Phys 3201 Fall 2018, Exam 2

October 17, 2018

We expect that you will complete this exam on your own without help from classmates, and without unfair use of the internet. The exam is, however, open notes and open book. Email dyanni3@gatech.edu with any questions or clarifications about what is being asked. The exam will be due at 5:00 pm, Monday Oct 22. To turn it in, follow the same guidelines as homework submissions- either drop it off in prof Goldman's office or email it to both of us. Good luck!

Problem 1: Conservative forces save us a headache

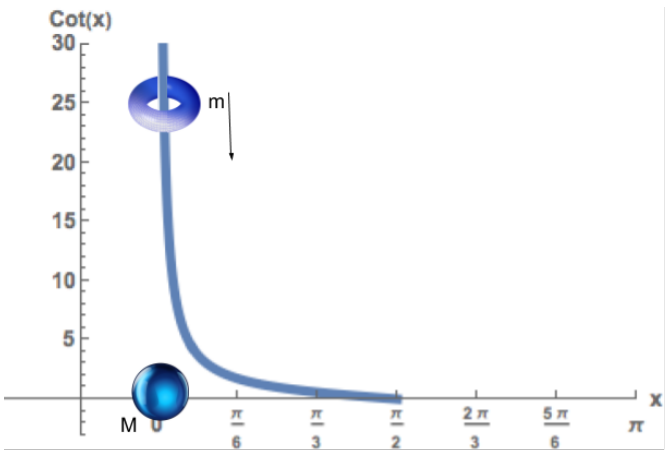

Figure 1: Torus constrained to move on the frictionless track, $y = cot(x)$.

A large mass M is stuck at the origin, and exerts a gravitational pull on a torus of mass m. The gravitational force is given by $\vec{F} = -\frac{GMm}{r^3}\vec{r}$, where $\vec{r}$ is the position vector of the torus (*from* the origin *to* the torus). The torus is constrained to move along a frictionless track given by the equation $y_{track} = \cot x$. The torus starts with zero speed at $(x = 0,\ y = \infty)$.

a Show that:

$$\vec{F}\cdot d\vec{s} = -GMm\left(\frac{x}{(x^2+cot^2(x))^{\frac{3}{2}}} - \frac{cot(x)csc^2(x)}{(x^2+cot^2(x))^{\frac{3}{2}}}\right)dx$$

where $d\vec{s}$ is a differential vector directed along the track.

b If we wish to know the velocity of the torus when it reaches the point $(x = \frac{\pi}{2},\ y = 0)$, we *could* use the work-energy theorem directly.

$$\int_a^b \vec{F}\cdot d\vec{s} = \frac{1}{2}m\left(v_b^2 - v_a^2\right)$$

As an optional but recommended exercise: evaluate this integral numerically (e.g. with Mathematica) and compare with the answers you get for part c. Instead, we'll use the fact that work done by a conservative force is path independent. **Prove that $\vec{F}$ is conservative.**

c Since $\vec{F}$ is conservative, it can be written as $\vec{F}(\vec{r}) = -\nabla U(\vec{r})$, where U is a potential energy. Find difference in potential energy between the points $(x = 0,\ y = \infty)$ and $(x = \frac{\pi}{2},\ y = 0)$.

d Find the speed of the torus when it reaches the point $(x = \frac{\pi}{2},\ y = 0)$.

Problem 2: Calibrating an AFM cantilever

An atomic force microscope is a powerful instrument based on incredibly simple principles. There is a laser beam that reflects off of the cantilever and hits a photodiode (see figure 2). This allows us to track the deflection of the cantilever over time. We're going to treat the cantilever as a simple harmonic oscillator—a spring of stiffness k with a mass m attached to the end, suspended in a fluid of damping coefficient b. We don't know b or k but wish to measure them. We can track the position of the mass $x(t)$ using the photodiode.

2.png

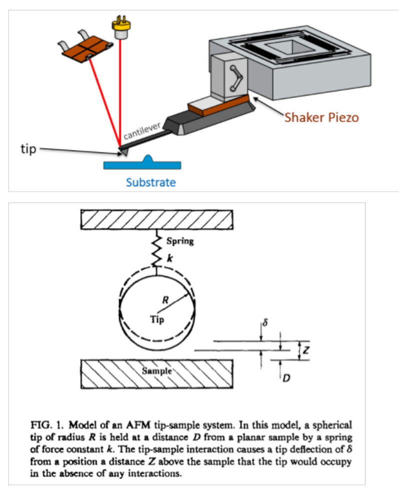

Figure 2: Simplified cartoon of AFM

a The first idea is to use the piezoelectric actuator to just "flick" the cantilever. We'll displace it by some unknown deflection X_0 and then let go and watch how it moves. The equation of motion for a damped harmonic oscillator as given in class is:

$$m\ddot{x} = -kx - b\dot{x}$$

Show that a solution to this differential equation has the form $x(t) = x_0 e^{-\alpha t}\cdot cos(\omega_1 t + \phi)$ and find α, and ω_1 in terms of k, b, and m. Plot or sketch $x(t)$ for your own choice of b, m, k. **hint:** see Kleppner and Kolenkow chapter 11. Also note that the provided "ansatz" is the underdamped solution, and is not the only solution to this differential equation! (good catch @MichaelRyan). Final note: x_0 and ϕ will be determined by initial conditions.

b Now show that

$$log(x(t_2)) - log(x(t_1)) = -\frac{1}{2}\gamma \cdot (t_2 - t_1)$$

for two times t_1 and t_2 after the "flick", **if** $t_2 - t_1 = \frac{2\pi}{\omega_1}$**.** Note that x and t and ω_1 are quantities we can measure from our trace of $x(t)$ from the photodiode. Now describe how you could measure ω_1 from the graph of $x(t)$. Find γ in terms of b and m. Find k in terms of m. So now if we knew the cantilever mass we could calculate the spring stiffness!

