I. S. GRADSHTEYN
I. M. RYZHIK

CD INCLUDED

TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION

Edited by Alan Jeffrey and Daniel Zwillinger

Table of Integrals, Series, and Products

Seventh Edition

This page intentionally left blank

Table of Integrals, Series, and Products

Seventh Edition

I.S. Gradshteyn and I.M. Ryzhik

Alan Jeffrey, Editor
University of Newcastle upon Tyne, England

Daniel Zwillinger, Editor
Rensselaer Polytechnic Institute, USA

Translated from Russian by Scripta Technica, Inc.

ELSEVIER

AMSTERDAM • BOSTON • HEIDELBERG • LONDON
NEW YORK • OXFORD • PARIS • SAN DIEGO
SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO

Academic Press is an imprint of Elsevier

ACADEMIC PRESS

Academic Press is an imprint of Elsevier
30 Corporate Drive, Suite 400, Burlington, MA 01803, USA
525 B Street, Suite 1900, San Diego, California 92101-4495, USA
84 Theobald's Road, London WC1X 8RR, UK

This book is printed on acid-free paper. ♾

Copyright © 2007, Elsevier Inc. All rights reserved.

No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without permission in writing from the publisher.

Permissions may be sought directly from Elsevier's Science & Technology Rights Department in Oxford, UK: phone: (+44) 1865 843830, fax: (+44) 1865 853333, E-mail: permissions@elsevier.com. You may also complete your request online via the Elsevier homepage (http://elsevier.com), by selecting "Support & Contact" then "Copyright and Permission" and then "Obtaining Permissions."

For information on all Elsevier Academic Press publications
visit our Web site at www.books.elsevier.com

ISBN-13: 978-0-12-373637-6
ISBN-10: 0-12-373637-4

PRINTED IN THE UNITED STATES OF AMERICA

07 08 09 10 11 9 8 7 6 5 4 3 2 1

Working together to grow
libraries in developing countries
www.elsevier.com | www.bookaid.org | www.sabre.org
ELSEVIER BOOK AID International Sabre Foundation

Contents

Preface to the Seventh Edition xxi
Acknowledgments xxiii
The Order of Presentation of the Formulas xxvii
Use of the Tables xxxi
Index of Special Functions xxxix
Notation xliii
Note on the Bibliographic References xlvii

0 Introduction **1**
0.1 Finite Sums . . . 1
0.11 Progressions . . . 1
0.12 Sums of powers of natural numbers . . . 1
0.13 Sums of reciprocals of natural numbers . . . 3
0.14 Sums of products of reciprocals of natural numbers . . . 3
0.15 Sums of the binomial coefficients . . . 3
0.2 Numerical Series and Infinite Products . . . 6
0.21 The convergence of numerical series . . . 6
0.22 Convergence tests . . . 6
0.23–0.24 Examples of numerical series . . . 8
0.25 Infinite products . . . 14
0.26 Examples of infinite products . . . 14
0.3 Functional Series . . . 15
0.30 Definitions and theorems . . . 15
0.31 Power series . . . 16
0.32 Fourier series . . . 19
0.33 Asymptotic series . . . 21
0.4 Certain Formulas from Differential Calculus . . . 21
0.41 Differentiation of a definite integral with respect to a parameter . . . 21
0.42 The nth derivative of a product (Leibniz's rule) . . . 22
0.43 The nth derivative of a composite function . . . 22
0.44 Integration by substitution . . . 23

1 Elementary Functions **25**
1.1 Power of Binomials . . . 25
1.11 Power series . . . 25
1.12 Series of rational fractions . . . 26
1.2 The Exponential Function . . . 26

1.21	Series representation	26
1.22	Functional relations	27
1.23	Series of exponentials	27
1.3–1.4	Trigonometric and Hyperbolic Functions	28
1.30	Introduction	28
1.31	The basic functional relations	28
1.32	The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)	31
1.33	The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions	33
1.34	Certain sums of trigonometric and hyperbolic functions	36
1.35	Sums of powers of trigonometric functions of multiple angles	37
1.36	Sums of products of trigonometric functions of multiple angles	38
1.37	Sums of tangents of multiple angles	39
1.38	Sums leading to hyperbolic tangents and cotangents	39
1.39	The representation of cosines and sines of multiples of the angle as finite products	41
1.41	The expansion of trigonometric and hyperbolic functions in power series	42
1.42	Expansion in series of simple fractions	44
1.43	Representation in the form of an infinite product	45
1.44–1.45	Trigonometric (Fourier) series	46
1.46	Series of products of exponential and trigonometric functions	51
1.47	Series of hyperbolic functions	51
1.48	Lobachevskiy's "Angle of Parallelism" $\Pi(x)$	51
1.49	The hyperbolic amplitude (the Gudermannian) $\operatorname{gd} x$	52
1.5	The Logarithm	53
1.51	Series representation	53
1.52	Series of logarithms (cf. 1.431)	55
1.6	The Inverse Trigonometric and Hyperbolic Functions	56
1.61	The domain of definition	56
1.62–1.63	Functional relations	56
1.64	Series representations	60

2 Indefinite Integrals of Elementary Functions 63

2.0	Introduction	63
2.00	General remarks	63
2.01	The basic integrals	64
2.02	General formulas	65
2.1	Rational Functions	66
2.10	General integration rules	66
2.11–2.13	Forms containing the binomial $a + bx^k$	68
2.14	Forms containing the binomial $1 \pm x^n$	74
2.15	Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$	78
2.16	Forms containing the trinomial $a + bx^k + cx^{2k}$	78
2.17	Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of x	79
2.18	Forms containing the quadratic trinomial $a + bx + cx^2$ and the binomial $\alpha + \beta x$	81
2.2	Algebraic Functions	82
2.20	Introduction	82
2.21	Forms containing the binomial $a + bx^k$ and $\sqrt{x}$	83

	2.22–2.23	Forms containing $\sqrt[n]{(a+bx)^k}$	84
	2.24	Forms containing $\sqrt{a+bx}$ and the binomial $\alpha+\beta x$	88
	2.25	Forms containing $\sqrt{a+bx+cx^2}$	92
	2.26	Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x	94
	2.27	Forms containing $\sqrt{a+cx^2}$ and integral powers of x	99
	2.28	Forms containing $\sqrt{a+bx+cx^2}$ and first- and second-degree polynomials	103
	2.29	Integrals that can be reduced to elliptic or pseudo-elliptic integrals	104
2.3		The Exponential Function	106
	2.31	Forms containing e^{ax}	106
	2.32	The exponential combined with rational functions of x	106
2.4		Hyperbolic Functions	110
	2.41–2.43	Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$	110
	2.44–2.45	Rational functions of hyperbolic functions	125
	2.46	Algebraic functions of hyperbolic functions	132
	2.47	Combinations of hyperbolic functions and powers	139
	2.48	Combinations of hyperbolic functions, exponentials, and powers	148
2.5–2.6		Trigonometric Functions	151
	2.50	Introduction	151
	2.51–2.52	Powers of trigonometric functions	151
	2.53–2.54	Sines and cosines of multiple angles and of linear and more complicated functions of the argument	161
	2.55–2.56	Rational functions of the sine and cosine	171
	2.57	Integrals containing $\sqrt{a \pm b\sin x}$ or $\sqrt{a \pm b\cos x}$	179
	2.58–2.62	Integrals reducible to elliptic and pseudo-elliptic integrals	184
	2.63–2.65	Products of trigonometric functions and powers	214
	2.66	Combinations of trigonometric functions and exponentials	227
	2.67	Combinations of trigonometric and hyperbolic functions	231
2.7		Logarithms and Inverse-Hyperbolic Functions	237
	2.71	The logarithm	237
	2.72–2.73	Combinations of logarithms and algebraic functions	238
	2.74	Inverse hyperbolic functions	240
2.8		Inverse Trigonometric Functions	241
	2.81	Arcsines and arccosines	241
	2.82	The arcsecant, the arccosecant, the arctangent, and the arccotangent	242
	2.83	Combinations of arcsine or arccosine and algebraic functions	242
	2.84	Combinations of the arcsecant and arccosecant with powers of x	244
	2.85	Combinations of the arctangent and arccotangent with algebraic functions	244

3–4 Definite Integrals of Elementary Functions **247**

3.0		Introduction	247
	3.01	Theorems of a general nature	247
	3.02	Change of variable in a definite integral	248
	3.03	General formulas	249
	3.04	Improper integrals	251
	3.05	The principal values of improper integrals	252
3.1–3.2		Power and Algebraic Functions	253
	3.11	Rational functions	253

3.12	Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials	254
3.13–3.17	Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions	254
3.18	Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions	313
3.19–3.23	Combinations of powers of x and powers of binomials of the form $(\alpha + \beta x)$	315
3.24–3.27	Powers of x, of binomials of the form $\alpha + \beta x^p$ and of polynomials in x	322
3.3–3.4	Exponential Functions	334
3.31	Exponential functions	334
3.32–3.34	Exponentials of more complicated arguments	336
3.35	Combinations of exponentials and rational functions	340
3.36–3.37	Combinations of exponentials and algebraic functions	344
3.38–3.39	Combinations of exponentials and arbitrary powers	346
3.41–3.44	Combinations of rational functions of powers and exponentials	353
3.45	Combinations of powers and algebraic functions of exponentials	363
3.46–3.48	Combinations of exponentials of more complicated arguments and powers	364
3.5	Hyperbolic Functions	371
3.51	Hyperbolic functions	371
3.52–3.53	Combinations of hyperbolic functions and algebraic functions	375
3.54	Combinations of hyperbolic functions and exponentials	382
3.55–3.56	Combinations of hyperbolic functions, exponentials, and powers	386
3.6–4.1	Trigonometric Functions	390
3.61	Rational functions of sines and cosines and trigonometric functions of multiple angles	390
3.62	Powers of trigonometric functions	395
3.63	Powers of trigonometric functions and trigonometric functions of linear functions	397
3.64–3.65	Powers and rational functions of trigonometric functions	401
3.66	Forms containing powers of linear functions of trigonometric functions	405
3.67	Square roots of expressions containing trigonometric functions	408
3.68	Various forms of powers of trigonometric functions	411
3.69–3.71	Trigonometric functions of more complicated arguments	415
3.72–3.74	Combinations of trigonometric and rational functions	423
3.75	Combinations of trigonometric and algebraic functions	434
3.76–3.77	Combinations of trigonometric functions and powers	436
3.78–3.81	Rational functions of x and of trigonometric functions	447
3.82–3.83	Powers of trigonometric functions combined with other powers	459
3.84	Integrals containing $\sqrt{1 - k^2 \sin^2 x}$, $\sqrt{1 - k^2 \cos^2 x}$, and similar expressions	472
3.85–3.88	Trigonometric functions of more complicated arguments combined with powers	475
3.89–3.91	Trigonometric functions and exponentials	485
3.92	Trigonometric functions of more complicated arguments combined with exponentials	493
3.93	Trigonometric and exponential functions of trigonometric functions	495
3.94–3.97	Combinations involving trigonometric functions, exponentials, and powers	497
3.98–3.99	Combinations of trigonometric and hyperbolic functions	509
4.11–4.12	Combinations involving trigonometric and hyperbolic functions and powers	516
4.13	Combinations of trigonometric and hyperbolic functions and exponentials	522

4.14	Combinations of trigonometric and hyperbolic functions, exponentials, and powers	525
4.2–4.4	Logarithmic Functions	527
4.21	Logarithmic functions	527
4.22	Logarithms of more complicated arguments	529
4.23	Combinations of logarithms and rational functions	535
4.24	Combinations of logarithms and algebraic functions	538
4.25	Combinations of logarithms and powers	540
4.26–4.27	Combinations involving powers of the logarithm and other powers	542
4.28	Combinations of rational functions of $\ln x$ and powers	553
4.29–4.32	Combinations of logarithmic functions of more complicated arguments and powers	555
4.33–4.34	Combinations of logarithms and exponentials	571
4.35–4.36	Combinations of logarithms, exponentials, and powers	573
4.37	Combinations of logarithms and hyperbolic functions	578
4.38–4.41	Logarithms and trigonometric functions	581
4.42–4.43	Combinations of logarithms, trigonometric functions, and powers	594
4.44	Combinations of logarithms, trigonometric functions, and exponentials	599
4.5	Inverse Trigonometric Functions	599
4.51	Inverse trigonometric functions	599
4.52	Combinations of arcsines, arccosines, and powers	600
4.53–4.54	Combinations of arctangents, arccotangents, and powers	601
4.55	Combinations of inverse trigonometric functions and exponentials	605
4.56	A combination of the arctangent and a hyperbolic function	605
4.57	Combinations of inverse and direct trigonometric functions	605
4.58	A combination involving an inverse and a direct trigonometric function and a power	607
4.59	Combinations of inverse trigonometric functions and logarithms	607
4.6	Multiple Integrals	607
4.60	Change of variables in multiple integrals	607
4.61	Change of the order of integration and change of variables	608
4.62	Double and triple integrals with constant limits	610
4.63–4.64	Multiple integrals	612

5 Indefinite Integrals of Special Functions 619

5.1	Elliptic Integrals and Functions	619
5.11	Complete elliptic integrals	619
5.12	Elliptic integrals	621
5.13	Jacobian elliptic functions	623
5.14	Weierstrass elliptic functions	626
5.2	The Exponential Integral Function	627
5.21	The exponential integral function	627
5.22	Combinations of the exponential integral function and powers	627
5.23	Combinations of the exponential integral and the exponential	628
5.3	The Sine Integral and the Cosine Integral	628
5.4	The Probability Integral and Fresnel Integrals	629
5.5	Bessel Functions	629

6–7 Definite Integrals of Special Functions **631**
6.1 Elliptic Integrals and Functions . . . 631
6.11 Forms containing $F(x,k)$ . . . 631
6.12 Forms containing $E(x,k)$ . . . 632
6.13 Integration of elliptic integrals with respect to the modulus . . . 632
6.14–6.15 Complete elliptic integrals . . . 632
6.16 The theta function . . . 633
6.17 Generalized elliptic integrals . . . 635
6.2–6.3 The Exponential Integral Function and Functions Generated by It . . . 636
6.21 The logarithm integral . . . 636
6.22–6.23 The exponential integral function . . . 638
6.24–6.26 The sine integral and cosine integral functions . . . 639
6.27 The hyperbolic sine integral and hyperbolic cosine integral functions . . . 644
6.28–6.31 The probability integral . . . 645
6.32 Fresnel integrals . . . 649
6.4 The Gamma Function and Functions Generated by It . . . 650
6.41 The gamma function . . . 650
6.42 Combinations of the gamma function, the exponential, and powers . . . 652
6.43 Combinations of the gamma function and trigonometric functions . . . 655
6.44 The logarithm of the gamma function* . . . 656
6.45 The incomplete gamma function . . . 657
6.46–6.47 The function $\psi(x)$ . . . 658
6.5–6.7 Bessel Functions . . . 659
6.51 Bessel functions . . . 659
6.52 Bessel functions combined with x and x^2 . . . 664
6.53–6.54 Combinations of Bessel functions and rational functions . . . 670
6.55 Combinations of Bessel functions and algebraic functions . . . 674
6.56–6.58 Combinations of Bessel functions and powers . . . 675
6.59 Combinations of powers and Bessel functions of more complicated arguments . 689
6.61 Combinations of Bessel functions and exponentials . . . 694
6.62–6.63 Combinations of Bessel functions, exponentials, and powers . . . 699
6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers . . . 708
6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers . . . 711
6.66 Combinations of Bessel, hyperbolic, and exponential functions . . . 713
6.67–6.68 Combinations of Bessel and trigonometric functions . . . 717
6.69–6.74 Combinations of Bessel and trigonometric functions and powers . . . 727
6.75 Combinations of Bessel, trigonometric, and exponential functions and powers . 742
6.76 Combinations of Bessel, trigonometric, and hyperbolic functions . . . 747
6.77 Combinations of Bessel functions and the logarithm, or arctangent . . . 747
6.78 Combinations of Bessel and other special functions . . . 748
6.79 Integration of Bessel functions with respect to the order . . . 749
6.8 Functions Generated by Bessel Functions . . . 753
6.81 Struve functions . . . 753
6.82 Combinations of Struve functions, exponentials, and powers . . . 754
6.83 Combinations of Struve and trigonometric functions . . . 755

6.84–6.85	Combinations of Struve and Bessel functions	756
6.86	Lommel functions	760
6.87	Thomson functions	761
6.9	Mathieu Functions	763
6.91	Mathieu functions	763
6.92	Combinations of Mathieu, hyperbolic, and trigonometric functions	763
6.93	Combinations of Mathieu and Bessel functions	767
6.94	Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems	767
7.1–7.2	Associated Legendre Functions	769
7.11	Associated Legendre functions	769
7.12–7.13	Combinations of associated Legendre functions and powers	770
7.14	Combinations of associated Legendre functions, exponentials, and powers	776
7.15	Combinations of associated Legendre and hyperbolic functions	778
7.16	Combinations of associated Legendre functions, powers, and trigonometric functions	779
7.17	A combination of an associated Legendre function and the probability integral	781
7.18	Combinations of associated Legendre and Bessel functions	782
7.19	Combinations of associated Legendre functions and functions generated by Bessel functions	787
7.21	Integration of associated Legendre functions with respect to the order	788
7.22	Combinations of Legendre polynomials, rational functions, and algebraic functions	789
7.23	Combinations of Legendre polynomials and powers	791
7.24	Combinations of Legendre polynomials and other elementary functions	792
7.25	Combinations of Legendre polynomials and Bessel functions	794
7.3–7.4	Orthogonal Polynomials	795
7.31	Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers	795
7.32	Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions	797
7.325*	Complete System of Orthogonal Step Functions	798
7.33	Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index	798
7.34	Combinations of Chebyshev polynomials and powers	800
7.35	Combinations of Chebyshev polynomials and elementary functions	802
7.36	Combinations of Chebyshev polynomials and Bessel functions	803
7.37–7.38	Hermite polynomials	803
7.39	Jacobi polynomials	806
7.41–7.42	Laguerre polynomials	808
7.5	Hypergeometric Functions	812
7.51	Combinations of hypergeometric functions and powers	812
7.52	Combinations of hypergeometric functions and exponentials	814
7.53	Hypergeometric and trigonometric functions	817
7.54	Combinations of hypergeometric and Bessel functions	817
7.6	Confluent Hypergeometric Functions	820
7.61	Combinations of confluent hypergeometric functions and powers	820
7.62–7.63	Combinations of confluent hypergeometric functions and exponentials	822
7.64	Combinations of confluent hypergeometric and trigonometric functions	829
7.65	Combinations of confluent hypergeometric functions and Bessel functions	830

	7.66	Combinations of confluent hypergeometric functions, Bessel functions, and powers	831
	7.67	Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers	834
	7.68	Combinations of confluent hypergeometric functions and other special functions	839
	7.69	Integration of confluent hypergeometric functions with respect to the index	841
7.7		Parabolic Cylinder Functions	841
	7.71	Parabolic cylinder functions	841
	7.72	Combinations of parabolic cylinder functions, powers, and exponentials	842
	7.73	Combinations of parabolic cylinder and hyperbolic functions	843
	7.74	Combinations of parabolic cylinder and trigonometric functions	844
	7.75	Combinations of parabolic cylinder and Bessel functions	845
	7.76	Combinations of parabolic cylinder functions and confluent hypergeometric functions	849
	7.77	Integration of a parabolic cylinder function with respect to the index	849
7.8		Meijer's and MacRobert's Functions (G and E)	850
	7.81	Combinations of the functions G and E and the elementary functions	850
	7.82	Combinations of the functions G and E and Bessel functions	854
	7.83	Combinations of the functions G and E and other special functions	856

8–9 Special Functions **859**

8.1		Elliptic Integrals and Functions	859
	8.11	Elliptic integrals	859
	8.12	Functional relations between elliptic integrals	863
	8.13	Elliptic functions	865
	8.14	Jacobian elliptic functions	866
	8.15	Properties of Jacobian elliptic functions and functional relationships between them	870
	8.16	The Weierstrass function $\wp(u)$	873
	8.17	The functions ζ(u) and σ(u)	876
	8.18–8.19	Theta functions	877
8.2		The Exponential Integral Function and Functions Generated by It	883
	8.21	The exponential integral function Ei(x)	883
	8.22	The hyperbolic sine integral $\operatorname{shi} x$ and the hyperbolic cosine integral $\operatorname{chi} x$	886
	8.23	The sine integral and the cosine integral: $\operatorname{si} x$ and $\operatorname{ci} x$	886
	8.24	The logarithm integral $\operatorname{li}(x)$	887
	8.25	The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$ and $C(x)$, the error function $\operatorname{erf}(x)$, and the complementary error function $\operatorname{erfc}(x)$	887
	8.26	Lobachevskiy's function $L(x)$	891
8.3		Euler's Integrals of the First and Second Kinds	892
	8.31	The gamma function (Euler's integral of the second kind): $\Gamma(z)$	892
	8.32	Representation of the gamma function as series and products	894
	8.33	Functional relations involving the gamma function	895
	8.34	The logarithm of the gamma function	898
	8.35	The incomplete gamma function	899
	8.36	The psi function $\psi(x)$	902
	8.37	The function $\beta(x)$	906
	8.38	The beta function (Euler's integral of the first kind): $\mathrm{B}(x,y)$	908
	8.39	The incomplete beta function $\mathrm{B}_x(p,q)$	910
8.4–8.5		Bessel Functions and Functions Associated with Them	910

8.40	Definitions	910
8.41	Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$	912
8.42	Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$	914
8.43	Integral representations of the functions $I_\nu(z)$ and $K_\nu(z)$	916
8.44	Series representation	918
8.45	Asymptotic expansions of Bessel functions	920
8.46	Bessel functions of order equal to an integer plus one-half	924
8.47–8.48	Functional relations	926
8.49	Differential equations leading to Bessel functions	931
8.51–8.52	Series of Bessel functions	933
8.53	Expansion in products of Bessel functions	940
8.54	The zeros of Bessel functions	941
8.55	Struve functions	942
8.56	Thomson functions and their generalizations	944
8.57	Lommel functions	945
8.58	Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$	948
8.59	Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$	949
8.6	Mathieu Functions	950
8.60	Mathieu's equation	950
8.61	Periodic Mathieu functions	951
8.62	Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$	951
8.63	Mathieu functions with a purely imaginary argument	952
8.64	Non-periodic solutions of Mathieu's equation	953
8.65	Mathieu functions for negative q	953
8.66	Representation of Mathieu functions as series of Bessel functions	954
8.67	The general theory	957
8.7–8.8	Associated Legendre Functions	958
8.70	Introduction	958
8.71	Integral representations	960
8.72	Asymptotic series for large values of $\|\nu\|$	962
8.73–8.74	Functional relations	964
8.75	Special cases and particular values	968
8.76	Derivatives with respect to the order	969
8.77	Series representation	970
8.78	The zeros of associated Legendre functions	972
8.79	Series of associated Legendre functions	972
8.81	Associated Legendre functions with integer indices	974
8.82–8.83	Legendre functions	975
8.84	Conical functions	980
8.85	Toroidal functions	981
8.9	Orthogonal Polynomials	982
8.90	Introduction	982
8.91	Legendre polynomials	983
8.919	Series of products of Legendre and Chebyshev polynomials	988
8.92	Series of Legendre polynomials	988
8.93	Gegenbauer polynomials $C_n^\lambda(t)$	990
8.94	The Chebyshev polynomials $T_n(x)$ and $U_n(x)$	993

	8.95	The Hermite polynomials $H_n(x)$	996
	8.96	Jacobi's polynomials	998
	8.97	The Laguerre polynomials	1000
9.1		Hypergeometric Functions	1005
	9.10	Definition	1005
	9.11	Integral representations	1005
	9.12	Representation of elementary functions in terms of a hypergeometric functions	1006
	9.13	Transformation formulas and the analytic continuation of functions defined by hypergeometric series	1008
	9.14	A generalized hypergeometric series	1010
	9.15	The hypergeometric differential equation	1010
	9.16	Riemann's differential equation	1014
	9.17	Representing the solutions to certain second-order differential equations using a Riemann scheme	1017
	9.18	Hypergeometric functions of two variables	1018
	9.19	A hypergeometric function of several variables	1022
9.2		Confluent Hypergeometric Functions	1022
	9.20	Introduction	1022
	9.21	The functions $\Phi(\alpha,\gamma;z)$ and $\Psi(\alpha,\gamma;z)$	1023
	9.22–9.23	The Whittaker functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$	1024
	9.24–9.25	Parabolic cylinder functions $D_p(z)$	1028
	9.26	Confluent hypergeometric series of two variables	1031
9.3		Meijer's G-Function	1032
	9.30	Definition	1032
	9.31	Functional relations	1033
	9.32	A differential equation for the G-function	1034
	9.33	Series of G-functions	1034
	9.34	Connections with other special functions	1034
9.4		MacRobert's E-Function	1035
	9.41	Representation by means of multiple integrals	1035
	9.42	Functional relations	1035
9.5		Riemann's Zeta Functions $\zeta(z,q)$ and $\zeta(z)$, and the Functions $\Phi(z,s,v)$ and $\xi(s)$	1036
	9.51	Definition and integral representations	1036
	9.52	Representation as a series or as an infinite product	1037
	9.53	Functional relations	1037
	9.54	Singular points and zeros	1038
	9.55	The Lerch function $\Phi(z,s,v)$	1039
	9.56	The function $\xi(s)$	1040
9.6		Bernoulli Numbers and Polynomials, Euler Numbers	1040
	9.61	Bernoulli numbers	1040
	9.62	Bernoulli polynomials	1041
	9.63	Euler numbers	1043
	9.64	The functions $\nu(x)$, $\nu(x,\alpha)$, $\mu(x,\beta)$, $\mu(x,\beta,\alpha)$, and $\lambda(x,y)$	1043
	9.65	Euler polynomials	1044
9.7		Constants	1045
	9.71	Bernoulli numbers	1045
	9.72	Euler numbers	1045

9.73 Euler's and Catalan's constants . . . 1046
9.74 Stirling numbers . . . 1046

10 Vector Field Theory 1049
10.1–10.8 Vectors, Vector Operators, and Integral Theorems . . . 1049
10.11 Products of vectors . . . 1049
10.12 Properties of scalar product . . . 1049
10.13 Properties of vector product . . . 1049
10.14 Differentiation of vectors . . . 1050
10.21 Operators grad, div, and curl . . . 1050
10.31 Properties of the operator ∇ . . . 1051
10.41 Solenoidal fields . . . 1052
10.51–10.61 Orthogonal curvilinear coordinates . . . 1052
10.71–10.72 Vector integral theorems . . . 1055
10.81 Integral rate of change theorems . . . 1057

11 Algebraic Inequalities 1059
11.1–11.3 General Algebraic Inequalities . . . 1059
11.11 Algebraic inequalities involving real numbers . . . 1059
11.21 Algebraic inequalities involving complex numbers . . . 1060
11.31 Inequalities for sets of complex numbers . . . 1061

12 Integral Inequalities 1063
12.11 Mean Value Theorems . . . 1063
12.111 First mean value theorem . . . 1063
12.112 Second mean value theorem . . . 1063
12.113 First mean value theorem for infinite integrals . . . 1063
12.114 Second mean value theorem for infinite integrals . . . 1064
12.21 Differentiation of Definite Integral Containing a Parameter . . . 1064
12.211 Differentiation when limits are finite . . . 1064
12.212 Differentiation when a limit is infinite . . . 1064
12.31 Integral Inequalities . . . 1064
12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals . . . 1064
12.312 Hölder's inequality for integrals . . . 1064
12.313 Minkowski's inequality for integrals . . . 1065
12.314 Chebyshev's inequality for integrals . . . 1065
12.315 Young's inequality for integrals . . . 1065
12.316 Steffensen's inequality for integrals . . . 1065
12.317 Gram's inequality for integrals . . . 1065
12.318 Ostrowski's inequality for integrals . . . 1066
12.41 Convexity and Jensen's Inequality . . . 1066
12.411 Jensen's inequality . . . 1066
12.412 Carleman's inequality for integrals . . . 1066
12.51 Fourier Series and Related Inequalities . . . 1066
12.511 Riemann-Lebesgue lemma . . . 1067
12.512 Dirichlet lemma . . . 1067
12.513 Parseval's theorem for trigonometric Fourier series . . . 1067
12.514 Integral representation of the nth partial sum . . . 1067

12.515 Generalized Fourier series . . . 1067
12.516 Bessel's inequality for generalized Fourier series . . . 1068
12.517 Parseval's theorem for generalized Fourier series . . . 1068

13 Matrices and Related Results 1069
13.11–13.12 Special Matrices . . . 1069
13.111 Diagonal matrix . . . 1069
13.112 Identity matrix and null matrix . . . 1069
13.113 Reducible and irreducible matrices . . . 1069
13.114 Equivalent matrices . . . 1069
13.115 Transpose of a matrix . . . 1069
13.116 Adjoint matrix . . . 1070
13.117 Inverse matrix . . . 1070
13.118 Trace of a matrix . . . 1070
13.119 Symmetric matrix . . . 1070
13.120 Skew-symmetric matrix . . . 1070
13.121 Triangular matrices . . . 1070
13.122 Orthogonal matrices . . . 1070
13.123 Hermitian transpose of a matrix . . . 1070
13.124 Hermitian matrix . . . 1070
13.125 Unitary matrix . . . 1071
13.126 Eigenvalues and eigenvectors . . . 1071
13.127 Nilpotent matrix . . . 1071
13.128 Idempotent matrix . . . 1071
13.129 Positive definite . . . 1071
13.130 Non-negative definite . . . 1071
13.131 Diagonally dominant . . . 1071
13.21 Quadratic Forms . . . 1071
13.211 Sylvester's law of inertia . . . 1072
13.212 Rank . . . 1072
13.213 Signature . . . 1072
13.214 Positive definite and semidefinite quadratic form . . . 1072
13.215 Basic theorems on quadratic forms . . . 1072
13.31 Differentiation of Matrices . . . 1073
13.41 The Matrix Exponential . . . 1074
3.411 Basic properties . . . 1074

14 Determinants 1075
14.11 Expansion of Second- and Third-Order Determinants . . . 1075
14.12 Basic Properties . . . 1075
14.13 Minors and Cofactors of a Determinant . . . 1075
14.14 Principal Minors . . . 1076
14.15* Laplace Expansion of a Determinant . . . 1076
14.16 Jacobi's Theorem . . . 1076
14.17 Hadamard's Theorem . . . 1077
14.18 Hadamard's Inequality . . . 1077
14.21 Cramer's Rule . . . 1077
14.31 Some Special Determinants . . . 1078

14.311 Vandermonde's determinant (alternant) . . . 1078
14.312 Circulants . . . 1078
14.313 Jacobian determinant . . . 1078
14.314 Hessian determinants . . . 1079
14.315 Wronskian determinants . . . 1079
14.316 Properties . . . 1079
14.317 Gram-Kowalewski theorem on linear dependence . . . 1080

15 Norms **1081**
15.1–15.9 Vector Norms . . . 1081
15.11 General Properties . . . 1081
15.21 Principal Vector Norms . . . 1081
15.211 The norm $||\mathbf{x}||_1$ . . . 1081
15.212 The norm $||\mathbf{x}||_2$ (Euclidean or L_2 norm) . . . 1081
15.213 The norm $||\mathbf{x}||_\infty$ . . . 1081
15.31 Matrix Norms . . . 1082
15.311 General properties . . . 1082
15.312 Induced norms . . . 1082
15.313 Natural norm of unit matrix . . . 1082
15.41 Principal Natural Norms . . . 1082
15.411 Maximum absolute column sum norm . . . 1082
15.412 Spectral norm . . . 1082
15.413 Maximum absolute row sum norm . . . 1083
15.51 Spectral Radius of a Square Matrix . . . 1083
15.511 Inequalities concerning matrix norms and the spectral radius . . . 1083
15.512 Deductions from Gerschgorin's theorem (see **15.814**) . . . 1083
15.61 Inequalities Involving Eigenvalues of Matrices . . . 1084
15.611 Cayley-Hamilton theorem . . . 1084
15.612 Corollaries . . . 1084
15.71 Inequalities for the Characteristic Polynomial . . . 1084
15.711 Named and unnamed inequalities . . . 1085
15.712 Parodi's theorem . . . 1086
15.713 Corollary of Brauer's theorem . . . 1086
15.714 Ballieu's theorem . . . 1086
15.715 Routh-Hurwitz theorem . . . 1086
15.81–15.82 Named Theorems on Eigenvalues . . . 1087
15.811 Schur's inequalities . . . 1087
15.812 Sturmian separation theorem . . . 1087
15.813 Poincare's separation theorem . . . 1087
15.814 Gerschgorin's theorem . . . 1088
15.815 Brauer's theorem . . . 1088
15.816 Perron's theorem . . . 1088
15.817 Frobenius theorem . . . 1088
15.818 Perron–Frobenius theorem . . . 1088
15.819 Wielandt's theorem . . . 1088
15.820 Ostrowski's theorem . . . 1089
15.821 First theorem due to Lyapunov . . . 1089
15.822 Second theorem due to Lyapunov . . . 1089

15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov . . . 1089
15.91 Variational Principles . . . 1091
15.911 Rayleigh quotient . . . 1091
15.912 Basic theorems . . . 1091

16 Ordinary Differential Equations 1093
16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations . . . 1093
16.11 First-Order Equations . . . 1093
16.111 Solution of a first-order equation . . . 1093
16.112 Cauchy problem . . . 1093
16.113 Approximate solution to an equation . . . 1093
16.114 Lipschitz continuity of a function . . . 1094
16.21 Fundamental Inequalities and Related Results . . . 1094
16.211 Gronwall's lemma . . . 1094
16.212 Comparison of approximate solutions of a differential equation . . . 1094
16.31 First-Order Systems . . . 1094
16.311 Solution of a system of equations . . . 1094
16.312 Cauchy problem for a system . . . 1095
16.313 Approximate solution to a system . . . 1095
16.314 Lipschitz continuity of a vector . . . 1095
16.315 Comparison of approximate solutions of a system . . . 1096
16.316 First-order linear differential equation . . . 1096
16.317 Linear systems of differential equations . . . 1096
16.41 Some Special Types of Elementary Differential Equations . . . 1097
16.411 Variables separable . . . 1097
16.412 Exact differential equations . . . 1097
16.413 Conditions for an exact equation . . . 1097
16.414 Homogeneous differential equations . . . 1097
16.51 Second-Order Equations . . . 1098
16.511 Adjoint and self-adjoint equations . . . 1098
16.512 Abel's identity . . . 1098
16.513 Lagrange identity . . . 1099
16.514 The Riccati equation . . . 1099
16.515 Solutions of the Riccati equation . . . 1099
16.516 Solution of a second-order linear differential equation . . . 1100
16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations . . . 1100
16.611 First basic comparison theorem . . . 1100
16.622 Second basic comparison theorem . . . 1101
16.623 Interlacing of zeros . . . 1101
16.624 Sturm separation theorem . . . 1101
16.625 Sturm comparison theorem . . . 1101
16.626 Szegö's comparison theorem . . . 1101
16.627 Picone's identity . . . 1102
16.628 Sturm-Picone theorem . . . 1102
16.629 Oscillation on the half line . . . 1102
16.71 Two Related Comparison Theorems . . . 1103
16.711 Theorem 1 . . . 1103

16.712	Theorem 2	1103
16.81–16.82	Non-Oscillatory Solutions	1103
16.811	Kneser's non-oscillation theorem	1103
16.822	Comparison theorem for non-oscillation	1104
16.823	Necessary and sufficient conditions for non-oscillation	1104
16.91	Some Growth Estimates for Solutions of Second-Order Equations	1104
16.911	Strictly increasing and decreasing solutions	1104
16.912	General result on dominant and subdominant solutions	1104
16.913	Estimate of dominant solution	1105
16.914	A theorem due to Lyapunov	1105
16.92	Boundedness Theorems	1106
16.921	All solutions of the equation	1106
16.922	If all solutions of the equation	1106
16.923	If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of	1106
16.924	Consider the equation	1106
16.93	Growth of maxima of $\lvert y \rvert$	1106

17 Fourier, Laplace, and Mellin Transforms **1107**

17.1–17.4	Integral Transforms	1107
17.11	Laplace transform	1107
17.12	Basic properties of the Laplace transform	1107
17.13	Table of Laplace transform pairs	1108
17.21	Fourier transform	1117
17.22	Basic properties of the Fourier transform	1118
17.23	Table of Fourier transform pairs	1118
17.24	Table of Fourier transform pairs for spherically symmetric functions	1120
17.31	Fourier sine and cosine transforms	1121
17.32	Basic properties of the Fourier sine and cosine transforms	1121
17.33	Table of Fourier sine transforms	1122
17.34	Table of Fourier cosine transforms	1126
17.35	Relationships between transforms	1129
17.41	Mellin transform	1129
17.42	Basic properties of the Mellin transform	1130
17.43	Table of Mellin transforms	1131

18 The z-Transform **1135**

18.1–18.3	Definition, Bilateral, and Unilateral z-Transforms	1135
18.1	Definitions	1135
18.2	Bilateral z-transform	1136
18.3	Unilateral z-transform	1138

References 1141

Supplemental references 1145

Index of Functions and Constants 1151

General Index of Concepts 1161

This page intentionally left blank

Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript "11" added to its entry reference number, where the equivalent superscript number for the sixth edition was "10." Similarly, an asterisk on an entry's reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O'Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site `www.az-tec.com/gr/errata`.

Alan Jeffrey
`Alan.Jeffrey@newcastle.ac.uk`

Daniel Zwillinger
`zwillinger@alum.mit.edu`

This page intentionally left blank

Acknowledgments

The publisher and editors would like to take this opportunity to express their gratitude to the following users of the *Table of Integrals, Series, and Products* who, either directly or through errata published in *Mathematics of Computation*, have generously contributed corrections and addenda to the original printing.

