

AS.110.415 Honors Analysis I

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0.0 Section.

Introduction.

1.0 Section.

The real numbers.

1.1 Definition.

Denote the natural numbers \mathbb{N} , the integers \mathbb{Z} , and the rational numbers \mathbb{Q} .

1.2 Remark.

An axiomatic treatment of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is beyond the scope of this course.

1.3 Remark.

For many purposes, the rationals are not "big enough".

1.4 Example.

There is no $x \in \mathbb{Q}$ such that $x^2 = 2$. Indeed, assume there are $p, q \in \mathbb{Z}$ coprime such that $(p/q)^2 = 2$. Then $p^2 = 2q^2$, which means that p^2 is even, so p is even as well. But if p is even, then $p^2 = 2q^2$ is divisible by 4. But this is only possible if q is even as well, a contradiction of the coprime-ness of p, q .

1.5 Example.

In fact, there is not even a "best approximation" in \mathbb{Q} of the solution to $x^2 = 2$. Consider the sets

$$A = \{a \in \mathbb{Q} \mid a^2 < 2, a > 0\} \text{ and } B = \{b \in \mathbb{Q} \mid b^2 > 2, b > 0\}.$$

A has no largest element. Indeed, suppose $p \in A$ with $p > 0$. Then setting

$$q = \frac{2p + 2}{p + 2}$$

has that $q \in A$ and $q > p$. A similar argument shows that B has no smallest element.

1.6 Definition.

An order on a set S is a relation $<$ such that

(1) (Trichotomy) For $x, y \in S$, exactly one of the statements hold:

$$x < y \text{ or } x = y \text{ or } y < x.$$

(2) (Transitivity) For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

If a set S is equipped with an order $<$, then $(S, <)$ or simply S is called an ordered set.

1.7 Example.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all ordered by $x < y$ iff $y - x$ is positive.

1.8 Definition.

Let S be an ordered set. $E \subseteq S$ is bounded above if there exists some $M \in S$ such that $x \leq M$ for all $x \in E$. Similarly, $E \subseteq S$ is bounded below if there exists some $L \in S$ such that $x \geq L$ for all $x \in E$.

1.9 Example.

\mathbb{N} has the well-ordering principle, in that every $E \subseteq \mathbb{N}$ nonempty has a least element.

1.10 Definition.

Let S be an ordered set and let E be bounded above. If there exists an α such that

- (1) α is an upper bound of E ,
- (2) If $\beta < \alpha$, then β is not an upper bound of E ,

then $\alpha = \sup E$ is the supremum or least upper bound of E . Define symmetrically $\inf E$ to be the infimum or greatest lower bound of E .

1.11 Example.

A supremum/infimum, when it exists, need not be a member of the subset.

- (1) The sets $A, B \subseteq \mathbb{Q}$ defined in Example 1.4 have no supremum or infimum, respectively.
- (2) $\mathbb{N} \subseteq \mathbb{Z}$ is bounded below but has no least upper bound.
- (3) $E = \{1/n \mid n \in \mathbb{Z}_+\}$ has that $\inf E = 0 \notin E$.

1.12 Definition.

An ordered set has the least upper bound property if whenever $E \subseteq S$ is nonempty and bounded above, then $\sup E$ exists. Define symmetrically the greatest lower bound property.

1.13 Example.

\mathbb{N}, \mathbb{Z} have the greatest upper bound property. \mathbb{Q} does not.

1.14 Theorem.

Let S be an ordered set with the least upper bound property and let $B \subseteq S$ is nonempty and bounded below. Let L be the set of all lower bounds of B . Then $\sup L$ exists and is equal to $\inf B$, which exists.

1.15 Proof.

$L \neq \emptyset$ since B is bounded below. Since $y \leq x$ for every $y \in L, x \in B$, then every $x \in B$ is an upper bound of L . Thus L is bounded above since B is nonempty, so $\sup L$ exists by

the least upper bound property. Since every $x \in B$ is an upper bound of L , $\sup L \leq x$ for every $x \in B$, so $\sup L$ is a lower bound of B . For any lower bound $y \in L$ of B , $y \leq \sup L$, so $\sup L = \inf B$.

1.16 Remark.

A set S has the least upper bound property iff it has the greatest lower bound property.

