STAT111: Mathematical Statistics II **Spring 2024**

Homework 3

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1. **Suppose a planet has m days in a year, and life forms have equal probability of being hatched on any of these days. For a random group of n lifeworms, find the expected proportion of the m possible hatch days that are represented. Find the value when $m = 365$ and $n = 365$ (hint: use indicator variables; the answer is close to $1 - e^{-1}$.)**

   Solution. Define an indicator variable I_k, for k from 1 to m, to represent whether or not an egg hatches on day k:

$$I_k = \begin{cases} 1 & \text{if egg hatches on day k} \\ 0 & \text{otherwise.} \end{cases}$$

   Note that by construction,

$$\begin{aligned} E(I_k) &= P(\text{egg hatching on day } k) \\ &= 1 - P(\text{ no egg hatches on day } k) \\ &= 1 - \left(\frac{m-1}{m}\right)^n. \end{aligned}$$

   Since the expected number of hatch days represented is simply $E(I_1 + I_2 + \cdots + I_m)$, which is equivalent to $mE(I_1)$ by the linearity of expectation. Thus, the expected proportion of hatch days represented is $\frac{mE(I_1)}{m} = E(I_1)$, or

$$1 - \left(\frac{m-1}{m}\right)^n = 1 - \left(\frac{364}{365}\right)^{365} \approx \boxed{0.6236}.$$

   Note that this matches what we expect, as $\frac{1}{e} \approx 0.632$. ■

2. **Suppose X has pmf $P(X=k)=\frac{c}{(1+|k|)^2}$ for $k=0,\pm1,\pm2,\ldots$. The constant $c=(2\psi'(1)-1)^{-1}$, where $\psi'(\alpha)=\frac{d^2}{d\alpha^2}\log\Gamma(\alpha)$ is the trigamma function. Explain why $E(X)$ is not 0, despite the symmetry of this pmf.**

Solution. The expected value of X, $E(X)$, is not 0 as the expectation of $|X|$, $E(|X|)$, is not defined. Note that by definition (and symmetry),

$$E(|X|)=\sum_{i=-\infty}^{\infty}\frac{c|x|}{(1+x)^2}=2c\sum_{i=0}^{\infty}\frac{|x|}{(1+x)^2}=2c\sum_{i=1}^{\infty}\frac{x}{(1+x)^2}$$

We claim that for all positive integers x, $(1+x)^2\le 5x^2$, or equivalently, $5x^2-(1+x)^2\ge 0$, in an attempt to bound this expected value. Note that upon completing the square, we find that

$$5x^2-(1+x)^2=4\left(x-\frac{1}{4}\right)^2-\frac{5}{4}.$$

Since $4\left(x-\frac{1}{4}\right)^2-\frac{5}{4}$ is increasing and greater than 0 when $x=1$, it follows that $(1+x)^2\le 5x^2$.

The above inequality gives us a lower bound for the expected value of $|X|$:

$$\begin{aligned}E(|X|)&=2c\sum_{i=1}^{\infty}\frac{x}{(1+x)^2}\\&>2c\sum_{i=1}^{\infty}\frac{x}{5x^2}=\frac{2c}{5}\sum_{i=1}^{\infty}\frac{1}{x}.\end{aligned}$$

Since the harmonic series diverges and is a lower bound for $E(|X|)$, it follows that $E(|X|)$ diverges and is consequently not defined. Thus, $E(X)\neq 0$. ■

3. **Suppose** $X \sim \text{Gamma}(\alpha, \lambda)$**, with pdf** $f_x(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I(x > 0)$ **for** $\alpha > 0$ **and** $\lambda > 0$**.**

   a) **Find an expression for** $E(X^k)$**, for** $k = 1, 2, \ldots$**, using integration by recognition.**

   Solution. By definition,

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx \\ &= \int_0^\infty x^{\alpha+k-1} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x}\, dx \\ &= \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} \frac{\lambda^{\alpha+k}}{\Gamma(\alpha+k)} e^{-\lambda x}\, dx \end{aligned}$$

