Homework 4

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1. **Suppose V_1 and V_2 are independent $\text{Gamma}(1, \lambda)$ random variables that represent waiting times in a Poisson process with rate λ events per unit time. Let $X = V_1$ be the time of the first event and let $Y = V_1 + V_2$ be the time of the second event.**

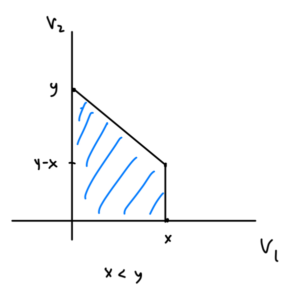
    a) **Find the joint CDF for X and Y: $F_{xy}(x, y) = P(X \leq x, Y \leq y)$. Hint: Graph the positive quadrant of the plane with axes V_1 and V_2, and mark the region where $V_1 \leq x$ and $V_1 + V_2 \leq y$. Integrate the joint pdf for V_1 and V_2 over this region to obtain the function $F_{xy}(x, y)$. Note that if there are restrictions on the arguments x and y that you do not specify, then you have failed to define the function.**

    Solution. Note that since $X = V_1$ and $Y = V_1 + V_2$, then

    $$F_{xy}(x, y) = P(X \leq x, Y \leq y) = P(V_1 \leq x, V_1 + V_2 \leq y) = P(V_1 \leq x, V_2 \leq y - V_1)$$

    with the restriction that $x \leq y$.

    Graphing the positive quadrant of the plane with axes V_1 and V_2 and integrating the joint pdf for V_1 and V_2 over the region where $V_1 \leq x$ and $V_1 + V_2 \leq y$ for $x < y$ (shaded),

    we see that

    $$\begin{aligned}
    F_{xy}(x, y) &= \int_0^x \int_0^{y-V_1} \left(\lambda e^{-\lambda V_1}\right) \left(\lambda e^{-\lambda V_2}\right) dV_2\, dV_1 \\
    &= \int_0^x \left[-\lambda e^{-\lambda(V_1+V_2)}\right]_0^{y-V_1} dV_1 \\
    &= \int_0^x -\lambda e^{-\lambda y} + \lambda e^{-\lambda V_1}\, dV_1 \\
    &= \left[-V_1 \lambda e^{-\lambda y} - e^{-\lambda V_1}\right]_0^x \\
    &= 1 - e^{-\lambda x} - \lambda x e^{-\lambda y},
    \end{aligned}$$

    so the joint CDF for X and Y (with the extra restriction $0 < x < y$) is

    $$\boxed{F_{xy}(x, y) = I(0 < x < y)\left(1 - e^{-\lambda x} - \lambda x e^{-\lambda y}\right)}. \qquad \blacksquare$$

b) **Show how to get the marginal CDF F_x by taking the upper limit for y, and F_y by taking the upper limit for x. Differentiate each marginal CDF to get the marginal pdfs.**

Solution. There are no restrictions on y, so the marginal CDF F_x is simply

$$\begin{aligned} F_x(x) &= \lim_{y\to\infty} I(0<x<y)\left(1-e^{-\lambda x}-\lambda x e^{-\lambda y}\right) \\ &= \boxed{1-e^{-\lambda x}}. \end{aligned}$$

Differentiating, we get the marginal pdf

$$\boxed{f_x(x) = \lambda e^{-\lambda x}}$$

which intuitively is the pdf for a Gamma$(1,\lambda)$ random variable.

On the other hand, the upper limit for x is y, as $0<x<y$. Thus, the marginal CDF F_y is

$$\begin{aligned} F_y(y) &= \lim_{x\to y} I(0<x<y)\left(1-e^{-\lambda x}-\lambda x e^{-\lambda y}\right) \\ &= \boxed{1-e^{-\lambda y}-\lambda y e^{-\lambda y}} \end{aligned}$$

Differentiating, we can get the marginal pdf

$$\begin{aligned} f_y(y) &= \frac{d}{dy}\left(1-e^{-\lambda y}-\lambda y e^{-\lambda y}\right) \\ &= \lambda e^{-\lambda y}-\lambda e^{-\lambda y}+\lambda^2 y e^{-\lambda y} \\ &= \boxed{\lambda^2 y e^{-\lambda y}} \end{aligned}$$

which intuitively is the pdf for a Gamma$(2,\lambda)$ random variable. ■

c) **Show that taking partial derivatives of F_{xy} with respect to x and y yields the joint pdf:**

$$\frac{\partial^2}{\partial x \partial y} F_{xy}(x,y) = \lambda^2 e^{-\lambda y} I(0<x<y).$$

