On the structure of vector-valued IRF_0

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1 Preliminaries

Let $\{X(s),\ s \in \mathbb{R}^d\}$ be a continuous in probability zero mean (Gaussian) random field with stationary increments taking values in $\mathbb{R}^k$. That is, such that

$$\{X(t+s) - X(s),\ t \in \mathbb{R}^d\} \stackrel{d}{=} \{X(t) - X(0),\ t \in \mathbb{R}^d\}$$

Such processes are known as IRF_0.

We shall assume that we are in the multivariate situation where

$$k \geq 2.$$

Our goal is to understand (characterize!) such classes of models under perhaps additional conditions.

Recall that rhe process X is also H-self-similar if

$$\{X(ct),\ t \in \mathbb{R}^d\} \stackrel{d}{=} \{c^H X(t),\ t \in \mathbb{R}^d\}, \quad \text{for all } c > 0.$$

1.1 The case $d = 1$

It is interesting (and it might have been done!) to completely characterize the $\mathbb{R}^k$-valued fractional Brownian motions. That is, the class of all Gaussian self-similar IRF_0 processes. This is not that easy and in principle should follow from Jinqi's work, but I am not so sure the results there are as explicit as possible. In the case when the increments are time-reversible, however, is solved in this section.

Suppose that X is an H-sssi process on the line and $X(0) = 0$. Then, it is well-known that $X_a[t] := \langle X(t), a \rangle$ is a fractional Brownian motion process for every vector $a \in \mathbb{R}^k$. It is not clear though, apriori, how these fBm's are tied together. Let

$$C(s,t) := \mathbb{E}[X(s)X(t)^\top], \quad s,t \in \mathbb{R}.$$

Observe that, for all $s, t \in \mathbb{R}$,

$$C(t-s, t-s) = \mathbb{E}[(X(t) - X(s))(X(t) - X(s))^\top] = C(t,t) + C(s,s) - C(s,t) - C(t,s). \tag{1}$$

Observe that

$$C(s,t) = \mathbb{E}[X(s)X(t)^\top] = C(t,s)^\top.$$

Condition 1.1 (Condition R)**.** *A vector-valued stationary increments process X is said to be covariance-reversible, if $C(s,t) = C(t,s)$, for all $s,t \in \mathbb{R}$. That is, the cross-covariance function is always symmetric.*

It is argued in the next section that for stationary processes, covariance-reversibility is equivalent to time (space) reversibility.

The stationarity of the increments and the fact that $X(0) = 0$ entails that

$$X(t) - X(0) \stackrel{d}{=} X(0) - X(-t) \quad \Leftrightarrow X(t) \stackrel{d}{=} -X(-t) \stackrel{d}{=} X(-t).$$

Since also the process X is Gaussian and hence we have $-X \stackrel{d}{=} X$. The above relation and the H-self-similarity entails that (just in terms of equality of the marginal distributions):

$$X(-t) \stackrel{d}{=} t^H X(-1) \stackrel{d}{=} t^H X(1) \stackrel{d}{=} X(t), \quad \forall t > 0.$$

That is,

$$C(t,t) = |t|^{2H} C(1,1) \equiv |t|^{2H} C(-1,-1) =: |t|^{2H} C_1,$$

where C is a psd matrix. These elementary observations lead us to the following.

Proposition 1.2. *Let $X = \{X(t),\ t \in \mathbb{R}\}$ be a zero-mean $\mathbb{R}^k$-valued Gaussian process with stationary increments and such that $X(0) = 0$. Assume that X is H-sssi. Then:*

(i) *We have that, for some psd $k \times k$ matrix C_1:*

$$C(s,t) + C(t,s)^\top = C_1 \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad \forall s,t \in \mathbb{R}.$$

(ii) *Condition R holds if and only if*

$$\mathbb{E}[X(s)X(t)^\top] \equiv C(s,t) = \frac{C_1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad \forall s,t \in \mathbb{R}.$$

Part **(ii)** is trivial. Condition R trivially implies or in fact, assumes away the challenges. The 'only if' part also follows trivially. This result shows, however, that the in the case when this (perhaps stringent) condition holds, the structure of the H-sssi $\mathbb{R}^k$-valued process is rather simple. Specifically, since C_1 is psd, it is diagonal in a suitable orthonormal basis. Therefore, we obtain

