AS.110.413 Introduction to Topology

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0.0 Section.

Introduction.

1.0 Section.

Connectedness and Compactness.

1.1 Definition.

Let X be a topological space. A separation of X is a paire (U, V) of disjoint nonempty dubsets of X whose union is X. A space X is said to be connected if there exists no separation of X.

1.2 Remark.

An equivalent definition is that X is connected if the only clopen sets of X are $\emptyset$ and X.

1.3 Lemma.

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y,neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

1.4 Proof.

Suppose that A and B form a sepration of Y. Then A is both open and closed in Y. A is closed, so $A = \overline{A} \cap Y$, that is, $\overline{A} \cap B = \emptyset$. So B contains no limit points of A. Similarly, A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonemtpy sets whose unioin is Y, neither of which contians a limit point of the other. Then $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$, so $overline A \cap Y = A$ and $\overline{B} \cap Y = B$. Thus A and B are both closed in Y and therefore both open in Y.

1.5 Lemma.

If the sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely in C or entirely in D.

1.6 Proof.

The sets $C \cap Y$ and $D \cap Y$ are disjoint nonempty sets whose union is Y. If btoh are nonempty, then Y is not connected, a contradiction.

1.7 Theorem.

The union of a collection of connected subspacesof X that have a point in common is connected.

1.8 Proof.

Let $\{A_\alpha\}$ be a collection of connected subspaces of X that share a point p. Let U denote the union of the collection. Suppose, by way of contradiction, that $Y = C \cup D$ is a separation of Y. Without loss of generality, suppose that $p \in C$. Since A_α is connected, it must lie entirely in C. Hence $A_\alpha \subseteq C$ for all α, so $Y \subseteq C$, a contradiction of the assumption that D is nonempty.

1.9 Theorem.

Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is connected.

1.10 Proof.

Let A be connected and let $A \subseteq B \subseteq \overline{A}$. Suppose that $B = C \cup D$ is a separation of B. Assume, without loss of generality, that A lies entirely in C by Lemma 1.5. Then $\overline{A}$ lies entirely in $\overline{C}$, for $\overline{C}$ is disjoint from D by Lemma 1.3. So B does not intersect D, a contradiction that D is nonempty.

1.11 Remark.

That is, if A is a connected subspace, then adjoining some or all of its limit points yields another connected subspace.

1.12 Theorem.

The image of a connected space under a continuous map is connected.

1.13 Proof.

Let $f : X \to Y$ be a continuous map. Let X be connected. Let $Z = f(X)$ be the image of f. Let $g : X \to Z$ be the restriction of f to the codomain Z. It is surjective. Suppose, by way of contradiction, that $Z = A \cup B$ is a separation of Z. Then $g^{-1}(A) \cup g^{-1}(B)$ is a separation of X, a contradiction.

1.14 Theorem.

A finite product of connected spaces is connected.

1.15 Proof.

TODO!

1.16 Theorem.

A product of connected spaces is connected in the product topology.

1.17 Proof.

TODO!

1.18 Example.

$\mathbb{R}^\omega$ in the box topology is not connected, for A the set of bounded sequences and B the set of unbounded sequences form a separation of $\mathbb{R}^\omega$.

1.19 Definition.

A simply ordered set L with more than one element is a linear continuum if

(a) L has the least upper bound property, and

(b) if $x < y$, there exists $x < z < y$.

1.20 Theorem.

If L is a linear continuum in the order topology, then L is connected. Moreoever, intervals and rays are connected in L.

1.21 Proof.

TOOD!

1.22 Corollary.

The real line $\mathbb{R}$ is connected, and so are intervals and rays in $\mathbb{R}$.

1.23 Proof.

$\mathbb{R}$ is a linear continuum, so Theorem 1.20 applies.

1.24 Theorem.

(Intermediate Value Theorem) Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y between $f(a)$ and $f(b)$, then there exists a c in X such that $f(c) = r$.

1.25 Proof.

The sets $A = f(x) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$ are disjoint nonemty open sets in $f(X)$. If there exists no $f(c) = r$, then A and B would form a separation of $f(X)$, a contradiction of Theorem 1.12.

1.26 Example.

The ordered square is a linear continuum.

1.27 Example.

If X is a well-ordered set, then $X \times [0, 1)$ is a linear continuum.

1.28 Definition.

A path in X between x and y is a continuous map $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.

1.29 Definition.

A space X is saidt to be path connected if every pair of points x and y are connected by a path in X.

1.30 Theorem.

A path connected space is connected.

1.31 Proof.

Suppose, by way of contradiction, that $X = A \cup B$ is a separation of X. Let $f : I \to X$ be a path in X. $f(I)$ must be connected by Theorem 1.12, so it lies entirely in A or B, a contradiction of the path-connectedness of X.

1.32 Example.

The ordered square is connected but not path connected.

1.33 Example.

Let $S \subseteq \mathbb{R}^2$ be the set

$$S = \{x \times \sin(1/x) \mid 0 < x \leq 1\}.$$

S is connected as the image of a connected space under a continuous map by Theorem 1.12. So it's closure

$$\overline{S} = S \cup (0 \times I)$$

is also connected by Theorem 1.9. However, $\overline{S}$ is not path connected. Suppose, by way of contradiction, that there exists a path $f : I \to \overline{S}$ such that $f(0) = 0 \times 0$ and $f(1) \in S$. The preimage $f^{-1}(0 \times I)$ is closed, so it has a maximum element b. Let $f : [b, 1] \to \overline{S}$ be a path that maps b to $0 \times I$ and maps all other points of $[b, 1]$ to S. Replace $[b, 1]$ with $[0, 1]$ for convenience. Let $f(t) = (x(t), y(t))$. There exists a sequence of points $t_n \to 0$ such that $y(t_n) = (-1)^n$, contradicting the continuitiy of f (for the sequence does not converge).

Indeed, given n, choose $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then use the intermediate value theorem to choose $0 < t_n < 1/n$ such that $x(t_n) = u$.

1.34 Definition.

Let X be a topological space. Define the equivalence relation $\sim$ on X, where $x \sim y$ if there is a connected subspace of X that cotnains btoh x and y. The equivalence classes of $\sim$ are called the (connected) components of X.

1.35 Theorem.

The componenets of X are disjoint subspaces of X whose union is X such that each nonempty connected subspaoce of X intersects only one component of X.

1.36 Proof.

TODO!

1.37 Definition.

Let X be a topological space. Define teh equivalence relation $\sim_p$ on X where $x \sim_p y$ if there exists a path in X between x and y. The equivalence classes of $\sim_p$ are called the path components of X.

1.38 Theorem.

The path components of X are path-connected disjoint subspaces of X whose union is X such that each nonempty path-connected subspace of X intersects only one path component of X.

1.39 Proof.

TODO!

1.40 Theorem.

Let X be a topological space. The components of X are closed. If X has finitely many components, then the components are open as well.

1.41 Proof.

The components are closed for they are connected to their closures by Theorem 1.9. If X has finitely many components, then the components are open as well, for they are the complements of a finite union of closed sets.

1.42 Definition.

A space X is said to be locally connected at x if every enighborhood of U of x contains a connected neighborhood V of x. A space X is said to be locally connected if it is locally connected at each of its points.

1.43 Defintiion.

A space X is said to be locally path connected at x if every neighborhood of U of x contains a path connected neighborhood V of x. A space X is said to be locally path connected if it is locally path connected at each of its points.

1.44 Example.

The topologist's sine curve is connected but not locally connected. For a small enighborhood of $x \in 0 \times I$ contains infinitely many disjoint pieces of the sine curve, so it is not connected.

1.45 Theorem.

A space X is locally connected if and only if for every open set U of X, each component of Y is open in X.

