AS.110.413 Introduction to Topology

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2.1 Definition.

A space X is said to have a countable basis at x if there exists a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

2.2 Definition.

A space X is said to be first-countable if it has a countable basis at each of its points.

2.3 Lemma.

(Sequence lemma) Let X be a topological space. Let A be a subset of X. If there exists a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X if first-countable.

2.4 Proof.

($\Rightarrow$) Let (x_n) be a sequence of points of A converging to x. Then, every neighborhood U of x contains all but finitely many points of (x_n), so $U \cap A$ is nonempty, which means precisely that $x \in \overline{A}$.

($\Leftarrow$) Let X be first-countable. Let $B_1, B_2, \ldots$ be a countable basis at x. Define a new countable basis with $B_1' = B_1$ and $B_{n+1}' = B_{n+1} \cap B_n'$ for $n = 1, 2, \ldots$. For each n, choose a point $x_n \in B_n' \cap A$. Then, (x_n) is a sequence of points of A converging to x.

2.5 Theorem.

Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

2.6 Proof.

($\Rightarrow$) Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x that contains all but finitely many points of (x_n). Thus, all but finitely many points of $(f(x_n))$ are in V, so $f(x_n) \to f(x)$.

($\Leftarrow$) Let X be first-countable. Let $A \subseteq X$. We want to show $f(\overline{A}) \subseteq \overline{f(A)}$. By the sequence lemma, if $x \in \overline{A}$, then there exists a sequence (x_n) of points of A converging to x and hence a sequence of points $f(x_n)$ of $f(A)$ converging to $f(x)$. So $f(x) \in \overline{f(A)}$ by the sequence lemma, which means that $f(\overline{A}) \subseteq \overline{f(A)}$.

2.7 Definition.

A space X is said to be second-countable if it has a countable basis.

2.8 Lemma.

If X is second-countable, then any discrete subspace of X is countable.

2.9 Proof.

Let Y be a discrete subspace of X. For each $y \in Y$, choose a basic neighborhood B_y that contains only y. Then the map $y \mapsto B_y$ is injective, so Y is countable.

2.10 Example.

The uniform topology $\mathbb{R}^\omega$ is first-countable but not second-countable. $\mathbb{R}^\omega$ is metrizable, so it is first-countable. Let Y be the set of all sequences of 0s and 1s. Then Y is a discrete subspace, but Y is uncountable, so $\mathbb{R}^\omega$ is not second-countable in the uniform topology.

2.11 Theorem.

(a) A subspace of a first-countable space is first-countable.

(b) A countable product of first-countable spaces is first-countable.

(c) A subspace of a second-countable space is second-countable.

(d) A countable product of second-countable spaces is second-countable.

2.12 Proof.

The proof is obvious.

2.13 Definition.

A space X is said to satisfy the Lindlof property if every open cover of X has a countable subcover.

2.14 Theorem.

A second-countable space is Lindlof.

2.15 Proof.

Let $\mathcal{A}$ be an open cover of X. Let $\{B_n\}$ be a countable basis for X. For each n for which it is possible, choose $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. Then the collection $\{A_n\}$ is a countable subcover of $\mathcal{A}$. Indeed, for all $x \in X$, there exists an open $A \in \mathcal{A}$ that contains x, which in turn contains a basic neighborhood B_n of x. So A_n is defined and contains x.

2.16 Definition.

A space X is said to be separable if it has a countable dense subset.

2.17 Theorem.

A second-countable space is separable.

2.18 Proof.

Let $\{B_n\}$ be a countable basis for X. For each n, choose an $x_n \in B_n$. Then the set D of all such x_n is countable and dense.

2.19 Theorem.

Second-countability, Lindlofness, and separabilty are equivalent for a metrizable space.

2.20 Proof.

See Munkres Section 30 Exercise 5.

2.21 Definition.

A space X is said to be T_1 is for every pair of distinct points x and y, there exists a neighborhood U of x such that $y \notin U$.

2.22 Lemma.

A space X is T_1 if and only if every singleton set $\{x\}$ is closed.

2.23 Proof.

($\Rightarrow$) Suppose that X is T_1. $\{x\}$ is its own closure, for any other point y has a neighborhood U disjoint from $\{x\}$. So $\{x\}$ is closed.

($\Leftarrow$) Suppose that $\{x\}$ is closed. Let $y \in X$ be distinct from x. Then there exists a neighborhood U of y that does not contain x. So X is T_1.

2.24 Definition.

A space X is said to be T_2, or Hausdorff, if every pair of distinct points x and y have distinct neighborhoods U and V, respectively.

2.25 Definition.

A space X is said to be regular if for every point x and closed set A such that $x \notin A$, there exists disjoint neighborhoods U and V, respectively,

2.26 Definition.

A space X is said to be T_3 if it is T_1 and regular.

2.27 Definition.

A space X is said to be normal if for every pair of disjoint closed sets A and B, there exist disjoint neighborhoods U and V of A and B, respectively.