Dr. A. Abbas
Dr. P. B. Abraham
Dr. Ari Abramson
Dr. Jose Adachi
Dr. R. J. Adler
Dr. N. Agmon
Dr. M. Ahmad
Dr. S. A. Ahmad
Dr. Luis Alvarez-Ruso
Dr. Maarten H P Ambaum
Dr. R. K. Amiet
Dr. L. U. Ancarani
Dr. M. Antoine
Dr. C. R. Appledorn
Dr. D. R. Appleton
Dr. Mitsuhiro Arikawa
Dr. P. Ashoshauvati
Dr. C. L. Axness
Dr. E. Badralexe
Dr. S. B. Bagchi
Dr. L. J. Baker
Dr. R. Ball
Dr. M. P. Barnett
Dr. Florian Baumann
Dr. Norman C. Beaulieu
Dr. Jerome Benoit
Mr. V. Bentley
Dr. Laurent Berger
Dr. M. van den Berg
Dr. N. F. Berk
Dr. C. A. Bertulani
Dr. J. Betancort-Rijo
Dr. P. Bickerstaff
Dr. Iwo Bialynicki-Birula
Dr. Chris Bidinosti
Dr. G. R. Bigg
Dr. Ian Bindloss
Dr. L. Blanchet
Dr. Mike Blaskiewicz
Dr. R. D. Blevins
Dr. Anders Blom
Dr. L. M. Blumberg
Dr. R. Blumel
Dr. S. E. Bodner
Dr. M. Bonsager
Dr. George Boros
Dr. S. Bosanac
Dr. B. Van den Bossche
Dr. A. Boström
Dr. J. E. Bowcock
Dr. T. H. Boyer
Dr. K. M. Briggs
Dr. D. J. Broadhurst
Dr. Chris Van Den Broeck
Dr. W. B. Brower
Dr. H. N. Browne
Dr. Christoph Bruegger
Dr. William J. Bruno
Dr. Vladimir Bubanja
Dr. D. J. Buch
Dr. D. J. Bukman
Dr. F. M. Burrows
Dr. R. Caboz
Dr. T. Calloway
Dr. F. Calogero
Dr. D. Dal Cappello
Dr. David Cardon
Dr. J. A. Carlson Gallos
Dr. B. Carrascal
Dr. A. R. Carr
Dr. S. Carter
Dr. G. Cavalleri
Mr. W. H. L. Cawthorne
Dr. A. Cecchini
Dr. B. Chan
Dr. M. A. Chaudhry
Dr. Sabino Chavez-Cerda
Dr. Julian Cheng
Dr. H. W. Chew
Dr. D. Chin
Dr. Young-seek Chung
Dr. S. Ciccariello
Dr. N. S. Clarke
Dr. R. W. Cleary
Dr. A. Clement
Dr. P. Cochrane
Dr. D. K. Cohoon
Dr. L. Cole
Dr. Filippo Colomo
Dr. J. R. D. Copley
Dr. D. Cox
Dr. J. Cox
Dr. J. W. Criss

Dr. A. E. Curzon
Dr. D. Dadyburjor
Dr. D. Dajaputra
Dr. C. Dal Cappello
Dr. P. Daly
Dr. S. Dasgupta
Dr. John Davies
Dr. C. L. Davis
Dr. A. Degasperis
Dr. B. C. Denardo
Dr. R. W. Dent
Dr. E. Deutsch
Dr. D. deVries
Dr. P. Dita
Dr. P. J. de Doelder
Dr. Mischa Dohler
Dr. G. Dôme
Dr. Shi-Hai Dong
Dr. Balazs Dora
Dr. M. R. D'Orsogna
Dr. Adrian A. Dragulescu
Dr. Eduardo Duenez
Mr. Tommi J. Dufva
Dr. E. B. Dussan, V
Dr. C. A. Ebner
Dr. M. van der Ende
Dr. Jonathan Engle
Dr. G. Eng
Dr. E. S. Erck
Dr. Jan Erkelens
Dr. Olivier Espinosa
Dr. G. A. Estévez
Dr. K. Evans
Dr. G. Evendon
Dr. V. I. Fabrikant
Dr. L. A. Falkovsky
Dr. K. Farahmand
Dr. Richard J. Fateman
Dr. G. Fedele
Dr. A. R. Ferchmin
Dr. P. Ferrant
Dr. H. E. Fettis
Dr. W. B. Fichter
Dr. George Fikioris
Mr. J. C. S. S. Filho
Dr. L. Ford
Dr. Nicolao Fornengo
Dr. J. France
Dr. B. Frank
Dr. S. Frasier
Dr. Stefan Fredenhagen
Dr. A. J. Freeman
Dr. A. Frink
Dr. Jason M. Gallaspy
Dr. J. A. C. Gallas
Dr. J. A. Carlson Gallas
Dr. G. R. Gamertsfelder
Dr. T. Garavaglia
Dr. Jaime Zaratiegui Garcia
Dr. C. G. Gardner
Dr. D. Garfinkle
Dr. P. N. Garner
Dr. F. Gasser
Dr. E. Gath
Dr. P. Gatt
Dr. D. Gay
Dr. M. P. Gelfand
Dr. M. R. Geller
Dr. Ali I. Genc
Dr. Vincent Genot
Dr. M. F. George
Dr. P. Germain
Dr. Ing. Christoph Gierull
Dr. S. P. Gill
Dr. Federico Girosi
Dr. E. A. Gislason
Dr. M. I. Glasser
Dr. P. A. Glendinning
Dr. L. I. Goldfischer
Dr. Denis Golosov
Dr. I. J. Good
Dr. J. Good
Mr. L. Gorin
Dr. Martin Götz
Dr. R. Govindaraj
Dr. M. De Grauf
Dr. L. Green
Mr. Leslie O. Green
Dr. R. Greenwell
Dr. K. D. Grimsley
Dr. Albert Groenenboom
Dr. V. Gudmundsson
Dr. J. Guillera
Dr. K. Gunn
Dr. D. L. Gunter
Dr. Julio C. Gutiérrez-Vega
Dr. Roger Haagmans
Dr. H. van Haeringen
Dr. B. Hafizi
Dr. Bahman Hafizi
Dr. T. Hagfors
Dr. M. J. Haggerty
Dr. Timo Hakulinen
Dr. Einar Halvorsen
Dr. S. E. Hammel
Dr. E. Hansen
Dr. Wes Harker
Dr. T. Harrett
Dr. D. O. Harris
Dr. Frank Harris
Mr. Mazen D. Hasna
Dr. Joel G. Heinrich
Dr. Sten Herlitz
Dr. Chris Herzog
Dr. A. Higuchi
Dr. R. E. Hise
Dr. Henrik Holm
Dr. Helmut Hölzler
Dr. N. Holte
Dr. R. W. Hopper
Dr. P. N. Houle
Dr. C. J. Howard
Dr. J. H. Hubbell
Dr. J. R. Hull
Dr. W. Humphries
Dr. Jean-Marc Huré
Dr. Ben Yu-Kuang Hu
Dr. Y. Iksbe
Dr. Philip Ingenhoven
Mr. L. Iossif
Dr. Sean A. Irvine
Dr. Óttar Ísberg
Dr. Cyril-Daniel Iskander
Dr. S. A. Jackson
Dr. John David Jackson
Dr. Francois Jaclot
Dr. B. Jacobs
Dr. E. C. James
Dr. B. Jancovici
Dr. D. J. Jeffrey
Dr. H. J. Jensen

Dr. Edwin F. Johnson
Dr. I. R. Johnson
Dr. Steven Johnson
Dr. I. Johnstone
Dr. Y. P. Joshi
Dr. Jae-Hun Jung
Dr. Damir Juric
Dr. Florian Kaempfer
Dr. S. Kanmani
Dr. Z. Kapal
Dr. Dave Kasper
Dr. M. Kaufman
Dr. B. Kay
Dr. Avinash Khare
Dr. Ilki Kim
Dr. Youngsun Kim
Dr. S. Klama
Dr. L. Klingen
Dr. C. Knessl
Dr. M. J. Knight
Dr. Mel Knight
Dr. Yannis Kohninos
Dr. D. Koks
Dr. L. P. Kok
Dr. K. S. Kölbig
Dr. Y. Komninos
Dr. D. D. Konowalow
Dr. Z. Kopal
Dr. I. Kostyukov
Dr. R. A. Krajcik
Dr. Vincent Krakoviack
Dr. Stefan Kramer
Dr. Tobias Kramer
Dr. Hermann Krebs
Dr. J. W. Krozel
Dr. E. D. Krupnikov
Dr. Kun-Lin Kuo
Dr. E. A. Kuraev
Dr. Konstantinos Kyritsis
Dr. Velimir Labinac
Dr. A. D. J. Lambert
Dr. A. Lambert
Dr. A. Larraza
Dr. K. D. Lee
Dr. M. Howard Lee
Dr. M. K. Lee
Dr. P. A. Lee
Dr. Todd Lee
Dr. J. Legg
Dr. Armando Lemus
Dr. S. L. Levie
Dr. D. Levi
Dr. Michael Lexa
Dr. Kuo Kan Liang
Dr. B. Linet
Dr. M. A. Lisa
Dr. Donald Livesay
Dr. H. Li
Dr. Georg Lohoefer
Dr. I. M. Longman
Dr. D. Long
Dr. Sylvie Lorthois
Dr. Y. L. Luke
Dr. W. Lukosz
Dr. T. Lundgren
Dr. E. A. Luraev
Dr. R. Lynch
Dr. R. Mahurin
Dr. R. Mallier
Dr. G. A. Mamon
Dr. A. Mangiarotti
Dr. I. Manning
Dr. J. Marmur
Dr. A. Martin
Sr. Yuzo Maruyama
Dr. David J. Masiello
Dr. Richard Marthar
Dr. H. A. Mavromatis
Dr. M. Mazzoni
Dr. K. B. Ma
Dr. P. McCullagh
Dr. J. H. McDonnell
Dr. J. R. McGregor
Dr. Kim McInturff
Dr. N. McKinney
Dr. David McA McKirdy
Dr. Rami Mehrem
Dr. W. N. Mei
Dr. Angelo Melino
Mr. José Ricardo Mendes
Dr. Andy Mennim
Dr. J. P. Meunier
Dr. Gerard P. Michon
Dr. D. F. R. Mildner
Dr. D. L. Miller
Dr. Steve Miller
Dr. P. C. D. Milly
Dr. S. P. Mitra
Dr. K. Miura
Dr. N. Mohankumar
Dr. M. Moll
Dr. Victor H. Moll
Dr. D. Monowalow
Mr. Tony Montagnese
Dr. Jim Morehead
Dr. J. Morice
Dr. W. Mueck
Dr. C. Muhlhausen
Dr. S. Mukherjee
Dr. R. R. Müller
Dr. Pablo Parmezani Munhoz
Dr. Paul Nanninga
Dr. A. Natarajan
Dr. Stefan Neumeier
Dr. C. T. Nguyen
Dr. A. C. Nicol
Dr. M. M. Nieto
Dr. P. Noerdlinger
Dr. A. N. Norris
Dr. K. H. Norwich
Dr. A. H. Nuttall
Dr. Frank O'Brien
Dr. R. P. O'Keeffe
Dr. A. Ojo
Dr. P. Olsson
Dr. M. Ortner
Dr. S. Ostlund
Dr. J. Overduin
Dr. J. Pachner
Dr. John D. Paden
Mr. Robert A. Padgug
Dr. D. Papadopoulos
Dr. F. J. Papp
Mr. Man Sik Park
Dr. Jong-Do Park
Dr. B. Patterson
Dr. R. F. Pawula
Dr. D. W. Peaceman
Dr. D. Pelat
Dr. L. Peliti
Dr. Y. P. Pellegrini

Dr. G. J. Pert
Dr. Nicola Pessina
Dr. J. B. Peterson
Dr. Rickard Petersson
Dr. Andrew Plumb
Dr. Dror Porat
Dr. E. A. Power
Dr. E. Predazzi
Dr. William S. Price
Dr. Paul Radmore
Dr. F. Raynal
Dr. X. R. Resende
Dr. J. M. Riedler
Dr. Thomas Richard
Dr. E. Ringel
Dr. T. M. Roberts
Dr. N. I. Robinson
Dr. P. A. Robinson
Dr. D. M. Rosenblum
Dr. R. A. Rosthal
Dr. J. R. Roth
Dr. Klaus Rottbrand
Dr. D. Roy
Dr. E. Royer
Dr. D. Rudermann
Dr. Sanjib Sabhapandit
Dr. C. T. Sachradja
Dr. J. Sadiku
Dr. A. Sadiq
Dr. Motohiko Saitoh
Dr. Naoki Saito
Dr. A. Salim
Dr. J. H. Samson
Dr. Miguel A. Sanchis-Lozano
Dr. J. A. Sanders
Dr. M. A. F. Sanjun
Dr. P. Sarquiz
Dr. Avadh Saxena
Dr. Vito Scarola
Dr. O. Schärpf
Dr. A. Scherzinger
Dr. B. Schizer
Dr. Martin Schmid
Dr. J. Scholes
Dr. Mel Schopper
Dr. H. J. Schulz
Dr. G. J. Sears
Dr. Kazuhiko Seki
Dr. B. Seshadri
Dr. A. Shapiro
Dr. Masaki Shigemori
Dr. J. S. Sheng
Dr. Kenneth Ing Shing
Dr. Tomohiro Shirai
Dr. S. Shlomo
Dr. D. Siegel
Dr. Matthew Stapleton
Dr. Steven H. Simon
Dr. Ashok Kumar Singal
Dr. C. Smith
Dr. G. C. C. Smith
Dr. Stefan Llewellyn Smith
Dr. S. Smith
Dr. G. Solt
Dr. J. Sondow
Dr. A. Sørenssen
Dr. Marcus Spradlin
Dr. Andrzej Staruszkiewicz
Dr. Philip C. L. Stephenson
Dr. Edgardo Stockmeyer
Dr. J. C. Straton
Mr. H. Suraweera
Dr. N. F. Svaiter
Dr. V. Svaiter
Dr. R. Szmytkowski
Dr. S. Tabachnik
Dr. Erik Talvila
Dr. G. Tanaka
Dr. C. Tanguy
Dr. G. K. Tannahill
Dr. B. T. Tan
Dr. C. Tavard
Dr. Gonçalo Tavares
Dr. Aba Teleki
Dr. Arash Dahi Taleghani
Dr. D. Temperley
Dr. A. J. Tervoort
Dr. Theodoros Theodoulidis
Dr. D. J. Thomas
Dr. Michael Thorwart
Dr. S. T. Thynell
Dr. D. C. Torney
Dr. R. Tough
Dr. B. F. Treadway
Dr. Ming Tsai
Dr. N. Turkkan
Dr. Sandeep Tyagi
Dr. J. J. Tyson
Dr. S. Uehara
Dr. M. Vadacchino
Dr. O. T. Valls
Dr. D. Vandeth
Mr. Andras Vanyolos
Dr. D. Veitch
Mr. Jose Lopez Vicario
Dr. K. Vogel
Dr. J. M. M. J. Vogels
Dr. Alexis De Vos
Dr. Stuart Walsh
Dr. Reinhold Wannemacher
Dr. S. Wanzura
Dr. J. Ward
Dr. S. I. Warshaw
Dr. R. Weber
Dr. Wei Qian
Dr. D. H. Werner
Dr. E. Wetzel
Dr. Robert Whittaker
Dr. D. T. Wilton
Dr. C. Wiuf
Dr. K. T. Wong
Mr. J. N. Wright
Dr. J. D. Wright
Dr. D. Wright
Dr. D. Wu
Dr. Michel Daoud Yacoub
Dr. Yu S. Yakovlev
Dr. H.-C. Yang
Dr. J. J. Yang
Dr. Z. J. Yang
Dr. J. J. Wang
Dr. Peter Widerin
Mr. Chun Kin Au Yeung
Dr. Kazuya Yuasa
Dr. S. P. Yukon
Dr. B. Zhang
Dr. Y. C. Zhang
Dr. Y. Zhao
Dr. Ralf Zimmer

The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x}\,\ln\tan x\,dx = -\frac{\pi}{p}\operatorname{cosec} p\pi. \tag{0.1}$$

By making the natural substitution $u = \cot x - 1$, we obtain

$$\int_0^\infty u^{p-1}\ln(1+u)\,du = \frac{\pi}{p}\operatorname{cosec} p\pi. \tag{0.2}$$

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln\left(\tan^p x + \cot^p x\right) \ln \tan x \, dx = 0.$$

The substitution $u = \tan x$ yields

$$I = \int_0^{\infty} \frac{\ln\left(u^p + u^{-p}\right) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{v e^v}{1 + e^{2v}} \ln\left(e^{pv} + e^{-pv}\right) \, dv = \int_{-\infty}^{\infty} v \, \frac{\ln\left(2 \cosh pv\right)}{2 \cosh v} \, dv.$$

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel's integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer's and MacRobert's functions.
21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear

in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x+\cos x}{\sin x-\cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$] are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of $\varphi_2(x)$ in the list] after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions [that is, $\varphi_2(x) = x$]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \dfrac{1}{x}$ are equally complex since $\ln \dfrac{1}{x} = -\ln x$. In the section on "powers and algebraic functions," polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

*For any natural number n, the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

This page intentionally left blank

Use of the Tables*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function, and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Gradshteyn and Ryzhik's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α, β, γ, δ, t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions $\operatorname{arcsinh} z$, $\operatorname{arccosh} z$, $\operatorname{arctanh} z$, and $\operatorname{arccoth} z$ are introduced through the definitions

$$\begin{aligned}
\arcsin z &= \frac{1}{i} \operatorname{arcsinh}(iz) \\
\arccos z &= \frac{1}{i} \operatorname{arccosh}(z) \\
\arctan z &= \frac{1}{i} \operatorname{arctanh}(iz) \\
\operatorname{arccot} z &= i \operatorname{arccoth}(iz)
\end{aligned}$$

*Prepared by Alan Jeffrey for the English language edition.

or

$$\operatorname{arcsinh} z = \frac{1}{i}\arcsin(iz)$$
$$\operatorname{arccosh} z = i\arccos z$$
$$\operatorname{arctanh} z = \frac{1}{i}\arctan(iz)$$
$$\operatorname{arccoth} z = \frac{1}{i}\operatorname{arccot}(-iz)$$

The numerical constants $\boldsymbol{C}$ and $\boldsymbol{G}$ which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant $\boldsymbol{C}$ is defined by the limit

$$\boldsymbol{C} = \lim_{s\to\infty}\left(\sum_{m=1}^{s}\frac{1}{m} - \ln s\right) = 0.577215\ldots.$$

On occasion, other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

$$\gamma = e^{\boldsymbol{C}} = 1.781072\ldots.$$

Catalan's constant $\boldsymbol{G}$ is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

by the expression

$$\boldsymbol{G} = \frac{1}{2}\int_0^1 \mathbf{K}\,dk = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots.$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for } n = 0, 1, \ldots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots.$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n\, E_n\left(\frac{1}{2}\right) \qquad \text{for } n = 0, 1, \ldots$$

The E_n are all integral, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, ...

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^*\frac{t^2}{2!} - B_2^*\frac{t^4}{4!} + \ldots.$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66,$$
$$B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \quad \ldots.$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} B_{n-k}\, E_k(2x) \qquad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n}\left\{B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right\}$$

or

$$E_{n-1}(x) = \frac{2}{n}\left\{B_n(x) - 2^n\, B_n\left(\frac{x}{2}\right)\right\} \qquad n = 1, 2, \ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{2}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}\left(2^{n-k} - 1\right) B_{n-k}\, B_n(x) \qquad n = 2, 3, \ldots$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right)\left\{B_n(x) - 2^n\, B_n\left(\frac{x}{2}\right)\right\}$$

the Euler polynomial of order n.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$

$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}$$

$$\operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

$$\begin{aligned}\Pi\left(\varphi, n^2, k\right) &= \int_0^{\varphi} \frac{da}{\left(1-n^2\sin^2 a\right)\sqrt{1-k^2\sin^2 a}} \\ &= \int_0^{\sin\varphi} \frac{dx}{\left(1-n^2x^2\right)\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}\end{aligned} \qquad \left(-\infty < n^2 < \infty\right)$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\mathrm{zn}(u,k)$, frequently written $Z(u)$, is defined by the relation

$$\mathrm{zn}(u,k) = Z(u) = \int_0^u \left\{\mathrm{dn}^2 \upsilon - \frac{\boldsymbol{E}}{\boldsymbol{K}}\right\} d\upsilon = \boldsymbol{E}(u) - \frac{\boldsymbol{E}}{\boldsymbol{K}} u.$$

This is related to the theta functions by the relationship

$$\mathrm{zn}(u,k) = \frac{\partial}{\partial u} \ln \Theta(u)$$

giving

(i). $$\mathrm{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_1'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}$$

(ii). $$\mathrm{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_2'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\mathrm{dn}\, u\, \mathrm{sn}\, u}{\mathrm{cn}\, u}$$

(iii). $$\mathrm{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_3'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_3\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - k^2 \frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

(iv). $$\mathrm{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_4'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts, the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions, the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d\left(\ln z!\right)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -\boldsymbol{C} + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n}\, dt \qquad (n = 0, 1, \ldots, \quad \operatorname{Re} z > 0)\,.$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\operatorname{Ei}(z)$ through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^{\infty} e^{-t} t^{-1}\, dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \qquad [z > 1]\,.$$

The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \left\{ e^{-z} - z\, E_{n-1}(z) \right\} \qquad [n > 1]$$

and

$$E_n'(z) = -\, E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots \right\} \qquad \left[|\arg z| < \frac{3\pi}{2} \right]$$

while for large n,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n\left(6x^2 - 8nx + n^2\right)}{(x+n)^6} + R(n,x) \right\},$$

where

$$-0.36 n^{-4} \le R(n,x) \le \left(1 + \frac{1}{x+n-1} \right) n^{-4} \qquad [x > 0]\,.$$

The sine and cosine integrals $\operatorname{si}(x)$ and $\operatorname{ci}(x)$ are related to the functions $\operatorname{Si}(x)$ and $\operatorname{Ci}(x)$ by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\mathrm{Ci}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t}\, dt.$$

The hyperbolic sine and cosine integrals $\mathrm{shi}(x)$ and $\mathrm{chi}(x)$ are defined by the relations

$$\mathrm{shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt$$

and

$$\mathrm{chi}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t}\, dt.$$

Some authors write

$$\mathrm{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t}\, dt$$

so that

$$\mathrm{Cin}(x) = -\,\mathrm{Ci}(x) + \ln x + \boldsymbol{C}.$$

The error function $\mathrm{erf}(x)$ is defined by the relation

$$\mathrm{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and the complementary error function $\mathrm{erfc}(x)$ is related to the error function $\mathrm{erfc}(x)$ and to $\Phi(x)$ by the expression

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Gradshteyn and Ryzhik as

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2\, dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2\, dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2}\, dt, \qquad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2}\, dt,$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\, dt, \qquad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\, dt.$$

These are related by the expressions

$$S(x) = S_1\left(x\sqrt{\frac{2}{\pi}}\right) = S_2\left(x^2\right)$$

and

$$C(x) = C_1\left(x\sqrt{\frac{2}{\pi}}\right) = C_2\left(x^2\right)$$

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $He_n(x)$ by the relations

$$He_n(x) = 2^{-n/2}\, H_n\left(\frac{x}{\sqrt{2}}\right)$$

and

$$H_n(x) = 2^{n/2}\, He_n\left(x\sqrt{2}\right).$$

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x\frac{d H_n}{dx} + 2n H_n = 0$$

and

$$\frac{d^2 He_n}{dx^2} - x\frac{d He_n}{dx} + n He_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2x H_n - 2n H_{n-1}$$

and

$$He_{n+1} = x He_n - n He_{n-1}.$$

The first six orthogonal polynomials He_n are

$$He_0 = 1, \quad He_1 = x, \quad He_2 = x^2 - 1, \quad He_3 = x^3 - 3x, \quad He_4 = x^4 - 6x^2 + 3, \quad He_5 = x^5 - 10x^3 + 15x.$$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n} \Gamma(n+1)\, J_n(z).$$

In the book by Gray, Mathews, and MacRobert, the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi\, Y_n(z) + (\ln 2 - \mathbf{C})\, J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $Hs_\nu(z)$ and $Hi_\nu(z)$, and some authors write $G_\nu(z) = \left(\frac{1}{2}\right)\pi i\, H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n+1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z)\, O_0(t) + 2\sum_{k=1}^{\infty} J_k(z)\, O_k(t),$$

giving

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{[n/2]} \frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \ldots,$$

where $\left[\frac{1}{2}n\right]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1)\, O_{n+1}(t) + (n+1)\, O_{n-1}(t) - \frac{2\left(n^2-1\right)}{t}\, O_n(t) = \frac{2n}{t}\sin^2\frac{n\pi}{2}.$$

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\mathrm{Ai}(z) = \frac{1}{3}\sqrt{z}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$$

$$\mathrm{Ai}(-z) = \frac{1}{3}\sqrt{z}\left\{J_{1/3}\left(\frac{2}{3}z^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}$$

and by

$$\mathrm{Bi}(z) = \sqrt{\frac{z}{3}}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\},$$

$$\mathrm{Bi}(-z) = \sqrt{\frac{z}{3}}\left\{J_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - J_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}.$$

Parabolic Cylinder Functions and Whittaker Functions

The differential equation

$$\frac{d^2y}{dz^2} + \left(az^2 + bz + c\right)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left(\frac{1}{4}z^2 + a\right)y = 0 \text{ and } \frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0,$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0$$

are sometimes written $U(a,z)$ and $V(a,z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a,z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a,z) = \frac{1}{\pi}\,\Gamma\left(\frac{1}{2} + a\right)\left\{D_{-a-\frac{1}{2}}(-z) + (\sin\pi a)\,D_{-a-\frac{1}{2}}(z)\right\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is

$$\frac{d^2y}{dz^2} + \left(a - 2k^2\cos 2z\right)y = 0 \qquad \text{with } k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ, λ and h^2, and b and $c = 2\sqrt{q}$, respectively. The periodic solutions $\mathrm{se}_n(z,q)$ and $\mathrm{ce}_n(z,q)$ and the modified periodic solutions $\mathrm{Se}_n(z,q)$ and $\mathrm{Ce}_n(z,q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions*. National Bureau of Standards, Columbia University Press, New York, 1951.

Index of Special Functions

Notation	Name of the function and the number of the formula containing its definition	
$\beta(x)$		8.37
$\Gamma(z)$	Gamma function	8.31–8.33
$\gamma(a,x), \quad \Gamma(a,x)$	Incomplete gamma functions	8.35
$\Delta(n-k)$	Unit integer pulse function	18.1
$\xi(s)$		9.56
$\lambda(x,y)$		9.640
$\mu(x,\beta), \quad \mu(x,\beta,\alpha)$		9.640
$\nu(x), \quad \nu(x,\alpha)$		9.640
$\Pi(x)$	Lobachevskiy's angle of parallelism	1.48
$\Pi(\varphi,n,k)$	Elliptic integral of the third kind	8.11
$\zeta(u)$	Weierstrass zeta function	8.17
$\zeta(z,q), \quad \zeta(z)$	Riemann's zeta functions	9.51–9.54
$\Theta(u)=\vartheta_4\left(\frac{\pi u}{2\mathbf{K}}\right), \quad \Theta_1(u)=\vartheta_3\left(\frac{\pi u}{2\mathbf{K}}\right)$	Jacobian theta function	8.191–8.196
$\left\{\begin{array}{l}\vartheta_0(\upsilon \mid \tau)=\vartheta_4(\upsilon \mid \tau),\\ \vartheta_1(\upsilon \mid \tau), \quad \vartheta_2(\upsilon \mid \tau),\\ \vartheta_3(\upsilon \mid \tau)\end{array}\right\}$	Elliptic theta functions	8.18, 8.19
$\sigma(u)$	Weierstrass sigma function	8.17
$\Phi(x)$	See probability integral	8.25
$\Phi(z,s,\upsilon)$	Lerch function	9.55
$\Phi(a,c;x)={}_1F_1(\alpha;\ \gamma;\ x)$	Confluent hypergeometric function	9.21
$\left\{\begin{array}{l}\Phi_1(\alpha,\beta,\gamma,x,y)\\ \Phi_2(\beta,\beta',\gamma,x,y)\\ \Phi_3(\beta,\gamma,x,y)\end{array}\right\}$	Degenerate hypergeometric series in two variables	9.26
$\psi(x)$	Euler psi function	8.36
$\wp(u)$	Weierstrass elliptic function	8.16
$\mathrm{am}(u,k)$	Amplitude (of an elliptic function)	8.141
B_n	Bernoulli numbers	9.61, 9.71
$B_n(x)$	Bernoulli polynomials	9.620
$\mathrm{B}(x,y)$	Beta functions	8.38
$\mathrm{B}_x(p,q)$	Incomplete beta functions	8.39
$\mathrm{bei}(z), \quad \mathrm{ber}(z)$	Thomson functions	8.56

continued on next page

continued from previous page

Notation	Name of the function and the number of the formula containing its definition	
$\boldsymbol{C}$	Euler constant	9.73, 8.367
$C(x)$	Fresnel cosine integral	8.25
$C_\nu(a)$	Young functions	3.76
$C_n^\lambda(t)$	Gegenbauer polynomials	8.93
$C_n^\lambda(x)$	Gegenbauer functions	8.932 1
$\mathrm{ce}_{2n}(z,q), \quad \mathrm{ce}_{2n+1}(z,q)$	Periodic Mathieu functions (Mathieu functions of the first kind)	8.61
$\mathrm{Ce}_{2n}(z,q), \quad \mathrm{Ce}_{2n+1}(z,q)$	Associated (modified) Mathieu functions of the first kind	8.63
$\mathrm{chi}(x)$	Hyperbolic cosine integral function	8.22
$\mathrm{ci}(x)$	Cosine integral	8.23
$\mathrm{cn}(u)$	Cosine amplitude	8.14
$D(k) \equiv \boldsymbol{D}$	Elliptic integral	8.112
$D(\varphi,k)$	Elliptic integral	8.111
$D_n(z), \quad D_p(z)$	Parabolic cylinder functions	9.24–9.25
$\mathrm{dn}\, u$	Delta amplitude	8.14
e_1, e_2, e_3	(used with the Weierstrass function)	8.162
E_n	Euler numbers	9.63, 9.72
$E(\varphi,k)$	Elliptic integral of the second kind	8.11–8.12
$\left\{\begin{array}{c}\boldsymbol{E}(k)=\boldsymbol{E}\\ \boldsymbol{E}(k')=\boldsymbol{E}'\end{array}\right\}$	Complete elliptic integral of the second kind	8.11-8.12
$E(p;a_r:q;\varrho_s:x)$	MacRobert's function	9.4
$\mathbf{E}_\nu(z)$	Weber function	8.58
$\mathrm{Ei}(z)$	Exponential integral function	8.21
$\mathrm{erf}(x)$	Error function	8.25
$\mathrm{erfc}(x)=1-\mathrm{erf}(x)$	Complementary error function	8.25
$F(\varphi,k)$	Elliptic integral of the first kind	8.11–8.12
${}_pF_q\left(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z\right)$	Generalized hypergeometric series	9.14
${}_2F_1\left(\alpha,\beta;\gamma;z\right)=F(\alpha,\beta;\gamma;z)$	Gauss hypergeometric function	9.10–9.13
${}_1F_1\left(\alpha;\gamma;z\right)=\Phi(\alpha,\gamma;z)$	Degenerate hypergeometric function	9.21
$F_\Lambda(\alpha:\beta_1,\dots,\beta_n;$ $\gamma_1,\dots\dots,\gamma_n:z_1,\dots,z_n)$	Hypergeometric function of several variables	9.19
F_1,F_2,F_3,F_4	Hypergeometric functions of two variables	9.18
$\left\{\begin{array}{c}\mathrm{fe}_n(z,q),\mathrm{Fe}_n(z,q)\dots\\ \mathrm{Fey}_n(z,q),\mathrm{Fek}_n(z,q)\dots\end{array}\right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$\boldsymbol{G}$	Catalan constant	9.73
g_2,g_3	Invariants of the $\wp(u)$-function	8.161
$\mathrm{gd}\, x$	Gudermannian	1.49
$\left\{\begin{array}{c}\mathrm{ge}_n(z,q),\mathrm{Ge}_n(z,q)\\ \mathrm{Gey}_n(z,q),\mathrm{Gek}_n(z,q)\end{array}\right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$G_{p,q}^{m,n}\left(x\left\vert\begin{array}{c}a_1,\dots,a_p\\ b_1,\dots,b_q\end{array}\right.\right)$	Meijer's functions	9.3

continued on next page

continued from previous page

Notation	Name of the function and the number of the formula containing its definition	
$h(n)$	Unit integer function	18.1
$\mathrm{hei}_\nu(z), \quad \mathrm{her}_\nu(z)$	Thomson functions	8.56
$H_\nu^{(1)}(z), \quad H_\nu^{(2)}(z)$	Hankel functions of the first and second kinds	8.405, 8.42
$H(u) = \vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)$	Theta function	8.192
$H_1(u) = \vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)$	Theta function	8.192
$H_n(z)$	Hermite polynomials	8.95
$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Bessel functions of an imaginary argument	8.406, 8.43
$I_x(p,q)$	Normalized incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger function	8.58
$\mathrm{k}_\nu(x)$	Bateman's function	9.210 3
$\mathbf{K}(\mathbf{k}) = \boldsymbol{K}, \quad \mathbf{K}(\mathbf{k}') = \boldsymbol{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Bessel functions of imaginary argument	8.407, 8.43
$\mathrm{kei}(z), \quad \mathrm{ker}(z)$	Thomson functions	8.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\mathrm{li}(x)$	Logarithm integral	8.24
$M_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$O_n(x)$	Neumann's polynomials	8.59
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_\nu(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$	Riemann's differential equation	9.160
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	8.96
$Q_\nu^\mu(z), \quad Q_\nu^\mu(x)$	Associated Legendre functions of the second kind	8.7, 8.8
$Q_\nu(z), \quad Q_\nu(x)$	Legendre functions of the second kind	8.82, 8.83
$S(x)$	Fresnel sine integral	8.25
$S_n(x)$	Schläfli's polynomials	8.59
$s_{\mu,\nu}(z), \quad S_{\mu,\nu}(z)$	Lommel functions	8.57
$\mathrm{se}_{2n+1}(z,q), \quad \mathrm{se}_{2n+2}(z,q)$	Periodic Mathieu functions	8.61
$\mathrm{Se}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+2}(z,q)$	Mathieu functions of an imaginary argument	8.63
$\mathrm{shi}(x)$	Hyperbolic sine integral	8.22
$\mathrm{si}(x)$	Sine integral	8.23
$\mathrm{sn}\, u$	Sine amplitude	8.14
$T_n(x)$	Chebyshev polynomial of the 1st kind	8.94
$U_n(x)$	Chebyshev polynomials of the 2nd kind	8.94

continued on next page

continued from previous page

Notation	Name of the function and the number of the formula containing its definition	
$U_\nu(w,z), \quad V_\nu(w,z)$	Lommel functions of two variables	8.578
$W_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$Y_\nu(z)$	Neumann functions	8.403, 8.41
$Z_\nu(z)$	Bessel functions	8.401
$\mathfrak{Z}_\nu(z)$	Bessel functions	

Notation

Symbol	Meaning
$\lfloor x \rfloor$	The integral part of the real number x (also denoted by $[x]$)
$\int_a^{(b+)} \quad \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a, proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve C
$\mathrm{PV}\int$	Principal value integral
$\overline{z} = x - iy$	The complex conjugate of $z = x + iy$
$n!$	$= 1 \cdot 2 \cdot 3 \ldots n, \qquad 0! = 1$
$(2n+1)!!$	$= 1 \cdot 3 \ldots (2n+1)$. (double factorial notation)
$(2n)!!$	$= 2 \cdot 4 \ldots (2n)$. (double factorial notation)
$0!! = 1$ and $(-1)!! = 1$	(cf. 3.372 for $n = 0$)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$)
$\binom{p}{n}$	$= \dfrac{p(p-1)\ldots(p-n+1)}{1 \cdot 2 \ldots n} = \dfrac{p!}{n!(p-n)!}, \quad \binom{p}{0} = 1, \quad \binom{p}{n} = \dfrac{p!}{n!(p-n)!}$ $[n = 1, 2, \ldots, p \geq n]$
$\binom{x}{n}$	$= x(x-1)\ldots(x-n+1)/n!$ $[n = 0, 1, \ldots]$
$(a)_n$	$= a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Pochhammer symbol)
$\sum_{k=m}^{n} u_k$	$= u_m + u_{m+1} + \ldots + u_n$. If $n < m$, we define $\sum_{k=m}^{n} u_k = 0$
$\sum_n{}', \quad \sum_{m,n}{}'$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$\sum, \quad \prod$	An empty $\sum$ has value 0, and an empty $\prod$ has value 1

continued on next page

continued from previous page

Symbol	Meaning
$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$	Kronecker delta
τ	Theta function parameter (cf. 8.18)
$\times$ and $\wedge$	Vector product (cf. 10.11)
$\cdot$	Scalar product (cf. 10.11)
∇ or "del"	Vector operator (cf. 10.21)
∇^2	Laplacian (cf. 10.31)
$\sim$	Asymptotically equal to
$\arg z$	The argument of the complex number $z = x + iy$
curl or rot	Vector operator (cf. 10.21)
div	Vector operator (divergence) (cf. 10.21)
$\mathcal{F}$	Fourier transform (cf. 17.21)
$\mathcal{F}_c$	Fourier cosine transform (cf. 17.31)
$\mathcal{F}_s$	Fourier sine transform (cf. 17.31)
grad	Vector operator (gradient) (cf. 10.21)
h_i and g_{ij}	Metric coefficients (cf. 10.51)
^{H}	Hermitian transpose of a vector or matrix (cf. 13.123)
$\mathrm{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	Heaviside step function
$\operatorname{Im} z \equiv y$	The imaginary part of the complex number $z = x + iy$
k	The letter k (when not used as an index of summation) denotes a number in the interval [0, 1]. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1-k^2}$ is denoted by k'.
$\mathcal{L}$	Laplace transform (cf. 17.11)
$\mathcal{M}$	Mellin transform (cf. 17.41)
$\mathbb{N}$	The natural numbers $(0, 1, 2, \dots)$
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0. If there exists an $M > 0$ such that $\lvert g(z)\rvert \leq M\lvert f(z)\rvert$ in some sufficiently small neighborhood of the point z_0, we write $g(z) = O(f(z))$.

continued on next page

continued from previous page	
Symbol	Meaning
q	The nome, a theta function parameter (cf. 8.18)
$\mathbb{R}$	The real numbers
$R(x)$	A rational function
$\operatorname{Re} z \equiv x$	The real part of the complex number $z = x + iy$
S_n^m	Stirling number of the first kind (cd. 9.74)
$\mathfrak{S}_n^m$	Stirling number of the second kind (cd. 9.74)
$\operatorname{sign} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$	The sign (signum) of the real number x
^{T}	Transpose of a vector or matrix (cf. 13.115)
$\mathbb{Z}$	The integers $(0, \pm 1, \pm 2, \dots)$
Z_b	Bilateral z transform (cf. 18.1)
Z_u	Unilateral z transform (cf. 18.1)

This page intentionally left blank

Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter a appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

ET I 118 (1) a

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms.*

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the a indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

This page intentionally left blank

0 Introduction

0.1 Finite Sums

0.11 Progressions

0.111 Arithmetic progression.

$$\sum_{k=0}^{n-1}(a+kr) = \frac{n}{2}[2a+(n-1)r] = \frac{n}{2}(a+l) \qquad [l = a+(n-1)r \text{ is the last term}]$$

0.112 Geometric progression.

$$\sum_{k=1}^{n} aq^{k-1} = \frac{a\left(q^n-1\right)}{q-1} \qquad [q \neq 1]$$

0.113 Arithmetic-geometric progression.

$$\sum_{k=0}^{n-1}(a+kr)q^k = \frac{a-[a+(n-1)r]q^n}{1-q} + \frac{rq\left(1-q^{n-1}\right)}{(1-q)^2}$$

$$[q \neq 1, \quad n > 1] \qquad \text{JO (5)}$$

0.114[8] $$\sum_{k=1}^{n-1} k^2x^k = \frac{\left(-n^2+2n-1\right)x^{n+2} + \left(2n^2-2n-1\right)x^{n+1} - n^2x^n + x^2 + x}{(1-x)^3}$$

0.12 Sums of powers of natural numbers

0.121

$$\sum_{k=1}^{n} k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2}\binom{q}{1}B_2n^{q-1} + \frac{1}{4}\binom{q}{3}B_4n^{q-3} + \frac{1}{6}\binom{q}{5}B_6n^{q-5} + \cdots$$

$$= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720}n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240}n^{q-5} - \cdots$$

[last term contains either n or n^2] CE 332

1. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ CE 333

2. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$ CE 333

3. $$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$ CE 333

4. $\displaystyle\sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$ CE 333

5. $\displaystyle\sum_{k=1}^{n} k^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1)$ CE 333

6. $\displaystyle\sum_{k=1}^{n} k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4+6n^3-3n+1)$ CE 333

7. $\displaystyle\sum_{k=1}^{n} k^7 = \frac{1}{24} n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$ CE 333

0.122 $$\sum_{k=1}^{n}(2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2}\binom{q}{1} 2^{q-1} B_2 n^{q-1} - \frac{1}{4}\binom{q}{3} 2^{q-3}\left(2^3-1\right) B_4 n^{q-3} - \cdots$$

[last term contains either n or n^2.]

1. $\displaystyle\sum_{k=1}^{n}(2k-1) = n^2$

2. $\displaystyle\sum_{k=1}^{n}(2k-1)^2 = \frac{1}{3} n(4n^2-1)$ JO (32a)

3. $\displaystyle\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$ JO (32b)

4.[11] $\displaystyle\sum_{k=1}^{n}(mk-1) = \frac{n}{2}[m(n+1)-2]$

5.[10] $\displaystyle\sum_{k=1}^{n}(mk-1)^2 = \frac{1}{6} n[m^2(n+1)(2n+1)-6m(n+1)+6]$

6.[10] $\displaystyle\sum_{k=1}^{n}(mk-1)^3 = \frac{1}{4} n[m^3 n(n+1)^2 - 2m^2(n+1)(2n+1)+6m(n+1)-4]$

0.123 $\displaystyle\sum_{k=1}^{n} k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$

0.124

1. $\displaystyle\sum_{k=1}^{q} k\left(n^2-k^2\right) = \frac{1}{4} q(q+1)\left(2n^2-q^2-q\right)$ $[q = 1, 2, \ldots]$

2.[10] $\displaystyle\sum_{k=1}^{n} k(k+1)^3 = \frac{1}{60} n(n+1)\left(12n^3+63n^2+107n+58\right)$

0.125 $\displaystyle\sum_{k=1}^{n} k!\cdot k = (n+1)! - 1$ AD (188.1)

0.126 $\displaystyle\sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}}\, K_{n+\frac{1}{2}}\left(\frac{1}{2}\right)$ WA 94

0.13 Sums of reciprocals of natural numbers

0.131[11] $$\sum_{k=1}^{n}\frac{1}{k}=\boldsymbol{C}+\ln n+\frac{1}{2n}-\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\dots(n+k-1)},$$ JO (59), AD (1876)

where

$$A_k=\frac{1}{k}\int_0^1 x(1-x)(2-x)(3-x)\cdots(k-1-x)\,dx$$
$$A_2=\frac{1}{12},\qquad A_3=\frac{1}{12}$$
$$A_4=\frac{19}{120},\qquad A_5=\frac{9}{20},$$

0.132[7] $$\sum_{k=1}^{n}\frac{1}{2k-1}=\frac{1}{2}\left(\boldsymbol{C}+\ln n\right)+\ln 2+\frac{B_2}{8n^2}+\frac{\left(2^3-1\right)B_4}{64n^4}+\dots$$ JO (71a)a

0.133 $$\sum_{k=2}^{n}\frac{1}{k^2-1}=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$$ JO (184f)

0.14 Sums of products of reciprocals of natural numbers

1. $$\sum_{k=1}^{n}\frac{1}{[p+(k-1)q](p+kq)}=\frac{n}{p(p+nq)}$$ GI III (64)a

2. $$\sum_{k=1}^{n}\frac{1}{[p+(k-1)q](p+kq)[p+(k+1)q]}=\frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]}$$ GI III (65)a

3. $$\sum_{k=1}^{n}\frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]}$$
$$=\frac{1}{(l+1)q}\left\{\frac{1}{p(p+q)\dots(p+lq)}-\frac{1}{(p+nq)[p+(n+1)q]\dots[p+(n+l)q]}\right\}$$
AD (1856)a

4. $$\sum_{k=1}^{n}\frac{1}{[1+(k-1)q][1+(k-l)q+p]}=\frac{1}{p}\left[\sum_{k=1}^{n}\frac{1}{1+(k-1)q}-\sum_{k=1}^{n}\frac{1}{1+(k-1)q+p}\right]$$
GI III (66)a

0.142 $$\sum_{k=1}^{n}\frac{k^2+k-1}{(k+2)!}=\frac{1}{2}-\frac{n+1}{(n+2)!}$$ JO (157)

0.15 Sums of the binomial coefficients

Notation: n is a natural number.