Solution. Finally, we can derive the joint pdf. Note that

$$\begin{aligned} f_{xy}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{xy}(x,y) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F_{xy}(x,y)\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F_{xy}(x,y)\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(I(0<x<y)\left(1-e^{-\lambda x}-\lambda x e^{-\lambda y}\right)\right)\right) \\ &= \frac{\partial}{\partial x}\left(I(0<x<y)\lambda^2 x e^{-\lambda y}\right) \\ &= \boxed{\lambda^2 e^{-\lambda y} I(0<x<y)} \end{aligned}$$

as desired. ■

2. **Suppose Z_1 and Z_2 have joint pdf**

$$f_{12}(z_1, z_2) = \exp\left[-\log(\pi) - 2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right].$$

a) **Identify this as a bivariate Normal density by specifying the means μ_1 and μ_2, standard deviations σ_1 and σ_2, and the correlation ρ.**

Solution. Note that

$$\begin{aligned} f_{12}(z_1, z_2) &= \exp\left[-\log(\pi) - 2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right]. \end{aligned}$$

Rewriting this expression, we have that

$$\begin{aligned} f_{12}(z_1, z_2) &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{2\pi\left(\frac{1}{2}\right)}\exp\left[-\frac{1}{2\left(\frac{1}{4}\right)}\left(z_1^2 + z_2^2 - 2\left(-\frac{\sqrt{3}}{2}\right)z_1 z_2\right)\right]. \end{aligned}$$

We recognize this as a bivariate Normal density, with

$$\boxed{\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1, \text{ and } \rho = -\frac{\sqrt{3}}{2}}. \qquad \blacksquare$$

b) **Any joint pdf may be expressed as a marginal pdf multiplied by a conditional pdf. Show that $f_{12}(z_1, z_2)$ may be written as the product of a standard Normal density for Z_1 and a Normal density for $Z_2 \mid z_1$ that depends on z_1. Give the conditional mean and variance for $Z_2 \mid z_1$ and show that they agree with the formulas $\mathbb{E}[Z_2 \mid z_1] = \beta_0 + \beta_1 z_1$ with $\beta_1 = \rho\frac{\sigma_2}{\sigma_1}$, $\beta_0 = \mu_2 - \beta_1\mu_1$, and $\mathrm{Var}\left[Z_2 \mid z_1\right] = (1-\rho^2)\sigma_2^2$.**

Solution. Let $Z_1 \sim N(0,1)$. We know that $f_{Z_1}(z_1) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z_1^2}{2}\right]$.

$$\begin{aligned} f_{12}(z_1, z_2) &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z_1^2}{2}\right]\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right] \end{aligned}$$

Recognizing the first term as $f_{Z_1}(z_1)$, the pdf for Z_1, we have that

$$f_{12}(z_1, z_2) = f_{Z_1}(z_1)\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right].$$

Rearranging the second term and simplifying, we get that

$$\begin{aligned} f_{12}(z_1, z_2) &= f_{Z_1}(z_1)\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right] \\ &= f_{Z_1}(z_1)\frac{\sqrt{2}}{\pi}\exp\left[-2\left(z_2^2 + \sqrt{3}z_1 z_2 + \frac{3}{4}z_1^2\right)\right] \\ &= f_{Z_1}(z_1)\frac{1}{\sqrt{2\pi\cdot\frac{1}{4}}}\exp\left[-\frac{\left(z_2 + \frac{\sqrt{3}}{2}z_1\right)^2}{2\left(\frac{1}{4}\right)}\right]. \end{aligned}$$

We recognize the second term now as the pdf for $Z_2 \mid z_1 \sim N\left(-\frac{\sqrt{3}}{2}z_1, \frac{1}{4}\right)$, so we conclude

$$f_{12}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2|z_1}(z_2 \mid z_1).$$

Note that $\mathbb{E}\left[Z_2 \mid z_1\right] = -\frac{\sqrt{3}}{2}z_1$, and

$$\begin{aligned}
\beta_0 + \beta_1 z_1 &= \left(\mu_2 - \left(\rho\frac{\sigma_2}{\sigma_1}\right)\mu_1\right) + \rho\frac{\sigma_2}{\sigma_1}z_1 \\
&= (0 - 0) - \frac{\sqrt{3}}{2}z_1 \\
&= -\frac{\sqrt{3}}{2}z_1.
\end{aligned}$$