2.28 Definition.

A space X is said to be T_4 if it is T_1 and normal.

2.29 Theorem.

T_4 implies T_3 implies T_2 implies T_1.

2.30 Proof.

The proof is obvious.

2.31 Remark.

For convenience, regular shall refer to T_3 and normal shall refer to T_4.

2.32 Lemma.

Let X be a T_1 space. X is regular if and only if for every $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$.

2.33 Proof.

($\Rightarrow$) Let $x \in X$ and U be a neighborhood of x. Let $B = X - U$. Then A is closed, so there exists a neighborhood V of x that is disjoint from a neighborhood W of B. Moreover, $\bar{V}$ is disjoint from B, and $\bar{V} \subseteq U$, as desired.

($\Leftarrow$) Let $x \in X$ and $B \subseteq X$ be a closed subset disjoint from x. Let $U = X - B$. By hypothesis, there exists a neighborhood V of x such that $\bar{V} \subseteq U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively.

2.34 Lemma.

Let X be a T_1 space. X is normal if and only if for every closed subset A of X and every neighborhood U of A, there exists a neighborhood V of A such that $\overline{V} \subseteq U$.

2.35 Proof.

The proof is exactly the same as for the previous theorem.

2.36 Theorem.

(a) A subspace of a Hausdorff space is Hausdorff.

(b) A product of Hausdorff spaces is Hausdorff.

(c) A subspace of a regular space is regular.

(d) a product of regular spaces is regular.

2.37 Proof.

(a) Let X be Hausdorf and let Y be a subspace of X. Let x and y be distinct points of Y. There exist disjoint neighborhoods U and V of x and y, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y, respectively, in Y.

(b) Let $\{X_\alpha\}$ be a collection of Hausdorff spaces and let X be the product space. Let x and y be distinct points of X. Then there exists at least one β such that $x_\beta \neq y_\beta$. Let U_β and V_β be disjoint neighborhoods of x_β and y_β, respectively, in X_β. The sets $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are disjoint neighborhoods of x and y, respectively, in X.

(c) Let X be regular and let Y be a subspace of X. Let $x \in Y$ and let B be a closed subset of Y that does not contain x. Let $\overline{B}$ be the closure of B in X. Then $\overline{B} \cap Y = B$, so $x \notin \overline{B}$. By the regularity of X, the exist disjoint neighborhoods U and V of x and $\overline{B}$, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and B, respectively, in Y.

(d) Let $\{X_\alpha\}$ be a collection of regular spaces and let X be the product space. It follows immediately that X is T_1 by Theorem 2.36a. Let $x \in X$ have the neighborhood U equals the product of U_α. For each U_α, let V_α be a neighborhood of x_α such that $\overline{V_\alpha} \subseteq U_\alpha$ by Lemma 2.32. If $U_\alpha = X_\alpha$, then let $V_\alpha = X_\alpha$. Let V be the product of V_α. Then $\overline{V} \subseteq U$, so X is regular.

2.38 Example.

$\mathbb{R}_K$ is Hausdorff but not regular.

2.39 Example.

$\mathbb{R}_l$ is normal.

2.40 Example.

$\mathbb{R}_l \times \mathbb{R}_l$ is regular but not normal.

2.41 Theorem.

A regular space with a countable basis is normal.

2.42 Proof.

Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. For each $x \in A$, there exists a neighborhood U of x and a neighborhood V of x such that $\overline{V} \subseteq U$. For each such V, choose a basic neighborhood U_x of x contained in V. This is a countable covering of A by sets whose closures do not intersect B. Index these sets as $\{U_n\}$. Similarly, cover B with a countable collection of neighborhoods $\{V_n\}$.

Define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V_i}$$

and

$$V'_n = V_n - \bigcup_{i=1}^{n} \overline{U_i}.$$

Each such set is open because it is the difference between an open set and a closed set. Furthermore, the collection $\{U'_n\}$ is an open cover of A and the collection $\{V'_n\}$ is an open cover of B. The open sets

$$U' = \bigcup_{n=1}^{\infty} U'_n$$

and

$$V' = \bigcup_{n=1}^{\infty} V'_n$$

are disjoint open sets that contain A and B. Indeed, if $x \in U' \cap V'$, then $x \in U'_i \cap V'_j$ for some i and j. Without loss of generality, assume that $i \leq j$. Then $x \in U_i$, but $x \notin \overline{U_i}$, hence a contradiction.

2.43 Theorem.

A metrizable space is normal.

2.44 Proof.

Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B(a, \varepsilon_a)$ is disjoint from B, and similarly, choose ε_b for each $b \in B$. The open sets

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2)$$

and

$$V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$$

are disjoint and contain A and B, respectively.

2.45 Theorem.

A compact Hausdorff space is normal.

