Metropolis-Hastings Algorithm. Suppose that we have a Markov chain with transition matrix P with a limiting distribution π. A sufficient condition for such to hold is (1) *detailed balance*

$$P(x)P(x' \mid x) = P(x')P(x \mid x')$$

and (2) *ergodicity*, i.e., that the chain be aperiodic and positive recurrent. We have that

$$\begin{aligned} P(x' \mid x)P(x) &= P(x' \mid x)P(x) \\ \frac{P(x' \mid x)}{P(x \mid x')} &= \frac{P(x')}{P(x)}. \end{aligned}$$

Denote $g(x' \mid x)$ the proposal distribution and $A(x', x)$ the acceptance probability. We then have that

$$P(x' \mid x) = g(x' \mid x)A(x', x),$$

so

$$\frac{A(x', x)}{A(x, x')} = \frac{P(x')}{P(x)} \frac{g(x' \mid x)}{g(x \mid x')}.$$

Take the choice

$$A(x', x) = \min\left(1, \frac{P(x')}{P(x)} \frac{g(x' \mid x)}{g(x \mid x')}\right).$$

Observe then that

$$0 \leq A(x', x) \leq 1.$$

The Metroplis-Hastings algorithm is then as follows.

(1) Intialize.
   (i) Pick an initial state x_0.
   (ii) Set $t = 0$.
(2) Iterate.
   (i) Generate a random candidate state x' according to $g(x' \mid x_t)$.
   (ii) Calculate the acceptance probability $A(x', x_t)$.
   (iii) Generate a uniform random number $u \in [0, 1]$.
   (iv) If $u \leq A(x', x_t)$, then accept the new state and set $x_{t+1} = x'$.
   (v) Otherwise, reject the new state and set $x_{t+1} = x_t$.
   (vi) Increment $t = t + 1$.

In particular, we are interested in sampling from the Gibbs distribution

$$\pi(x) = \frac{1}{Z_\beta} e^{-\beta H(x)}$$

for $x \in S$ where Z_β is the partition function

$$Z_\beta = \sum_{y \in S} e^{-\beta E(y)}$$

for an energy function $E : S \to \mathbb{R}$ and an inverse temperature $\beta > 0$. Recall that we have the likelihood function

$$L(\sigma) = P(\sigma^{-1}(b_1)) \prod_{j=1}^{n-1} Q(\sigma^{-1}(b_{j+1}) \mid \sigma^{-1}(b_j)).$$

Define the energy function

$$\begin{aligned} E(\sigma) &= -\log L(\sigma) \\ &= -\log P(\sigma^{-1}(b_1)) - \sum_{j=1}^{n-1} \log Q(\sigma^{-1}(b_{j+1}) \mid \sigma^{-1}(b_j)). \end{aligned}$$