AS.110.413 Introduction to Topology

Daniel Yao

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Contents

0 Introduction. 4
0.0 Section. 4
1 Countability and Separation Axioms. 5
1.0 Section. 5
1.1 Definition. 5
1.2 Definition. 5
1.3 Lemma. 5
1.4 Proof. 5
1.5 Theorem. 5
1.6 Proof. 5
1.7 Definition. 5
1.8 Lemma. 5
1.9 Proof. 5
1.10 Example. 5
1.11 Theorem. 6
1.12 Proof. 6
1.13 Definition. 6
1.14 Theorem. 6
1.15 Proof. 6
1.16 Definition. 6
1.17 Theorem. 6
1.18 Proof. 6
1.19 Theorem. 6
1.20 Proof. 6
1.21 Definition. 6
1.22 Lemma. 6
1.23 Proof. 7
1.24 Definition. 7
1.25 Definition. 7
1.26 Definition. 7
1.27 Definition. 7
1.28 Definition. 7
1.29 Theorem. 7
1.30 Proof. 7
1.31 Remark. 7
1.32 Lemma. 7
1.33 Proof. 7
1.34 Lemma. 8
1.35 Proof. 8
1.36 Theorem. 8
1.37 Proof. 8
1.38 Example. 8
1.39 Example. 8
1.40 Example. 8
1.41 Theorem. 8
1.42 Proof. 9
1.43 Theorem. 9
1.44 Proof. 9
1.45 Theorem. 9
1.46 Proof. 9
1.47 Theorem. 10
1.48 Proof. 10

1.49 Remark. 10
1.50 Example. 10
1.51 Example. 10
1.52 Lemma. 10
1.53 Proof. 10
1.54 Definition. 11
1.55 Remark. 11
1.56 Definition. 11
1.57 Remark. 11
1.58 Theorem. 11
1.59 Proof. 11
1.60 Theorem. 11
1.61 Proof. 11
1.62 Theorem. 11
1.63 Proof. 12
1.64 Theorem. 12
1.65 Theorem. 12
1.66 Proof. 12
2 Fundamental Group. 13
2.0 Section. 13
2.1 Definition. 13
2.2 Definition. 13
2.3 Definition. 13
2.4 Lemma . 13
2.5 Proof. 13
2.6 Definition. 13
2.7 Example. 13
2.8 Definition. 13
2.9 Definition. 13
2.10 Lemma. 14
2.11 Proof. 14
2.12 Lemma. 14
2.13 Proof. 14
2.14 Theorem. 14
2.15 Proof. 14
2.16 Theorem. 14
2.17 Proof. 15
2.18 Definition. 15
2.19 Definition. 15
2.20 Definition. 15
2.21 Definition. 15
2.22 Definition. 15
2.23 Definition. 15
2.24 Defintion. 15
2.25 Definition. 15
2.26 Definition. 15
2.27 Example. 15
2.28 Definition. 16
2.29 Theorem. 16
2.30 Proof. 16
2.31 Corollary. 16
2.32 Proof. 16
2.33 Remark. 16
2.34 Definition. 16

2.35 Lemma. 16
2.36 Proof. 16
2.37 Definition. 17
2.38 Theorem. 17
2.39 Proof. 17
2.40 Theorem. 17
2.41 Proof. 17
2.42 Corollary. 17
2.43 Proof. 17
2.44 Definition. 17
2.45 Definition. 17
2.46 Definition. 18
2.47 Example. 18
2.48 Theorem. 18
2.49 Proof. 18

0.0 Section.

Introduction.

1.0 Section.

Countability and Separation Axioms.

1.1 Definition.

A space X is said to have a countable basis at x if there exists a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

1.2 Definition.

A space X is said to be first-countable if it has a countable basis at each of its points.

1.3 Lemma.

(Sequence lemma) Let X be a topological space. Let A be a subset of X. If there exists a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X if first-countable.

1.4 Proof.

($\Rightarrow$) Let (x_n) be a sequence of points of A converging to x. Then, every neighborhood U of x contains all but finitely many points of (x_n), so $U \cap A$ is nonempty, which means precisely that $x \in \overline{A}$.

