1 Multivariate Model for scores

Given a model for Y, the above gives a relevant functional model. We consider the problem here on how to model the scores Y.

Following Stilian?s formulation, when $d = 1$ and s and t are scalar, we consider

$$Y(t) = \int_{\mathbb{R}} e^{itx}((1+ix)^{-\nu-1/2}A1_{\{x>0\}} + (1+ix)^{-\nu-1/2}\overline{A}1_{\{x<0\}}\tilde{B}(dx)$$

where $\tilde{B}(dx)$ is $\mathbb{C}^k$-valued Brownian motion with

$$\tilde{B}(x) = \overline{\tilde{B}(-x)} \qquad\qquad \mathbb{E}[\tilde{B}(dx)\tilde{B}(dx)^*] = \mathbb{I}_k dx$$

and A is a complex-valued matrix. When $\nu = \nu\mathbb{I}_k$ is a scalar, this above integral gives the covariance

$$\mathbb{E}[Y(t)Y(s)^\top] = \int_{\mathbb{R}} e^{i(s-t)x}(1+x^2)^{-\nu-1/2}(AA^*1_{\{x>0\}} + \overline{AA}^*1_{\{x<0\}})dx.$$

2 Simple Model

We consider a more simple model and aim to derive its validity for any dimension d. In particular, we let AA^* be a constant matrix. Also, specify a hyperplane that goes through the origin in $d-1$ dimensions that is the plane of reflection of the nonreversibility. When a vector lies on the hyperplane, the model is reversible; when one lies perpendicular to the hyperplane, the model is at its most nonreversible direction. This formulation is considerably less flexible than the model described above with $\sigma(\theta)$ and $AA^\star(\theta)$.

2.1 Formulation for general d

Let $\boldsymbol{h}, \boldsymbol{\omega}$ be vectors in $\mathbb{R}^d$, and let $AA^* = R + iM$ where R is a $k \times k$ real positive definite matrix and $M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$ for some $m \in \mathbb{R}$. Let $\boldsymbol{a}$ be a vector in $\mathbb{R}^d$ that describes the plane through the origin for which the non-reversibility is reflected, defined by all $\boldsymbol{\omega}$ such that $\boldsymbol{a}^\top\boldsymbol{\omega} = 0$.

We want to consider the covariance of

$$C(\mathbf{0}, \boldsymbol{h}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^\top\boldsymbol{h}}(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}}\left(AA^*1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}>0\}} + \overline{AA^*}1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}<0\}}\right)d\boldsymbol{\omega}$$

Plugging in $AA^* = R + iM$ gives

$$C(\mathbf{0}, \boldsymbol{h}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^\top\boldsymbol{h}}(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}}\left(R + iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}>0\}} - iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}<0\}}\right)d\boldsymbol{\omega}$$

and by breaking up the integral we have

$$\begin{aligned}C(\mathbf{0}, \boldsymbol{h}) = &\int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top\boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}}\left(R + iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}>0\}} - iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}<0\}}\right)d\boldsymbol{\omega} \\ &+ i\int_{\mathbb{R}^d} \sin(\boldsymbol{\omega}^\top\boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}}\left(R + iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}>0\}} - iM1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}<0\}}\right)d\boldsymbol{\omega}.\end{aligned}$$

Using the even and odd properties of the cosine and sine functions, respectively, gives

$$C(\mathbf{0}, \boldsymbol{h}) = R\int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top\boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}}d\boldsymbol{\omega}$$

$$
\begin{aligned}
&+ i^2 M \int_{\mathbb{R}^d} \sin(\boldsymbol{\omega}^\top \boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}} \left(1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}>0\}} - 1_{\{\boldsymbol{a}^\top\boldsymbol{\omega}<0\}}\right) d\boldsymbol{\omega} \\
= {} & R \int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top \boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}} d\boldsymbol{\omega} \\
&- 2M \int_{\boldsymbol{\omega}|\boldsymbol{a}^\top\boldsymbol{\omega}>0} \sin(\boldsymbol{\omega}^\top \boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}} d\boldsymbol{\omega}.
\end{aligned}
$$

When M is the 0 matrix, each component is Matern, and the above is a simplified version of Gneiting et al (2010) with scale parameter 1, which is a valid covariance iff R is positive definite and $\nu > 0$. The derivation of the Matern covariance in the univariate case in arbitrary dimension is given in Stein 1999 *Interpolation of Spatial Data*.

