STAT111: Mathematical Statistics II **Spring 2024**

Homework 3

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1. **Suppose V_1 and V_2 are independent $\mathrm{Gamma}(1, \lambda)$ random variables that represent waiting times in a Poisson process with rate λ events per unit time. Let $X = V_1$ be the time of the first event and let $Y = V_1 + V_2$ be the time of the second event.**

   a) **Find the joint CDF for X and Y: $F_{xy}(x, y) = P(X \leq x, Y \leq y)$. Hint: Graph the positive quadrant of the plane with axes V_1 and V_2, and mark the region where $V_1 \leq x$ and $V_1 + V_2 \leq y$. Integrate the joint pdf for V_1 and V_2 over this region to obtain the function $F_{xy}(x, y)$. Note that if there are restrictions on the arguments x and y that you do not specify, then you have failed to define the function.**

   b) **Show how to get the marginal CDF F_x by taking the upper limit for y, and F_y by taking the upper limit for x. Differentiate each marginal CDF to get the marginal pdfs.**

   c) **Show that taking partial derivatives of F_{xy} with respect to x and y yields the joint pdf:**

$$\frac{\partial^2}{\partial x \partial y} F_{xy}(x, y) = \lambda^2 e^{-\lambda y} I(0 < x < y).$$

2. **Suppose Z_1 and Z_2 have joint pdf**

$$f_{12}(z_1, z_2) = \exp\left[-\log(\pi) - 2(z_1^2 + z_2^2 + \sqrt{3} z_1 z_2)\right].$$

   a) **Identify this as a bivariate Normal density by specifying the means μ_1 and μ_2, standard deviations σ_1 and σ_2, and the correlationρ.**

   b) **Any joint pdf may be expressed as a marginal pdf multiplied by a conditional pdf. Show that $f_{12}(z_1, z_2)$ may be written as the product of a standard Normal density for Z_1 and a Normal density for $Z_2 \mid z_1$ that depends on z_1. Give the conditional mean and variance for $Z_2 \mid z_1$ and show that they agree with the formulas $\mathbb{E}[Z_2 \mid z_1] = \beta_0 + \beta_1 z_1$ with $\beta_1 = \rho \frac{\sigma_2}{\sigma_1}$, $\beta_0 = \mu_2 - \beta_1 \mu_1$, and $\mathrm{Var}\left[Z_2 \mid z_1\right] = (1 - \rho^2)\sigma_2^2$.**

   c) **You can also show conditional results using representation. For $Z_o \sim N(0, 1)$ independent of Z_2, define $Z_1 = \rho Z_2 + \sqrt{1 - \rho^2} Z_o$ to have correlation ρ with Z_2. Show that conditioning on $Z_2 = z_2$ and treating this as constant in the representation of Z_1 results in a conditional distribution $Z_1 \mid z_2$ that mirrors that of $Z_2 \mid z_1$ from part (b).**

3. **Suppose X and Y have joint pdf $f_{xy}(x, y) = I(0 < x < 1, -x < y < x)$.**

   a) **Explain how you can tell, without finding the marginal densities, that the conditional densities are Uniform. Write out the conditional densities $f_{x|y}(x \mid y)$ and $f_{y|x}(y \mid x)$.**

b) **Explain how you can tell, without finding the marginal densities, that X and Y are not independent. Find the marginal pdf's $f_x(x)$ and $f_y(y)$ and verify that $f_{xy}(x, y) \neq f_x(x)f_y(y)$.**

c) **Show that X and Y are uncorrelated.**

a) **Suppose X_1 and X_2 are Bernoulli random variables with expectations p_1 and p_2. Show that X_1 and X_2 are independent if and only if they are uncorrelated. This shows the Bernoulli distribution is special like the multivariate Normal distribution in that uncorrelated implies independence.**

b) **Suppose $Y = X_1 + X_2$ with X_1 and X_2 independent. If you learn that Y and X_1 are Normal variables, prove that X_2 is also a Normal random variable.**