1.46 Proof.

($\Rightarrow$) Suppose that X is locally connected. Let U be an open set of X. Let C be a component of U. If $x \in C$, then there exists a connected neighborhood V of x such that $V \subseteq U$. V is connected, so it lies entirely in C, so C is open in X.

($\Leftarrow$) Let U be an open set in X. Suppose that the components of U are open in X. Given $x \in X$ and a neighborhood U of x, let C be the component of U that contains x. C is connected, so it is open in X by hypothesis. So X is locally connected at x.

1.47 Theorem.

A space X is locally path connected if and only if for every open set U of X, each path component of Y is open in X.

1.48 Proof.

The proof is similar to that of Theorem 1.45.

1.49 Theorem.

If X is a topological space, each path comoponent of X lies in a component of X. If X is locally path connected, then the components and path comopnents are the same.

1.50 Theorem.

Let $x \in X$ and let C its component and P be its path component. $P \subseteq C$ because P is connected.

Suppose, by way of contradiction, that $P \neq C$. Let Q denote the union of all path components of X taht are different from P and intersect C. Each lies in C, so that $C = P \cup Q$. X is locally path connected, so P and Q are disjoint nonempty open sets in C, a contradiction of the connectedness of C.

1.51 Definition.

A collection $\mathcal{A}$ of subsets of X is said to be a covering of X if its union is equal to X. It is said to be an open covering if each of its elements is open in X.

1.52 Definition.

A space X is said to be compact if every open covering of X admits a finite subcovering of X.

1.53 Definition.

Let Y be a subspace of X. A collection $\mathcal{A}$ of subsets of X is said to be a covering of Y if its union contains Y.

1.54 Lemma.

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by open sets of X admits a fintie subcovering of Y.

1.55 Proof.

($\Rightarrow$) Let Y be compact and let $\mathcal{A} = \{A_\alpha\}$ for $\alpha \in J$ be a covering of Y by sets open in X. The collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y, hence a finite subcollection of $\mathcal{A}$ covers Y.

($\Leftarrow$) Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose an A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

Then $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by setes open in X. By hypothesis, some finite sobcollection $A_1, \ldots, A_n$ covers Y, so $A'_1, \ldots, A'_n$ covers Y.

1.56 Theorem.

A closed subspace of a compact space is compact.

1.57 Proof.

Let Y be a closed subspace of a compact space X. Let $\mathcal{A}$ be a covering of Y by open sets of X. Define

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\},$$

which is an open covering of X. Some finite subcollection of $\mathcal{B}$ covers X. If it includes $X \setminus Y$, discard $X \setminus Y$. The remaining collection is a finite subcollection of $\mathcal{A}$ that covers Y, so Y is compact by Lemma 1.54.

1.58 Theorem.

A compact subspace of a Hausdorff space is closed.

1.59 Proof.

Let Y be a compact subspace of a Hausdorff space X. Let $x_0 \in X \setminus Y$. There exists a neighborhood of x_0 that is disjoint from Y. Indeed, for each $y \in Y$, choose disjoint neighborhoods U_y and V_y of x_0 and y, respectively. The collection $\{V_y\}$ is an open covering of Y, so some finite collection $V_{y_1}, \ldots, V_{y_n}$ covers Y. The set

$$V = \bigcup_{i=1}^{n} V_{y_i}$$

is open and contains Y and is disjoint from

$$U = \bigcap_{i=1}^{n} U_{y_i}.$$

So U is a neighborhood of x_0 that is disjoint from Y, so $X \setminus Y$ is open, so Y is closed.

1.60 Lemma.

If Y is a compact subspace of a Hausdorff space X and if $x_0 \in X \setminus Y$, then there exist disjoint open sets U and V containing x_0 and Y, respectively.

1.61 Proof.

The proof is the same as that of Theorem 1.58.

1.62 Theorem.

The image of a compact space under a continuous map is compact.

1.63 Proof.

Let $f : X \to Y$ be continuous and let X be compact. Let $\mathcal{A}$ a covering of $f(X)$ by open sets of Y. The collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is an open covering of X, so there exists a finnite subcollection $f^{-1}(A_1), \ldots, f^{-1}(A_n)$ that covers X. Then the collection $\{A_1, \ldots, A_n\}$ is a finite subcollection of $\mathcal{A}$ that covers $f(X)$.

1.64 Theorem.

Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

1.65 Proof.

We show that f is a closed map. Let A be a closed subset of X. Then A is compact, so $f(A)$ is compact, by Theorem 1.62. But Y is Hausdorff, so $f(A)$ is closed, by Theorem 1.58.

1.66 Lemma.

(Tube Lemma) Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.

1.67 Proof.

Let X and Y be spaces, where Y is compact. Suppose that $x_0 \in X$ and N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$. Cover $x_0 \times Y$ by basis elements $U \times V$ lying in N. The space $x_0 \times Y$ is compact, being homeomorphic to Y, so there is a finite subcover of $x_0 \times Y$ of basis elements $U_1 \times V_1, \ldots, U_n \times V_n$. (Assume each basis element intersects $x_0 \times Y$.) Define

$$W = \bigcap_{i=1}^{n} U_i,$$

which is open and contains x_0.

$U_1 \times V_1, \ldots, U_n \times V_n$ cover $W \times Y$. Indeed, let $x \times y \in W \times Y$. Consider $x_0 \times y \in x_0 \times Y$. Now $x_0 \times y \in U_i \times V_i$, for some i, so that $y \in V_i$. But $x \in U_j$ for every j because $x \in W$, so $x \times y \in U_i \times V_i$, as desired.

$U_1 \times V_1, \ldots, U_n \times V_n$ lie in N, and they cover $W \times Y$, so $W \times Y$ lies in N.

1.68 Theorem.

A product of finitely many compact spaces is compact.

1.69 Proof.

Let X and Y be commpact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$, being homeomorphic to Y, so it is covered by a finite subcollection $A_1, \ldots, A_m$ of $\mathcal{A}$. The union

$$N = \bigcup_{i=1}^{m} A_i$$

is an open set containing $x_0 \times Y$. By the tube lemma, N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathcal{A}$.

For each $x \in X$, choose a neighborhood W_x such that $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection $\{W_x\}$ is an open covering of X, so there exists a finite subcollection $W_{x_1}, \ldots, W_{x_n}$ that covers X. The collection $W_1 \times Y, \ldots, W_n \times Y$ is a finite subcollection of $\mathcal{A}$ that covers $X \times Y$, so $X \times Y$ is compact. It follows by induction that a product of finitely many compact spaces is compact.

1.70 Remark.

The product of infinitely many compact spaces is compact by the Tychonoff theorem, though this result is difficult to prove.

1.71 Definition.

A collection $\mathcal{C}$ of subsets of X is said to have the fintie intesection property if the intersection of every finite subcollection of $\mathcal{C}$ is nonempty.

1.72 Theorem.

Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection of all the elements of $\mathcal{C}$ is nonempty.

1.73 Proof.

TODO!

1.74 Theorem.

Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

1.75 Proof.

TODO!

1.76 Corollary.

Every closed interval in $\mathbb{R}$ is compact.

1.77 Proof.

$\mathbb{R}$ is a simply ordered set having the least upper bound property, so the result follows from Theorem 1.74.

1.78 Theorem.

(Heine-Borel Theorem) A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.

1.79 Proof.

It suffices to only consider ρ, for

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

implies that A is bounded under d if and only if it is bounded under ρ.

($\Rightarrow$) Suppose that A is compact. Then A is closed by Theorem 1.58. Consider the collection of open sets

$$\{B_\rho(0, m) \mid m \in \mathbb{Z}_+\},$$

which union is all of $\mathbb{R}^n$. Some finite subcollection covers A, so $A \subseteq B_\rho(0, M)$ for some finite M. Then $\rho(x, y) \leq 2M$ for all $x, y \in A$, so A is bounded.