2.46 Proof.

Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. For each $x \in A$, choose disjoint neighborhoods U_x and V_x of x and B, respectively. The collection $\{U_x\}$ is an open cover of A, so there exists a finite subcover $U_1, \ldots, U_n$. It follows that the sets

$$U = \bigcup_{i=1}^{n} U_i$$

and

$$V = \bigcap_{i=1}^{n} V_i$$

are disjoint neighborhoods of A and B, respectively.

2.47 Theorem.

Every well-ordered set is normal in the order topology.

2.48 Proof.

First observe that any interval $(x, y]$ is open, for if y is not the maximum element, then $(x, y] = (x, y')$ where y' is the immediate successor of y. Let A and B be disjoint closed sets in X.

Assume that neither A nor B contains the minimum element a_0 of X. For each $a \in A$, choose a neighborhood $(x_a, a]$ disjoint from B. For each $b \in B$, choose a neighborhood $(y_b, b]$ disjoint from A. The sets

$$U = \bigcup_{a \in A} (x_a, a]$$

and

$$V = \bigcup_{b \in B} (y_b, b]$$

are disjoint open sets containing A and B, respectively. Indeed, if $z \in U \cap V$, then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. Without loss of generality, assume that $a < b$. If $a \leq y_b$, then the intervals are disjoint. If $a > y_b$, then $(y_b, b]$ is not disjoint from A, a contradiction.

If A contains a_0, then the set $A - \{a_0\}$ is closed and disjoint from B, so it admits disjoint open intervals U and V of $A - \{a_0\}$ and B, respectively. Now the sets $U \cup \{a_0\}$ and V are disjoint neighborhoods of A and B, respectively.

2.49 Remark.

Indeed, every ordered set is normal in the order topology.

2.50 Example.

If J is uncountable, then the product space $\mathbb{R}^J$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $\mathbb{R}^J$ is homeomorphic to a subspace of $[0, 1]^J$).

2.51 Example.

The product space $S_\Omega \times \overline{S_\Omega}$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $overline S_\Omega \times \overline{S_\Omega}$ compact Hasudorff and therefore normal).

2.52 Lemma.

(Urysohn Lemma) Let X be a normal space. Let A and B be disjoint closed subsets of X. THen there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

2.53 Proof.

TODO!

2.54 Definition.

If A and B are two subsets of X, and if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$, then we say that A and B are separated by a continuous function.

2.55 Remark.

The Urysohn lemma says that if every pair A and B of disjoint closed subsets can be separated by open sets, then they can be separated by a continuous function. The converse is trivial.

2.56 Definition.

A space X is said to be completely regular if it is T_1 and if for every $x_0 \in X$ and every closed set A that does not contain x_0, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

2.57 Remark.

Complete regularity is also known as $T_{3+1/2}$.

2.58 Theorem.

(a) A subspace of a completely regular space is completely regular.

(b) A product of completely regular spaces is completely regular.

2.59 Proof.

(a) Let X be completely regular and let Y be a subspace of X. Let $x_0 \in Y$ and let A be a closed subset of Y that does not contain x_0. Then $A = \overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X, and moreoever, $x_0 \notin \overline{A}$. There exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in \overline{A}$. Then $f \mid Y$ is a continuous function from Y to $[0,1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

(b) Let X be the product of completely regular spaces $\{X_\alpha\}$. Let $b \in X$ and let A be a closed set of X that does not contain b. Choose a basic neighborhood U of b, and denote $U_1, \ldots, U_n$ the basic neighborhoods of b_α in X_α that are not all of X_α. For each U_i, choose a continuous function $f_i : X_\alpha \to [0,1]$ such that $f_i(b_\alpha) = 0$ and $f_i(x) = 1$ for all $x \in A$. The function $f : X \to [0,1]$ defined by

$$f(x) = \prod_{i=1}^{n} f_i(x_\alpha)$$

is continuous and satisfies the desired properties.

2.60 Theorem.

(Urysohn Metrization Theorem) A regular space with a countable basis is metrizable.

2.61 Proof.

TODO!

2.62 Theorem.

(Embedding Theorem) Let X be a a T_1 space. Suppose that $\{f_\alpha\}$ is a family of continuous functions indexed for $\alpha \in J$. If $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ for all $x \notin U$. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = \prod_{\alpha \in J} f_\alpha(x)$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$, then F is an imbedding of X in $[0,1]^J$.

2.63 Proof.

TODO!

2.64 Theorem.

A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.

2.65 Theorem.

(Tietze Extension Theorem) Let X be a normal space. Let A be a closed subspace of X. Any continuous map of A into $[0,1]$ or A into $\mathbb{R}$ can be extended to a continuous map of X into $[0,1]$ or $\mathbb{R}$, respectively.

2.66 Proof.

TODO!

3.0 Section.

Fundamental Group.

3.1 Definition.

If f and f' are continuous maps of X into Y, we say that f is homotopic to f' if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$. We denote $f \simeq f'$ and call F a homotopy between f and f'.

