STAT111: Mathematical Statistics II **Spring 2024**

Homework 2

David Yang

1. **Evaluate the integral**

$$\int_0^\infty x^5 e^{-2x}\, dx$$

using integration by recognition. That is, recognize this function as proportional to a standard pdf and identify the constant multiplier needed to make the integral equal 1. Then take the reciprocal of that constant.

Solution. The integrand is the kernel of a Gamma$(6, 2)$ random variable.

Using this fact, we know that the PDF integrates to 1, i.e.

$$\int_0^\infty \frac{2^6}{\Gamma(6)} x^5 e^{-2x}\, dx = 1.$$

Solving for the integral we want to evaluate, we find that

$$\int_0^\infty x^5 e^{-2x}\, dx = \frac{\Gamma(6)}{2^6} = \frac{(6-1)!}{64} = \boxed{\frac{15}{8}}.$$

■

2. **Suppose that $X \sim \text{Gamma}\left(\alpha, \frac{\alpha}{\mu}\right)$ is parameterized so that the mean is μ.**

a) **Identify the mode of the pdf for X as a function of α and μ. That is, for what value of x is $f_X(x)$ (or $\ln f_X(x)$) maximized?**

Solution. $X \sim \text{Gamma}\left(\alpha, \frac{\alpha}{\mu}\right)$ has pdf

$$f_X(x) = \frac{\left(\frac{\alpha}{\mu}\right)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{\alpha}{\mu}\right)x}.$$

$f_X(x)$ is maximized when $\ln(f_X(x))$ is maximized. For convenience, let $\ell(x)$ denote $\ln(f_X(x))$. Note that

$$\ell(x) = \ln(f_X(x)) = \ln\left(\frac{\left(\frac{\alpha}{\mu}\right)^\alpha}{\Gamma(\alpha)}\right) + (\alpha - 1)\ln(x) - \left(\frac{\alpha}{\mu}\right)x.$$

We take the derivative of $\ell(x)$ and set it to 0 to solve for the maximum:

$$\ell'(x) = \frac{\alpha - 1}{x} - \frac{\alpha}{\mu}.$$

Note that $\ell'(x) = 0$ when $\frac{\alpha-1}{x} = \frac{\alpha}{\mu}$. Solving for x, we find that $x = \frac{\mu(\alpha-1)}{\alpha}$.

However, note that this is the mode only when $\alpha > 1$ (for which $x > 0$); when $\alpha \leq 1$, the mode occurs at $x = 0$. Thus, the mode of the pdf for X as a function of α and μ is

$$\boxed{x = \begin{cases} 0 & \text{if } \alpha \leq 1; \\ \frac{\mu(\alpha-1)}{\alpha} & \text{if } \alpha > 1 \end{cases}}$$

■

b) **Let $Y = \frac{1}{X}$, so that Y follows a reciprocal-Gamma$\left(\alpha, \frac{\alpha}{\mu}\right)$ distribution. Find the pdf for Y, and identify its mode as a function of α and u.**

Solution. Let $Y = \frac{1}{X}$. By the change of variables formula for pdfs, we know that

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$

where $g^{-1}(y) = \frac{1}{y}$ and $\frac{d}{dy}\left(g^{-1}(y)\right) = \frac{d}{dy}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$.

Plugging these back into our above formula, we have that

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{d}{dy}\left(g^{-1}(y)\right)\right| \\ &= \left[\frac{\left(\frac{\alpha}{\mu}\right)^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\left(\frac{\alpha}{\mu}\right)\left(\frac{1}{y}\right)}\right] \left(\frac{1}{y^2}\right) \\ &= \frac{\left(\frac{\alpha}{\mu}\right)^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\frac{\alpha}{\mu y}}. \end{aligned}$$

Thus, the pdf for Y is

$$\boxed{f_Y(y) = \frac{\left(\frac{\alpha}{\mu}\right)^{\alpha}}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\frac{\alpha}{\mu y}}}.$$

