Circle Packings from Tilings of the Plane

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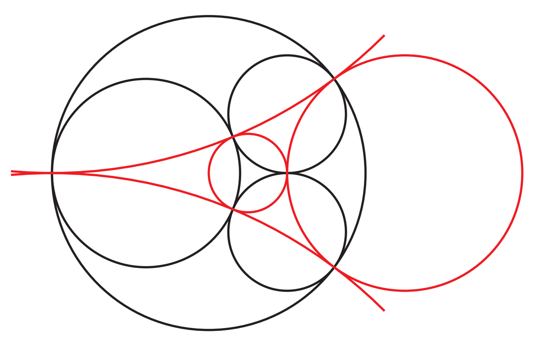
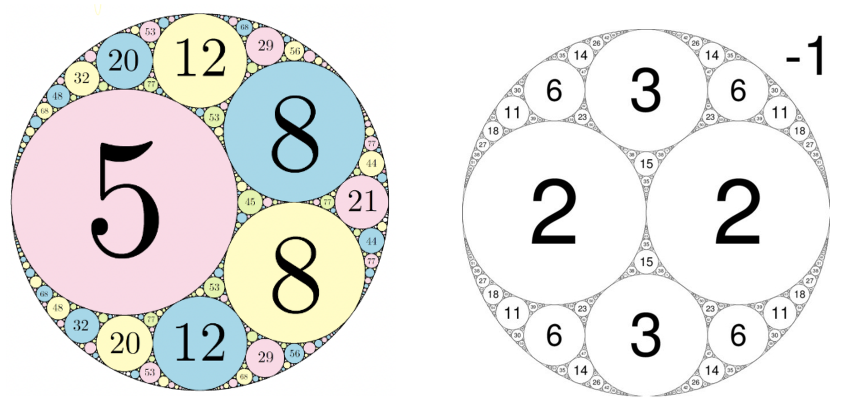
Introduction

We study circle packings that arise from limits of polyhedral packings, which are generalized versions of the well-known Apollonian packing. These include the first known nonpolyhedral superintegral circle packings. We also prove theorems that characterize the symmetry groups of all packings.

The most well-known circle packing begins with a Descartes configuration of four mutually tangent circles. From this base configuration, inversions across **dual circles**, circles orthogonal to a ring of mutually tangent circles at their points of tangencies, generate the Apollonian packing.



Descartes Configuration and Dual Circles

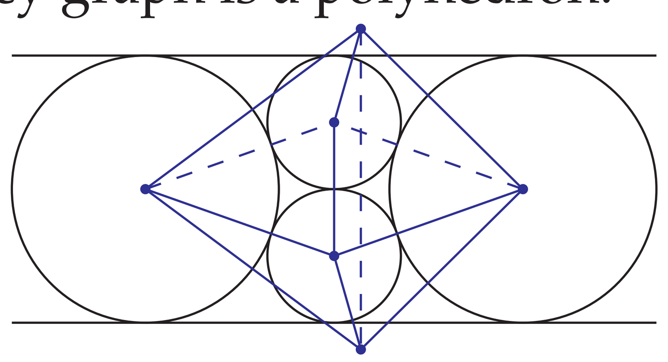


Apollonian Circle Packings

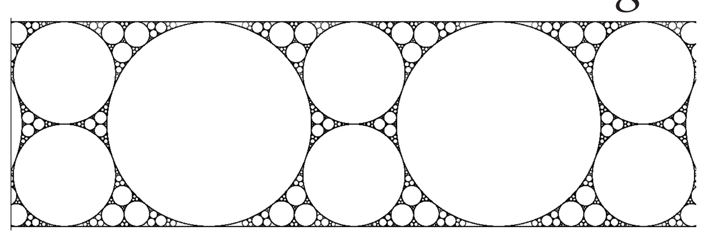
Each circle is labeled with their curvature, the inverse of their radius.

Polyhedral Packings

The **tangency graph** of a circle packing is a graph with a vertex at the center of each circle and an edge between a pair of vertices if the corresponding circles are tangent. A packing is **polyhedral** if the tangency graph is a polyhedron.



