Vectors and Kinematics

September 7, 2018

Problem 1- Triple Product 1

Show that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Solution

let $\vec{D} := \vec{A} \times (\vec{B} \times \vec{C})$. By the properties of the cross product we know that:

i $\vec{D} \perp \vec{A}$

ii $\vec{D} \perp (\vec{B} \times \vec{C})$

(ii) implies that the vector $\vec{D}$ lies in the plane defined by $\vec{B}$ and $\vec{C}$. Therefore, we can write $\vec{D}$ as a linear combination of $\vec{B}$ and $\vec{C}$.

iii $\vec{D} = x\vec{B} + y\vec{C}$

(i) implies that $\vec{A} \cdot \vec{D} = 0$. Then we can substitute in our expression (iii) for $\vec{D}$:

$$\begin{aligned} \vec{A} \cdot \vec{D} &= 0 \\ \vec{A} \cdot (x\vec{B} + y\vec{C}) &= 0 \\ x(\vec{A} \cdot \vec{B}) &= -y(\vec{C} \cdot \vec{A}) \end{aligned}$$

The last line is a single equation in two unknowns and so admits of a continuous family of solutions of the form

$$\begin{aligned} x &= k(\vec{A} \cdot \vec{C}) \\ y &= k(\vec{A} \cdot \vec{B}) \end{aligned}$$

for an arbitrary constant k. Combining everything we have so far:

iv $\vec{A} \times (\vec{B} \times \vec{C}) = k\left(\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})\right)$

Finally, we need to prove that k must be 1. We need another equation to get rid of the final unknown. Without loss of generality, we can pick any vectors for $\vec{A}$, $\vec{B}$, and $\vec{C}$ and obtain such an equation. Consider the case when $\vec{A} = \vec{B} = \hat{i}$ and $\vec{C} = \hat{j}$. Substitution into (iv) yields:

$$\begin{aligned} \hat{i} \times (\hat{i} \times \hat{j}) &= k\left(\hat{i}(\hat{i} \cdot \hat{j}) - \hat{j}(\hat{i} \cdot \hat{i})\right) \\ -\hat{j} &= -k \cdot \hat{j} \\ k &= 1 \end{aligned}$$

Problem 2- Fidget Spinner

Last night I put an LED in one of the three holes of a fidget spinner, and then set the fidget spinner spinning at an angular velocity ω. I then dropped the spinner from the top of Howey – height h. Note that I dropped the spinner so that its rotation was in the horizontal, xy plane, not the xz plane. In other words the LED will corkscrew as it falls, not tumble.

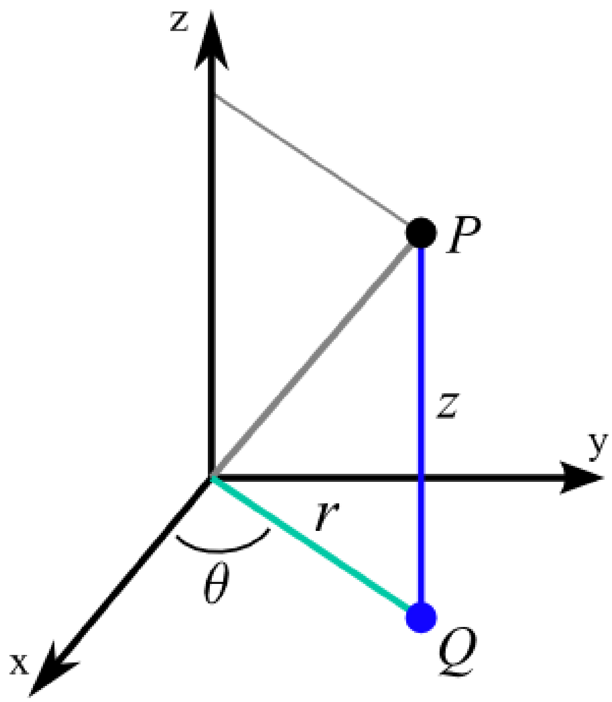
a

Describe the position and velocity of the LED at an arbitrary time after the drop, t_0, in the reference frame of someone standing on the ground watching the LED.

Solution 2a

Using the cylindrical coordinate system with r, θ, and z, we can easily write the position vector $\vec{P}(t)$:

$$\vec{P}(t) = R\hat{r} + (h - \frac{1}{2}gt^2)\hat{k}$$



To find the velocity, we need to take the time derivative $\frac{d\vec{P}}{dt}$.