($\Leftarrow$) Suppose that A is closed and bounded under ρ. Suppose that $\rho(x, y) \leq N$ for all $x, y \in A$. Choose a point of $x_0 \in A$ and let $\rho(x_0, 0) = b$. Then $\rho(x, 0) \leq N + b$ for all $x \in A$. If $P = N + b$, then $A \subseteq [-P, P]^n$, which is compact by Theorem 1.68. A is also compact by Theorem 1.56.

1.80 Theorem.

(Extreme Value Theorem) Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in X$.

1.81 Proof.

The image $A = f(X)$ is compact by Theorem 1.62. A has a smalllest element m and a largest element M, so there exist $f(c) = m$ and $f(c) = M$.

Indeed, if A has no largest element, then teh collection

$$\{(-\infty, a) \mid a \in A\}$$

is an open covering of A. A is compact, so some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is the largest of the $a_1, \ldots, a_n$ then a_i belongs to none of these sets, which contradicts the assumption that the sets cover A. Hence, A has a largest element. A similar argument shows that A has a smallest element.

1.82 Definition.

Let (X, d) be a metric space. Let A be a nonempty subset of X. For each $x \in X$, define the distance from x to A by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

1.83 Theorem.

The diameter of a bounded subset A of a metric space (X, d) is

$$\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

1.84 Lemma.

(Lebesgue Number Lemma) Let $\mathcal{A}$ be an open covering of a metric space (X, d). If X is compact, then there exists a $\delta > 0$ such that every subset of X having diameter less than δ is contained in some element of $\mathcal{A}$. δ is called the Lebesgue number for the covering $\mathcal{A}$.

1.85 Proof.

Let $\mathcal{A}$ be an open covering of X. If $X \in \mathcal{A}$, then any $\delta > 0$ is a Lebesgue number, so assume that $X \notin \mathcal{A}$.

A finite subcollection $A_1, \ldots, A_n$ covers X. For each i, let $C_i = X \setminus A_i$ and define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

$f(x) > 0$ for all x. Indeed, given $x \in X$, choose i such taht $x \in A_i$. Then choose ε such that $B(x, \varepsilon) \subseteq A_i$. Then $d(x, C_i) \geq \varepsilon$, so $f(x) > \varepsilon/n$.

Since f is continuous and X is compact, f has a minumum value $\delta > 0$ by the extreme value theorem. δ is the Lebesgue number. Indeed, let B be a subset of X with diameter less than δ. Choose a point $x_0 \in B$. Then $B \subseteq B(x_0, \delta)$. Now

$$\delta \leq f(x_0 \leq d(x_0, C_m))$$

where $d(x_0, C_m)$ is the largest such $d(x_0, C_i)$. Then $B(x_0, \delta) \subseteq X - C_m = A_m$.

1.86 Theorem.

(Uniform Continuity Theorem) Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

1.87 Proof.

Given $\varepsilon > 0$, cover Y by open balls $B(y, \varepsilon/2)$ for $y \in Y$. Let $\mathcal{A}$ be the open covering of X by the preimages of the open balls in Y. Choose δ to be the a Lebesgue number for $\mathcal{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, then $f(x_1)$ and $f(x_2)$ are in the same open ball, the images $f(x_1)$ and $f(x_2)$ lie in some ball $B(y, \varepsilon/2)$. so $d_Y(f(x_1), f(x_2)) < \varepsilon$.

1.88 Definition.

A point $x \in X$ is said to be an isolated point of X if the one-pint set $\{x\}$ is open in X.

1.89 Theorem.

Let X be a nonempty compact Hasudorff space. If X has no isolated points, then X is uncountable.

1.90 Proof.

TODO!

1.91 Corollary.

Every closed interval in $\mathbb{R}$ is uncountable.

1.92 Proof.

The closed interval $[a, b]$ is a nonempty compact Hausdorff space with no isolated points, so it is uncountable by Theorem 1.89.

1.93 Definition.

A space X is said to be limit point compact if every finite subset of X has a limit point.

1.94 Theorem.

Compactness implies limit piotn compactness.

1.95 Proof.

Let X be a compact space. If A has no limit point, then A is finite.

Indeed, if A has no limit points, then A is closed. For each $a \in A$, choose a neighborhood U_a of a such that U_a intersects a alone. The space X is covered by the open set $X - A$ and the open sets $\{U_a\}$, so a finite subcollection of these sets covers X. So A must be finite for each $a \in A$ is contained only in U_a.

1.96 Example.

Let Y consist of two points and give Y the indiscrete topology. The space $X = Z_+ \times Y$ is limit point compact, for every nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ for $n \in Z_+$ has no finite subcovering.

1.97 Definition.

Let X be a topological space. If (x_n) is a sequecne of points of X and if $n_1 < n_2 < \ldots$ is a sequence of positive integers, then let $(y_i) = (x_{n_i})$ be a subsequence of (x_n). A space X is said to be sequentiall ycompact if every sequence of poitns of X has a convergent subsequence.

1.98 Theorem.

Let X be a metrizable space. Then the following are equivalent.

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

1.99 Proof.

TODO!

1.100 Definition.

A space X is said to be locally compact at x if there is some compact subspace C of X that contains a neighborhood of x. A space X is said to be locally compact if it is locally compact at each of its points.

1.101 Theorem.

A compact space is locally compact.

1.102 Proof.

The proof is trivial.

1.103 Example.

$\mathbb{R}^n$ is locally compact. $\mathbb{R}^\omega$ is not locally compact.

1.104 Example.

A simply ordered set X with the least upper bound is locally compact.

1.105 Theorem.

Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y that satisfies the following properties.

(a) X is a subspace of Y.

(b) The set $Y \setminus X$ consists of one point.

(3) Y is compact Hausdorff.

If Y and Y' are two such spaces, then there is a homeomorphism of Y with Y' that equals the indentity map on X.

1.106 Proof.

TODO!

1.107 Definition.

If Y is commpact Hausdorff and X is a proper subspaec of Y whose closure equals Y, then Y is said to bea compactification of X. If $Y \setminus X$ consists of one point, then Y is said to be a one-point compactification of X.

1.108 Theorem.

Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood V of x such taht $\overline{V}$ is compact and $\overline{V} \subseteq U$.

1.109 Proof.

($\Rightarrow$) $C = \overline{V}$ is a compact set that contains the neighborhood V of X, so X is locally compact.

($\Leftarrow$) Suppose that X is locally compact and let $x \in X$ have the neighborhood U. Take the one-point compactification Y of X and let $C = Y \setminus U$. C is closed in Y, so C is a compact subspace of Y. By Lemma 1.60, there exist disjoint open sets V and W that contain x and C, respectively. The closure $\overline{V}$ of V in Y is compact. Moreover, $\overline{V}$ is disjoint from C, so $\overline{V} \subseteq U$.

1.110 Corollary.

Let X be a locally compact Hausdorff space. Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

1.111 Proof.

Suppose that A is closed in X. Given $x \in A$, let C be a compact subspace of X that contains a neighborhood U of x. Then $C \cap A$ is closed in C and thus compact by Theorem 1.56. It contains a neighborhood $U \cap A$ of $x \in A$. (We need not assume that X is Hasudorff.)

Suppose that A is open in X. Given $x \in A$, apply Theorem 1.108 to choose a neighborhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq A$. Then $C = \overline{V}$ is a compact subspace of A that contains the neighborhood V of x in A.

1.112 Corollary.

A space X is homeomorphic to an open subspace of a compact Hasudroff space if and only if X is locally compact Hasudorff.

1.113 Proof.

Apply Theorem 1.105 and Corollary 1.112.

2.0 Section.

Countability and Separation Axioms.