To identify the mode of y, we can once again work to maximize the log of the pdf for Y, denoted $\ell(y)$ for convenience. Note that

$$\ell(y) = \ln\left(\frac{\left(\frac{\alpha}{\mu}\right)^{\alpha}}{\Gamma(\alpha)}\right) + (-\alpha - 1)\ln(y) - \frac{\alpha}{\mu y}.$$

Taking the derivative of $\ell(y)$, we find

$$\ell'(y) = \frac{-\alpha - 1}{y} + \frac{\alpha}{\mu y^2}.$$

The mode occurs when $\ell'(y) = 0$, or when $\frac{\alpha+1}{y} = \frac{\alpha}{\mu y^2}$. Solving for y, we find that the mode is at

$$\boxed{y = \frac{\alpha}{\mu(\alpha + 1)}}$$

for all $\alpha > 0$. ■

3. **Let** $F(x) = \frac{x}{x+2} I(x > 0)$**.**

   a) **Show that** $F_X(x)$ **is a CDF and find the corresponding pdf.**

   Solution. First, note that $F_X(x)$ is nondecreasing as

$$F'_x(x) = \frac{2}{(x+2)^2} > 0$$

   for any x.

   Furthermore,

$$\lim_{x \to -\infty} F(x) = 0$$

   as $F(x) = 0$ for any $x \leq 0$.

   Finally, note that

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \frac{x}{x+2} = 1.$$

   Thus, $F_X(x)$ satisfies all the properties of a CDF. The corresponding pdf for X is simply $F'_X(x)$, or

$$\boxed{f_X(x) = \frac{2}{(x+2)^2} I(x > 0)}. \qquad \blacksquare$$

   b) **Identify these as the CDF and pdf for an** F^* **random variable (give the parameter values** a**,** b**, and** c**).**

   Solution. Recall that if $X \sim F^*(a, b, c)$, then

$$f_X(x) \propto \frac{x^{a-1}}{(c+x)^{a+b}} I(x > 0).$$

   Comparing the derived PDF $f_X = \frac{2}{(x+2)^2} I(x > 0)$ to an F^* distribution kernel, we find that $a = 1$, $b = 1$, and $c = 2$, so

$$\boxed{X \sim F^*(1, 1, 2)}. \qquad \blacksquare$$

   c) **For a random variable** X **that follows this** F^* **distribution, represent** X **in terms of two independent Gamma random variables and a positive constant** c**. Use this representation to identify the distributions of** $Y = \frac{1}{X}$ **and of** $R = \frac{X}{2+X}$**.**

   Solution. Recall that $F^*(a, b, c) = c\frac{V_1}{V_2}$ where $V_1 \sim \text{Gamma}(a, 1)$ and $V_2 \sim \text{Gamma}(b, 1)$.

   It follows that since $X \sim F^*(1, 1, 2)$,

$$\boxed{X = 2\frac{V_1}{V_2}}$$

   where $V_1 \sim \text{Gamma}(1, 1)$ and $V_2 \sim \text{Gamma}(1, 1)$ are independent.

Furthermore, note that for $Y = \frac{1}{X} = \frac{1}{2}\frac{V_2}{V_1}$, we have that

$$\boxed{Y \sim F^*\left(1, 1, \frac{1}{2}\right)}$$

and for $R = \frac{X}{2+X}$,

$$\boxed{R \sim \text{Beta}(1, 1)}$$

by properties of the F^* distribution. ■

4. **Suppose $X \mid \theta \sim \text{Poisson}(\theta)$, with $\theta \sim \text{Gamma}(\alpha, \lambda)$. Find the marginal pmf for X by integrating θ out of the joint pmf/pdf. Show that this is a Negative Binomial distribution that represents the count of successes at the time of our αth failure (if α happens to be an integer) and identify the success probability.**

   Solution. By rules of marginal pmfs, we know that the marginal pmf of X, $f_X(x)$, is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X\mid\theta}(x \mid \theta) f_\theta(\theta)\, d\theta$$

