Exam 2 Solutions

October 14, 2018

Problem 1

part a

First we find the differential element $d\vec{s}$ along the track.

$$d\vec{s} = dx\,\hat{x} + dy\,\hat{y}$$
$$d\vec{s} = dx\,\hat{x} + \frac{dy}{dx}\cdot dx\,\hat{y}$$
$$d\vec{s} = dx\,\hat{x} + \frac{d\left(\cot(x)\right)}{dx}\cdot dx\,\hat{y}$$
$$d\vec{s} = dx\,\hat{x} - \csc^2(x)\cdot dx\,\hat{y}$$

Next we write $\vec{F}$ in 2D cartesian coordinates.

$$\vec{F} = -\frac{GMm}{r^3}\cdot\vec{r}$$
$$\vec{F} = -\frac{GMm}{r^2}\cdot\hat{r}$$
$$\vec{F} = -\frac{GMm}{r^2}\cdot\left(\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\right)$$
$$\vec{F} = -\frac{GMm}{(x^2+y^2)}\cdot\left(\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\right)$$
$$\vec{F} = -\frac{GMm}{(x^2+y^2)}\cdot\left(\frac{x}{r}\,\hat{x} + \frac{y}{r}\,\hat{y}\right)$$
$$\vec{F} = -\frac{GMm}{(x^2+y^2)}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\,\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{y}\right)$$
$$\vec{F} = -\frac{GMm}{(x^2+y^2)^{\frac{3}{2}}}\cdot\left(x\,\hat{x} + y\,\hat{y}\right)$$
$$\vec{F} = -\frac{GMmx}{(x^2+y^2)^{\frac{3}{2}}}\,\hat{x} - \frac{GMmy}{(x^2+y^2)^{\frac{3}{2}}}\,\hat{y}$$

Next we take the dot product $\vec{F} \cdot d\vec{s}$

$$\vec{F} \cdot d\vec{s} = F_x \, ds_x + F_y \, ds_y$$
$$\vec{F} \cdot d\vec{s} = -\frac{GMmx}{(x^2+y^2)^{\frac{3}{2}}} \, dx + \frac{GMmy}{(x^2+y^2)^{\frac{3}{2}}} \, \csc^2(x)$$
$$\vec{F} \cdot d\vec{s} = -GMm \left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}} - \frac{y}{(x^2+y^2)^{\frac{3}{2}}} \, \csc^2(x) \right) dx$$

Finally, plugging in $y = \cot(x)$ yields

$$\vec{F} \cdot d\vec{s} = -GMm \left(\frac{x}{(x^2+\cot^2(x))^{\frac{3}{2}}} - \frac{\cot(x)\csc^2(x)}{(x^2+\cot^2(x))^{\frac{3}{2}}} \right) dx$$

part b

To prove that $\vec{F}$ is conservative it is sufficient to show that $\nabla \times \vec{F} = 0$. (See Kleppner and Kolenkow and/or course notes for derivation).

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMmx}{(x^2+y^2)^{\frac{3}{2}}} & -\frac{GMmy}{(x^2+y^2)^{\frac{3}{2}}} & 0 \end{vmatrix} \\
&= \hat{z} \left(-\frac{\partial}{\partial x} \frac{GMmy}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\partial}{\partial y} \frac{GMmx}{(x^2+y^2)^{\frac{3}{2}}} \right) \\
&= -GMm \, \hat{z} \left(\frac{\partial}{\partial x} \frac{y}{r^{\frac{3}{2}}} - \frac{\partial}{\partial y} \frac{x}{r^{\frac{3}{2}}} \right) \\
&= -GMm \, \hat{z} \left(y \cdot \left(-\frac{3}{2} (r^{\frac{-5}{2}}) \right) \cdot 2x - x \cdot \left(-\frac{3}{2} (r^{\frac{-5}{2}}) \right) \cdot 2y \right) \\
&= 0
\end{aligned}$$

part c

To find $U(r)$ we can guess and check.

