CIRCLE PACKINGS FROM TILINGS OF THE PLANE

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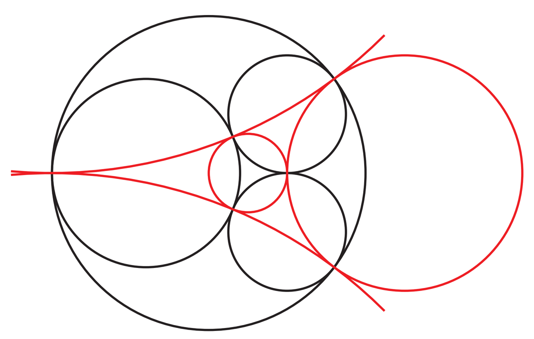
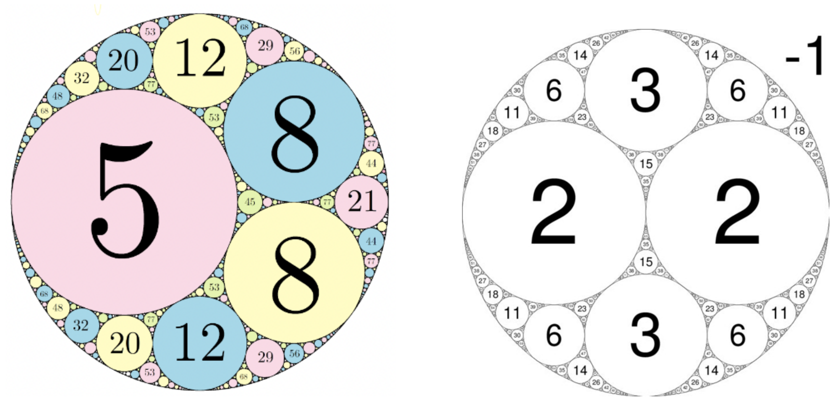
INTRODUCTION

We study circle packings that arise from the limit of polyhedral packings, which are generalized versions of the well-known Apollonian packing. We discover the first known nonpolyhedral superintegral circle packings and prove a number of theorems that relate the symmetry groups of any given circle packing.

The most well-known circle packing begins with a Descartes configuration of four mutually tangent circles. From this base configuration, inversions across **dual circles**, circles orthogonal to a ring of mutually tangent circles at their points of tangencies, generate the Apollonian packing.



Descartes Configuration and Dual Circles

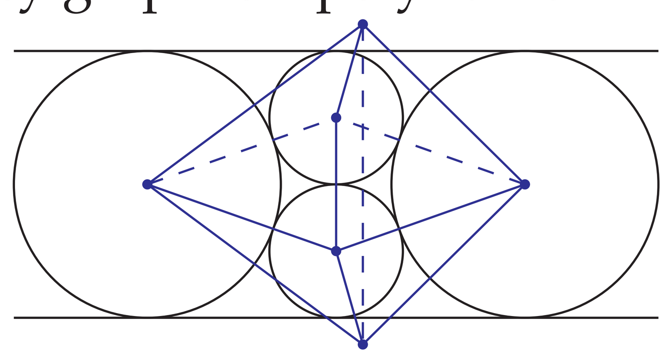


Apollonian Circle Packings

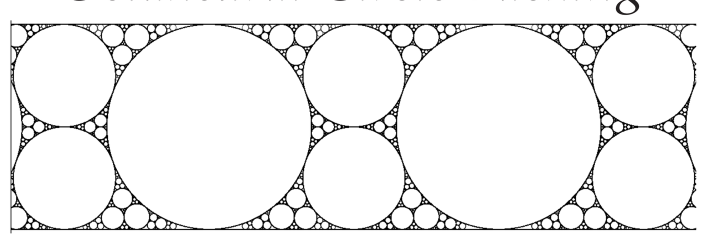
Each circle is labeled with their curvature, the inverse of their radius.

POLYHEDRAL PACKINGS

The **tangency graph** of a circle packing is a graph with a vertex at the center of each circle and an edge between a pair of vertices if the corresponding circles are tangent. A packing is **polyhedral** if the tangency graph is a polyhedron.



