Exploring Projective Set

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Abstract

We explore Projective SET, a variation on the classic card game SET introduced by Benjamin Lent Davis and Diane Maclagan in 2003 [1] and expanded upon by Cathy Hsu, Jonah Ostroff, and Lucas Van Meter in 2020 [2], in $\mathbb{F}_2$. We discover properties of collections of projective set cards and enumerate multisets through the Orbit-Stabilizer Theorem. Finally, we present an optimal way of play based on the probabilities of certain set sizes in a given deal.

Keywords: projective set, n-set, multiset, Orbit-Stabilizer Theorem

Projective Set Cards

The popular card game SET consists of a deck of 81 cards, each of which has a distinct combination of four attributes — number, shading, color, and shape — that assume one of three possible values. In **Projective SET**, each card comes with six dots, each of which is a different color, that can be filled in or not filled in. In total, the projective SET deck consists of 63 cards, with the card that has six empty dots removed from the deck.

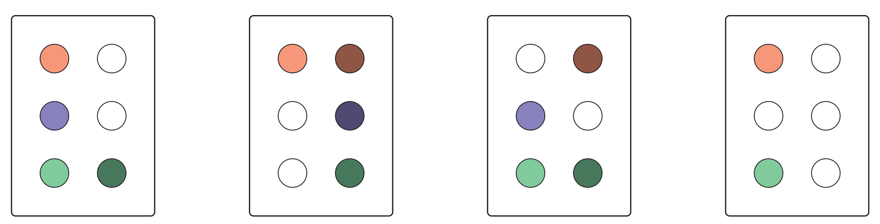

Figure 3: Example 4 card collection

What is a Projective SET?

The Projective SET Rule: A collection of cards forms a **SET** if for each color, there is an even number of dots of that color.

Each individual card can be represented by a point in $\mathbb{F}_2^6$, a 6-tuple with each component a 0 or a 1, where a 0 represents an empty dot and a 1 represents a colored one.

Projective sets follow the same geometric patterns as sets in the normal SET deck: cards form a projective SET if and only if the associated points in $\mathbb{F}_2^6$ are collinear. Projective sets also preserve the Affine Collinearity Rule, first presented by Davis and Maclagan in their work on the household version of SET. [1]

Projective SET Theorem: Cards form a projective SET if and only if their associated points in $\mathbb{F}_2^6$ sum to 0.

Introduction

Our main goal was to understand and identify algebraic and geometric properties of projective SET. We find that there is always at least one projective set in a collection of 7 or more projective set cards, so we focus on deals of 7 cards and:

- Enumerate multisets (combinations of sets) using the Orbit-Stabilizer Theorem
- Calculate the frequency of a n-set
- Determine optimal play and compare results to the household SET game

Main Result

Multiset of SETs	Fraction of total	Order of Stabilizer
3	0.2530	$3! \cdot 4!$
4	0.2530	$4! \cdot 3!$
5	0.1518	$5! \cdot 2!$
6	0.0506	$6! \cdot 1!$
7	0.007229	$7!$
3,3,4	0.07116	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2^5$
3,3,6	0.01581	$2 \cdot 3! \cdot 3! \cdot 2^5$
3,4,5	0.09488	$2! \cdot 3! \cdot 2^5$
4,4,4	0.02372	$3! \cdot 2! \cdot 2! \cdot 2! \cdot 2^5$
3,4,7	0.007907	$3! \cdot 4! \cdot 2^5$
3,5,6	0.02372	$2! \cdot 4! \cdot 2^5$
4,4,6	0.01581	$3! \cdot 3! \cdot 2^5$
4,5,5	0.02372	$2! \cdot 2! \cdot 3! \cdot 2^5$
3,3,3,3,4,4,4	0.0009884	$4! \cdot 3! \cdot 2^8$
3,3,4,4,4,5,5	0.002965	$2 \cdot 4 \cdot 6 \cdot 2^8$
3,3,3,4,4,4,7	0.0004941	$2! \cdot 2! \cdot 2! \cdot 3! \cdot 6 \cdot 2^8$
3,3,3,4,4,5,6	0.002965	$2 \cdot 2 \cdot 2 \cdot 6 \cdot 2^8$
4,4,4,4,4,4,4	0.0001412	$7 \cdot 6 \cdot 2^2 \cdot 6 \cdot 2^8$
3,3,3,3,3,3,3,4,4,4,4,4,4,4,7	0.000002521	$168 \cdot 168 \cdot 2^9$