$$U(r) = -\frac{GMm}{r}$$
$$-\nabla U(r) = \frac{\partial}{\partial r} \frac{GMm}{r} \hat{r}$$
$$-\nabla U(r) = -\frac{GMm}{r^2} \hat{r} = \vec{F}(\vec{r})$$

So this potential satisfies our requirements. The value of the potential at point $(x = 0, y = \infty)$ is 0. The value of the potential at the point $(x = \frac{\pi}{2}, y = 0)$ is

$$\begin{aligned} U &= -\frac{GMm}{r} \\ &= -\frac{GMm}{\sqrt{x^2 + y^2}} \\ &= -\frac{2GMm}{\pi} \end{aligned}$$

So the difference in potential is

$$-\frac{2GMm}{\pi} - 0 = -\frac{2GMm}{\pi}$$

part d

$$\begin{aligned} -\Delta U &= \Delta KE \\ \frac{2GMm}{\pi} &= \frac{1}{2}mv^2 - 0 \\ \frac{4GM}{\pi} &= v^2 \\ \sqrt{\frac{4GM}{\pi}} &= v \end{aligned}$$

Problem 2

part a

Using the provided form, we can find $\dot{x}$ and $\ddot{x}$. Let's name the suggested solution.

$$x(t) = x_0 e^{-\alpha t} \cos(\omega_1 t + \phi) \qquad \text{(eq. 0)}$$

To determine whether eq. 0 is a solution, first, we find $\dot{x}$

$$\begin{aligned} x(t) &= x_0 e^{-\alpha t} \cos(\omega_1 t + \phi) \\ \dot{x}(t) &= -\alpha x_0 e^{-\alpha t} \cos(\omega_1 t + \phi) - \omega_1 x_0 e^{-\alpha t} \sin(\omega_1 t + \phi) \\ \dot{x}(t) &= -\alpha x(t) - \omega_1 x_0 e^{-\alpha t} \sin(\omega_1 t + \phi) && \text{eq. 1} \end{aligned}$$

To find $\ddot{x}$, starting with eq. 1

$$\begin{aligned} \dot{x}(t) &= -\alpha x(t) - \omega_1 x_0 e^{-\alpha t} \sin(\omega_1 t + \phi) \\ \ddot{x}(t) &= -\alpha \dot{x}(t) + \omega_1 x_0 \alpha e^{-\alpha t} \sin(\omega_1 t + \phi) + \omega_1^2 x_0 e^{-\alpha t} \cos(\omega_1 t + \phi) \\ \ddot{x}(t) &= -\alpha \dot{x}(t) + \omega_1 x_0 \alpha e^{-\alpha t} \sin(\omega_1 t + \phi) + \omega_1^2 x(t) && \text{eq. 2} \end{aligned}$$

Now, using eq. 1 we can write

$$\omega_1 x_0 e^{-\alpha t}\sin(\omega_1 t + \phi) = -\alpha x(t) - \dot{x}(t)$$

And plugging this back into eq. 2.

$$\begin{aligned}
\ddot{x}(t) &= -\alpha\dot{x}(t) + \omega_1 x_0 \alpha e^{-\alpha t}\sin(\omega_1 t + \phi) + \omega_1^2 x(t) \\
\ddot{x}(t) &= -\alpha\dot{x}(t) + \alpha\left(-\alpha x(t) - \dot{x}(t)\right) + \omega_1^2 x(t) \\
\ddot{x}(t) &= (\omega_1^2 - \alpha^2)x(t) - 2\alpha\dot{x}(t) && \text{eq. 3}
\end{aligned}$$

Now, by comparing eq. 3 to the original differential equation

$$\begin{aligned}
m\ddot{x} &= -kx - b\dot{x} \\
\ddot{x} &= -\frac{k}{m}x - \frac{b}{m}\dot{x} \\
\ddot{x} &= (\omega_1^2 - \alpha^2)x - 2\alpha\dot{x} && \text{eq. 3}
\end{aligned}$$