Therefore, in the following, we focus on evaluating the second integral:

$$
\int_{\boldsymbol{\omega}|\boldsymbol{a}^\top\boldsymbol{\omega}>0} \sin(\boldsymbol{\omega}^\top \boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-\frac{d}{2}} d\boldsymbol{\omega} \tag{1}
$$

2.2 $d = 1$

Consider the case $d = 1$ where ω and h are scalars and we have

$$
\begin{aligned}
&\int_{\omega>0} \sin(\omega h)(1+\omega^2)^{-\nu-\frac{1}{2}} d\omega \\
&= \operatorname{sign}(h)|h|^\nu 2^{-\nu-1}\sqrt{\pi}\Gamma(-\nu+1/2)\left(I_\nu(|h|) - \boldsymbol{L}_{-\nu}(|h|)\right)
\end{aligned}
$$

given on page 332 of Watson: *Theory of Bessel Functions* and I_ν is the modified Bessel function and $\boldsymbol{L}_\nu$ is the modified Struve function. Note that the above is undefined for $\nu = 1/2, 3/2, \ldots$ due to the gamma function, where we have extended the gamma function to the negative nonintegers. $\boxed{\text{What happens when } \nu = 1/2, 3/2, \ldots?}$

2.3 $d = 2$

For the 2-dimensional case, we switch to polar coordinates. Note that $\sin(\boldsymbol{\omega}^\top\boldsymbol{h}) = \sin(r\,\|\boldsymbol{h}\|\cos(\theta))$ where θ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{h}$ and $r = \|\boldsymbol{\omega}\|$. Assume these polar coordinates, and let c be the angle that $\boldsymbol{h}$ makes with the x axis. Finally, assume without loss of generality that $a = (0,1)^\top$ so that we integrate over the upper half of $\mathbb{R}^2$. See the figure below for a visual representation. Then the integral (1) is

$$
\begin{aligned}
&\int_{\boldsymbol{\omega}|\boldsymbol{a}^\top\boldsymbol{\omega}>0} \sin(\boldsymbol{\omega}^\top \boldsymbol{h})(1+\boldsymbol{\omega}^\top\boldsymbol{\omega})^{-\nu-1} d\boldsymbol{\omega} \\
&= \int_0^\infty \int_{-\pi+c}^{c} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))(1+r^2)^{-\nu-1} r\, d\theta\, dr.
\end{aligned}
$$

Now, by 12.1.7 of Abramowitz and Stegun *Handbook of Mathematical Functions*,

$$
\int_0^{\pi/2} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))d\theta = \frac{\pi}{2}\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|)
$$

where $\boldsymbol{H}_\nu$ is the Struve function. One can further see that

$$
\int_{\pi/2}^{\pi} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))d\theta = -\frac{\pi}{2}\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|) \qquad \int_{-\pi/2}^{0} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))d\theta = \frac{\pi}{2}\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|)
$$

based on properties of sin and cos. We can now evaluate for two values of c:

1. $c = \pi$: the above integral is 0, which makes sense since this is the reversible direction as defined. This non-reversible part of the covariance is 0.

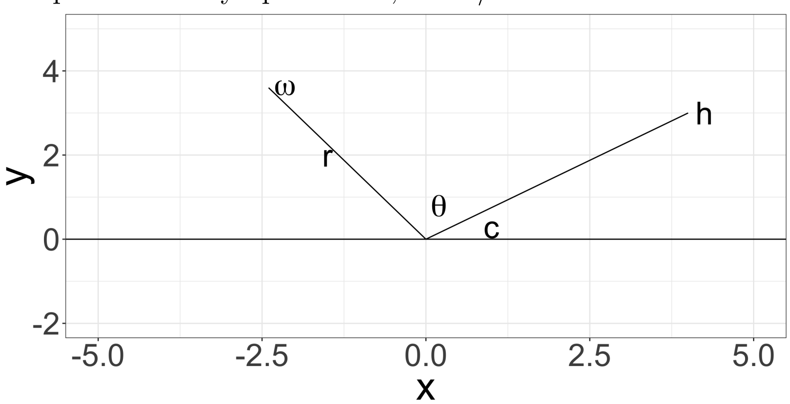
2. $c = \pi/2$: We have

$$\int_0^\infty \int_{-\pi/2}^{\pi/2} \sin(r \|\boldsymbol{h}\| \cos(\theta))(1+r^2)^{-\nu-1} r d\theta dr = \int_0^\infty \pi \boldsymbol{H}_0(r\|\boldsymbol{h}\|)(1+r^2)^{-\nu-1} r dr$$
$$= \pi \frac{2^{-\nu-1}\pi \|\boldsymbol{h}\|^\nu}{\Gamma(\nu+1)\cos(\nu\pi)} \left(I_\nu(\|\boldsymbol{h}\|) - \boldsymbol{L}_{-\nu}(\|\boldsymbol{h}\|)\right) \tag{2}$$

by 6.814 of I.S. Gradshteyn and I.M. Ryzhik *Tables of Integrals, Series, and Products.* This gives the non-reversible part perpendicular to the reversible direction. It looks similar to the 1-d case.

