AS.110.415 Honors Analysis I

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Fall 2025

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0.0 Section.

Introduction.

1.0 Section.

The real numbers.

1.1 Definition.

Denote the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$.

1.2 Remark.

An axiomatic treatment of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ is beyond the scope of this course.

1.3 Remark.

For many purposes, the rationls are not "big enough".

1.4 Example.

There is no $x \in \mathbb{Q}$ such that $x^2 = 2$. Indeed, assume there are $p, q \in \mathbb{Z}$ coprime such that $(p/q)^2 = 2$. Then $p^2 = 2q^2$, which means that p^2 is even, so p is even as well. But if p is even, then $p^2 = 2q^2$ is divisible by 4. But this is only possible if q is even as well, a contradiction of the coprime-ness of p, q.

1.5 Example.

In fact, there is not even a "best approximation" in $\mathbb{Q}$ of the solution to $x^2 = 2$. Consider the sets

$$A = \{a \in \mathbb{Q} \mid a^2 < 2, a > 0\} \text{ and } B = \{b \in \mathbb{Q} \mid b^2 > 2, b > 0\}.$$

A has no largest element. Indeed, suppose $p \in A$ with $p > 0$. Then setting

$$q = \frac{2p + 2}{p + 2}$$

has that $q \in A$ and $q > p$. A similar argument shows that B has no smallest element.

1.6 Definition.

An order on a set S is a relation $<$ such that

(1) (Trichotomy) For $x, y \in S$, exactly one of the statements hold:

$$x < y \text{ or } x = y \text{ or } y < x.$$

(2) (Transitivity) For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

If a set S is equipped with an order $<$, then $(S, <)$ or simply S is called an ordered set.

1.7 Example.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all ordered by $x < y$ iff $y - x$ is positive.

1.8 Definition.

Let S be an ordered set. $E \subseteq S$ is bounded above if there exists some $M \in S$ such that $x \leq B$ for all $x \in E$. Similarly, $E \subseteq S$ is bounded below if there exists some $L \in S$ such that $x \geq B$ for all $x \in E$.

1.9 Example.

$\mathbb{N}$ has the well-ordering principle, in that every $E \subseteq \mathbb{N}$ nonempty has a least element.

1.10 Definition.

Let S be an ordered set and let E be bounded above. If there exists an α such that

(1) α is an upper bound of E,

(2) If $\beta < \alpha$, then β is not an upper bound of E,

then $\alpha = \sup E$ is the supremum or least upper bound of E. Define symmetrically $\inf E$ to be the infimum or greatest lower bound of E.

1.11 Example.

A supremum/infimum, when it exists, need not be a member of the subset.

(1) The sets $A, B \subseteq \mathbb{Q}$ defined in Example 1.4 have no supremum or infimum, respectively.

(2) $\mathbb{N} \subseteq \mathbb{Z}$ is bounded below but has no least upper bound.

(3) $E = \{1/n \mid n \in \mathbb{Z}_+\}$ has that $\inf E = 0 \notin E$.

1.12 Definition.

An ordered set has the least upper bound property if whenever $E \subseteq S$ is nonempty and bounded above, then $\sup E$ exists. Define symmetrically the greatest lower bound property.

1.13 Example.

$\mathbb{N}, \mathbb{Z}$ have the greatest upper bound property. $\mathbb{Q}$ does not.

1.14 Theorem.

Let S be an ordered set with the least upper bound property and let $B \subseteq S$ is nonempty and bounded below. Let L be the set of all lower bounds of B. Then $\sup L$ exists and is equal to $\inf B$, which exists.

1.15 Proof.

$L \neq \emptyset$ since B is bounded below. Since $y \leq x$ for every $y \in L, x \in B$, then every $x \in B$ is an upper bound of L. Thus L is bounded above since B is nonempty, so $\sup L$ exists by the least upper bound property. Since every $x \in B$ is an upper bound of L, $\sup L \le x$ for every $x \in B$, so $\sup L$ is a lower bound of B. For any lower bound $y \in L$ of B, $y \le \sup L$, so $\sup L = \inf B$.

1.16 Remark.

A set S has the least upper bound property iff it has the greatest lower bound property.

1.17 Definition.

A field $(F, +, \cdot)$ is a set F equipped with two binary operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ that satisfy the field axioms: For any $x, y, z \in F$,

(1) (Closure) $x + y \in F$ and $x \cdot y \in F$.

(2) (Commutativity) $x + y = y + x$ and $x \cdot y = y \cdot x$.

(3) (Associativity) $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(4) (Identity) There exists two symbols $0, 1 \in F$ with $0 \neq 1$ such that $0 + x = x$ and $1 \cdot x = x$.

(5) (Inverse) There exists $-x \in F$ such that $x + (-x) = 0$ for any x, and there exists $1/x \in F$ such that $x \cdot 1/x = 1$ for any $x \neq 0$.

(6) (Distibutivity) $x \cdot (y + z) = xy + xz$.

1.18 Example.

(1) $\mathbb{Q}$ with $+$ and $\cdot$ defined normally as a field.

(2) $\{0, 1\}$ is the trivial field.

(3) $\mathbb{Z}$ is not a field because it does not have multiplicative inverses.

1.19 Remark.

Fields are interesting because any statement proven about a general field F must hold in any field such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (to be defined later).

1.20 Definition.

An ordered field is a field F equipped with an order $<$ such that for $x, y, z \in F$,

(1) $x + y < x + z$ if $y < z$.

(2) $xy > 0$ if $x, y > 0$.

$x > 0$ is said to be positive and $x < 0$ is said to be negative.

1.21 Theorem.

There exists an ordered field $\mathbb{R}$ with the least upper bound property. Moreover, $\mathbb{Q} \subseteq \mathbb{R}$. The elements of $\mathbb{R}$ are called the real numbers.

1.22 Proof.

The proof is delayed until the end of the course (though the tools already exist to prove it).

1.23 Theorem.

$\mathbb{R}$ has the follow properties:

(1) (Archimedean property) Lorem ipsum.

(2) (Density of $\mathbb{Q}$) Lorem ipsum.

(3) (Existence of roots) Lorem ipsum.

1.24 Proof.

Lorem ipsum.

2.0 Section.

Sequences and series of functions.

2.1 Definition.

Given a sequence of (f_n) of $\mathbb{C}$-valued functions on a metric space (X, d) such that $\lim f_n(x)$ exists for every $x \in X$, then define the limit $\lim f_n$ to be the function $f : X \to \mathbb{C}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every $x \in X$. (f_n) is said to converge pointwise to f

2.2 Remark.

Do limits/sums of functions preserve the properties of the sequence? If (f_n) is a sequence of continuous/differentiable functions, then is the limit/sum continuous/differentiable? Moreover, is (f_n') related to f'?

2.3 Example.