1.17 Definition.

A field $(F, +, \cdot)$ is a set F equipped with two binary operations $+: F \times F \rightarrow F$ and $\cdot: F \times F \rightarrow F$ that satisfy the field axioms: For any $x, y, z \in F$,

- (1) (Closure) $x + y \in F$ and $x \cdot y \in F$.
- (2) (Commutativity) $x + y = y + x$ and $x \cdot y = y \cdot x$.
- (3) (Associativity) $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (4) (Identity) There exists two symbols $0, 1 \in F$ with $0 \neq 1$ such that $0 + x = x$ and $1 \cdot x = x$.
- (5) (Inverse) There exists $-x \in F$ such that $x + (-x) = 0$ for any x , and there exists $1/x \in F$ such that $x \cdot 1/x = 1$ for any $x \neq 0$.
- (6) (Distributivity) $x \cdot (y + z) = xy + xz$.

1.18 Example.

- (1) \mathbb{Q} with $+$ and \cdot defined normally as a field.
- (2) $\{0, 1\}$ is the trivial field.
- (3) \mathbb{Z} is not a field because it does not have multiplicative inverses.

1.19 Remark.

Fields are interesting because any statement proven about a general field F must hold in any field such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (to be defined later).

1.20 Definition.

An ordered field is a field F equipped with an order $<$ such that for $x, y, z \in F$,

- (1) $x + y < x + z$ if $y < z$.
- (2) $xy > 0$ if $x, y > 0$.

$x > 0$ is said to be positive and $x < 0$ is said to be negative.

1.21 Theorem.

There exists an ordered field \mathbb{R} with the least upper bound property. Moreover, $\mathbb{Q} \subseteq \mathbb{R}$. The elements of \mathbb{R} are called the real numbers.

1.22 Proof.

The proof is delayed until the end of the course (though the tools already exist to prove it).

1.23 Theorem.

\mathbb{R} has the follow properties:

- (1) (Archimedean property) Lorem ipsum.
- (2) (Density of \mathbb{Q}) Lorem ipsum.
- (3) (Existence of roots) Lorem ipsum.

1.24 Proof.

Lorem ipsum.

2.0 Section.

Sequences and series of functions.

2.1 Definition.

Given a sequence of (f_n) of \mathbb{C} -valued functions on a metric space (X, d) such that $\lim f_n(x)$ exists for every $x \in X$, then define the limit $\lim f_n$ to be the function $f : X \rightarrow \mathbb{C}$ such that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

for every $x \in X$. (f_n) is said to converge pointwise to f

2.2 Remark.

Do limits/sums of functions preserve the properties of the sequence? If (f_n) is a sequence of continuous/differentiable functions, then is the limit/sum continuous/differentiable? Moreover, is (f'_n) related to f' ?

Recall that f is continuous at x iff $f(t) \rightarrow f(x)$ as $t \rightarrow x$. Thus, asking whether the limit of continuous functions is continuous is the same as asking if

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t),$$

namely if it is possible to "swap limits".

2.3 Example.

Pointwise convergence is not sufficient to swap limits. Let $S_{m,n} = m/(m+n)$ for each $m, n \in \mathbb{Z}_+$. Then,

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} S_{m,n} = \lim_{n \rightarrow \infty} 1 = 1$$

while

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} S_{m,n} = \lim_{m \rightarrow \infty} 0 = 0.$$

2.4 Example.

Pointwise convergence is not enough to guarantee continuity of limits! For $x \in \mathbb{R}$, let

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Since $f_n(0) = 0$ for every $n \in \mathbb{N}$, $f(0) = 0$. If $x \neq 0$, then

$$f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots = 1 + x^2,$$

so

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + x^2 & \text{otherwise.} \end{cases},$$

so f is not continuous!

2.5 Example.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$g_n(x) = \frac{\sin(nx)}{\sqrt{n}}$$

so that

$$g(x) = \lim_{n \rightarrow \infty} g_n(x) = 0$$

for every $x \in \mathbb{R}$. But

$$g'_n = \sqrt{n} \cos(nx)$$

is such that

$$\lim_{n \rightarrow \infty} g'_n(0) = \sqrt{n},$$

so g'_n does not converge pointwise to g' .

2.6 Definition.

Let (f_n) be a sequence of \mathbb{C} -valued functions on a metric space (X, d) . (f_n) converges uniformly on $E \subseteq X$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies that

$$|f_n(x) - f(x)| < \varepsilon$$

for every $x \in E$.

2.7 Remark.

Uniform convergence is stronger than pointwise convergence: $f_n \rightarrow f$ uniformly implies that $f_n \rightarrow f$ pointwise.