3.2 Definition.

If $f \simeq f'$ and f' is a constant map, then we say that f is nullhomotopic.

3.3 Definition.

If f and f' are paths in X with initial point x_0 and final point x_1, then we say that f is path homotopic to f' if there exists a continuous map $F : I \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in I$ and $F(0, t) = x_0$ and $F(1, t) = x_1$ for all $t \in I$. We denote $f \simeq_p f'$ and call F a path homotopy between f and f'.

3.4 Lemma

The relations $\simeq$ and $\simeq_p$ are equivalence relations.

3.5 Proof.

The proof is obvious.

3.6 Definition.

Denote $[f]$ the equivalence class of f under the relation $\simeq_p$. $[f]$ is called the path homotopy class of f.

3.7 Example.

If A is a convex subspace of $\mathbb{R}^n$, then any two paths in A with the same endpoints are homotopic. For the straihgt-line homotopy

$$F(x, t) = (1 - t)f(x) + tf'(x)$$

is a homotopy between f and f'.

3.8 Definition.

If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, then the product $h = f * g$ is defined as

$$h(s) = \begin{cases} f(2s) & \text{if } s \in [0, 1/2] \\ g(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases}$$

3.9 Definition.

The product between path-homotopy classes is defined as $[f] * [g] = [f * g]$. Indeed, let F be a path homotopy between f and f' and let G be a path homotpy between g and g'. Define

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } s \in [0, 1/2] \\ G(2s - 1, t) & \text{if } s \in [1/2, 1]. \end{cases}$$

3.10 Lemma.

If $k : X \to Y$ is a continuous map and if F is a path homotopy in X between f and f', then $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ f'$.

3.11 Proof.

The proof is obvious.

3.12 Lemma.

If $k : X \to Y$ is a continuous map and if f and g are paths in X such that $f(1) = g(0)$, then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

3.13 Proof.

The proof is obvious.

3.14 Theorem.

The operation $*$ satisfies the following properties.

(a) (Associativity) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$ whenever the products are defined.

(b) (Identity) Let e_x denote the constant path at point x. If f is a path from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(c) (Inverse) If f is a path from x_0 to x_1, then let its reverse $\overline{f}$ be defined as $\overline{f}(s) = f(1-s)$ for $s \in I$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

3.15 Proof.

(a) Let $[a, b]$ and $[c, d]$ be two intervals in I. There exists an unique continuous map $p : [a, b] \to [c, d]$ of the form $p(x) = mx + k$ called the positive linear map. The inverse of a positive linear map is a positive linear map, and the composition of two positive linear maps is a positive linear map.

When the triple product $f * g * h$ is defined, it is the path k_{ab} in X where on $[0, a]$, it is the positive linear map of $[0, a]$ to $[0, 1]$ followed by f and similarly for $[a, b]$ and $[b, 1]$. The path homotopy class of k_{ab} is independent of the choice of a and b. Indeed, $[f] * ([g] * [h])$ is the path homotopy class of k_{ab} where $a = 1/2$ and $b = 3/4$ whille $([f] * [g]) * [h]$ is the path homotopy class of k_{ab} where $a = 1/4$ and $b = 1/2$. These are equivalent.

(b) Let e_0 denote the constant path in I at 0 and let i denote the identity path in I. Then $e_0 * i$ is a path in I from 0 to 1. I is convex, so there is a path homotopy G between i and $e_0 * i$, so $f \circ G$ is a path homotopy between $f \circ i = f$ and

$$f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f.$$

The proof for right identity is entirely similar.

(c) Let i denote the identity path in I and $\bar{i}$ denote its reverse. I is convex, so there is a path homotopy H between e_0 and $i * \bar{i}$. Then $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and

$$f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \overline{f}.$$

The proof for left inverse is entirely similar.

3.16 Theorem.

Let f be a path in X. Let $a_0, \ldots, a_n$ such that that $0 < a_0 < \ldots < a_n < 1$. Let f_i be the path in X defined as the positive linear map of I to $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * \ldots * [f_n].$$

3.17 Proof.

The proof is sketched in Proof 3.15a.

3.18 Definition.

Let G and G' be two groups with the operation $\cdot$. A homomorphism $f : G \to G'$ is such that

$$f(x \cdot y) = f(x) \cdot f(y)$$

for all $x, y \in G$. f satisfies $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.

3.19 Definition.

The kernel of f is the set $f^{-1}(e')$. It is a subgroup of G.

3.20 Definition.

The image of f is the set $f(G)$. It is a subgroup of G'.

3.21 Definition.

A homomorphism is called a monomorphism if it is injective (or equivalently of $f^{-1}(e') = e$). It is called an epimorphism if it is surjective. It is called an isomorphism if it is bijective.

3.22 Definition.

Let H be a subgroup of G. Let xH denote the set of products xh for all $h \in H$. It is called the left coset of H in G, and the collection of all such xH for $x \in G$ is a partition of G. Similarly, let Hx denote the right coset of H in G.