Octahedral Circle Packing



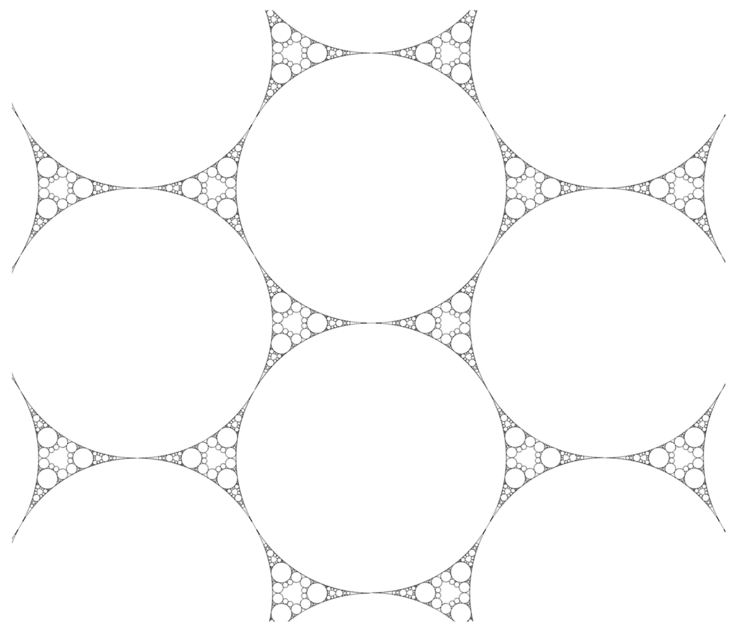
Formal Definition of Circle Packings

Let B and $\hat{B}$ be two collections of oriented circles. Denote the tangency graphs of B and $\hat{B}$ by G_B and $G_{\hat{B}}$, respectively. Then B is the **base configuration** and $\hat{B}$ is the **dual configuration** if the following properties hold:

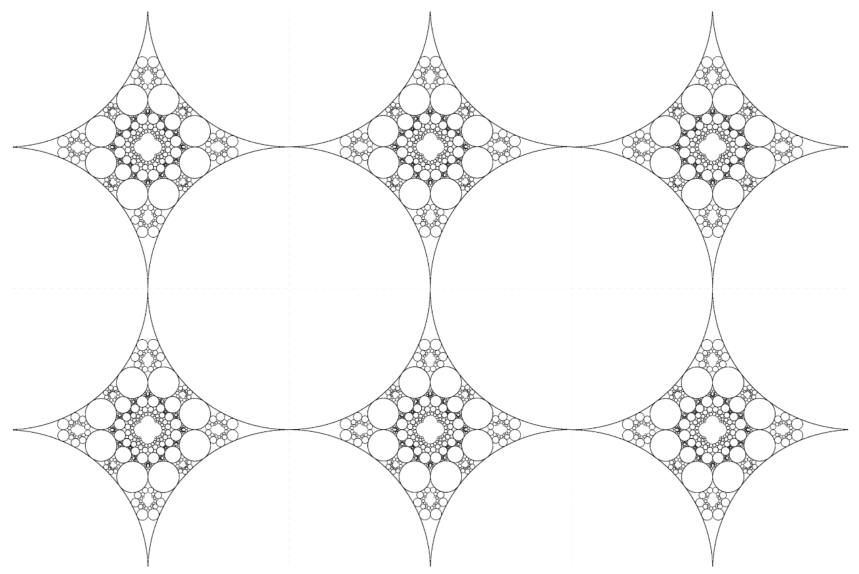
1. The interiors of the circles in B are pairwise disjoint. The same holds for $\hat{B}$.
2. The tangency graphs G_B and $G_{\hat{B}}$ are nontrivial, connected, dual graphs.
3. Circles in B and $\hat{B}$ intersect orthogonally for each face-vertex pair in the tangency graph.
4. $B \cup \hat{B}$ has at most one accumulation point.

The **packing** $\mathscr{P}$ is the orbit of B under the group generated by reflections across circles in $\hat{B}$. The **dual packing** $\hat{\mathscr{P}}$ is the orbit of $\hat{B}$ under the same group. The **superpacking** is the orbit of B under the group generated by reflections across circles in B and $\hat{B}$.

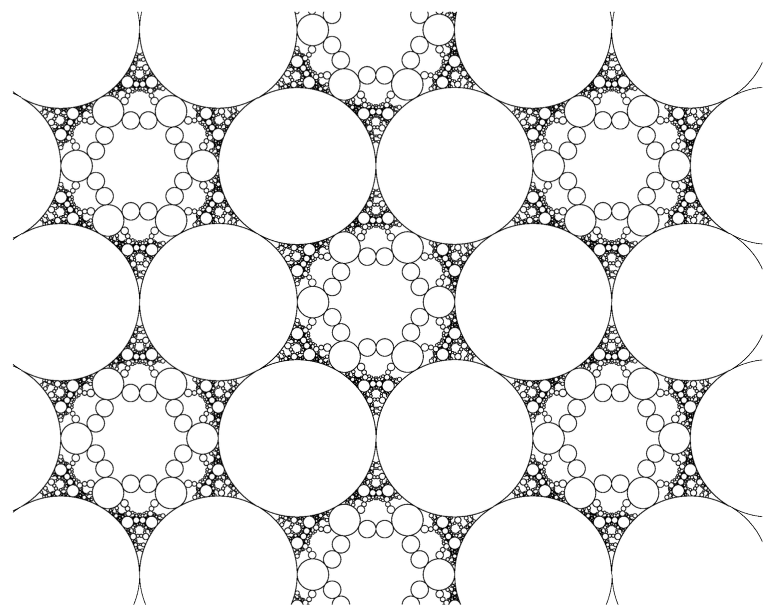
Circle Packings Accumulating at Infinity