We can clearly see that the provided form, eq. 0, is a solution if we let

$$\begin{aligned}
k &= m(\alpha^2 - \omega_1^2) \\
b &= 2m\alpha
\end{aligned}$$

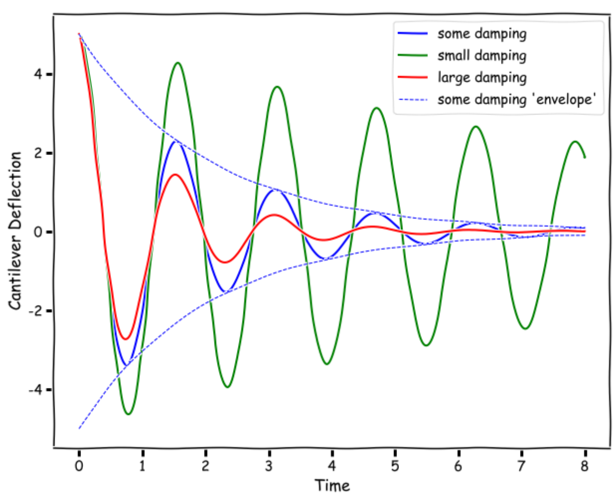

Figure 1: Showing some solutions of the form of eq. 0, with different damping parameters. The dotted blue line shows the exponentially decaying “envelope” of the oscillation.

part b

Using eq. 0,

$$
\begin{aligned}
x(t_1) &= x_0 e^{-\alpha t_1}\cos(\omega_1 t_1 + \phi) \\
x(t_2) &= x_0 e^{-\alpha t_2}\cos(\omega_1 t_2 + \phi) \\
\log(x(t_1)) &= -\alpha t_1 + \log(x_0) + \log(\cos(\omega_1 t_1 + \phi)) \\
\log(x(t_2)) &= -\alpha t_2 + \log(x_0) + \log(\cos(\omega_1 t_2 + \phi)) \\
\log(x(t_2)) - \log(x(t_1)) &= -\alpha(t_2 - t_1) + \log\left(\frac{\cos(\omega_1 t_2 + \phi)}{\cos(\omega_1 t_1 + \phi)}\right)
\end{aligned}
$$

Now if $t_2 - t_1 = \frac{2\pi}{\omega_1}$, then $\frac{\cos(\omega_1 t_2+\phi)}{\cos(\omega_1 t_1+\phi)} = 1$. So then we are left with

$$\log(x(t_2)) - \log(x(t_1)) = -\alpha(t_2 - t_1)$$

From which it is clear that $\gamma = 2\alpha = \frac{b}{m}$. If we want to use the graph of $x(t)$ after the "flick" to measure k, b, and m, we can do the following. First, we measure the time between successive peaks in $x(t)$. This time is one period of oscillation, τ. To find ω_1 we use $\omega_1 = \frac{2\pi}{\tau}$. Next we can use the equation $\log(x(t_2)) - \log(x(t_1)) = -\alpha(t_2 - t_1)$ to find α, provided that we choose t_2 and t_1 to be separated by τ.

Once we have measured α and ω_1, we can use our results from part a to convert to k, b, and m.

Problem 3

First to find the unknown constant A in terms of M_{sphere}

$$
\begin{aligned}
M_s &= \int_{\text{region where mass is}} dm \\
M_s &= \int_{\text{half sphere}} \rho_s dV \\
M_s &= 2\pi \int_0^{R_s} \int_0^{\frac{\pi}{2}} Ar \cdot r^2 \sin(\theta) d\theta dr \\
M_s &= \frac{A\pi R_s^4}{2} \int_0^{\frac{\pi}{2}} \sin(\theta) d\theta \\
M_s &= \frac{A\pi R_s^4}{2} \\
A &= \frac{2M_s}{\pi R_s^4}
\end{aligned}
$$

Next to find the center of mass of the half-sphere. I'll set the origin of coordinates at the origin of the sphere with the positive z-direction pointing toward the top of the mushroom, as shown.

