Homework 4

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1. **Suppose** V_1 **and** V_2 **are independent** $\text{Gamma}(1, \lambda)$ **random variables that represent waiting times in a Poisson process with rate** λ **events per unit time. Let** $X = V_1$ **be the time of the first event and let** $Y = V_1 + V_2$ **be the time of the second event.**

   a) **Find the joint CDF for** X **and** Y**:** $F_{xy}(x, y) = P(X \leq x, Y \leq y)$**. Hint: Graph the positive quadrant of the plane with axes** V_1 **and** V_2**, and mark the region where** $V_1 \leq x$ **and** $V_1 + V_2 \leq y$**. Integrate the joint pdf for** V_1 **and** V_2 **over this region to obtain the function** $F_{xy}(x, y)$**. Note that if there are restrictions on the arguments** x **and** y **that you do not specify, then you have failed to define the function.**

   b) **Show how to get the marginal CDF** F_x **by taking the upper limit for** y**, and** F_y **by taking the upper limit for** x**. Differentiate each marginal CDF to get the marginal pdfs.**

   c) **Show that taking partial derivatives of** F_{xy} **with respect to** x **and** y **yields the joint pdf:**
   $$\frac{\partial^2}{\partial x \partial y} F_{xy}(x, y) = \lambda^2 e^{-\lambda y} I(0 < x < y).$$

2. **Suppose Z_1 and Z_2 have joint pdf**

$$f_{12}(z_1, z_2) = \exp\left[-\log(\pi) - 2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right].$$

   a) **Identify this as a bivariate Normal density by specifying the means μ_1 and μ_2, standard deviations σ_1 and σ_2, and the correlation ρ.**

   Solution. Note that

$$\begin{aligned} f_{12}(z_1, z_2) &= \exp\left[-\log(\pi) - 2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right]. \end{aligned}$$

   Rewriting this expression, we have that

$$\begin{aligned} f_{12}(z_1, z_2) &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{2\pi\left(\frac{1}{2}\right)}\exp\left[-\frac{1}{2\left(\frac{1}{4}\right)}\left(z_1^2 + z_2^2 - 2\left(-\frac{\sqrt{3}}{2}\right)z_1 z_2\right)\right]. \end{aligned}$$

   We recognize this as a bivariate Normal density, with

$$\boxed{\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1, \text{ and } \rho = -\frac{\sqrt{3}}{2}}.$$

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   b) **Any joint pdf may be expressed as a marginal pdf multiplied by a conditional pdf. Show that $f_{12}(z_1, z_2)$ may be written as the product of a standard Normal density for Z_1 and a Normal density for $Z_2 \mid z_1$ that depends on z_1. Give the conditional mean and variance for $Z_2 \mid z_1$ and show that they agree with the formulas $\mathbb{E}[Z_2 \mid z_1] = \beta_0 + \beta_1 z_1$ with $\beta_1 = \rho\frac{\sigma_2}{\sigma_1}$, $\beta_0 = \mu_2 - \beta_1\mu_1$, and $\mathrm{Var}\left[Z_2 \mid z_1\right] = (1-\rho^2)\sigma_2^2$.**

   Solution. Let $Z_1 \sim N(0,1)$. We know that $f_{Z_1}(z_1) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z_1^2}{2}\right]$.

$$\begin{aligned} f_{12}(z_1, z_2) &= \frac{1}{\pi}\exp\left[-2(z_1^2 + z_2^2 + \sqrt{3}z_1 z_2)\right] \\ &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z_1^2}{2}\right]\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right] \end{aligned}$$

   Recognizing the first term as $f_{Z_1}(z_1)$, the pdf for Z_1, we have that

$$f_{12}(z_1, z_2) = f_{Z_1}(z_1)\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right].$$

   Rearranging the second term and simplifying, we get that

$$\begin{aligned} f_{12}(z_1, z_2) &= f_{Z_1}(z_1)\frac{\sqrt{2}}{\sqrt{\pi}}\exp\left[-\left(\frac{3}{2}z_1^2 + 2z_2^2 + 2\sqrt{3}z_1 z_2\right)\right] \\ &= f_{Z_1}(z_1)\frac{\sqrt{2}}{\pi}\exp\left[-2\left(z_2^2 + \sqrt{3}z_1 z_2 + \frac{3}{4}z_1^2\right)\right] \\ &= f_{Z_1}(z_1)\frac{1}{\sqrt{2\pi\cdot\frac{1}{4}}}\exp\left[-\frac{\left(z_2 + \frac{\sqrt{3}}{2}z_1\right)^2}{2\left(\frac{1}{4}\right)}\right]. \end{aligned}$$

We recognize the second term now as the pdf for $Z_2 \mid z_1 \sim N\left(-\frac{\sqrt{3}}{2}z_1, \frac{1}{4}\right)$, so we conclude

$$f_{12}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2|z_1}(z_2 \mid z_1).$$