2.8 Theorem.

Let (f_n) be a sequence of \mathbb{C} -valued functions on a metric space (X, d) . (f_n) converges uniformly on $E \subseteq X$ iff (f_n) is uniformly Cauchy on E , namely iff for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies that

$$|f_m(x) - f_n(x)| < \varepsilon$$

for every $x \in E$.

2.9 Proof.

(\Rightarrow) If $f_n \rightarrow f$ uniformly on E and $\varepsilon > 0$, then there is an $N \in \mathbb{N}$ such that $n \geq n$ implies that $|f_n(x) - f(x)| < \varepsilon/2$ for every $x \in E$. Hence, for $m, n \geq N$,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon/2 + \varepsilon/2$$

for every $x \in E$, so (f_n) is uniformly Cauchy in E .

(\Leftarrow) If (f_n) is uniformly Cauchy on E and $\varepsilon > 0$, then there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $|f_m - f_n| < \varepsilon/2$ for every $x \in E$. The sequence $(f_m(x))$ is Cauchy in \mathbb{C} for every $x \in E$, so

$$f(x) = \lim_{m \rightarrow \infty} f_m(x)$$

exists for every $x \in E$. This means that if $m, n \geq N$, then for each $x \in E$,

$$|f_m(x) - f_n(x)| < \varepsilon/2 \implies -\varepsilon/2 < f_m(x) - f_n(x) < \varepsilon/2,$$

so that

$$-\varepsilon/2 \leq \lim_{m \rightarrow \infty} (f_m(x) - f_n(x)) = f(x) - f_n(x) \leq \varepsilon/2,$$

which means that

$$|f(x) - f_n(x)| < \varepsilon$$

for every $x \in E$, so $f_n \rightarrow f$ uniformly on E .

2.10 Theorem.

Suppose that $f_n \rightarrow f$ pointwise on $E \subseteq X$. Then $f_n \rightarrow f$ uniformly iff $f_n \rightarrow f$ in the supremum norm, namely, if

$$\sup_{x \in E} |f_n(x) - f| \rightarrow 0$$

as $n \rightarrow \infty$.

2.11 Proof.

(\Rightarrow) If $f_n \rightarrow f$ uniformly then $\sup |f_n(x) - f(x)| \rightarrow 0$ by definition.

(\Leftarrow) If $\sup |f_n(x) - f(x)| \rightarrow 0$, then for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies that $\sup |f_n(x) - f(x)| < \varepsilon$. Hence, for $n \geq N$,

$$|f_n(x) - f(x)| \leq \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$$

for every $x \in E$, so $f_n \rightarrow f$ uniformly.

2.12 Example.

The sequence of functions $f_n(x) = 1/(nx + 1)$ on $(0, 1) \supseteq \mathbb{R}$ for $n \in \mathbb{N}$ is such that $f_n \rightarrow 0$ pointwise but for every n ,

$$|0 - f_n(x)| = \left| \frac{1}{nx + 1} \right|,$$

so choosing $x = 1/n \in (0, 1)$ has that

$$\left| 0 - f_n\left(\frac{1}{n}\right) \right| = \frac{1}{2}$$

for every $n \in \mathbb{N}$. So f_n does not converge to 0 uniformly.

2.13 Theorem.

(Weierstrauss M-test) If (f_n) is a sequence of \mathbb{C} -valued functions on $E \subseteq X$ for a metric space (X, d) with $|f_n(x)| \leq M_n$ for every $x \in E, n \in \mathbb{N}$, then $\sum f_n$ converges uniformly if $\sum M_n$ converges.

2.14 Proof.

If $\sum M_n$ converges in \mathbb{R} , then its partial sums are Cauchy in \mathbb{R} . Hence, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies by the triangle equality that

$$\left| \sum_{i=m}^n f_i(x) \right| \leq \sum_{i=m}^n |f_i(x)| \leq \sum_{i=m}^n M_i < \varepsilon$$

so that the partial sums of $\sum f_n$ are uniformly Cauchy and hence uniformly convergent.

2.15 Example.

The converse statement to the Weierstrauss M-test fails in general. Choose the sequence of functions on \mathbb{R} defined by

$$f_n(x) = \frac{(-1)^{n+1}}{n}$$

for $n \in \mathbb{Z}_+$. Then

$$\sum_{n=1}^{\infty} f_n(x) = \log 2$$

for every $x \in \mathbb{R}$. But $|f_n(x)| = 1/n$ for every $x \in X$, so setting $M_n = 1/n$ has that $\sum M_n$ diverges. Hence $\sum f_n$ converges uniformly but $\sum M_n$ diverges, so the converse fails.