3.23 Definition.

H is said to be a normal subgroup of G if $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. In this case, $xH = Hx$ for all $x \in G$. The partition G/H is called the quotient group of G by H with the operation $(xH)(yH) = xyH$.

3.24 Defintion.

The map $f : G \to G/H$ defined by $f(x) = xH$ is epimorphism with kernel H. Conversely, if $f : G \to G'$ is an epimorphism and N is a normal subgroup of G, then f induces an isomorphism $g : G/N \to G'$ defined by $g(xN) = f(x)$.

3.25 Definition.

If H is not normal, then G/H denotes the collection of right cosets of H in G.

3.26 Definition.

Let X be a topological space and let $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ relative to the base point x_0 is the group of path homotopy classes of loops in X based at x_0 with the operation $*$.

3.27 Example.

Any convex subspace of $\mathbb{R}^n$ has a trivial fundamental group.

3.28 Definition.

Let α be a path in X from x_0 to x_1. Define the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha].$$

If f is a loop at x_0, then $\overline{\alpha} * f * \alpha$ is a loop at x_1.

3.29 Theorem.

The map $\hat{\alpha}$ is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

3.30 Proof.

We compute that

$$\begin{aligned}\hat{\alpha}([f]) * \hat{\alpha}([g]) &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= [\overline{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f] * [g]),\end{aligned}$$

so $\hat{\alpha}$ is a homomorphism. We show that $\hat{\alpha}$ has a left inverse and a right inverse. Let $\beta = \overline{\alpha}$. Then

$$\begin{aligned}\hat{\beta}(\hat{\alpha}([f])) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [e_{x_0}] * [f] * [e_{x_1}] \\ &= [f]\end{aligned}$$

for all $[f] \in \pi_1(X, x_0)$. A similar computation shows that $\hat{\beta}$ is also a right inverse of $\hat{\alpha}$. So $\hat{\alpha}$ is an isomorphism.

3.31 Corollary.

If X is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

3.32 Proof.

The proof is trivial.

3.33 Remark.

We must still specify the base point x_0 in the definition of the fundamental group, for the isomorphism $\hat{\alpha}$ depends on the choice of α.

3.34 Definition.

A space X is said to be simply connected if it is path connected and $\pi_1(X, x_0)$ is trivial for one (and hence all) $x_0 \in X$.

3.35 Lemma.

If X is simply connected, then any two paths in X with the same endpoints are path homotopic.

3.36 Proof.

Let α and β be two paths in X from x_0 to x_1. We compute that

$$\begin{aligned}[\alpha] &= [\alpha] * [\overline{\beta}] * [\beta] \\ &= [\alpha * \overline{\beta}] * [\beta] \\ &= [e_{x_0}] * [\beta] \\ &= [\beta].\end{aligned}$$

3.37 Definition.

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. The homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by h, relative to base point x_0, is defined as

$$h_*([f]) = [h \circ f].$$

h_* is a homomorphism because

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

h_* depends on the choice of x_0, so we denote $(h_{x_0})_*$ when necessary.

3.38 Theorem.

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism of $\pi_1(X, x_0)$.

3.39 Proof.

We compute that

$$\begin{aligned} i_*([f]) &= [i \circ f] \\ &= [f]. \end{aligned}$$

3.40 Theorem.

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps, then

$$(k \circ h)_* = k_* \circ h_*.$$

3.41 Proof.

We compute that

$$\begin{aligned} (k \circ h)_*([f]) &= [(k \circ h) \circ f] \\ &= [k \circ (h \circ f)] \\ &= k_*([h \circ f]) \\ &= (k_* \circ h_*)([f]). \end{aligned}$$

3.42 Corollary.

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

3.43 Proof.

Let k be the inverse of h. Then $(k \circ h)_* = i_*$ is the identity homomorphism of $\pi_1(X, x_0)$, and $(h \circ k)_* = j_*$ is the identity homomorphism of $\pi_1(Y, y_0)$. So h_* is an isomorphism.

3.44 Definition.

Let $p : E \to B$ be a continuous surjective map. An open set U of B is said to be evenly covered by p if $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ such that $p \mid V_\alpha$ is a homeomorphism onto U for each α. The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices.

3.45 Definition.

Let $p : E \to B$ be a continuous surjective map. If each point $b \in B$ has a neighborhood U that is evenly covered by p, then p is said to be a covering map and E is said to be a covering space of B.

3.46 Definition.

A map $p : E \to B$ is a local homeomorphism if for every point $e \in E$ has a neighborhood U that is homeomorphic to a neighborhood of $p(e)$ in B. A covering map is a local homeomorphism, but the converse does not hold necessarily.