Corollary 1.3. *Under the conditions of Proposition 1.2.***(ii)***, we have that*

$$\{X(t),\ t \in \mathbb{R}\} \stackrel{d}{=} \left\{ \sum_{j=1}^k \sqrt{\lambda_j} f_j Z_j(t), \quad t \in \mathbb{R} \right\},$$

where $f_1, \cdots, f_k \in \mathbb{R}^k$ are orthonormal and $Z_j = \{Z_j(t),\ t \in \mathbb{R}\}$ are iid standard fBm's with Hurst parameter H. In fact, we have

$$C_1 = \sum_{j=1}^k \lambda_j f_j f_j^\top.$$

This means that, in the orthonormal basis $f_1, \cdots, f_k$, the process X consists of independent fBm's in each coordinate (with possibly different) variances. Once again, this happens if and only if the cross-covariance matrices $C(s,t) = C(t,s)$ are symmetric.

1.2 The case $d \geq 2$.

As before, we obtain $X(t) \stackrel{d}{=} X(-t)$ and therefore

$$C(t,t) = \|t\|^{2H} C(t_0, t_0), \quad \text{where } t_0 := \frac{t}{\|t\|}, \ \ t \in \mathbb{R}^k \setminus \{0\}.$$

Suppose, yet again, that $C(s,t) = C(t,s)$, i.e., Condition R holds. Then, the covariance structure of X is completely determined by the M-variogram (M, for matrix):

$$C(s,t) = \frac{1}{2} \left(\gamma(t) + \gamma(s) - \gamma(t-s) \right),$$

where

$$\gamma(t) = C(t,t) = \mathbb{E}[X(t)X(t)^\top].$$

Note that the M-variogram determines the covariance structure if and only if $C(s,t) = C(t,s)$ Condition R holds.

A covariance reversible model. This discussion suggest the following model, which is H-sssi. It is very likely that *not all* H-sssi $\mathbb{R}^k$-valued processes can be represented using this model. Namely, let

$$X(t) := \int_S \int_{\mathbb{R}} \Big((\langle t, \theta \rangle - x)_+^{H-1/2} - (-x)_+^{H-1/2} \Big) B(\theta) W(dx, d\theta),$$

where W is a vector-valued independently scattered Gaussian random measure with the standard control measure, i.e.,

$$\mathbb{E}[W(dx, d\theta) W(dy, d\eta)^\top] = \delta(x-y) dx \delta(\theta - \eta) d\theta \mathbb{I}_k.$$

Here, $B(\theta)$ is an arbitrary (measurable) $k \times k$ matrix-valued function.

2 The stationary (IRF$_{-1}$) case

Here we take an important detour to the case where X is *stationary* aka IRF$_{-1}$ and $\mathbb{R}^k$-valued.

Proposition 2.1. *The stationary zero-mean Gaussian process X is time-reversible if and only if Condition R holds.*

Remark. Before we proceed, observe that in the scalar case ($k = 1$) Condition R always holds and hence all 1D stationary Gaussian random field are (time) reversible.

Proof. 'if' part: Assume Condition R. By Gaussianity, to prove reversibility, i.e.,

$$\{X(t), \ t \in \mathbb{R}^d\} \stackrel{d}{=} \{X(-t), \ t \in \mathbb{R}^d\}$$

it is enough to show that

$$\mathbb{E}[X(t)X(s)^\top] = \mathbb{E}[X(-t)X(-s)^\top], \quad \text{for all } s, t \in \mathbb{R}^d. \tag{2}$$

Stationarity entails however that

$$\mathbb{E}[X(t)X(s)^\top] = \mathbb{E}[X(t-s)X(0)^\top] \quad \text{and} \quad \mathbb{E}[X(-t)X(-s)^\top] = \mathbb{E}[X(0)X(t-s)^\top].$$

Condition R implies, however, that $\mathbb{E}[X(t-s)X(0)^\top] = \mathbb{E}[X(0)X(t-s)^\top]$, which in view of the last display yields (2).