1. $$\sum_{k=0}^{m}\binom{n+k}{n}=\binom{n+m+1}{n+1}$$ KR 64 (70.1)

2. $$1+\binom{n}{2}+\binom{n}{4}+\dots=2^{n-1}$$ KR 62 (58.1)

3. $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots=2^{n-1}$ KR 62 (58.1)

4. $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$ $[n\geq 1]$ KR 64 (70.2)

0.152

1. $\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$ KR 62 (59.1)

2. $\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right)$ KR 62 (59.2)

3. $\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-4)\pi}{3}\right)$ KR 62 (59.3)

0.153

1. $\binom{n}{0}+\binom{n}{4}+\binom{n}{8}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right)$ KR 63 (60.1)

2. $\binom{n}{1}+\binom{n}{5}+\binom{n}{9}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right)$ KR 63 (60.2)

3. $\binom{n}{2}+\binom{n}{6}+\binom{n}{10}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right)$ KR 63 (60.3)

4. $\binom{n}{3}+\binom{n}{7}+\binom{n}{11}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right)$ KR 63 (60.4)

0.154

1. $\sum_{k=0}^{n}(k+1)\binom{n}{k}=2^{n-1}(n+2)$ $[n\geq 0]$ KR 63 (66.1)

2. $\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k}=0$ $[n\geq 2]$ KR 63 (66.2)

3. $\sum_{k=0}^{N}(-1)^k\binom{N}{k}k^{n-1}=0$ $[N\geq n\geq 1;\quad 0^0\equiv 1]$

4. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n=(-1)^n n!$ $[n\geq 0;\quad 0^0\equiv 1]$

5. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^n=(-1)^n n!$ $[n\geq 0;\quad 0^0\equiv 1]$

6. $\sum_{k=0}^{N}(-1)^k\binom{N}{k}(\alpha+k)^{n-1}=0$ $[N\geq n\geq 1,\quad 0^0\equiv 1\quad N,n\in N^+]$

0.155

1. $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k}=\frac{n}{n+1}$ KR 63 (67)

2. $$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$ KR 63 (68.1)

3. $$\sum_{k=0}^{n} \frac{\alpha^{k+1}}{k+1}\binom{n}{k} = \frac{(\alpha+1)^{n+1}-1}{n+1}$$ KR 63 (68.2)

4. $$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k} = \sum_{m=1}^{n} \frac{1}{m}$$ KR 64 (69)

0.156

1. $$\sum_{k=0}^{p} \binom{n}{k}\binom{m}{p-k} = \binom{n+m}{p} \qquad [m \text{ is a natural number}]$$ KR 64 (71.1)

2. $$\sum_{k=0}^{n-p} \binom{n}{k}\binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}$$ KR 64 (71.2)

0.157

1. $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$ KR 64 (72.1)

2. $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$$ KR 64 (72.2)

3. $$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0$$ KR 64 (72.3)

4. $$\sum_{k=1}^{n} k\binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}$$ KR 64 (72.4)

0.158[10]

1. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1}\binom{2n-k-1}{n-k-1}\right] k = 4^n - \binom{2n}{n}$$

2. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1}\binom{2n-k-1}{n-k-1}\right] k^2 = 4^n - \binom{2n}{n} 3\cdot 4^n$$

3. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1}\binom{2n-k-1}{n-k-1}\right] k^3 = (6n+13)4^n - 18n\binom{2n}{n}$$

4. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1}\binom{2n-k-1}{n-k-1}\right] k^4 = (32n^2+104n)\binom{2n}{n} - (60n+75)4^n$$

0.159[10]

1. $$\sum_{k=0}^{n} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1}\right] k = \frac{1}{2}\left[4^n - \binom{2n}{n}\right]$$

2. $$\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^2=\frac{1}{2}\left[(2n+1)\binom{2n}{n}-4^n\right]$$

3. $$\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^3=\frac{(3n+2)}{4}\cdot 4^n-\frac{1}{2}\binom{2n}{n}(3n+1)$$

0.160[10]

1. $$\sum_{k=n+1}^{2n}\binom{2n}{k}\alpha^k+\frac{1}{2}\binom{2n}{n}\alpha^n+\frac{(1+\alpha)^{2n-1}(1-\alpha)}{2}\sum_{k=0}^{n-1}\binom{2k}{k}\left[\frac{\alpha}{(1+\alpha)^2}\right]^k=\frac{1}{2}(1+\alpha)^{2n}$$

2. $$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{\Gamma(r+b)}{\Gamma(r+a)}=\frac{\mathrm{B}(n+a-b,b)}{\Gamma(a-b)}$$

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

The series

0.211 $$\sum_{k=1}^{\infty}u_k=u_1+u_2+u_3+\dots$$

is said to *converge absolutely* if the series

0.212 $$\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\cdots,$$

composed of the absolute values of its terms converges. If the series **0.211** converges and the series **0.212** diverges, the series **0.211** is said to *converge conditionally.* Every absolutely convergent series converges.

0.22 Convergence tests

Suppose that

$$\lim_{k\to\infty}|u_k|^{1/k}=q$$

If $q<1$, the series **0.211** converges absolutely. On the other hand, if $q>1$, the series **0.211** diverges. (Cauchy)

0.222 Suppose that

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|=q$$

Here, if $q<1$, the series **0.211** converges absolutely. If $q>1$, the series **0.211** diverges. If $\left|\frac{u_{k+1}}{u_k}\right|$ approaches 1 but remains greater than unity, then the series **0.211** diverges. (d'Alembert)

0.223 Suppose that

$$\lim_{k\to\infty}k\left\{\left|\frac{u_k}{u_{k+1}}\right|-1\right\}=q$$

Here, if $q>1$, the series **0.211** converges absolutely. If $q<1$, the series **0.211** diverges. (Raabe)

0.224 Suppose that $f(x)$ is a positive decreasing function and that

$$\lim_{k\to\infty} \frac{e^k f\left(e^k\right)}{f(k)} = q$$

for natural k. If $q < 1$, the series $\sum_{k=1}^{\infty} f(k)$ converges. If $q > 1$, this series diverges. (Ermakov)

0.225 Suppose that

$$\left|\frac{u_k}{u_{k+1}}\right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

where $p > 1$ and the $|v_k|$ are bounded, that is, the $|v_k|$ are all less than some M, which is independent of k. Here, if $q > 1$, the series **0.211** converges absolutely. If $q \le 1$, this series diverges. (Gauss)

0.226 Suppose that a function $f(x)$ defined for $x \ge q \ge 1$ is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges accordingly as the integral

$$\int_q^{\infty} f(x)\,dx$$

converges or diverges (the Cauchy integral test).

0.227 Suppose that all terms of a sequence $u_1, u_2, \ldots, u_n$ are positive. In such a case, the series

1. $$\sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \ldots$$

   is called an *alternating series.*

   If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

2. $$u_{k+1} < u_k \text{ and } \lim_{k\to\infty} u_k = 0,$$

   the series **0.227** 1 converges. Here, the remainder of the series is

3. $$\sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left|\sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^{n} (-1)^{k+1} u_k < u_{n+1}\right|$$ (Leibniz)

0.228 If the series

1. $$\sum_{k=1}^{\infty} v_k = v_1 + v_2 + \ldots + v_k + \ldots$$

   converges and the numbers u_k form a monotonic bounded sequence, that is, if $|u_k| < M$ for some number M and for all k, the series

2. $$\sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \ldots + u_k v_k + \ldots$$ FI II 354

   converges. (Abel)

0.229 If the partial sums of the series **0.228** 1 are bounded and if the numbers u_k constitute a monotonic sequence that approaches zero, that is, if

$$\left|\sum_{k=1}^{n} v_k\right| < M \qquad [n = 1, 2, \ldots] \qquad \text{and } \lim_{k\to\infty} u_k = 0, \qquad \text{FI II 355}$$

then the series **0.228** 2 converges (Dirichlet).

0.23–0.24 Examples of numerical series

0.231 Progressions

1. $$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} \qquad [|q| < 1]$$

2. $$\sum_{k=0}^{\infty} (a+kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} \qquad [|q| < 1] \qquad \text{(cf. } \mathbf{0.113}\text{)}$$

0.232

1. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = \ln 2 \qquad \text{(cf. } \mathbf{1.511}\text{)}$$

2. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{2k-1} = 1 - 2\sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4}$$

   (cf. **1.643**)

3.* $$\sum_{k=1}^{\infty} \frac{k^a}{b^k} = \frac{1}{(b-1)^{a+1}} \sum_{i=1}^{a} \left[\frac{1}{b^{a-i}} \sum_{j=0}^{i} \frac{(-1)^j (a+1)!(i-j)^a}{j!(a+1-j)!}\right]$$

   $[a = 1, 2, 3, \ldots, \quad b \neq 1]$

0.233

1. $$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots = \zeta(p) \qquad [\operatorname{Re} p > 1] \qquad \text{WH}$$

2. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k^p} = (1 - 2^{1-p})\,\zeta(p) \qquad [\operatorname{Re} p > 0] \qquad \text{WH}$$

3.[10] $$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{FI II 721}$$

4. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k^{2n}} = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}| \qquad \text{JO (165)}$$

5. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2\cdot(2n)!}|B_{2n}| \qquad \text{JO (184b)}$$

6. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}|E_{2n}| \qquad \text{JO (184d)}$$

0.234

1. $$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}=\frac{\pi^2}{12}$$ EU

2. $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$$ EU

3. $$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=\boldsymbol{G}$$ FI II 482

4. $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3}=\frac{\pi^3}{32}$$ EU

5. $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}$$ EU

6. $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^5}=\frac{5\pi^5}{1536}$$ EU

7. $$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{(k+1)^2}=\frac{\pi^2}{12}-\ln 2$$

8.[6] $$\sum_{k=1}^{\infty}\frac{1}{k(2k+1)}=2-2\ln 2$$

9.* $$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n^2\,\Gamma(n)}=\sqrt{\pi}\ln 4$$

0.235 $$S_n=\sum_{k=1}^{\infty}\frac{1}{\left(4k^2-1\right)^n}$$

$$S_1=\frac{1}{2},\quad S_2=\frac{\pi^2-8}{16},\quad S_3=\frac{32-3\pi^2}{64},\quad S_4=\frac{\pi^4+30\pi^2-384}{768}$$

JO (186)

0.236

1. $$\sum_{k=1}^{\infty}\frac{1}{k\left(4k^2-1\right)}=2\ln 2-1$$ BR 51a

2. $$\sum_{k=1}^{\infty}\frac{1}{k\left(9k^2-1\right)}=\frac{3}{2}\left(\ln 3-1\right)$$ BR 51a

3. $$\sum_{k=1}^{\infty}\frac{1}{k\left(36k^2-1\right)}=-3+\frac{3}{2}\ln 3+2\ln 2$$ BR 52, AD (6913.3)

4. $$\sum_{k=1}^{\infty}\frac{k}{\left(4k^2-1\right)^2}=\frac{1}{8}$$ BR 52

5. $$\sum_{k=1}^{\infty}\frac{1}{k\left(4k^2-1\right)^2}=\frac{3}{2}-2\ln 2$$ BR 52

6. $$\sum_{k=1}^{\infty} \frac{12k^2-1}{k\left(4k^2-1\right)^2} = 2\ln 2$$ AD (6917.3), BR 52

7.[6] $$\sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2\ln 2$$

0.237

1. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}$$ AD (6917.2), BR 52

2. $$\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}$$

3. $$\sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4}$$ [cf. **0.133**],

4. $$\sum_{k=1,k\neq m}^{\infty}{}' \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2}$$ [m is an integer] AD (6916.1)

5. $$\sum_{k=1,k\neq m}^{\infty}{}' \frac{(-1)^{k-1}}{(m-k)(m+k)} = \frac{3}{4m^2}$$ [m is an even number] AD (6916.2)

0.238

1. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2}$$ GI III (93)

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2}\left(1-\ln 2\right)$$ GI III (94)a

3. $$\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4}\ln 3 + \frac{\pi}{12\sqrt{3}}$$ GI III (95)

0.239

1.[11] $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-2} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2\right)$$ GI III (85), BR* 161 (1)

2.[7] $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} - \ln 2\right)$$ BR* 161 (1)

3. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}}\left[\pi + 2\ln\left(\sqrt{2}+1\right)\right]$$ BR* 161 (1)

4. $$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2}\ln 2$$ GI III (87)

5. $$\sum_{k=1}^{\infty}(-1)^{\left[\frac{k+3}{2}\right]}\frac{1}{2k-1}=\frac{\pi}{2\sqrt{2}}$$

6. $$\sum_{k=1}^{\infty}(-1)^{\left[\frac{k+5}{3}\right]}\frac{1}{2k-1}=\frac{5\pi}{12}$$ GI III (88)

7. $$\sum_{k=1}^{\infty}\frac{1}{(8k-1)(8k+1)}=\frac{1}{2}-\frac{\pi}{16}\left(\sqrt{2}+1\right)$$

0.241

1. $$\sum_{k=1}^{\infty}\frac{1}{2^k k}=\ln 2$$ JO (172g)

2. $$\sum_{k=1}^{\infty}\frac{1}{2^k k^2}=\frac{\pi^2}{12}-\frac{1}{2}(\ln 2)^2$$ JO (174)

3.[11] $$\sum_{n=0}^{\infty}\binom{2n}{n}p^n=\frac{1}{\sqrt{1-4p}}\qquad\left[0\le p<\tfrac{1}{4}\right]$$

4.[10] $$\sum_{n=1}^{\infty}\frac{p^n}{n^2}=\frac{\pi^2}{6}-\int_1^p\frac{\ln(1-x)}{x}\,dx\qquad[0\le p\le 1]$$

5.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j=4^i-\binom{2i}{i}$$
$$\left[\binom{n}{m}=0,\quad m<0\right]$$

6.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^2=4i\binom{2i}{i}-3\cdot 4^i$$
$$\left[\binom{n}{m}=0,\quad m<0\right]$$

7.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^3=(6i+13)4^i-18i\binom{2i}{i}$$
$$\left[\binom{n}{m}=0,\quad m<0\right]$$

8.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^4=\left(32i^2+104i\right)\binom{2i}{i}-(60i+75)4^i$$

9.[10] $$\sum_{j=n+1}^{2n}\binom{2n}{j}k^j+\frac{1}{2}\binom{2n}{n}k^n+\frac{(1+k)^{2n-1}(1-k)}{2}\sum_{i=0}^{n-1}\binom{2i}{i}\left[\frac{k}{(1+k)^2}\right]^i=\frac{1}{2}(1+k)^{2n}$$

10.[10] $$\sum_{k=0}^{i}\binom{i+k}{k}2^{i-k}=4^i$$

11.[10] $$\sum_{k=0}^{i}\binom{i+k}{h}^{i-k}k=(i+1)4^i-(2i+1)\binom{2i}{i}$$

12.[10] $$\sum_{k=0}^{i}\binom{2i}{k}=\frac{1}{2}\left[4^i+\binom{2i}{i}\right]$$

13.[10] $$\sum_{k=0}^{i}\binom{2i}{k}k=\frac{i}{2}4^i$$

14.[10] $$\sum_{k=0}^{i}\binom{2i}{k}k^2=(2i+1)i4^{i-1}-\frac{i^2}{2}\binom{2i}{i}$$

0.242 $$\sum_{k=0}^{\infty}(-1)^k\frac{1}{n^{2k}}=\frac{n^2}{n^2+1} \qquad [n>1]$$

0.243

1. $$\sum_{k=1}^{\infty}\frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]}=\frac{1}{(l+1)q}\frac{1}{p(p+q)\dots(p+lq)}$$

(see also **0.141** 3)

2.[7] $$\sum_{k=1}^{\infty}\frac{x^{k-1}}{[p+(k-1)q][p+(k-1)q+1][p+(k-1)q+2]\dots[p+(k-1)q+l]}=\frac{1}{l!}\int_0^i\frac{t^{p-1}(1-t)^t}{1-xt^q}\,dt$$

$[p>0,\quad x^2<1]$ BR* 161 (2), AD (6.704)

3. $$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\left(\frac{1}{x}\tanh\left[\frac{(2k+1)\pi x}{2}\right]+x\tanh\left[\frac{(2k+1)\pi}{2x}\right]\right)=\frac{\pi^3}{16}$$

0.244

1. $$\sum_{k=1}^{\infty}\frac{1}{(k+p)(k+q)}=\frac{1}{q-p}\int_0^1\frac{x^p-x^q}{1-x}\,dx \qquad [p>-1,\quad q>-1,\quad p\neq q]$$ GI III (90)

2. $$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{p+(k-1)q}=\int_0^1\frac{t^{p-1}}{1+t^q}\,dt \qquad [p>0,\quad q>0]$$ BR* 161 (1)

3.[10] $$\sum_{k=1}^{\infty}\frac{1}{(k+p)(k+q)}=\frac{1}{q-p}\sum_{m=p+1}^{q}\frac{1}{m} \qquad [q>p>-1,\quad p \text{ and } q \text{ integers}]$$

Summations of reciprocals of factorials

0.245

1. $$\sum_{k=0}^{\infty}\frac{1}{k!}=e=2.71828\dots$$

2.[11] $$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=\frac{1}{2e}\approx 0.1839397\dots$$

3. $$\sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787\ldots$$

4. $$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$$

5. $$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2}\left(e + \frac{1}{e}\right) = 1.54308\ldots$$

6. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2}\left(e - \frac{1}{e}\right) = 1.17520\ldots$$

7. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030\ldots$$

8. $$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147\ldots$$

0.246

1. $$\sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530\ldots$$

2. $$\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855\ldots$$

3. $$\sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2)$$

4. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078\ldots$$

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} = J_1(2) = 0.57672481\ldots$$

6. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} = J_n(2)$$

0.247 $$\sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2)\cdot(n-1)!}$$

0.248 $$\sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$S_1 = e, \qquad S_2 = 2e, \qquad S_3 = 5e, \qquad S_4 = 15e$$
$$S_5 = 52e, \qquad S_6 = 203e, \qquad S_7 = 877e, \qquad S_8 = 4140e$$

0.249[7] $$\sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e$$

0.25 Infinite products

0.250 Suppose that a sequence of numbers $a_1, a_2, \ldots, a_k, \ldots$ is given. If the limit $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} (1+a_k)$ exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product* $\prod\limits_{k=1}^{\infty} (1+a_k)$, and we write

1. $$\lim_{n\to\infty} \prod_{k=1}^{n} (1+a_k) = \prod_{k=1}^{\infty} (1+a_k)$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no a_k is equal to -1. FI II 400

0.251 For the infinite product **0.250** 1. to converge, it is necessary that $\lim\limits_{k\to\infty} a_k = 0$. FI II 403

0.252 If $a_k > 0$ or $a_k < 0$ for all values of the index k starting with some particular value, then, for the product **0.250** 1 to converge, it is necessary and sufficient that the series $\sum_{k=1}^{\infty} a_k$ converge.

0.253 The product $\prod\limits_{k=1}^{\infty} (1+a_k)$ is said to converge absolutely if the product $\prod\limits_{k=1}^{\infty} (1+|a_k|)$ converges. FI II 403

0.254 Absolute convergence of an infinite product implies its convergence.

0.255 The product $\prod\limits_{k=1}^{\infty} (1+a_k)$ converges absolutely if, and only if, the series $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely. FI II 406

0.26 Examples of infinite products

0.261 $$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}$$ EU

0.262

1. $$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$$ FI II 401

2. $$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi}$$ FI II 401

3. $$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4}$$ FI II 401

0.263

1. $$e = \frac{2}{1} \cdot \left(\frac{4}{3}\right)^{1/2} \left(\frac{6\cdot 8}{5\cdot 7}\right)^{1/4} \left(\frac{10\cdot 12\cdot 14\cdot 16}{9\cdot 11\cdot 13\cdot 15}\right)^{1/8} \cdots$$

2.* $$e = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/3} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/4} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/5} \cdots$$

3.* $$\frac{\pi}{2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/4} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/8} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/16} \cdots$$

where the nth factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}$.

0.264

1. $$e^{C} = \prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1+\frac{1}{k}}$$ FI II 402

2.* $$e^{C} = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/3} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/4} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/5} \cdots$$

where the nth factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^{n}(k+1)^{(-1)^{k+1}\binom{n}{k}}$. Here $\boldsymbol{C}$ is the Euler constant, denoted in other works by γ.

0.265 $$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$ FI II 402

0.266[8] $$\prod_{k=0}^{\infty}\left(1+x^{2^k}\right) = \frac{1}{1-x} \qquad [0<x<1]$$ FI II 401

0.3 Functional Series

0.30 Definitions and theorems

0.301 The series

1. $$\sum_{k=1}^{\infty} f_k(x),$$

the terms of which are functions, is called a *functional series*. The set of values of the independent variable x for which the series **0.301** 1 converges constitutes what is called the *region of convergence* of that series.

0.302 A series that converges for all values of x in a region M is said to *converge uniformly* in that region if, for every $\varepsilon \geq 0$, there exists a number N such that, for $n > N$, the inequality

$$\left|\sum_{k=n+1}^{\infty} f_k(x)\right| < \varepsilon$$

holds for *all* x in M.

0.303 If the terms of the functional series **0.301** 1 satisfy the inequalities:

$$|f_k(x)| < u_k \quad (k = 1, 2, 3, \ldots),$$

throughout the region M, where the u_k are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_k + \ldots,$$

the series **0.301** 1 converges uniformly in M. (Weierstrass)

0.304 Suppose that the series **0.301** 1 converges uniformly in a region M and that a set of functions $g_k(x)$ constitutes (for each x) a monotonic sequence, and that these functions are uniformly bounded; that is, suppose that a number L exists such that the inequalities

1. $|g_n(x)| \leq L$

   hold for all n and x. Then, the series

2. $$\sum_{k=1}^{\infty} f_k(x) g_k(x)$$

   converges uniformly in the region M. (Abel) FI II 451

0.305 Suppose that the partial sums of the series **0.301** 1 are uniformly bounded; that is, suppose that, for some L and for all n and x in M, the inequalities

$$\left| \sum_{k=1}^{n} f_k(x) \right| < L$$

hold. Suppose also that for each x the functions $g_n(x)$ constitute a monotonic sequence that approaches zero uniformly in the region M. Then, the series **0.304** 2 converges uniformly in the region M. (Dirichlet) FI II 451

0.306[6] If the functions $f_k(x)$ (for $k = 1, 2, 3, \dots$) are integrable on the interval $[a, b]$ and if the series **0.301** 1 made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\, dx \qquad [a \leq x \leq b]$$ FI II 459

0.307 Suppose that the functions $f_k(x)$ (for $k = 1, 2, 3, \dots$) have continuous derivatives $f_k'(x)$ on the interval $[a, b]$. If the series **0.301** 1 converges on this interval and if the series $\sum_{k=1}^{\infty} f_k'(x)$ of these derivatives converges uniformly, the series **0.301** 1 may be differentiated termwise; that is,

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f_k'(x)$$ FI II 460

0.31 Power series

0.311 A functional series of the form

1. $$\sum_{k=0}^{\infty} a_k (x - \xi)^k = a_0 + a_1(x - \xi) + a_2(x - \xi)^2 + \dots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point ξ and a radius equal to R; at every interior point of this circle, the power series **0.311** 1 converges absolutely, and outside this circle, it diverges. This circle is called the *circle of convergence*, and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ($R = +\infty$).

0.312 Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^{x}\left\{\sum_{k=0}^{\infty}a_k(x-\xi)^k\right\}dx=\sum_{k=0}^{\infty}\frac{a_k}{k+1}(x-\xi)^{k+1},$$
$$\frac{d}{dx}\left\{\sum_{k=0}^{\infty}a_k(x-\xi)^k\right\}=\sum_{k=1}^{\infty}ka_k(x-\xi)^{k-1}.$$

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

Operations on power series

0.313 Division of power series.

$$\frac{\sum_{k=0}^{\infty}b_kx^k}{\sum_{k=0}^{\infty}a_kx^k}=\frac{1}{a_0}\sum_{k=0}^{\infty}c_kx^k,$$

where

$$c_n+\frac{1}{a_0}\sum_{k=1}^{n}c_{n-k}a_k-b_n=0,$$

or

$$c_n=\frac{(-1)^n}{a_0^n}\begin{bmatrix} a_1b_0-a_0b_1 & a_0 & 0 & \cdots & 0\\ a_2b_0-a_0b_2 & a_1 & a_0 & & 0\\ a_3b_0-a_0b_3 & a_2 & a_1 & & 0\\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1}b_0-a_0b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0\\ a_nb_0-a_0b_n & a_{n-1} & a_{n-2} & \cdots & a_1\end{bmatrix} \qquad \text{AD (6360)}$$

0.314 Power series raised to powers.

$$\left(\sum_{k=0}^{\infty}a_kx^k\right)^n=\sum_{k=0}^{\infty}c_kx^k,$$

where

$$c_0=a_0^n,\quad c_m=\frac{1}{ma_0}\sum_{k=1}^{m}(kn-m+k)a_kc_{m-k}\qquad \text{for } m\geq 1\qquad [n \text{ is a natural number}] \qquad \text{AD (6361)}$$

0.315 The substitution of one series into another.

$$\sum_{k=1}^{\infty}b_ky^k=\sum_{k=1}^{\infty}c_kx^k\qquad y=\sum_{k=1}^{\infty}a_kx^k;$$
$$c_1=a_1b_1,\quad c_2=a_2b_1+a_1^2b_2,\quad c_3=a_3b_1+2a_1a_2b_2+a_1^3b_3,$$
$$c_4=a_4b_1+a_2^2b_2+2a_1a_3b_2+3a_1^2a_2b_3+a_1^4b_4,\quad \ldots \qquad \text{AD (6362)}$$

0.316 Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$ FI II 372

Taylor series

0.317 If a function $f(x)$ has derivatives of all orders throughout a neighborhood of a point ξ, then we may write the series

1. $$f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots ,$$

   which is known as the *Taylor series* of the function $f(x)$.

   The Taylor series converges to the function $f(x)$ if the remainder

2. $$R_n(x) = f(x) - f(\xi) - \sum_{k=1}^{n} \frac{(x-\xi)^k}{k!} f^{(k)}(\xi)$$

   approaches zero as $n \to \infty$.

   The following are different forms for the remainder of a Taylor series:

3. $$R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < \theta < 1]$$ (Lagrange)

4. $$R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < \theta < 1]$$ (Cauchy)

5. $$R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi + \theta(x-\xi))$$

   $$[0 < \theta < 1]\,,$$ (Schlömilch)

   where $\psi(x)$ is an arbitrary function satisfying the following two conditions: (1) It and its derivative $\psi'(x)$ are continuous in the interval $(0, x-\xi)$; and (2) the derivative $\psi'(x)$ does not change sign in that interval. If we set $\psi(x) = x^{p+1}$, we obtain the following form for the remainder:

   $$R_n(x) = \frac{(x-\xi)^{n+1}(1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < p \le n; \quad 0 < \theta < 1]$$ (Rouché)

6. $$R_n(x) = \frac{1}{n!} \int_{\xi}^{x} f^{(n+1)}(t)(x-t)^n \, dt$$

0.318 Other forms in which a Taylor series may be written:

1.[11] $$f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

2. $$\sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$ (Maclaurin series)

0.319 The Taylor series of functions of several variables:

$$f(x,y) = f(\xi,\eta) + (x-\xi)\frac{\partial f(\xi,\eta)}{\partial x} + (y-\eta)\frac{\partial f(\xi,\eta)}{\partial y}$$
$$+ \frac{1}{2!}\left\{(x-\xi)^2\frac{\partial^2 f(\xi,\eta)}{\partial x^2} + 2(x-\xi)(y-\eta)\frac{\partial^2 f(\xi,\eta)}{\partial x\,\partial y} + (y-\eta)^2\frac{\partial^2 f(\xi,\eta)}{\partial y^2}\right\} + \dots$$

0.32 Fourier series

0.320 Suppose that $f(x)$ is a *periodic* function of period $2l$ and that it is absolutely integrable (possibly improperly) over the interval $(-l,l)$. The following trigonometric series is called the *Fourier series* of $f(x)$:

1. $$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi x}{l} + b_k\sin\frac{k\pi x}{l},\right)$$

   the coefficients of which (the Fourier coefficients) are given by the formulas

2. $$a_k = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\cos\frac{k\pi t}{l}\,dt \quad (k=0,1,2,\dots)$$

3.[11] $$b_k = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\sin\frac{k\pi t}{l}\,dt \quad (k=1,2,\dots)$$

Convergence tests

0.321 The Fourier series of a function $f(x)$ at a point x_0 converges to the number

$$\frac{f(x_0+0)+f(x_0-0)}{2},$$

if, for some $h>0$, the integral

$$\int_0^h \frac{|f(x_0+t)+f(x_0-t)-f(x_0+0)-f(x_0-0)|}{t}\,dt$$

exists. Here, it is assumed that the function $f(x)$ either is continuous at the point x_0 or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits $f(x_0+0)$ and $f(x_0-0)$ exist. (Dini) FI III 524

0.322 The Fourier series of a periodic function $f(x)$ that satisfies the Dirichlet conditions on the interval $[a,b]$ converges at every point x_0 to the value $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$. (Dirichlet)

We say that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a,b]$ if it is bounded on that interval and if the interval $[a,b]$ can be partitioned into a finite number of subintervals inside each of which the function $f(x)$ is continuous and monotonic.

0.323 The Fourier series of a function $f(x)$ at a point x_0 converges to $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$ if $f(x)$ is of bounded variation in some interval (x_0-h, x_0+h) with center at x_0. (Jordan–Dirichlet) FI III 528

The definition of a function of bounded variation. Suppose that a function $f(x)$ is defined on some interval $[a,b]$, where $z<b$. Let us partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

and let us form the sum

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

Different partitions of the interval $[a, b]$ (that is, different choices of points of division x_i) yield, generally speaking, different sums. If the set of these sums is bounded above, we say that the function $f(x)$ is *of bounded variation* on the interval $[a, b]$. The least upper bound of these sums is called the *total variation* of the function $f(x)$ on the interval $[a, b]$.

0.324 Suppose that a function $f(x)$ is piecewise-continuous on the interval $[a, b]$ and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point x_0 of the interval $[a, b]$, the Fourier series of the function $f(x)$ converges to $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$.

0.325 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a cosine series of the form

1. $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

   where

2. $$a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l}\, dt$$

0.326 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a sine series of the form

1. $$\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

   where

2. $$b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l}\, dt$$

The convergence tests for the series **0.325** 1 and **0.326** 1 are analogous to the convergence tests for the series **0.320** 1 (see **0.321**–**0.324**).

0.327 The Fourier coefficients a_k and b_k (given by formulas **0.320** 2 and **0.320** 3) of an absolutely integrable function approach zero as $k \to \infty$.

If a function $f(x)$ is square-integrable on the interval $(-l, l)$, the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) = \frac{1}{l} \int_{-l}^{l} f^2(x)\, dx \qquad \text{(A. M. Lyapunov)}$$ FI III 705

0.328 Suppose that $f(x)$ and $\varphi(x)$ are two functions that are square-integrable on the interval $(-l, l)$ and that a_k, b_k and α_k, β_k are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty} (a_k\alpha_k + b_k\beta_k) = \frac{1}{l} \int_{-l}^{l} f(x)\varphi(x)\, dx$$ FI III 709

For examples of Fourier series, see **1.44** and **1.45**.

0.33 Asymptotic series

0.330 Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. **0.227** 3).

Asymptotic series have many properties that are analogous to the properties of convergent series, and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}$$

This is the definition of an asymptotic expansion. The divergent series $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ is called the *asymptotic expansion* of a function $f(z)$ in a given region of values of $\arg z$ if the expression $R_n(z) = z^n\,[f(z) - S_n(z)]$, where $S_n(z) = \sum_{k=0}^{n} \frac{A_k}{z^k}$, satisfies the condition $\lim_{|z|\to\infty} R_n(z) = 0$ for fixed n. FI II 820

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series*.

0.331 Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.
2. One asymptotic series can be divided by another, provided that the first term A_0 of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic. FI II 823-825
3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible. FI II 824
4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an asymptotic series in only one manner.

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

0.410 $$\frac{d}{da}\int_{\psi(a)}^{\varphi(a)} f(x,a)\,dx = f(\varphi(a),a)\frac{d\varphi(a)}{da} - f(\psi(a),a)\frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x,a)\,dx$$ FI II 680

0.411 In particular,

1. $$\frac{d}{da}\int_b^a f(x)\,dx = f(a)$$
2. $$\frac{d}{db}\int_b^a f(x)\,dx = -f(b)$$

0.42 The nth derivative of a product (Leibniz's rule)

Suppose that u and v are n-times-differentiable functions of x. Then,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^nv}{dx^n} + \binom{n}{1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \binom{n}{2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \binom{n}{3}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \cdots + v\frac{d^nu}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u+v)^{(n)}$$ FI I 272

0.43 The nth derivative of a composite function

0.430 If $f(x) = F(y)$and $y = \varphi(x)$, then

1. $$\frac{d^n}{dx^n}f(x) = \frac{U_1}{1!}F'(y) + \frac{U_2}{2!}F''(y) + \frac{U_3}{3!}F'''(y) + \ldots + \frac{U_n}{n!}F^{(n)}(y),$$
where
$$U_k = \frac{d^n}{dx^n}y^k - \frac{k}{1!}y\frac{d^n}{dx^n}y^{k-1} + \frac{k(k-1)}{2!}y^2\frac{d^n}{dx^n}y^{k-2} - \ldots + (-1)^{k-1}ky^{k-1}\frac{d^ny}{dx^n}$$ AD (7361) GO

2. $$\frac{d^n}{dx^n}f(x) = \sum \frac{n!}{i!j!h!\ldots k!}\frac{d^mF}{dy^m}\left(\frac{y'}{1!}\right)^i\left(\frac{y''}{2!}\right)^j\left(\frac{y'''}{3!}\right)^h\cdots\left(\frac{y^{(l)}}{l!}\right)^k,$$
Here, the symbol $\sum$ indicates summation over all solutions in non-negative integers of the equation $i + 2j + 3h + \ldots + lk = n$ and $m = i + j + h + \ldots + k$.

0.431

1. $$(-1)^n\frac{d^n}{dx^n}F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}}F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}}\frac{n}{1!}F^{(n-1)}\left(\frac{1}{x}\right) + \frac{(n-1)(n-2)}{x^{2n-2}}\frac{n(n-1)}{2!}F^{(n-2)}\left(\frac{1}{x}\right) + \ldots$$
AD (7362.1)

2. $$(-1)^n\frac{d^n}{dx^n}e^{\frac{a}{x}} = \frac{1}{x^n}e^{\frac{a}{x}}\left[\left(\frac{a}{x}\right)^n + (n-1)\binom{n}{1}\left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2)\binom{n}{2}\left(\frac{a}{x}\right)^{n-2} + (n-1)(n-2)(n-3)\binom{n}{3}\left(\frac{a}{x}\right)^{n-3} + \ldots\right]$$
AD (7362.2)

0.432

1. $$\frac{d^n}{dx^n}F\left(x^2\right) = (2x)^nF^{(n)}\left(x^2\right) + \frac{n(n-1)}{1!}(2x)^{n-2}F^{(n-1)}\left(x^2\right) + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}F^{(n-2)}\left(x^2\right) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}(2x)^{n-6}F^{(n-3)}\left(x^2\right) + \ldots$$
AD (7363.1)

2. $$\frac{d^n}{dx^n}e^{ax^2} = (2ax)^n e^{ax^2}\left[1 + \frac{n(n-1)}{1!\,(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!\,(4ax^2)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!\,(4ax^2)^3} + \dots\right]$$

AD (7363.2)

3. $$\frac{d^n}{dx^n}\left(1+ax^2\right)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times\left\{1 + \frac{n(n-1)}{1!(p-n+1)}\frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)}\left(\frac{1+ax^2}{4ax^2}\right)^2 + \dots\right\},$$

AD (7363.3)

4. $$\frac{d^{m-1}}{dx^{m-1}}\left(1-x^2\right)^{m-\frac{1}{2}} = (-1)^{m-1}\frac{(2m-1)!!}{m}\sin\left(m\arccos x\right)$$ AD (7363.4)

5. $$(-1)^n\frac{\partial^n}{\partial a^n}\left(\frac{a}{a^2+b^2}\right) = n!\left(\frac{a}{a^2+b^2}\right)^{n+1}\sum_{0\le 2k\le n+1}(-1)^k\binom{n+1}{2k}\left(\frac{b}{a}\right)^{2k}$$ (3.944.12)

6. $$(-1)^n\frac{\partial^n}{\partial a^n}\left(\frac{b}{a^2+b^2}\right) = n!\left(\frac{a}{a^2+b^2}\right)^{n+1}\sum_{0\le 2k\le n}(-1)^k\binom{n+1}{2k+1}\left(\frac{b}{a}\right)^{2k+1}$$ (3.944.11)

0.433

1. $$\frac{d^n}{dx^n}F\left(\sqrt{x}\right) = \frac{F^{(n)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^n} - \frac{n(n-1)}{1!}\frac{F^{(n-1)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!}\frac{F^{(n-2)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^{n+2}} - \dots$$

AD (7364.1)

2. $$\frac{d^n}{dx^n}\left(1+a\sqrt{x}\right)^{2n-1} = \frac{(2n-1)!!}{2^n}\frac{a}{\sqrt{x}}\left(a^2-\frac{1}{x}\right)^{n-1}$$ AD (7364.2)

0.434 $$\frac{d^n}{dx^n}y^p = p\binom{n-p}{n}\left\{-\binom{n}{1}\frac{1}{p-1}y^{p-1}\frac{d^ny}{dx^n} + \binom{n}{2}\frac{1}{p-2}y^{p-2}\frac{d^n\left(y^2\right)}{dx^n} - \dots\right\}$$ AD (737.1)

0.435 $$\frac{d^n}{dx^n}\ln y = \left\{\binom{n}{1}\frac{1}{1\cdot y}\frac{d^ny}{dx^n} - \binom{n}{2}\frac{1}{2\cdot y^2}\frac{d^n\left(y^2\right)}{dx^n} + \frac{d^n\left(y^3\right)}{dx^n}x^n - \dots\right\}$$ AD (737.2)

0.44 Integration by substitution

0.440[11] Let $f(g(x))$ and $g(x)$ be continuous in $[a,b]$. Further, let $g'(x)$ exist and be continuous there. Then $\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

This page intentionally left blank

1 Elementary Functions

1.1 Power of Binomials

1.11 Power series

1.110 $(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\ldots(q-k+1)}{k!}x^k + \cdots = \sum_{k=0}^{\infty} \binom{q}{k} x^k$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series converges for $q > -1$ and diverges for $q \leq -1$. For $x = 1$, the series converges absolutely for $q > 0$. For $x = -1$, it converges absolutely for $q > 0$ and diverges for $q < 0$. If $q = n$ is a natural number, the series **1.110** is reduced to the finite sum **1.111**. FI II 425

1.111 $(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$

1.112

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$

   (see also **1.121** 2)

2. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}$

3.[11] $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \ldots$

4. $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ldots$

1.113 $\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k$ $\quad [x^2 < 1]$

1.114

1. $\left(1+\sqrt{1+x}\right)^q = 2^q \left[1 + \frac{q}{1!}\left(\frac{x}{4}\right) + \frac{q(q-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{q(q-4)(q-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots\right]$

   $[x^2 < 1, \quad q \text{ is a real number}]$

   AD (6351.1)

2. $$\left(x+\sqrt{1+x^2}\right)^q = 1+\sum_{k=0}^{\infty}\frac{q^2\left(q^2-2^2\right)\left(q^2-4^2\right)\ldots\left[q^2-(2k)^2\right]x^{2k+2}}{(2k+2)!}$$
$$+qx+q\sum_{k=1}^{\infty}\frac{\left(q^2-1^2\right)\left(q^2-3^2\right)\ldots\left[q^2-(2k-1)^2\right]}{(2k+1)!}x^{2k+1}$$
$$\left[x^2<1,\quad q \text{ is a real number}\right]$$ AD(6351.2)

1.12 Series of rational fractions

1.121

1. $$\frac{x}{1-x}=\sum_{k=1}^{\infty}\frac{2^{k-1}x^{2^{k-1}}}{1+x^{2^{k-1}}}=\sum_{k=1}^{\infty}\frac{x^{2^{k-1}}}{1-x^{2^k}}\qquad\left[x^2<1\right]$$ AD (6350.3)

2. $$\frac{1}{x-1}=\sum_{k=1}^{\infty}\frac{2^{k-1}}{x^{2^{k-1}}+1}\qquad\left[x^2>1\right]$$ AD (6350.3)

1.2 The Exponential Function

1.21 Series representation

1.211

1.[11] $$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

2. $$a^x=\sum_{k=0}^{\infty}\frac{(x\ln a)^k}{k!}$$

3. $$e^{-x^2}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}$$

4.* $$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

1.212 $$e^x(1+x)=\sum_{k=0}^{\infty}\frac{x^k(k+1)}{k!}$$

1.213 $$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k)!}\qquad[x<2\pi]$$ FI II 520

1.214 $$e^{e^x}=e\left(1+x+\frac{2x^2}{2!}+\frac{5x^3}{3!}+\frac{15x^4}{4!}+\ldots\right)$$ AD (6460.3)

1.215

1. $$e^{\sin x}=1+x+\frac{x^2}{2!}-\frac{3x^4}{4!}-\frac{8x^5}{5!}-\frac{3x^6}{6!}+\frac{56x^7}{7!}+\ldots$$ AD (6460.4)

2. $$e^{\cos x}=e\left(1-\frac{x^2}{2!}+\frac{4x^4}{4!}-\frac{31x^6}{6!}+\ldots\right)$$ AD (6460.5)

3. $e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots$ AD (6460.6)

1.216

1. $e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots$ AD (6460.7)

2. $e^{\arctan x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots$ AD (6460.8)

1.217

1. $\pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{1}{x^2 + k^2}$ (cf. **1.421** 3) AD (6707.1)

2. $\frac{2\pi}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 + k^2}$ (cf. **1.422** 3) AD (6707.2)

1.22 Functional relations

1.221

1. $a^x = e^{x \ln a}$

2. $a^{\log_a x} = a^{\frac{1}{\log_x a}} = x$

1.222

1. $e^x = \cosh x + \sinh x$

2. $e^{ix} = \cos x + i \sin x$

1.223 $e^{ax} - e^{bx} = (a-b)x \exp\left[\frac{1}{2}(a+b)x\right] \prod_{k=1}^{\infty} \left[1 + \frac{(a-b)^2 x^2}{2k^2 \pi^2}\right]$ MO 216

1.23 Series of exponentials

1.231 $\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1 - a^x}$ $[a > 1 \text{ and } x < 0 \text{ or } 0 < a < 1 \text{ and } x > 0]$

1.232

1. $\tanh x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx}$ $[x > 0]$

2. $\operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x}$ $[x > 0]$

3. $\operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x}$ $[x > 0]$

4.* $\sin x = \exp\left[-\sum_{n=1}^{\infty} \frac{\cos^{2n} x}{2n}\right]$ $[0 \le x \le \pi]$

1.3–1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\sinh x = \frac{1}{i}\sin(ix), \qquad \sin x = \frac{1}{i}\sinh(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\cosh x = \cos(ix), \qquad \cos x = \cosh(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful "pairs" are included in the list.