($\Leftarrow$) Let X be first-countable. Let $B_1, B_2, \ldots$ be a countable basis at x. Define a new countable basis with $B_1' = B_1$ and $B_{n+1}' = B_{n+1} \cap B_n'$ for $n = 1, 2, \ldots$. For each n, choose a point $x_n \in B_n' \cap A$. Then, (x_n) is a sequence of points of A converging to x.

1.5 Theorem.

Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

1.6 Proof.

($\Rightarrow$) Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x that contains all but finitely many points of (x_n). Thus, all but finitely many points of $(f(x_n))$ are in V, so $f(x_n) \to f(x)$.

($\Leftarrow$) Let X be first-countable. Let $A \subseteq X$. We want to show $f(\overline{A}) \subseteq \overline{f(A)}$. By the sequence lemma, if $x \in \overline{A}$, then there exists a sequence (x_n) of points of A converging to x and hence a sequence of points $f(x_n)$ of $f(A)$ converging to $f(x)$. So $f(x) \in \overline{f(A)}$ by the sequence lemma, which means that $f(\overline{A}) \subseteq \overline{f(A)}$.

1.7 Definition.

A space X is said to be second-countable if it has a countable basis.

1.8 Lemma.

If X is second-countable, then any discrete subspace of X is countable.

1.9 Proof.

Let Y be a discrete subspace of X. For each $y \in Y$, choose a basic neighborhood B_y that contains only y. Then the map $y \mapsto B_y$ is injective, so Y is countable.

1.10 Example.

The uniform topology $\mathbb{R}^\omega$ is first-countable but not second-countable. $\mathbb{R}^\omega$ is metrizable, so it is first-countable. Let Y be the set of all sequences of 0s and 1s. Then Y is a discrete subspace, but Y is uncountable, so $\mathbb{R}^\omega$ is not second-countable in the uniform topology.

1.11 Theorem.

(a) A subspace of a first-countable space is first-countable.

(b) A countable product of first-countable spaces is first-countable.

(c) A subspace of a second-countable space is second-countable.

(d) A countable product of second-countable spaces is second-countable.

1.12 Proof.

The proof is obvious.

1.13 Definition.

A space X is said to satisfy the Lindlof property if every open cover of X has a countable subcover.

1.14 Theorem.

A second-countable space is Lindlof.

1.15 Proof.

Let $\mathcal{A}$ be an open cover of X. Let $\{B_n\}$ be a countable basis for X. For each n for which it is possible, choose $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. Then the collection $\{A_n\}$ is a countable subcover of $\mathcal{A}$. Indeed, for all $x \in X$, there exists an open $A \in \mathcal{A}$ that contains x, which in turn contains a basic neighborhood B_n of x. So A_n is defined and contains x.

1.16 Definition.

A space X is said to be separable if it has a countable dense subset.

1.17 Theorem.

A second-countable space is separable.

1.18 Proof.

Let $\{B_n\}$ be a countable basis for X. For each n, choose an $x_n \in B_n$. Then the set D of all such x_n is countable and dense.

1.19 Theorem.

Second-countability, Lindlofness, and separabilty are equivalent for a metrizable space.

1.20 Proof.

See Munkres Section 30 Exercise 5.

1.21 Definition.

A space X is said to be T_1 is for every pair of distinct points x and y, there exists a neighborhood U of x such that $y \notin U$.

1.22 Lemma.

A space X is T_1 if and only if every singleton set $\{x\}$ is closed.

1.23 Proof.

($\Rightarrow$) Suppose that X is T_1. $\{x\}$ is its own closure, for any other point y has a neighborhood U disjoint from $\{x\}$. So $\{x\}$ is closed.

($\Leftarrow$) Suppose that $\{x\}$ is closed. Let $y \in X$ be distinct from x. Then there exists a neighborhood U of y that does not contain x. So X is T_1.

1.24 Definition.

A space X is said to be T_2, or Hausdorff, if every pair of distinct points x and y have distinct neighborhoods U and V, respectively.

1.25 Definition.

A space X is said to be regular if for every point x and closed set A such that $x \notin A$, there exists disjoint neighborhoods U and V, respectively,

1.26 Definition.

A space X is said to be T_3 if it is T_1 and regular.