Similarly, $\mathrm{Var}\left[Z_2 \mid z_1\right] = \frac{1}{4}$ and

$$\begin{aligned}
\left(1 - \rho^2\right)\sigma_2^2 &= \left(1 - \left(-\frac{\sqrt{3}}{2}\right)^2\right)1^2 \\
&= \frac{1}{4},
\end{aligned}$$

matching the results from the given formulas. ■

c) **You can also show conditional results using representation. For $Z_o \sim N(0,1)$ independent of Z_2, define $Z_1 = \rho Z_2 + \sqrt{1-\rho^2}Z_o$ to have correlation ρ with Z_2. Show that conditioning on $Z_2 = z_2$ and treating this as constant in the representation of Z_1 results in a conditional distribution $Z_1 \mid z_2$ that mirrors that of $Z_2 \mid z_1$ from part (b).**

Solution. For $Z_o \sim N(0,1)$ independent of Z_2, define $Z_1 = \rho Z_2 + \sqrt{1-\rho^2}Z_o$ as stated. Note that $\sqrt{1-\rho^2}Z_o \sim N(0, 1-\rho^2)$. Conditioning on $Z_2 \mid z_2$, we find that

$$Z_1 \mid z_2 \sim N(\rho Z_2, 1 - \rho^2)$$

For $\rho = -\frac{\sqrt{3}}{2}$ (giving $1 - \rho^2 = \frac{1}{4}$), we have

$$Z_1 \mid z_2 \sim N\left(-\frac{\sqrt{3}}{2}z_2, \frac{1}{4}\right)$$

which matches the distribution of $Z_2 \mid z_1$ in part (b). ■

3. **Suppose X and Y have joint pdf $f_{xy}(x,y) = I(0 < x < 1, -x < y < x)$.**

   a) **Explain how you can tell, without finding the marginal densities, that the conditional densities are Uniform. Write out the conditional densities $f_{x|y}(x \mid y)$ and $f_{y|x}(y \mid x)$.**

   Solution. The conditional densities are uniform as they are proportional to the joint density, which can be thought of as Uniform (as it is an indicator variable).

   Since by construction, $|y| < x$ and $x < 1$ it follows that the conditional density is

$$f_{x|y}(x \mid y) = \frac{1}{1-|y|} I(|y| < x < 1).$$

   Similarly, we must have that $-x < y < x$, so

$$f_{x|y}(y \mid x) = \frac{1}{2x} I(-x < y < x).$$

   ■

   b) **Explain how you can tell, without finding the marginal densities, that X and Y are not independent. Find the marginal pdf's $f_x(x)$ and $f_y(y)$ and verify that $f_{xy}(x,y) \neq f_x(x) f_y(y)$.**

   Solution. Note that very clearly, $f_{x|y}(x,y)$ are not equal for different values of y, so X and Y cannot be independent. We can verify this with the marginal pdfs:

$$f_x(x) = \frac{f_{xy}(x,y)}{f_{y|x}(y \mid x)} = \frac{I(0 < x < 1, -x < y, x)}{\frac{1}{2x} I(-x < y < x)} = 2xI(0 < x < 1).$$

   Similarly,

$$f_x(y) = \frac{f_{xy}(x,y)}{f_{x|y}(x \mid y)} = \frac{I(0 < x < 1, -x < y, x)}{\frac{1}{1-|y|} I(|y| < x < 1)} = (1 - |y|) I(-1 < y < 1).$$

   We check that

$$\begin{aligned} f_X(x) f_Y(y) &= 2x(1 - |y|) I(0 < x < 1) I(-1 < y < 1) \\ &\neq I(0 < x < 1, -x < y, x) = f_{XY}(x,y) \end{aligned}$$

   confirming that X and Y are not independent. ■

   c) **Show that X and Y are uncorrelated.**

   Solution. Note that

$$\mathbb{E}[XY] = \int_0^1 \int_{-x}^{x} xyI(0 < x < 1, -x < y < x)\, dy\, dx = 0$$

   by symmetry. Similarly,

$$\mathbb{E}[X] = \int_0^1 x(2x)\, dx = 1$$

   and

$$\mathbb{E}[Y] = \int_{-1}^{1} y(1 - |y|)\, dy = 0$$

   by symmetry. Thus, we have that

$$\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 0$$

   so X and Y are uncorrelated, as desired. ■

4. a) **Suppose X_1 and X_2 are Bernoulli random variables with expectations p_1 and p_2. Show that X_1 and X_2 are independent if and only if they are uncorrelated. This shows the Bernoulli distribution is special like the multivariate Normal distribution in that uncorrelated implies independence.**

Solution. Note that if X_1 and X_2 are independent, then we have that $P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1)$. Note that

$$X_1X_2 = \begin{cases} 1 & \text{if } X_1 = 1 \text{ and } X_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

so $\mathbb{E}[X_1X_2] = P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1) = \mathbb{E}[X_1]\mathbb{E}[X_2]$.