Conversely, if (2) holds, then by reversing the above argument, one concludes that $\mathbb{E}[X(h)X(0)^\top] = \mathbb{E}[X(0)X(h)^\top]$ for all $h \in \mathbb{R}^d$ (by taking $h := t-s$). This, in view of the stationarity of X, yields Condition R. $\square$

Remark. This result shows (or rather suggests) that there are *non-reversible* $\mathbb{R}^k$-valued stationary Gaussian processes. This does not seem to contradict, however, the classic Bochner representation theorem that we have in our IRF paper (Section 5). There may be some issues with Theorem 5.4 (and possibly Theorem 5.5). The main issue may be with Definition 5.1, where we explicitly assume the generalized covariance to be *symmetric*. In the $\mathbb{R}^k$-valued and certainly $\mathbb{H}$-valued case, however, the cross covariance function

$$C(t,s) = C(t-s,0) = \mathbb{E}[X(t-s)X(0)^\top] \equiv \mathcal{K}(t-s)$$

need not be symmetric. That is, $\mathcal{K}(-h) = \mathcal{K}(h)$ may not always hold. Indeed, Proposition 2.1 implies that the symmetry of the covariance function is equivalent to time-reversibility, which we need not have in general (I guess). Having looked at the Bochner's theorem, I see nothing wrong with it. That is, one can still have

$$\mathcal{K}(h) = \int e^{ih^\top x}\mu(dx), \quad h \in \mathbb{R}^d,$$

for some $\mathbb{T}_+$-valued measure μ. **But** the notion of generalized covariance function needs to be extended. Specifically, I don't think we should assume symmetry. Then, I am not very confident in Theorems 5.4 and 5.5, but the proofs may just be valid if we **fix Definition 5.1**. **Note:** The spectral measure in Theorems 5.4 and 5.5 should not (and it is not!) assumed to be symmetric.

3 The local structure of stationary processes through an example

Suppose that $X = \{X(t),\ t \in \mathbb{R}^d\}$ is an $\mathbb{R}^k$-valued stationary zero-mean Gaussian process. Suppose that

$$C(t,s) = \mathbb{E}[X(t) \otimes X(s)] \equiv \mathbb{E}[X(t-s)X(0)^\top] = \mathcal{K}(t-s), \quad t,s \in \mathbb{R}^d$$

is the covariance function of X. Here $\mathcal{K}(\cdot)$ is a $(k \times k)$-matrix-valued psd function. That is,

$$\sum_{i,j} \langle a_i, K(t_i - t_j)a_j\rangle \equiv \sum_{i,j} \overline{a}_i^\top \mathcal{K}(t_i - t_j)a_j \geq 0,$$

for all choices of $a_i \in \mathbb{C}^k$ and $t_i \in \mathbb{R}^d,\ i = 1, \ldots, d$. This can be readily shown by noting that

$$\mathrm{Var}\left(\sum_{i=1}^n \overline{a}_i^\top X(t_i)\right) = \sum_{i,j=1}^n \overline{a_i}^\top \underbrace{\mathbb{E}[X(t_i)X(t_j)^\top]}_{\mathcal{K}(t_i - t_j)} a_j \geq 0.$$

Assuming that $\mathcal{K}(\cdot)$ is continuous at 0, an extension of the classic Bochner theorem entails:

$$\mathcal{K}(h) = \int_{\mathbb{R}^d} e^{ih^\top x}\mu(dx),$$

where $\mu(dx)$ is a $(k \times k)$-matrix-valued measure, such that

$$\mu(A) = \overline{\mu}(-A) \quad \text{and} \quad \mu(A) \text{ is Hermitian and psd},$$

for all Borel $A \subset \mathbb{R}^d$. Recall that a complex-valued matrix B is said to be Hermitian if $B = \overline{B}^\top$. It is psd iff $\overline{a}^\top Ba \geq 0$, for all $a \in \mathbb{C}^k$.