1.27 Definition.

A space X is said to be normal if for every pair of disjoint closed sets A and B, there exist disjoint neighborhoods U and V of A and B, respectively.

1.28 Definition.

A space X is said to be T_4 if it is T_1 and normal.

1.29 Theorem.

T_4 implies T_3 implies T_2 implies T_1.

1.30 Proof.

The proof is obvious.

1.31 Remark.

For convenience, regular shall refer to T_3 and normal shall refer to T_4.

1.32 Lemma.

Let X be a T_1 space. X is regular if and only if for every $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$.

1.33 Proof.

($\Rightarrow$) Let $x \in X$ and U be a neighborhood of x. Let $B = X - U$. Then A is closed, so there exists a neighborhood V of x that is disjoint from a neighborhood W of B. Moreover, $\bar{V}$ is disjoint from B, and $\bar{V} \subseteq U$, as desired.

($\Leftarrow$) Let $x \in X$ and $B \subseteq X$ be a closed subset disjoint from x. Let $U = X - B$. By hypothesis, there exists a neighborhood V of x such that $\bar{V} \subseteq U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively.

1.34 Lemma.

Let X be a T_1 space. X is normal if and only if for every closed subset A of X and every neighborhood U of A, there exists a neighborhood V of A such that $\overline{V} \subseteq U$.

1.35 Proof.

The proof is exactly the same as for the previous theorem.

1.36 Theorem.

(a) A subspace of a Hausdorff space is Hausdorff.

(b) A product of Hausdorff spaces is Hausdorff.

(c) A subspace of a regular space is regular.

(d) a product of regular spaces is regular.

1.37 Proof.

(a) Let X be Hausdorf and let Y be a subspace of X. Let x and y be distinct points of Y. There exist disjoint neighborhoods U and V of x and y, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y, respectively, in Y.

(b) Let $\{X_\alpha\}$ be a collection of Hausdorff spaces and let X be the product space. Let x and y be distinct points of X. Then there exists at least one β such that $x_\beta \neq y_\beta$. Let U_β and V_β be disjoint neighborhoods of x_β and y_β, respectively, in X_β. The sets $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are disjoint neighborhoods of x and y, respectively, in X.

(c) Let X be regular and let Y be a subspace of X. Let $x \in Y$ and let B be a closed subset of Y that does not contain x. Let $\overline{B}$ be the closure of B in X. Then $\overline{B} \cap Y = B$, so $x \notin \overline{B}$. By the regularity of X, the exist disjoint neighborhoods U and V of x and $\overline{B}$, respectively, in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and B, respectively, in Y.

(d) Let $\{X_\alpha\}$ be a collection of regular spaces and let X be the product space. It follows immediately that X is T_1 by theorem 1.36a. Let $x \in X$ have the neighborhood U equals the product of U_α. For each U_α, let V_α be a neighborhood of x_α such that $\overline{V_\alpha} \subseteq U_\alpha$ by Lemma 1.32. If $U_\alpha = X_\alpha$, then let $V_\alpha = X_\alpha$. Let V be the product of V_α. Then $\overline{V} \subseteq U$, so X is regular.

1.38 Example.

$\mathbb{R}_K$ is Hausdorff but not regular.

1.39 Example.

$\mathbb{R}_l$ is normal.

1.40 Example.

$\mathbb{R}_l \times \mathbb{R}_l$ is regular but not normal.

1.41 Theorem.

A regular space with a countable basis is normal.

1.42 Proof.

Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. For each $x \in A$, there exists a neighborhood U of x and a neighborhood V of x such that $\overline{V} \subseteq U$. For each such V, choose a basic neighborhood U_x of x contained in V. This is a countable covering of A by sets whose closures do not intersect B. Index these sets as $\{U_n\}$. Similarly, cover B with a countable collection of neighborhoods $\{V_n\}$.