2.1 Definition.

A space X is said to have a countable basis at x if there exists a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

2.2 Definition.

A space X is said to be first-countable if it has a countable basis at each of its points.

2.3 Lemma.

(Sequence lemma) Let X be a topological space. Let A be a subset of X. If there exists a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X if first-countable.

2.4 Proof.

($\Rightarrow$) Let (x_n) be a sequence of points of A converging to x. Then, every neighborhood U of x contains all but finitely many points of (x_n), so $U \cap A$ is nonempty, which means precisely that $x \in \overline{A}$.

($\Leftarrow$) Let X be first-countable. Let $B_1, B_2, \ldots$ be a countable basis at x. Define a new countable basis with $B_1' = B_1$ and $B_{n+1}' = B_{n+1} \cap B_n'$ for $n = 1, 2, \ldots$. For each n, choose a point $x_n \in B_n' \cap A$. Then, (x_n) is a sequence of points of A converging to x.

2.5 Theorem.

Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

2.6 Proof.

($\Rightarrow$) Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x that contains all but finitely many points of (x_n). Thus, all but finitely many points of $(f(x_n))$ are in V, so $f(x_n) \to f(x)$.

($\Leftarrow$) Let X be first-countable. Let $A \subseteq X$. We want to show $f(\overline{A}) \subseteq \overline{f(A)}$. By the sequence lemma, if $x \in \overline{A}$, then there exists a sequence (x_n) of points of A converging to x and hence a sequence of points $f(x_n)$ of $f(A)$ converging to $f(x)$. So $f(x) \in \overline{f(A)}$ by the sequence lemma, which means that $f(\overline{A}) \subseteq \overline{f(A)}$.

2.7 Definition.

A space X is said to be second-countable if it has a countable basis.

2.8 Lemma.

If X is second-countable, then any discrete subspace of X is countable.

2.9 Proof.

Let Y be a discrete subspace of X. For each $y \in Y$, choose a basic neighborhood B_y that contains only y. Then the map $y \mapsto B_y$ is injective, so Y is countable.

2.10 Example.

The uniform topology $\mathbb{R}^\omega$ is first-countable but not second-countable. $\mathbb{R}^\omega$ is metrizable, so it is first-countable. Let Y be the set of all sequences of 0s and 1s. Then Y is a discrete subspace, but Y is uncountable, so $\mathbb{R}^\omega$ is not second-countable in the uniform topology.

2.11 Theorem.

(a) A subspace of a first-countable space is first-countable.

(b) A countable product of first-countable spaces is first-countable.

(c) A subspace of a second-countable space is second-countable.

(d) A countable product of second-countable spaces is second-countable.

2.12 Proof.

The proof is obvious.

2.13 Definition.

A space X is said to satisfy the Lindlof property if every open cover of X has a countable subcover.

2.14 Theorem.

A second-countable space is Lindlof.

2.15 Proof.

Let $\mathcal{A}$ be an open cover of X. Let $\{B_n\}$ be a countable basis for X. For each n for which it is possible, choose $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. Then the collection $\{A_n\}$ is a countable subcover of $\mathcal{A}$. Indeed, for all $x \in X$, there exists an open $A \in \mathcal{A}$ that contains x, which in turn contains a basic neighborhood B_n of x. So A_n is defined and contains x.

2.16 Definition.

A space X is said to be separable if it has a countable dense subset.

2.17 Theorem.

A second-countable space is separable.

2.18 Proof.

Let $\{B_n\}$ be a countable basis for X. For each n, choose an $x_n \in B_n$. Then the set D of all such x_n is countable and dense.

2.19 Theorem.

Second-countability, Lindlofness, and separabilty are equivalent for a metrizable space.

2.20 Proof.

See Munkres Section 30 Exercise 5.

2.21 Definition.

A space X is said to be T_1 is for every pair of distinct points x and y, there exists a neighborhood U of x such that $y \notin U$.

2.22 Lemma.

A space X is T_1 if and only if every singleton set $\{x\}$ is closed.

2.23 Proof.

($\Rightarrow$) Suppose that X is T_1. $\{x\}$ is its own closure, for any other point y has a neighborhood U disjoint from $\{x\}$. So $\{x\}$ is closed.

($\Leftarrow$) Suppose that $\{x\}$ is closed. Let $y \in X$ be distinct from x. Then there exists a neighborhood U of y that does not contain x. So X is T_1.

2.24 Definition.

A space X is said to be T_2, or Hausdorff, if every pair of distinct points x and y have distinct neighborhoods U and V, respectively.

2.25 Definition.

A space X is said to be regular if for every point x and closed set A such that $x \notin A$, there exists disjoint neighborhoods U and V, respectively,

2.26 Definition.

A space X is said to be T_3 if it is T_1 and regular.

2.27 Definition.

A space X is said to be normal if for every pair of disjoint closed sets A and B, there exist disjoint neighborhoods U and V of A and B, respectively.

2.28 Definition.

A space X is said to be T_4 if it is T_1 and normal.

2.29 Theorem.

T_4 implies T_3 implies T_2 implies T_1.

2.30 Proof.

The proof is obvious.

2.31 Remark.

For convenience, regular shall refer to T_3 and normal shall refer to T_4.

2.32 Lemma.

Let X be a T_1 space. X is regular if and only if for every $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$.

2.33 Proof.

($\Rightarrow$) Let $x \in X$ and U be a neighborhood of x. Let $B = X - U$. Then A is closed, so there exists a neighborhood V of x that is disjoint from a neighborhood W of B. Moreover, $\bar{V}$ is disjoint from B, and $\bar{V} \subseteq U$, as desired.

($\Leftarrow$) Let $x \in X$ and $B \subseteq X$ be a closed subset disjoint from x. Let $U = X - B$. By hypothesis, there exists a neighborhood V of x such that $\bar{V} \subseteq U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively.

2.34 Lemma.

Let X be a T_1 space. X is normal if and only if for every closed subset A of X and every neighborhood U of A, there exists a neighborhood V of A such that $\overline{V} \subseteq U$.

2.35 Proof.

The proof is exactly the same as for the previous theorem.

2.36 Theorem.

(a) A subspace of a Hausdorff space is Hausdorff.

(b) A product of Hausdorff spaces is Hausdorff.

(c) A subspace of a regular space is regular.

(d) a product of regular spaces is regular.

2.37 Proof.

(a) Let X be Hausdorf and let Y be a subspace of X. Let x and y be distinct points of Y. There exist disjoint neighborhoods U and V of x and y, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y, respectively, in Y.

(b) Let $\{X_\alpha\}$ be a collection of Hausdorff spaces and let X be the product space. Let x and y be distinct points of X. Then there exists at least one β such that $x_\beta \neq y_\beta$. Let U_β and V_β be disjoint neighborhoods of x_β and y_β, respectively, in X_β. The sets $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are disjoint neighborhoods of x and y, respectively, in X.

(c) Let X be regular and let Y be a subspace of X. Let $x \in Y$ and let B be a closed subset of Y that does not contain x. Let $\overline{B}$ be the closure of B in X. Then $\overline{B} \cap Y = B$, so $x \notin \overline{B}$. By the regularity of X, the exist disjoint neighborhoods U and V of x and $\overline{B}$, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and B, respectively, in Y.

(d) Let $\{X_\alpha\}$ be a collection of regular spaces and let X be the product space. It follows immediately that X is T_1 by Theorem 2.36a. Let $x \in X$ have the neighborhood U equals the product of U_α. For each U_α, let V_α be a neighborhood of x_α such that $\overline{V_\alpha} \subseteq U_\alpha$ by Lemma 2.32. If $U_\alpha = X_\alpha$, then let $V_\alpha = X_\alpha$. Let V be the product of V_α. Then $\overline{V} \subseteq U$, so X is regular.

2.38 Example.

$\mathbb{R}_K$ is Hausdorff but not regular.