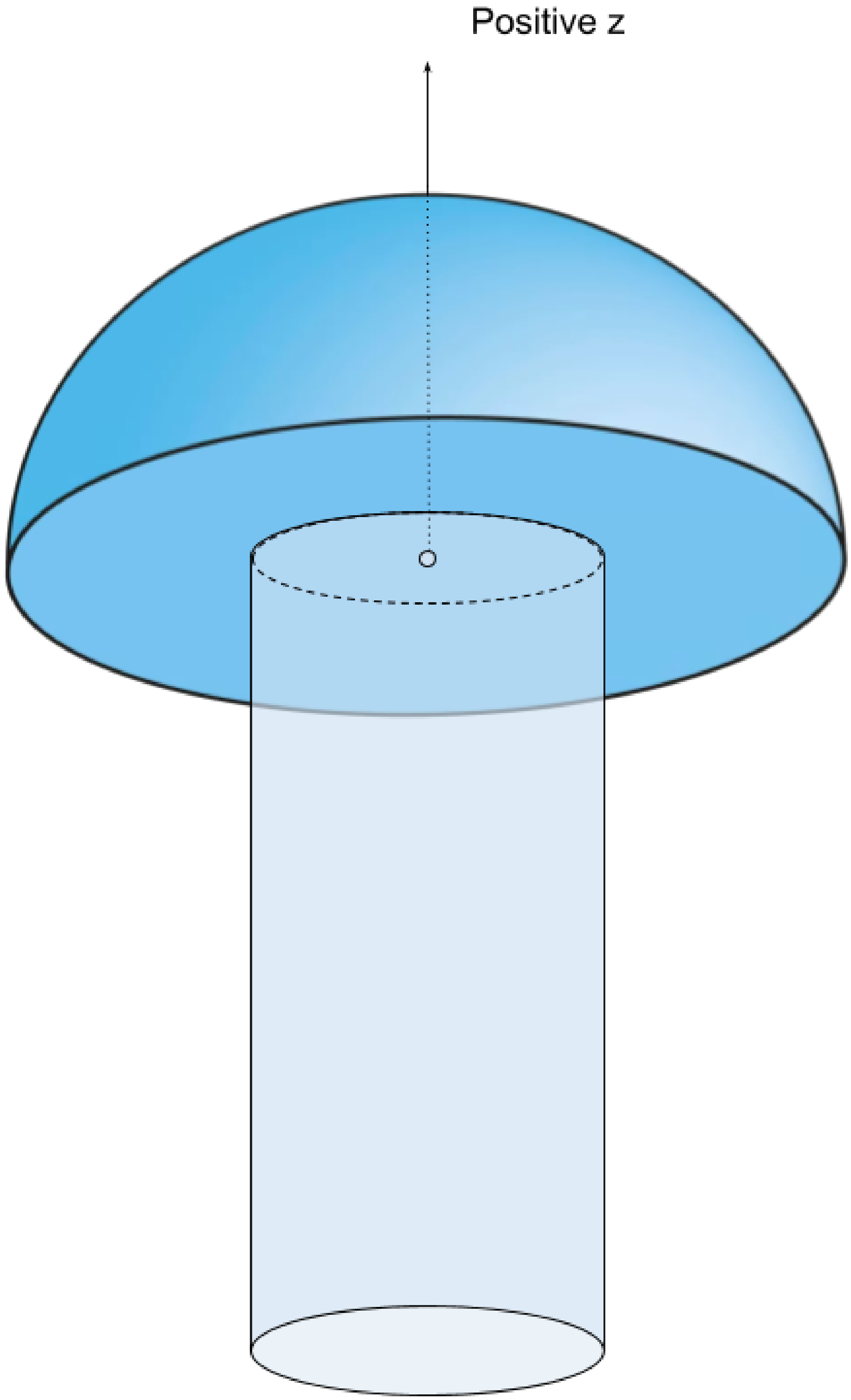

Figure 2: origin of coordinates chosen

By symmetry, the center of mass of the half sphere will lie on the z-axis, so we can write $\vec{R}_{\text{sphere COM}} = Z_{COM}\hat{z}$. To calculate the center of mass we use