Observe that if $\mathcal{K}(h)$ is real, then

$$\mathcal{K}(h)^\top = \overline{\mathcal{K}}(h)^\top = \int e^{-ih^\top x}\nu(dx),$$

where $\nu(dx) = \overline{\mu}(dx)^\top$. Since $\mu(A) = \overline{\mu}(A)^\top$, then it follows that $\mu = \nu$, which, as expected, yields

$$\mathcal{K}(h)^\top = \mathcal{K}(-h).$$

Now, if the measure μ was also symmetric, i.e., $\mu(A) = \mu(-A)$, then we would have

$$\mathcal{K}(-h) = \mathcal{K}(h).$$

Notice that in this case since $\mu(A) = \overline{\mu}(-A) = \overline{\mu}(A)$, we get that μ takes real values. This essentially completes the proof of the following.

Corollary 3.1. *The process X is reversible, i.e., $\mathcal{K}(h) = \mathcal{K}(-h)$, $\forall h \in \mathbb{R}^d$, if and only if the spectral measure μ is symmetric and afortori, $\mathbb{R}^{k\times k}$-valued.*

Some asymptotics. WLOG, let $K(0) = \mathbb{I}_k$ and suppose that

$$\lim_{c\downarrow 0} \frac{\mathbb{I}_k - \mathcal{K}(ch)}{r(c)} = \gamma(h), \tag{3}$$

for some $r(c) \downarrow 0$ as $c \downarrow 0$, where $\gamma(\cdot)$ is a $(k \times k)$-valued function.

The psd-ness of $\mathcal{K}$ implies the negative-definiteness of γ.

Definition 3.2. A function $\gamma(\cdot)$ is said to be negative definite, if

$$\sum_{i,j} \overline{a}_i^\top \gamma(t_i - t_j)a_j \leq 0, \tag{4}$$

for all $a_i \in \mathbb{C}^k$, such that $\sum_i a_i = 0$.

It is easy to see that for a_i's as in the above definition, we have

$$\sum_{i,j} \overline{a}_i^\top (\mathbb{I}_k - \mathcal{K}(t_i - t_j))a_j = -\sum_{i,j} \overline{a}_i^\top \mathcal{K}(t_i - t_j)a_j \leq 0,$$

where the latter inequality follows from the psd-ness of $\mathcal{K}$ and the former by the fact that $\sum_i a_i = 0$. Since the above is non-positive, so is its limit (normalized by $r(c) > 0$). Thus, (4) holds, i.e., γ is nsd.

Consequences: It can be shown that the following are necessarily true:

- If γ is non-trivial, it follows that $r(c) = c^\alpha \ell(c)$ for some

$$0 < \alpha \le 2$$

and a slowly varying function ℓ.

- The limit γ is α-homogeneous, i.e.

$$\gamma(ch) = c^\alpha \gamma(h), \quad \forall c > 0,\ h \in \mathbb{R}^d.$$

- As shown γ is *negative semidefinite* (nsd).

Tangent fields. Consider the increments $Y(t) := X(t) - X(0)$ and let us compute their cross-covariance

$$C_Y(t,s) := \mathbb{E}[Y(t) \otimes Y(s)] = \mathcal{K}(t-s) - \mathcal{K}(t) - \mathcal{K}(-s) + \mathcal{K}(0)$$

Thus, by adding and subtracting $\mathcal{K}(0) = \mathbb{I}_k$, in view of (3), we obtain

$$\frac{1}{r(c)} C_Y(ct, cs) \to \gamma(t) + \gamma(-s) - \gamma(t-s), \quad \text{as } c \downarrow 0.$$

(Note that $\mathcal{K}(-s) = \mathcal{K}(s)^\top$ and hence $\gamma(-s) = \gamma(s)^\top$.) The convergence of the covariance function and Gaussianity entail process convergence in the sense of fdd's. Namely, we have

$$\left\{ \frac{1}{r(c)}(X(ct) - X(0)),\ t \in \mathbb{R}^d \right\} \xrightarrow{fdd} \{Z(t),\ t \in \mathbb{R}^d\}, \qquad \text{as } c \downarrow 0,$$

where Z is a zero-mean Gaussian process taking values in $\mathbb{R}^k$ with cross-covariance:

$$C_Z(t,s) = \gamma(t) + \gamma(s)^\top - \gamma(t-s), \quad t, s \in \mathbb{R}^d. \tag{5}$$

(Sanity check: verify that the above formula does indeed satisfy $C_Z(s,t) = C_Z(t,s)^\top$. Since $C_Z(t,s)$ is a limit cross-covariance, it is also a valid cross-covariance function itself.)