Thus,

$$\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$$

and so X_1 and X_2 are uncorrelated.

On the other hand, suppose that X_1 and X_2 are independent, so that $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2] = p_1p_2$.

By the Law of Total Probability, we have that

$$P(X_1 = 1) = P(X_1 = 1 \mid X_2 = 1)P(X_2 = 1) + P(X_1 = 1 \mid X_2 = 0)P(X_2 = 0).$$

Since $P(X_1 = 1) = \mathbb{E}[X_1] = p_1$, $P(X_1 = 1 \mid X_2 = 1)P(X_2 = 1) = \mathbb{E}[X_1X_2] = p_1p_2$, and $P(X_2 = 0) = 1 - \mathbb{E}[X_2] = 1 - p_2$, we must have that

$$\begin{aligned} P(X_1 = 1 \mid X_2 = 0) &= \frac{P(X_1 = 1) - P(X_1 = 1 \mid X_2 = 1)}{P(X_2 = 0)} \\ &= \frac{p_1 - p_1p_2}{1 - p_2} \\ &= p_1. \end{aligned}$$

Similarly, by the Law of Total Probability, we have that

$$P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 0)P(X_1 = 0)$$

and so

$$\begin{aligned} P(X_2 = 1 \mid X_1 = 0) &= \frac{P(X_2 = 1) - P(X_2 = 1 \mid X_1 = 1)}{P(X_1 = 0)} \\ &= \frac{p_2 - p_1p_2}{1 - p_1} \\ &= p_2. \end{aligned}$$

We see that both $P(X_1 = 1) = P(X_1 = 1 \mid X_2 = 1) = P(X_1 = 1 \mid X_2 = 0) = p_1$ and $P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 1) = P(X_2 = 1 \mid X_1 = 0) = p_2$.

Through a similar line of reasoning, we can conclude that $P(X_1 = 0) = P(X_1 = 0 \mid X_2 = 1) = P(X_1 = 0 \mid X_2 = 1) = 1 - p_1$ and $P(X_2 = 0) = P(X_2 = 0 \mid X_1 = 1) = P(X_2 = 0 \mid X_1 = 0) = 1 - p_2$.

Thus, X_1 and X_2 are independent, as desired.

It follows that X_1 and X_2 are independent if and only if they are uncorrelated, as desired. ■

b) **Suppose $Y = X_1 + X_2$ with X_1 and X_2 independent. If you learn that Y and X_1 are Normal variables, prove that X_2 is also a Normal random variable.**

Solution. Suppose that $Y \sim N(\mu_Y, \sigma_Y^2)$ and $X_1 \sim N(\mu_1, \sigma_1^2)$. Since $Y = X_1 + X_2$, we know that the MGF of Y and $X_1 + X_2$ are the same, so

$$M_{X_1+X_2}(t) = M_Y(t)$$

meaning that $M_{X_1}(t)M_{X_2}(t) = M_Y(t)$ and

$$M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)}.$$

Since $Y \sim N(\mu_Y, \sigma_Y^2)$ and $X_1 \sim N(\mu_1, \sigma_1^2)$, we know that

$$M_Y(t) = e^{t\mu_Y + \frac{1}{2}\sigma_Y^2 t^2} \text{ and } M_{X_1}(t) = e^{t\mu_1 + \frac{1}{2}\sigma_1^2 t^2}.$$

Thus,

$$\begin{aligned} M_{X_2}(t) &= \frac{M_Y(t)}{M_{X_1}(t)} \\ &= \frac{e^{t\mu_Y + \frac{1}{2}\sigma_Y^2 t^2}}{e^{t\mu_1 + \frac{1}{2}\sigma_1^2 t^2}} \\ &= e^{t(\mu_Y - \mu_1) + \frac{1}{2}(\sigma_Y^2 - \sigma_1^2)t^2} \end{aligned}$$

which is the MGF of the Normal random variable $N(\mu_Y - \mu_1, \sigma_Y^2 - \sigma_1^2)$.

Thus, if Y and X_1 are Normal variables and $Y = X_1 + X_2$ with X_1 and X_2 independent, it follows that X_2 is also a Normal random variable. ■