3.47 Example.

The map $p : \mathbb{R}_+ \to \S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is surjective and a local homomorphism, but it is not a coverign map. For the point $b_0 = (1, 0)$ has no neighborhood U that is evenly covered by p. This example also shows that the restriction of a covering map need not be a covering map.

3.48 Theorem.

The map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

3.49 Proof.

Let U be the subset of S^1 consisting of points with positive first coordinate, i.e., the open right semicircle. The preimage $p^{-1}(U)$ is the union of the open intervals $V_n = (n - 1/4, n + 1/4)$ for $n \in \mathbb{Z}$. The restriction $p \mid V_n$ is injective because sin $\mid V_n$ is strictly monotonic, and it is surjective by the intermediate value theorem. $p \mid V_n$ is a continuous bijective map between a compact space and a Hausdorff space, so it is a homeomorphism of V_n and U, so U is evenly covered by p. A similar argument can be made for the left, upper, and lower semicircles. Thesse semicircles form an open cover of S^1, and each one is evenly covered by p, so p is a covering map.

3.50 Theorem.

Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ is a covering map, where p_0 is the restriction of p to E_0.

3.51 Proof.

Let $b_0 \in B_0$. Let U be a neighborhood of b_0 in B that is evenly covered by p. Let $\{V_\alpha\}$ be a partion of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $\{V_\alpha \cap E_0\}$ are a partition of $p_0^{-1}(U \cap B_0)$ into slices.

3.52 Theorem.

If $p : E \to B$ and $p' LE' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.

3.53 Proof.

Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b', respectively, that are evenly covered by p and p', respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ into slices. Then the preimage of $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets in $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. So $p \times p'$ is a covering map.

3.54 Example.

Let $T = S^1 \times S^1$ be the torus. The map $p \times p : R^2 \to T$ is a covering map, where p is the covering map $p : \mathbb{R} to S^1$ of Theorem 3.48. Intuitively, each unit square of $\mathbb{R}^2$ is mapped homeomorphically onto a unit disk of T.

3.55 Example.

Let $p \times p : \mathbb{R}^2 \to T$ be the covering map of the previous example. Let $b_0 = p(0)$, and let B_0 denote the figure-eight

$$B_0 = (S^1 \times b_0) \cup (b_0 \times S^1).$$

The preimage of B_0 is

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}),$$

and the map $P_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is a covering map.

3.56 Example.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. Let $p \times i : R \times R_+ \to S^1 \times R_+$ be a covering map, where i is the identity map of R_+. Let $x \times t \mapsto tx$ be a homeomorphism of $S^1 \times R_+$ with $R^2 \setminus \{0\}$. The composition yields a covering map $\mathbb{R} \times R_+ \to R^2 \setminus \{0\}$.

3.57 Definition.

Let $p : E \to B$ be a map. If $f : X \to B$ is continuous, a lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

3.58 Example.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. The path $f : I \to S^1$ defined by $f(t) = (\cos 2\pi f t, \sin 2\pi f t)$ lifts to the path $\tilde{f} : I \to \mathbb{R}$ defined by $\tilde{f}(t) = ft$.

3.59 Lemma.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Any path $f : I \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f} : I \to E$ beginning at e_0.

3.60 Proof.

Cover B by open sets U each of which is evenly covered by p. Find a subdivision of I, say $s_0, \ldots, s_n$ such that every $f([s_i, s_{i+1}])$ lies in a U. (Use the Lebesgue number lemma.)

Define $\tilde{f}(0) = e_0$. Suppose that $\tilde{f}(s)$ is defined for $0 \leq s \leq s_i$. Define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows. The set $f([s_i, s_{i+1}])$ is contained in a U that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Now $\tilde{f}(s_i)$ lies in one of these sets, say V_0. Define $\tilde{f}(s)$ for $s_i \leq s \leq s_{i+1}$ by

$$\tilde{f}(s) = (p \mid V_0)^{-1}(f(s)).$$

$p \mid V_0 : V_0 \to U$ is a homeomorphism, so $tildef$ is continuous on $[s_i, s_{i+1}]$. Hence, $\tilde{f}$ may be defined on I by induction. It is continuous by the pasting lemma, and $p \circ \tilde{f} = f$.

Suppose that $\tilde{f}'$ is another lifting of f beginning at e_0. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$. Suppose that $\tilde{f}'(s) = \tilde{f}(s)$ for all $0 \leq s \leq s_i$. By construction, $\tilde{f}'(s_i)$ lies in the same slice V_0 as $\tilde{f}(s_i)$. The map $p \mid V_0$ is a homeomorphism, so $\tilde{f}'(s_i) = \tilde{f}(s_i)$. By induction, $\tilde{f}'(s) = \tilde{f}(s)$ for all $s \in I$, so the lifting is unique.