1.31 The basic functional relations

1.311

1. $$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -i\sinh(ix)$$

2. $$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) = -i\sin(ix)$$

3. $$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cosh(ix)$$

4. $$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cos(ix)$$

5. $$\tan x = \frac{\sin x}{\cos x} = \frac{1}{i}\tanh(ix)$$

6. $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{1}{i}\tan(ix)$$

7. $$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = i\coth(ix)$$

8. $$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = i\cot(ix)$$

1.312

1. $$\cos^2 x + \sin^2 x = 1$$

2. $\cosh^2 x - \sinh^2 x = 1$

1.313

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
2. $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$
3. $\sin(x \pm iy) = \sin x \cosh y \pm i \sinh y \cos x$
4. $\sinh(x \pm iy) = \sinh x \cos y \pm i \sin y \cosh x$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
7. $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$
8. $\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$
9. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
10. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
11. $\tan(x \pm iy) = \dfrac{\tan x \pm i \tanh y}{1 \mp i \tan x \tanh y}$
12. $\tanh(x \pm iy) = \dfrac{\tanh x \pm i \tan y}{1 \pm i \tanh x \tan y}$

1.314

1. $\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$
2. $\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$
3. $\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$
4. $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cosh \frac{1}{2}(x-y)$
5. $\cos x - \cos y = 2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(y-x)$
6. $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \sinh \frac{1}{2}(x-y)$
7. $\tan x \pm \tan y = \dfrac{\sin(x \pm y)}{\cos x \cos y}$
8. $\tanh x \pm \tanh y = \dfrac{\sinh(x \pm y)}{\cosh x \cosh y}$
9.* $$\begin{aligned} \sin x \pm \cos y &= \pm 2 \sin\left[\frac{1}{2}(x+y) \pm \frac{\pi}{4}\right] \sin\left[\frac{1}{2}(x-y) \pm \frac{\pi}{4}\right] \\ &= \pm 2 \cos\left[\frac{1}{2}(x+y) \mp \frac{\pi}{4}\right] \cos\left[\frac{1}{2}(x-y) \mp \frac{\pi}{4}\right] \\ &= 2 \sin\left[\frac{1}{2}(x \pm y) \pm \frac{\pi}{4}\right] \cos\left[\frac{1}{2}(x \mp y) \mp \frac{\pi}{4}\right] \end{aligned}$$

10.* $$a\sin x \pm b\cos x = a\sqrt{1+\left(\frac{b}{a}\right)^2}\sin\left[x \pm \arctan\left(\frac{b}{a}\right)\right]$$

$[a \neq 0]$

11.* $$\pm a\sin x + b\cos x = b\sqrt{1+\left(\frac{a}{b}\right)^2}\cos\left[x \mp \arctan\left(\frac{a}{b}\right)\right]$$

$[b \neq 0]$

12.* $$a\sin x \pm b\cos y = q\sqrt{1+\left(\frac{r}{q}\right)^2}\sin\left[\frac{1}{2}(x \pm y) + \arctan\left(\frac{r}{q}\right)\right]$$

$$q = (a+b)\cos\left[\frac{1}{2}(x \mp y)\right], \quad r = (a-b)\sin\left[\frac{1}{2}(x \mp y)\right] \qquad [q \neq 0]$$

13.* $$a\cos x + b\cos y = t\sqrt{1+\left(\frac{s}{t}\right)^2}\cos\left[\frac{1}{2}(x \mp y) + \arctan\left(\frac{s}{t}\right)\right] \qquad [t \neq 0]$$

$$= -s\sqrt{1+\left(\frac{t}{s}\right)^2}\cos\left[\frac{1}{2}(x \mp y) - \arctan\left(\frac{t}{s}\right)\right] \qquad [s \neq 0]$$

$$s = (a-b)\sin\left[\frac{1}{2}(x \pm y)\right], \quad t = (a+b)\cos\left[\frac{1}{2}(x \pm y)\right]$$

1.315

1. $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$
2. $\sinh^2 x - \sinh^2 y = \sinh(x+y)\sinh(x-y) = \cosh^2 x - \cosh^2 y$
3. $\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y) = \cos^2 y - \sin^2 x$
4. $\sinh^2 x + \cosh^2 y = \cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$

1.316

1. $(\cos x + i\sin x)^n = \cos nx + i\sin nx$ [n is an integer]
2. $(\cosh x + \sinh x)^n = \sinh nx + \cosh nx$ [n is an integer]

1.317

1. $\sin\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1-\cos x)}$
2. $\sinh\frac{x}{2} = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$
3. $\cos\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1+\cos x)}$
4. $\cosh\frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)}$
5. $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

6. $$\tanh\frac{x}{2}=\frac{\cosh x-1}{\sinh x}=\frac{\sinh x}{\cosh x+1}$$

The signs in front of the radical in formulas **1.317** 1, **1.317** 2, and **1.317** 3 are taken so as to agree with the signs of the left-hand members. The sign of the left hand members depends in turn on the value of x.

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.320

1. $$\sin^{2n}x=\frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1}(-1)^{n-k}2\binom{2n}{k}\cos 2(n-k)x+\binom{2n}{n}\right\}$$ KR 56 (10, 2)

2. $$\sinh^{2n}x=\frac{(-1)^n}{2^{2n}}\left\{\sum_{k=0}^{n-1}(-1)^{n-k}2\binom{2n}{k}\cosh 2(n-k)x+\binom{2n}{n}\right\}$$

3. $$\sin^{2n-1}x=\frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}(-1)^{n+k-1}\binom{2n-1}{k}\sin(2n-2k-1)x$$ KR 56 (10, 4)

4. $$\sinh^{2n-1}x=\frac{(-1)^{n-1}}{2^{2n-2}}\sum_{k=0}^{n-1}(-1)^{n+k-1}\binom{2n-1}{k}\sinh(2n-2k-1)x$$

5. $$\cos^{2n}x=\frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1}2\binom{2n}{k}\cos 2(n-k)x+\binom{2n}{n}\right\}$$ KR 56 (10, 1)

6. $$\cosh^{2n}x=\frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1}2\binom{2n}{k}\cosh 2(n-k)x+\binom{2n}{n}\right\}$$

7. $$\cos^{2n-1}x=\frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)x$$ KR 56 (10, 3)

8. $$\cosh^{2n-1}x=\frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cosh(2n-2k-1)x$$

Special cases

1.321

1. $$\sin^2 x=\frac{1}{2}\left(-\cos 2x+1\right)$$

2. $$\sin^3 x=\frac{1}{4}\left(-\sin 3x+3\sin x\right)$$

3. $$\sin^4 x=\frac{1}{8}\left(\cos 4x-4\cos 2x+3\right)$$

4. $$\sin^5 x=\frac{1}{16}\left(\sin 5x-5\sin 3x+10\sin x\right)$$

5. $\sin^6 x = \frac{1}{32}(-\cos 6x + 6\cos 4x - 15\cos 2x + 10)$

6. $\sin^7 x = \frac{1}{64}(-\sin 7x + 7\sin 5x - 21\sin 3x + 35\sin x)$

1.322

1. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$

2. $\sinh^3 x = \frac{1}{4}(\sinh 3x - 3\sinh x)$

3. $\sinh^4 x = \frac{1}{8}(\cosh 4x - 4\cosh 2x + 3)$

4. $\sinh^5 x = \frac{1}{16}(\sinh 5x - 5\sinh 3x + 10\sinh x)$

5. $\sinh^6 x = \frac{1}{32}(\cosh 6x - 6\cosh 4x + 15\cosh 2x + 10)$

6. $\sinh^7 x = \frac{1}{64}(\sinh 7x - 7\sinh 5x + 21\sinh 3x + 35\sinh x)$

1.323

1. $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

2. $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

3. $\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$

4. $\cos^5 x = \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x)$

5. $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$

6. $\cos^7 x = \frac{1}{64}(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x)$

1.324

1. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

2. $\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$

3. $\cosh^4 x = \frac{1}{8}(\cosh 4x + 4\cosh 2x + 3)$

4. $\cosh^5 x = \frac{1}{16}(\cosh 5x + 5\cosh 3x + 10\cosh x)$

5. $\cosh^6 x = \frac{1}{32}(\cosh 6x + 6\cosh 4x + 15\cosh 2x + 10)$

6. $\cosh^7 x = \frac{1}{64}(\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x)$

1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

1.331

1.[7] $$\sin nx = n\cos^{n-1}x\sin x - \binom{n}{3}\cos^{n-3}x\sin^3 x + \binom{n}{5}\cos^{n-5}x\sin^5 x - \dots;$$
$$= \sin x\left\{2^{n-1}\cos^{n-1}x - \binom{n-2}{1}2^{n-3}\cos^{n-3}x + \binom{n-3}{2}2^{n-5}\cos^{n-5}x - \binom{n-4}{3}2^{n-7}\cos^{n-7}x + \dots\right\}$$

AD (3.175)

2. $$\sinh nx = x\sum_{k=1}^{[(n+1)/2]}\binom{n}{2k-1}\sinh^{2k-2}x\cosh^{n-2k+1}x$$
$$= \sinh x\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n-k-1}{k}2^{n-2k-1}\cosh^{n-2k-1}x$$

3. $$\cos nx = \cos^n x - \binom{n}{2}\cos^{n-2}x\sin^2 x + \binom{n}{4}\cos^{n-4}x\sin^4 x - \dots;$$
$$= 2^{n-1}\cos^n x - \frac{n}{1}2^{n-3}\cos^{n-2}x + \frac{n}{2}\binom{n-3}{1}2^{n-5}\cos^{n-4}x - \frac{n}{3}\binom{n-4}{2}2^{n-7}\cos^{n-6}x + \dots$$

AD (3.175)

4.[3] $$\cosh nx = \sum_{k=0}^{[n/2]}\binom{n}{2k}\sinh^{2k}x\cosh^{n-2k}x$$
$$= 2^{n-1}\cosh^n x + n\sum_{k=1}^{[n/2]}(-1)^k\frac{1}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\cosh^{n-2k}x$$

1.332

1. $$\sin 2nx = 2n\cos x\left\{\sin x - \frac{4n^2-2^2}{3!}\sin^3 x + \frac{(4n^2-2^2)(4n^2-4^2)}{5!}\sin^5 x - \dots\right\}$$ AD (3.171)
$$= (-1)^{n-1}\cos x\left\{2^{2n-1}\sin^{2n-1}x - \frac{2n-2}{1!}2^{2n-3}\sin^{2n-3}x + \frac{(2n-3)(2n-4)}{2!}2^{2n-5}\sin^{2n-5}x - \frac{(2n-4)(2n-5)(2n-6)}{3!}2^{2n-7}\sin^{2n-7}x + \dots\right\}$$ AD (3.173)

2. $$\sin(2n-1)x = (2n-1)\left\{\sin x - \frac{(2n-1)^2-1^2}{3!}\sin^3 x + \frac{\left[(2n-1)^2-1^2\right]\left[(2n-1)^2-3^2\right]}{5!}\sin^5 x - \dots\right\}$$ AD (3.172)

$$= (-1)^{n-1}\left\{2^{2n-2}\sin^{2n-1} x - \frac{2n-1}{1!}2^{2n-4}\sin^{2n-3} x + \frac{(2n-1)(2n-4)}{2!}2^{2n-6}\sin^{2n-5} x - \frac{(2n-1)(2n-5)(2n-6)}{3!}2^{2n-8}\sin^{2n-7} x + \dots\right\}$$ AD (3.174)

3. $$\cos 2nx = 1 - \frac{4n^2}{2!}\sin^2 x + \frac{4n^2\left(4n^2-2^2\right)}{4!}\sin^4 x - \frac{4n^2\left(4n^2-2\right)\left(4n^2-4^2\right)}{6!}\sin^6 x + \dots$$ AD (3.171)

$$= (-1)^n\left\{2^{2n-1}\sin^{2n} x - \frac{2n}{1!}2^{2n-3}\sin^{2n-2} x + \frac{2n(2n-3)}{2!}2^{2n-5}\sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!}2^{2n-7}\sin^{2n-6} x + \dots\right\}$$ AD (3.173)a

4. $$\cos(2n-1)x = \cos x\left\{1 - \frac{(2n-1)^2-1^2}{2!}\sin^2 x + \frac{\left[(2n-1)^2-1^2\right]\left[(2n-1)^2-3^2\right]}{4!}\sin^4 x - \dots\right\}$$ AD (3.172)

$$= (-1)^{n-1}\cos x\left\{2^{2n-2}\sin^{2n-2} x - \frac{2n-3}{1!}2^{2n-4}\sin^{2n-4} x + \frac{(2n-4)(2n-5)}{2!}2^{2n-6}\sin^{2n-6} x - \frac{(2n-5)(2n-6)(2n-7)}{3!}2^{2n-8}\sin^{2n-8} x + \dots\right\}$$ AD (3.174)

By using the formulas and values of **1.30**, we can write formulas for $\sinh 2nx$, $\sinh(2n-1)x$, $\cosh 2nx$, and $\cosh(2n-1)x$ that are analogous to those of **1.332**, just as was done in the formulas in **1.331**.

Special cases

1.333

1. $\sin 2x = 2\sin x\cos x$
2. $\sin 3x = 3\sin x - 4\sin^3 x$
3. $\sin 4x = \cos x\left(4\sin x - 8\sin^3 x\right)$
4. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$
5. $\sin 6x = \cos x\left(6\sin x - 32\sin^3 x + 32\sin^5 x\right)$

6. $\sin 7x = 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x$

1.334

1. $\sinh 2x = 2\sinh x\cosh x$
2. $\sinh 3x = 3\sinh x + 4\sinh^3 x$
3.[11] $\sinh 4x = \cosh x\left(4\sinh x + 8\sinh^3 x\right)$
4. $\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x$
5.[11] $\sinh 6x = \cosh x\left(6\sinh x + 32\sinh^3 x + 32\sinh^5 x\right)$
6. $\sinh 7x = 7\sinh x + 56\sinh^3 x + 112\sinh^5 x + 64\sinh^7 x$

1.335

1. $\cos 2x = 2\cos^2 x - 1$
2. $\cos 3x = 4\cos^3 x - 3\cos x$
3. $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$
4. $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$
5. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
6. $\cos 7x = 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

1.336

1. $\cosh 2x = 2\cosh^2 x - 1$
2. $\cosh 3x = 4\cosh^3 x - 3\cosh x$
3. $\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$
4. $\cosh 5x = 16\cosh^5 x - 20\cosh^3 x + 5\cosh x$
5. $\cosh 6x = 32\cosh^6 x - 48\cosh^4 x + 18\cosh^2 x - 1$
6. $\cosh 7x = 64\cosh^7 x - 112\cosh^5 x + 56\cosh^3 x - 7\cosh x$

1.337

1.* $\dfrac{\cos 3x}{\cos^3 x} = 1 - 3\tan^2 x$
2.* $\dfrac{\cos 4x}{\cos^4 x} = 1 - 6\tan^2 x + \tan^4 x$
3.* $\dfrac{\cos 5x}{\cos^5 x} = 1 - 10\tan^2 x + 5\tan^4 x$
4.* $\dfrac{\cos 6x}{\cos^6 x} = 1 - 15\tan^2 x + 15\tan^4 x - \tan^6 x$
5.* $\dfrac{\sin 3x}{\cos^3 x} = 3\tan x - \tan^3 x$
6.* $\dfrac{\sin 4x}{\cos^4 x} = 4\tan x - 4\tan^3 x$

7.* $\dfrac{\sin 5x}{\cos^5 x} = 5\tan x - 10\tan^3 x + \tan^5 x$

8.* $\dfrac{\sin 6x}{\cos^6 x} = 6\tan x - 20\tan^3 x + 6\tan^5 x$

9.* $\dfrac{\cos 3x}{\sin^3 x} = \cot^3 x - 3\cot x$

10.* $\dfrac{\cos 4x}{\sin^4 x} = \cot^4 x - 6\cot^2 x + 1$

11.* $\dfrac{\cos 5x}{\sin^5 x} = \cot^5 x - 10\cot^3 x + 5\cot x$

12.* $\dfrac{\cos 6x}{\sin^6 x} = \cot^6 x - 15\cot^4 x + 15\cot^2 x - 1$

13.* $\dfrac{\sin 3x}{\sin^3 x} = 3\cot^2 x - 1$

14.* $\dfrac{\sin 4x}{\sin^4 x} = 4\cot^3 x - 4\cot x$

15.* $\dfrac{\sin 5x}{\sin^5 x} = 5\cot^4 x - 10\cot^2 x + 1$

16.* $\dfrac{\sin 6x}{\sin^6 x} = 6\cot^5 x - 20\cot^3 x + 6\cot x$

1.34 Certain sums of trigonometric and hyperbolic functions

1.341

1. $$\sum_{k=0}^{n-1} \sin(x+ky) = \sin\left(x + \frac{n-1}{2}y\right)\sin\frac{ny}{2}\operatorname{cosec}\frac{y}{2}$$ AD (361.8)

2. $$\sum_{k=0}^{n-1} \sinh(x+ky) = \sinh\left(x + \frac{n-1}{2}y\right)\sinh\frac{ny}{2}\frac{1}{\sinh\frac{y}{2}}$$

3. $$\sum_{k=0}^{n-1} \cos(x+ky) = \cos\left(x + \frac{n-1}{2}y\right)\sin\frac{ny}{2}\operatorname{cosec}\frac{y}{2}$$ AD (361.9)

4. $$\sum_{k=0}^{n-1} \cosh(x+ky) = \cosh\left(x + \frac{n-1}{2}y\right)\sinh\frac{ny}{2}\frac{1}{\sinh\frac{y}{2}}$$

5. $$\sum_{k=0}^{2n-1} (-1)^k\cos(x+ky) = \sin\left(x + \frac{2n-1}{2}y\right)\sin ny\sec\frac{y}{2}$$ JO (202)

6. $$\sum_{k=0}^{n-1} (-1)^k\sin(x+ky) = \sin\left(x + \frac{n-1}{2}(y+\pi)\right)\sin\frac{n(y+\pi)}{2}\sec\frac{y}{2}$$ AD (202a)

Special cases

1.342

1. $$\sum_{k=1}^{n} \sin kx = \sin \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$$ AD (361.1)

2.[10] $$\sum_{k=0}^{n} \cos kx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 = \cos \frac{nx}{2} \sin \frac{n+1}{2} x \operatorname{cosec} \frac{x}{2} = \frac{1}{2} \left(1 + \frac{\sin \left(n + \frac{1}{2}\right) x}{\sin \frac{x}{2}} \right)$$ AD (361.2)

3. $$\sum_{k=1}^{n} \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x$$ AD (361.7)

4. $$\sum_{k=1}^{n} \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x$$ JO (207)

1.343

1. $$\sum_{k=1}^{n} (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos\left(\frac{2n+1}{2} x\right)}{2 \cos \frac{x}{2}}$$ AD (361.11)

2. $$\sum_{k=1}^{n} (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x}$$ AD (361.10)

3. $$\sum_{k=1}^{n} \cos(4k-3)x + \sum_{k=1}^{n} \sin(4k-1)x = \sin 2nx \left(\cos 2nx + \sin 2nx\right) \left(\cos x + \sin x\right) \operatorname{cosec} 2x$$ JO (208)

1.344

1. $$\sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \cot \frac{\pi}{2n}$$ AD (361.19)

2. $$\sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right)$$ AD (361.18)

3. $$\sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right)$$ AD (361.17)

1.35 Sums of powers of trigonometric functions of multiple angles

1.351

1. $$\sum_{k=1}^{n} \sin^2 kx = \frac{1}{4} \left[(2n+1) \sin x - \sin(2n+1)x\right] \operatorname{cosec} x = \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x}$$ AD (361.3)

2. $$\sum_{k=1}^{n} \cos^2 kx = \frac{n-1}{2} + \frac{1}{2}\cos nx \sin(n+1)x \operatorname{cosec} x = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2\sin x}$$ AD (361.4)a

3. $$\sum_{k=1}^{n} \sin^3 kx = \frac{3}{4}\sin\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2} - \frac{1}{4}\sin\frac{3(n+1)x}{2}\sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}$$ JO (210)

4. $$\sum_{k=1}^{n} \cos^3 kx = \frac{3}{4}\cos\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2} + \frac{1}{4}\cos\frac{3(n+1)}{2}x \sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}$$ JO (211)a

5. $$\sum_{k=1}^{n} \sin^4 kx = \frac{1}{8}\left[3n - 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x\right]$$ JO (212)

6. $$\sum_{k=1}^{n} \cos^4 kx = \frac{1}{8}\left[3n + 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x\right]$$ JO (213)

1.352

1.[11] $$\sum_{k=1}^{n-1} k\sin kx = \frac{\sin nx}{4\sin^2\frac{x}{2}} - \frac{n\cos\left(\frac{2n-1}{2}x\right)}{2\sin\frac{x}{2}}$$ AD (361.5)

2.[11] $$\sum_{k=1}^{n-1} k\cos kx = \frac{n\sin\left(\frac{2n-1}{2}x\right)}{2\sin\frac{x}{2}} - \frac{1-\cos nx}{4\sin^2\frac{x}{2}}$$ AD (361.6)

1.353

1. $$\sum_{k=1}^{n-1} p^k \sin kx = \frac{p\sin x - p^n \sin nx + p^{n+1}\sin(n-1)x}{1-2p\cos x + p^2}$$ AD (361.12)a

2. $$\sum_{k=1}^{n-1} p^k \sinh kx = \frac{p\sinh x - p^n \sinh nx + p^{n+1}\sinh(n-1)x}{1-2p\cosh x + p^2}$$

3. $$\sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p\cos x - p^n \cos nx + p^{n+1}\cos(n-1)x}{1-2p\cos x + p^2}$$ AD (361.13)a¡

4. $$\sum_{k=0}^{n-1} p^k \cosh kx = \frac{1 - p\cosh x - p^n \cosh nx + p^{n+1}\cosh(n-1)x}{1-2p\cosh x + p^2}$$ JO (396)

1.36 Sums of products of trigonometric functions of multiple angles

1.361

1. $$\sum_{k=1}^{n} \sin kx \sin(k+1)x = \frac{1}{4}\left[(n+1)\sin 2x - \sin 2(n+1)x\right]\operatorname{cosec} x$$ JO (214)

2. $$\sum_{k=1}^{n} \sin kx \sin(k+2)x = \frac{n}{2}\cos 2x - \frac{1}{2}\cos(n+3)x \sin nx \operatorname{cosec} x$$ JO (216)

3. $$2\sum_{k=1}^{n}\sin kx\cos(2k-1)y=\sin\left(ny+\frac{n+1}{2}x\right)\sin\frac{n(x+2y)}{2}\operatorname{cosec}\frac{x+2y}{2}$$
$$-\sin\left(ny-\frac{n+1}{2}x\right)\sin\frac{n(2y-x)}{2}\operatorname{cosec}\frac{2y-x}{2}$$
JO (217)

1.362

1. $$\sum_{k=1}^{n}\left(2^k\sin^2\frac{x}{2^k}\right)^2=\left(2^n\sin\frac{x}{2^n}\right)^2-\sin^2 x$$ AD (361.15)

2. $$\sum_{k=1}^{n}\left(\frac{1}{2^k}\sec\frac{x}{2^k}\right)^2=\operatorname{cosec}^2 x-\left(\frac{1}{2^n}\operatorname{cosec}\frac{x}{2^n}\right)^2$$ AD (361.14)

1.37 Sums of tangents of multiple angles

1.371

1. $$\sum_{k=0}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}=\frac{1}{2^n}\cot\frac{x}{2^n}-2\cot 2x$$ AD (361.16)

2. $$\sum_{k=0}^{n}\frac{1}{2^{2k}}\tan^2\frac{x}{2^k}=\frac{2^{2n+2}-1}{3\cdot 2^{2n-1}}+4\cot^2 2x-\frac{1}{2^{2n}}\cot^2\frac{x}{2^n}$$ AD (361.20)

1.38 Sums leading to hyperbolic tangents and cotangents

1.381

1. $$\sum_{k=0}^{n-1}\frac{\tanh\left(x\dfrac{1}{n\sin^2\left(\dfrac{2k+1}{4n}\pi\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{2k+1}{4n}\pi\right)}}=\tanh(2nx)$$ JO (402)a

2. $$\sum_{k=1}^{n-1}\frac{\tanh\left(x\dfrac{1}{n\sin^2\left(\dfrac{k\pi}{2n}\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{k\pi}{2n}\right)}}=\coth(2nx)-\frac{1}{2n}(\tanh x+\coth x)$$ JO (403)

3. $$\sum_{k=0}^{n-1} \frac{\tanh\left(x \dfrac{2}{(2n+1)\sin^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}} = \tanh(2n+1)\,x - \frac{\tanh x}{2n+1}$$ JO (404)

4. $$\sum_{k=1}^{n} \frac{\tanh\left(x \dfrac{2}{(2n+1)\sin^2\left(\dfrac{k\pi}{2(2n+1)}\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{k\pi}{(2n+1)}\right)}} = \coth(2n+1)\,x - \frac{\coth x}{2n+1}$$ JO (405)

1.382

1. $$\sum_{k=0}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{2k+1}{4n}\pi\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = 2n\tanh(nx)$$ JO (406)

2. $$\sum_{k=1}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{k\pi}{2n}\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = 2n\coth(nx) - 2\coth x$$ JO (407)

3. $$\sum_{k=0}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = (2n+1)\tanh\left(\frac{(2n+1)x}{2}\right) - \tanh\frac{x}{2}$$ JO (408)

4. $$\sum_{k=1}^{n} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{k\pi}{2n+1}\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = (2n+1)\coth\left(\frac{(2n+1)x}{2}\right) - \coth\frac{x}{2}$$ JO (409)

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.391

1. $$\sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \qquad [n \text{ is even}]$$ JO (568)

2. $$\cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}}\right) \qquad [n \text{ is even}]$$ JO (569)

3. $$\sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \qquad [n \text{ is odd}]$$ JO (570)

4. $$\cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}}\right) \qquad [n \text{ is odd}]$$ JO (571)a

1.392

1. $$\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(x + \frac{k\pi}{n}\right)$$ JO (548)

2. $$\cos nx = 2^{n-1} \prod_{k=1}^{n} \sin\left(x + \frac{2k-1}{2n}\pi\right)$$ JO (549)

1.393

1. $$\begin{aligned} \prod_{k=0}^{n-1} \cos\left(x + \frac{2k}{n}\pi\right) &= \frac{1}{2^{n-1}} \cos nx && [n \text{ odd}] \\ &= \frac{1}{2^{n-1}} \left[(-1)^{\frac{n}{2}} - \cos nx\right] && [n \text{ even}] \end{aligned}$$ JO (543)

2.[11] $$\begin{aligned} \prod_{k=0}^{n-1} \sin\left(x + \frac{2k}{n}\pi\right) &= \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx && [n \text{ odd}] \\ &= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) && [n \text{ even}] \end{aligned}$$ JO (544)

1.394 $$\prod_{k=0}^{n-1} \left\{x^2 - 2xy \cos\left(\alpha + \frac{2k\pi}{n}\right) + y^2\right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}$$ JO (573)

1.395

1. $$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{\cos x - \cos\left(y + \frac{2k\pi}{n}\right)\right\}$$ JO (573)

2. $$\cosh nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cosh x - \cos\left(y + \frac{2k\pi}{n}\right) \right\}$$ JO (538)

1.396

1. $$\prod_{k=1}^{n-1} \left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right) = \frac{x^{2n}-1}{x^2-1}$$ KR 58 (28.1)

2. $$\prod_{k=1}^{n} \left(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\right) = \frac{x^{2n+1}-1}{x-1}$$ KR 58 (28.2)

3. $$\prod_{k=1}^{n} \left(x^2 + 2x\cos\frac{2k\pi}{2n+1} + 1\right) = \frac{x^{2n+1}-1}{x+1}$$ KR 58 (28.3)

4. $$\prod_{k=0}^{n-1} \left(x^2 - 2x\cos\frac{(2k+1)\pi}{2n} + 1\right) = x^{2n} + 1$$ KR 58 (28.4)

1.41 The expansion of trigonometric and hyperbolic functions in power series

1.411

1. $$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

2. $$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

3. $$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

4. $$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

5. $$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)}{(2k)!} |B_{2k}| x^{2k-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$ FI II 523

6.[11] $$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315}x^7 + \cdots = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)}{(2k)!} B_{2k} x^{2k-1}$$
$$\left[x^2 < \frac{\pi^2}{4}\right]$$

7. $$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} \qquad \left[x^2 < \pi^2\right]$$ FI II 523a

8. $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1}$$
$$\left[x^2 < \pi^2\right]$$ FI II 522a

9. $\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k}$ $\left[x^2 < \frac{\pi^2}{4}\right]$ CE 330a

10. $\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}$

$\left[x^2 < \frac{\pi^2}{4}\right]$ CE 330

11. $\operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1} - 1\right)|B_{2k}| x^{2k-1}}{(2k)!}$ $\left[x^2 < \pi^2\right]$ CE 329a

12. $\operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1} - 1\right) B_{2k}}{(2k)!} x^{2k-1}$

$\left[x^2 < \pi^2\right]$ JO (418)

1.412

1. $\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}$ JO (452)a

2. $\cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}$ JO (443)

3. $\sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1} - 3}{(2k+1)!} x^{2k+1}$ JO (452a)a

4. $\cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^{k} \frac{\left(3^{2k} + 3\right) x^{2k}}{(2k)!}$ JO (443a)

1.413

1. $\sinh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!}$ JO (508)

2. $\cosh x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^{k} \frac{2^{2k} x^{4k}}{(4k)!}$ JO (507)

3. $\sinh x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}$ JO (510)

4. $\cosh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}$ JO (509)

1.414

1. $\cos\left[n \ln\left(x + \sqrt{1+x^2}\right)\right] = 1 - \sum_{k=0}^{\infty} (-1)^{k} \frac{\left(n^2 + 0^2\right)\left(n^2 + 2^2\right) \ldots \left[n^2 + (2k)^2\right]}{(2k+2)!} x^{2k+2}$

$\left[x^2 < 1\right]$ AD (6456.1)

2. $$\sin\left[n\ln\left(x+\sqrt{1+x^2}\right)\right] = nx - n\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(n^2+1^2\right)\left(n^2+3^2\right)\ldots\left[n^2+(2k-1)^2\right]x^{2k+1}}{(2k+1)!}$$
$$\left[x^2<1\right]$$ AD (6456.2)

Power series for $\ln\sin x$, $\ln\cos x$, and $\ln\tan x$ see **1.518**.

1.42 Expansion in series of simple fractions

1.421

1. $$\tan\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}$$ BR* (191), AD (6495.1)

2.[10] $$\tanh\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2+x^2}$$

3. $$\cot\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2-k^2} = \frac{1}{\pi x}+\frac{x}{\pi}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{k(x-k)}$$ AD (6495.2), JO (450a)

4. $$\coth\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2+k^2}$$ (cf. **1.217** 1)

5. $$\tan^2\frac{\pi x}{2} = x^2\sum_{k=1}^{\infty}\frac{2(2k-1)^2-x^2}{\left(1^2-x^2\right)^2\left(3^2-x^2\right)^2\ldots\left[(2k-1)^2-x^2\right]^2}$$ JO (450)

1.422

1. $$\sec\frac{\pi x}{2} = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k-1}{(2k-1)^2-x^2}$$ AD (6495.3)a

2. $$\sec^2\frac{\pi x}{2} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\left\{\frac{1}{(2k-1-x)^2}+\frac{1}{(2k-1+x)^2}\right\}$$ JO (451)a

3. $$\operatorname{cosec}\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{x^2-k^2}$$ (see also **1.217** 2) AD (6495.4)a

4. $$\operatorname{cosec}^2\pi x = \frac{1}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(x-k)^2} = \frac{1}{\pi^2x^2}+\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{x^2+k^2}{\left(x^2-k^2\right)^2}$$ JO (446)

5. $$\frac{1+x\operatorname{cosec}x}{2x^2} = \frac{1}{x^2}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\left(x^2-k^2\pi^2\right)}$$ JO (449)

6. $$\operatorname{cosec}\pi x = \frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{x^2-k^2}$$ JO (450b)

1.423 $$\frac{\pi^2}{4m^2}\operatorname{cosec}^2\frac{\pi}{m}+\frac{\pi}{4m}\cot\frac{\pi}{m}-\frac{1}{2} = \sum_{k=1}^{\infty}\frac{1}{\left(1-k^2m^2\right)^2}$$ JO (477)

1.43 Representation in the form of an infinite product

1.431

1. $\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)$ EU

2. $\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right)$ EU

3. $\cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right)$ EU

4. $\cosh x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2\pi^2}\right)$ EU

1.432

1.[11] $\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{(2k\pi+y)^2}\right)\left(1 - \frac{x^2}{(2k\pi-y)^2}\right)$ AD (653.2)

2. $\cosh x - \cos y = 2\left(1 + \frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left(1 + \frac{x^2}{(2k\pi+y)^2}\right)\left(1 + \frac{x^2}{(2k\pi-y)^2}\right)$ AD (653.1)

1.433 $\cos\frac{\pi x}{4} - \sin\frac{\pi x}{4} = \prod_{k=1}^{\infty}\left[1 + \frac{(-1)^k x}{2k-1}\right]$ BR* 189

1.434 $\cos^2 x = \frac{1}{4}(\pi+2x)^2\prod_{k=1}^{\infty}\left[1 - \left(\frac{\pi+2x}{2k\pi}\right)^2\right]^2$ MO 216

1.435 $\frac{\sin\pi(x+a)}{\sin\pi a} = \frac{x+a}{a}\prod_{k=1}^{\infty}\left(1 - \frac{x}{k-a}\right)\left(1 + \frac{x}{k+a}\right)$ MO 216

1.436 $1 - \frac{\sin^2\pi x}{\sin^2\pi a} = \prod_{k=-\infty}^{\infty}\left[1 - \left(\frac{x}{k-a}\right)^2\right]$ MO 216

1.437 $\frac{\sin 3x}{\sin x} = -\prod_{k=-\infty}^{\infty}\left[1 - \left(\frac{2x}{x+k\pi}\right)^2\right]$ MO 216

1.438 $\frac{\cosh x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty}\left[1 + \left(\frac{x}{2k\pi+a}\right)^2\right]$ MO 216

1.439

1. $\sin x = x\prod_{k=1}^{\infty}\cos\frac{x}{2^k}$ $[|x| < 1]$ AD (615), MO 216

2. $\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left[1 - \frac{4}{3}\sin^2\left(\frac{x}{3^k}\right)\right]$ MO 216

1.44–1.45 Trigonometric (Fourier) series

1.441

1. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad [0 < x < 2\pi]$$ FI III 539

2. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln [2(1 - \cos x)] \qquad [0 < x < 2\pi]$$ FI III 530a, AD (6814)

3. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \qquad [-\pi < x < \pi]$$ FI III 542

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left(2 \cos \frac{x}{2}\right) \qquad [-\pi < x < \pi]$$ FI III 550

1.442

1.[11] $$\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x \qquad [-\pi < x < \pi]$$ FI III 541

2. $$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2} \qquad [0 < x < \pi]$$

BR* 168, JO (266), GI III(195)

3. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$ BR* 168, JO (268)a

4.[10] $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \begin{cases} \dfrac{\pi}{4} & \left[-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right] \\ -\dfrac{\pi}{4} & \left[\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}\right] \end{cases}$$

BR* 168, JO (269)

1.443

1.[8] $$\sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k = (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}\left(\frac{x}{2}\right)$$

$$\left[0 \le x \le 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340, GE 71

2. $$\sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k = (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{x}{2}\right)$$

$$\left[0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340

3. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}=\frac{\pi^2}{6}-\frac{\pi x}{2}+\frac{x^2}{4}$ $[0\le x\le 2\pi]$ FI III 547

4. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos kx}{k^2}=\frac{\pi^2}{12}-\frac{x^2}{4}$ $[-\pi\le x\le \pi]$ FI III 544

5. $\displaystyle\sum_{k=1}^{\infty}\frac{\sin kx}{k^3}=\frac{\pi^2 x}{6}-\frac{\pi x^2}{4}+\frac{x^3}{12}$ $[0\le x\le 2\pi]$

6. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos kx}{k^4}=\frac{\pi^4}{90}-\frac{\pi^2 x^2}{12}+\frac{\pi x^3}{12}-\frac{x^4}{48}$ $[0\le x\le 2\pi]$ AD (6617)

7. $\displaystyle\sum_{k=1}^{\infty}\frac{\sin kx}{k^5}=\frac{\pi^4 x}{90}-\frac{\pi^2 x^3}{36}+\frac{\pi x^4}{48}-\frac{x^5}{240}$ $[0\le x\le 2\pi]$ AD (6818)

1.444

1. $\displaystyle\sum_{k=1}^{\infty}\frac{\sin 2(k+1)x}{k(k+1)}=\sin 2x-(\pi-2x)\sin^2 x-\sin x\cos x\ln\left(4\sin^2 x\right)$

$[0\le x\le \pi]$ BR* 168, GI III (190)

2. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2(k+1)x}{k(k+1)}=\cos 2x-\left(\frac{\pi}{2}-x\right)\sin 2x+\sin^2 x\ln\left(4\sin^2 x\right)$

$[0\le x\le \pi]$ BR* 168

3. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\sin(k+1)x}{k(k+1)}=\sin x-\frac{x}{2}(1+\cos x)-\sin x\ln\left|2\cos\frac{x}{2}\right|$ MO 213

4. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k+1)x}{k(k+1)}=\cos x-\frac{x}{2}\sin x-(1+\cos x)\ln\left|2\cos\frac{x}{2}\right|$ MO 213

5. $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{\sin(2k+1)x}{(2k+1)^2}=\frac{\pi}{4}x$ $\left[-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right]$

$\displaystyle=\frac{\pi}{4}(\pi-x)$ $\left[\frac{\pi}{2}\le x\le\frac{3}{2}\pi\right]$

MO 213

6.[6] $\displaystyle\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}=\frac{\pi}{4}\left(\frac{\pi}{2}-|x|\right)$ $[-\pi\le x\le \pi]$ FI III 546

7. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos 2kx}{(2k-1)(2k+1)}=\frac{1}{2}-\frac{\pi}{4}\sin x$ $\left[0\le x\le\frac{\pi}{2}\right]$ JO (591)

1.445

1. $\displaystyle\sum_{k=1}^{\infty}\frac{k\sin kx}{k^2+\alpha^2}=\frac{\pi}{2}\frac{\sinh\alpha(\pi-x)}{\sinh\alpha\pi}$ $[0<x<2\pi]$ BR* 157, JO (411)

2. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos kx}{k^2+\alpha^2}=\frac{\pi}{2\alpha}\frac{\cosh\alpha(\pi-x)}{\sinh\alpha\pi}-\frac{1}{2\alpha^2}$ $[0\le x\le 2\pi]$ BR* 257, JO (410)

3. $$\sum_{k=1}^{\infty}\frac{(-1)^k\cos kx}{k^2+\alpha^2}=\frac{\pi}{2\alpha}\frac{\cosh\alpha x}{\sinh\alpha\pi}-\frac{1}{2\alpha^2}\qquad[-\pi\le x\le\pi]$$ FI III 546

4. $$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k\sin kx}{k^2+\alpha^2}=\frac{\pi}{2}\frac{\sinh\alpha x}{\sinh\alpha\pi}\qquad[-\pi<x<\pi]$$ FI III, 546

5. $$\sum_{k=1}^{\infty}\frac{k\sin kx}{k^2-\alpha^2}=\pi\frac{\sin\{\alpha[(2m+1)\pi-x]\}}{2\sin\alpha\pi}\qquad\left[\text{if } x=2m\pi,\text{ then }\sum\cdots=0\right]$$
$[2m\pi<x<(2m+2)\pi,\quad \alpha \text{ not an integer}]$ MO 213

6. $$\sum_{k=1}^{\infty}\frac{\cos kx}{k^2-\alpha^2}=\frac{1}{2\alpha^2}-\frac{\pi}{2}\frac{\cos\left[\alpha\left\{(2m+1)\pi-x\right\}\right]}{\alpha\sin\alpha\pi}$$
$[2m\pi\le x\le(2m+2)\pi,\quad \alpha \text{ not an integer}]$ MO 213

7. $$\sum_{k=1}^{\infty}(-1)^k\frac{k\sin kx}{k^2-\alpha^2}=\pi\frac{\sin[\alpha(2m\pi-x)]}{2\sin\alpha\pi}\qquad\left[\text{if } x=(2m+1)\pi,\text{ then }\sum\cdots=0\right],$$
$[(2m-1)\pi<x<(2m+1)\pi,\alpha \text{ not an integer}]$ FI III 545a

8. $$\sum_{k=1}^{\infty}(-1)^k\frac{\cos kx}{k^2-\alpha^2}=\frac{1}{2\alpha^2}-\frac{\pi}{2}\frac{\cos[\alpha(2m\pi-x)]}{\alpha\sin\alpha\pi}$$
$[(2m-1)\pi\le x\le(2m+1)\pi,\alpha \text{ not an integer}]$ FI III 545a

9.* $$\sum_{n=-\infty}^{\infty}\frac{e^{in\alpha}}{(n-\beta)^2+\gamma^2}=\frac{\pi}{\gamma}\frac{e^{i\beta(\alpha-2\pi)}\sinh(\gamma\alpha)+e^{i\beta\alpha}\sinh\left[\gamma(2\pi-\alpha)\right]}{\cosh(2\pi\gamma)-\cos(2\pi\beta)}$$
$[0\le\alpha\le 2\pi]$

1.446 $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(2k+1)x}{(2k-1)(2k+1)(2k+3)}=\frac{\pi}{8}\cos^2 x-\frac{1}{3}\cos x$$
$\left[-\frac{\pi}{2}\le x\le\frac{\pi}{2}\right]$ BR* 256, GI III (189)

1.447

1. $$\sum_{k=1}^{\infty}p^k\sin kx=\frac{p\sin x}{1-2p\cos x+p^2}$$
$[|p|<1]$ FI II 559

2. $$\sum_{k=0}^{\infty}p^k\cos kx=\frac{1-p\cos x}{1-2p\cos x+p^2}$$
$[|p|<1]$ FI II 559

3. $$1+2\sum_{k=1}^{\infty}p^k\cos kx=\frac{1-p^2}{1-2p\cos x+p^2}$$
$[|p|<1]$ FI II 559a, MO 213

1.448

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$
$\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ FI II 559

2. $$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln \left(1 - 2p \cos x + p^2\right)$$
$\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ FI II 559

3. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$
$\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ JO (594)

4. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}$$
$\left[0 < x < 2\pi, \quad p^2 \le 1\right]$ JO (259)

5. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \sin x + p^2}{1 - 2p \sin x + p^2}$$
$\left[0 < x < \pi, \quad p^2 \le 1\right]$ JO (261)

6. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$
$\left[0 < x < \pi, \quad p^2 \le 1\right]$ JO (597)

1.449

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin (p \sin x)$$
$\left[p^2 \le 1\right]$ JO (486)

2. $$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos (p \sin x)$$
$\left[p^2 \le 1\right]$ JO (485)

Let $S(x) = -\frac{1}{x} \cos x + \frac{1}{x}$ and $C(x) = \frac{1}{x} \sin x$.