The limit process Z is said to be the *tangent process* of X at 0. By stationarity [of X], the tangent process will have the same distribution if instead at 0 we zoomed-in at another point u and considered the local asymptotic behavior of $(X(ct+u) - X(u))$, as $c \downarrow 0$. Since γ is α-homogeneous, it necessarily follows that the tangent process Z is H-self-similar with $H := \alpha/2$.

Indeed, to prove that $\{Z(ct),\ t \in \mathbb{R}^d\} \stackrel{d}{=} \{c^H Z(t),\ t \in \mathbb{R}^d\}$ it is enough to show that

$$C_Z(ct, cs) = c^{2H} C_Z(t,s), \quad \forall c > 0, t, s \in \mathbb{R}^d.$$

This, however, automatically follows from the fact that $\gamma(ct) = c^{2H}\gamma(t)$ for all $c > 0$ and $t \in \mathbb{R}^d$.

The tangent process Z has also stationary increments. Indeed, for all $s, t, h \in \mathbb{R}^d$, using (5), we obtain

$$\begin{aligned}
\mathbb{E}(Z(t+h) - Z(h)) \otimes (Z(s+h) - Z(h)) &= \gamma(t+h) + \gamma(s+h)^\top - \gamma(t-s) + \gamma(h) + \gamma(h)^\top \\
&\quad - \gamma(h) - \gamma(s+h)^\top + \gamma(s)^\top - \gamma(t+h) - \gamma(h)^\top + \gamma(t) \\
&= \gamma(t) + \gamma(s)^\top - \gamma(t-s) \equiv C_Z(t,s).
\end{aligned}$$

This calculation shows that the covariance function of $\{Z(t+h) - Z(h),\ t \in \mathbb{R}^d\}$ is the same as that of Z and hence (by Gaussianity) it follows that Z has stationary increments.

Observe that since $\gamma(0) = 0$ and γ is continuous at 0 (must be verified),

$$\mathbb{E}[Z(t)Z(t)^\top] = \gamma(t) + \gamma(-t).$$

Note that, given the developments in the previous sections, $\gamma(t) = \gamma(-t) \equiv \gamma(t)^\top$ if and only if Z satisfies Condition R. In particular, if X is reversible (satisfies Condition R), then $\mathcal{K}(h) = \mathcal{K}(-h) \equiv \mathcal{K}(h)^\top$ and hence $\gamma(t) = \gamma(t)^\top$. In this case, the structure of the limit process in the case when $d = 1$ is established in Proposition 1.2.

Comments and questions to be addressed:

- As we have seen, under the rather general condition (3), we have that the stationary $\mathbb{R}^k$-valued Gaussian process X has a tangent process Z, which is H-sssi, where $H = \alpha/2$. This underscores the importance of the class of self-similar processes with stationary increments (i.e., self-similar IRF_0). Basically, locally very many spatial models look like fractional Brownian sheets.

- If the process X is *smooth*, then condition (3) does not hold. In this case, higher order or generalized increments can be taken to study the local behavior of the process leading to self-similar IRF_k models. For simplicity, we stick with the IRF_0 case here.

- In the vector-valued case, the notion of reversibility plays an important role. In the special case when X (or the tangent process Z) satisfy Condition R, then can one characterize more explicitly (through convenient stochastic integral representations) all possible tangent processes (fBm's). This should follow from the existing spectral representations, but needs to be written up. Specifically, an efficient method for the simulation of such fBm's could be developed that extends the scalar version.

- Understand the non-reversible case when $d = 1$. Then, it can be used as in the previous point to provide a turning-band-type representation of a general $\mathbb{R}^k$ (or eventually Hilbert-space)-valued fBm. A certain notion of skewness should play a role.

- **Question:** In the case $d = 1$, we have that $\gamma(t) = t^{2H}C$ for all $t > 0$, where the matrix C is such that $C + C^\top$ is psd. Does every choice of such C lead to a valid covariance?

