**STAT111: Mathematical Statistics II** **Spring 2024**

# Homework 1

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1. **Evaluate the integral**

$$\int_0^\infty x^5 e^{-2x}\,dx$$

**using integration by recognition. That is, recognize this function as proportional to a standard pdf and identify the constant multiplier needed to make the integral equal 1. Then take the reciprocal of that constant.**

*Solution.* The integrand is the kernel of a $\text{Gamma}(6,2)$ random variable.

Using this fact, we know that the PDF integrates to 1, i.e.

$$\int_0^\infty \frac{2^6}{\Gamma(6)} x^5 e^{-2x}\,dx = 1.$$

Solving for the integral we want to evaluate, we find that

$$\int_0^\infty x^5 e^{-2x}\,dx = \frac{\Gamma(6)}{2^6} = \frac{(6-1)!}{64} = \boxed{\frac{15}{8}}.$$

■

2. **Suppose that $X \sim \text{Gamma}\left(\alpha, \frac{\alpha}{u}\right)$ is parameterized so that the mean is $\mu$.**

   a) **Identify the mode of the pdf for $X$ as a function of $\alpha$ and $u$. That is, for what value of $x$ is $f_x(x)$ (or $\ln f_x(x)$) maximized?**

   *Solution.* $X \sim \text{Gamma}\left(\alpha, \frac{\alpha}{u}\right)$ has pdf

   $$f_X(x) = \frac{\left(\frac{\alpha}{u}\right)^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{\alpha}{u}\right)x}.$$

   $f_X(x)$ is maximized when $\ln(f_X(x))$ is maximized. For convenience, let $\ell(x)$ denote $\ln(f_X(x))$. Note that

   $$\ell(x) = \ln(f_X(x)) = \ln\left(\frac{\left(\frac{\alpha}{\mu}\right)^{\alpha}}{\Gamma(\alpha)}\right) + (\alpha-1)\ln(x) - \left(\frac{\alpha}{\mu}\right)x.$$

   We take the derivative of $\ell(x)$ and set it to 0 to solve for the maximum:

   $$\ell'(x) = \frac{\alpha-1}{x} - \frac{\alpha}{\mu}.$$

   Note that $\ell'(x) = 0$ when $\frac{\alpha-1}{x} - \frac{\alpha}{\mu}$. Solving for $x$, we find that $x = \frac{\mu(\alpha-1)}{\alpha}$. ■

   b) **Let $Y = \frac{1}{X}$, so that $Y$ follows a reciprocal-Gamma$\left(\alpha, \frac{\alpha}{u}\right)$ distribution. Find the pdf for $Y$, and identify its mode as a function of $\alpha$ and $u$.**

3. **Let $F(x) = \frac{x}{x+2} I_{(x>0)}$.**

   a) **Show that $F_x(x)$ is a CDF and find the corresponding pdf.**

   b) **Identify these as the CDF and pdf for an $F^*$ random variable (give the parameter values $a$, $b$, and $c$).**

   c) **For a random variable $X$ that follows this $F^*$ distribution, represent $X$ in terms of two independent Gamma random variables and a positive constant $c$. Use this representation to identify the distributions of $Y = \frac{1}{X}$ and of $R = \frac{X}{2+X}$.**

4. **Suppose $X \mid \theta \sim \text{Poisson}(\theta)$, with $\theta \sim \text{Gamma}(\alpha, \lambda)$. Find the marginal pmf for $X$ by integrating $\theta$ out of the joint pmf/pdf. Show that this is a Negative Binomial distribution that represents the count of successes at the time of our $\alpha$th failure (if $\alpha$ happens to be an integer) and identify the success probability.**