3.* $$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \left[C(ax) - \cot(\pi a) S(ax)\right]$$
$[0 < x < 2\pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

4.* $$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a} \left[S(ax) - \cot(\pi a) C(ax)\right]$$
$[0 \le x \le 2\pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

5.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$
$[-\pi < x < \pi, \quad a \ne 0, \pm 1, \pm 2, \ldots]$

6.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax) \qquad [-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

7.* $$\sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4} \left[C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax) \right]$$

$$[0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

8.* $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-a^2} C(nx) = -\frac{\pi}{4a} \left[S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax) \right]$$

$$[0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$$

9.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax) \qquad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$$

10.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax) \qquad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$$

Fourier expansions of hyperbolic functions

1.451

1. $$\sinh x = \cos x \sum_{k=0}^{\infty} \frac{\left(1^2+0^2\right)\left(1^2+2^2\right)\ldots\left[1^2+(2k)^2\right]}{(2k+1)!} \sin^{2k+1} x$$ JO (504)

2. $$\cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{\left(1^2+1^2\right)\left(1^2+3^2\right)\ldots\left[1^2+(2k-1)^2\right]}{(2k)!} \sin^{2k} x$$ JO (503)

1.452

1. $$\sinh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\cos(2k+1)\theta}{(2k+1)!}$$

$$\left[x^2 < 1\right]$$ JO (391)

2. $$\cosh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k}\cos 2k\theta}{(2k)!}$$

$$\left[x^2 < 1\right]$$ JO (390)

3. $$\sinh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=1}^{\infty} \frac{x^{2k}\sin 2k\theta}{(2k)!}$$

$$\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$$ JO (393)

4. $$\cosh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\sin(2k+1)\theta}{(2k+1)!}$$

$$\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$$ JO (392)

1.46 Series of products of exponential and trigonometric functions

1.461

1. $$\sum_{k=0}^{\infty} e^{-kt} \sin kx = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x} \qquad [t > 0]$$ MO 213

2. $$1 + 2 \sum_{k=1}^{\infty} e^{-kt} \cos kx = \frac{\sinh t}{\cosh t - \cos x} \qquad [t > 0]$$ MO 213

1.462[9] $$\sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[\frac{\sin^2 \dfrac{x+y}{2} + \sinh^2 t}{\sin^2 \dfrac{x-y}{2} + \sinh^2 t} \right]$$ MO 214

1.463

1. $$e^{x \cos \varphi} \cos (x \sin \varphi) = \sum_{n=0}^{\infty} \frac{x^n \cos n\varphi}{n!} \qquad \left[x^2 < 1\right]$$ AD (6476.1)

2. $$e^{x \cos \varphi} \sin (x \sin \varphi) = \sum_{n=1}^{\infty} \frac{x^n \sin n\varphi}{n!} \qquad \left[x^2 < 1\right]$$ AD (6476.2)

1.47 Series of hyperbolic functions

1.471

1. $$\sum_{k=1}^{\infty} \frac{\sinh kx}{k!} = e^{\cosh x} \sinh (\sinh x) .$$ JO (395)

2. $$\sum_{k=0}^{\infty} \frac{\cosh kx}{k!} = e^{\cosh x} \cosh (\sinh x) .$$ JO (394)

3. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \tanh \frac{(2m+1)\pi x}{2} + x \tanh \frac{(2m+1)\pi}{2x} \right] = \frac{\pi^3}{16}$$

1.472

1. $$\sum_{k=1}^{\infty} p^k \sinh kx = \frac{p \sinh x}{1 - 2p \cosh x + p^2} \qquad \left[p^2 < 1\right]$$ JO (396)

2. $$\sum_{k=0}^{\infty} p^k \cosh kx = \frac{1 - p \cosh x}{1 - 2p \cosh x + p^2} \qquad \left[p^2 < 1\right]$$ JO (397)a

1.48 Lobachevskiy's "Angle of Parallelism" $\Pi(x)$

1.480 Definition.

1. $$\Pi(x) = 2 \operatorname{arccot} e^x = 2 \arctan e^{-x} \qquad [x \geq 0]$$ LO III 297, LOI 120

2. $\Pi(x) = \pi - \Pi(-x)$ $[x < 0]$ LO III 183, LOI 193

1.481 Functional relations

1. $\sin \Pi(x) = \dfrac{1}{\cosh x}$ LO III 297
2. $\cos \Pi(x) = \tanh x$ LO III 297
3. $\tan \Pi(x) = \dfrac{1}{\sinh x}$ LO III 297
4. $\cot \Pi(x) = \sinh x$ LO III 297
5. $\sin \Pi(x+y) = \dfrac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$ LO III 297
6. $\cos \Pi(x+y) = \dfrac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$ LO III 183

1.482 Connection with the Gudermannian.

$$\operatorname{gd}(-x) = \Pi(x) - \frac{\pi}{2}$$

(Definite) integral of the angle of parallelism: cf. **4.581** and **4.561**.

1.49 The hyperbolic amplitude (the Gudermannian) $\operatorname{gd} x$

1.490 Definition.

1. $\operatorname{gd} x = \displaystyle\int_0^x \frac{dt}{\cosh t} = 2 \arctan e^x - \frac{\pi}{2}$ JA
2. $x = \displaystyle\int_0^{\operatorname{gd} x} \frac{dt}{\cos t} = \ln \tan \left(\frac{\operatorname{gd} x}{2} + \frac{\pi}{4} \right)$ JA

1.491 Functional relations.

1. $\cosh x = \sec(\operatorname{gd} x)$ AD (343.1), JA
2. $\sinh x = \tan(\operatorname{gd} x)$ AD (343.2), JA
3. $e^x = \sec(\operatorname{gd} x) + \tan(\operatorname{gd} x) = \tan \left(\dfrac{\pi}{4} + \dfrac{\operatorname{gd} x}{2} \right) = \dfrac{1 + \sin(\operatorname{gd} x)}{\cos(\operatorname{gd} x)}$ AD (343.5), JA
4. $\tanh x = \sin(\operatorname{gd} x)$ AD (343.3), JA
5. $\tanh \dfrac{x}{2} = \tan \left(\dfrac{1}{2} \operatorname{gd} x \right)$ AD (343.4), JA
6. $\arctan(\tanh x) = \dfrac{1}{2} \operatorname{gd} 2x$ AD (343.6a)

1.492 If $\gamma = \operatorname{gd} x$, then $ix = \operatorname{gd} i\gamma$ JA

1.493 Series expansion.

1. $\dfrac{\operatorname{gd} x}{2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tanh^{2k+1} \frac{x}{2}$ JA

2. $$\frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \tan^{2k+1}\left(\frac{1}{2}\operatorname{gd} x\right)$$ JA

3. $$\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \cdots$$ JA

4. $$x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \ldots \qquad \left[\operatorname{gd} x < \frac{\pi}{2}\right]$$ JA

1.5 The Logarithm

1.51 Series representation

1.511 $$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$$

$$[-1 < x \leq 1]$$

1.512

1. $$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$$

$$[0 < x \leq 2]$$

2. $$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right] = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}$$

$$[0 < x]$$

3. $$\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x-1}{x}\right)^k$$

$$\left[x \geq \tfrac{1}{2}\right]$$ AD (644.6)

4.* $$\ln x = \lim_{\epsilon \to 0}\left(\frac{x^\epsilon - 1}{\epsilon}\right)$$

1.513

1. $$\ln\frac{1+x}{1-x} = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}x^{2k-1} \qquad [x^2 < 1]$$ FI II 421

2. $$\ln\frac{x+1}{x-1} = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)x^{2k-1}} \qquad [x^2 > 1]$$ AD (644.9)

3. $$\ln\frac{x}{x-1} = \sum_{k=1}^{\infty}\frac{1}{kx^k} \qquad [x \leq -1 \text{ or } x > 1]$$ JO (88a)

4. $$\ln\frac{1}{1-x} = \sum_{k=1}^{\infty}\frac{x^k}{k} \qquad [-1 \leq x < 1]$$ JO (88b)

5. $$\frac{1-x}{x}\ln\frac{1}{1-x} = 1 - \sum_{k=1}^{\infty}\frac{x^k}{k(k+1)} \qquad [-1 \leq x < 1]$$ JO (102)

6. $$\frac{1}{1-x}\ln\frac{1}{1-x}=\sum_{k=1}^{\infty}x^k\sum_{n=1}^{k}\frac{1}{n}\qquad [x^2<1]$$ JO (88e)

7. $$\frac{(1-x)^2}{2x^3}\ln\frac{1}{1-x}=\frac{1}{2x^2}-\frac{3}{4x}+\sum_{k=1}^{\infty}\frac{x^{k-1}}{k(k+1)(k+2)}\qquad [-1\le x<1]$$ AD (6445.1)

1.514 $$\ln\left(1-2x\cos\varphi+x^2\right)=-2\sum_{k=1}^{\infty}\frac{\cos k\varphi}{k}x^k;\quad \ln\left(x+\sqrt{1+x^2}\right)=\operatorname{arcsinh}x$$

(see **1.631, 1.641, 1.642, 1.646**) $[x^2\le 1,\quad x\cos\varphi\ne 1]$ MO 98, FI II 485

1.515

1.[11] $$\ln\left(1+\sqrt{1+x^2}\right)=\ln 2+\frac{1\cdot 1}{2\cdot 2}x^2-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}x^4+\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6-\dots$$
$$=\ln 2-\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!}{2^{2k}\,(k!)^2}x^{2k}$$
$[x^2\le 1]$ JO (91)

2. $$\ln\left(1+\sqrt{1+x^2}\right)=\ln x+\frac{1}{x}-\frac{1}{2\cdot 3x^3}+\frac{1\cdot 3}{2\cdot 4\cdot 5x^5}-\dots$$
$$=\ln x+\frac{1}{x}+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!}{2^{2k-1}\cdot k!(k-1)!(2k+1)x^{2k+1}}$$
$[x^2\ge 1]$ AD (644.4)

3. $$\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right)=x-\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k-1}(k-1)!k!}{(2k+1)!}x^{2k+1}$$
$[x^2\le 1]$ JO (93)

4. $$\frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}=\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}\,(k!)^2}{(2k+1)!}x^{2k+1}\qquad [x^2\le 1]$$ JO (94)

1.516

1. $$\frac{1}{2}\left\{\ln(1\pm x)\right\}^2=\sum_{k=1}^{\infty}\frac{(\mp 1)^{k+1}\,x^{k+1}}{k+1}\sum_{n=1}^{k}\frac{1}{n}\qquad [x^2<1]$$ JO (86), JO (85)

2. $$\frac{1}{6}\left\{\ln(1+x)\right\}^3=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k+2}}{k+2}\sum_{n=1}^{k}\frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m}\qquad [x^2<1]$$ AD (644.14)

3. $$-\ln(1+x)\cdot\ln(1-x)=\sum_{k=1}^{\infty}\frac{x^{2k}}{k}\sum_{n=1}^{2k-1}\frac{(-1)^{n+1}}{n}\qquad [x^2<1]$$ JO (87)

4. $$\frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}+2\ln(1-x)\right\}=\frac{1}{2x}+\sum_{k=1}^{\infty}\frac{x^{k-1}}{(2k-1)2k(2k+1)}$$
$[0<x<1]$ AD (6445.2)

1.517

1.[6] $$\frac{1}{2x}\left\{1-\ln(1+x)-\frac{1-x}{\sqrt{x}}\arctan\sqrt{x}\right\}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k-1}}{(2k-1)2k(2k+1)}$$ $[0<x\le 1]$ AD (6445.3)

2. $$\frac{1}{2}\arctan x\ln\frac{1+x}{1-x}=\sum_{k=1}^{\infty}\frac{x^{4k-2}}{2k-1}\sum_{n=1}^{2k-1}\frac{(-1)^{n-1}}{2n-1}$$ $\left[x^2<1\right]$ BR* 163

3. $$\frac{1}{2}\arctan x\ln\left(1+x^2\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k+1}}{2k+1}\sum_{n=1}^{2k}\frac{1}{n}$$ $\left[x^2\ge 1\right]$ AD (6455.3)

1.518

1. $$\ln\sin x=\ln x-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\dots$$ $$=\ln x+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}B_{2k}x^{2k}}{k(2k)!}$$ $[0<x<\pi]$ AD (643.1)a

2.[3] $$\ln\cos x=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\dots$$ $$=-\sum_{k=1}^{\infty}\frac{2^{2k-1}\left(2^{2k}-1\right)|B_{2k}|}{k(2k)!}x^{2k}=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin^{2k}x}{k}$$ $\left[x^2<\frac{\pi^2}{4}\right]$ FI II 524

3. $$\ln\tan x=\ln x+\frac{x^2}{3}+\frac{7}{90}x^4+\frac{62}{2835}x^6+\frac{127}{18,900}x^8+\dots$$ $$=\ln x+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(2^{2k-1}-1\right)2^{2k}B_{2k}x^{2k}}{k(2k)!}$$ $\left[0<x<\frac{\pi}{2}\right]$ AD (643.3)a

1.52 Series of logarithms (cf. 1.431)

1.521

1. $$\sum_{k=1}^{\infty}\ln\left(1-\frac{4x^2}{(2k-1)^2\pi^2}\right)=\ln\cos x$$ $\left[-\frac{\pi}{2}<x<\frac{\pi}{2}\right]$

2. $$\sum_{k=1}^{\infty}\ln\left(1-\frac{x^2}{k^2\pi^2}\right)=\ln\sin x-\ln x$$ $[0<x<\pi]$

1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

1. $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}; \quad 0 \le \arccos x \le \pi$ $\quad [-1 \le x \le 1]$ FI II 553
2. $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}; \quad 0 < \operatorname{arccot} x < \pi$ $\quad [-\infty < x < +\infty]$ FI II 552

1.62–1.63 Functional relations

1.621 The relationship between the inverse and the direct trigonometric functions.

1. $\arcsin(\sin x) = x - 2n\pi \qquad \left[2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}\right]$

   $= -x + (2n+1)\pi \qquad \left[(2n+1)\pi - \frac{\pi}{2} \le x \le (2n+1)\pi + \frac{\pi}{2}\right]$
2. $\arccos(\cos x) = x - 2n\pi \qquad [2n\pi \le x \le (2n+1)\pi]$

   $= -x + 2(n+1)\pi \qquad [(2n+1)\pi \le x \le 2(n+1)\pi]$
3. $\arctan(\tan x) = x - n\pi \qquad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right]$
4. $\operatorname{arccot}(\cot x) = x - n\pi \qquad [n\pi < x < (n+1)\pi]$

1.622 The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

1. $\arcsin z = \frac{1}{i} \ln\left(iz + \sqrt{1-z^2}\right) = \frac{1}{i} \operatorname{arcsinh}(iz)$
2. $\arccos z = \frac{1}{i} \ln\left(z + \sqrt{z^2-1}\right) = \frac{1}{i} \operatorname{arccosh} z$
3. $\arctan z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} = \frac{1}{i} \operatorname{arctanh}(iz)$
4. $\operatorname{arccot} z = \frac{1}{2i} \ln \frac{iz-1}{iz+1} = i \operatorname{arccoth}(iz)$
5. $\operatorname{arcsinh} z = \ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{i} \arcsin(iz)$
6. $\operatorname{arccosh} z = \ln\left(z + \sqrt{z^2-1}\right) = i \arccos z$
7. $\operatorname{arctanh} z = \frac{1}{2} \ln \frac{1+z}{1-z} = \frac{1}{i} \arctan(iz)$
8. $\operatorname{arccoth} z = \frac{1}{2} \ln \frac{z+1}{z-1} = \frac{1}{i} \operatorname{arccot}(-iz)$

Relations between different inverse trigonometric functions

1.623

1. $\arcsin x + \arccos x = \dfrac{\pi}{2}$ NV 43

2. $\arctan x + \operatorname{arccot} x = \dfrac{\pi}{2}$ NV 43

1.624

1. $\arcsin x = \arccos\sqrt{1-x^2} \qquad [0 \le x \le 1]$ NV 47 (5)

   $= -\arccos\sqrt{1-x^2} \qquad [-1 \le x \le 0]$ NV 46 (2)

2. $\arcsin x = \arctan\dfrac{x}{\sqrt{1-x^2}} \qquad [x^2 < 1]$

3. $\arcsin x = \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} \qquad [0 < x \le 1]$

   $= \operatorname{arccot}\dfrac{\sqrt{1-x^2}}{x} - \pi \qquad [-1 \le x < 0]$ NV 49 (10)

4. $\arccos x = \arcsin\sqrt{1-x^2} \qquad [0 \le x \le 1]$

   $= \pi - \arcsin\sqrt{1-x^2} \qquad [-1 \le x \le 0]$ NV 48 (6)

5. $\arccos x = \arctan\dfrac{\sqrt{1-x^2}}{x} \qquad [0 < x \le 1]$

   $= \pi + \arctan\dfrac{\sqrt{1-x^2}}{x} \qquad [-1 \le x < 0]$ NV 48 (8)

6. $\arccos x = \operatorname{arccot}\dfrac{x}{\sqrt{1-x^2}} \qquad [-1 \le x < 1]$ NV 46 (4)

7. $\arctan x = \arcsin\dfrac{x}{\sqrt{1+x^2}}$ NV 6 (3)

8. $\arctan x = \arccos\dfrac{1}{\sqrt{1+x^2}} \qquad [x \ge 0]$

   $= -\arccos\dfrac{1}{\sqrt{1+x^2}} \qquad [x \le 0]$ NV 48 (7)

9. $\arctan x = \operatorname{arccot}\dfrac{1}{x} \qquad [x > 0]$

   $= -\operatorname{arccot}\dfrac{1}{x} - \pi \qquad [x < 0]$ NV 49 (9)

10.[11] $\operatorname{arccot} x = \arcsin\dfrac{1}{\sqrt{1+x^2}} \qquad [x > 0]$

   $= \pi - \arcsin\dfrac{1}{\sqrt{1+x^2}} \qquad [x < 0]$ NV 49 (11)

11. $\operatorname{arccot} x = \arccos\dfrac{x}{\sqrt{1+x^2}}$ NV 46 (4)

12. $$\operatorname{arccot} x = \arctan \frac{1}{x} \qquad [x > 0]$$
$$= \pi + \arctan \frac{1}{x} \qquad [x < 0]$$
NV 49 (12)

1.625

1. $$\arcsin x + \arcsin y = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad \left[xy \le 0 \text{ or } x^2+y^2 \le 1\right]$$
$$= \pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad \left[x > 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right]$$
$$= -\pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \qquad \left[x < 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right]$$
NV 54(1), GI I (880)

2. $$\arcsin x + \arcsin y = \arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) \qquad [x \ge 0, \quad y \ge 0]$$
$$= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) \qquad [x < 0, \quad y < 0]$$
NV 55

3. $$\arcsin x + \arcsin y = \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} \qquad \left[xy \le 0 \text{ or } x^2+y^2 < 1\right]$$
$$= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi \qquad \left[x > 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right]$$
$$= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi \qquad \left[x < 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right]$$
NV 56

4. $$\arcsin x - \arcsin y = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad \left[xy \ge 0 \text{ or } x^2+y^2 \le 1\right]$$
$$= \pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad \left[x > 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right]$$
$$= -\pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \qquad \left[x < 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right]$$
NV 55(2)

5. $$\arcsin x - \arcsin y = \arccos\left(x\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) \qquad [xy > y]$$
$$= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) \qquad [x < y]$$
NV 56

6. $$\arccos x + \arccos y = \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x + y \ge 0]$$
$$= 2\pi - \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x + y < 0]$$
NV 57 (3)

7.[11] $$\arccos x - \arccos y = -\arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x \ge y]$$
$$= \arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \qquad [x < y]$$
NV 57 (4)

8. $$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy} \qquad [xy < 1]$$
$$= \pi + \arctan \frac{x+y}{1-xy} \qquad [x > 0, \quad xy > 1]$$
$$= -\pi + \arctan \frac{x+y}{1-xy} \qquad [x < 0, \quad xy > 1]$$

NV 59(5), GI I (879)

9. $$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy} \qquad [xy > -1]$$
$$= \pi + \arctan \frac{x-y}{1+xy} \qquad [x > 0, \quad xy < -1]$$
$$= -\pi + \arctan \frac{x-y}{1+xy} \qquad [x < 0, \quad xy < -1]$$

NV 59(6)

1.626

1. $$2 \arcsin x = \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[|x| \le \frac{1}{\sqrt{2}}\right]$$
$$= \pi - \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[\frac{1}{\sqrt{2}} < x \le 1\right]$$
$$= -\pi - \arcsin \left(2x\sqrt{1-x^2}\right) \qquad \left[-1 \le x < -\frac{1}{\sqrt{2}}\right]$$

NV 61 (7)

2. $$2 \arccos x = \arccos \left(2x^2 - 1\right) \qquad [0 \le x \le 1]$$
$$= 2\pi - \arccos \left(2x^2 - 1\right) \qquad [-1 \le x < 0]$$

NV 61 (8)

3. $$2 \arctan x = \arctan \frac{2x}{1-x^2} \qquad [|x| < 1]$$
$$= \arctan \frac{2x}{1-x^2} + \pi \qquad [x > 1]$$
$$= \arctan \frac{2x}{1-x^2} - \pi \qquad [x < -1]$$

NV 61 (9)

1.627

1. $$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2} \qquad [x > 0]$$
$$= -\frac{\pi}{2} \qquad [x < 0]$$

GI I (878)

2. $$\arctan x + \arctan \frac{1-x}{1+x} = \frac{\pi}{4} \qquad [x > -1]$$
$$= -\frac{3}{4}\pi \qquad [x < -1]$$

NV 62, GI I (881)

1.628

1. $$\arcsin\frac{2x}{1+x^2} = -\pi - 2\arctan x \qquad [x \le -1]$$
$$= 2\arctan x \qquad [-1 \le x \le 1]$$
$$= \pi - 2\arctan x \qquad [x \ge 1]$$
NV 65

2. $$\arccos\frac{1-x^2}{1+x^2} = 2\arctan x \qquad [x \ge 0]$$
$$= -2\arctan x \qquad [x \le 0]$$
NV 66

1.629 $$\frac{2x-1}{2} - \frac{1}{\pi}\arctan\left(\tan\frac{2x-1}{2}\pi\right) = E(x)$$ GI (886)

1.631 Relations between the inverse hyperbolic functions.

1. $$\operatorname{arcsinh} x = \operatorname{arccosh}\sqrt{x^2+1} = \operatorname{arctanh}\frac{x}{\sqrt{x^2+1}}$$ JA

2. $$\operatorname{arccosh} x = \operatorname{arcsinh}\sqrt{x^2-1} = \operatorname{arctanh}\frac{\sqrt{x^2-1}}{x}$$ JA

3. $$\operatorname{arctanh} x = \operatorname{arcsinh}\frac{x}{\sqrt{1-x^2}} = \operatorname{arccosh}\frac{1}{\sqrt{1-x^2}} = \operatorname{arccoth}\frac{1}{x}$$ JA

4. $$\operatorname{arcsinh} x \pm \operatorname{arcsinh} y = \operatorname{arcsinh}\left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right)$$ JA

5. $$\operatorname{arccosh} x \pm \operatorname{arccosh} y = \operatorname{arccosh}\left(xy \pm \sqrt{(x^2-1)(y^2-1)}\right)$$ JA

6. $$\operatorname{arctanh} x \pm \operatorname{arctanh} y = \operatorname{arctanh}\frac{x \pm y}{1 \pm xy}$$ JA

1.64 Series representations

1.641

1. $$\arcsin x = \frac{\pi}{2} - \arccos x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots$$
$$= \sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}x^{2k+1} = x\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right)$$
$$[x^2 \le 1]$$ FI II 479

2. $$\operatorname{arcsinh} x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \ldots;$$
$$= \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}x^{2k+1}$$
$$= x\,F\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-x^2\right)$$
$$[x^2 \le 1]$$ FI II 480

1.642

1. $$\operatorname{arcsinh} x = \ln 2x + \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} + \dots$$
$$= \ln 2x + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2k)!x^{-2k}}{2^{2k}\,(k!)^2\,2k} \qquad [x \geq 1]$$
AD (6480.2)a

2. $$\operatorname{arccosh} x = \ln 2x - \sum_{k=1}^{\infty}\frac{(2k)!x^{-2k}}{2^{2k}\,(k!)^2\,2k} \qquad [x \geq 1]$$
AD (6480.3)a

1.643

1. $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$$
$$[x^2 \leq 1]$$
FI II 479

2. $$\operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1} \qquad [x^2 < 1]$$
AD (6480.4)

1.644

1. $$\arctan x = \frac{x}{\sqrt{1+x^2}}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}\,(k!)^2\,(2k+1)}\left(\frac{x^2}{1+x^2}\right)^k$$
$$= \frac{x}{\sqrt{1+x^2}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{x^2}{1+x^2}\right) \qquad [x^2 < \infty]$$
AD (641.3)

2. $$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)x^{2k+1}}$$
AD (641.4)

1.645

1. $$\operatorname{arcsec} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2\cdot 3x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{(2k)!x^{-(2k+1)}}{(k!)^2\,2^{2k}(2k+1)}$$
$$= \frac{\pi}{2} - \frac{1}{x}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{1}{x^2}\right) \qquad [x^2 > 1]$$
AD (641.5)

2. $$(\arcsin x)^2 = \sum_{k=0}^{\infty}\frac{2^{2k}\,(k!)^2\,x^{2k+2}}{(2k+1)!(k+1)} \qquad [x^2 \leq 1]$$
AD (642.2), GI III (152)a

3. $$(\arcsin x)^3 = x^3 + \frac{3!}{5!}3^2\left(1+\frac{1}{3^2}\right)x^5 + \frac{3!}{7!}3^2\cdot 5^2\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)x^7 + \dots$$
$$[x^2 \leq 1]$$
BR* 188, AD (642.2), GI III (153)a

1.646

1. $$\operatorname{arcsinh}\frac{1}{x} = \operatorname{arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}\,(k!)^2\,(2k+1)} x^{-2k-1} \qquad \left[x^2 \geq 1\right]$$ AD (6480.5)

2. $$\operatorname{arccosh}\frac{1}{x} = \operatorname{arcsech} x = \ln\frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}\,(k!)^2\,2k} x^{2k} \qquad [0 < x \leq 1]$$ AD (6480.6)

3. $$\operatorname{arcsinh}\frac{1}{x} = \operatorname{arcosech} x = \ln\frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{2^{2k}\,(k!)^2\,2k} x^{2k} \qquad [0 < x \leq 1]$$ AD (6480.7)a

4. $$\operatorname{arctanh}\frac{1}{x} = \operatorname{arccoth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1} \qquad \left[x^2 > 1\right]$$ AD (6480.8)

1.647

1. $$\sum_{k=1}^{\infty} \frac{\tanh(2k-1)\,(\pi/2)}{(2k-1)^{4n+3}} = \frac{\pi^{4n+3}}{2} \left(2\sum_{j=1}^{n} \frac{(-1)^{j-1}\left(2^{2j}-1\right)\left(2^{4n-2j+4}-1\right) B^*_{2j-1} B^*_{4n-2j+3}}{(2j)!(4n-2j+4)!} + \frac{(-1)^n \left(2^{2n+2}-1\right)^2 {B^*_{2n+1}}^2}{\left[(2n+2)!\right]^2} \right) \qquad n = 0, 1, 2, \ldots,$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \operatorname{sech}(2k-1)\,(\pi/2)}{(2k-1)^{4n+1}} = \frac{\pi^{4n+1}}{2^{4n+3}} \left(2\sum_{j=1}^{n-1} \frac{(-1)^j B^*_{2j} B^*_{4n-2j}}{(2j)!(4n-2j)!} + \frac{2B^*_{4n}}{(4n)!} + \frac{(-1)^n {B^*_{2n}}^2}{[(2n)]!^2} \right), \qquad n = 1, 2, \ldots$$

(The summation term on the right is to be omitted for $n = 1$.) (See page xxxiii for the definition of B^*_r.)

2 Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign (=) means that the functions on the left and right of this symbol differ by a constant. For example (see **201** 15), we write

$$\int \frac{dx}{1+x^2} = \arctan x = -\arctan x$$

although

$$\arctan x = -\arctan x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example, $\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{1+x^2}\right|$). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula **2.02** 8 for $n = -1$ or formula **2.02** 15 for $a = b$). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right-hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters x, y, t, ... denote independent variables; f, g, φ, ... denote functions of x, y, t, ...; f', g', φ', ..., f'', g'', φ'', ... denote their first, second, etc., derivatives; a, b, m, p, ... denote constants, by which we generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is made, provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas **2.148** 4 and **2.424** 6, we make no remark since it is clear from the form of these formulas themselves that n must be a natural number (that is, a positive integer).

2.01 The basic integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln x$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6.[11] $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x$

8.[11] $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x$

9. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \sec x$

10. $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = -\operatorname{cosec} x$

11. $\displaystyle\int \tan x\,dx = -\ln\cos x$

12. $\displaystyle\int \cot x\,dx = \ln\sin x$

13. $\displaystyle\int \frac{dx}{\sin x} = \ln\tan\frac{x}{2}$

14. $\displaystyle\int \frac{dx}{\cos x} = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \ln(\sec x + \tan x)$

15. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x = \frac{\pi}{2} - \operatorname{arccot} x$

16. $\displaystyle\int \frac{dx}{1-x^2} = \operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$

17. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x = -\arccos x$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$

19. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x = \ln\left(x+\sqrt{x^2-1}\right)$

20. $\displaystyle\int \sinh x\,dx = \cosh x$

21. $\displaystyle\int \cosh x\,dx = \sinh x$

22.[11] $\displaystyle\int \frac{dx}{\sinh^2 x} = -\coth x$

23. $\displaystyle\int \frac{dx}{\cosh^2 x} = \tanh x$

24. $\displaystyle\int \tanh x\,dx = \ln\cosh x$

25. $\displaystyle\int \coth x\,dx = \ln\sinh x$

26. $\displaystyle\int \frac{dx}{\sinh x} = \ln\tanh\frac{x}{2}$

2.02 General formulas

1. $$\int af\,dx = a\int f\,dx$$

2. $$\int [af \pm b\varphi \pm c\psi \pm \ldots]\,dx = a\int f\,dx \pm b\int \varphi\,dx \pm c\int \psi\,dx \pm \ldots$$

3. $$\frac{d}{dx}\int f\,dx = f$$

4. $$\int f'\,dx = f$$

5. $$\int f'\varphi\,dx = f\varphi - \int f\varphi'\,dx$$ [integration by parts]

6. $$\int f^{(n+1)}\varphi\,dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \ldots + (-1)^n\varphi^{(n)} f + (-1)^{n+1}\int \varphi^{(n+1)} f\,dx$$

7. $$\int f(x)\,dx = \int f[\varphi(y)]\varphi'(y)\,dy$$ $[x = \varphi(y)]$ [change of variable]

8.[11] $$\int (f)^n f'\,dx = \frac{(f)^{n+1}}{n+1}$$ $[n \neq -1]$

For $n = -1$

$$\int \frac{f'\,dx}{f} = \ln f$$

9. $$\int (af+b)^n f'\,dx = \frac{(af+b)^{n+1}}{a(n+1)}$$

10. $$\int \frac{f'\,dx}{\sqrt{af+b}} = \frac{2\sqrt{af+b}}{a}$$

11. $$\int \frac{f'\varphi - \varphi' f}{\varphi^2}\,dx = \frac{f}{\varphi}$$

12. $$\int \frac{f'\varphi - \varphi' f}{f\varphi}\,dx = \ln\frac{f}{\varphi}$$

13. $$\int \frac{dx}{f\,(f \pm \varphi)} = \pm\int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi\,(f \pm \varphi)}$$

14. $$\int \frac{f'\,dx}{\sqrt{f^2+a}} = \ln\left(f + \sqrt{f^2+a}\right)$$

15. $$\int \frac{f\,dx}{(f+a)(f+b)} = \frac{a}{a-b}\int \frac{dx}{(f+a)} - \frac{b}{a-b}\int \frac{dx}{(f+b)}$$

For $a = b$

$$\int \frac{f\,dx}{(f+a)^2} = \int \frac{dx}{f+a} - a\int \frac{dx}{(f+a)^2}$$

16. $$\int \frac{f\,dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi\,dx}{(f+\varphi)^n}$$

17. $$\int \frac{f'\,dx}{p^2+q^2f^2} = \frac{1}{pq}\arctan\frac{qf}{p}$$

18. $$\int \frac{f'\,dx}{q^2f^2-p^2} = \frac{1}{2pq}\ln\frac{qf-p}{qf+p}$$

19. $$\int \frac{f\,dx}{1-f} = -x + \int \frac{dx}{1-f}$$

20. $$\int \frac{f^2\,dx}{f^2-a^2} = \frac{1}{2}\int \frac{f\,dx}{f-a} + \frac{1}{2}\int \frac{f\,dx}{f+a}$$

21. $$\int \frac{f'\,dx}{\sqrt{a^2-f^2}} = \arcsin\frac{f}{a}$$

22. $$\int \frac{f'\,dx}{af^2+bf} = \frac{1}{b}\ln\frac{f}{af+b}$$

23. $$\int \frac{f'\,dx}{f\sqrt{f^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{f}{a}$$

24. $$\int \frac{(f'\varphi - f\varphi')\,dx}{f^2+\varphi^2} = \arctan\frac{f}{\varphi}$$

25. $$\int \frac{(f'\varphi - f\varphi')\,dx}{f^2-\varphi^2} = \frac{1}{2}\ln\frac{f-\varphi}{f+\varphi}$$

2.1 Rational Functions

2.10 General integration rules

2.101 To integrate an arbitrary rational function $\dfrac{F(x)}{f(x)}$, where $F(x)$ and $f(x)$ are polynomials with no common factors, we first need to separate out the integral part $E(x)$ [where $E(x)$ is a polynomial], if there is an integral part, and then to integrate separately the integral part and the remainder; thus: $\displaystyle\int \frac{F(x)\,dx}{f(x)} = \int E(x)\,dx + \int \frac{\varphi(x)}{f(x)}\,dx.$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions*.

2.102 If $a, b, c, \ldots, m$ are roots of the equation $f(x)=0$ and if $\alpha, \beta, \gamma, \ldots, \mu$ are their corresponding multiplicities, so that $f(x) = (x-a)^\alpha(x-b)^\beta\ldots(x-m)^\mu$, then $\dfrac{\varphi(x)}{f(x)}$ can be decomposed into the following partial fractions:

$$\frac{\varphi(x)}{f(x)} = \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \ldots + \frac{A_1}{x-a} + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \ldots + \frac{B_1}{x-b} + \ldots$$
$$+\frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \ldots + \frac{M_1}{x-m},$$

where the numerators of the individual fractions are determined by the following formulas:

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \qquad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \qquad \ldots, \qquad M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$\psi_1(x) = \frac{\varphi(x)(x-a)^\alpha}{f(x)}, \qquad \psi_2(x) = \frac{\varphi(x)(x-b)^\beta}{f(x)}, \qquad \ldots, \qquad \psi_m(x) = \frac{\varphi(x)(x-m)^\mu}{f(x)}$$

TI 51a

If $a, b, \ldots, m$ are simple roots, that is, if $\alpha = \beta = \ldots = \mu = 1$, then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \qquad B = \frac{\varphi(b)}{f'(b)}, \qquad \ldots, \qquad M = \frac{\varphi(m)}{f'(m)}.$$

If some of the roots of the equation $f(x) = 0$ are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1 x + N_1}{x^2 + 2Bx + C} + \frac{M_2 x + N_2}{(x^2 + 2Bx + C)^2} + \ldots + \frac{M_p x + N_p}{(x^2 + 2Bx + C)^p}.$$

2.103 Thus, the integration of a proper rational fraction $\dfrac{\varphi(x)}{f(x)}$ reduces to integrals of the form $\displaystyle\int \frac{g\,dx}{(x-a)^\alpha}$ or $\displaystyle\int \frac{Mx+N}{(A+2Bx+Cx^2)^p}\,dx$. Fractions of the first form yield rational functions for $\alpha > 1$ and logarithms for $\alpha = 1$. Fractions of the second form yield rational functions and logarithms or arctangents:

1. $$\int \frac{g\,dx}{(x-a)^\alpha} = g\int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}$$

2. $$\int \frac{g\,dx}{x-a} = g\int \frac{d(x-a)}{x-a} = g\ln|x-a|$$

3. $$\int \frac{Mx+N}{(A+2Bx+Cx^2)^p}\,dx = \frac{NB - MA + (NC-MB)x}{2(p-1)\,(AC-B^2)\,(A+2Bx+Cx^2)^{p-1}} + \frac{(2p-3)(NC-MB)}{2(p-1)\,(AC-B^2)}\int \frac{dx}{(A+2Bx+Cx^2)^{p-1}}$$

4. $$\int \frac{dx}{A+2Bx+Cx^2} = \frac{1}{\sqrt{AC-B^2}}\arctan\frac{Cx+B}{\sqrt{Ac-B^2}} \qquad \text{for } \left[AC > B^2\right]$$
$$= \frac{1}{2\sqrt{B^2-AC}}\ln\left|\frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}}\right| \qquad \text{for } \left[AC < B^2\right]$$

5. $$\int \frac{(Mx+N)\,dx}{A+2Bx+Cx^2}$$
$$= \frac{M}{2C}\ln\left|A+2Bx+Cx^2\right| + \frac{NC-MB}{C\sqrt{AC-B^2}}\arctan\frac{Cx+B}{\sqrt{AC-B^2}} \qquad \text{for } \left[AC > B^2\right]$$
$$= \frac{M}{2C}\ln\left|A+2Bx+Cx^2\right| + \frac{NC-MB}{2C\sqrt{B^2-AC}}\ln\left|\frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}}\right| \qquad \text{for } \left[AC < B^2\right]$$

The Ostrogradskiy–Hermite method

2.104 By means of the Ostrogradskiy-Hermite method, we can find the rational part of $\displaystyle\int \frac{\varphi(x)}{f(x)}\,dx$ without finding the roots of the equation $f(x) = 0$ and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)}\,dx = \frac{M}{D} + \int \frac{N\,dx}{Q}$$ FI II 49

Here, M, N, D, and Q are rational functions of x. Specifically, D is the greatest common divisor of the function $f(x)$ and its derivative $f'(x)$; $Q = \dfrac{f(x)}{D}$; M is a polynomial of degree no higher than $m-1$, where m is the degree of the polynomial D; N is a polynomial of degree no higher than $n-1$, where n is the degree of the polynomial Q. The coefficients of the polynomials M and N are determined by equating the coefficients of like powers of x in the following identity:

$$\varphi(x) = M'Q - M\,(T - Q') + ND$$

where $T = \dfrac{f'(x)}{D}$ and M' and Q' are the derivatives of the polynomials M and Q.