Figure 1: Multisets in 7 card deal

Using the Orbit Stabilizer Theorem, we enumerate all possible multisets in a 7 card collection. We calculate both the fraction of total deals a given multiset covers and the order of the stabilizer for that multiset.

n	Proportion of deals with at least 1 n-set	Average Number of n-sets
3	0.4739292877	0.5737704918
4	0.4977915321	0.5974915395
5	0.2763502062	0.3267574326
6	0.1089191442	0.1089191442
7	0.01563299747	0.01563299747

Figure 2: 7 card deal frequencies

We also calculate the proportion of 7 card deals that contain a 3-set, 4-set, 5-set, 6-set, and 7-set. We find that a 4-set is the most common set found in a 7 card deal, both in terms of the proportion of deals that contain at least one 4-set and in terms of the average number of 4-sets found per deal.

In terms of optimal play for projective SET, we determine that it is most efficient to look for 4-sets, then 3-sets, 5-sets, 6-sets, and 7-sets.

Comparison to Set

Deal Size	Normal SET	3-set	4-set	5-set	6-set	7-set
7	0.39	0.47	0.50	0.28	0.11	0.016

Figure 4: Comparison of frequencies in a 7 card deal

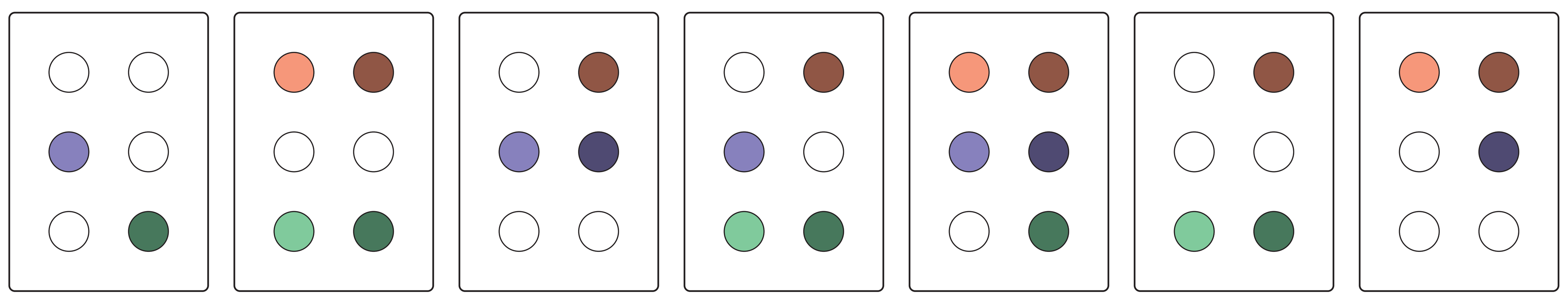

Figure 5: Example 7 card collection: Can you find all the SETs?

Methods

Use Orbit-Stabilizer Theorem on $\mathbb{PGL}_6$ to count the number of projective sets of certain sizes. Our general strategy when applying the Orbit-Stabilizer Theorem is as follows:

1. Pick an example of a collection of cards that has the property we want to count
2. Find the general form of a stabilizer for that given collection of cards
3. Count the number of stabilizers
4. Apply the Orbit-Stabilizer Theorem to count the number of orbits, which corresponds to the number of collections of cards of the a given property

Future Work

- Generalize findings to larger deal sizes and larger projective sets as well.
- Discover new algebraic and geometric interpretations of Projective SET
- Find optimal way of play for larger deal sizes
- Popularize projective SET game

References

[1] Benjamin Lent Davis and Diane Maclagan. The card game set.

[2] Cathy Hsu, Jonah Ostroff, and Lucas Van Meter. Projectivizing set. *Math Horizons*, 27(4):12–15, 2020.

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