Problem 3: Mushroom

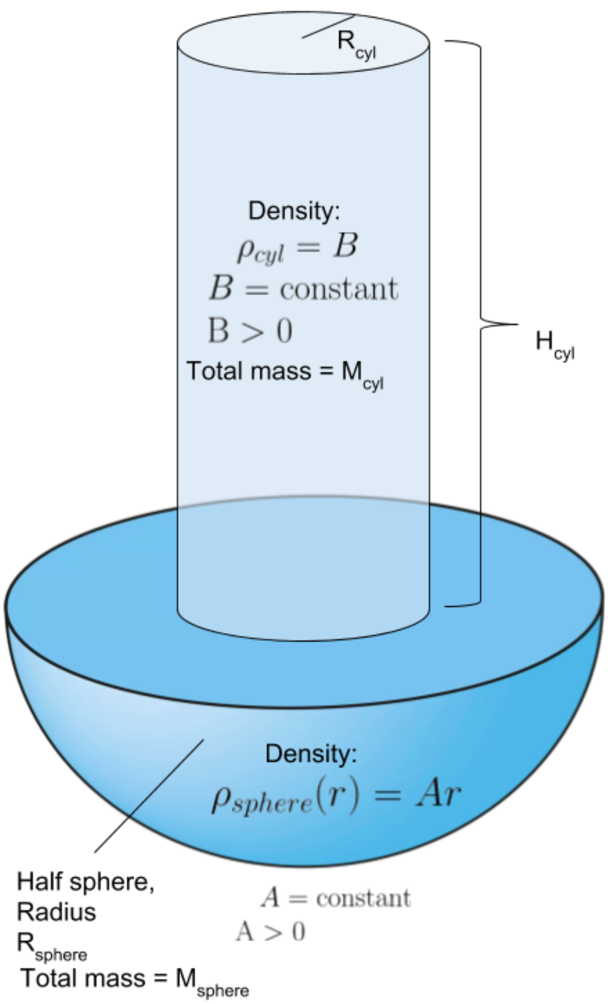

Figure 3:

Calculate the center of mass of the mushroom!
Use any coordinate system you like, but be sure to label it.

Problem 4: Any Potential

A particle of mass m moves in along an unknown potential energy landscape. We only know that there is a minimum at x_0.

a Use a Taylor expansion to write an approximation for $U(x)$ in the vicinity of x_0. Keep only the terms to second order.

b We approximated the cantilever in problem 1 as a simple spring with a mass attached. The reality is more complicated— the cantilever is closer to a diving board. Use your result from part (a) to argue that our simplifying assumption is likely valid for small deflections of the cantilever.

c The mass in part (a) is initially at rest at x_0 and then gets bumped to a position x_1 which is not too far away from x_0. If the motion of the mass is weakly damped by friction so that $m\ddot{x} \propto F_{spring-like} - b\dot{x}$, find the frequency that the mass oscillates. Leave your answer in terms of b, m, and $\left.\frac{d^2U(x)}{dx^2}\right|_{x_0}$ (the curvature of the potential evaluated at x_0.)

Problem 5: Optical Tweezers

Optical tweezers link to wiki are a very useful tool for biophysical exeriments. A small (usually $\approx 1\,\mu$m) bead is trapped by laser light in an "optical trap". The trapped bead can then be used to push and pull other objects, like a pair of tweezers. The potential energy well that traps the bead arises from somewhat complicated electrodynamics, but near the origin we can sweep those details under the rug and pretend the trapped particle is simply a damped harmonic oscillator. see figure 4. Let's imagine that potential energy had the form $U(r) = Ae^{Br}$ for constants $A > 0$, and $B > 0$, and r is the *distance* from the origin (the potential energy minimum), $r \geq 0$. Note, the exponential form was chosen to make the test doable, it is **not** really the form of the potential energy for an optical trap. See Introduction to optical trapping for details.

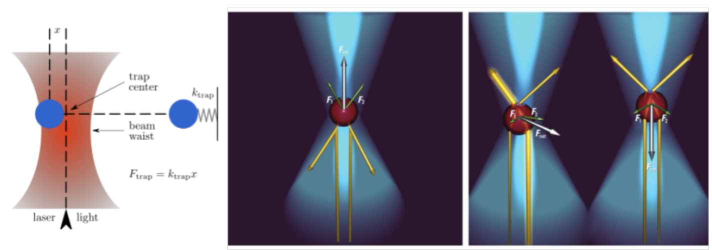

Figure 4: Optical tweezer basics

We know from statistical mechanics that $\frac{1}{2}k_BT = \frac{1}{2}k\langle r^2\rangle$. If in a fluid temperature T, what value do you expect for $\langle r^2(t)\rangle$, the time averaged value of the the trapped particle's position $r(t)$, if we pretend that this is all happening in one dimension? Report your answer in terms of A, B, and k_BT. One popular method to calibrate optical tweezers for force measurements is, again, the "thermal noise method".

Problem 6: Elastic collision

Ball 1 of mass m and initial velocity $\vec{u}$ collides elastically with another ball, (ball 2) of mass m that is initially at rest. Ball 2 flies off in a direction θ_2 from the direction $\frac{\vec{u}}{|\vec{u}|}$.

Find $\vec{v}_1$ the final velocity of ball 1, and $\vec{v}_2$, the final velocity of ball 2.

Report your answer in whatever coordinate system you like, but be sure to label/draw it.