Define

$$U'_n = U_n - \bigcup_{i=1}^{n} \overline{V_i}$$

and

$$V'_n = V_n - \bigcup_{i=1}^{n} \overline{U_i}.$$

Each such set is open because it is the difference between an open set and a closed set. Furthermore, the collection $\{U'_n\}$ is an open cover of A and the collection $\{V'_n\}$ is an open cover of B. The open sets

$$U' = \bigcup_{n=1}^{\infty} U'_n$$

and

$$V' = \bigcup_{n=1}^{\infty} V'_n$$

are disjoint open sets that contain A and B. Indeed, if $x \in U' \cap V'$, then $x \in U'_i \cap V'_j$ for some i and j. Without loss of generality, assume that $i \leq j$. Then $x \in U_i$, but $x \notin \overline{U_i}$, hence a contradiction.

1.43 Theorem.

A metrizable space is normal.

1.44 Proof.

Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B(a, \varepsilon_a)$ is disjoint from B, and similarly, choose ε_b for each $b \in B$. The open sets

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2)$$

and

$$V = \bigcup_{b \in B} B(b, \varepsilon_b/2)$$

are disjoint and contain A and B, respectively.

1.45 Theorem.

A compact Hausdorff space is normal.

1.46 Proof.

Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. For each $x \in A$, choose disjoint neighborhoods U_x and V_x of x and B, respectively. The collection $\{U_x\}$ is an open cover of A, so there exists a finite subcover $U_1, \ldots, U_n$. It follows that the sets

$$U = \bigcup_{i=1}^{n} U_i$$

and

$$V = \bigcap_{i=1}^{n} V_i$$

are disjoint neighborhoods of A and B, respectively.

1.47 Theorem.

Every well-ordered set is normal in the order topology.

1.48 Proof.

First observe that any interval $(x, y]$ is open, for if y is not the maximum element, then $(x, y] = (x, y')$ where y' is the immediate successor of y. Let A and B be disjoint closed sets in X.

Assume that neither A nor B contains the minimum element a_0 of X. For each $a \in A$, choose a neighborhood $(x_a, a]$ disjoint from B. For each $b \in B$, choose a neighborhood $(y_b, b]$ disjoint from A. The sets

$$U = \bigcup_{a \in A} (x_a, a]$$

and

$$V = \bigcup_{b \in B} (y_b, b]$$

are disjoint open sets containing A and B, respectively. Indeed, if $z \in U \cap V$, then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and $b \in B$. Without loss of generality, assume that $a < b$. If $a \leq y_b$, then the intervals are disjoint. If $a > y_b$, then $(y_b, b]$ is not disjoint from A, a contradiction.

If A contains a_0, then the set $A - \{a_0\}$ is closed and disjoint from B, so it admits disjoint open intervals U and V of $A - \{a_0\}$ and B, respectively. Now the sets $U \cup \{a_0\}$ and V are disjoint neighborhoods of A and B, respectively.

1.49 Remark.

Indeed, every ordered set is normal in the order topology.

1.50 Example.

If J is uncountable, then the product space $\mathbb{R}^J$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $\mathbb{R}^J$ is homeomorphic to a subspace of $[0, 1]^J$).

1.51 Example.

The product space $S_\Omega \times \overline{S_\Omega}$ is not normal. So the product of normal spaces need not be normal. Nor does the subspace of a normal space need be normal (for $overline S_\Omega \times \overline{S_\Omega}$ compact Hasudorff and therefore normal).

1.52 Lemma.

(Urysohn Lemma) Let X be a normal space. Let A and B be disjoint closed subsets of X. THen there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

1.53 Proof.

TODO

1.54 Definition.

If A and B are two subsets of X, and if there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$, then we say that A and B are separated by a continuous function.

1.55 Remark.

The Urysohn lemma says that if every pair A and B of disjoint closed subsets can be separated by open sets, then they can be separated by a continuous function. The converse is trivial.

1.56 Definition.

A space X is said to be completely regular if it is T_1 and if for every $x_0 \in X$ and every closed set A that does not contain x_0, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

1.57 Remark.

Complete regularity is also known as $T_{3+1/2}$.

1.58 Theorem.

(a) A subspace of a completely regular space is completely regular.

(b) A product of completely regular spaces is completely regular.

1.59 Proof.

(a) Let X be completely regular and let Y be a subspace of X. Let $x_0 \in Y$ and let A be a closed subset of Y that does not contain x_0. Then $A = \overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X, and moreoever, $x_0 \notin \overline{A}$. There exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in \overline{A}$. Then $f \mid Y$ is a continuous function from Y to $[0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in A$.