However, this does not work in general because $c \notin \{\pi, \pi/2\}$ necessarily.

An example of what is going on below. For a fixed $\boldsymbol{h}$, one integrates $\boldsymbol{\omega}$ against the upper half plane. When $\boldsymbol{h}$ is in line with the x axis, the integral is 0 and the model is reversible across this axis; when $\boldsymbol{h}$ points directly up or down, $c = \pi/2$ and the model is at its most non-reversible direction.



By (1.8) on page 24 of *Theory of incomplete cylindrical functions and their applications* by M. M. Agrest M. S. Maksimov, we have

$$\int_0^c \sin(r\|\boldsymbol{h}\|\cos(\theta))d\theta = \frac{\pi}{2}\boldsymbol{H}_0(c, r\|\boldsymbol{h}\|)$$

where $\boldsymbol{H}_\nu(c,z)$ is the incomplete Struve function. Thus, we are left to evaluate

$$\int_0^\infty \frac{\pi}{2}\left(\boldsymbol{H}_0(c, r\|\boldsymbol{h}\|) + \boldsymbol{H}_0(\pi - c, r\|\boldsymbol{h}\|)\right)(1+r^2)^{-\nu-1} r dr$$

By (5.10) on page 183 of *Theory of incomplete cylindrical functions and their applications* and setting $\mu = 2$ and $\nu = 0$, we have

$$\int_0^\infty \frac{-\boldsymbol{H}_0(c, ax)}{(x^2+k^2)^{m+1}} x dx = \frac{\pi}{m!}(-1)^{m+1}\left(\frac{d}{dk^2}\right)^m F_0^-(c, ak)$$

where

$$F_0^-(c, ak) = \left\{\frac{I_0(c, ak) - \boldsymbol{L}_0(c, ak)}{2}\right\}$$

and $I_0(\cdot,\cdot)$ is the incomplete modified Bessel function of the first kind and $\boldsymbol{L}_0(\cdot,\cdot)$ is the incomplete modified Struve function. When $c=\pi/2$ these reduce to their "complete" versions.

$\boxed{\textbf{special case of } \nu = 0}$ Consider the special case where $\nu = 0$. Then, the entire integral is

$$\begin{aligned}&\int_0^\infty \frac{\pi}{2}\left(\boldsymbol{H}_0(c, r\|\boldsymbol{h}\|) + \boldsymbol{H}_0(\pi - c, r\|\boldsymbol{h}\|)\right)(1+r^2)^{-1} r dr \\ &= \frac{\pi}{2}\frac{\pi}{2}\left(I_0(c,\|\boldsymbol{h}\|) - \boldsymbol{L}_0(c,\|\boldsymbol{h}\|) + I_0(\pi - c,\|\boldsymbol{h}\|) - \boldsymbol{L}_0(\pi - c,\|\boldsymbol{h}\|)\right)\end{aligned}$$

When $c = \pi/2$, we reduce to

$$\frac{\pi^2}{2}\left(I_0(\|\boldsymbol{h}\|) - \boldsymbol{L}_0(\|\boldsymbol{h}\|)\right)$$

which matches with (2). When $c = \pi$, this reduces to 0 as expected.

$\boxed{\textbf{different values of } \nu}$

Assuming that fractional derivatives exist and that we can replace the factorial with the gamma function, we have that , we have

$$\int_0^\infty \frac{-\boldsymbol{H}_0(c, r\|\boldsymbol{h}\|)}{(1+r^2)^{\nu+1}} r dr = \frac{\pi}{\Gamma(\nu+1)}(-1)^{\nu+1}\left(\frac{d}{d\|\boldsymbol{h}\|}\right)^{2\nu} F_0^-(c,\|\boldsymbol{h}\|)$$

I'm not sure how to deal with $(-1)^{\nu+1}$. Consider $\nu = 1/2$. Then this integral is

$$\frac{1}{2}\frac{d}{d\|\boldsymbol{h}\|}E_0^+(c, i\|\boldsymbol{h}\|)$$

which is

$$\frac{i}{2}\left(-E_1^+(c, i\|\boldsymbol{h}\|) + \frac{2i\sin(c)}{\pi}e^{i\|\boldsymbol{h}\|\cos(c)}\right)$$

by using pages 31 and 25. This then becomes

$$\frac{i}{2}\left(-2e^{i\pi/2}F_1^-(c,\|\boldsymbol{h}\|) + \frac{2i\sin(c)}{\pi}e^{i\|\boldsymbol{h}\|\cos(c)}\right)$$

which is

$$F_1^-(c,\|\boldsymbol{h}\|) - \frac{\sin(c)}{\pi}e^{i\|\boldsymbol{h}\|\cos(c)}$$

which is

$$\frac{I_0(c,\|\boldsymbol{h}\|) - L_0(c,\|\boldsymbol{h}\|)}{2} - \frac{\sin(c)}{\pi}e^{i\|\boldsymbol{h}\|\cos(c)}$$