2.11–2.13 Forms containing the binomial $a + bx^k$

2.110 Reduction formulas for $z_k = a + bx^k$ and an explicit expression for the general case.

1. $$\int x^n z_k^m\,dx = \frac{x^{n+1} z_k^m}{km+n+1} + \frac{amk}{km+n+1}\int x^n z_k^{m-1}\,dx$$
$$= \frac{x^{n+1}}{m+1}\sum_{s=0}^{p}\frac{(ak)^s(m+1)m(m-1)\dots(m-s+1)z_k^{m-s}}{[mk+n+1][(m-1)k+n+1]\dots[(m-s)k+n+1]}$$
$$+\frac{(ak)^{p+1}m(m-1)\dots(m-p+1)(m-p)}{[mk+n+1][(m-1)k+n+1]\dots[(m-p)k+n+1]}\int x^n z_k^{m-p-1}\,dx$$
LA 126(4)

2. $$\int x^n z_k^m\,dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)}\int x^n z_k^{m+1}\,dx$$ LA 126 (6)

3. $$\int x^n z_k^m\,dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1}\int x^{n+k} z_k^{m-1}\,dx$$

4. $$\int x^n z_k^m\,dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)}\int x^{n-k} z_k^{m+1}\,dx$$ LA 125 (2)

5. $$\int x^n z_k^m\,dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)}\int x^{n-k} z_k^{m}\,dx$$ LA 126 (3)

6. $$\int x^n z_k^m\,dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)}\int x^{n+k} z_k^{m}\,dx$$ LA 126 (5)

7.* $$\int x^n \left(nx^b + c\right)^k\,dx = \frac{n^k}{b}\sum_{i=0}^{k}\frac{(-1)^i\,k!\,\Gamma\left(\frac{a+1}{b}\right)\left(n^b + \frac{c}{n}\right)^{k-i}}{(k-i)!\,\Gamma\left(\frac{a+1}{b}+i+1\right)}x^{a+1+ib}$$

[a, b, $k \geq 0$ are all integers]

8.* $$\int x^n z_k^m\,dx = \frac{b^m}{k}\sum_{i=0}^{m}\frac{(-1)^i m!J!\left(x^k + \frac{a}{b}\right)^{m-i} x^{k(J+i+1)}}{(m-i)!(J+i+1)!}$$

$$J = \frac{n+1}{k} - 1 \qquad [a, b, k, m, n \text{ real}, \quad k \neq 0, \quad m \geq 0 \text{ an integer}]$$

Forms containing the binomial $z_1 = a + bx$

2.111

1. $$\int z_1^m\,dx = \frac{z_1^{m+1}}{b(m+1)}$$
For $m = -1$
$$\int \frac{dx}{z_1} = \frac{1}{b}\ln z_1$$

2. $$\int \frac{x^n\,dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b}\int \frac{x^{n-1}\,dx}{z_1^m}$$
For $n = m - 1$, we may use the formula

3.[8] $$\int \frac{x^{m-1}\,dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b}\int \frac{x^{m-2}\,dx}{z_1^{m-1}}$$
For $m = 1$
$$\int \frac{x^n\,dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2x^{n-2}}{(n-2)b^3} - \ldots + (-1)^{n-1}\frac{a^{n-1}x}{1\cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}}\ln z_1$$

4. $$\int \frac{x^n\,dx}{z_1^2} = \sum_{k=1}^{n-1}(-1)^{k-1}\frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1}\frac{a^n}{b^{n+1}z_1} + (-1)^{n+1}\frac{na^{n-1}}{b^{n+1}}\ln z_1$$

5. $$\int \frac{x\,dx}{z_1} = \frac{x}{b} - \frac{a}{b^2}\ln z_1$$

6. $$\int \frac{x^2\,dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3}\ln z_1$$

2.113

1. $$\int \frac{dx}{z_1^2} = -\frac{1}{bz_1}$$

2. $$\int \frac{x\,dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2}\ln z_1 = \frac{a}{b^2z_1} + \frac{1}{b^2}\ln z_1$$

3. $$\int \frac{x^2\,dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3z_1} - \frac{2a}{b^3}\ln z_1$$

2.114

1. $$\int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}$$

2. $$\int \frac{x\,dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2}\right]\frac{1}{z_1^2}$$

3. $$\int \frac{x^2\,dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right]\frac{1}{z_1^2} + \frac{1}{b^3}\ln z_1$$

4.[6] $$\int \frac{x^3\,dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2}x^2 - 2\frac{a^2}{b^3}x - \frac{5}{2}\frac{a^3}{b^4}\right]\frac{1}{z_1^2} - 3\frac{a}{b^4}\ln z_1$$

2.115

1. $$\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}$$

2. $$\int \frac{x\,dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right]\frac{1}{z_1^3}$$

3. $$\int \frac{x^2\,dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right]\frac{1}{z_1^3}$$

4. $$\int \frac{x^3\,dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right]\frac{1}{z_1^3} + \frac{1}{b^4}\ln z_1$$

2.116

1. $$\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}$$

2. $$\int \frac{x\,dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right]\frac{1}{z_1^4}$$

3. $$\int \frac{x^2\,dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right]\frac{1}{z_1^4}$$

4. $$\int \frac{x^3\,dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right]\frac{1}{z_1^4}$$

2.117

1. $$\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)}\int \frac{dx}{x^{n-1}z_1^m}$$

2. $$\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}$$

3. $$\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a}\int \frac{dx}{xz_1^{m-1}}$$

4. $$\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n}\ln\frac{z_1}{x}$$

2.118

1. $$\int \frac{dx}{xz_1} = -\frac{1}{a}\ln\frac{z_1}{x},$$

2. $$\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2x} - \frac{b^2}{a^3}\ln\frac{z_1}{x}$$

2.119

1. $$\int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^2} = -\left[\frac{1}{ax} + \frac{2b}{a^2}\right]\frac{1}{z_1} + \frac{2b}{a^3}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^2} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3}\right]\frac{1}{z_1} - \frac{3b^2}{a^4}\ln\frac{z_1}{x}$$

2.121

1. $$\int \frac{dx}{x z_1^3} = \left[\frac{3}{2a} + \frac{bx}{a^2}\right]\frac{1}{z_1^2} - \frac{1}{a^3}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^3} = -\left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3}\right]\frac{1}{z_1^2} + \frac{3b}{a^4}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4}\right]\frac{1}{z_1^2} - \frac{6b^2}{a^5}\ln\frac{z_1}{x}$$

2.122

1. $$\int \frac{dx}{x z_1^4} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3}\right]\frac{1}{z_1^3} - \frac{1}{a^4}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^4} = -\left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4}\right]\frac{1}{z_1^3} + \frac{4b}{a^5}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^4} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5}\right]\frac{1}{z_1^3} - \frac{10b^2}{a^6}\ln\frac{z_1}{x}$$

2.123

1.[11] $$\int \frac{dx}{x z_1^5} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4}\right]\frac{1}{z_1^4} - \frac{1}{a^5}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^5} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2x}{3a^3} - \frac{35b^3x^2}{2a^4} - \frac{5b^4x^3}{a^5}\right]\frac{1}{z_1^4} + \frac{5b}{a^6}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^5} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2x} + \frac{125b^2}{4a^3} + \frac{65b^3x}{a^4} + \frac{105b^4x^2}{2a^5} + \frac{15b^5x^3}{a^6}\right]\frac{1}{z_1^4} - \frac{15b^2}{a^7}\ln\frac{z_1}{x}$$

2.124 Forms containing the binomial $z_2 = a + bx^2$.

1. $$\int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}}\arctan x\sqrt{\frac{b}{a}} \qquad \text{if } [ab>0] \qquad \text{(see also } \mathbf{2.141}\ 2)$$
$$= \frac{1}{2i\sqrt{ab}}\ln\frac{a+xi\sqrt{ab}}{a-xi\sqrt{ab}} \qquad \text{if } [ab<0] \qquad \text{(see also } \mathbf{2.143}\ 2 \text{ and } \mathbf{2.143}3)$$

2. $$\int \frac{x\,dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \qquad \text{(see also } \mathbf{2.145}\ 2,\ \mathbf{2.145}\ 6, \text{ and } \mathbf{2.18})$$

Forms containing the binomial $z_3 = a + bx^3$

Notation: $\alpha = \sqrt[3]{\dfrac{a}{b}}$

2.125

1. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)}\int \frac{x^{n-3}\,dx}{z_3^m}$$

2. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)}\int \frac{x^n\,dx}{z_3^{m-1}}$$ LA 133 (1)

2.126

1. $$\int \frac{dx}{z_3} = \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{x\sqrt{3}}{2\alpha - x}\right\}$$
$$= \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$
(see also **2.141** 3 and **2.143**)

2. $$\int \frac{x\,dx}{z_3} = -\frac{1}{3b\alpha}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} - \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$
(see also **2.145** 3. and **2.145** 7)

3. $$\int \frac{x^2\,dx}{z_3} = \frac{1}{3b}\ln\left(1+x^3\alpha^{-3}\right) = \frac{1}{3b}\ln z_3$$

4. $$\int \frac{x^3\,dx}{z_3} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{z_3}$$ (see **2.126** 1)

5. $$\int \frac{x^4\,dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

2.127

1. $$\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a}\int\frac{dx}{z_3}$$ (see **2.126** 1)

2. $$\int \frac{x\,dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{x^2\,dx}{z_3^2} = -\frac{1}{3bz_3}$$

4. $$\int \frac{x^3\,dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b}\int\frac{dx}{z_3}$$ (see **2.126** 1)

2.128

1. $$\int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)}\int\frac{dx}{x^{n-3}z_3^m}$$

2. $$\int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)}\int\frac{dx}{x^n z_3^{m-1}}$$ LA 133 (2)

2.129

1. $$\int \frac{dx}{xz_3} = \frac{1}{3a}\ln\frac{x^3}{z_3}$$

2. $$\int \frac{dx}{x^2z_3} = -\frac{1}{ax} - \frac{b}{a}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{dx}{x^3z_3} = -\frac{1}{2ax^2} - \frac{b}{a}\int\frac{dx}{z_3}$$ (see **2.126** 1)

2.131

1. $$\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2}\ln\frac{x^3}{z_3}$$

2. $$\int \frac{dx}{x^2z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2}\right]\frac{1}{z_3} - \frac{4b}{3a^2}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{dx}{x^3z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2}\right]\frac{1}{z_3} - \frac{5b}{3a^2}\int\frac{dx}{z_3}$$ (see **2.126** 1)

Forms containing the binomial $z_4 = a + bx^4$

Notation: $\alpha = \sqrt[4]{\frac{a}{b}} \qquad \alpha' = \sqrt[4]{-\frac{a}{b}}$

2.132

1.[8] $$\int \frac{dx}{z_4} = \frac{\alpha}{4a\sqrt{2}}\left\{\ln\frac{x^2+\alpha x\sqrt{2}+\alpha^2}{x^2-\alpha x\sqrt{2}+\alpha^2} + 2\arctan\frac{\alpha x\sqrt{2}}{\alpha^2-x^2}\right\}$$ for $ab>0$ (see also **2.141** 4)

$$= \frac{\alpha'}{4a}\left\{\ln\frac{x+\alpha'}{x-\alpha'} + 2\arctan\frac{x}{\alpha'}\right\}$$ for $ab<0$ (see also **2.143** 5)

2. $$\int \frac{x\,dx}{z_4} = \frac{1}{2\sqrt{ab}}\arctan x^2\sqrt{\frac{b}{a}}$$ for $ab>0$ (see also **2.145** 4)

$$= \frac{1}{4i\sqrt{ab}}\ln\frac{a+x^2i\sqrt{ab}}{a-x^2i\sqrt{ab}}$$ for $ab<0$ (see also **2.145** 8)

3. $$\int \frac{x^2\,dx}{z_4} = \frac{1}{4b\alpha\sqrt{2}}\left\{\ln\frac{x^2-\alpha x\sqrt{2}+\alpha^2}{x^2+\alpha x\sqrt{2}+\alpha^2} + 2\arctan\frac{\alpha x\sqrt{2}}{\alpha^2-x^2}\right\}$$ for $ab>0$

$$= -\frac{1}{4b\alpha'}\left\{\ln\frac{x+\alpha'}{x-\alpha'} - 2\arctan\frac{x}{\alpha'}\right\}$$ for $ab<0$

4. $$\int \frac{x^3\,dx}{z_4} = \frac{1}{4b}\ln z_4$$

2.133

1. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)}\int\frac{x^n\,dx}{z_4^{m-1}}$$ LA 134 (1)

2. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)}\int\frac{x^{n-4}\,dx}{z_4^m}$$

2.134

1. $$\int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a}\int \frac{dx}{z_4}$$ (see **2.132** 1)

2. $$\int \frac{x\,dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a}\int \frac{x\,dx}{z_4}$$ (see **2.132** 2)

3. $$\int \frac{x^2\,dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a}\int \frac{x^2\,dx}{z_4}$$ (see **2.132** 3)

4. $$\int \frac{x^3\,dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}$$

2.135 $$\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a}\int \frac{dx}{x^{n-4}z_4^m}$$

For $n = 1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a}\int \frac{dx}{xz_4^{m-1}} - \frac{b}{a}\int \frac{dx}{x^{-3}z_4^m}$$

2.136

1. $$\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a}\ln\frac{x^4}{z_4}$$

2. $$\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x^2\,dx}{z_4}$$ (see **2.132** 3)

2.14 Forms containing the binomial $1 \pm x^n$

2.141

1. $$\int \frac{dx}{1+x} = \ln(1+x)$$

2.[11] $$\int \frac{dx}{1+x^2} = \arctan x = -\arctan\left(\frac{1}{x}\right)$$ (see also **2.124** 1)

3. $$\int \frac{dx}{1+x^3} = \frac{1}{3}\ln\frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2-x}$$ (see also **2.126** 1)

4. $$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2}$$

(see also **2.132** 1)

2.142 $$\int \frac{dx}{1+x^n} = -\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} P_k \cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} Q_k \sin\left(\frac{2k+1}{n}\pi\right)$$

for n a positive even number

TI (43)a

$$= \frac{1}{n}\ln(1+x) - \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k \cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k \sin\left(\frac{2k+1}{n}\pi\right)$$

for n a positive odd number

TI (45)

where

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\left(\frac{2k+1}{n}\pi\right) + 1\right)$$

$$Q_k = \arctan\frac{x\sin\left(\frac{2k+1}{n}\pi\right)}{1 - x\cos\left(\frac{2k+1}{n}\pi\right)} = \arctan\frac{x - \cos\left(\frac{2k+1}{n}\pi\right)}{\sin\left(\frac{2k+1}{n}\pi\right)}$$

2.143

1. $$\int\frac{dx}{1-x} = -\ln(1-x)$$

2. $$\int\frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} = \operatorname{arctanh} x \qquad [-1 < x < 1] \qquad \text{(see also } \mathbf{2.141}\ 1)$$

3. $$\int\frac{dx}{x^2-1} = \frac{1}{2}\ln\frac{x-1}{x+1} = -\operatorname{arccoth} x \qquad [x > 1, \quad x < -1]$$

4. $$\int\frac{dx}{1-x^3} = \frac{1}{3}\ln\frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2+x} \qquad \text{(see also } \mathbf{2.126}\ 1)$$

5. $$\int\frac{dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x}{1-x} + \frac{1}{2}\arctan x = \frac{1}{2}(\operatorname{arctanh} x + \arctan x)$$

(see also **2.132** 1)

2.144

1. $$\int\frac{dx}{1-x^n} = \frac{1}{n}\ln\frac{1+x}{1-x} - \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} P_k\cos\frac{2k}{n}\pi + \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} Q_k\sin\frac{2k}{n}\pi$$

for n a positive even number TI (47)

where $P_k = \frac{1}{2}\ln\left(x^2 + 2x\cos\frac{2k+1}{n}\pi + 1\right), \qquad Q_k = \arctan\frac{x + \cos\frac{2k+1}{n}\pi}{\sin\frac{2k+1}{n}\pi}$

2. $$\int\frac{dx}{1-x^n} = -\frac{1}{n}\ln(1-x) + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k\cos\frac{2k+1}{n}\pi + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k\sin\frac{2k+1}{n}\pi$$

for n a positive odd number TI (49)

where $P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\frac{2k}{n}\pi + 1\right), \qquad Q_k = \arctan\frac{x - \cos\frac{2k}{n}\pi}{\sin\frac{2k}{n}\pi}$

2.145

1. $$\int\frac{x\,dx}{1+x} = x - \ln(1+x)$$

2. $$\int\frac{x\,dx}{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right)$$

3. $$\int\frac{x\,dx}{1+x^3} = -\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} \qquad \text{(see also } \mathbf{2.126}\ 2)$$

4. $$\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\arctan x^2$$

5. $$\int \frac{x\,dx}{1-x} = -\ln(1-x) - x$$

6. $$\int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\ln\left(1-x^2\right)$$

7. $$\int \frac{x\,dx}{1-x^3} = -\frac{1}{6}\ln\frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$ (see also **2.126** 2)

8. $$\int \frac{x\,dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x^2}{1-x^2}$$ (see also **2.132** 2)

2.146 For m and n natural numbers.

1. $$\int \frac{x^{m-1}\,dx}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n}\ln\left\{1-2x\cos\frac{2k-1}{2n}\pi+x^2\right\} + \frac{1}{n}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n}\arctan\frac{x-\cos\frac{2k-1}{2n}\pi}{\sin\frac{2k-1}{2n}\pi}$$
$[m<2n]$ TI (44)a

2. $$\int \frac{x^{m-1}\,dx}{1+x^{2n+1}} = (-1)^{m+1}\frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left\{1-2x\cos\frac{2k-1}{2n+1}\pi+x^2\right\} + \frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\frac{x-\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}$$
$[m\le 2n]$ TI (46)a

3.[11] $$\int \frac{x^{m-1}\,dx}{1-x^{2n}} = \frac{1}{2n}\left\{(-1)^{m+1}\ln(1+x)-\ln(1-x)\right\} - \frac{1}{2n}\sum_{k=1}^{n-1}\cos\frac{km\pi}{n}\ln\left(1-2x\cos\frac{k\pi}{n}+x^2\right) + \frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{km\pi}{n}\arctan\left(\frac{x-\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right)$$
$[m<2n]$ TI (48)

4. $$\int \frac{x^{m-1}\,dx}{1-x^{2n+1}} = -\frac{1}{2n+1}\ln(1-x) + (-1)^{m+1}\frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left(1+2x\cos\frac{2k-1}{2n+1}\pi+x^2\right) + (-1)^{m+1}\frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\frac{x+\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}$$
$[m\le 2n]$ TI (50)

2.147

1. $$\int \frac{x^m\,dx}{1-x^{2n}} = \frac{1}{2}\int\frac{x^m\,dx}{1-x^n} + \frac{1}{2}\int\frac{x^m\,dx}{1+x^n}$$

2. $$\int \frac{x^m\,dx}{(1+x^2)^n} = -\frac{1}{2n-m-1}\cdot\frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1}\int\frac{x^{m-2}\,dx}{(1+x^2)^n}$$ LA 139 (28)

3. $$\int \frac{x^m}{1+x^2}\,dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2}\,dx$$

4. $$\int \frac{x^m\,dx}{(1-x^2)^n} = \frac{1}{2n-m-1}\frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-m-1}\int \frac{x^{m-2}\,dx}{(1-x^2)^n}$$
$$= \frac{1}{2n-2}\frac{x^{m-1}}{(1-x^2)^{n-1}} - \frac{m-1}{2n-2}\int \frac{x^{m-2}\,dx}{(1-x^2)^{n-1}}$$

LA 139 (33)

5. $$\int \frac{x^m\,dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2}\,dx}{1-x^2}$$

2.148

1. $$\int \frac{dx}{x^m\,(1+x^2)^n} = -\frac{1}{m-1}\frac{1}{x^{m-1}\,(1+x^2)^{n-1}} - \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}\,(1+x^2)^n}$$ LA 139 (29)

For $m=1$
$$\int \frac{dx}{x\,(1+x^2)^n} = \frac{1}{2n-2}\frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x\,(1+x^2)^{n-1}}$$ LA 139 (31)

For $m=1$ and $n=1$
$$\int \frac{dx}{x\,(1+x^2)} = \ln\frac{x}{\sqrt{1+x^2}}$$

2. $$\int \frac{dx}{x^m\,(1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2}\,(1+x^2)}$$

3. $$\int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2}\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)^{n-1}}$$ FI II 40

4. $$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)\,(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!}\arctan x$$

TI (91)

2.149

1. $$\int \frac{dx}{x^m\,(1-x^2)^n} = -\frac{1}{(m-1)x^{m-1}\,(1-x^2)^{n-1}} + \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}\,(1-x^2)^n}$$ LA 139 (34)

For $m=1$
$$\int \frac{dx}{x\,(1-x^2)^n} = \frac{1}{2(n-1)\,(1-x^2)^{n-1}} + \int \frac{dx}{x\,(1-x^2)^{n-1}}$$ LA 139 (36)

For $m=1$ and $n=1$
$$\int \frac{dx}{x\,(1-x^2)} = \ln\frac{x}{\sqrt{1-x^2}}$$

2. $$\int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2}\frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1-x^2)^{n-1}}$$ LA 139 (35)

3. $$\int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)\,(1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n\cdot(n-1)!}\ln\frac{1+x}{1-x}$$

TI (91)

2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

Notation: $z = a + bx; \quad t = \alpha + \beta x; \quad \Delta = a\beta - \alpha b$

2.151 $$\int z^n t^m\,dx = \frac{z^{n+1}t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b}\int z^n t^{m-1}\,dx$$

2.152

1. $$\int \frac{z}{t}\,dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2}\ln t$$

2. $$\int \frac{t}{z}\,dx = \frac{\beta x}{b} - \frac{\Delta}{b^2}\ln z$$

2.153 $$\begin{aligned}\int \frac{t^m\,dx}{z^n} &= \frac{1}{(m-n+1)b}\frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b}\int \frac{t^{m-1}\,dx}{z^n}\\ &= \frac{1}{(n-1)\Delta}\frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta}\int \frac{t^m\,dx}{z^{n-1}}\\ &= -\frac{1}{(n-1)b}\frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b}\int \frac{t^{m-1}}{z^{n-1}}\,dx\end{aligned}$$

2.154 $$\int \frac{dx}{zt} = \frac{1}{\Delta}\ln\frac{t}{z}$$

2.155 $$\begin{aligned}\int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{t^{m-1}z^n}\\ &= \frac{1}{(n-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta}\int \frac{dx}{t^m z^{n-1}}\end{aligned}$$

2.156 $$\int \frac{x\,dx}{zt} = \frac{1}{\Delta}\left(\frac{a}{b}\ln z - \frac{\alpha}{\beta}\ln t\right)$$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160 Reduction formulas for $R_k = a + bx^k + cx^{2k}$.

1. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma}\int x^{m+k-1}R_k^n\,dx - \frac{(m+2k+2kn)c}{ma}\int x^{m+2k-1}R_k^n\,dx$$

2. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m}\int x^{m+k-1}R_k^{n-1}\,dx - \frac{2ckn}{m}\int x^{m+2k-1}R_k^{n-1}\,dx$$

3. $$\begin{aligned}\int x^{m-1}R_k^n\,dx &= \frac{x^{m-2k}R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c}\int x^{m-2k-1}R_k^n\,dx - \frac{(m-k+kn)b}{(m+2kn)c}\int x^{m-k-1}R_k^n\,dx\\ &= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn}\int x^{m-1}R_k^{n-1}\,dx + \frac{bkn}{m+2kn}\int x^{m+k-1}R_k^{n-1}\,dx\end{aligned}$$

2.161 Forms containing the trinomial $R_2 = a + bx^2 + cx^4$.

Notation: $f = \frac{b}{2} - \frac{1}{2}\sqrt{b^2-4ac}, \quad g = \frac{b}{2} + \frac{1}{2}\sqrt{b^2-4ac},$

$$h = \sqrt{b^2-4ac}, \quad q = \sqrt[4]{\frac{a}{c}}, \quad l = 2a(n-1)\left(b^2-4ac\right), \quad \cos\alpha = -\frac{b}{2\sqrt{ac}}$$

1. $$\int \frac{dx}{R_2}$$
$$= \frac{c}{h}\left\{\int \frac{dx}{cx^2+f} - \int \frac{dx}{cx^2+g}\right\} \qquad \left[h^2>0\right] \qquad \text{LA 146 (5)}$$
$$= \frac{1}{4cq^3\sin\alpha}\left\{\sin\frac{\alpha}{2}\ln\frac{x^2+2qx\cos\frac{\alpha}{2}+q^2}{x^2-2qx\cos\frac{\alpha}{2}+q^2} + 2\cos\frac{\alpha}{2}\arctan\frac{x^2-q^2}{2qx\sin\frac{\alpha}{2}}\right\} \qquad \left[h^2<0\right] \qquad \text{LA 146 (8)a}$$

2. $$\int \frac{x\,dx}{R_2} = \frac{1}{2h}\ln\frac{cx^2+f}{cx^2+g} \qquad \left[h^2>0\right] \qquad \text{LA 146 (6)}$$
$$= \frac{1}{2cq^2\sin\alpha}\arctan\frac{x^2-q^2\cos\alpha}{q^2\sin\alpha} \qquad \left[h^2<0\right] \qquad \text{LA 146 (9)a}$$

3. $$\int \frac{x^2\,dx}{R_2} = \frac{g}{h}\int\frac{dx}{cx^2+g} - \frac{f}{h}\int\frac{dx}{cx^2+f} \qquad \left[h^2>0\right] \qquad \text{LA 146 (7)}$$

4. $$\int \frac{dx}{R_2^2} = \frac{bcx^3+\left(b^2-2ac\right)x}{lR_2} + \frac{b^2-6ac}{l}\int\frac{dx}{R_2} + \frac{bc}{l}\int\frac{x^2\,dx}{R_2}$$

5. $$\int \frac{dx}{R_2^n} = \frac{bcx^3+\left(b^2-2ac\right)x}{lR_{n-1}^2} + \frac{(4n-7)bc}{l}\int\frac{x^2\,dx}{R_2^{n-1}} + \frac{2(n-1)h^2+2ac-b^2}{l}\int\frac{dx}{R_2^{n-1}}$$
$$[n>1] \qquad \text{LA 146}$$

6.[9] $$\int \frac{dx}{x^mR_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a}\int\frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)bc}{(m-1)a}\int\frac{dx}{x^{m-4}R_2^n}$$
LA 147 (12)a

2.17 Forms containing the quadratic trinomial $a+bx+cx^2$ and powers of x

Notation: $R=a+bx+cx^2$; $\quad \Delta = 4ac-b^2$

2.171

1. $$\int x^{m+1}R^n\,dx = \frac{x^mR^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)}\int x^{m-1}R^n\,dx - \frac{b(m+n+1)}{c(m+2n+2)}\int x^mR^n\,dx$$
TI (97)

2. $$\int \frac{R^n\,dx}{x^{m+1}} = -\frac{R^{n+1}}{amx^m} + \frac{b(n-m+1)}{am}\int\frac{R^n\,dx}{x^m} + \frac{c(2n-m+2)}{am}\int\frac{R^n\,dx}{x^{m-1}} \qquad \text{LA 142(3), TI (96)a}$$

3. $$\int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta}\int\frac{dx}{R^n} \qquad \text{TI (94)a}$$

4. $$\int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1}\sum_{k=0}^{n-1}\frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\cdots(n-k)\Delta^{k+1}R^{n-k}} + 2^n\frac{(2n-1)!!c^n}{n!\Delta^n}\int\frac{dx}{R}$$
TI (96)a

2.172[11] $$\int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}}\ln\frac{\sqrt{-\Delta}-(b+2cx)}{(b+2cx)+\sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}}\operatorname{arctanh}\frac{b+2cx}{\sqrt{-\Delta}} \qquad \text{for } [\Delta<0]$$
$$= \frac{-2}{b+2cx} \qquad \text{for } [\Delta=0,\ b \text{ and } c \text{ non-zero}]$$
$$= \frac{2}{\sqrt{\Delta}}\arctan\frac{b+2cx}{\sqrt{\Delta}} \qquad \text{for } [\Delta>0]$$

2.173

1. $$\int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

2. $$\int \frac{dx}{R^3} = \frac{b+2cx}{\Delta}\left\{\frac{1}{2R^2} + \frac{3c}{\Delta R}\right\} + \frac{6c^2}{\Delta^2}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

2.174

1. $$\int \frac{x^m\,dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c}\int \frac{x^{m-1}\,dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c}\int \frac{x^{m-2}\,dx}{R^n}$$

For $m = 2n-1$, this formula is inapplicable. Instead, we may use

2. $$\int \frac{x^{2n-1}\,dx}{R^n} = \frac{1}{c}\int \frac{x^{2n-3}\,dx}{R^{n-1}} - \frac{a}{c}\int \frac{x^{2n-3}\,dx}{R^n} - \frac{b}{c}\int \frac{x^{2n-2}\,dx}{R^n}$$

2.175

1. $$\int \frac{x\,dx}{R} = \frac{1}{2c}\ln R - \frac{b}{2c}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

2. $$\int \frac{x\,dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

3. $$\int \frac{x\,dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

4. $$\int \frac{x^2\,dx}{R} = \frac{x}{c} - \frac{b}{2c^2}\ln R + \frac{b^2-2ac}{2c^2}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

5. $$\int \frac{x^2\,dx}{R^2} = \frac{ab+\left(b^2-2ac\right)x}{c\Delta R} + \frac{2a}{\Delta}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

6. $$\int \frac{x^2\,dx}{R^3} = \frac{ab+\left(b^2-2ac\right)x}{2c\Delta R^2} + \frac{\left(2ac+b^2\right)(b+2cx)}{2c\Delta^2 R} + \frac{2ac+b^2}{\Delta^2}\int \frac{dx}{R}$$

(see **2.172**)

7. $$\int \frac{x^3\,dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^3}\ln R - \frac{b\left(b^2-3ac\right)}{2c^3}\int \frac{dx}{R}$$

(see **2.172**)

8. $$\int \frac{x^3\,dx}{R^2} = \frac{1}{2c^2}\ln R + \frac{a\left(2ac-b^2\right)+b\left(3ac-b^2\right)x}{c^2\Delta R} - \frac{b\left(6ac-b^2\right)}{2c^2\Delta}\int \frac{dx}{R}$$

(see **2.172**)

9. $$\int \frac{x^3\,dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right)\frac{1}{2R^2} - \frac{3ab}{2c\Delta}\int \frac{dx}{R^2} \qquad \text{(see } \mathbf{2.173}\ 1\text{)}$$

2.176 $$\int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)}\int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)}\int \frac{dx}{x^{m-2}R^n}$$

2.177

1. $$\int \frac{dx}{xR} = \frac{1}{2a}\ln\frac{x^2}{R} - \frac{b}{2a}\int \frac{dx}{R} \qquad \text{(see } \mathbf{2.172}\text{)}$$

2. $$\int \frac{dx}{xR^2} = \frac{1}{2a^2}\ln\frac{x^2}{R} + \frac{1}{2aR}\left\{1 - \frac{b(b+2cx)}{\Delta}\right\} - \frac{b}{2a^2}\left(1 + \frac{2ac}{\Delta}\right)\int\frac{dx}{R}$$

(see **2.172**)

3. $$\int \frac{dx}{xR^3} = \frac{1}{4aR^2} + \frac{1}{2a^2R} + \frac{1}{2a^3}\ln\frac{x^2}{R} - \frac{b}{2a}\int\frac{dx}{R^3} - \frac{b}{2a^2}\int\frac{dx}{R^2} - \frac{b}{2a^3}\int\frac{dx}{R}$$

(see **2.172**, **2.173**)

4. $$\int \frac{dx}{x^2R} = -\frac{b}{2a^2}\ln\frac{x^2}{R} - \frac{1}{ax} + \frac{b^2-2ac}{2a^2}\int\frac{dx}{R}$$ (see **2.172**)

5. $$\int \frac{dx}{x^2R^2} = -\frac{b}{a^3}\ln\frac{x^2}{R} - \frac{a+bx}{a^2xR} + \frac{\left(b^2-3ac\right)(b+2cx)}{a^2\Delta R} - \frac{1}{\Delta}\left(\frac{b^4}{a^3} - \frac{6b^2c}{a^2} + \frac{6c^2}{a}\right)\int\frac{dx}{R}$$

(see **2.172**)

6. $$\int \frac{dx}{x^2R^3} = -\frac{1}{axR^2} - \frac{3b}{a}\int\frac{dx}{xR^3} - \frac{5c}{a}\int\frac{dx}{R^3}$$ (see **2.173** and **2.177** 3)

7. $$\int \frac{dx}{x^3R} = -\frac{ac-b^2}{2a^3}\ln\frac{x^2}{R} + \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b\left(3ac-b^2\right)}{2a^3}\int\frac{dx}{R}$$

(see **2.172**)

8. $$\int \frac{dx}{x^3R^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x}\right)\frac{1}{R} + \left(\frac{3b^2}{a^2} - \frac{2c}{a}\right)\int\frac{dx}{xR^2} + \frac{9bc}{2a^2}\int\frac{dx}{R^2}$$

(see **2.173** 1 and **2.177** 2)

9. $$\int \frac{dx}{x^3R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2x}\right)\frac{1}{R^2} + \left(\frac{6b^2}{a^2} - \frac{3c}{a}\right)\int\frac{dx}{xR^3} + \frac{10bc}{a^2}\int\frac{dx}{R^3}$$

(see **2.173** 2 and **2.177** 3)

2.18 Forms containing the quadratic trinomial $a+bx+cx^2$ and the binomial $\alpha+\beta x$

Notation: $R = a+bx+cx^2$; $z = \alpha+\beta x$; $A = a\beta^2 - \alpha b\beta + c\alpha^2$;
$B = b\beta - 2c\alpha$; $\Delta = 4ac - b^2$

1. $$\int z^m R^n\,dx = \frac{\beta z^{m-1}R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c}\int z^{m-1}R^n\,dx - \frac{(m-1)A}{(m+2n+1)c}\int z^{m-2}R^n\,dx$$

2. $$\int\frac{R^n\,dx}{z^m} = -\frac{1}{(m-2n-1)\beta}\frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2}\int\frac{R^{n-1}\,dx}{z^m}$$
$$-\frac{nB}{(m-2n-1)\beta^2}\int\frac{R^{n-1}\,dx}{z^{m-1}};$$ LA 184 (4)a
$$= \frac{-\beta}{(m-1)A}\frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A}\int\frac{R^n\,dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A}\int\frac{R^n\,dx}{z^{m-2}}$$ LA 148 (5)
$$= -\frac{1}{(m-1)\beta}\frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2}\int\frac{R^{n-1}\,dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2}\int\frac{R^{n-1}\,dx}{z^{m-2}}$$ LA 418 (6)

3. $$\int \frac{z^m\,dx}{R^n} = \frac{\beta}{(m-2n+1)c}\frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c}\int \frac{z^{m-1}\,dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c}\int \frac{z^{m-2}\,dx}{R^n}$$

LA 147 (1)

$$= \frac{b+2cx}{(n-1)\Delta}\frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta}\int \frac{z^m\,dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta}\int \frac{z^{m-1}\,dx}{R^{n-1}}$$

LA 148 (3)

4.[3] $$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m-1)A}\frac{1}{z^{m-1}R^{n-1}} - \frac{(m+n-2)B}{(m-1)A}\int \frac{dx}{z^{m-1}R^n} - \frac{(m+2n-3)c}{(m-1)A}\int \frac{dx}{z^{m-2}R^n}$$

LA 148 (7)

$$= \frac{\beta}{2(n-1)A}\frac{1}{z^{m-1}R^{n-1}} - \frac{B}{2A}\int \frac{dx}{z^{m-1}R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A}\int \frac{dx}{z^m R^{n-1}}$$

LA 148 (8)

For $m=1$ and $n=1$

$$\int \frac{dx}{zR} = \frac{\beta}{2A}\ln\frac{z^2}{R} - \frac{B}{2A}\int \frac{dx}{R}$$

For $A=0$

$$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m+n-1)B}\frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B}\int \frac{dx}{z^{m-1}R^n}$$ LA 148 (9)

2.2 Algebraic Functions

2.20 Introduction

2.201 The integrals $\int R\left(x, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^r, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^s, \ldots\right) dx$, where $r, s, \ldots$ are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x+\beta}{\gamma x+\delta} = t^m,$$ FI II 57

where m is the common denominator of the fractions $r, s, \ldots$.

2.202 Integrals of the form $\int x^m (a+bx^n)^p\,dx$,* where m, n, and p are rational numbers, can be expressed in terms of elementary functions only in the following cases:

(a) When p is an integer; then, this integral takes the form of a sum of the integrals shown in **2.201**;

(b) When $\frac{m+1}{n}$ is an integer: by means of the substitution $x^n = z$, this integral can be transformed to the form $\frac{1}{n}\int (a+bz)^p z^{\frac{m+1}{n}-1}\,dz$, which we considered in **2.201**;

(c) When $\frac{m+1}{n}+p$ is an integer; by means of the same substitution $x^n = z$, this integral can be reduced to an integral of the form $\frac{1}{n}\int \left(\frac{a+bz}{z}\right)^p z^{\frac{m+1}{n}+p-1}\,dz$, considered in **2.201**;

For reduction formulas for integrals of binomial differentials, see **2.110**.

*Translator: The authors term such integrals "integrals of binomial differentials."

2.21 Forms containing the binomial $a+bx^k$ and $\sqrt{x}$

Notation: $z_1 = a + bx$.

2.211 $$\int \frac{dx}{z_1\sqrt{x}} = \frac{2}{\sqrt{ab}} \arctan\sqrt{\frac{bx}{a}} \qquad [ab>0]$$
$$= \frac{1}{i\sqrt{ab}} \ln \frac{a - bx + 2i\sqrt{xab}}{z_1} \qquad [ab<0]$$

2.212 $$\int \frac{x^m\sqrt{x}}{z_1}\,dx = 2\sqrt{x}\sum_{k=0}^{m} \frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1}\frac{a^{m+1}}{b^{m+1}}\int \frac{dx}{z_1\sqrt{x}}$$
(see **2.211**)

2.213

1. $$\int \frac{\sqrt{x}\,dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

2. $$\int \frac{x\sqrt{x}\,dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right)2\sqrt{x} + \frac{a^2}{b^2}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

3. $$\int \frac{x^2\sqrt{x}\,dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3}\right)2\sqrt{x} - \frac{a^3}{b^3}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

4. $$\int \frac{dx}{z_1^2\sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

5. $$\int \frac{\sqrt{x}\,dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

6. $$\int \frac{x\sqrt{x}\,dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b}\int \frac{\sqrt{x}\,dx}{z_1^2}$$ (see **2.213** 5)

7. $$\int \frac{x^2\sqrt{x}\,dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right)\frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2}\int \frac{\sqrt{x}\,dx}{z_1^2}$$ (see **2.213** 5)

8. $$\int \frac{dx}{z_1^3\sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2z_1}\right)\sqrt{x} + \frac{3}{8a^2}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

9. $$\int \frac{\sqrt{x}\,dx}{z_1^3} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1}\right)\sqrt{x} + \frac{1}{8ab}\int \frac{dx}{z_1\sqrt{x}}$$ (see **2.211**)

10. $$\int \frac{x\sqrt{x}\,dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b}\int \frac{\sqrt{x}\,dx}{z_1^3}$$ (see **2.213** 9)

11. $$\int \frac{x^2\sqrt{x}\,dx}{z_1^3} = \left(\frac{x^2}{b} + \frac{5ax}{b^2}\right)\frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2}\int \frac{\sqrt{x}\,dx}{z_1^3}$$ (see **2.213** 9)

Notation: $z_2 = a + bx^2, \quad \alpha = \sqrt[4]{\frac{a}{b}}, \quad \alpha' = \sqrt[4]{-\frac{a}{b}}$.

2.214 $$\int \frac{dx}{z_2\sqrt{x}} = \frac{1}{b\alpha^3\sqrt{2}}\left[\ln\frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2 - x}\right] \qquad \left[\frac{a}{b}>0\right]$$
$$= \frac{1}{2b\alpha'^3}\left(\ln\frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2\arctan\frac{\sqrt{x}}{\alpha'}\right) \qquad \left[\frac{a}{b}<0\right]$$

2.215 $$\int \frac{\sqrt{x}\,dx}{z_2} = \frac{1}{b\alpha\sqrt{2}}\left[-\ln\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{z_2}} + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2-x}\right] \qquad \left[\frac{a}{b}>0\right]$$
$$= \frac{1}{2b\alpha'}\left[\ln\frac{\alpha'-\sqrt{x}}{\alpha'+\sqrt{x}} + 2\arctan\frac{\sqrt{x}}{\alpha'}\right] \qquad \left[\frac{a}{b}<0\right]$$

2.216

1. $$\int \frac{x\sqrt{x}\,dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

2. $$\int \frac{x^2\sqrt{x}\,dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

3. $$\int \frac{dx}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

4. $$\int \frac{\sqrt{x}\,dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

5. $$\int \frac{x\sqrt{x}\,dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

6. $$\int \frac{x^2\sqrt{x}\,dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

7. $$\int \frac{dx}{z_2^3\sqrt{x}} = \left(\frac{1}{4az_2^2} + \frac{7}{16a^2z_2}\right)\sqrt{x} + \frac{21}{32a^2}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

8. $$\int \frac{\sqrt{x}\,dx}{z_2^3} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2z_2}\right)x\sqrt{x} + \frac{5}{32a^2}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

9. $$\int \frac{x\sqrt{x}\,dx}{z_2^3} = \frac{\left(bx^2-3a\right)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

10. $$\int \frac{x^2\sqrt{x}\,dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b}\int\frac{\sqrt{x}\,dx}{z_2^3}$$ (see **2.216** 8)

2.22–2.23 Forms containing $\sqrt[n]{(a+bx)^k}$

Notation: $z = a + bx$.