(b) Let X be the product of completely regular spaces $\{X_\alpha\}$. Let $b \in X$ and let A be a closed set of X that does not contain b. Choose a basic neighborhood U of b, and denote $U_1, \dots, U_n$ the basic neighborhoods of b_α in X_α that are not all of X_α. For each U_i, choose a continuous function $f_i : X_\alpha \to [0, 1]$ such that $f_i(b_\alpha) = 0$ and $f_i(x) = 1$ for all $x \in A$. The function $f : X \to [0, 1]$ defined by

$$f(x) = \prod_{i=1}^{n} f_i(x_\alpha)$$

is continuous and satisfies the desired properties.

1.60 Theorem.

(Urysohn Metrization Theorem) A regular space with a countable basis is metrizable.

1.61 Proof.

TODO

1.62 Theorem.

(Embedding Theorem) Let X be a a T_1 space. Suppose that $\{f_\alpha\}$ is a family of continuous functions indexed for $\alpha \in J$. If $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha(x) = 0$ for all $x \notin U$. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = \prod_{\alpha \in J} f_\alpha(x)$$

is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$, then F is an imbedding of X in $[0,1]^J$.

1.63 Proof.

TODO

1.64 Theorem.

A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.

1.65 Theorem.

(Tietze Extension Theorem) Let X be a normal space. Let A be a closed subspace of X. Any continuous map of A into $[0,1]$ or A into $\mathbb{R}$ can be extended to a continuous map of X into $[0,1]$ or $\mathbb{R}$, respectively.

1.66 Proof.

TODO

2.0 Section.

Fundamental Group.

2.1 Definition.

If f and f' are continuous maps of X into Y, we say that f is homotopic to f' if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$. We denote $f \simeq f'$ and call F a homotopy between f and f'.

2.2 Definition.

If $f \simeq f'$ and f' is a constant map, then we say that f is nullhomotopic.

2.3 Definition.

If f and f' are paths in X with initial point x_0 and final point x_1, then we say that f is path homotopic to f' if there exists a continuous map $F : I \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in I$ and $F(0, t) = x_0$ and $F(1, t) = x_1$ for all $t \in I$. We denote $f \simeq_p f'$ and call F a path homotopy between f and f'.

2.4 Lemma

The relations $\simeq$ and $\simeq_p$ are equivalence relations.

2.5 Proof.

The proof is obvious.

2.6 Definition.

Denote $[f]$ the equivalence class of f under the relation $\simeq_p$. $[f]$ is called the path homotopy class of f.

2.7 Example.

If A is a convex subspace of $\mathbb{R}^n$, then any two paths in A with the same endpoints are homotopic. For the straihgt-line homotopy

$$F(x, t) = (1 - t)f(x) + tf'(x)$$

is a homotopy between f and f'.

2.8 Definition.

If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, then the product $h = f * g$ is defined as

$$h(s) = \begin{cases} f(2s) & \text{if } s \in [0, 1/2] \\ g(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases}$$

2.9 Definition.

The product between path-homotopy classes is defined as $[f] * [g] = [f * g]$. Indeed, let F be a path homotopy between f and f' and let G be a path homotpy between g and g'. Define

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } s \in [0, 1/2] \\ G(2s - 1, t) & \text{if } s \in [1/2, 1]. \end{cases}$$

2.10 Lemma.

If $k : X \to Y$ is a continuous map and if F is a path homotopy in X between f and f', then $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ f'$.

2.11 Proof.

The proof is obvious.

2.12 Lemma.

If $k : X \to Y$ is a continuous map and if f and g are paths in X such that $f(1) = g(0)$, then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$

2.13 Proof.

The proof is obvious.

2.14 Theorem.

The operation $*$ satisfies the following properties.

(a) (Associativity) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$ whenever the products are defined.

(b) (Identity) Let e_x denote the constant path at point x. If f is a path from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(c) (Inverse) If f is a path from x_0 to x_1, then let its reverse $\overline{f}$ be defined as $\overline{f}(s) = f(1-s)$ for $s \in I$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

2.15 Proof.

(a) Let $[a, b]$ and $[c, d]$ be two intervals in I. There exists an unique continuous map $p : [a, b] \to [c, d]$ of the form $p(x) = mx + k$ called the positive linear map. The inverse of a positive linear map is a positive linear map, and the composition of two positive linear maps is a positive linear map.