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(R\hat{r}) + \frac{d}{dt}(h - \frac{1}{2}gt^2)\hat{k}$$
$$\frac{d\vec{P}}{dt} = R\frac{d\hat{r}}{dt} - gt\hat{k}$$
$$\frac{d\vec{P}}{dt} = R\frac{d\hat{r}}{dt} - gt\hat{k}$$

We can write the unit vector $\hat{r}$ as:

$$\hat{r} = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$$

Then

$$\frac{d\hat{r}}{dt} = \dot{\theta}\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right)$$
$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

And we know that the angular velocity $\omega = \dot{\theta}$. So finally:

$$\frac{d\vec{P}}{dt} = R\omega\hat{\theta} - gt\hat{k}$$

b

Describe the position and velocity of the LED at an arbitrary time after the drop, t_0, in the reference frame of the fidget spinner's center of mass. Is this an inertial reference frame? How could you prove that it is or isn't?

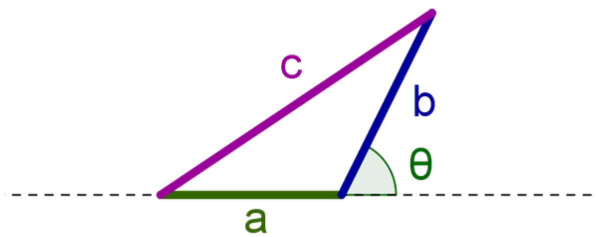
Solution 2b

In the c.o.m (center of mass) reference frame, we can describe the motion simply with plane polar coordinates:

$$\vec{r} = R\hat{r} \qquad\qquad \vec{v} = \frac{d}{dt}\vec{r} = R\frac{d}{dt}\hat{r} = R\dot{\theta}\hat{\theta}$$

However, this is not an inertial reference frame. The easiest way to see this is by noticing that the origin of coordinates is accelerating along a certain direction. But this could be proved by noting the violation of Newton's third law in this reference frame. In the c.o.m. frame, the ground, Howey physics building, and everything else appear to be accelerating upward at a rate g, and yet there can not be found a mirror force causing a commensurate acceleration of mass downward.

Problem 3- Law of Cosines



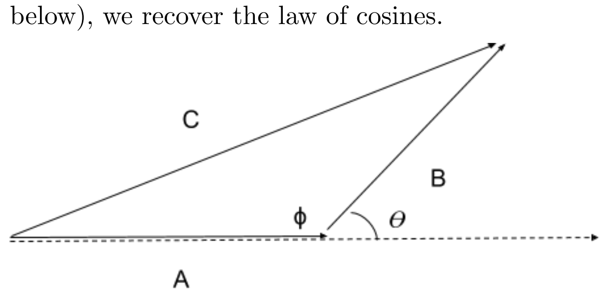
Find the length of C using the dot product.

Solution 3

Let the sides of the triangle be represented by vectors, such that $\vec{c} = \vec{a} + \vec{b}$. Then to find the squared magnitude $|\vec{c}|^2$, we can use the dot product:

$$|\vec{c}|^2 = \vec{c}\cdot\vec{c}$$
$$|\vec{c}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$
$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$
$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

Noticing that $\cos\theta = -\cos\pi - \theta$, and naming the triangle's interior angle ϕ (the angle formed by sides a and b, see figure below), we recover the law of cosines.



$$c^2 = a^2 + b^2 - 2ab\cos\phi.$$

Problem 4- Triple Product 2

a

Show that the volume of a parallelepiped formed by the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ is given by

$$V = \vec{A} \cdot (\vec{B} \times \vec{C})$$

Solution 4a

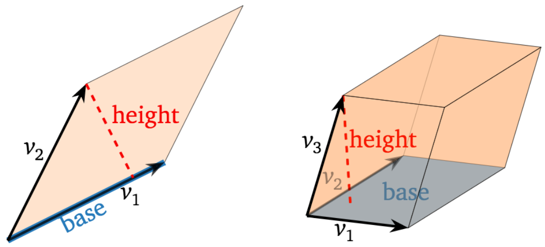


Figure 1: The area of a parallelogram is base $\times$ height. Similarly, the volume of a parallelepiped is the area of the base times the height.