2.16 Theorem.

Suppose that $f_n \rightarrow f$ uniformly on $E \subseteq X$ for a metric space (X, d) and that $x \in X$ is a limit point of E and that $f_n(t) \rightarrow A_n$ as $t \rightarrow x$ for every $n \in \mathbb{N}$. Then (A_n) converges and

$$\lim_{t \rightarrow x} f(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{n \rightarrow \infty} A_n.$$

Moreover, if (f_n) is a sequence of continuous functions, then f is continuous also.

2.17 Proof.

Since $f_n \rightarrow f$ uniformly, the sequence is uniformly Cauchy, so for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $|f_m(t) - f_n(t)| < \varepsilon/2$ for every $t \in E$. Sending $t \rightarrow x$, $m, n \geq N$ implies that

$$|A_m - A_n| \leq \varepsilon/2 < \varepsilon,$$

so (A_n) is uniformly Cauchy in \mathbb{C} and thus converges to $A \in \mathbb{C}$. Note that for every $n \in \mathbb{N}$ and $t \in E$,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

by the triangle inequality. For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for $n \geq N$, both

$$|f(t) - f_n(t)| < \varepsilon/3$$

for every $t \in E$ (since $f_n \rightarrow f$ uniformly) and

$$|A_n - A| < \varepsilon/3$$

since $A_n \rightarrow A$. Finally, choose some neighborhood U of x in X such that

$$|f_n(t) - A_n| < \varepsilon/3$$

for all $t \in (U \cap E) \setminus \{x\}$. Combining these three inequalities,

$$|f(t) - A| < \varepsilon,$$

for every $t \in (U \cap E) \setminus \{x\}$, to get the desired conclusions.

2.18 Theorem.

(Dini's) Suppose that (f_n) be a sequence of \mathbb{R} -valued functions on a compact subset $K \subseteq X$ for a metric space (X, d) . If $f_n \rightarrow f$ pointwise, f_n continuous for every $n \in \mathbb{N}$, f is continuous, and $f_n \geq f_{n+1}$ or \leq for every $n \in \mathbb{N}$, then $f_n \rightarrow f$ uniformly on K .

2.19 Proof.

Consider the sequence $(g_n) = (f_n - f)$. Then g_n is continuous for every $n \in \mathbb{N}$ and $g_n \rightarrow 0$ pointwise and $g_n \geq g_{n+1}$ for every $n \in \mathbb{N}$. For $\varepsilon > 0$, let

$$K_n = g_n^{-1}([\varepsilon, \infty))$$

(which is closed since g_n is continuous) for every $n \in \mathbb{N}$. Since K is compact, $K_n \subseteq K$ is compact also. If $x \in K_{n+1}$, then $\varepsilon \leq g_{n+1}(x) \leq g_n(x)$, so $x \in K_n$ also. Thus the $K_{n+1} \subseteq K_n$ are nested for $n \in \mathbb{N}$. Now for each $x \in K$, $g_n(x) \rightarrow 0$ so that $x \notin K_n$ eventually. Thus $\bigcap K_n = \emptyset$, which means that $K_n = \emptyset$ eventually since the K_n are compact and nested. This means that there is some $N \in \mathbb{N}$ such that $K_n = \emptyset$ for all $n \geq N$, so $g_n < \varepsilon$ for all $n \geq N$ for all $x \in K$. Since $g_n \geq 0$ for every $n \geq 1$, this implies that $g_n \rightarrow 0$ uniformly which implies that $f_n \rightarrow f$ uniformly.

2.20 Remark.

(1) Compactness is necessary for Dini's Theorem as exhibited by the sequence $(f_n(x)) = (1/(nx + 1))$ on $(0, 1)$ with $f_n \geq f_{n+1}$ for every $n \in \mathbb{N}$.

(2) Monotonicity is necessary for Dini's Theorem as exhibited by the example of the sequence (g_n) on $[0, 1]$ defined by

$$g_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ 2 - nx & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{otherwise.} \end{cases}$$

Here, $g_n \rightarrow 0$ pointwise, but g_n does not converge to 0 uniformly.

2.21 Theorem.

The set of bounded continuous functions $\mathcal{C}(X)$ on a metric space (X, d) under the supremum norm is a complete metric space.