When the triple product $f * g * h$ is defined, it is the path k_{ab} in X where on $[0, a]$, it is the positive linear map of $[0, a]$ to $[0, 1]$ followed by f and similarly for $[a, b]$ and $[b, 1]$. The path homotopy class of k_{ab} is independent of the choice of a and b. Indeed, $[f] * ([g] * [h])$ is the path homotopy class of k_{ab} where $a = 1/2$ and $b = 3/4$ whille $([f] * [g]) * [h]$ is the path homotopy class of k_{ab} where $a = 1/4$ and $b = 1/2$. These are equivalent.

(b) Let e_0 denote the constant path in I at 0 and let i denote the identity path in I. Then $e_0 * i$ is a path in I from 0 to 1. I is convex, so there is a path homotopy G between i and $e_0 * i$, so $f \circ G$ is a path homotopy between $f \circ i = f$ and

$$f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f.$$

The proof for right identity is entirely similar.

(c) Let i denote the identity path in I and $\bar{i}$ denote its reverse. I is convex, so there is a path homotopy H between e_0 and $i * \bar{i}$. Then $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and

$$f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \overline{f}.$$

The proof for left inverse is entirely similar.

2.16 Theorem.

Let f be a path in X. Let $a_0, \ldots, a_n$ such that that $0 < a_0 < \ldots < a_n < 1$. Let f_i be the path in X defined as the positive linear map of I to $[a_{i-1}, a_i]$ followed by f. Then

$$[f] = [f_1] * \ldots * [f_n].$$

2.17 Proof.

The proof is sketched in Proof 2.15a.

2.18 Definition.

Let G and G' be two groups with the operation $\cdot$. A homomorphism $f : G \to G'$ is such that

$$f(x \cdot y) = f(x) \cdot f(y)$$

for all $x, y \in G$. f satisfies $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$ for all $x \in G$.

2.19 Definition.

The kernel of f is the set $f^{-1}(e')$. It is a subgroup of G.

2.20 Definition.

The image of f is the set $f(G)$. It is a subgroup of G'.

2.21 Definition.

A homomorphism is called a monomorphism if it is injective (or equivalently of $f^{-1}(e') = e$). It is called an epimorphism if it is surjective. It is called an isomorphism if it is bijective.

2.22 Definition.

Let H be a subgroup of G. Let xH denote the set of products xh for all $h \in H$. It is called the left coset of H in G, and the collection of all such xH for $x \in G$ is a partition of G. Similarly, let Hx denote the right coset of H in G.

2.23 Definition.

H is said to be a normal subgroup of G if $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$. In this case, $xH = Hx$ for all $x \in G$. The partition G/H is called the quotient group of G by H with the operation $(xH)(yH) = xyH$.

2.24 Defintion.

The map $f : G \to G/H$ defined by $f(x) = xH$ is epimorphism with kernel H. Conversely, if $f : G \to G'$ is an epimorphism and N is a normal subgroup of G, then f induces an isomorphism $g : G/N \to G'$ defined by $g(xN) = f(x)$.

2.25 Definition.

If H is not normal, then G/H denotes the collection of right cosets of H in G.

2.26 Definition.

Let X be a topological space and let $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ relative to the base point x_0 is the group of path homotopy classes of loops in X based at x_0 with the operation $*$.

2.27 Example.

Any convex subspace of $\mathbb{R}^n$ has a trivial fundamental group.

2.28 Definition.

Let α be a path in X from x_0 to x_1. Define the map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha].$$

If f is a loop at x_0, then $\overline{\alpha} * f * \alpha$ is a lopo at x_1.