   Note that the integral is 1, as it is the integrand of a *Gamma*$(\alpha+k, \lambda)$ random variable. Thus, we find that

$$\boxed{E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}}. \qquad \blacksquare$$

   b) $Y = \frac{1}{X}$ **follows a reciprocal Gamma distribution. Find** $\mathbb{E}[Y]$**,** $\mathbb{E}[Y^2]$**, and** $\text{Var}[Y]$**, first using integration by recognition with the pdf found in HW2, and again using LOTUS with the pdf for** X**. Be sure to say if there are conditions when these are not defined.**

   Solution. Note that $Y = \frac{1}{X}$ has pdf

$$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\frac{\lambda}{y}}.$$

   Consequently, the expected value of Y, $E(Y)$, is

$$\begin{aligned} \mathbb{E}[Y] &= \int_0^\infty y \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\frac{\lambda}{y}}\, dy \\ &= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-\alpha} e^{-\frac{\lambda}{y}}\, dy \\ &= \frac{\lambda\Gamma(\alpha-1)}{\Gamma(\alpha)} \int_0^\infty y^{-(\alpha-1+1)} \tfrac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} e^{-\frac{\lambda}{y}}\, dy. \end{aligned}$$

   Note that the integral is 1, as it is the integrand of a Reciprocal-Gamma$(\alpha-1, \lambda)$ random variable. Thus, we find that (defined for any $\alpha > 1$),

$$\boxed{\mathbb{E}[Y] = \frac{\lambda\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{\lambda}{\alpha-1}}.$$

   We can follow a similar procedure to identify $\mathbb{E}[Y^2]$. Note that

$$\begin{aligned} \mathbb{E}[Y^2] &= \int_0^\infty y^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\frac{\lambda}{y}}\, dy \\ &= \frac{\lambda^2\Gamma(\alpha-2)}{\Gamma(\alpha)} \int_0^\infty y^{-(\alpha-2+1)} \tfrac{\lambda^{\alpha-2}}{\Gamma(\alpha-2)} e^{-\frac{\lambda}{y}}\, dy. \end{aligned}$$

Note that the integral is 1, as it is the integrand of a Reciprocal-Gamma$(\alpha-2, \lambda)$ random variable. Thus, we find that (defined for any $\alpha > 2$),

$$\boxed{\mathbb{E}\left[Y^2\right] = \frac{\lambda^2 \Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{\lambda^2}{(\alpha-1)(\alpha-2)}}.$$

For the variance of Y, we know

$$\begin{aligned}
\operatorname{Var}[Y] &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 \\
&= \frac{\lambda^2}{(\alpha-1)(\alpha-2)} - \left(\frac{\lambda}{\alpha-1}\right)^2 \\
&= \frac{\lambda^2(\alpha-1) - \lambda^2(\alpha-2)}{(\alpha-1)^2(\alpha-2)} \\
&= \frac{\lambda^2}{(\alpha-1)^2(\alpha-2)}.
\end{aligned}$$

Thus, the variance of Y, defined for $\alpha > 2$, is

$$\boxed{\operatorname{Var}[Y] = \frac{\lambda^2}{(\alpha-1)^2(\alpha-2)}}.$$

We can also use LOTUS and the pdf of X to calculate these expected values. Note that

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx.$$

Simplifying, we find that

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^\infty \frac{1}{x} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-2} e^{-\lambda x}\, dx \\
&= \frac{\Gamma(\alpha-1)\lambda}{\Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} x^{(\alpha-1)-1} e^{-\lambda x}\, dx
\end{aligned}$$

Once again, the integrand integrates to 1 as it is the pdf of a Gamma$(\alpha-1, \lambda)$ variable. Thus, for $\alpha > 1$,

$$\boxed{\mathbb{E}[Y] = \frac{\Gamma(\alpha-1)\lambda}{\Gamma(\alpha)} = \frac{\lambda}{\alpha-1}}.$$