$$\begin{aligned}
\vec{R}_{\text{sphere COM}} &= \frac{1}{M_s}\int_{\text{half sphere}} \vec{r}\,dm \\
\vec{R}_{\text{sphere COM}} &= \frac{1}{M_s}\int_{\text{half sphere}} r\rho_s\,dV\,\hat{r} \\
Z_{COM} &= \frac{1}{M_s}\int_{\text{half sphere}} r\rho_s\,dV\,\hat{r}\cdot\hat{z} \\
Z_{COM} &= \frac{1}{M_s}\int_{\text{half sphere}} r\cos(\theta)\rho_s\,dV \\
Z_{COM} &= \frac{2\pi}{M_s}\int_0^{R_s}\int_0^{\frac{\pi}{2}} r\cos(\theta)\rho_s\,r^2\sin(\theta)d\theta\,dr \\
Z_{COM} &= \frac{2\pi}{M_s}\int_0^{R_s}\int_0^{\frac{\pi}{2}} Ar^4dr\;\sin(\theta)\cos(\theta)d\theta \\
Z_{COM} &= \frac{2\pi AR_s^5}{5M_s}\int_0^{\frac{\pi}{2}}\sin(\theta)\cos(\theta)d\theta \\
Z_{COM} &= \frac{2\pi AR_s^5}{5M_s}\cdot\left(\frac{1}{2}\sin^2(\theta)\bigg|_0^{\frac{\pi}{2}}\right) \\
Z_{COM} &= \frac{\pi AR_s^5}{5M_s} \\
Z_{COM} &= \frac{2R_s}{5}
\end{aligned}$$

Now the density of the cylinder is constant, and so we know its center of mass also lies on the z-axis at a location $(0, 0, -\frac{h_{cyl}}{2})$. So to find the center of mass of the entire mushroom we can treat the "cap" and the "stem" as point particles with all of their respective mass stationed at their COMs.

$$Z_{\text{mushroom COM}} = \frac{\frac{2M_sR_s}{5} - \frac{M_{cyl}h_{cyl}}{2}}{M_s + M_{cyl}}$$

$$Z_{\text{mushroom COM}} = \frac{4M_sR_s - 5M_{cyl}h_{cyl}}{10\,(M_s + M_{cyl})}$$

Problem 4

part a

$$U(x) = U(x_0) + U'\Big|_{x_0}(x - x_0) \; + \; \frac{1}{2}U''\Big|_{x_0}(x - x_0)^2 \; + \; \frac{1}{6}U'''\Big|_{x_0}(x - x_0)^3 \; + ...$$
$$U(x) \approx U(x_0) + U'\Big|_{x_0}(x - x_0) + \frac{1}{2}U''\Big|_{x_0}(x - x_0)^2$$

$U(x_0)$ is an arbitrary constant that won't affect any of the physics. We are only interested in *changes* in U and *derivatives* of U. So we might as well set $U(x_0$ to 0. Since x_0 is a minimum, $U'\Big|_{x_0} = 0$, and so the first order term drops out. We are left with:

$$U(x) = \frac{1}{2}U''\Big|_{x_0}(x - x_0)^2$$

part b

In the above problem, if we name the curvature of the potential near the minimum k, $\frac{1}{2}U''\Big|_{x_0} = k$, and choose our coordinates so that $x_0 = 0$, then the formula in part a looks like

$$U(x) = \frac{1}{2}kx^2$$

But also we didn't know anything at all about the potential in part a! We only knew that it had a minimum at x_0. The potential energy of the cantilever in problem 2 has a minimum at zero deflection (like a diving board), but we don't know the exact form of the restoring force when we start to push it away from its equilibrium position at zero deflection. However, if we're dealing with small deflections we'll have a small x^2, which means that x^3 will be even smaller, and so most of the terms in our Taylor expansion for U will be *tiny*! For small x, the dominant contribution to the potential is the harmonic part. This means that for small deflections of our cantilever, treating it like a spring of stiffness k is just fine.