2.220 $$\int x^n\sqrt[l]{z^{lm+f}}\,dx = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{ln-lk+l(m+1)+f}\right\}\frac{l\sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}$$

The square root

2.221 $$\int x^n\sqrt{z^{2m-1}}\,dx = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{2n-2k+2m+1}\right\}\frac{2\sqrt{z^{2m+1}}}{b^{n+1}}$$

2.222

1. $$\int\frac{dx}{\sqrt{z}} = \frac{2}{b}\sqrt{z}$$

2. $$\int \frac{x\,dx}{\sqrt{z}} = \left(\frac{1}{3}z - a\right)\frac{2\sqrt{z}}{b^2}$$

3. $$\int \frac{x^2\,dx}{\sqrt{z}} = \left(\frac{1}{5}z^2 - \frac{2}{3}az + a^2\right)\frac{2\sqrt{z}}{b^3}$$

2.223

1. $$\int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}$$

2. $$\int \frac{x\,dx}{\sqrt{z^3}} = (z+a)\frac{2}{b^2\sqrt{z}}$$

3. $$\int \frac{x^2\,dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2\right)\frac{2}{b^3\sqrt{z}}$$

2.224

1. $$\int \frac{z^m\,dx}{x^n\sqrt{z}} = -\frac{z^m\sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)}\frac{b}{a}\int \frac{z^m\,dx}{x^{n-1}\sqrt{z}}$$

2. $$\int \frac{z^m\,dx}{x^n\sqrt{z}} = -z^m\sqrt{z}\left\{\frac{1}{(n-1)ax^{n-1}} + \sum_{k=1}^{n-2}\frac{(2m-2n+3)(2m-2n+5)\ldots(2m-2n+2k+1)}{2^k(n-1)(n-2)\ldots(n-k-1)x^{n-k-1}}\frac{b^k}{a^{k+1}}\right\} + \frac{(2m-2n+3)(2m-2n+5)\ldots(2m-3)(2m-1)}{2^{n-1}(n-1)!x}\frac{b^{n-1}}{a^{n-1}}\int \frac{z^m\,dx}{x\sqrt{z}}$$

For $n=1$

3. $$\int \frac{z^m}{x\sqrt{z}}\,dx = \frac{2z^m}{(2m-1)\sqrt{z}} + a\int \frac{z^{m-1}}{x\sqrt{z}}\,dx$$

4. $$\int \frac{z^m}{x\sqrt{z}}\,dx = \sum_{k=1}^{m}\frac{2a^{m-k}z^k}{(2k-1)\sqrt{z}} + a^m\int \frac{dx}{x\sqrt{z}}$$

5.[6] $$\int \frac{dx}{x\sqrt{z}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{z}-\sqrt{a}}{\sqrt{z}+\sqrt{a}}\right| \qquad [a>0]$$
$$= \frac{2}{\sqrt{-a}}\arctan\frac{\sqrt{z}}{\sqrt{-a}} \qquad [a<0]$$

2.225

1. $$\int \frac{\sqrt{z}\,dx}{x} = 2\sqrt{z} + a\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{\sqrt{z}\,dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

3. $$\int \frac{\sqrt{z}\,dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.226

1. $$\int \frac{\sqrt{z^3}\,dx}{x} = \left(\frac{z}{3}+a\right)2\sqrt{z}+a^2\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{\sqrt{z^3}\,dx}{x^2} = -\frac{\sqrt{z^5}}{ax}+\frac{3b}{2a}\int\frac{\sqrt{z^3}\,dx}{x}$$ (see **2.226** 1)

3. $$\int \frac{\sqrt{z^3}\,dx}{x^3} = -\left(\frac{1}{2ax^2}+\frac{b}{4a^2x}\right)\sqrt{z^5}+\frac{3b^2}{8a^2}\int\frac{\sqrt{z^3}\,dx}{x}$$

(see **2.226** 1)

2.227 $$\int \frac{dx}{xz^m\sqrt{z}} = \sum_{k=0}^{m-1}\frac{2}{(2k+1)a^{m-k}z^k\sqrt{z}}+\frac{1}{a^m}\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.228

1. $$\int \frac{dx}{x^2\sqrt{z}} = -\frac{\sqrt{z}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{dx}{x^3\sqrt{z}} = \left(-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right)\sqrt{z}+\frac{3b^2}{8a^2}\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.229

1. $$\int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}}+\frac{1}{a}\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{dx}{x^2\sqrt{z^3}} = \left(-\frac{1}{ax}-\frac{3b}{a^2}\right)\frac{1}{\sqrt{z}}-\frac{3b}{2a^2}\int\frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

3. $$\int \frac{dx}{x^3\sqrt{z^3}} = \left(-\frac{1}{2ax^2}+\frac{5b}{4a^2x}+\frac{15b^2}{4a^3}\right)\frac{1}{\sqrt{z}}+\frac{15b^2}{8a^3}\int\frac{dx}{x\sqrt{z}}$$

(see **2.224** 4)

Cube root

2.231

1. $$\int \sqrt[3]{z^{3m+1}}x^n\,dx = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{3n-3k+3(m+1)+1}\right\}\frac{3\sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}$$

2. $$\int \frac{x^n\,dx}{\sqrt[3]{z^{3m+2}}} = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{3n-3k-3(m-1)-2}\right\}\frac{3}{b^{n+1}\sqrt[3]{z^{3(m-1)+2}}}$$

3. $$\int \sqrt[3]{z^{3m+2}}x^n\,dx = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{3n-3k+3(m+1)+2}\right\}\frac{3\sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}$$

4. $$\int \frac{x^n\,dx}{\sqrt[3]{z^{3m+1}}} = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{3n-3k-3(m-1)-1}\right\}\frac{3}{b^{n+1}\sqrt[3]{z^{3(m-1)+1}}}$$

5. $$\int \frac{z^n\,dx}{x^m\sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1)ax^{m-1}} + \frac{3n-3m+4}{3(m-1)}\frac{b}{a}\int \frac{z^n\,dx}{x^{m-1}\sqrt[3]{z^2}}$$

For $m=1$

$$\int \frac{z^n\,dx}{x\sqrt[3]{z^2}} = \frac{3z^n}{(3n-2)\sqrt[3]{z^2}} + a\int \frac{z^{n-1}\,dx}{x\sqrt[3]{z^2}}$$

6. $$\int \frac{dx}{xz^n\sqrt[3]{z^2}} = \frac{3\sqrt[3]{z}}{(3n-1)az^n} + \frac{1}{a}\int \frac{\sqrt[3]{z}\,dx}{xz^n}$$

2.232 $$\int \frac{dx}{x\sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}}\left\{\frac{3}{2}\ln\frac{\sqrt[3]{z}-\sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3}\arctan\frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z}+2\sqrt[3]{a}}\right\}$$

2.233

1. $$\int \frac{\sqrt[3]{z}\,dx}{x} = 3\sqrt[3]{z} + a\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

2. $$\int \frac{\sqrt[3]{z}\,dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

3. $$\int \frac{\sqrt[3]{z}\,dx}{x^3} = \left(-\frac{1}{2ax^2} + \frac{b}{3a^2x}\right)z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z^2}}$$

(see **2.232**)

4. $$\int \frac{dx}{x^2\sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

5. $$\int \frac{dx}{x^3\sqrt[3]{z^2}} = \left[-\frac{1}{2ax^2} + \frac{5b}{6a^2x}\right]\sqrt[3]{z} + \frac{5b^2}{9a^2}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

2.234

1. $$\int \frac{z^n\,dx}{x^m\sqrt[3]{z^2}} = -\frac{z^n\sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)}\frac{b}{a}\int \frac{z^n\,dx}{x^{m-1}\sqrt[3]{z}}$$

For $m=1$:

2. $$\int \frac{z^n\,dx}{x\sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a\int \frac{z^{n-1}\,dx}{x\sqrt[3]{z}}$$

3. $$\int \frac{dx}{xz^n\sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a}\int \frac{\sqrt[3]{z^2}\,dx}{xz^n}$$

2.235 $$\int \frac{dx}{x\sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}}\left\{\frac{3}{2}\ln\frac{\sqrt[3]{z}-\sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3}\arctan\frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z}+2\sqrt[3]{a}}\right\}$$

2.236

1. $$\int \frac{\sqrt[3]{z^2}\,dx}{x} = \frac{3}{2}\sqrt[3]{z^2} + a\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

2. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a}\sqrt[3]{z^2} + \frac{2b}{3}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

3. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^3} = \left[-\frac{1}{2ax^2} + \frac{b}{6a^2x}\right] z^{5/3} - \frac{b^2}{6a^2}\sqrt[3]{z^2} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

4. $$\int \frac{dx}{x^2\sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

5. $$\int \frac{dx}{x^3\sqrt[3]{z}} = \left[-\frac{1}{2ax^2} + \frac{2b}{3a^2x}\right]\sqrt[3]{z} + \frac{2b^2}{9a^2}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

2.24 Forms containing $\sqrt{a+bx}$ and the binomial $\alpha+\beta x$

Notation: $z = a + bx, \quad t = \alpha + \beta\mathrm{x}, \quad \Delta = a\beta - b\alpha.$

2.241

1. $$\int \frac{z^m t^n\,dx}{\sqrt{z}} = \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1}\sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta}\int \frac{z^{m-1}t^n\,dx}{\sqrt{z}}$$ LA 176 (1)

2. $$\int \frac{t^n z^m\,dx}{\sqrt{z}} = 2\sqrt{z^{2m+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p+2m+1}$$

2.242

1.[11] $$\int \frac{t\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z}}{b} + \beta\left(\frac{z}{3} - a\right)\frac{2\sqrt{z}}{b^2}$$

2. $$\int \frac{t^2\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z}}{b} + 2\alpha\beta\left(\frac{z}{3} - a\right)\frac{2\sqrt{z}}{b^2} + \beta^2\left(\frac{z^2}{5} - \frac{2}{3}za + a^2\right)\frac{2\sqrt{z}}{b^3}$$

3. $$\int \frac{t^3\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z}}{b} + 3\alpha^2\beta\left(\frac{z}{3} - \alpha\right)\frac{2\sqrt{z}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{5} - \frac{2}{3}za + a^2\right)\frac{2\sqrt{z}}{b^3} + \beta^\alpha\left(\frac{z^3}{7} - \frac{3z^2a}{5} + za^2 - a^3\right)\frac{2\sqrt{z}}{b^4}$$

4. $$\int \frac{tz\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^3}}{3b} + \beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2}$$

5. $$\int \frac{t^2z\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^3}}{3b} + 2\alpha\beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2} + \beta^2\left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3}\right)\frac{2\sqrt{z^3}}{b^3}$$

6. $$\int \frac{t^3z\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^3}}{3b} + 3\alpha^2\beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3}\right)\frac{2\sqrt{z^3}}{b^3} + \beta^3\left(\frac{z^3}{9} - \frac{3z^2a}{7} + \frac{3za^2}{5} - \frac{a^3}{3}\right)\frac{2\sqrt{z^3}}{b^4}$$

7. $$\int \frac{tz^2\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^5}}{5b} + \beta\left(\frac{z}{7} - \frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2}$$

8. $$\int \frac{t^2z^2\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^5}}{5b} + 2\alpha\beta\left(\frac{z}{7} - \frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2} + \beta^2\left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5}\right)\frac{2\sqrt{z^5}}{b^3}$$

9. $$\int \frac{t^3 z^2\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^5}}{5b} + 3\alpha^2\beta\left(\frac{z}{7}-\frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{9}-\frac{2za}{7}+\frac{a^2}{5}\right)\frac{2\sqrt{z^5}}{b^3} + \beta^3\left(\frac{z^3}{11}-\frac{3z^2a}{9}+\frac{3za^2}{7}-\frac{a^3}{5}\right)\frac{2\sqrt{z^5}}{b^4}$$

10. $$\int \frac{tz^3\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^7}}{7b} + \beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2}$$

11. $$\int \frac{t^2 z^3\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^7}}{7b} + 2\alpha\beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2} + \beta^2\left(\frac{z^2}{11}-\frac{2za}{9}+\frac{a^2}{7}\right)\frac{2\sqrt{z^7}}{b^3}$$

12. $$\int \frac{t^3 z^3\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^7}}{7b} + 3\alpha^2\beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{11}-\frac{2za}{9}+\frac{a^2}{7}\right)\frac{2\sqrt{z^7}}{b^3} + \beta^3\left(\frac{z^3}{13}-\frac{3z^2a}{11}+\frac{3za^2}{9}-\frac{a^3}{7}\right)\frac{2\sqrt{z^7}}{b^4}$$

2.243

1. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{t^{n+1}}{z^m}\sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta}\int\frac{t^n\,dx}{z^{m-1}\sqrt{z}} = -\frac{2}{(2m-1)b}\frac{t^n}{z^m}\sqrt{z} + \frac{2n\beta}{(2m-1)b}\int\frac{t^{n-1}\,dx}{z^{m-1}\sqrt{z}}$$

LA 176 (2)

2. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}}\sum_{k=0}^{n}\binom{n}{k}\frac{a^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p-2m+1}$$

2.244

1. $$\int \frac{t\,dx}{z\sqrt{z}} = -\frac{2a}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2\sqrt{z}}$$

2. $$\int \frac{t^2\,dx}{z\sqrt{z}} = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2\sqrt{z}} + \frac{2\beta^2\left(\frac{z^2}{3}-2za-a^2\right)}{b^3\sqrt{z}}$$

3. $$\int \frac{t^3\,dx}{z\sqrt{z}} = -\frac{2\alpha^3}{b\sqrt{z}} + \frac{6\alpha^2\beta(z+a)}{b^2\sqrt{z}} + \frac{6\alpha\beta^2\left(\frac{z^2}{3}-2za-a^2\right)}{b^3\sqrt{z}} + \frac{2\beta^3\left(\frac{z^3}{5}-z^2a+3za^2+a^3\right)}{b^4\sqrt{z}}$$

4. $$\int \frac{t\,dx}{z^2\sqrt{z}} = -\frac{2a}{3b\sqrt{z^3}} - \frac{2\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}}$$

5. $$\int \frac{t^2\,dx}{z^2\sqrt{z}} = -\frac{2a^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{2\beta^2\left(z^2+2az-\frac{a^2}{3}\right)}{b^3\sqrt{z^3}}$$

6. $$\int \frac{t^3\,dx}{z^2\sqrt{z}} = -\frac{2\alpha^3}{3b\sqrt{z^3}} - \frac{6\alpha^2\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{6\alpha\beta^2\left(z^2+2za-\frac{a^2}{3}\right)}{b^3\sqrt{z^3}} + \frac{2\beta^3\left(\frac{z^3}{3}-3z^2a-3za^2+\frac{a^3}{3}\right)}{b^4\sqrt{z^3}}$$

7. $$\int \frac{t\,dx}{z^3\sqrt{z}} = -\frac{2\alpha}{5b\sqrt{z^5}} - \frac{2\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}}$$

8. $$\int \frac{t^2\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{2\beta^2\left(z^2-\frac{2za}{3}+\frac{a^2}{5}\right)}{b^3\sqrt{z^5}}$$

9. $$\int \frac{t^3\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{6\alpha\beta^2\left(z^2-\frac{2za}{3}+\frac{a^2}{5}\right)}{b^3\sqrt{z^5}} + \frac{2\beta^3\left(z^3+3z^2a-za^2+\frac{a^3}{5}\right)}{b^4\sqrt{z^5}}$$

2.245

1. $$\int \frac{z^m\,dx}{t^n\sqrt{z}} = -\frac{2}{(2n-2m-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta}\int\frac{z^{m-1}\,dx}{t^n\sqrt{z}}$$ LA 176 (3)
$$= -\frac{1}{(n-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta}\int\frac{z^{m-1}}{t^{n-1}\sqrt{z}}\,dx$$
$$= -\frac{1}{(n-1)\Delta}\frac{z^m}{t^{n-1}}\sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta}\int\frac{z^m\,dx}{t^{n-1}\sqrt{z}}$$

2. $$\int \frac{z^m\,dz}{t^n\sqrt{z}} = -z^m\sqrt{z}\left[\frac{1}{(n-1)\Delta}\frac{1}{t^{n-1}} + \sum_{k=2}^{n-1}\frac{(2n-2m-3)(2n-2m-5)\ldots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\ldots(n-k)\Delta^k}\frac{1}{t^{n-k}}\Bigg\}\right] - \frac{(2n-2m-3)(2n-2m-5)\ldots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}\cdot(n-1)!\Delta^n}\int\frac{z^m\,dx}{t\sqrt{z}}$$

For $n=1$

3. $$\int \frac{z^m\,dx}{t\sqrt{z}} = \frac{2}{(2m-1)\beta}\frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta}\int\frac{z^{m-1}\,dx}{t\sqrt{z}}$$

4. $$\int \frac{z^m\,dx}{t\sqrt{z}} = 2\sum_{k=0}^{m-1}\frac{\Delta^k}{(2m-2k-1)\beta^{k+1}}\frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m}\int\frac{dx}{t\sqrt{z}}$$

2.246 $$\int \frac{dx}{t\sqrt{z}} \quad \frac{1}{\sqrt{\beta\Delta}}\ln\frac{\beta\sqrt{z}-\sqrt{\beta\Delta}}{\beta\sqrt{z}+\sqrt{\beta\Delta}} \qquad [\beta\Delta>0]$$
$$= \frac{2}{\sqrt{-\beta\Delta}}\arctan\frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} \qquad [\beta\Delta<0]$$
$$= -\frac{2\sqrt{z}}{bt} \qquad [\Delta=0]$$

2.247 $$\int \frac{dx}{tz^m\sqrt{z}} = \frac{2}{z^{m-1}\sqrt{z}} + \sum_{k=1}^{m}\frac{\beta^{k-1}z^k}{\Delta^k(2m-2k+1)} + \frac{\beta^m}{\Delta^m}\int\frac{dx}{t\sqrt{z}}$$
(see **2.246**)

2.248

1. $$\int \frac{dx}{tz\sqrt{z}} = \frac{2}{\Delta\sqrt{z}} + \frac{\beta}{\Delta}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

2. $$\int \frac{dx}{tz^2\sqrt{z}} = \frac{2}{3\Delta z\sqrt{z}} + \frac{2\beta}{\Delta^2\sqrt{z}} + \frac{\beta^2}{\Delta^2}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

3. $$\int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

4. $$\int \frac{dx}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

5. $$\int \frac{dx}{t^2 z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

6. $$\int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

7. $$\int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

8. $$\int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

9. $$\int \frac{dx}{t^3 z\sqrt{z}} = -\frac{1}{2\Delta t^2\sqrt{z}} + \frac{5b}{4\Delta^2 t\sqrt{z}} + \frac{15b^2}{4\Delta^3\sqrt{z}} + \frac{15b^2\beta}{8\Delta^3}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

10. $$\int \frac{dx}{t^3 z^2\sqrt{z}} = -\frac{1}{2\Delta t^2 z\sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 t z\sqrt{z}} + \frac{35b^2}{12\Delta^2 z\sqrt{z}} + \frac{35b^2\beta}{4\Delta^4\sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

11. $$\int \frac{dx}{t^3 z^3\sqrt{z}} = -\frac{1}{2\Delta t^2 z^2\sqrt{z}} + \frac{9b}{4\Delta^2 t z^2\sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2\sqrt{z}} + \frac{21b^2\beta}{4\Delta^4 z\sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5\sqrt{z}} + \frac{63b^2\beta^3}{8\Delta^5}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

12. $$\int \frac{z\,dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

13. $$\int \frac{z^2\,dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

14. $$\int \frac{z^3\,dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

15. $$\int \frac{z\,dx}{t^2\sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

16. $$\int \frac{z^2\,dx}{t^2\sqrt{z}} = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

17. $$\int \frac{z^3\,dx}{t^2\sqrt{z}} = -\frac{z^3\sqrt{z}}{\Delta t} + \frac{bz^2\sqrt{z}}{\beta\Delta} + \frac{5bz\sqrt{z}}{3\beta^2} + \frac{5b\Delta\sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

18.[3] $$\int \frac{z\,dx}{t^3\sqrt{z}} = -\frac{z\sqrt{z}}{2\Delta t^2} + \frac{bz\sqrt{z}}{4\Delta^2 t} - \frac{b^2\sqrt{z}}{4\beta\Delta^2} + \frac{b^2}{8\beta\Delta}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

19. $$\int \frac{z^2\,dx}{t^3\sqrt{z}} = -\frac{z^2\sqrt{z}}{2\Delta t^2} + \frac{bz^2\sqrt{z}}{4\Delta^2 t} + \frac{b^2 z\sqrt{z}}{4\beta\Delta^2} + \frac{3b^2\sqrt{z}}{4\beta^2\Delta} + \frac{3b^2}{8\beta^2}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

20. $$\int \frac{z^3\,dx}{t^3\sqrt{z}} = -\frac{z^3\sqrt{z}}{2\Delta t^2} + \frac{3bz^3\sqrt{z}}{\Delta^2 t} + \frac{3b^2z^2\sqrt{z}}{4\beta\Delta^2} + \frac{5b^2z\sqrt{z}}{4\beta^2\Delta} + \frac{15b^2\sqrt{z}}{4\beta^3} + \frac{15b^2\Delta}{8\beta^3}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

2.249

1. $$\int \frac{dx}{z^m t^n\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{\sqrt{z}}{t^{n-1}z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta}\int \frac{dx}{t^n z^{m-1}\sqrt{z}}$$ LA 177 (4)
$$= -\frac{1}{(n-1)\Delta}\frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta}\int \frac{dx}{t^{n-1}z^m\sqrt{z}}$$

2. $$\int \frac{dx}{z^m t^n\sqrt{z}} = \frac{\sqrt{z}}{z^m}\left[\frac{-1}{(n-1)\Delta}\frac{1}{t^{n-1}}\right.$$
$$\left. + \sum_{k=2}^{n-1}(-1)^k\frac{(2n+2m-3)(2n+2m-5)\ldots(2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\ldots(n-k)\Delta^k}\cdot\frac{1}{t^{n-k}}\right]$$
$$+(-1)^{n-1}\frac{(2n+2m-3)(2n+2m-5)\ldots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}}\int \frac{dx}{tz^m\sqrt{z}}$$

For $n=1$
$$\int \frac{dx}{z^m t\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{1}{z^{m-1}\sqrt{z}} + \frac{\beta}{\Delta}\int \frac{dx}{tz^{m-1}\sqrt{z}}$$

2.25 Forms containing $\sqrt{a+bx+cx^2}$

Integration techniques

2.251 It is possible to rationalize the integrand in integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right)dx$ by using one or more of the following three substitutions, known as the "Euler substitutions":

1. $\sqrt{a+bx+cx^2} = xt \pm \sqrt{a}$ for $a>0$;
2. $\sqrt{a+bx+cx^2} = t \pm x\sqrt{c}$ for $c>0$;
3. $\sqrt{c\,(x-x_1)\,(x-x_2)} = t\,(x-x_1)$ when x_1 and x_2 are real roots of the equation $a+bx+cx^2=0$.

2.252 Besides the Euler substitutions, there is also the following method of calculating integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right)\,dx$. By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of x and an expression of the form $\dfrac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, where $P_1(x)$ and $P_2(x)$ are both polynomials. By separating the integral portion of the rational function $\dfrac{P_1(x)}{P_2(x)}$ from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals, each of which is in one of the following three forms:

1. $\displaystyle\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$, where $P(x)$ is a polynomial of some degree r;
2. $\displaystyle\int \frac{dx}{(x+p)^k\sqrt{a+bx+cx^2}}$;
3. $\displaystyle\int \frac{(Mx+N)\,dx}{(a+\beta x+x^2)^m\sqrt{c\,(a_1+b_1x+x^2)}}, \qquad \left(a_1=\frac{a}{c}, \quad b_1=\frac{b}{c}\right).$

In more detail:

1. $\displaystyle\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}} = Q(x)\sqrt{a+bx+cx^2} + \lambda\int \frac{dx}{\sqrt{a+bx+cx^2}}$, where $Q(x)$ is a polynomial of degree $(r-1)$. Its coefficients, and also the number λ, can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)\left(a+bx+cx^2\right) + \frac{1}{2}Q(x)(b+2cx) + \lambda \qquad \text{LI II 77}$$

   Integrals of the form $\displaystyle\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$ (where $r \le 3$) can also be calculated by use of formulas **2.26**.

2. Integrals of the form $\displaystyle\int \frac{P(x)\,dx}{(x+p)^k\sqrt{a+bx+cx^2}}$, where the degree n of the polynomial $P(x)$ is lower than k can, by means of the substitution $t=\dfrac{1}{x+p}$, be reduced to an integral of the form $\displaystyle\int \frac{P(t)\,dt}{\sqrt{a+\beta t+\gamma t^2}}$. (See also **2.281**).

3. Integrals of the form $\displaystyle\int \frac{(Mx+N)\,dx}{(\alpha+\beta x+x^2)^m\sqrt{c\,(a_1+b_1x+x^2)}}$ can be calculated by the following procedure:

   - If $b_1 \ne \beta$, by using the substitution

$$x = \frac{a_1-\alpha}{\beta b_1} + \frac{t-1}{t+1}\,\frac{\sqrt{(a_1-\alpha)^2-(\alpha b_1-a_1\beta)(\beta-b_1)}}{\beta-b_1}$$

we can reduce this integral to an integral of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c\,(t^2+q)}}$, where $P(t)$ is a polynomial of degree no higher than $2m-1$. The integral $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{t^2+q}}$ can be reduced to the sum of integrals of the forms $\int \frac{t\,dt}{(t^2+p)^k\sqrt{t^2+q}}$ and $\int \frac{dt}{(t^2+p)^k\sqrt{t^2+q}}$.

- If $b_1=\beta$, we can reduce it to integrals of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c\,(t^2+q)}}$ by means of the substitution $t=x+\frac{b_1}{2}$.
  The integral $\int \frac{t\,dt}{(t^2+p)^k\sqrt{c\,(t^2+q)}}$ can be evaluated by means of the substitution $t^2+q=u^2$.
  The integral $\int \frac{dt}{(t^2+p)^k\sqrt{c\,(t^2+q)}}$ can be evaluated by means of the substitution $\frac{t}{\sqrt{t^2+q}}=v$ (see also **2.283**). FI II 78-82

2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x

Notation: $R=a+bx+cx^2$, $\quad \Delta=4ac-b^2$

For simplified formulas for the case $b=0$, see **2.27**.

2.260

1. $$\int x^m\sqrt{R^{2n+1}}\,dx=\frac{x^{m-1}\sqrt{R^{2n+3}}}{(m+2n+2)c}-\frac{(2m+2n+1)b}{2(m+2n+2)c}\int x^{m-1}\sqrt{R^{2n+1}}\,dx -\frac{(m-1)a}{(m+2n+2)c}\int x^{m-2}\sqrt{R^{2n+1}}\,dx$$ TI (192)a

2. $$\int \sqrt{R^{2n+1}}\,dx=\frac{2cx+b}{4(n+1)c}\sqrt{R^{2n+1}}+\frac{2n+1}{8(n+1)}\frac{\Delta}{c}\int\sqrt{R^{2n-1}}\,dx$$ TI (188)

3. $$\int \sqrt{R^{2n+1}}\,dx=\frac{(2cx+b)\sqrt{R}}{4(n+1)c}\left\{R^n+\sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)}\left(\frac{\Delta}{c}\right)^{k+1}R^{n-k-1}\right\} +\frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{\Delta}{c}\right)^{n+1}\int\frac{dx}{\sqrt{R}}$$ TI (190)

2.261[11] For $n=-1$

$$\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{cR}+2cx+b}{\sqrt{\Delta}}\right)\qquad [c>0]$$ TI (127)

$$=\frac{1}{\sqrt{c}}\operatorname{arcsinh}\left(\frac{2cx+b}{\sqrt{\Delta}}\right)\qquad [c>0,\quad \Delta>0]$$ DW

$$=\frac{1}{\sqrt{c}}\ln(2cx+b)\qquad [c>0,\quad \Delta=0]$$ DW

$$=\frac{-1}{\sqrt{-c}}\arcsin\left(\frac{2cx+b}{\sqrt{-\Delta}}\right)\qquad [c<0,\quad \Delta<0]$$ TI (128)

2.262

1. $$\int \sqrt{R}\,dx = \frac{(2cx+b)\sqrt{R}}{4c} + \frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2. $$\int x\sqrt{R}\,dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx+b)b}{8c^2}\sqrt{R} - \frac{b\Delta}{16c^2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

3. $$\int x^2\sqrt{R}\,dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right)\sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{(2cx+b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

4. $$\int x^3\sqrt{R}\,dx = \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right)\sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{(2cx+b)\sqrt{R}}{4c} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

5. $$\int \sqrt{R^3}\,dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right)(2cx+b)\sqrt{R} + \frac{3\Delta^2}{128c^2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

6. $$\int x\sqrt{R^3}\,dx = \frac{\sqrt{R^5}}{5c} - (2cx+b)\left(\frac{b}{16c^2}\sqrt{R^3} + \frac{3\Delta b}{128c^3}\sqrt{R}\right) - \frac{3\Delta^2 b}{256c^3}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

7. $$\int x^2\sqrt{R^3}\,dx = \left(\frac{x}{6c} - \frac{7b}{60c^2}\right)\sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) + \left(\frac{7b^2}{4c} - a\right)\frac{\Delta^2}{256c^3}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

8. $$\int x^3\sqrt{R^3}\,dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2}\right)\sqrt{R^5} - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) - \left(\frac{3b^2}{4c} - a\right)\frac{3\Delta^2 b}{512c^4}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2.263

1. $$\int \frac{x^m\,dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c}\int \frac{x^{m-1}\,dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c}\int \frac{x^{m-2}\,dx}{\sqrt{R^{2n+1}}}$$ TI (193)a

For $m = 2n$

2. $$\int \frac{x^{2n}\,dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c}\int \frac{x^{2n-1}}{\sqrt{R^{2n+1}}}\,dx + \frac{1}{c}\int \frac{x^{2n-2}}{\sqrt{R^{2n-1}}}\,dx$$ TI (194)a

3. $$\int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta}\int \frac{dx}{\sqrt{R^{2n-1}}}$$ TI (189)

4. $$\int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}}\left\{1+\sum_{k=1}^{n-1}\frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\frac{c^k}{\Delta^k}R^k\right\}$$
$$[n \geq 1]\,.$$ TI (191)

2.264

1. $$\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2. $$\int \frac{x\,dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

3. $$\int \frac{x^2\,dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{R} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right)\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

4. $$\int \frac{x^3\,dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^2}\right)\int \frac{dx}{\sqrt{R}}$$
(see **2.261**)

5. $$\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$$

6. $$\int \frac{x\,dx}{\sqrt{R^3}} = -\frac{2(2a+bx)}{\Delta\sqrt{R}}$$

7. $$\int \frac{x^2\,dx}{\sqrt{R^3}} = -\frac{\left(\Delta - b^2\right)x - 2ab}{c\Delta\sqrt{R}} + \frac{1}{c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

8. $$\int \frac{x^3\,dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b\left(10ac - 3b^2\right)x + a\left(8ac - 3b^2\right)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2}\int \frac{dx}{\sqrt{R}}$$
(see **2.261**)

2.265 $$\int \frac{\sqrt{R^{2n+1}}}{x^m}\,dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-1}}\,dx + \frac{(2n-m+4)c}{(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-2}}\,dx$$
TI (195)

For $m = 1$
$$\int \frac{\sqrt{R^{2n+1}}}{x}\,dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2}\int \sqrt{R^{2n-1}}\,dx + a\int \frac{\sqrt{R^{2n-1}}}{x}\,dx$$ TI (198)
For $a = 0$
$$\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m}\,dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b}\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}}$$ LA 169 (3)
For $m = 0$ see **2.260** 2 and **2.260** 3.
For $n = -1$ and $m = 1$:

2.266[8]

$$\int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \qquad [a>0] \qquad \text{TI (137)}$$

$$= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \qquad [a<0, \quad \Delta<0] \qquad \text{TI (138)}$$

$$= \frac{1}{\sqrt{-a}} \arctan \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \qquad [a<0] \qquad \text{LA 178 (6)a}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arcsinh} \frac{2a+bx}{x\sqrt{\Delta}} \qquad [a>0, \quad \Delta>0] \qquad \text{DW}$$

$$= -\frac{1}{\sqrt{a}} \operatorname{arctanh} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \qquad [a>0]$$

$$= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} \qquad [a>0, \quad \Delta=0]$$

$$= -\frac{2\sqrt{bx+cx^2}}{bx} \qquad [a=0, \quad b\neq 0] \qquad \text{LA 170 (16)}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arccosh}\left(\frac{2a+bx}{x\sqrt{-\Delta}}\right) \qquad [a>0, \Delta<0]$$

2.267

1. $$\int \frac{\sqrt{R}\,dx}{x} = \sqrt{R} + a\int \frac{dx}{x\sqrt{R}} + \frac{b}{2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261** and **2.266**)

2. $$\int \frac{\sqrt{R}\,dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{R}} + c\int \frac{dx}{\sqrt{R}}$$ (see **2.261** and **2.266**)

   For $a=0$

   $$\int \frac{\sqrt{bx+cx^2}}{x^2}\,dx = -\frac{2\sqrt{bx+cx^2}}{x} + c\int \frac{dx}{\sqrt{bx+cx^2}}$$ (see **2.261**)

3. $$\int \frac{\sqrt{R}\,dx}{x^3} = -\left(\frac{1}{2x^2}+\frac{b}{4ax}\right)\sqrt{R} - \left(\frac{b^2}{8a}-\frac{c}{2}\right)\int \frac{dx}{x\sqrt{R}}$$

   (see **2.266**)

   For $a=0$

   $$\int \frac{\sqrt{bx+cx^2}}{x^3}\,dx = -\frac{2\sqrt{(bx+cx^2)^3}}{3bx^3}$$

4. $$\int \frac{\sqrt{R^3}}{x}\,dx = \frac{\sqrt{R^3}}{3} + \frac{2bcx+b^2+8ac}{8c}\sqrt{R} + a^2\int \frac{dx}{x\sqrt{R}} + \frac{b\left(12ac-b^2\right)}{16c}\int \frac{dx}{\sqrt{R}}$$

   (see **2.261** and **2.266**)

5. $$\int \frac{\sqrt{R^3}}{x^2}\,dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx+b}{a}\sqrt{R^3} + \frac{3}{4}(2cx+3b)\sqrt{R} + \frac{3}{2}ab\int \frac{dx}{x\sqrt{R}} + \frac{3\left(4ac+b^2\right)}{8}\int \frac{dx}{\sqrt{R}}$$

   (see **2.261** and **2.266**)

   For $a=0$

   $$\int \frac{\sqrt{(bx+cx^2)^3}}{x^2} = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b}{4}\sqrt{bx+cx^2} + \frac{3b^2}{8}\int \frac{dx}{\sqrt{bx+cx^2}}$$

   (see **2.261**)

6. $$\int \frac{\sqrt{R^3}}{x^3}\,dx = -\left(\frac{1}{2ax^2}+\frac{b}{4a^2x}\right)\sqrt{R^5}+\frac{bcx+2ac+b^2}{4a^2}\sqrt{R^3}+\frac{3\left(bcx+2ac+b^2\right)}{4a}\sqrt{R}$$
$$+\frac{3}{8}\left(4ac+b^2\right)\int\frac{dx}{x\sqrt{R}}+\frac{3}{2}bc\int\frac{dx}{\sqrt{R}}$$
(see **2.261** and **2.266**)

For $a=0$
$$\int\frac{\sqrt{\left(bx+cx^2\right)^3}}{x^3}\,dx=\left(c-\frac{2b}{x}\right)\sqrt{bx+cx^2}+\frac{3bc}{2}\int\frac{dx}{\sqrt{bx+cx^2}}$$
(see **2.261**)

2.268 $$\int\frac{dx}{x^m\sqrt{R^{2n+1}}}=-\frac{1}{(m-1)ax^{m-1}\sqrt{R^{2n-1}}}$$
$$-\frac{(2n+2m-3)b}{2(m-1)a}\int\frac{dx}{x^{m-1}\sqrt{R^{2n+1}}}-\frac{(2n+m-2)c}{(m-1)a}\int\frac{dx}{x^{m-2}\sqrt{R^{2n+1}}}$$
TI (196)

For $m=1$
$$\int\frac{dx}{x\sqrt{R^{2n+1}}}=\frac{1}{(2n-1)a\sqrt{R^{2n-1}}}-\frac{b}{2a}\int\frac{dx}{\sqrt{R^{2n+1}}}+\frac{1}{a}\int\frac{dx}{x\sqrt{R^{2n-1}}}$$
TI (199)

For $a=0$
$$\int\frac{dx}{x^m\sqrt{\left(bx+cx^2\right)^{2n+1}}}=-\frac{2}{(2n+2m-1)bx^m\sqrt{\left(bx+cx^2\right)^{2n-1}}}$$
$$-\frac{(4n+2m-2)c}{(2n+2m-1)b}\int\frac{dx}{x^{m-1}\sqrt{\left(bx+cx^2\right)^{2n+1}}}$$
(cf. **2.265**)

2.269

1. $$\int\frac{dx}{x\sqrt{R}}$$ (see **2.266**)

2. $$\int\frac{dx}{x^2\sqrt{R}}=-\frac{\sqrt{R}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{R}}$$ (see **2.266**)

For $a=0$
$$\int\frac{dx}{x^2\sqrt{bx+cx^2}}=\frac{2}{3}\left(-\frac{1}{bx^2}+\frac{2c}{b^2x}\right)\sqrt{bx+cx^2}$$

3. $$\int\frac{dx}{x^3\sqrt{R}}=\left(-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right)\sqrt{R}+\left(\frac{3b^2}{8a^2}-\frac{c}{2a}\right)\int\frac{dx}{x\sqrt{R}}$$
(see **2.266**)

For $a=0$
$$\int\frac{dx}{x^3\sqrt{bx+cx^2}}=\frac{2}{5}\left(-\frac{1}{bx^3}+\frac{4c}{3b^2x^2}-\frac{8c^2}{3b^3x}\right)\sqrt{bx+cx^2}$$

4. $$\int\frac{dx}{x\sqrt{R^3}}=-\frac{2\left(bcx-2ac+b^2\right)}{a\Delta\sqrt{R}}+\frac{1}{a}\int\frac{dx}{x\sqrt{R}}$$ (see **2.266**)

For $a=0$

$$\int \frac{dx}{x\sqrt{(bx+cx^2)^3}} = \frac{2}{3}\left(-\frac{1}{bx}+\frac{4c}{b^2}+\frac{8c^2x}{b^3}\right)\frac{1}{\sqrt{bx+cx^2}}$$

5.[11] $$\int \frac{dx}{x^2\sqrt{R^3}} = -\frac{A}{\sqrt{R}} - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{R}}$$

$$\text{where } A = \left(-\frac{1}{ax} - \frac{b\left(10ac-3b^2\right)}{a^2\Delta} - \frac{c\left(8ac-3b^2\right)x}{a^2\Delta}\right) \qquad \text{(see } \mathbf{2.266}\text{)}$$

For $a=0$

$$\int \frac{dx}{x^2\sqrt{(bx+cx^2)^3}} = \frac{2}{5}\left(-\frac{1}{bx^2}+\frac{2c}{b^2x}-\frac{8c^2}{b^3}-\frac{16c^3x}{b^4}\right)\frac{1}{\sqrt{bx+cx^2}}$$

6. $$\int \frac{dx}{x^3\sqrt{R^3}}$$

$$= \left(-\frac{1}{ax^2}+\frac{5b}{2a^2x}-\frac{15b^4-62acb^2+24a^2c^2}{2a^3\Delta}-\frac{bc\left(15b^2-52ac\right)x}{2a^3\Delta}\right)\frac{1}{2\sqrt{R}}+\frac{15b^2-12ac}{8a^3}\int\frac{dx}{x\sqrt{R}}$$

(see **2.266**)

For $a=0$

$$\int \frac{dx}{x^3\sqrt{(bx+cx^2)^3}} = \frac{2}{7}\left(-\frac{1}{bx^3}+\frac{8c}{5b^2x^2}-\frac{16c^2}{5b^3x}+\frac{64c^3}{5b^4}+\frac{128c^4x}{5b^5}\right)\frac{1}{\sqrt{bx+cx^2}}$$

2.27 Forms containing $\sqrt{a+cx^2}$ and integral powers of x

Notation: $u=\sqrt{a+cx^2}$.

$$I_1 = \frac{1}{\sqrt{c}}\ln\left(x\sqrt{c}+u\right) \qquad [c>0]$$

$$= \frac{1}{\sqrt{-c}}\arcsin x\sqrt{-\frac{c}{a}} \qquad [c<0 \text{ and } a>0]$$

$$I_2 = \frac{1}{2\sqrt{a}}\ln\frac{u-\sqrt{a}}{u+\sqrt{a}} \qquad [a>0 \text{ and } c>0]$$

$$= \frac{1}{2\sqrt{a}}\ln\frac{\sqrt{a}-u}{\sqrt{a}+u} \qquad [a>0 \text{ and } c>0]$$

$$= \frac{1}{\sqrt{-a}}\operatorname{arcsec} x\sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}}\arccos\frac{1}{x}\sqrt{-\frac{a}{c}} \qquad [a<0 \text{ and } c>0]$$

2.271

1. $$\int u^5\,dx = \frac{1}{6}xu^5+\frac{5}{24}axu^3+\frac{5}{16}a^2xu+\frac{5}{16}a^3I_1$$ DW

2. $$\int u^3\,dx = \frac{1}{4}xu^3+\frac{3}{8}axu+\frac{3}{8}a^2I_1$$ DW

3. $$\int u\,dx = \frac{1}{2}xu+\frac{1}{2}aI_1$$ DW

4. $$\int \frac{dx}{u} = I_1$$ DW

5. $$\int \frac{dx}{u^3} = \frac{1}{a}\frac{x}{u}$$ DW

6. $$\int \frac{dx}{u^{2n+1}} = \frac{1}{a^n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}\binom{n-1}{k}\frac{c^k x^{2k+1}}{u^{2k+1}}$$

7. $$\int \frac{x\,dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}}$$ DW

2.272

1. $$\int x^2u^3\,dx = \frac{1}{6}\frac{xu^5}{c} - \frac{1}{24}\frac{axu^3}{c} - \frac{1}{16}\frac{a^2xu}{c} - \frac{1}{16}\frac{a^3}{c}I_1$$ DW