4 Operator Matérn

This is inspired by the characterization results of the operator fractional Brownian motion by Didier and Pipiras [**?**]. The focus here (for now) is on the case $d = 1$. Let $H \in M_k(\mathbb{R})$, i.e., H is an $k \times k$ real matrix and for all $c > 0$ interpret c^H as $\exp\{\log(c)H\}$, where the matrix exponent is defined in the usual way.

An $\mathbb{R}^k$-valued stochastic process $X = \{X(t),\ t \in \mathbb{R}\}$ is said to be operator self-similar with exponent H, if

$$\{X(ct),\ t \in \mathbb{R}\} \stackrel{d}{=} \{c^H X(t),\ t \in \mathbb{R}\}.$$

An zero-mean Gaussian $\mathbb{R}^k$-valued process X with stationary increments which is H-self-similar will be referred to as an operator fBm. We refer to these processes as to H-fBm's.

Didier and Pipiras [?] provide a characterization of all possible operator fBm processes under the natural condition that

$$0 < \mathrm{Re}(h_j) < 1,\ j = 1, \dots, k,$$

where the h_j's are the (possibly complex) eigenvalues of the matrix H. Namely, under this condition, their Theorem 3.1 entails that every H-fBm has the representation

$$\{X(t),\ t \in \mathbb{R}\} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} (x_+^{-D} A + x_-^{-D} \overline{A}) \tilde{B}(dx) \right\} \tag{6}$$

where

$$D = H - (1/2)\mathbb{I}_k,$$

A is a complex-valued $(k \times k)$ matrix, $\overline{A}$ is its conjugate and $\tilde{B}(dx)$ denotes a $\mathbb{C}^k-$valued Brownian motion such that

$$\tilde{B}(x) = \overline{\tilde{B}(-x)} \quad \text{and} \quad \mathbb{E}[\tilde{B}(dx)\tilde{B}(dx)^*] = \mathbb{I}_k dx.$$

Here unless stated otherwise all our vectors are column vectors and for a matrix (or a vector) C, we denote $C^* := \overline{C}^\top$ – the adjoint matrix.

Inspired by these results and the fact that operator fBm's can arise as tangent processes for stationary $\mathbb{R}^k$-valued models (under operator normalization), we would like to introduce an extension of the Matérn model. This extension can be considered as *canonical* or *complete*, if its tangent processes can realize all possible operator fBm models.

To this end, consider the process

$$Y(t) := \int_{\mathbb{R}} e^{itx} ((1 + ix)^{-\nu - 1/2} A 1_{\{x>0\}} + (1 + ix)^{-\nu - 1/2} \overline{A} 1_{\{x<0\}}) \tilde{B}(dx), \tag{7}$$

where ν is a real $(k \times k)$ matrix and $A \in M_k(\mathbb{C})$. The matrix $\nu = (\nu_{ij})$ will play the role of the exponent of the Matérn model. Note that ν will essentially correspond to the operator H of the tangent operator fBm process associated with the above model. Indeed, to gain some intuition, suppose that $\nu = \nu\mathbb{I}_k$ is scalar. In this case, using commutativity of multiplication with scalars, for the cross-covariance of $Y(t)$, we obtain:

$$\mathbb{E}[Y(t)Y(s)^\top] \equiv \mathbb{E}[Y(t)Y(s)^*] = \int_{\mathbb{R}} e^{i(t-s)x} (1 + x^2)^{-\nu - 1/2} (AA^* 1_{\{x>0\}} + \overline{AA}^* 1_{\{x<0\}}) dx. \tag{8}$$

Indeed, this follows from the fact that $(1 + ix)^{-\nu-1/2}\overline{(1 + ix)^{-\nu-1/2}} = (1 + x^2)^{-\nu-1/2}$. (Note: I am being cavalier with complex exponents – they can be defined properly in every neighborhood of a point on the complex plane that does not contain zero and extended analytically.).

Notice that the latter complex integral is in fact real-valued since $\boxed{\text{check!}}$ $\mathbb{E}[Y(t)Y(s)^\top] = \overline{\mathbb{E}[Y(t)Y(s)^\top]}$. We do not generally have however that $\mathbb{E}[Y(t)Y(s)^\top] = \mathbb{E}[Y(s)Y(t)^\top]$, unless

$$AA^* = \overline{AA}^*.$$

That is, the so-proposed Matérn model is *time reversible* only if the matrix $\Sigma := AA^*$ is real-valued and a fortori psd. In this latter case, as we have seen above, in a suitable orthonormal basis, we have that $Y(t)$ consists of independent scalar stationary time series with Matérn auto-covariance. Notice that time reversibility is a very restrictive property shared by many existing extensions of the Matern models.