part c

The spring-like part of the force will be $F = -\frac{dU}{dx} = -kx$. Then we can write:

$$m\ddot{x} = -kx - b\dot{x}.$$

But this is exactly the same equation from problem 2 a. Using our results from earlier

$$\omega_1 = \sqrt{\alpha^2 - \frac{k}{m}} = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

By identifying k with the curvature of the potential—

$$\omega_1 = \sqrt{\frac{b^2}{4m^2} - \frac{U''|_{x_0}}{m}}$$

Problem 5

Using the same trick of Taylor expanding about the energy minimum, we can write

$$\begin{aligned}
U(r) &\approx \frac{1}{2}U''\Big|_{r=0} r^2 \\
U(r) &\approx \frac{1}{2}\left(\frac{d^2}{dr^2}Ae^{Br}\right)\Big|_{r=0} r^2 \\
U(r) &\approx \frac{1}{2}AB^2e^{Br}\Big|_0 r^2 \\
U(r) &\approx \frac{1}{2}\cdot AB^2\cdot r^2
\end{aligned}$$

Next, using the provided equipartition equation

$$\begin{aligned}
\frac{1}{2}k_BT &= \frac{1}{2}k\langle r^2\rangle \\
k_BT &= AB^2\cdot\langle r^2\rangle \\
\frac{k_BT}{AB^2} &= \langle r^2\rangle
\end{aligned}$$

Problem 6

$$\begin{aligned}
m\vec{u} &= m\vec{v}_1 + m\vec{v}_2 && \text{conservation of momentum} \\
\vec{u} &= \vec{v}_1 + \vec{v}_2
\end{aligned}$$

Conservation of momentum implies that the vectors $\vec{u}$, $\vec{v}_1$, and $\vec{v}_2$ form a triangle. Now, since the collision is elastic, conservation of energy gives us

$$\begin{aligned}
\frac{1}{2}m(\vec{u}\cdot\vec{u}) &= \frac{1}{2}m(\vec{v}_1\cdot\vec{v}_1) + \frac{1}{2}m(\vec{v}_2\cdot\vec{v}_2) && \text{conservation of energy} \\
\vec{u}\cdot\vec{u} &= \vec{v}_1\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2
\end{aligned}$$

On the other hand, since we know $\vec{u} = \vec{v}_1 + \vec{v}_2$, we can write

$$\begin{aligned}
\vec{u}\cdot\vec{u} &= (\vec{v}_1 + \vec{v}_2)\cdot(\vec{v}_1 + \vec{v}_2) \\
\vec{u}\cdot\vec{u} &= \vec{v}_1\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2 + 2\vec{v}_1\cdot\vec{v}_2
\end{aligned}$$

which implies that $\vec{v}_1\cdot\vec{v}_2 = 0$. In other words, the vectors $\vec{u}$, $\vec{v}_1$, and $\vec{v}_2$ form a *right* triangle.

I will choose the coordinate system that $\hat{x}$ lies in the direction of $\vec{v}_1$, so that $\hat{y}$ is perpendicular, and is the direction of $\vec{v}_2$, as shown in Fig. 3.