2.39 Example.

$\mathbb{R}_l$ is normal.

2.40 Example.

$\mathbb{R}_l \times \mathbb{R}_l$ is regular but not normal.

2.41 Theorem.

A regular space with a countable basis is normal.

2.42 Proof.

Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. For each $x \in A$, there exists a neighborhood U of x and a neighborhood V of x such that $\overline{V} \subseteq U$. For each such V, choose a basic neighborhood U_x of x contained in V. This is a countable covering of A by sets whose closures do not intersect B. Index these sets as $\{U_n\}$. Similarly, cover B with a countable collection of neighborhoods $\{V_n\}$.

Define

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i}$$

and

$$V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}.$$

Each such set is open because it is the difference between an open set and a closed set. Furthermore, the collection $\{U_n'\}$ is an open cover of A and the collection $\{V_n'\}$ is an open cover of B. The open sets

$$U' = \bigcup_{n=1}^{\infty} U_n'$$

and

$$V' = \bigcup_{n=1}^{\infty} V_n'$$

are disjoint open sets that contain A and B. Indeed, if $x \in U' \cap V'$, then $x \in U_i' \cap V_j'$ for some i and j. Without loss of generality, assume that $i \leq j$. Then $x \in U_i$, but $x \notin \overline{U_i}$, hence a contradiction.

2.43 Theorem.

A metrizable space is normal.

2.44 Proof.

Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B(a, \varepsilon_a)$ is disjoint from B, and similarly, choose ε_b for each $b \in B$. The open sets

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2)$$

and

$$V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$$

are disjoint and contain A and B, respectively.

2.45 Theorem.

A compact Hausdorff space is normal.

2.46 Proof.

Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. For each $x \in A$, choose disjoint neighborhoods U_x and V_x of x and B, respectively. The collection $\{U_x\}$ is an open cover of A, so there exists a finite subcover $U_1, \ldots, U_n$. It follows that the sets

$$U = \bigcup_{i=1}^{n} U_i$$

and

$$V = \bigcap_{i=1}^{n} V_i$$

are disjoint neighborhoods of A and B, respectively.

2.47 Theorem.

Every well-ordered set is normal in the order topology.

2.48 Proof.

First observe that any interval $(x, y]$ is open, for if y is not the maximum element, then $(x, y] = (x, y')$ where y' is the immediate successor of y. Let A and B be disjoint closed sets in X.

Assume that neither A nor B contains the minimum element a_0 of X. For each $a \in A$, choose a neighborhood $(x_a, a]$ disjoint from B. For each $b \in B$, choose a neighborhood $(y_b, b]$ disjoint from A. The sets

$$U = \bigcup_{a \in A} (x_a, a]$$

and

$$V = \bigcup_{b \in B} (y_b, b]$$

are disjoint open sets containing A and B, respectively. Indeed, if $z \in U \cap V$, then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. Without loss of generality, assume that $a < b$. If $a \leq y_b$, then the intervals are disjoint. If $a > y_b$, then $(y_b, b]$ is not disjoint from A, a contradiction.

If A contains a_0, then the set $A - \{a_0\}$ is closed and disjoint from B, so it admits disjoint open intervals U and V of $A - \{a_0\}$ and B, respectively. Now the sets $U \cup \{a_0\}$ and V are disjoint neighborhoods of A and B, respectively.

2.49 Remark.

Indeed, every ordered set is normal in the order topology.

2.50 Example.

If J is uncountable, then the product space $\mathbb{R}^J$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $\mathbb{R}^J$ is homeomorphic to a subspace of $[0, 1]^J$).

2.51 Example.

The product space $S_\Omega \times \overline{S_\Omega}$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $overline S_\Omega \times \overline{S_\Omega}$ compact Hasudorff and therefore normal).

2.52 Lemma.

(Urysohn Lemma) Let X be a normal space. Let A and B be disjoint closed subsets of X. THen there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

2.53 Proof.

TODO!

2.54 Definition.

If A and B are two subsets of X, and if there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$, then we say that A and B are separated by a continuous function.

2.55 Remark.

The Urysohn lemma says that if every pair A and B of disjoint closed subsets can be separated by open sets, then they can be separated by a continuous function. The converse is trivial.

2.56 Definition.

A space X is said to be completely regular if it is T_1 and if for every $x_0 \in X$ and every closed set A that does not contain x_0, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

2.57 Remark.

Complete regularity is also known as $T_{3+1/2}$.

2.58 Theorem.

(a) A subspace of a completely regular space is completely regular.

(b) A product of completely regular spaces is completely regular.

2.59 Proof.

(a) Let X be completely regular and let Y be a subspace of X. Let $x_0 \in Y$ and let A be a closed subset of Y that does not contain x_0. Then $A = \overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X, and moreoever, $x_0 \notin \overline{A}$. There exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in \overline{A}$. Then $f \mid Y$ is a continuous function from Y to $[0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

(b) Let X be the product of completely regular spaces $\{X_\alpha\}$. Let $b \in X$ and let A be a closed set of X that does not contain b. Choose a basic neighborhood U of b, and denote $U_1, \ldots, U_n$ the basic neighborhoods of b_α in X_α that are not all of X_α. For each U_i, choose a continuous function $f_i : X_\alpha \to [0, 1]$ such that $f_i(b_\alpha) = 0$ and $f_i(x) = 1$ for all $x \in A$. The function $f : X \to [0, 1]$ defined by

$$f(x) = \prod_{i=1}^{n} f_i(x_\alpha)$$

is continuous and satisfies the desired properties.

2.60 Theorem.

(Urysohn Metrization Theorem) A regular space with a countable basis is metrizable.

2.61 Proof.

TODO!

2.62 Theorem.

(Embedding Theorem) Let X be a a T_1 space. Suppose that $\{f_\alpha\}$ is a family of continuous functions indexed for $\alpha \in J$. If $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ for all $x \notin U$. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = \prod_{\alpha \in J} f_\alpha(x)$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$, then F is an imbedding of X in $[0,1]^J$.

2.63 Proof.

TODO!

2.64 Theorem.

A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.

2.65 Theorem.

(Tietze Extension Theorem) Let X be a normal space. Let A be a closed subspace of X. Any continuous map of A into $[0,1]$ or A into $\mathbb{R}$ can be extended to a continuous map of X into $[0,1]$ or $\mathbb{R}$, respectively.

2.66 Proof.

TODO!

3.0 Section.

Fundamental Group.

3.1 Definition.

If f and f' are continuous maps of X into Y, we say that f is homotopic to f' if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$. We denote $f \simeq f'$ and call F a homotopy between f and f'.

3.2 Definition.

If $f \simeq f'$ and f' is a constant map, then we say that f is nullhomotopic.

3.3 Definition.

If f and f' are paths in X with initial point x_0 and final point x_1, then we say that f is path homotopic to f' if there exists a continuous map $F : I \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in I$ and $F(0, t) = x_0$ and $F(1, t) = x_1$ for all $t \in I$. We denote $f \simeq_p f'$ and call F a path homotopy between f and f'.

3.4 Lemma

The relations $\simeq$ and $\simeq_p$ are equivalence relations.

3.5 Proof.

The proof is obvious.

3.6 Definition.

Denote $[f]$ the equivalence class of f under the relation $\simeq_p$. $[f]$ is called the path homotopy class of f.

3.7 Example.

If A is a convex subspace of $\mathbb{R}^n$, then any two paths in A with the same endpoints are homotopic. For the straihgt-line homotopy

$$F(x, t) = (1 - t)f(x) + tf'(x)$$

is a homotopy between f and f'.