Octahedral Circle Packing



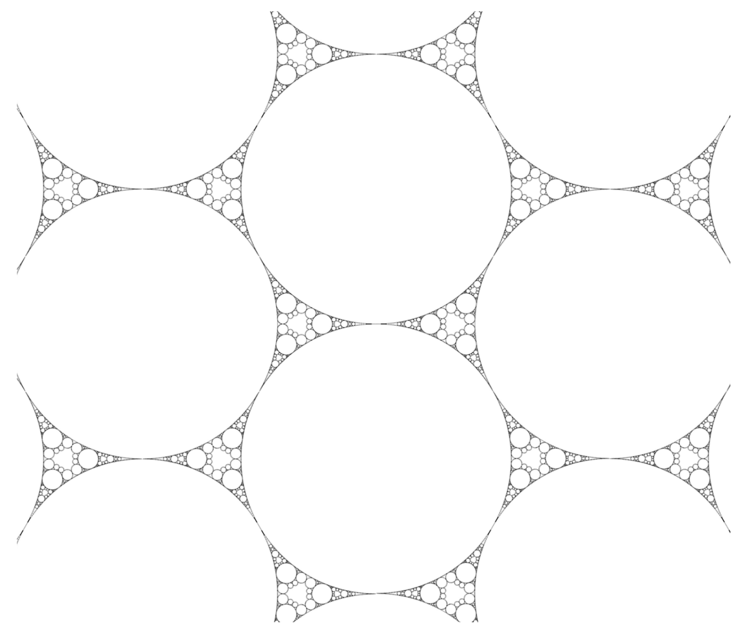
FORMAL DEFINITION OF CIRCLE PACKINGS

A collection of oriented circles **accumulates at a point** x if any circle with radius $\delta > 0$ centered at x contains infinitely many circles from this collection. Let B and $\hat{B}$ be two collections of oriented circles. Denote the tangency graphs of B and $\hat{B}$ by G_B and $G_{\hat{B}}$, respectively. Then B is the **base configuration** and $\hat{B}$ is the **dual configuration** if the following properties hold:

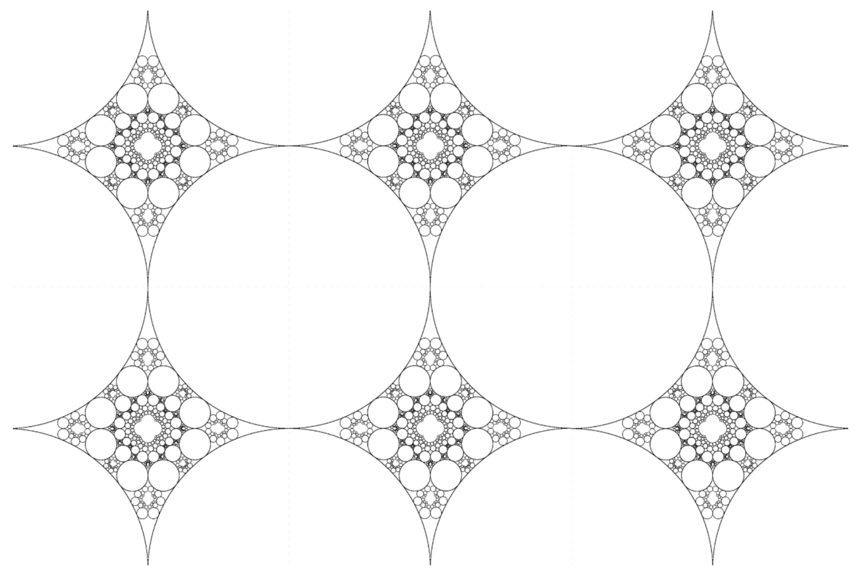
1. The interiors of the circles in B are pairwise disjoint and the interiors of the circles in $\hat{B}$ are pairwise disjoint.
2. The tangency graphs G_B and $G_{\hat{B}}$ are each nontrivial, connected, and are duals of each other.
3. If circles in B and $\hat{B}$ intersect, they do so orthogonally and they correspond to a face-vertex pair in the tangency graph.
4. B accumulates at a single point.