Then the magnitude and direction of $\vec{v}_1$, and $\vec{v}_2$ are both known.

$$\begin{aligned}
\vec{v}_1 &= |\vec{u}|\sin(\theta_2)\hat{x} \\
\vec{v}_2 &= |\vec{u}|\cos(\theta_2)\hat{y}
\end{aligned}$$

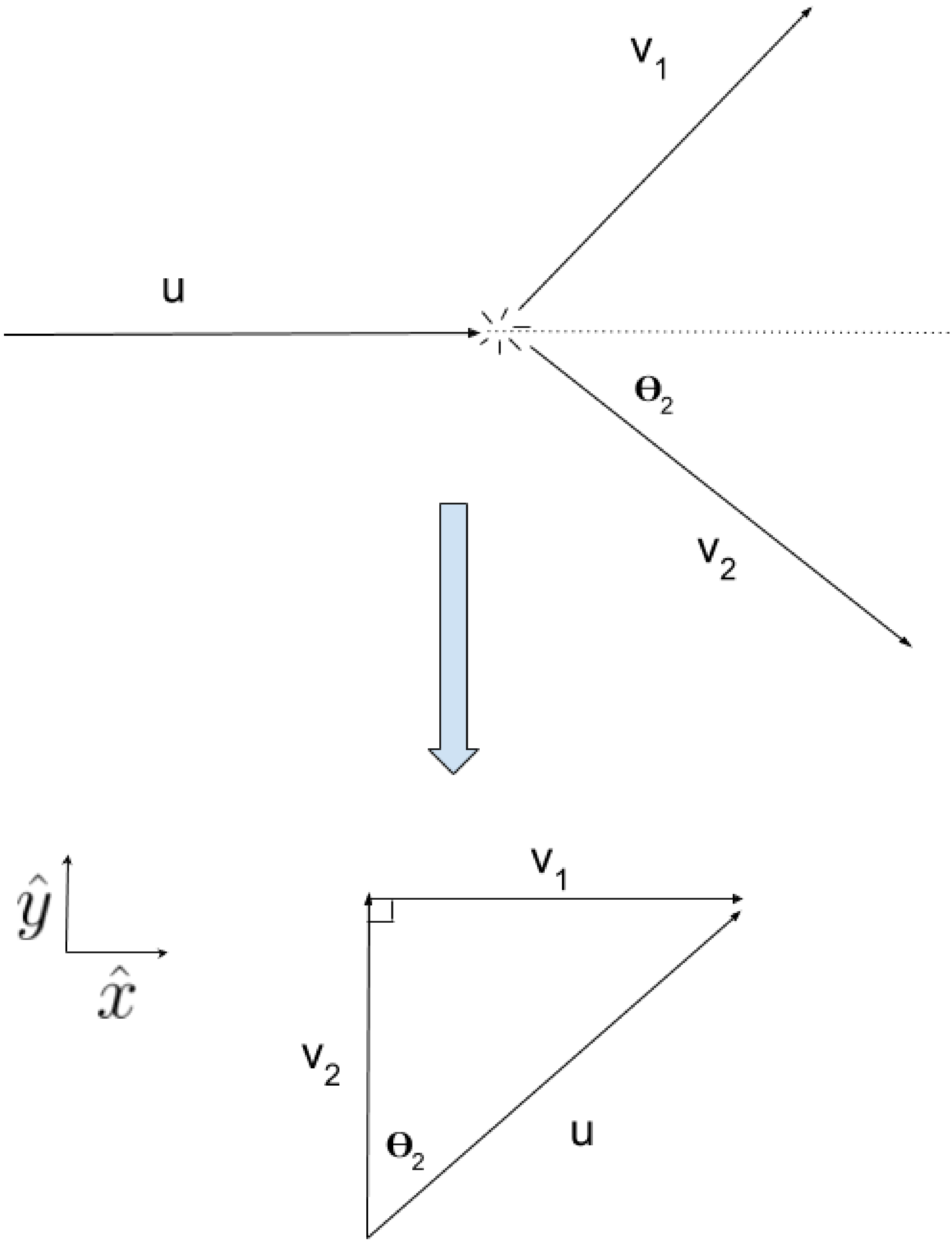

Figure 3: vectors $\vec{u}$, $\vec{v}_1$, and $\vec{v}_2$ form a right triangle.