Similarly, note that

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[\frac{1}{X^2}\right] = \int_0^\infty \frac{1}{x^2} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx.$$

Simplifying, we find that

$$\begin{aligned}
\mathbb{E}\left[Y\right] &= \int_0^\infty \frac{1}{x^2} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-3} e^{-\lambda x}\, dx \\
&= \frac{\Gamma(\alpha-2)\lambda^2}{\Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha-2}}{\Gamma(\alpha-2)} x^{(\alpha-2)-1} e^{-\lambda x}\, dx
\end{aligned}$$

Once again, the integrand integrates to 1 as it is the pdf of a Gamma$(\alpha-2, \lambda)$ variable. Thus, for $\alpha > 2$,

$$\boxed{\mathbb{E}\left[Y^2\right] = \frac{\Gamma(\alpha-2)\lambda^2}{\Gamma(\alpha)} = \frac{\lambda^2}{(\alpha-1)(\alpha-2)}}.$$

These results both match our previous integration-by-recognition calculations. ■

4. **Suppose $V \sim \text{Gamma}(b, 1)$ and $X \mid V \sim \text{Gamma}(a, V)$.**

   a) **Show that $X \sim F^*(a, b, 1)$.**

   Solution. Let $U \sim \text{Gamma}(a, 1)$. Then by the scaling property of Gamma random variables, we know that, by treating V as a constant, $\frac{U}{V} \sim \text{Gamma}(a, V)$. Consequently, treating V as a random variable, we have that

$$\frac{U}{V} \sim X,$$

   where $X \sim F^*(a, b, 1)$ by the definition of a F^* random variable. ■

   b) **Use the laws of total expectation and variance to find the mean and variance of the $F^*(a, b, 1)$ distribution. Be sure to say if there are conditions when these are not defined.**

   Solution. By Adam's Law (The Law of Total Expectation), since $X \sim F^*(a, b, 1)$, we have that

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid V]] \\ &= \mathbb{E}\left[\frac{a}{V}\right] \\ &= a\mathbb{E}\left[\frac{1}{V}\right]. \end{aligned}$$

   By problem 2, since $V \sim \text{Gamma}(b, 1)$, we know that $\mathbb{E}\left[\frac{1}{V}\right] = \frac{1}{b-1}$. Thus, we conclude that (for $b > 1$, when this is defined)

$$\boxed{\mathbb{E}[X] = \frac{a}{b-1}}.$$

   Similarly, by Eve's Law (The Law of Total Variance), we know that

$$\text{Var}[X] = \mathbb{E}[\text{Var}[X \mid V]] + \text{Var}[\mathbb{E}[X \mid V]].$$

   Since $X \mid V \sim \text{Gamma}(a, V)$, we know that $\text{Var}[X \mid V] = \frac{a}{V^2}$. Similarly, we know that $\mathbb{E}[X \mid V] = \frac{a}{V}$. Plugging these in, we find that

$$\begin{aligned} \text{Var}[X] &= \mathbb{E}[\text{Var}[X \mid V]] + \text{Var}[\mathbb{E}[X \mid V]] \\ &= \mathbb{E}\left[\frac{a}{V^2}\right] + \text{Var}\left[\frac{a}{V}\right] \\ &= a\mathbb{E}\left[\frac{1}{V^2}\right] + a^2\text{Var}\left[\frac{1}{V}\right]. \end{aligned}$$

   By problem 2, since $V \sim \text{Gamma}(b, 1)$, we know that $\text{Var}\left[\frac{1}{V}\right] = \frac{1^2}{(b-1)^2(b-2)}$. Similarly, by problem 2, we know $\mathbb{E}\left[\frac{1}{V^2}\right] = \frac{1}{(b-1)(b-2)}$. Plugging these values in, we get that (for $b > 2$, when this is defined)

$$\boxed{\text{Var}[X] = \frac{a}{(b-1)(b-2)} + \frac{a^2}{(b-1)^2(b-2)}}.$$

   ■