Homework 5

David Yang

1. **Suppose $X_1, \ldots, X_n$ are i.i.d. Normal with mean μ and variance σ^2.**

   a) **Show the MLE for σ is the square root of the MLE for σ^2 (in general, if $\hat{\theta}$ is the MLE of θ, then $\hat{\theta}$ is the MLE of θ, then $\hat{\varphi} = g(\hat{\theta})$ is the MLE for $\varphi = g(\theta)$).**

   Solution. If $X_1, \ldots, X_n$ are i.i.d. Normal with mean μ and variance σ^2,

$$\begin{aligned} L(\mu, \sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x_i-\mu)^2}{2\sigma^2}} \\ &= \left(2\pi\sigma^2\right)^{-\frac{n}{2}} e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}. \end{aligned}$$

   The log-likelihood is

$$\begin{aligned} l(\mu, \sigma^2) &= n \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\sum(x_i-\mu)^2}{2\sigma^2} \\ &= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{\sum(x_i-\mu)^2}{2\sigma^2}. \end{aligned}$$

   Setting the partials to 0 to solve for the MLEs, we get that

$$\begin{aligned} \frac{\partial l}{\partial \mu} &= 2\frac{\sum(x_i-\mu)}{2\sigma^2} \\ \frac{\partial l}{\partial \sigma^2} &= -\frac{n}{2} \cdot \frac{1}{2\pi\sigma^2} \cdot 2\pi - \frac{1}{2} \cdot \left(\sum(x_i-\mu)^2\right) \cdot \left(-\frac{1}{\sigma^4}\right) \\ &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 \end{aligned}$$

   The first equation is equivalent to $\sum(x_i - \mu) = 0$, so solving for μ gives

$$\boxed{\hat{\mu} = \frac{\sum x_i}{n} = \overline{x}}.$$

   The second equation is equivalent to $\frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum(x_i-\mu)^2$, and solving for σ^2 gives

$$\sigma^2 = \frac{\sum(x_i-\mu)^2}{n}.$$

   Plugging in the MLE for μ, we get the MLE for $\hat{\sigma}^2$:

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\sum(x_i-\mu)^2}{n} \\ &= \boxed{\frac{1}{n}\left(\sum(x_i-\overline{x})^2\right)}. \ \blacksquare \end{aligned}$$

b) **Show that $\hat{\sigma}^2 = \sum \frac{(X_i - \bar{X})^2}{n}$ has smaller mean square error than the unbiased estimate $s^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$.**

Solution. ∎

c) **Find an expression for the expected value of the sample standard deviation $s = \sqrt{s^2}$. Use this to construct an unbiased estimate for σ. You could check your answer by generating sample variances from the Gamma distribution implied by a given n and σ.**

2. **Suppose $X_1, \ldots, X_n$ are i.i.d. Poisson(θ).**

a) **Let $Y = \sum X_i$. If $n = 3$, find $P(X_1 = 2, X_2 = 3, X_3 = 0 \mid Y = 5, \theta = 2)$. How does this change if $\theta = 3$?**

Solution. ∎

b) **Show that $\overline{X}$ is the MLE for θ, and that the reciprocal Fisher information gives the exact variance of $\hat{\theta}$. Verify that Poisson(θ) is a 1-parameter exponential family, so you know you can use the second derivative formula for the Fisher information.**

Solution. ∎

c) **If $n = 10$ and $\theta = 10$, check the Normal approximation to the distribution of $\overline{X}$. Use the continuity correction to approximate $P(\overline{X} = 10)$ and compare this to the exact probability based on the associated Poisson distribution.**

Solution. ∎

d) **Show that $\theta \sim \text{Gamma}\left(\alpha, \frac{\alpha}{u}\right)$ defines a conjugate family of prior distributions for θ, and that the posterior mean for $\theta \mid \overline{x}$ is a weighted average of $\overline{x}$ and the prior mean μ, with more weight on $\overline{x}$ as n increases.**

Solution. ∎