3.8 Definition.

If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, then the product $h = f * g$ is defined as

$$h(s) = \begin{cases} f(2s) & \text{if } s \in [0, 1/2] \\ g(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases}$$

3.9 Definition.

The product between path-homotopy classes is defined as $[f] * [g] = [f * g]$. Indeed, let F be a path homotopy between f and f' and let G be a path homotpy between g and g'. Define

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } s \in [0, 1/2] \\ G(2s - 1, t) & \text{if } s \in [1/2, 1]. \end{cases}$$

3.10 Lemma.

If $k : X \to Y$ is a continuous map and if F is a path homotopy in X between f and f', then $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ f'$.

3.11 Proof.

The proof is obvious.

3.12 Lemma.

If $k : X \to Y$ is a continuous map and if f and g are paths in X such that $f(1) = g(0)$, then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

3.13 Proof.

The proof is obvious.

3.14 Theorem.

The operation $*$ satisfies the following properties.

(a) (Associativity) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$ whenever the products are defined.

(b) (Identity) Let e_x denote the constant path at point x. If f is a path from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(c) (Inverse) If f is a path from x_0 to x_1, then let its reverse $\overline{f}$ be defined as $\overline{f}(s) = f(1-s)$ for $s \in I$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

3.15 Proof.

(a) Let $[a, b]$ and $[c, d]$ be two intervals in I. There exists an unique continuous map $p : [a, b] \to [c, d]$ of the form $p(x) = mx + k$ called the positive linear map. The inverse of a positive linear map is a positive linear map, and the composition of two positive linear maps is a positive linear map.

When the triple product $f * g * h$ is defined, it is the path k_{ab} in X where on $[0, a]$, it is the positive linear map of $[0, a]$ to $[0, 1]$ followed by f and similarly for $[a, b]$ and $[b, 1]$. The path homotopy class of k_{ab} is independent of the choice of a and b. Indeed, $[f] * ([g] * [h])$ is the path homotopy class of k_{ab} where $a = 1/2$ and $b = 3/4$ whille $([f] * [g]) * [h]$ is the path homotopy class of k_{ab} where $a = 1/4$ and $b = 1/2$. These are equivalent.

(b) Let e_0 denote the constant path in I at 0 and let i denote the identity path in I. Then $e_0 * i$ is a path in I from 0 to 1. I is convex, so there is a path homotopy G between i and $e_0 * i$, so $f \circ G$ is a path homotopy between $f \circ i = f$ and

$$f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f.$$

The proof for right identity is entirely similar.

(c) Let i denote the identity path in I and $\bar{i}$ denote its reverse. I is convex, so there is a path homotopy H between e_0 and $i * \bar{i}$. Then $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and

$$f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \overline{f}.$$

The proof for left inverse is entirely similar.

3.16 Theorem.

Let f be a path in X. Let $a_0, \ldots, a_n$ such that that $0 < a_0 < \ldots < a_n < 1$. Let f_i be the path in X defined as the positive linear map of I to $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * \ldots * [f_n].$$

3.17 Proof.

The proof is sketched in Proof 3.15a.

3.18 Definition.

Let G and G' be two groups with the operation $\cdot$. A homomorphism $f : G \to G'$ is such that

$$f(x \cdot y) = f(x) \cdot f(y)$$

for all $x, y \in G$. f satisfies $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.

3.19 Definition.

The kernel of f is the set $f^{-1}(e')$. It is a subgroup of G.

3.20 Definition.

The image of f is the set $f(G)$. It is a subgroup of G'.

3.21 Definition.

A homomorphism is called a monomorphism if it is injective (or equivalently of $f^{-1}(e') = e$). It is called an epimorphism if it is surjective. It is called an isomorphism if it is bijective.

3.22 Definition.

Let H be a subgroup of G. Let xH denote the set of products xh for all $h \in H$. It is called the left coset of H in G, and the collection of all such xH for $x \in G$ is a partition of G. Similarly, let Hx denote the right coset of H in G.

3.23 Definition.

H is said to be a normal subgroup of G if $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. In this case, $xH = Hx$ for all $x \in G$. The partition G/H is called the quotient group of G by H with the operation $(xH)(yH) = xyH$.

3.24 Defintion.

The map $f : G \to G/H$ defined by $f(x) = xH$ is epimorphism with kernel H. Conversely, if $f : G \to G'$ is an epimorphism and N is a normal subgroup of G, then f induces an isomorphism $g : G/N \to G'$ defined by $g(xN) = f(x)$.

3.25 Definition.

If H is not normal, then G/H denotes the collection of right cosets of H in G.

3.26 Definition.

Let X be a topological space and let $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ relative to the base point x_0 is the group of path homotopy classes of loops in X based at x_0 with the operation $*$.

3.27 Example.

Any convex subspace of $\mathbb{R}^n$ has a trivial fundamental group.

3.28 Definition.

Let α be a path in X from x_0 to x_1. Define the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha].$$

If f is a loop at x_0, then $\overline{\alpha} * f * \alpha$ is a loop at x_1.

3.29 Theorem.

The map $\hat{\alpha}$ is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

3.30 Proof.

We compute that

$$\begin{aligned} \hat{\alpha}([f]) * \hat{\alpha}([g]) &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= [\overline{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f] * [g]), \end{aligned}$$

so $\hat{\alpha}$ is a homomorphism. We show that $\hat{\alpha}$ has a left inverse and a right inverse. Let $\beta = \overline{\alpha}$. Then

$$\begin{aligned} \hat{\beta}(\hat{\alpha}([f])) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [e_{x_0}] * [f] * [e_{x_1}] \\ &= [f] \end{aligned}$$

for all $[f] \in \pi_1(X, x_0)$. A similar computation shows that $\hat{\beta}$ is also a right inverse of $\hat{\alpha}$. So $\hat{\alpha}$ is an isomorphism.

3.31 Corollary.

If X is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

3.32 Proof.

The proof is trivial.

3.33 Remark.

We must still specify the base point x_0 in the definition of the fundamental group, for the isomorphism $\hat{\alpha}$ depends on the choice of α.

3.34 Definition.

A space X is said to be simply connected if it is path connected and $\pi_1(X, x_0)$ is trivial for one (and hence all) $x_0 \in X$.

3.35 Lemma.

If X is simply connected, then any two paths in X with the same endpoints are path homotopic.

3.36 Proof.

Let α and β be two paths in X from x_0 to x_1. We compute that

$$\begin{aligned} [\alpha] &= [\alpha] * [\overline{\beta}] * [\beta] \\ &= [\alpha * \overline{\beta}] * [\beta] \\ &= [e_{x_0}] * [\beta] \\ &= [\beta]. \end{aligned}$$

3.37 Definition.

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. The homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by h, relative to base point x_0, is defined as

$$h_*([f]) = [h \circ f].$$

h_* is a homomorphism because

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

h_* depends on the choice of x_0, so we denote $(h_{x_0})_*$ when necessary.

3.38 Theorem.

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism of $\pi_1(X, x_0)$.

3.39 Proof.

We compute that

$$\begin{aligned} i_*([f]) &= [i \circ f] \\ &= [f]. \end{aligned}$$

3.40 Theorem.

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps, then

$$(k \circ h)_* = k_* \circ h_*.$$

3.41 Proof.

We compute that

$$\begin{aligned} (k \circ h)_*([f]) &= [(k \circ h) \circ f] \\ &= [k \circ (h \circ f)] \\ &= k_*([h \circ f]) \\ &= (k_* \circ h_*)([f]). \end{aligned}$$

3.42 Corollary.

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

3.43 Proof.

Let k be the inverse of h. Then $(k \circ h)_* = i_*$ is the identity homomorphism of $\pi_1(X, x_0)$, and $(h \circ k)_* = j_*$ is the identity homomorphism of $\pi_1(Y, y_0)$. So h_* is an isomorphism.

