Math 104: Topology

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Abstract

These notes arise from my studies in Math 104: Topology, taught by Professor Allison N. Miller, at Swarthmore College, following the material of Munkre's *Topology*. I am responsible for all faults in this document, mathematical or otherwise. Feel free to message me with any suggestions or corrections at dyang5@swarthmore.edu.

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1 Chapter 1: Set Theory and Logic

1.2 Functions

Definition 1.1 (Injective, Surjective, Bijection)

A function $f\colon A \to B$ is said to be **injective** (or **one-to-one**) if for each pair of distinct points of A, their images under f are distinct.

It is said to be **surjective** if every element of B is the image of some element of A under f.

If f is both **injective** and **surjective**, it said to be **bijective**.

1.3 Relations

Definition 1.2 (Relation)

A **relation** on a set A is a subset C of the Cartesian product $A \times A$.

Definition 1.3 (Equivalence Relation)

An **equivalence relation** $\sim$ on a set A is a relation C on A having the following three properties:

- (Reflexivity) $x \sim x$ for every x in A.
- (Symmetry) If $x \sim y$, then $y \sim x$.
- (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 1.4 (Order Relation)

A relation C on a set A is an **order relation** (also simple order, or linear order) if it has the following properties:

- (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
- (Nonreflexivity) For no x in A does the relation xCx hold.
- (Transitivity) If xCy and yCz, then xCz.

Note: the relation C is often replaced as $<$, just as how it is synonyous with $\sim$ in the case of an equivalence relation.

Remark. It follows that xCy and yCx cannot both be true. If so, then transitivity implies xCx, contradicting nonreflexivity.

Example. Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. The order relation $<$ on $A \times B$ defined by

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$ or if $a_1 = a_2$ and $b_1 <_B b_2$ is known as the **dictionary order relation** on $A \times B$.

Definition 1.5 (Supremum and Infimum)

Let A be an ordered set. The subset A_0 of A is **bounded above** if there is an element b of A such that $x \leq b$ for every $x \in A_0$: b is an **upper bound** for A_0. If the set of all upper bounds for A_0 has a smallest element, that elements is the **supremum** of A_0 (also the least upper bound).

The subset A_0 of A is **bounded below** if there is an element b of A such that $b \leq x$ for every $x \in A_0$: b is a **lower bound** for A_0. If the set of all lower bounds for A_0 has a largest element, that elements is the **infimum** of A_0 (also the greatest lower bound).

Definition 1.6 (Least Upper Bound and Greatest Lower Bound Properties)

An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound.

An ordered set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

1.4 The Integers and the Real Numbers

Theorem 1.1 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

1.5 Cartesian Products

This section contains definitions and examples of indexing functions (e.g. $\{1, \ldots, n\}$, $\mathbb{Z}_+$), tuples, sequences, and Cartesian products.

Definition 1.7 (ω-tuple / Sequence)

Given a set X, a ω-tuple of elements of X is a function

$$\mathbf{x} \colon \mathbb{Z}_+ \to X$$

also known as a **sequence** (or infinite sequence) of elements of X.

1.6 Finite Sets

This section contains basic definitions of finite sets, including cardinality and proof of a number of set axioms.

1.7 Countable and Uncountable Sets

Definition 1.8 (Countably Infinite)

A set A is infinite if it is not finite. It is **countably infinite** of there is a bijective correspondence

$$f \colon A \to \mathbb{Z}_+.$$

Example. The set $\mathbb{Z}$ of all integers is countably infinite. Similarly, $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof (Countability of $\mathbb{Z} \times \mathbb{Z}$). *Proof 1.* Consider the bijections $f \colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ and $g \colon A \to \mathbb{Z}_+$ defined as follows:

$$f(x, y) = (x + y - 1, y) \text{ and } g(x, y) = \frac{1}{2}(x - 1)x + y.$$

The composition $g \circ f$ is also a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, so $\mathbb{Z} \times \mathbb{Z}$ is countably infinite.

Proof 2. Consider $f(n, m) = 2^n 3^m$, an injective map from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. □

Definition 1.9 (Countable and Uncountable Sets)
A set is **countable** if it is either finite or countably infinite. A set that is not countable is **uncountable**.

Example. $\{0,1\}^\omega$, $\mathcal{P}(\mathbb{Z}_+)$, and $\mathbb{R}$ are examples of uncountable sets.

Theorem 1.2. Let B be a nonempty set. Then the following are equivalent:

1. B is countable.
2. There is a surjective function $f\colon \mathbb{Z}_+ \to B$.
3. There is an injective function $g\colon B \to \mathbb{Z}_+$.

Theorem 1.3 (Countable Union of Countable Sets)**.** A countable union of countable sets is countable.

1.8 Principle of Recursive Definition

This section contains recursion axioms and the introduction of the principle of recursion/recursion formula.

1.9 Infinite Sets and the Axiom of Choice