Triangular Packing



Square Packing



Hexagonal Packing

Group Structure Theorem

For each packing $\mathscr{P}$, we define the following symmetry groups:

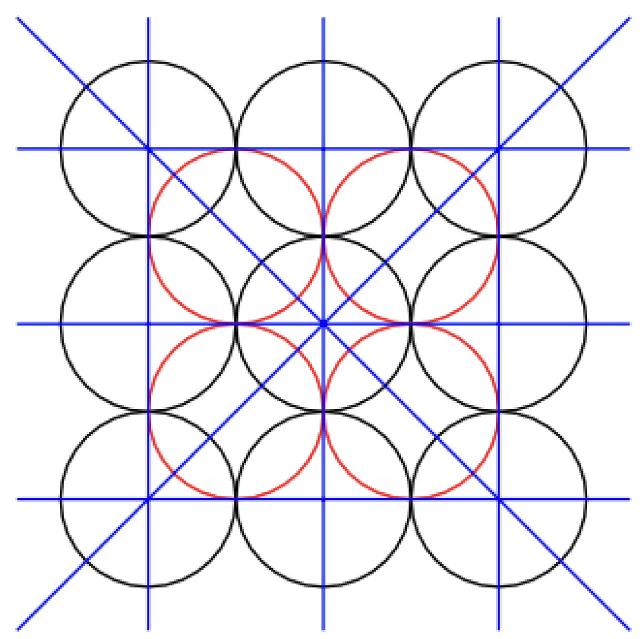
1. $\Gamma = \mathrm{SYM}(\mathscr{P}, \hat{\mathscr{P}})$: the group of Möbius transformations that preserve both packing and dual packing;
2. $\Gamma_1 = \langle \hat{B} \rangle$: the group generated by reflections across the dual circles;
3. $\Gamma_2 = \mathrm{SYM}(B, \hat{B})$: the group of Möbius transformations that preserve both base and dual configuration.

Γ_1 is a free Coxeter group. Γ_2 is the group of symmetries of a polyhedron or conjugate to a discrete group of isometries of the plane.

Theorem: $\Gamma \cong \Gamma_1 \rtimes \Gamma_2$.

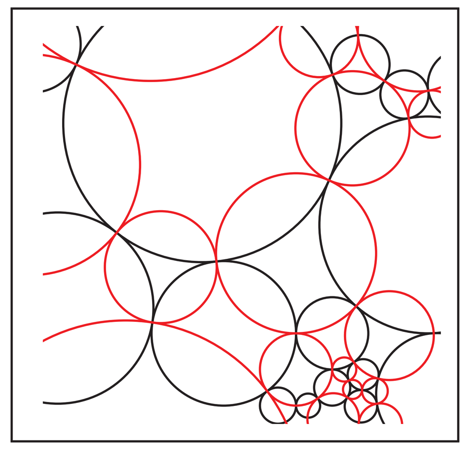
Symmetry Groups Example

Imagine infinitely many black circles tile the plane in a square lattice:

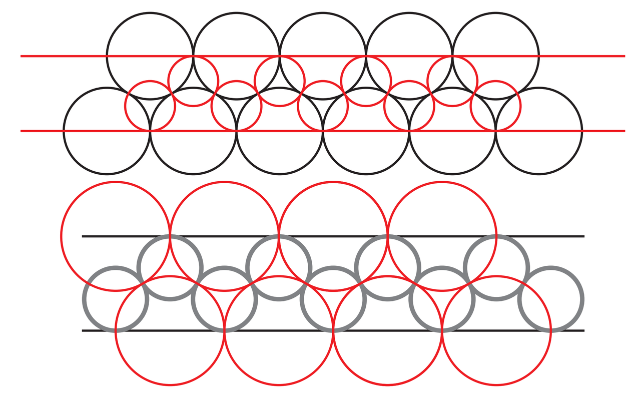


Symmetry Groups Example (Γ_1, Γ_2)

More Circle Packings



Base and Dual Configuration



Triangular and Trapezohedron Packings

Integrality

A packing $\mathscr{P}$ is **integral** if every circle in $\mathscr{P}$ has integral curvature. The packing is **superintegral** if every circle in the superpacking has integral curvature.

Proposition: The Triangular, Square, and Hexagonal packings are all integral and super-integral.

Möbius Rigidity

Koebe–Andreev–Thurston Theorem: For any polyhedron Π, there exists a packing with tangency graph isomorphic to Π. This packing is unique up to Möbius transformation (**Möbius rigid**).

Proposition: Triangular, Square, and Hexagonal packings are Möbius rigid.

Conjecture: Packings under our definition are all Möbius rigid.

Acknowledgments

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