3.44 Definition.

Let $p : E \to B$ be a continuous surjective map. An open set U of B is said to be evenly covered by p if $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ such that $p \mid V_\alpha$ is a homeomorphism onto U for each α. The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices.

3.45 Definition.

Let $p : E \to B$ be a continuous surjective map. If each point $b \in B$ has a neighborhood U that is evenly covered by p, then p is said to be a covering map and E is said to be a covering space of B.

3.46 Definition.

A map $p : E \to B$ is a local homeomorphism if for every point $e \in E$ has a neighborhood U that is homeomorphic to a neighborhood of $p(e)$ in B. A covering map is a local homeomorphism, but the converse does not hold necessarily.

3.47 Example.

The map $p : \mathbb{R}_+ \to \S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is surjective and a local homomorphism, but it is not a coverign map. For the point $b_0 = (1, 0)$ has no neighborhood U that is evenly covered by p. This example also shows that the restriction of a covering map need not be a covering map.

3.48 Theorem.

The map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

3.49 Proof.

Let U be the subset of S^1 consisting of points with positive first coordinate, i.e., the open right semicircle. The preimage $p^{-1}(U)$ is the union of the open intervals $V_n = (n - 1/4, n + 1/4)$ for $n \in \mathbb{Z}$. The restriction $p \mid V_n$ is injective because $\sin \mid V_n$ is strictly monotonic, and it is surjective by the intermediate value theorem. $p \mid V_n$ is a continuous bijective map between a compact space and a Hausdorff space, so it is a homeomorphism of V_n and U, so U is evenly covered by p. A similar argument can be made for the left, upper, and lower semicircles. Thesse semicircles form an open cover of S^1, and each one is evenly covered by p, so p is a covering map.

3.50 Theorem.

Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ is a covering map, where p_0 is the restriction of p to E_0.

3.51 Proof.

Let $b_0 \in B_0$. Let U be a neighborhood of b_0 in B that is evenly covered by p. Let $\{V_\alpha\}$ be a partion of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $\{V_\alpha \cap E_0\}$ are a partition of $p_0^{-1}(U \cap B_0)$ into slices.

3.52 Theorem.

If $p : E \to B$ and $p' L E' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.

3.53 Proof.

Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b', respectively, that are evenly covered by p and p', respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$ into slices. Then the preimage of $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets in $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. So $p \times p'$ is a covering map.

3.54 Example.

Let $T = S^1 \times S^1$ be the torus. The map $p \times p : R^2 \to T$ is a covering map, where p is the covering map $p : \mathbb{R} to S^1$ of Theorem 3.48. Intuitively, each unit square of $\mathbb{R}^2$ is mapped homeomorphically onto a unit disk of T.

3.55 Example.

Let $p \times p : \mathbb{R}^2 \to T$ be the covering map of the previous example. Let $b_0 = p(0)$, and let B_0 denote the figure-eight

$$B_0 = (S^1 \times b_0) \cup (b_0 \times S^1).$$

The preimage of B_0 is

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}),$$

and the map $P_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is a covering map.

3.56 Example.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. Let $p \times i : R \times R_+ \to S^1 \times R_+$ be a covering map, where i is the identity map of R_+. Let $x \times t \mapsto tx$ be a homeomorphism of $S^1 \times R_+$ with $R^2 \setminus \{0\}$. The composition yields a covering map $\mathbb{R} \times R_+ \to R^2 \setminus \{0\}$.

3.57 Definition.

Let $p : E \to B$ be a map. If $f : X \to B$ is continuous, a lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

3.58 Example.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. The path $f : I \to S^1$ defined by $f(t) = (\cos 2\pi f t, \sin 2\pi f t)$ lifts to the path $\tilde{f} : I \to \mathbb{R}$ defined by $\tilde{f}(t) = ft$.

3.59 Lemma.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Any path $f : I \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f} : I \to E$ beginning at e_0.

3.60 Proof.

Cover B by open sets U each of which is evenly covered by p. Find a subdivision of I, say $s_0, \ldots, s_n$ such that every $f([s_i, s_{i+1}])$ lies in a U. (Use the Lebesgue number lemma.)

Define $\tilde{f}(0) = e_0$. Suppose that $\tilde{f}(s)$ is defined for $0 \le s \le s_i$. Define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows. The set $f([s_i, s_{i+1}])$ is contained in a U that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Now $\tilde{f}(s_i)$ lies in one of these sets, say V_0. Define $\tilde{f}(s)$ for $s_i \le s \le s_{i+1}$ by

$$\tilde{f}(s) = (p \mid V_0)^{-1}(f(s)).$$

$p \mid V_0 : V_0 \to U$ is a homeomorphism, so $tildef$ is continuous on $[s_i, s_{i+1}]$. Hence, $\tilde{f}$ may be defined on I by induction. It is continuous by the pasting lemma, and $p \circ \tilde{f} = f$.

Suppose that $\tilde{f}'$ is another lifting of f beginning at e_0. Then $\tilde{f}'(0) = e_0 = \tilde{f}(0)$. Suppose that $\tilde{f}'(s) = \tilde{f}(s)$ for all $0 \le s \le s_i$. By construction, $\tilde{f}'(s_i)$ lies in the same slice V_0 as $\tilde{f}(s_i)$. The map $p \mid V_0$ is a homeomorphism, so $\tilde{f}'(s_i) = \tilde{f}(s_i)$. By induction, $\tilde{f}'(s) = \tilde{f}(s)$ for all $s \in I$, so the lifting is unique.

3.61 Lemma.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Let $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

3.62 Proof.

Given F, define $\tilde{F}(0,0) = e_0$. Use Lemma 3.59 to define $\tilde{F}$ for $\{0\} \times I$ and for $I \times \{0\}$. Choose subdivisions $s_0 < \ldots, < s_m$ and $t_0 < \ldots < t_n$ small enough such that each rectangle $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is contained in an open set U of B that is evenly covered by p. (Use the Lebesgue number lemma.)

Given i_0 and j_0, assume that $\tilde{F}$ is defined on A, the union of $\{0\} \times I$ and $I \times \{0\}$ and all the previous rectangles. Assume that $\tilde{F}$ is a continuous lifting of $F \mid A$. Choose an open set U of B that is evnely covered by p and contains $F(I_{i_0} \times J_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Now $\tilde{F}$ is already defined on $C = A \cap (I_{i_0} \times J_{j_0})$, which is connected, so $\tilde{F}(C)$ is connected, so it must lie entirely in, say, V_0. Let $p_0 : V_0 \to U$ denote the restriction of p to V_0. Since $\tilde{F}$ is a lifting of $F \mid A$, for $x \in C$, $p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x)$, so that $\tilde{F}(x) = p_0^{-1}(F(x))$. Define

$$\tilde{F}(x) = p_0^1(F(x))$$

for $x \in I_{i_0} \times J_{j_0}$. The lifting $\tilde{F}$ may be defined on $I \times I$ by induction. It is continuous by the pasting lemma, and $p \circ \tilde{F} = F$. It is unique for the same reason as for Lemma 3.59.

Suppose that F is a path homotopy. The map F carries $\{0\} \times I$ to b_0. Now $\tilde{F}$ is a lifting of F, so it carries this set to $p^{-1}(b_0)$. But $p^{-1}(b_0)$ has the discrete topology as a subspace of E. Since $\{0\} \times I$ is connected, $\tilde{F}(\{0\} \times I)$ is connected, so it must be a single point. So $\tilde{F}$ is a path homotopy.

3.63 Theorem.

Let $p : E \to B$ be a covering map. Let $p_{e_0} = b_0$. Let f and g be two paths in B from b_0 to b_1 and let $\tilde{f}$ and $\tilde{g}$ denote their respective liftings to paths in E, beginiing at e_0. If f and g are path homotpic, then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic.