The area of the parallelogram formed by vectors $\vec{B}$ and $\vec{C}$ can be found as follows. Let $\vec{B}$ be the "base", and θ be the angle between $\vec{B}$ and $\vec{C}$. Then the "height" of the paralellogram is clearly $|\vec{C}|\sin\theta$. Therefore the area is given by $|\vec{B}||\vec{C}|\sin\theta$. This is equal in magnitude to $\vec{B} \times \vec{C}$.

Then the volume of the parallelepiped will be this area (i.e. $|\vec{B} \times \vec{C}|$) times the "height". The height of the parallelepiped is given by the projection of the vector $\vec{A}$ onto the direction perpendicular to the plane formed by $\vec{B}$ and $\vec{C}$. In other words, the scalar product of $\vec{A}$ with a unit vector in the direction normal to $\vec{B}$ and $\vec{C}$: $\vec{A} \cdot \frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$.

Finally, the area of the base parallelogram multiplied by the height of the parallelepiped give the triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$.

b

Find $\vec{C} \cdot (\vec{A} \times \vec{B})$ in terms of V.

Solution 4b

This describes the same exact parallelepiped as in problem 4a, with the same volume V. So $\vec{C} \cdot (\vec{A} \times \vec{B}) = V$.

By stacking the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ into a matrix

$$M = \begin{bmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix}$$

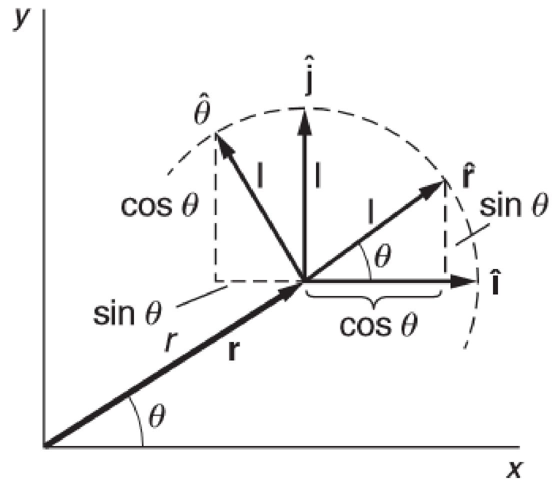
It is easy to see that $\det(M) = \vec{A} \cdot (\vec{B} \times \vec{C})$. The determinant of any matrix with linearly independent component vectors describes the volume of an n-dimensional parallelepiped. Furthermore, cyclic permutations of the rows of a matrix do not affect the determinant (and non-cyclic permutations only change the sign).

Problem 5- Polar Coordinates

a

Write the 2D polar coordinate system unit vectors, $\hat{r}$ and $\hat{\theta}$, in terms of the 2D Cartesian unit vectors $\hat{x}$ and $\hat{y}$, and in terms of θ.

Solution 5a



$$\hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$$
$$\hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y}$$

b

Write the time derivatives of $\hat{r}$ and $\hat{\theta}$ entirely in terms of θ, r, and $\dot{\theta}$.

Solution 5b

$$\frac{d}{dt}\hat{r} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$
$$\frac{d}{dt}\hat{r} = \dot{\theta}\frac{d}{d\theta}(\cos(\theta)\hat{x} + \sin(\theta)\hat{y})$$
$$\frac{d}{dt}\hat{r} = \dot{\theta}(-\sin(\theta)\hat{x} + \cos(\theta)\hat{y})$$
$$\frac{d}{dt}\hat{r} = \dot{\theta}\hat{\theta}$$

$$\frac{d}{dt}\hat{\theta} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}$$
$$\frac{d}{dt}\hat{\theta} = \dot{\theta}\frac{d}{d\theta}(-\sin(\theta)\hat{x} + \cos(\theta)\hat{y})$$
$$\frac{d}{dt}\hat{\theta} = \dot{\theta}(-\cos(\theta)\hat{x} - \sin(\theta)\hat{y})$$
$$\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{r}$$