3.61 Lemma.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Let $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

3.62 Proof.

Given F, define $\tilde{F}(0,0) = e_0$. Use Lemma 3.59 to define $\tilde{F}$ for $\{0\} \times I$ and for $I \times \{0\}$. Choose subdivisions $s_0 < \ldots, < s_m$ and $t_0 < \ldots < t_n$ small enough such that each rectangle $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is contained in an open set U of B that is evenly covered by p. (Use the Lebesgue number lemma.)

Given i_0 and j_0, assume that $\tilde{F}$ is defined on A, the union of $\{0\} \times I$ and $I \times \{0\}$ and all the previous rectangles. Assume that $\tilde{F}$ is a continuous lifting of $F \mid A$. Choose an open set U of B that is evnely covered by p and contains $F(I_{i_0} \times J_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Now $\tilde{F}$ is already defined on $C = A \cap (I_{i_0} \times J_{j_0})$, which is connected, so $\tilde{F}(C)$ is connected, so it must lie entirely in, say, V_0. Let $p_0 : V_0 \to U$ denote the restriction of p to V_0. Since $\tilde{F}$ is a lifting of $F \mid A$, for $x \in C$, $p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x)$, so that $\tilde{F}(x) = p_0^{-1}(F(x))$. Define

$$\tilde{F}(x) = p_0^1(F(x))$$

for $x \in I_{i_0} \times J_{j_0}$. The lifting $\tilde{F}$ may be defined on $I \times I$ by induction. It is continuous by the pasting lemma, and $p \circ \tilde{F} = F$. It is unique for the same reason as for Lemma 3.59.

Suppose that F is a path homotopy. The map F carries $\{0\} \times I$ to b_0. Now $\tilde{F}$ is a lifting of F, so it carries this set to $p^{-1}(b_0)$. But $p^{-1}(b_0)$ has the discrete topology as a subspace of E. Since $\{0\} \times I$ is connected, $\tilde{F}(\{0\} \times I)$ is connected, so it must be a single point. So $\tilde{F}$ is a path homotopy.

3.63 Theorem.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Let f and g be two paths in B from b_0 to b_1 and let $\tilde{f}$ and $\tilde{g}$ denote their respective liftings to paths in E, beginiing at e_0. If f and g are path homotpic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

3.64 Proof.

Let $F : I \times I \to B$ be the path homotoy between f and g. Then $F(0,0) = b_0$. Let $\tilde{F} : I \times I \to E$ be the lifting of F to E such that $\tilde{F}(0,0) = e_0$. By Lemma 3.61, $\tilde{F}$ is a path homotopy, so $\tilde{F}(\{0\} \times I) = \{e_0\}$ and $\tilde{F}(\{1\} \times I) = \{e_1\}$.

The restriction $\tilde{F} \mid (I \times \{0\})$ is a path on E from e_0 that is the lifting of $F \mid (I \times \{0\})$. By the uniqueness of path liftings, $\tilde{F}(s, 0) = \tilde{f}(s)$. Similarly, $\tilde{F}(s, 1) = \tilde{g}(s)$. So $\tilde{f}$ and $\tilde{g}$ both end at e_1, and $\tilde{F}$ is a path homotopy between them.

3.65 Definition.

Let $p : E \to B$ be a covering map. Let $b_0 \in B$. Choose e_0 such that $p(e_0) = b_0$. Given an element $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a well-defined map. We call ϕ the lifting correspondence derived from the covering map p. It depends on the choice of e_0.

3.66 Theorem.

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. If E is simply connected, then ϕ is bijective.

3.67 Proof.

If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$. So ϕ is surjective.

Suppose that E is simply connected. Let $[f]$ and $[g]$be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, to path sin E that begin at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotpy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. So $p \circ \tilde{F}$ is a path homopty in B between f and g. So $[f] = [g]$, and ϕ is injective.

3.68 Theorem.

The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$.

3.69 Proof.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. Let $e_0 = 0$ and let $b_0 = p(e_0)$. Then $p^{-1}(b_0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, the lifting correspondence $\phi : \pi_1(S^1, b_0) \to p^{-1}(b_0)$ is bijective. Indeed, ϕ is a homomorphism.

Given $[f]$ and $[g]$ in $\pi_1(S^1, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively. Let $m = \tilde{f}(1)$ and $n = \tilde{g}(1)$. So $\phi([f]) = m$ and $\phi([g]) = n$. Because $p(m + x) = p(x)$ for all $x \in \mathbb{R}$,

$$\tilde{g}'(s) = m + \tilde{g}(s)$$

is the lifting of g to $\mathbb{R}$ that begins at m. Then the product $\tilde{f} * \tilde{g}'$ is well-defined, and it is the lifting of $f * g$ to $\mathbb{R}$ that begins at 0. The end point is $\tilde{g}' = m + n$. So $\phi([f] * [g]) = m + n$. so ϕ is a homomorphism and hence an isomorphism.