The **packing** $\mathscr{P}$ is the orbit of B under the group generated by reflections across circles in $\hat{B}$. The **dual packing** $\hat{\mathscr{P}}$ is the orbit of $\hat{B}$ under the same group. The **superpacking** is the orbit of B under the group generated by reflections across circles in B and $\hat{B}$.

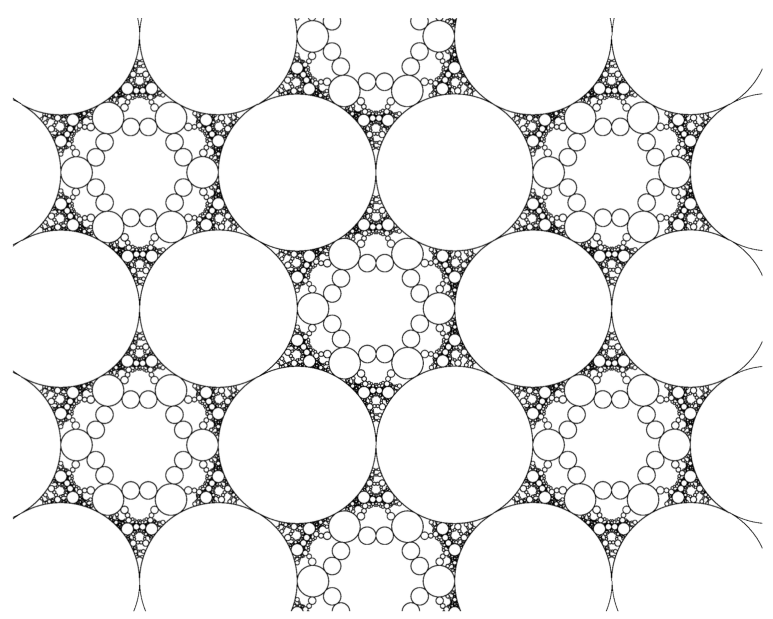
CIRCLE PACKINGS ACCUMULATING AT INFINITY



Triangular Packing



Square Packing



Hexagonal Packing

GROUP STRUCTURE THEOREMS

The Group Structure Theorems relate the symmetry groups of a packing to each other. We also discuss the relationship between the wallpaper groups and symmetry groups of our packings.

Theorem: For a circle packing $\mathscr{P}$ with base configuration B and dual configuration $\hat{B}$, we have

- $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$.
- Γ_1 is a normal subgroup of Γ.
- The intersection of Γ_1 and Γ_2 is trivial.
- $\Gamma \cong \Gamma_1 \rtimes \Gamma_2$.
- If B accumulates at infinity, then Γ_2 is either finite, a frieze group, or a wallpaper group.

Theorem: Any wallpaper group is the symmetry group Γ_2 of the base configuration B of some circle packing. Moreover, such B can be realized as the refinement of the base configuration of either the triangular or square packing.

SYMMETRY GROUPS

For each packing $\mathscr{P}$, we define the following symmetry groups:

1. $\Gamma = \mathrm{SYM}(\mathscr{P}, \hat{\mathscr{P}})$: the group of Möbius transformations that preserve both packing and dual packing;
2. $\Gamma_1 = \langle \hat{B} \rangle$: the group generated by reflections across the dual circles;
3. $\Gamma_2 = \mathrm{SYM}(B, \hat{B})$: the group of Möbius transformations that preserve both base and dual configuration.

INTEGRALITY

A packing $\mathscr{P}$ is **integral** if every circle in $\mathscr{P}$ has integral curvature. The packing is **superintegral** if every circle in the superpacking has integral curvature. If a class of packings contain at least one integral/superintegral packing, we say this class is integral/superintegral.

Proposition: The Triangular, Square, and Hexagonal packings are all integral and super-integral.

MÖBIUS RIGIDITY

Koebe–Andreev–Thurston Theorem: For any polyhedron Π, there exists a packing with tangency graph isomorphic to Π. Such a packing is unique up to Möbius transformation (or called **Möbius rigid**).

Proposition: Triangular, Square, and Hexagonal packings are Möbius rigid.

Conjecture: Packings under our definition are all Möbius rigid.

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