Now, this derivative can be evaluated using (1.22) on page 25 and (1.37) on page 27 such that

$$\begin{aligned}\left(\frac{d}{dk}\right)^{2m} F_0^-(c,ak) &= \frac{1}{2}\left(\frac{d}{dk}\right)^{2m} E_0^+(c,iak)\\ &= i^m \frac{A_m}{2\pi(ak)^m}\psi_m(c,iak)\end{aligned}$$

Therefore, the entire integral is

$$\begin{aligned}&\int_0^\infty \frac{\pi}{2}\left(\boldsymbol{H}_0(c,r\|\boldsymbol{h}\|) - \boldsymbol{H}_0(\pi-c,r\|\boldsymbol{h}\|)\right)(1+r^2)^{-\nu-1} r dr\\ &\quad= \frac{\pi}{\nu!}(-1)^{\nu+1} i^\nu \frac{A_\nu}{2\pi\|\boldsymbol{h}\|^\nu}\left(-\psi_\nu(c,i\|\boldsymbol{h}\|)+\psi_\nu(\pi-c,i\|\boldsymbol{h}\|)\right)\\ &\quad= \frac{\pi}{\nu!}(-1)^{\nu+1} i^\nu \frac{A_\nu}{\pi\|\boldsymbol{h}\|^\nu}\left(\frac{(i\|\boldsymbol{h}\|)^\nu}{A_\nu}\left(-\int_0^c e^{-\|\boldsymbol{h}\|\cos(t)}\cos^{2\nu}(t)dt + \int_0^{\pi-c} e^{-\|\boldsymbol{h}\|\cos(t)}\cos^{2\nu}(t)dt\right)\right)\\ &\quad= \frac{1}{\nu!}\left(\int_c^{\pi-c} e^{-\|\boldsymbol{h}\|\cos(t)}\cos^{2\nu}(t)dt\right)\\ &\quad= \frac{1}{\nu!}\left(\int_c^{\pi-c} e^{-\|\boldsymbol{h}\|\cos(t)}\cos^{2\nu}(t)dt\right)\\ &\quad= \frac{-2}{\nu!}\frac{A_\nu}{2\|\boldsymbol{h}\|^\nu}\left(E_\nu^-(c,\|\boldsymbol{h}\|)+E_\nu^-(\pi-c,\|\boldsymbol{h}\|)\right)\end{aligned}$$

The first step comes from using hte derivative. . The second step is using the definition of ϕ. The third step is cancelling and combining integrals.

USE PAGE 31

By (5.10) on page 183 of *Theory of incomplete cylindrical functions and their applications* and setting $\mu = 2$ and $\nu = 0$, we have

$$\int_0^\infty \frac{-\boldsymbol{H}_0(c,ax)}{(x^2+k^2)^{m+1}} x dx = \frac{\pi}{m!}(-1)^{m+1}\left(\frac{d}{dk^2}\right)^m F_0^-(c,ak)$$

where

$$F_0^-(c,ak) = \left\{\frac{I_0(c,ak) - \boldsymbol{L}_0(c,ak)}{2}\right\}$$

WE NEED AN i in both of theseand $I_0(\cdot,\cdot)$ is the incomplete modified Bessel function orf the first kind and $\boldsymbol{L}_0(\cdot,\cdot)$ is the incomplete modified Struve function. When $m = 0$, this is

$$\frac{\pi}{2}(L_0(c,ak) - I_0(c,ak)).$$

Therefore, the entire integral when $\nu = 0$ is

$$\begin{aligned}&\int_0^\infty \frac{\pi}{2}\left(\boldsymbol{H}_0(c,r\|\boldsymbol{h}\|) + \boldsymbol{H}_0(\pi-c,r\|\boldsymbol{h}\|)\right)(1+r^2)^{-1} r dr\\ &\quad= \frac{\pi}{4juhnjjnj}\left(L_0(c,\|\boldsymbol{h}\|) - I_0(c,\|\boldsymbol{h}\|) + L_0(\pi-c,\|\boldsymbol{h}\|) - I_0(\pi-c,\|\boldsymbol{h}\|)\right)\end{aligned}$$

This, when $c = \pi/2$ matches the formula for (2) where $\nu = 0$. Similarly, when $c = \pi$, the above is 0 as expected. This isn't quite right. (no h)