2.22 Proof.

$(f_n) \subseteq \mathcal{C}(X)$ is Cauchy iff it is uniformly Cauchy iff there is some $f : X \rightarrow \mathbb{C}$ such that $f_n \rightarrow f$ uniformly on X . Since (f_n) is a sequence of continuous functions, then f is continuous. Moreover, f is bounded because $f_n \rightarrow f$ uniformly implies that $|f_n(x) - f(x)| < 1$ for every $x \in X$ eventually. So $f \in \mathcal{C}(X)$ and since $f_n \rightarrow f$ uniformly, $d(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$.

2.23 Theorem.

Let (f_n) be a sequence of \mathbb{C} -valued functions differentiable on $[a, b]$ such that $f_n(x_0)$ converges for some $x_0 \in [a, b]$. If (f'_n) converges uniformly on $[a, b]$, then (f_n) converges uniformly on $[a, b]$ to a function f and

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

for every $x \in [a, b]$.

2.24 Proof.

Let $\varepsilon > 0$. Since $(f_n(x_0))$ converges, it is Cauchy, so there is some $N \in \mathbb{N}$ such that $m, n \geq N$ implies that

$$|f_m(x_0) - f_n(x_0)| < \varepsilon/2$$

and since (f'_n) is uniformly convergent, it is uniformly Cauchy, so taking N potentially larger, $m, n \geq N$ implies that

$$|f'_m(t) - f'_n(t)| < \frac{1}{2} \frac{\varepsilon}{b - a}$$

for every $t \in [a, b]$. By the Mean Value Theorem, for any $x, t \in [a, b]$ there is some $\xi \in (x, t)$ (or $\xi \in (t, x)$) such that if $m, n \geq N$, then

$$|(f_m(x) - f_n(x)) - (f_m(t) - f_n(t))| = |x - t| |f'_m(\xi) - f'_n(\xi)| \leq \frac{1}{2} \frac{|x - t|}{b - a} \varepsilon \leq \frac{\varepsilon}{2} \quad (*)$$

Hence for any $x \in [a, b]$, for $m, n \geq N$,

$$|f_m(x) - f_n(x)| \leq |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))| + |f_m(x_0) - f_n(x_0)| < \varepsilon/2 + \varepsilon/2$$

so that (f_n) is uniformly Cauchy and hence uniformly convergent on $[a, b]$. Let f be the limit of (f_n) and fix $x \in [a, b]$. Define for $t \in [a, b] \setminus \{x\}$ the functions

$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \text{ and } \varphi(t) = \frac{f(t) - f(x)}{t - x}.$$

Note then that

$$\lim_{t \rightarrow x} \varphi_n(t) = f'_n(x)$$

for every $n \in \mathbb{N}$. By (*),

$$|\varphi_m(t) - \varphi_m(x)| \leq \frac{1}{2} \frac{\varepsilon}{b-a}$$

for $m, n \geq \mathbb{N}$, so (φ_n) converges uniformly on $[a, b] \setminus \{x\}$ as it is uniformly Cauchy. As $f_n \rightarrow f$ uniformly, $\varphi_n \rightarrow \varphi$ on $[a, b] \setminus \{x\}$, and as x is a limit point of $[a, b] \setminus \{x\}$,

$$\lim_{t \rightarrow x} \varphi(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} \varphi_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \varphi_n(t) = \lim_{n \rightarrow \infty} f'_n(x)$$

so that

$$f'(x) = \lim_{t \rightarrow x} \varphi(t) = \lim_{n \rightarrow \infty} f'_n(x)$$

for every $x \in [a, b]$.

2.25 Example.

If $(f_n(x_0))$ converging for some $x_0 \in [a, b]$ is not given, then (f_n) may not even converge. Consider the $(f_n) = n$ on $[0, 1]$ such that $f'_n = 0$ for every $n \in \mathbb{N}$ but f_n diverges.

2.26 Theorem.

There is a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is nowhere differentiable.

2.27 Proof.

Lorem ipsum.

2.28 Definition.

A sequence (f_n) of \mathbb{C} -valued functions on a metric space (X, d) is

- (1) pointwise bounded on X if there is some $\varphi : X \rightarrow \mathbb{R}$ such that $|f_n| \leq \varphi$ for all $n \in \mathbb{N}$.
- (2) uniformly bounded on X if there is some $M \geq 0$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$.

2.29 Remark.

Uniformly convergent implies uniformly bounded.

2.30 Remark.

If X is countable, then one can use the Cantor diagonalization argument to find a subsequence converging pointwise on X if the sequence is pointwise bounded.