The interesting feature of the model in (7) is that even in the scalar case, with a suitable choice of a matrix A such that AA^* is not real-valued, we obtain *time-non-reversible* models. This is remarkable!

Connection to the characterization of Didier and Pipiras. In the following, we will argue that the tangent process to Y above is an operator fBm. Letting $c \downarrow 0$, consider a matrix exponent H (to be determined) and focus on the increment:

$$c^{-H}(Y(ct) - Y(0)) = c^{-H} \int_{\mathbb{R}} \frac{(e^{ictx} - 1)}{ix} \Big(ix(1+ix)^{-\nu-1/2} A 1_{\{x>0\}} + ix(1+ix)^{-\nu-1/2} \overline{A} 1_{\{x<0\}} \Big) \tilde{B}(dx).$$

By changing the variables $u := cx$, and using the facts that

$$iu(c+iu)^{-\nu-1/2} 1_{\pm u>0} \to (u)_{\pm}^{-\nu+1/2} e^{\pm i(1/2-\nu)\pi/2}, \quad \text{as } c \downarrow 0,$$

we obtain $\boxed{\text{justify the exchange of limits and integration}}$

$$\{c^{-H}(Y(ct) - Y(0))\} \stackrel{d}{=} \Big\{ c^{-H} \int_{\mathbb{R}} c \frac{(e^{itu} - 1)}{iu} \Big(iuc^{-1} c^{\nu+1/2} (c+iu)^{-\nu-1/2} \Big(A 1_{\{u>0\}} + \overline{A} 1_{\{u<0\}} \Big) \Big) c^{-1/2} \tilde{B}(du) \Big\}$$
$$\stackrel{fdd}{\to} \Big\{ \int_{\mathbb{R}} \frac{(e^{itu} - 1)}{iu} \Big((u)_+^{-D} A_\nu 1_{\{u>0\}} + (u)_-^{-D} \overline{A}_\nu 1_{\{u<0\}} \Big) \tilde{B}(du) \Big\},$$

where we used the fact that $\{B(du/c)\} \stackrel{d}{=} \{c^{-1/2} B(du)\}$ and $\stackrel{fdd}{\to}$ means convergence in fdd,

$$D \equiv H - 1/2 := \nu - 1/2 \quad \text{and} \quad A_\nu := e^{i(1/2-\nu)\pi/2} A.$$

(Note: observe that $(ab)^C = a^C b^C = b^C a^C$ for all $a, b \in \mathbb{C}$ (appropriately defined) while $c^A c^B$ does not always equal $c^B c^A$ – unless A and B commute.) This shows that if we choose $H := \nu$, we obtain that the tangent process to Y is the operator fBm with self-similarity exponent $H \equiv \nu$.

Now, the characterization of [**?**] in (6) shows that the model in (7) can achieve all possible tangent processes and in this sense it is a complete extension to the Matérn model, where the exponent ν is replaced by an operator ($k \times k$ matrix) whose eigenvalues have real parts in $(0, 1)$.

Exercise: Consider the spectral representation of the operator-Matern in the simple case of a scalar exponet (8). Letting $\mathbb{I} = (e_1 \ e_2 \ \cdots e_k)$ for the cross-covariance, between the i-th and j-th components of $Y(t)$, we obtain

$$\mathbb{E}[Y_i(t) Y_j(s)] = \int_{\mathbb{R}} e^{i(t-s)x} (1+x^2)^{-\nu-1/2} \Big(e_i^\top AA^* e_j 1_{\{x>0\}} + e_i^\top \overline{AA}^* e_j 1_{\{x<0\}} \Big) dx$$
$$= \int_{\mathbb{R}} e^{i(t-s)x} (1+x^2)^{-\nu-1/2} (z_{i,j} 1_{\{x>0\}} + \overline{z}_{i,j} 1_{\{x>0\}}) dx,$$

where

$$z_{i,j} = a_i \overline{a}_j^\top \in \mathbb{C},$$

with $a_1, \cdots, a_k$ being the *row*-vectors of the matrix A.

