STAT111: Mathematical Statistics II **Spring 2024**

Homework 5

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1. **Suppose $X_1, \ldots, X_n$ are i.i.d. Normal with mean μ and variance σ^2.**

   a) **Show the MLE for σ is the square root of the MLE for σ^2 (in general, if $\hat{\theta}$ is the MLE of θ, then $\hat{\theta}$ is the MLE of θ, then $\hat{\varphi} = g(\hat{\theta})$ is the MLE for $\varphi = g(\theta)$).**

   Solution. If $X_1, \ldots, X_n$ are i.i.d. Normal with mean μ and variance σ^2,

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x_i-\mu)^2}{2\sigma^2}}$$
$$= \left(2\pi\sigma^2\right)^{-\frac{n}{2}} e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}.$$

The log-likelihood is

$$l(\mu, \sigma^2) = n \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\sum(x_i-\mu)^2}{2\sigma^2}$$
$$= -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{\sum(x_i-\mu)^2}{2\sigma^2}.$$

Setting the partials to 0 to solve for the MLEs, we get that

$$\frac{\partial l}{\partial \mu} = 2\frac{\sum(x_i-\mu)}{2\sigma^2}$$
$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{2\pi\sigma^2}\cdot 2\pi - \frac{1}{2}\cdot\left(\sum(x_i-\mu)^2\right)\cdot\left(-\frac{1}{\sigma^4}\right)$$
$$= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2$$

The first equation is equivalent to $\sum(x_i - \mu) = 0$, so solving for μ gives

$$\boxed{\hat{\mu} = \frac{\sum x_i}{n} = \overline{x}}.$$

The second equation is equivalent to $\frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum(x_i-\mu)^2$, and solving for σ^2 gives

$$\sigma^2 = \frac{\sum(x_i-\mu)^2}{n}.$$

Plugging in the MLE for μ, we get the MLE for $\hat{\sigma}^2$:

$$\hat{\sigma}^2 = \frac{\sum(x_i-\mu)^2}{n}$$
$$= \boxed{\frac{1}{n}\left(\sum(x_i-\overline{x})^2\right)}. \blacksquare$$

b) **Show that** $\hat{\sigma}^2 = \sum \frac{(X_i - \bar{X})^2}{n}$ **has smaller mean square error than the unbiased estimate** $s^2 = \sum \frac{(X_i - \overline{X})^2}{n-1}$**.**

Solution. ■

c) **Find an expression for the expected value of the sample standard deviation** $s = \sqrt{s^2}$**. Use this to construct an unbiased estimate for** σ**. You could check your answer by generating sample variances from the Gamma distribution implied by a given** n **and** σ**.**

2. **Suppose** $X_1, \ldots, X_n$ **are i.i.d. Poisson(**θ**).**

a) **Let** $Y = \sum X_i$**. If** $n = 3$**, find** $P(X_1 = 2, X_2 = 3, X_3 = 0 \mid Y = 5, \theta = 2)$**. How does this change if** $\theta = 3$**?**

Solution. ■

b) **Show that** $\overline{X}$ **is the MLE for** θ**, and that the reciprocal Fisher information gives the exact variance of** $\hat{\theta}$**. Verify that Poisson(**θ**) is a** 1**-parameter exponential family, so you know you can use the second derivative formula for the Fisher information.**

c) **If** $n = 10$ **and** $\theta = 10$**, check the Normal approximation to the distribution of** $\overline{X}$**. Use the continuity correction to approximate** $P(\overline{X} = 10)$ **and compare this to the exact probability based on the associated Poisson distribution.**

d) **Show that** $\theta \sim \text{Gamma}\left(\alpha, \frac{\alpha}{u}\right)$ **defines a conjugate family of prior distributions for** θ**, and that the posterior mean for** $\theta \mid \overline{x}$ **is a weighted average of** $\overline{x}$ **and the prior mean** μ**, with more weight on** $\overline{x}$ **as** n **increases.**