinite, the only open set in (X,τ) containing $f(x)$ is X itself. Thus, any open set containing $g(x)$ must also intersect X, the only open set containing $f(x)$, so (X,τ) is not Hausdorff. ■

b) **Show that every subset $K \subseteq (X,\tau)$ that contains the identity function is compact.**

Solution. Let K be a subset of (X,τ) containing the identity function. Following the above reasoning, since the supremum of the identity function defined over the reals is infinite, the only open set in (X,τ) containing the identity function is X itself. Consequently, any open cover of K must include X. Clearly, any open cover of K will then have a finite subcover consisting of the open set X itself. Thus, every subset $K \subseteq (X,\tau)$ containing the identity function is compact. ■

c) **Let $Y \subseteq (X,\tau)$ be the subset of all constant functions. Is Y compact? Is it connected? Is it path-connected?**

Solution. Y is not compact. Consider the open cover $A = \{X_t \mid t \in \mathbb{R}\}$ of Y. Let $X_{t_1}, \ldots, X_{t_n}$ correspond to an arbitrary finite collection of open sets in A. Let us use f_a to denote the constant function $f(x) = a$ for all $x \in \mathbb{R}$. Note that the constant function $f_{\max(t_1,\ldots,t_n)+1}$ in Y is not covered by this finite collection of open sets, so A has no finite subcovers. Thus, by definition, Y is not compact.

Y is connected. Note that any two open sets in X must intersect in Y – the open sets X_a and X_b both include the constant functions f_c where $c < \min(a,b)$. Thus, there cannot be two open sets U and V of Y such that U and V are disjoint satisfying $U \cup V = Y$, or equivalently, there is no separation of Y. Thus, Y is connected.

Y is path-connected. Let f_a and f_b be an arbitrary pair of constant functions in Y. We will construct a path between them. Consider $\gamma\colon I \to Y$ defined by

$$\gamma(t) = f_{(1-t)a+tb}.$$

This path γ is constructed from the straight-line homotopy between f_a and f_b, which is the composition (sum and product) of continuous functions, so it is itself continuous. Furthermore, $\gamma(0) = f_a$, $\gamma(1) = f_b$, and $\gamma(t) \in Y$ for all $t \in [0,1]$. Thus, γ is a path from f_a to f_b in Y, so Y is path-connected.[1] ■

[1] note that since every path-connected space is connected, it also follows that Y is connected.