$\boxed{\text{Drew}}$ Can you compute explicitly (via perhaps Bessel functions) the last integral? This will be very nice since it will give the cross-covariance model for Matérn, which is non-reversible in general! In some sense, it will correct the misconceptions in the literature as to what one should call multivariate Matérn!

Preliminary

We consider the integral for the general Matern first as an exercise. To evaluate the integral

$$\int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}dx,$$

we use the fact that for the modified Bessel function of the second kind, $K_\nu(\cdot)$, we have

$$K_\nu(h) = \frac{\Gamma(\nu+1/2)2^\nu}{\pi^{1/2}h^\nu}\int_0^\infty \frac{\cos(hx)dx}{(1+x^2)^{\nu+1/2}}$$

(for example, see Abramowitz and Stegun 9.6.25). Then, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}dx &= \int_{\mathbb{R}} \frac{\cos(hx)+i\sin(hx)}{(1+x^2)^{\nu+1/2}}dt \\
&= 2K_\nu(h)\frac{\pi^{1/2}h^\nu}{2^\nu\Gamma(\nu+1/2)} + i\int_{\mathbb{R}}\frac{\sin(hx)}{(1+x^2)^{\nu+1/2}}dt \\
&= K_\nu(h)\frac{\pi^{1/2}h^\nu}{2^{\nu-1}\Gamma(\nu+1/2)} + 0
\end{aligned}$$

because the right integrand is an odd function. This gives the standard Matern class as expected.

Non-reversible

We return to the cross-covariance problem; the challenge is that the imaginary part will probably be nonzero. We have, letting $h = t-s$,

$$\begin{aligned}
\mathbb{E}(Y_i(s+h)Y_j(s)) &= \int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx \\
&= \int_{\mathbb{R}} (\cos(hx)+i\sin(hx))(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx
\end{aligned}$$

For simplicity, let $M(h,\nu)$ be the Matern covariance at lag h and parameter ν, so that we can write

$$\mathbb{E}(Y_i(s+h)Y_j(s)) = M(h,\nu)\frac{z_{i,j}+\overline{z}_{i,j}}{2} + i\int_{\mathbb{R}}\sin(hx)(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx$$

Now, if $z_{i,j} = \overline{z}_{i,j}$, then simply $\mathbb{E}(Y_i(s+h)Y_j(s)) = M(h,\nu)z_{i,j}$, and the reversible Matern cross-covariance falls out cleanly. We turn to when cross-covariance may not be reversible.

We can see that

$$\begin{aligned}
\mathbb{E}(Y_i(s+h)Y_j(s)) &= \frac{z_{i,j}+\overline{z}_{i,j}}{2}M(h,\nu) + i(z_{i,j}-\overline{z}_{i,j})\int_{(0,\infty)}\frac{\sin(hx)}{(1+x^2)^{\nu+1/2}}dx \\
&= \mathrm{Re}(z_{i,j})M(h,\nu) - \mathrm{Im}(z_{i,j})\int_{(0,\infty)}\frac{2\sin(hx)}{(1+x^2)^{\nu+1/2}}dx.
\end{aligned}$$

Using Wolfram-Alpha to evaluate the integral, we see that

$$\int_{(0,\infty)}\frac{2\sin(hx)}{(1+x^2)^{\nu+1/2}}dx = \frac{\mathrm{sign}(h)\pi^{3/2}2^{-\nu}|h|^\nu\sec(\pi\nu)(I_\nu(|h|)-L_{-\nu}(|h|))}{\Gamma(\nu+1/2)}$$

where I_ν is the modified Bessel function of the first kind and $L_{-\nu}$ is the modified Struve function (see https://www.wolframalpha.com/input/?i=integrate+from+x%3D0+to+x+%3D+infinity%3A+sin%28%7Cb%7Cx%29%2F%281%2Bx%5E2%29%5E%28%7Ca%7C%2B+1%2F2%29+dx).

Numerical testing seem to support that this formula is true.

References