3.70 Definition.

Let G be a group and let $x \in G$ have the inverse x^{-1}. If the set of all elements x^m for $m \in \mathbb{Z}$ equals G, then G is said to be a cyclic group with generator x. $|G|$ is said to be the order of G. G has infinite order if and only if it is isomorphic to the additive group of integers. G has order k if and only if it is isomorphic to the group $\mathbb{Z}/k$ of integers modulo k.

3.71 Remark.

If x is a generator of an inifite cyclic group G, and if y is an element of an arbitrary group H, then there exists a unique homomorphism $h : G \to H$ such that $h(x) = y$ defined by $h(x^n) = y^n$ for all $n \in \mathbb{Z}$.

3.72 Theorem.

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$.

(a) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism

(b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

3.73 Proof.

(a) Suppose that $\tilde{h}$ is a loop in E baesd at e_0 and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop. If $\tilde{F}$ is a lifting of F to E such that $\tilde{F}(0, 0) = e_0$, then $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(b) Given loops f and g in B, let $\tilde{f}$ and $\tilde{g}$ be the liftings of them to E that begin at e_0. Then $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. We show that $\phi([g]) = \phi([g])$ if and only if $[f] \in H * [g]$.

Suppose that $[g] \in H * [g]$. Then $[g] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Now the product $\tilde{h} * \tilde{g}$ is defined, and it is a lifting of $h * g$. Because $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point of E. So $\phi([f]) = \phi([g])$.

Now suppose that $\phi([f]) = \phi([g])$. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E. The product of $\tilde{f}$ and the reverse of $\tilde{g}$ is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy in

E between the loops $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy in B between $h * g$ and f, where $h = p \circ \tilde{h}$. Thus, $[f] \in H * [g]$ as desired.

If E is path connected, then ϕ is surjective, so Φ is surjective as well, and hence bijective.

(c) Injectivity of Φ implies that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$. Now $\phi[f] = e_0$ if and only if $[f] \in H * [e_0] = H$. But $\phi([f]) = e_0$ if and only if f lifts to a loop in E based at e_0.

3.74 Definition.

If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r \mid A$ is the identity map of A. If such an r exists, then A is said to be a retract of X.

3.75 Lemma.

If A is a retract of X, then the homomorphism of fundamental groups induced by the inclusion map $j : A \to X$ is injective.

3.76 Proof.

If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. It follows that $(rcircj)_* = r_* \circ j_*$ is the identity map of $\pi_1(A, a)$ by Theorem 3.38 and Theorem 3.40, so that j_* is injective.

3.77 Theorem.

(No-Retraction Theorem) There is no retraction of B^2 onto S^1.

3.78 Proof.

If S^1 were a retract of B^2, then the homomorphism induced by $j : S^1 \to B^2$ would be injective by Lemma 3.75. But the fundamental group of S^1 is nontrivial while the fundamental group of B^2 is trivial, a contradiction.

3.79 Lemma.

Let $h : S^1 \to X$ be a continuous map. Then teh following conditions are equivalent.

(a) h is nulhomotopic.

(b) h extends to a continuous map $k : B^2 \to X$.

(3) h_* is the trivial homomorphism of fundamental groups.

3.80 Proof.

TODO!

3.81 Corollary.

The inclusion map $j : S^1 \to R^2 \setminus \{0\}$ is not nulhomotopic. The identity map $i : S^1 \to S^1$ is not nulhomotopic.

3.82 Proof.

There is a retraction of $\mathbb{R}^2 \setminus \{0\}$ onto S^1. Therefore, j_* is injective and hence nontrivial. Similarly, i_* is the indentity homomorphism and is hence nontrivial.

3.83 Theorem.

Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector fiend points directly inward and a point of S^1 where the vector field points directly outward.

3.84 Proof.

TODO!

3.85 Theorem.

(Brouwer Fixed Point Theorem) If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such taht $f(x) = x$.

3.86 Proof.

Suppose, by way of contradiction, that $f(x) \neq x$ for all $x \in B^2$. Then define $v(x) = f(x) - x$, which yields a nonvanishing vector field $(x, v(x))$ on B^2. But the vector field cannot point directly outward at any point x of X^1, for otherwise $f(x) - x = ax$ for some $a > 0$, so $f(x) = (1 + a)x$ would lie outside the unit ball B^2, hence a contradiction.

3.87 Theorem.

(Seifert-van Kampen Theorem) Suppose that $X = U \cup V$ where U and V are open sets of X. Suppose that $U \cap V$is path connected and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respecitvely, into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$$

and

$$j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

3.88 Proof.

TODO!

3.89 Remark.

The Seifert-van Kampen theorem says that any loop f in X based at x_0 is homotopic to a product of the form $g_1 * (g_2 * (\ldots * g_n))$ where each g_i is a loop in X based at x_0 that lies entirely in either U or V.

3.90 Corollary.

Suppose that $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.

3.91 Proof.

The proof is obvious.

3.92 Theorem.

If $n = 2, 3, \ldots$, then S^n is simply connected.

3.93 Proof.

TODO!