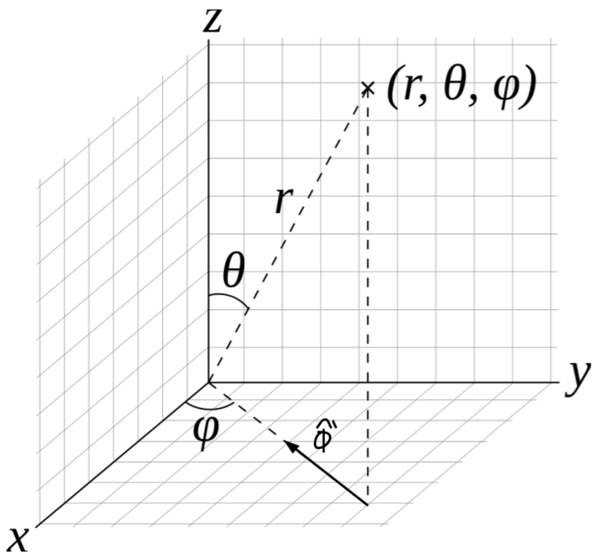
c

Do the same exercise in spherical polar coordinates

Solution 5c

The unit vectors in the spherical coordinate system are:

$$\hat{r} = \sin(\theta)\cos(\phi)\hat{x} + \sin(\theta)\sin(\phi)\hat{y} + \cos(\theta)\hat{z}$$
$$\hat{\theta} = \cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}$$
$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$



To find the time derivatives of the unit vectors we proceed in the same way as before.

$$\frac{d}{dt}\hat{r} = \dot{\theta}\left(\cos(\theta)\cos(\phi)\hat{x} + \cos(\theta)\sin(\phi)\hat{y} - \sin(\theta)\hat{z}\right) + \\ \dot{\phi}\left(-\sin(\theta)\sin(\phi)\hat{x} + \sin(\theta)\cos(\phi)\hat{y}\right)$$

$$\boxed{\frac{d}{dt}\hat{r} = \dot{\theta}\hat{\theta} + \sin(\theta)\dot{\phi}\hat{\phi}} \tag{1}$$

$$\frac{d}{dt}\hat{\theta} = \dot{\theta}\left(-\sin(\theta)\cos(\phi)\hat{x} - \sin(\theta)\sin(\phi)\hat{y} - \cos(\theta)\hat{z}\right) + \\ \dot{\phi}\left(-\cos(\theta)\sin(\phi)\hat{x} + \cos(\theta)\cos(\phi)\hat{y}\right)$$

$$\boxed{\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{r} + \cos\theta\dot{\phi}\hat{\phi}} \tag{2}$$

$$\frac{d}{dt}\hat{\phi} = \dot{\phi}\left(-\cos(\phi)\hat{x} - \sin(\phi)\hat{y}\right)$$
$$\frac{d}{dt}\hat{\phi} = \dot{\phi}\hat{\phi}'$$

Where $\hat{\phi}' = \frac{d}{d\phi}\hat{\phi} = -\cos(\phi)\hat{x} - \sin(\phi)\hat{y}$.

From here it is easiest to draw some pictures. Notice that $\hat{\phi}'$ lies in the same plane as $\hat{r}$, $\hat{\theta}$, and $\hat{z}$, as show in the figure above.

When drawn all in the same plane (see below), it is easier to see that

$$\boxed{\frac{d}{dt}\hat{\phi} = \dot{\phi}\left(-\cos(\theta)\hat{\theta} - \sin\theta\hat{r}\right)} \tag{3}$$

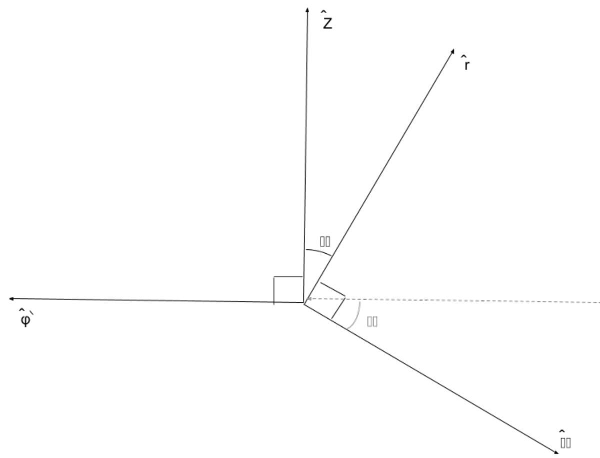


Figure 2: $\hat{\phi}'$ in the same plane as $\hat{r}$, $\hat{\theta}$, and $\hat{z}$.

This can be verified by doing the algebra.