   Since $X \mid \theta \sim \text{Poisson}(\theta)$, we know that

$$f_{X\mid\theta}(x \mid \theta) = \frac{e^{-\theta}\theta^x}{x!} I(x > 0).$$

   Similarly, since $\theta \sim \text{Gamma}(\alpha, \lambda)$, we know that

$$f_\theta(\theta) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta}.$$

   Substituting these expressions in our expression for $f_X(x)$ and simplifying, we find that

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X\mid\theta}(x \mid \theta) f_\theta(\theta)\, d\theta \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{-\theta}\theta^x}{x!} I(x > 0)\right) \left(\frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta}\right) d\theta \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \int_{-\infty}^{\infty} \left(e^{-\theta}\theta^x I(x > 0)\right) \left(\theta^{\alpha-1} e^{-\lambda\theta}\right) d\theta \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \int_0^{\infty} e^{-\theta(\lambda+1)} \theta^{x+\alpha-1}\, d\theta
\end{aligned}$$

   We will now apply the change of variables $y = \theta(\lambda + 1)$ so that $\theta = \frac{y}{\lambda+1}$. Note that $dy = \lambda + 1\, d\theta$, so $d\theta = \frac{1}{\lambda+1} dy$. Applying this change of variables to our expression for $f_X(x)$, we get

$$\begin{aligned}
f_X(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \int_0^{\infty} e^{-\theta(\lambda+1)} \theta^{x+\alpha-1}\, d\theta \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \int_0^{\infty} e^{-y} \left(\frac{y}{\lambda+1}\right)^{x+\alpha-1} \left(\frac{1}{\lambda+1} dy\right) \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \int_0^{\infty} e^{-y} y^{x+\alpha-1} \left(\frac{1}{\lambda+1}\right)^{x+\alpha} dy \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \left(\frac{1}{\lambda+1}\right)^{x+\alpha} \int_0^{\infty} e^{-y} y^{x+\alpha-1}\, dy.
\end{aligned}$$

   Note that by definition of the Gamma Function, $\Gamma(x + \alpha) = \int_0^\infty e^{-y} y^{x+\alpha-1}\, dy$. It follows upon substitution that

$$\begin{aligned}
f_X(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \left(\frac{1}{\lambda+1}\right)^{x+\alpha} \int_0^{\infty} e^{-y} y^{x+\alpha-1}\, dy \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)x!} \left(\frac{1}{\lambda+1}\right)^{x+\alpha} \Gamma(x + \alpha).
\end{aligned}$$

Rewriting this expression by grouping and rearranging terms, we find that

$$
\begin{aligned}
f_X(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)x!}\left(\frac{1}{\lambda+1}\right)^{x+\alpha}\Gamma(x+\alpha-1) \\
&= \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)x!}\left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\left(\frac{1}{\lambda+1}\right)^{\times}
\end{aligned}
$$

When α is an integer, note that $\Gamma(x+\alpha) = (x+\alpha-1)!$ and $\Gamma(\alpha) = (\alpha-1)!$. Substituting and simplifying once more, we get that

$$
\begin{aligned}
f_X(x) &= \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)x!}\left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\left(\frac{1}{\lambda+1}\right)^{x} \\
&= \frac{(x+\alpha-1)!}{(\alpha-1)!\,x!}\left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\left(\frac{1}{\lambda+1}\right)^{x} \\
&= \binom{x+\alpha-1}{x}\left(\frac{1}{\lambda+1}\right)^{x}\left(\frac{\lambda}{\lambda+1}\right)^{\alpha}.
\end{aligned}
$$

Note that $\frac{1}{\lambda+1} + \frac{\lambda}{\lambda+1} = 1$, so this is precisely the PDF for a Negative Binomial Random Variable counting the number of successes x at the time of our αth failure, for success probability $\boxed{p = \dfrac{1}{\alpha+1}}$. ■