3.64 Proof.

Let $F : I \times I \to B$ be the path homotoy between f and g. Then $F(0,0) = b_0$. Let $\tilde{F} : I \times I \to E$ be the lifting of F to E such that $\tilde{F}(0,0) = e_0$. By Lemma 3.61, $\tilde{F}$ is a path homotopy, so $\tilde{F}(\{0\} \times I) = \{e_0\}$ and $\tilde{F}(\{1\} \times I) = \{e_1\}$.

The restriction $\tilde{F} \mid (I \times \{0\})$ is a path on E from e_0 that is the lifting of $F \mid (I \times \{0\})$. By the uniqueness of path liftings, $\tilde{F}(s,0) = \tilde{f}(s)$. Similarly, $\tilde{F}(s,1) = \tilde{g}(s)$. So $\tilde{f}$ and $\tilde{g}$ both end at e_1, and $\tilde{F}$ is a path homotopy between them.

3.65 Definition.

Let $p : E \to B$ be a covering map. Let $b_0 \in B$. Choose e_0 such that $p(e_0) = b_0$. Given an element $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a well-defined map. We call ϕ the lifting correspondence derived from the covering map p. It depends on the choice of e_0.

3.66 Theorem.

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. If E is simply connected, then ϕ is bijective.

3.67 Proof.

If E is path connected, then, given $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$. So ϕ is surjective.

Suppose that E is simply connected. Let $[f]$ and $[g]$be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively, to path sin E that begin at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there is a path homotpy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. So $p \circ \tilde{F}$ is a path homopty in B between f and g. So $[f] = [g]$, and ϕ is injective.

3.68 Theorem.

The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$.

3.69 Proof.

Let $p : \mathbb{R} \to S^1$ be the covering map of Theorem 3.48. Let $e_0 = 0$ and let $b_0 = p(e_0)$. Then $p^{-1}(b_0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, the lifting correspondence $\phi : \pi_1(S^1, b_0) \to p^{-1}(b_0)$ is bijective. Indeed, ϕ is a homomorphism.

Given $[f]$ and $[g]$ in $\pi_1(S^1, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g, respectively. Let $m = \tilde{f}(1)$ and $n = \tilde{g}(1)$. So $\phi([f]) = m$ and $\phi([g]) = n$. Because $p(m + x) = p(x)$ for all $x \in \mathbb{R}$,

$$\tilde{g}'(s) = m + \tilde{g}(s)$$

is the lifting of g to $\mathbb{R}$ that begins at m. Then the product $\tilde{f} * \tilde{g}'$ is well-defined, and it is the lifting of $f * g$ to $\mathbb{R}$ that begins at 0. The end point is $\tilde{g}' = m + n$. So $\phi([f] * [g]) = m + n$. so ϕ is a homomorphism and hence an isomorphism.

3.70 Definition.

Let G be a group and let $x \in G$ have the inverse x^{-1}. If the set of all elements x^m for $m \in \mathbb{Z}$ equals G, then G is said to be a cyclic group with generator x. $|G|$ is said to be the order of G. G has infinite order if and only if it is isomorphic to the additive group of integers. G has order k if and only if it is isomorphic to the group $\mathbb{Z}/k$ of integers modulo k.

3.71 Remark.

If x is a generator of an inifite cyclic group G, and if y is an element of an arbitrary group H, then there exists a unique homomorphism $h : G \to H$ such that $h(x) = y$ defined by $h(x^n) = y^n$ for all $n \in \mathbb{Z}$.

3.72 Theorem.

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$.

(a) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism

(b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

3.73 Proof.

(a) Suppose that $\tilde{h}$ is a loop in E baesd at e_0 and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p \circ \tilde{h}$ and the constant loop. If $\tilde{F}$ is a lifting of F to E such that $\tilde{F}(0, 0) = e_0$, then $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(b) Given loops f and g in B, let $\tilde{f}$ and $\tilde{g}$ be the liftings of them to E that begin at e_0. Then $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. We show that $\phi([g]) = \phi([g])$ if and only if $[f] \in H * [g]$.

Suppose that $[g] \in H * [g]$. Then $[g] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Now the product $\tilde{h} * \tilde{g}$ is defined, and it is a lifting of $h * g$. Because $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point of E. So $\phi([f]) = \phi([g])$.

Now suppose that $\phi([f]) = \phi([g])$. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E. The product of $\tilde{f}$ and the reverse of $\tilde{g}$ is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy in

E between the loops $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy in B between $h * g$ and f, where $h = p \circ \tilde{h}$. Thus, $[f] \in H * [g]$ as desired.

If E is path connected, then ϕ is surjective, so Φ is surjective as well, and hence bijective.

(c) Injectivity of Φ implies that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$. Now $\phi[f] = e_0$ if and only if $[f] \in H * [e_0] = H$. But $\phi([f]) = e_0$ if and only if f lifts to a loop in E based at e_0.

3.74 Definition.

If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r \mid A$ is the identity map of A. If such an r exists, then A is said to be a retract of X.

3.75 Lemma.

If A is a retract of X, then the homomorphism of fundamental groups induced by the inclusion map $j : A \to X$ is injective.

3.76 Proof.

If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. It follows that $(rcircj)_* = r_* \circ j_*$ is the identity map of $\pi_1(A, a)$ by Theorem 3.38 and Theorem 3.40, so that j_* is injective.

3.77 Theorem.

(No-Retraction Theorem) There is no retraction of B^2 onto S^1.

3.78 Proof.

If S^1 were a retract of B^2, then the homomorphism induced by $j : S^1 \to B^2$ would be injective by Lemma 3.75. But the fundamental group of S^1 is nontrivial while the fundamental group of B^2 is trivial, a contradiction.

3.79 Lemma.

Let $h : S^1 \to X$ be a continuous map. Then teh following conditions are equivalent.

(a) h is nulhomotopic.

(b) h extends to a continuous map $k : B^2 \to X$.

(3) h_* is the trivial homomorphism of fundamental groups.

3.80 Proof.

TODO!

3.81 Corollary.

The inclusion map $j : S^1 \to R^2 \setminus \{0\}$ is not nulhomotopic. The identity map $i : S^1 \to S^1$ is not nulhomotopic.

3.82 Proof.

There is a retraction of $\mathbb{R}^2 \setminus \{0\}$ onto S^1. Therefore, j_* is injective and hence nontrivial. Similarly, i_* is the indentity homomorphism and is hence nontrivial.

3.83 Theorem.

Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector fiend points directly inward and a point of S^1 where the vector field points directly outward.

3.84 Proof.

TODO!

3.85 Theorem.

(Brouwer Fixed Point Theorem) If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such taht $f(x) = x$.

3.86 Proof.

Suppose, by way of contradiction, that $f(x) \neq x$ for all $x \in B^2$. Then define $v(x) = f(x) - x$, which yields a nonvanishing vector field $(x, v(x))$ on B^2. But the vector field cannot point directly outward at any point x of X^1, for otherwise $f(x) - x = ax$ for some $a > 0$, so $f(x) = (1 + a)x$ would lie outside the unit ball B^2, hence a contradiction.

3.87 Theorem.

(Seifert-van Kampen Theorem) Suppose that $X = U \cup V$ where U and V are open sets of X. Suppose that $U \cap V$is path connected and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respecitvely, into X. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$$

and

$$j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

3.88 Proof.

TODO!

3.89 Remark.

The Seifert-van Kampen theorem says that any loop f in X based at x_0 is homotopic to a product of the form $g_1 * (g_2 * (\ldots * g_n))$ where each g_i is a loop in X based at x_0 that lies entirely in either U or V.

3.90 Corollary.

Suppose that $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.

3.91 Proof.

The proof is obvious.

3.92 Theorem.

If $n = 2, 3, \ldots$, then S^n is simply connected.

3.93 Proof.

TODO!