Note that $\mathbb{E}\left[Z_2 \mid z_1\right] = -\frac{\sqrt{3}}{2}z_1$, and

$$\begin{aligned}
\beta_0 + \beta_1 z_1 &= \left(\mu_2 - \left(\rho\frac{\sigma_2}{\sigma_1}\right)\mu_1\right) + \rho\frac{\sigma_2}{\sigma_1}z_1 \\
&= (0 - 0) - \frac{\sqrt{3}}{2}z_1 \\
&= -\frac{\sqrt{3}}{2}z_1.
\end{aligned}$$

Similarly, $\text{Var}\left[Z_2 \mid z_1\right] = \frac{1}{4}$ and

$$\begin{aligned}
\left(1 - \rho^2\right)\sigma_2^2 &= \left(1 - \left(-\frac{\sqrt{3}}{2}\right)^2\right)1^2 \\
&= \frac{1}{4},
\end{aligned}$$

matching the results from the given formulas. ■

c) **You can also show conditional results using representation. For $Z_o \sim N(0,1)$ independent of Z_2, define $Z_1 = \rho Z_2 + \sqrt{1-\rho^2}Z_o$ to have correlation ρ with Z_2. Show that conditioning on $Z_2 = z_2$ and treating this as constant in the representation of Z_1 results in a conditional distribution $Z_1 \mid z_2$ that mirrors that of $Z_2 \mid z_1$ from part (b).**

Solution. For $Z_o \sim N(0,1)$ independent of Z_2, define $Z_1 = \rho Z_2 + \sqrt{1-\rho^2}Z_o$ as stated. Note that $\sqrt{1-\rho^2}Z_o \sim N(0, 1-\rho^2)$. Conditioning on $Z_2 \mid z_2$, we find that

$$Z_1 \mid z_2 \sim N(\rho Z_2, 1 - \rho^2)$$

For $\rho = -\frac{\sqrt{3}}{2}$ (giving $1 - \rho^2 = \frac{1}{4}$), we have

$$Z_1 \mid z_2 \sim N\left(-\frac{\sqrt{3}}{2}z_2, \frac{1}{4}\right)$$

which matches the distribution of $Z_2 \mid z_1$ in part (b). ■

3. **Suppose X and Y have joint pdf $f_{xy}(x, y) = I(0 < x < 1, -x < y < x)$.**

   a) **Explain how you can tell, without finding the marginal densities, that the conditional densities are Uniform. Write out the conditional densities $f_{x|y}(x \mid y)$ and $f_{y|x}(y \mid x)$.**

   Solution. The conditional densities are uniform as they are proportional to the joint density, which can be thought of as Uniform (as it is an indicator variable).

   Since by construction, $|y| < x$ and $x < 1$ it follows that the conditional density is

$$f_{x|y}(x \mid y) = \frac{1}{1 - |y|} I(|y| < x < 1).$$

   Similarly, we must have that $-x < y < x$, so

$$f_{x|y}(y \mid x) = \frac{1}{2x} I(-x < y < x).$$

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   b) **Explain how you can tell, without finding the marginal densities, that X and Y are not independent. Find the marginal pdf's $f_x(x)$ and $f_y(y)$ and verify that $f_{xy}(x, y) \neq f_x(x) f_y(y)$.**

   Solution. Note that very clearly, $f_{x|y}(x, y)$ are not equal for different values of y, so X and Y cannot be independent. We can verify this with the marginal pdfs:

$$f_x(x) = \frac{f_{xy}(x, y)}{f_{y|x}(y \mid x)} = \frac{I(0 < x < 1, -x < y, x)}{\frac{1}{2x} I(-x < y < x)} = 2xI(0 < x < 1).$$

   Similarly,

$$f_x(y) = \frac{f_{xy}(x, y)}{f_{x|y}(x \mid y)} = \frac{I(0 < x < 1, -x < y, x)}{\frac{1}{1-|y|} I(|y| < x < 1)} = (1 - |y|)I(-1 < y < 1).$$

   We check that

$$\begin{aligned} f_X(x) f_Y(y) &= 2x(1 - |y|)I(0 < x < 1)I(-1 < y < 1) \\ &\neq I(0 < x < 1, -x < y, x) = f_{XY}(x, y) \end{aligned}$$

   confirming that X and Y are not independent. ■

   c) **Show that X and Y are uncorrelated.**

   Solution. Note that

$$\mathbb{E}[XY] = \int_0^1 \int_{-x}^x xyI(0 < x < 1, -x < y < x)\, dy\, dx = 0$$

   by symmetry. Similarly,

$$\mathbb{E}[X] = \int_0^1 x(2x)\, dx = 1$$

   and

$$\mathbb{E}[Y] = \int_{-1}^1 y(1 - |y|)\, dy = 0$$

   by symmetry. Thus, we have that

$$\mathbb{E}[XY] - \mathbb{E}[X]\, \mathbb{E}[Y] = 0$$

   so X and Y are uncorrelated, as desired. ■

4. a) **Suppose X_1 and X_2 are Bernoulli random variables with expectations p_1 and p_2. Show that X_1 and X_2 are independent if and only if they are uncorrelated. This shows the Bernoulli distribution is special like the multivariate Normal distribution in that uncorrelated implies independence.**

   b) **Suppose $Y = X_1 + X_2$ with X_1 and X_2 independent. If you learn that Y and X_1 are Normal variables, prove that X_2 is also a Normal random variable.**