2.29 Theorem.

The map $\hat{\alpha}$ is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

2.30 Proof.

We compute that

$$\begin{aligned}\hat{\alpha}([f]) * \hat{\alpha}([g]) &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= [\overline{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f] * [g]),\end{aligned}$$

so $\hat{\alpha}$ is a homomorphism. We show that $\hat{\alpha}$ has a left inverse and a right inverse. Let $\beta = \overline{\alpha}$. Then

$$\begin{aligned}\hat{\beta}(\hat{\alpha}([f])) &= [\overline{\beta}] * ([\overline{\alpha}] * [f] * [\alpha]) * [\beta] \\ &= [e_{x_0}] * [f] * [e_{x_1}] \\ &= [f]\end{aligned}$$

for all $[f] \in \pi_1(X, x_0)$. A similar computation shows that $\hat{\beta}$ is also a right inverse of $\hat{\alpha}$. So $\hat{\alpha}$ is an isomorphism.

2.31 Corollary.

If X is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

2.32 Proof.

The proof is trivial.

2.33 Remark.

We must still specify the base point x_0 in the definition of the fundamental group, for the isomorphism $\hat{\alpha}$ depends on the choice of α.

2.34 Definition.

A space X is said to be simply connected if it is path connected and $\pi_1(X, x_0)$ is trivial for one (and hence all) $x_0 \in X$.

2.35 Lemma.

If X is simply connected, then any two paths in X with the same endpoints are path homotopic.

2.36 Proof.

Let α and β be two paths in X from x_0 to x_1. We compute that

$$\begin{aligned}[\alpha] &= [\alpha] * [\overline{\beta}] * [\beta] \\ &= [\alpha * \overline{\beta}] * [\beta] \\ &= [e_{x_0}] * [\beta] \\ &= [\beta].\end{aligned}$$

2.37 Definition.

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. The homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by h, relative to base point x_0, is defined as

$$h_*([f]) = [h \circ f].$$

h_* is a homomorphism because

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

h_* depends on the choice of x_0, so we denote $(h_{x_0})_*$ when necessary.

2.38 Theorem.

If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism of $\pi_1(X, x_0)$.

2.39 Proof.

We compute that

$$\begin{aligned} i_*([f]) &= [i \circ f] \\ &= [f]. \end{aligned}$$

2.40 Theorem.

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps, then

$$(k \circ h)_* = k_* \circ h_*.$$

2.41 Proof.

We compute that

$$\begin{aligned} (k \circ h)_*([f]) &= [(k \circ h) \circ f] \\ &= [k \circ (h \circ f)] \\ &= k_*([h \circ f]) \\ &= (k_* \circ h_*)([f]). \end{aligned}$$

2.42 Corollary.

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

2.43 Proof.

Let k be the inverse of h. Then $(k \circ h)_* = i_*$ is the identity homomorphism of $\pi_1(X, x_0)$, and $(h \circ k)_* = j_*$ is the identity homomorphism of $\pi_1(Y, y_0)$. So h_* is an isomorphism.

2.44 Definition.

Let $p : E \to B$ be a continuous surjective map. An open set U of B is said to be evenly covered by p if $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ such that $p \mid V_\alpha$ is a homeomorphism onto U for each α. The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices.

2.45 Definition.

Let $p : E \to B$ be a continuous surjective map. If each point $b \in B$ has a neighborhood U that is evenly covered by p, then p is said to be a covering map and E is said to be a covering space of B.

2.46 Definition.

A map $p : E \to B$ is a local homeomorphism if for every point $e \in E$ has a neighborhood U that is homeomorphic to a neighborhood of $p(e)$ in B. A covering map is a local homeomorphism, but the converse does not hold necessarily.

2.47 Example.

2.48 Theorem.

The map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.

2.49 Proof.

Let U be the subset of S^1 consisting of points with positive first coordinate, i.e., the open right semicircle. The preimage $p^{-1}(U)$ is the union of the open intervals $V_n = (n - 1/4, n + 1/4)$ for $n \in \mathbb{Z}$. The restriction $p \mid V_n$ is injective because $\sin \mid V_n$ is strictly monotonic, and it is surjective by the intermediate value theorem. $p \mid V_n$ is a continuous bijective map between a compact space and a Hausdorff space, so it is a homeomorphism of V_n and U, so U is evenly covered by p. A similar argument can be made for the left, upper, and lower semicircles. Thesse semicircles form an open cover of S^1, and each one is evenly covered by p, so p is a covering map.