2.31 Remark.

If (f_n) is uniformly bounded, it does not necessarily contain a pointwise convergent subsequence. This is shown in Rudin using the dominated convergence theorem on the sequence $(\sin(nx))$ on $[0, 2\pi]$.

2.32 Example.

Convergent uniformly bounded sequences do not necessarily contain uniformly convergent subsequences. Let (f_n) be defined by

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$$

on $[0, 1]$ for $n \in \mathbb{N}$. Then $|f_n(x)| \leq 1$ so (f_n) is uniformly bounded. Moreover, $f_n(x) \rightarrow 0$ for every $x \in [0, 1]$. But $f_n(1/n) = 1$ for every $n \in \mathbb{N}$, so no subsequence converges uniformly.

2.33 Theorem.

If (f_n) is a pointwise bounded sequence of \mathbb{C} -valued functions on a countable metric space (X, d) , then there is a subsequence (f_{n_k}) that converges for every $x \in X$.

2.34 Proof.

Enumerate $X = (x_i)$. As $(f_n(x_1))$ is bounded, there is a subsequence (f_k^1) of (f_n) such that $f_k^1(x_1)$ converges as $k \rightarrow \infty$. As $f_k^1(x_2)$ is bounded, there is a subsequence (f_k^2) of (f_k^1) such that $(f_k^2(x_2))$ converges as $k \rightarrow \infty$. Continuing inductively, consider the subsequence (f_k^{l+1}) of f_k^l such that $(f_k^{l+1})(x_{l+1})$ converges as $k \rightarrow \infty$. Choose the diagonal subsequence (f_k^k) of (f_n) such that $(f_k^k)(x_i)$ converges for every $i \in \mathbb{N}$ since (f_k^k) is a subsequence of (f_k^i) for $k \geq i$ by construction. Reindexing $(f_k^k) = (f_{n_k})$ for every $k \in \mathbb{N}$ gives the desired subsequence.

$$\begin{array}{c|cccc} x_1 & f_1^1 & f_2^1 & f_3^1 & \cdots \\ x_2 & f_1^2 & f_2^2 & f_3^2 & \cdots \\ x_3 & f_1^3 & f_2^3 & f_3^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

The i th row converges for x_i . Every row is a subsequence of the row above. The diagonal subsequence is such that $(f_k^k(x_i))$ is convergent for every $i \in \mathbb{N}$.

2.35 Definition.

A collection \mathcal{F} of \mathbb{C} -valued functions on a set $E \subseteq X$ for a metric space (X, d) is (uniformly) equicontinuous on E if for every $\varepsilon > 0$, there is some $\delta > 0$ such that for any $x, y \in E$, $d(x, y) < \delta$ implies that

$$|f(x) - f(y)| < \varepsilon$$

for every $f \in \mathcal{F}$.

2.36 Remark.

If \mathcal{F} is equicontinuous, then every $f \in \mathcal{F}$ is uniformly continuous.

2.37 Example.

(1) If $\mathcal{F} = (f_n)$, a sequence of differentiable functions on $[0, 1]$ with (f'_n) uniformly bounded, then \mathcal{F} is equicontinuous. Indeed, if $|f'_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$, then for $\varepsilon > 0$, choose $\delta = \varepsilon/(M + 1)$ so that by the Mean Value Theorem,

$$|x - y| < \frac{\varepsilon}{M + 1}$$

implies that

$$|f_n(x) - f_n(y)| \leq |x - y| \sup_{t \in [0, 1]} |f'_n(t)| < \frac{\varepsilon M}{M + 1} < \varepsilon.$$

(2) The sequence

$$\mathcal{G} = \left(\frac{x^2}{x^2 + (1 - nx)^2} \right)$$

on $[0, 1]$ is uniformly bounded, converges pointwise to 0, but has no uniformly convergent subsequence. \mathcal{G} is not equicontinuous as $g_n(1/n) = 1$ but $g_n = 0$ for every $n \in \mathbb{N}$, so there is no $\delta > 0$ for $\varepsilon \in (0, 1)$.

(3) $\mathcal{H} = (\arctan(nx))$ is not equicontinuous since $\arctan(nx) \rightarrow \pm\pi/2$ if $x > / < 0$.

2.38 Theorem.

Let (K, d) be compact and $(f_n) \subseteq \mathcal{C}(K)$. Then if (f_n) converges uniformly on K , then (f_n) is equicontinuous on K .