2. $$\int x^2u\,dx = \frac{1}{4}\frac{xu^3}{c} - \frac{1}{8}\frac{axu}{c} - \frac{1}{8}\frac{a^2}{c}I_1$$ DW

3. $$\int \frac{x^2}{u}\,dx = \frac{1}{2}\frac{xu}{c} - \frac{1}{2}\frac{a}{c}I_1$$ DW

4. $$\int \frac{x^2}{u^3}\,dx = -\frac{x}{cu} + \frac{1}{c}I_1$$ DW

5. $$\int \frac{x^2}{u^5}\,dx = \frac{1}{3}\frac{x^3}{au^3}$$ DW

6. $$\int \frac{x^2\,dx}{u^{2n+1}} = \frac{1}{a^{n-1}}\sum_{k=0}^{n-2}\frac{(-1)^k}{2k+3}\binom{n-2}{k}\frac{c^k x^{2k+3}}{u^{2k+3}}$$

7. $$\int \frac{x^3\,dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2u^{2n-3}} + \frac{a}{(2n-1)c^2u^{2n-1}}$$ DW

2.273

1. $$\int x^4u^3\,dx = \frac{1}{8}\frac{x^3u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2xu^3}{64c^2} + \frac{3a^3xu}{128c^2} + \frac{3a^4}{128c^2}I_1$$ DW

2. $$\int x^4u\,dx = \frac{1}{6}\frac{x^3u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2xu}{16c^2} + \frac{a^3}{16c^2}I_1$$ DW

3. $$\int \frac{x^4}{u}\,dx = \frac{1}{4}\frac{x^3u}{c} - \frac{3}{8}\frac{axu}{c^2} + \frac{3}{8}\frac{a^2}{c^2}I_1$$ DW

4. $$\int \frac{x^4}{u^3}\,dx = \frac{1}{2}\frac{xu}{c^2} + \frac{ax}{c^2u} - \frac{3}{2}\frac{a}{c^2}I_1$$ DW

5. $$\int \frac{x^4}{u^5}\,dx = -\frac{x}{c^2u} - \frac{1}{3}\frac{x^3}{cu^3} + \frac{1}{c^2}I_1$$ DW

6. $$\int \frac{x^4}{u^7}\,dx = \frac{1}{5}\frac{x^5}{au^5}$$ DW

7. $$\int \frac{x^4\,dx}{u^{2n+1}} = \frac{1}{a^{n-2}}\sum_{k=0}^{n-3}\frac{(-1)^k}{2k+5}\binom{n-3}{k}\frac{c^k x^{2k+5}}{u^{2k+5}}$$

8. $$\int \frac{x^5\,dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3u^{2n-5}} + \frac{2a}{(2n-3)c^{u2n-3}} - \frac{a^2}{(2n-1)c^3u^{2n-1}}$$ DW

2.274

1. $$\int x^6u^3\,dx = \frac{1}{10}\frac{x^5u^5}{c} - \frac{ax^3u^5}{16c^2} + \frac{a^2xu^5}{32c^3} - \frac{a^3xu^3}{128c^3} - \frac{3a^4xu}{256c^3} - \frac{3}{256}\frac{a^5}{c^3}I_1$$

2. $$\int x^6u\,dx = \frac{1}{8}\frac{x^5u^3}{c} - \frac{5}{48}\frac{ax^3u^3}{c^2} + \frac{5a^2xu^3}{64c^3} - \frac{5a^3xu}{128c^3} - \frac{5}{128}\frac{a^4}{c^3}I_1$$

3. $$\int \frac{x^6}{u}\,dx = \frac{1}{6}\frac{x^5u}{c} - \frac{5}{24}\frac{ax^3u}{c^2} + \frac{5}{16}\frac{a^2xu}{c^3} - \frac{5}{16}\frac{a^3}{c^3}I_1$$ DW

4. $$\int \frac{x^6}{u^3}\,dx = \frac{1}{4}\frac{x^5}{cu} - \frac{5}{8}\frac{ax^3}{c^2u} - \frac{15}{8}\frac{a^2x}{c^3u} + \frac{15}{8}\frac{a^2}{c^3}I_1$$ DW

5. $$\int \frac{x^6}{u^5}\,dx = \frac{1}{2}\frac{x^5}{cu^3} + \frac{10}{3}\frac{ax^3}{c^2u^3} + \frac{5}{2}\frac{a^2x}{c^3u^3} - \frac{5}{2}\frac{a}{c^3}I_1$$ DW

6. $$\int \frac{x^6}{u^7}\,dx = -\frac{23}{15}\frac{x^5}{cu^5} - \frac{7}{3}\frac{ax^3}{c^2u^5} - \frac{a^2x}{c^3u^5} + \frac{1}{c^3}I_1$$ DW

7. $$\int \frac{x^6}{u^9}\,dx = \frac{1}{7}\frac{x^7}{au^7}$$ DW

8. $$\int \frac{x^6\,dx}{u^{2n+1}} = \frac{1}{a^{n-3}}\sum_{k=0}^{n-4}\frac{(-1)^k}{2k+7}\binom{n-4}{k}\frac{c^kx^{2k+7}}{u^{2k+7}}$$

9. $$\int \frac{x^7\,dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4u^{2n-7}} + \frac{3a}{(2n-5)c^4u^{2n-5}} - \frac{3a^2}{(2n-3)c^4u^{2n-3}} + \frac{a^3}{(2n-1)c^4u^{2n-1}}$$ DW

2.275

1. $$\int \frac{u^5}{x}\,dx = \frac{u^5}{5} + \frac{1}{3}au^3 + a^2u + a^3I_2$$ DW

2. $$\int \frac{u^3}{x}\,dx = \frac{u^3}{3} + au + a^2I_2$$ DW

3. $$\int \frac{u}{x}\,dx = u + aI_2$$ DW

4. $$\int \frac{dx}{xu} = I_2$$ DW

5. $$\int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n}I_2 + \sum_{k=0}^{n-1}\frac{1}{(2k+1)a^{n-k}u^{2k+1}}$$

6. $$\int \frac{u^5}{x^2}\,dx = -\frac{u^5}{x} + \frac{5}{4}cxu^3 + \frac{15}{8}acxu + \frac{15}{8}a^2I_1$$ DW

7. $$\int \frac{u^3}{x^2}\,dx = -\frac{u^3}{x} + \frac{3}{2}cxu + \frac{3}{2}aI_1$$ DW

8. $$\int \frac{u}{x^2}\,dx = -\frac{u}{x} + cI_1$$ DW

9. $$\int \frac{dx}{x^2u^{2n+1}} = -\frac{1}{a^{n+1}}\left\{\frac{u}{x} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}\binom{n}{k} c^k \left(\frac{x}{u}\right)^{2k-1}\right\}$$

2.276

1. $$\int \frac{u^5}{x^3}\,dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2cI_2$$ DW

2. $$\int \frac{u^3}{x^3}\,dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2$$ DW

3. $$\int \frac{u}{x^3}\,dx = -\frac{u}{2x^2} + \frac{c}{2}I_2$$ DW

4. $$\int \frac{dx}{x^3u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2$$ DW

5. $$\int \frac{dx}{x^3u^3} = -\frac{1}{2ax^2u} - \frac{3c}{2a^2u} - \frac{3c}{2a^2}I_2$$ DW

6. $$\int \frac{dx}{x^3u^5} = -\frac{1}{2ax^2u^3} - \frac{5}{6}\frac{c}{a^2u^3} - \frac{5}{2}\frac{c}{a^3u} - \frac{5}{2}\frac{c}{a^3}I_2$$ DW

7. $$\int \frac{u^5}{x^4}\,dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2xu}{2} + \frac{5}{2}acI_1$$ DW

8. $$\int \frac{u^3}{x^4}\,dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + cI_1$$ DW

9. $$\int \frac{u}{x^4}\,dx = -\frac{u^3}{3ax^3}$$ DW

10. $$\int \frac{dx}{x^4u^{2n+1}} = \frac{1}{a^{n+2}}\left\{-\frac{u^3}{3x^3} + (n+1)\frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3}\binom{n+1}{k} c^k \left(\frac{x}{u}\right)^{2k-3}\right\}$$

2.277

1. $$\int \frac{u^3}{x^5}\,dx = -\frac{u^3}{4x^4} - \frac{3}{8}\frac{cu^3}{ax^2} + \frac{3}{8}\frac{c^2u}{a} + \frac{3}{8}c^2I_2$$ DW

2. $$\int \frac{u}{x^5}\,dx = -\frac{u}{4x^4} - \frac{1}{8}\frac{cu}{ax^2} - \frac{1}{8}\frac{c^2}{a}I_2$$ DW

3. $$\int \frac{dx}{x^5u} = -\frac{u}{4ax^4} + \frac{3}{8}\frac{cu}{a^2x^2} + \frac{3}{8}\frac{c^2}{a^2}I_2$$ DW

4. $$\int \frac{dx}{x^5u^3} = -\frac{1}{4ax^4u} + \frac{5}{8}\frac{c}{a^2x^2u} + \frac{15}{8}\frac{c^2}{a^3u} + \frac{15}{8}\frac{c^2}{a^3}I_2$$ DW

2.278

1. $$\int \frac{u^3}{x^6}\,dx = -\frac{u^5}{5ax^5}$$ DW

2. $$\int \frac{u}{x^6}\,dx = -\frac{u^3}{5ax^5} + \frac{2}{15}\frac{cu^3}{a^2x^3}$$ DW

3. $$\int \frac{dx}{x^6 u} = \frac{1}{a^3}\left(-\frac{u^5}{5x^5} + \frac{2}{3}\frac{cu^3}{x^3} - \frac{c^2 u}{x}\right)$$ DW

4. $$\int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}}\left\{-\frac{u^5}{5x^5} + \frac{1}{3}\binom{n+2}{1}\frac{cu^3}{x^3} - \binom{n+2}{2}\frac{c^2 u}{x} + \sum_{k=3}^{n+2}\frac{(-1)^k}{2k-5}\binom{n+2}{k}c^k\left(\frac{x}{u}\right)^{2k-5}\right\}$$

2.28 Forms containing $\sqrt{a+bx+cx^2}$ and first- and second-degree polynomials

Notation: $R = a + bx + cx^2$

See also 2.252

2.281[3] $$\int \frac{dx}{(x+p)^n\sqrt{R}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c+(b-2pc)t+(a-bp+cp^2)\,t^2}} \qquad \left[t = \frac{1}{x+p} > 0\right]$$

2.282

1.[3] $$\int \frac{\sqrt{R}\,dx}{x+p} = c\int \frac{x\,dx}{\sqrt{R}} + (b-cp)\int \frac{dx}{\sqrt{R}} + \left(a-bp+cp^2\right)\int \frac{dx}{(x+p)\sqrt{R}} \qquad [x+p>0]$$

2. $$\int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}$$

3. $$\int \frac{\sqrt{R}\,dx}{(x+p)(x+q)} = \frac{1}{q-p}\int \frac{\sqrt{R}\,dx}{x+p} + \frac{1}{p-q}\int \frac{\sqrt{R}\,dx}{x+q}$$

4. $$\int \frac{(x+p)\sqrt{R}\,dx}{x+q} = \int \sqrt{R}\,dx + (p-q)\int \frac{\sqrt{R}\,dx}{x+q}$$

5. $$\int \frac{(rx+s)\,dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}$$

2.283 $$\int \frac{(Ax+B)\,dx}{(p+R)^n\sqrt{R}} = \frac{A}{c}\int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c}\int \frac{\left(1-cv^2\right)^{n-1}\,dv}{\left[p+a-\dfrac{b^2}{4c}-cpv^2\right]^n},$$

where $u = \sqrt{R}$ and $v = \dfrac{b+2cx}{2c\sqrt{R}}$.

2.284 $$\int \frac{Ax+B}{(p+R)\sqrt{R}}\,dx = \frac{A}{c}I_1 + \frac{2Bc-Ab}{\sqrt{c^2p\,[b^2-4(a+p)c]}}I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}}\arctan\sqrt{\frac{R}{p}} \qquad [p>0]$$

$$= \frac{1}{2\sqrt{-p}}\ln\frac{\sqrt{-p}-\sqrt{R}}{\sqrt{-p}+\sqrt{R}} \qquad [p<0]$$

$$I_2 = \arctan\sqrt{\frac{p}{b^2-4(a+p)c}}\,\frac{b+2cx}{\sqrt{R}} \qquad \left[p\left\{b^2-4(a+p)c\right\}>0, \quad p<0\right]$$

$$= -\arctan\sqrt{\frac{p}{b^2-4(a+p)c}}\,\frac{b+2cx}{\sqrt{R}} \qquad \left[p\left\{b^2-4(a+p)c\right\}>0, \quad p>0\right]$$

$$= \frac{1}{2i}\ln\frac{\sqrt{4(a+p)c-b^2}\sqrt{R}+\sqrt{p}(b+2cx)}{\sqrt{4(a+p)c-b^2}\sqrt{R}-\sqrt{p}(b+2cx)} \qquad \left[p\left\{b^2-4(a+p)c\right\}<0, \quad p>0\right]$$

$$= \frac{1}{2i}\ln\frac{\sqrt{b^2-4(a+p)c}\sqrt{R}-\sqrt{-p}(b+2cx)}{\sqrt{b^2-4(a+p)c}\sqrt{R}+\sqrt{-p}(b+2cx)} \qquad \left[p\left\{b^2-4(a+p)c\right\}<0, \quad p<0\right]$$

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.290 Integrals of the form $\int R\left(x, \sqrt{P(x)}\right)\,dx$, where $P(x)$ is a third- or fourth-degree polynomial, can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see **8.11**). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see **3.13**, **3.17**).

2.291 Certain integrals of the form $\int R\left(x, \sqrt{P(x)}\right)\,dx$, where $P_n(x)$ is a polynomial of not more than fourth degree, can be reduced to integrals of the form $\int R\left(x, \sqrt[k]{P_n(x)}\right)\,dx$ with $k \geq 2$. Below are examples of this procedure.

1. $$\int\frac{dx}{\sqrt{1-x^6}} = -\int\frac{dz}{\sqrt{3+3z^2+z^4}} \qquad \left[x^2=\frac{1}{1+z^2}\right]$$

2. $$\int\frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2}\int\frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}}$$
$$\left[x^2=z\right]$$

3. $$\int\left(a+2bx+cx^2+gx^3\right)^{\pm 1/3}\,dx = \frac{3}{2}\int\frac{z^2A^{\pm\frac{1}{3}}\,dz}{B}$$
$$\left[a+2bx+cx^2=z^3, \quad A=g\left(\frac{-b+\sqrt{b^2+(z^3-a)\,c}}{c}\right)^3+z^3, \quad B=\sqrt{b^2+(z^3-a)\,c}\right]$$

4. $$\int\frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}}$$
$$= -\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{(z-1)p}} \qquad \left[x=z+\sqrt{z^2-1}\right]$$
$$= -\frac{1}{\sqrt{2}}\int\frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{(z-1)p}} \qquad \left[x=z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(4z^3-3z\right)+2b\left(2z^2-1\right)+2cz+d$.

5. $$\int \frac{dx}{\sqrt{a+bx^2+cx^4+bx^6+ax^8}} = \frac{1}{2}\int \frac{dy}{\sqrt{y}\sqrt{a+by+cy^2+by^3+ay^4}} \qquad [x=\sqrt{y}]$$
$$= -\frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[y = z+\sqrt{z^2-1}\right]$$
$$= \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[y = z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(2z^2-1\right)+2bz+c$.

6. $$\int \frac{dx}{\sqrt{a+bx^4+cx^8}} = \frac{1}{2}\sqrt[8]{\frac{a}{c}}\int \frac{dt}{\sqrt{t}\sqrt{ab_1t^2+at^4}} \qquad \left[x = \sqrt[8]{\frac{a}{c}}\sqrt{t}\right];$$
$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad \left[t = z+\sqrt{z^2-1}\right]$$
$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad \left[t = z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(2z^2-1\right)+b_1$; $b_1 = b\sqrt{\frac{a}{c}}$.

7. $$\int \frac{x\,dx}{\sqrt[4]{a+bx^2+cx^4}} = 2\int \frac{z^2\,dz}{\sqrt{A+Bz^4}} \qquad \left[a+bx^2+cx^4 = z^4, \quad A = b^2-4ac, \quad B = 4c\right]$$

8. $$\int \frac{dx}{\sqrt[4]{a+2bx^2+cx^4}} = \int \frac{\sqrt{b^2-a\left(c-z^4\right)}+b}{\left(c-z^4\right)\sqrt{b^2-a\left(c-z^4\right)}}z^2\,dz = \int R_1\left(z^4\right)z^2\,dz + \int \frac{R_2\left(z^4\right)z^2\,dz}{\sqrt{b^2-a\left(c-z^4\right)}},$$

where $R_1\left(z^4\right)$ and $R_2\left(z^4\right)$ are rational functions of z^4 and $a+2bx^2+cx^4 = x^4z^4$.

2.292 In certain cases, integrals of the form $\int R\left(x, \sqrt{P(x)}\right)dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = f_1\left(\frac{1}{k^2x}\right), \qquad f_2(x) = f_2\left(\frac{1-k^2x}{k^2(1-x)}\right), \qquad f_3(x) = f_3\left(\frac{1-x}{1-k^2x}\right),$$

hold, then

1. $$\int \frac{f_1(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_1(z)\,dz \qquad \left[z = \sqrt{x(1-x)\left(1-k^2x\right)}\right]$$

2. $$\int \frac{f_2(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_2(z)\,dz \qquad \left[z = \frac{\sqrt{x\left(1-k^2x\right)}}{\sqrt{1-x}}\right]$$

3. $$\int \frac{f_3(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_3(z)\,dz \qquad \left[z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}}\right]$$

where $R_1(z)$, $R_2(z)$, and $R_3(z)$ are rational functions of z.

2.3 The Exponential Function

2.31 Forms containing e^{ax}

2.311 $$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

2.312 a^x in the integrands should be replaced with $e^{x\ln a} = a^x$

2.313

1. $$\int \frac{dx}{a+be^{mx}} = \frac{1}{am}\left[mx - \ln\left(a+be^{mx}\right)\right]$$ PE (410)

2. $$\int \frac{dx}{1+e^x} = \ln\frac{e^x}{1+e^x} = x - \ln\left(1+e^x\right)$$ PE (409)

2.314 $$\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\arctan\left(e^{mx}\sqrt{\frac{a}{b}}\right) \qquad [ab>0]$$ PE (411)

$$= \frac{1}{2m\sqrt{-ab}}\ln\left|\frac{b+e^{mx}\sqrt{-ab}}{b-e^{mx}\sqrt{-ab}}\right| \qquad [ab<0]$$

2.315 $$\int \frac{dx}{\sqrt{a+be^{mx}}} = \frac{1}{m\sqrt{a}}\ln\frac{\sqrt{a+be^{mx}}-\sqrt{a}}{\sqrt{a+be^{mx}}+\sqrt{a}} \qquad [a>0]$$

$$= \frac{2}{m\sqrt{-a}}\arctan\frac{\sqrt{a+be^{mx}}}{\sqrt{-a}} \qquad [a<0]$$

2.32 The exponential combined with rational functions of x

2.321

1. $$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

2.[11] $$\int x^n e^{ax}\,dx = e^{ax}\left(\sum_{k=0}^{n}\frac{(-1)^k k!\binom{n}{k}}{a^{k+1}}x^{n-k}\right)$$

2.322

1. $$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)$$

2. $$\int x^2e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)$$

3. $$\int x^3e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a}-\frac{3x^2}{a^2}+\frac{6x}{a^3}-\frac{6}{a^4}\right)$$

4.[10] $$\int x^4e^{ax}\,dx = e^{ax}\left(\frac{x^4}{a}-\frac{4x^3}{a^2}+\frac{12x^2}{a^3}-\frac{24x}{a^4}+\frac{24}{a^5}\right)$$

2.323 $$\int P_m(x)e^{ax}\,dx = \frac{e^{ax}}{a}\sum_{k=0}^{m}(-1)^k\frac{P^{(k)}(x)}{a^k},$$

where $P_m(x)$ is a polynomial in x of degree m and $P^{(k)}(x)$ is the kth derivative of $P_m(x)$ with respect to x.

2.324

1. $$\int \frac{e^{ax}\,dx}{x^m} = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int \frac{e^{ax}\,dx}{x^{m-1}}\right]$$

2. $$\int \frac{e^{ax}}{x^n}\,dx = -e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!}\,\mathrm{Ei}(ax)$$

2.325

1. $$\int \frac{e^{ax}}{x}\,dx = \mathrm{Ei}(ax)$$

2. $$\int \frac{e^{ax}}{x^2}\,dx = -\frac{e^{ax}}{x} + a\,\mathrm{Ei}(ax)$$

3. $$\int \frac{e^{ax}}{x^3}\,dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2}\,\mathrm{Ei}(ax)$$

4.* $$\int \frac{e^{ax}}{x^4}\,dx = -\frac{e^{ax}}{3x^3} - \frac{ae^{ax}}{6x^2} - \frac{a^2e^{ax}}{6x} + \frac{a^3}{6}\,\mathrm{Ei}(ax)$$

5.* $$\int \frac{e^{\pm ax^n}}{x^m}\,dx = \frac{1}{m-1}\left[-\frac{e^{\pm ax^n}}{x^{m-1}} \pm na\int \frac{e^{\pm ax^n}}{x^{m-n}}\,dx\right] \qquad [m \neq 1]$$

6.* $$\int \frac{e^{ax^n}}{x^m}\,dx = \frac{(-1)^{z+1}a^z\,\Gamma(-z,-ax^n)}{n} = \frac{(-1)^{z+1}a^z}{n}\int_{-ax^n}^{\infty}\frac{e^{-t}}{t^{z+1}}\,dt$$

$$z = \frac{m-1}{n}, \quad \text{for } \Gamma(\alpha,x) \text{ see } 8.350.2 \quad [n \neq 0]$$

7.* $$\int \frac{e^{ax^n}}{x}\,dx = \frac{\mathrm{Ei}\,(ax^n)}{n} \qquad [a \neq 0, \quad n \neq 0]$$

8.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -e^{ax^n}\frac{\sum_{k=0}^{z-1} k!\,\frac{a^{z-k-1}}{x^{n(k+1)}}}{nz!} + \frac{a^z\,\mathrm{Ei}\,(ax^n)}{nz!}$$

$$\left[a \neq 0, \quad z = \frac{m-1}{n} = 1,2,\dots, \quad m = 2,3,\dots\right]$$

9.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{nx^n} + \frac{a\,\mathrm{Ei}\,(ax^n)}{n} \qquad \left[a \neq 0, \quad z = \frac{m-1}{n} = 1\right]$$

10.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{2nx^{2n}} - \frac{ae^{ax^n}}{2nx^n} + \frac{a^2\,\mathrm{Ei}\,(ax^n)}{2n} \qquad \left[a \neq 0, \quad z = \frac{m-1}{n} = 2\right]$$

11.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{3nx^{3n}} - \frac{e^{ax^n}}{6nx^{2n}} - \frac{a^2e^{ax^n}}{6nx^n} + \frac{a^3\,\mathrm{Ei}\,(ax^n)}{6n}$$

$$\left[a \neq 0, \quad z = \frac{m-1}{n} = 3\right]$$

12.* $$\int \frac{e^{ax^2}}{x^2}\,dx = -\frac{e^{ax^2}}{x} + \sqrt{a\pi}\,\mathrm{erfi}\left(\sqrt{a}x\right) \qquad \text{where } \mathrm{erfi}(z) = \frac{\mathrm{erf}(iz)}{i}$$

13.* $$\int e^{\left(ax^2+2bx+c\right)}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{ac-b^2}{a}\right)\operatorname{erfi}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$

$[a\neq 0]$

2.326 $$\int\frac{xe^{ax}\,dx}{(1+ax)^2}=\frac{e^{ax}}{a^2(1+ax)}$$ $[a\neq 0]$

2.33

1.[8] $$\int e^{-\left(ax^2+2bx+c\right)}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\operatorname{erf}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$

$[a\neq 0]$

2.* $$\int e^{ax^2}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\operatorname{erfi}\left(\sqrt{a}x\right)$$ where $\operatorname{erfi}(z)=\dfrac{\operatorname{erf}(iz)}{i}$ $[a\neq 0]$

3.* $$\int e^{ax^2+bx+c}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{ac-b^2}{a}\right)\operatorname{erfi}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$

where $\operatorname{erfi}(z)=\dfrac{\operatorname{erf}(iz)}{i}$ $[a\neq 0]$

4.* $$\int x^m e^{\pm ax^n}\,dx=\pm\frac{x^{m+1-n}}{na}\mp\frac{m+1-n}{na}\int x^{m-n}e^{\pm ax^n}\,dx$$

$[a\neq 0,\quad n\neq 0]$

5.* $$\int x^m e^{ax^n}\,dx=\frac{e^{ax^n}}{n}\left[(\gamma-1)!\sum_{k=0}^{\gamma-1}(-1)^{k+1-\gamma}\frac{x^{nk}}{k!a^{\gamma-k}}\right]$$

$\left[a\neq 0,\quad \gamma=\dfrac{m+1}{n}=1,2,\ldots\right]$

6.* $$\int x^m e^{ax^n}\,dx=\frac{e^{ax^n}}{na}$$ $\left[a\neq 0,\quad \gamma=\dfrac{m+1}{n}=1\right]$

7.* $$\int x^m e^{ax^n}\,dx=\frac{e^{ax^n}}{n}\left(\frac{x^n}{a}-\frac{1}{a^2}\right)$$ $\left[a\neq 0,\quad \gamma=\dfrac{m+1}{n}=2\right]$

8.* $$\int x^m e^{ax^n}\,dx=\frac{e^{ax^n}}{n}\left(\frac{x^{2n}}{a}-\frac{2x^n}{a^2}+\frac{2}{a^3}\right)$$ $\left[a\neq 0,\quad \gamma=\dfrac{m+1}{n}=3\right]$

9.* $$\int x^m e^{ax^n}\,dx=\frac{e^{ax^n}}{n}\left(\frac{x^{3n}}{a}-\frac{3x^{2n}}{a^2}+\frac{6x^n}{a^3}-\frac{6}{a^4}\right)$$ $\left[a\neq 0,\quad \gamma=\dfrac{m+1}{n}=4\right]$

10.* $$\int x^m e^{-\beta x^n}\,dx=-\frac{\Gamma\left(\gamma,\beta x^n\right)}{n\beta^\gamma}$$ for $\Gamma(\alpha,x)$ see 8.350.2

$$=-\frac{1}{n\beta^\gamma}\int_{\beta x^n}^{\infty}t^{\gamma-1}e^{-t}\,dx$$ $\left[\gamma=\dfrac{m+1}{n},\quad \beta\neq 0,\quad n\neq 0\right]$

11.* $$\int x^m\exp\left(-\beta x^n\right)\,dx=-\frac{(\gamma-1)!}{n}\exp\left(-\beta x^n\right)\left[\sum_{k=0}^{\gamma-1}\frac{x^{nk}}{k!\beta^{\gamma-k}}\right]$$

$\left[\gamma=\dfrac{m+1}{n}=1,2,\ldots\right]$

12.* $$\int x^m \exp(-\beta x^n)\, dx = -\frac{\exp(-\beta x^n)}{n\beta} \qquad \left[\gamma = \frac{m+1}{n} = 1\right]$$

13.* $$\int x^m \exp(-\beta x^n)\, dx = -\frac{\exp(-\beta x^n)}{n}\left(\frac{x^n}{\beta} + \frac{1}{\beta^2}\right) \qquad \left[\gamma = \frac{m+1}{n} = 2\right]$$

14.* $$\int x^m \exp(-\beta x^n)\, dx = -\frac{\exp(-\beta x^n)}{n}\left(\frac{x^{2n}}{\beta} + \frac{2x^n}{\beta^2} + \frac{2}{\beta^3}\right)$$
$$\left[\gamma = \frac{m+1}{n} = 3\right]$$

15.* $$\int x^m \exp(-\beta x^n)\, dx = -\frac{\exp(-\beta x^n)}{n}\left(\frac{x^{3n}}{\beta} + \frac{3x^{2n}}{\beta^2} + \frac{6x^n}{\beta^3} + \frac{6}{\beta^4}\right)$$
$$\left[\gamma = \frac{m+1}{n} = 4\right]$$

16.* $$\int e^{-\beta x^n}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,\mathrm{erf}\left(\sqrt{\beta}x\right) \qquad [\beta \neq 0]$$

17.* $$\int \frac{\exp(-\beta x^n)}{x^m}\, dx = -\frac{\beta^z\,\Gamma(-z, \beta x^n)}{n} = -\frac{\beta^z}{n}\int_{\beta x^n}^{\infty} \frac{e^{-t}}{t^{z+a}}\, dt$$
$$z = \frac{m-1}{n}$$

18.* $$\int \frac{\exp(-\beta x^n)}{x}\, dx = \frac{\mathrm{Ei}(-\beta x^n)}{n}$$

19.* $$\int \frac{\exp(-\beta x^n)}{x^m}\, dx = (-1)^z \frac{\exp(-\beta x^n)}{nz!}\sum_{k=0}^{z-1}(-1)^{k!}\frac{\beta^{z-k-1}}{x^{n(k+1)}} + (-1)^z\frac{\beta^z}{nz!}\,\mathrm{Ei}(-\beta x^n)$$
$$\left[z = \frac{m-1}{n} = 1, 2, \ldots, \quad m = 2, 3, \ldots\right]$$

20.* $$\int \frac{\exp(-\beta x^n)}{x^m}\, dx = -\frac{\exp(-\beta x^n)}{nx^n} - \frac{\beta\,\mathrm{Ei}(-\beta x^n)}{n} \qquad \left[z = \frac{m-1}{n} = 1\right]$$

21.* $$\int \frac{\exp(-\beta x^n)}{x^m}\, dx = -\frac{\exp(-\beta x^n)}{2nx^{2n}} + \frac{\beta\exp(-\beta x^n)}{2nx^n} + \frac{\beta^2\,\mathrm{Ei}(-\beta x^n)}{2n}$$
$$\left[z = \frac{m-1}{n} = 2\right]$$

22.* $$\int \frac{\exp(-\beta x^n)}{x^m}\, dx = -\frac{\exp(-\beta x^n)}{3nx^{3n}} + \frac{\beta\exp(-\beta x^n)}{6nx^{2n}} - \frac{\beta^2\exp(-\beta x^n)}{6nx^n} - \frac{\beta^3\,\mathrm{Ei}(-\beta x^n)}{6n}$$
$$\left[z = \frac{m-1}{n} = 3\right]$$

23.* $$\int \frac{\exp(-\beta x^2)}{x^2}\, dx = -\frac{\exp(-\beta x^2)}{x} - \sqrt{\beta\pi}\,\mathrm{erf}\left(\sqrt{\beta}x\right)$$

2.4 Hyperbolic Functions

2.41–2.43 Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$

2.411
$$\begin{aligned}\int \sinh^p x \cosh^q x\,dx &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sinh^p x \cosh^{q-2} x\,dx\\ &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{p+q} - \frac{p-1}{p+q}\int \sinh^{p-2} x \cosh^{q} x\,dx\\ &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{q+1} - \frac{p-1}{q+1}\int \sinh^{p-2} x \cosh^{q+2} x\,dx\\ &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+1} - \frac{q-1}{p+1}\int \sinh^{p+2} x \cosh^{q-2} x\,dx\\ &= \frac{\sinh^{p+1} x \cosh^{q+1} x}{p+1} - \frac{p+q+2}{p+1}\int \sinh^{p+2} x \cosh^{q} x\,dx\\ &= -\frac{\sinh^{p+1} x \cosh^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \sinh^{p} x \cosh^{q+2} x\,dx\end{aligned}$$

2.412

1. $$\begin{aligned}\int \sinh^p x \cosh^{2n} x\,dx = \frac{\sinh^{p+1} x}{2n+p}\Bigg[&\cosh^{2n-1} x\\ &+ \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \cosh^{2n-2k-1} x\Bigg]\\ &+ \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \sinh^p x\,dx\end{aligned}$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

2. $$\int \sinh^{2m} x\,dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\sinh(2m-2k)x}{2m-2k}$$ TI (543)

3. $$\int \sinh^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \frac{\cosh(2m-2k+1)x}{2m-2k+1};$$ TI (544)
$$= (-1)^n \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\cosh^{2k+1} x}{2k+1}$$ GU (351) (5)

4. $$\int \sinh^p x \cosh^{2n+1} x\,dx$$
$$= \frac{\sinh^{p+1} x}{2n+p+1}\left\{\cosh^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\cosh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.413

1. $$\int \cosh^p x \sinh^{2n} x\,dx = \frac{\cosh^{p+1} x}{2n+p}\left[\sinh^{2n-1} x + \sum_{k=1}^{n-1}(-1)^k \frac{(2n-1)(2n-3)\dots(2n-2k+1)\sinh^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right] + (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \cosh^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

2. $$\int \cosh^{2m} x\,dx = \binom{2m}{m}\frac{x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\frac{\sinh(2m-2k)x}{2m-2k}$$ TI (541)

3. $$\int \cosh^{2m+1} x\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\frac{\sinh(2m-2k+1)x}{2m-2k+1}$$ TI (542)
$$= \sum_{k=0}^{m}\binom{m}{k}\frac{\sinh^{2k+1} x}{2k+1}$$ GU (351) (8)

4. $$\int \cosh^p x \sinh^{2n+1} x\,dx = \frac{\cosh^{p+1} x}{2n+p+1}\left[\sinh^{2n} x + \sum_{k=1}^{n}(-1)^k \frac{2^k n(n-1)\dots(n-k+1)\sinh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right]$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.414

1. $$\int \sinh ax\,dx = \frac{1}{a}\cosh ax$$

2. $$\int \sinh^2 ax\,dx = \frac{1}{4a}\sinh 2ax - \frac{x}{2}$$

3. $$\int \sinh^3 x\,dx = -\frac{3}{4}\cosh x + \frac{1}{12}\cosh 3x = \frac{1}{3}\cosh^3 x - \cosh x$$

4. $$\int \sinh^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sinh 2x + \frac{1}{32}\sinh 4x = \frac{3}{8}x - \frac{3}{8}\sinh x\cosh x + \frac{1}{4}\sinh^3 x\cosh x$$

5. $$\int \sinh^5 x\,dx = \frac{5}{8}\cosh x - \frac{5}{48}\cosh 3x + \frac{1}{80}\cosh 5x = \frac{4}{5}\cosh x + \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\cosh^3 x$$

6. $$\int \sinh^6 x\,dx = -\frac{5}{16}x + \frac{15}{64}\sinh 2x - \frac{3}{64}\sinh 4x + \frac{1}{192}\sinh 6x = -\frac{5}{16}x + \frac{1}{6}\sinh^5 x\cosh x - \frac{5}{24}\sinh^3 x\cosh x + \frac{5}{16}\sinh x\cosh x$$

7. $$\int \sinh^7 x\,dx = -\tfrac{35}{64}\cosh x + \tfrac{7}{64}\cosh 3x - \tfrac{7}{320}\cosh 5x + \tfrac{1}{448}\cosh 7x$$
$$= -\tfrac{24}{35}\cosh x + \tfrac{8}{35}\cosh^3 x - \tfrac{6}{35}\cosh x \sinh^4 x + \tfrac{1}{7}\cosh x \sinh^6 x$$

8. $$\int \cosh ax\,dx = \frac{1}{a}\sinh ax$$

9. $$\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{1}{4a}\sinh 2ax$$

10. $$\int \cosh^3 x\,dx = \tfrac{3}{4}\sinh x + \tfrac{1}{12}\sinh 3x = \sinh x + \tfrac{1}{3}\sinh^3 x$$

11. $$\int \cosh^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sinh 2x + \tfrac{1}{32}\sinh 4x = \tfrac{3}{8}x + \tfrac{3}{8}\sinh x\cosh x + \tfrac{1}{4}\sinh x\cosh^3 x$$

12. $$\int \cosh^5 x\,dx = \tfrac{5}{8}\sinh x + \tfrac{5}{48}\sinh 3x + \tfrac{1}{80}\sinh 5x$$
$$= \tfrac{4}{5}\sinh x + \tfrac{1}{5}\cosh^4 x\sinh x + \tfrac{4}{15}\sinh^3 x$$

13. $$\int \cosh^6 x\,dx = \tfrac{5}{16}x + \tfrac{15}{64}\sinh 2x + \tfrac{3}{64}\sinh 4x + \tfrac{1}{192}\sinh 6x$$
$$= \tfrac{5}{16}x + \tfrac{5}{16}\sinh x\cosh x + \tfrac{5}{24}\sinh x\cosh^3 x + \tfrac{1}{6}\sinh x\cosh^5 x$$

14. $$\int \cosh^7 x\,dx = \tfrac{35}{64}\sinh x + \tfrac{7}{64}\sinh 3x + \tfrac{7}{320}\sinh 5x + \tfrac{1}{448}\sinh 7x$$
$$= \tfrac{24}{35}\sinh x + \tfrac{8}{35}\sinh^3 x + \tfrac{6}{35}\sinh x\cosh^4 x + \tfrac{1}{7}\sinh x\cosh^6 x$$

2.415

1. $$\int \sinh ax\cosh bx\,dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}$$

2. $$\int \sinh ax\cosh ax\,dx = \frac{1}{4a}\cosh 2ax$$

3. $$\int \sinh^2 x\cosh x\,dx = \tfrac{1}{3}\sinh^3 x$$

4. $$\int \sinh^3 x\cosh x\,dx = \tfrac{1}{4}\sinh^4 x$$

5. $$\int \sinh^4 x\cosh x\,dx = \tfrac{1}{5}\sinh^5 x$$

6. $$\int \sinh x\cosh^2 x\,dx = \tfrac{1}{3}\cosh^3 x$$

7. $$\int \sinh^2 x\cosh^2 x\,dx = -\frac{x}{8} + \frac{1}{32}\sinh 4x$$

8. $$\int \sinh^3 x\cosh^2 x\,dx = \tfrac{1}{5}\left(\sinh^2 x - \tfrac{2}{3}\right)\cosh^3 x$$

9. $$\int \sinh^4 x\cosh^2 x\,dx = \frac{x}{16} - \frac{1}{64}\sinh 2x - \frac{1}{64}\sinh 4x + \frac{1}{192}\sinh 6x$$

10. $$\int \sinh x \cosh^3 x\, dx = \tfrac{1}{4} \cosh^4 x$$

11. $$\int \sinh^2 x \cosh^3 x\, dx = \tfrac{1}{5} \left(\cosh^2 x + \tfrac{2}{3}\right) \sinh^3 x$$

12. $$\int \sinh^3 x \cosh^3 x\, dx = -\tfrac{3}{64} \cosh 2x + \tfrac{1}{192} \cosh 6x = \tfrac{1}{48} \cosh^3 2x - \tfrac{1}{16} \cosh 2x$$
$$= \frac{\sinh^6 x}{6} + \frac{\sinh^4 x}{4} = \frac{\cosh^6 x}{6} - \frac{\cosh^4 x}{4}$$

13. $$\int \sinh^4 x \cosh^3 x\, dx = \tfrac{1}{7} \sinh^3 x \left(\cosh^4 x - \tfrac{3}{5} \cosh^2 x - \tfrac{2}{5}\right) = \tfrac{1}{7} \left(\cosh^2 x + \tfrac{2}{5}\right) \sinh^5 x$$

14. $$\int \sinh x \cosh^4 x\, dx = \tfrac{1}{5} \cosh^5 x$$

15. $$\int \sinh^2 x \cosh^4 x\, dx = -\frac{x}{16} - \frac{1}{64} \sinh 2x + \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$$

16. $$\int \sinh^3 x \cosh^4 x\, dx = \tfrac{1}{7} \cosh^3 x \left(\sinh^4 x + \tfrac{3}{5} \sinh^2 x - \tfrac{2}{5}\right) = \tfrac{1}{7} \left(\sinh^2 x - \tfrac{2}{5}\right) \cosh^5 x$$

17. $$\int \sinh^4 x \cosh^4 x\, dx = \frac{3x}{128} - \frac{1}{128} \sinh 4x + \frac{1}{1024} \sinh 8x$$

2.416

1.[10] $$\int \frac{\sinh^p x}{\cosh^{2n} x}\, dx = \frac{\sinh^{p+1} x}{2n-1} \left[\operatorname{sech}^{2n-1} x \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \sec h^{2n-2k-1} x \right]$$
$$+ \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \sinh^p x\, dx$$

This formula is applicable for arbitrary real p. For $\int \sinh^p x\, dx$, where p is a natural number, see **2.412** 2 and **2.412** 3. For $n = 0$ and p a negative integer, we have for this integral:

2. $$\int \frac{dx}{\sinh^{2m} x} = \frac{\cosh x}{2m-1} \left[-\operatorname{cosech}^{2m-1} x \right.$$
$$\left. + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{cosec} h^{2m-2k-1} x \right]$$

3. $$\int \frac{dx}{\sinh^{2m+1} x} = \frac{\cosh x}{2m} \left[-\operatorname{cosech}^{2m} x \right.$$
$$\left. + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \operatorname{cosec} h^{2m-2k} x \right]$$
$$+ (-1)^m \frac{(2m-1)!!}{(2m)!!} \ln \tanh \frac{x}{2}$$

2.417

1. $$\int \frac{\sinh^p x}{\cosh^{2n+1} x}\,dx = \frac{\sinh^{p+1} x}{2n}\left[\operatorname{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \sec h^{2n-2k} x\right] + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\sinh^p x}{\cosh x}\,dx$$

This formula is applicable for arbitrary real p. For $n=0$ and p integral, we have

2. $$\int \frac{\sinh^{2m+1} x}{\cosh x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\sinh^{2k} x + (-1)^m \ln\cosh x = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\binom{m}{k}\cosh^{2k} x + (-1)^m \ln\cosh x \qquad [m \geq 1]$$

3. $$\int \frac{\sinh^{2m} x}{\cosh x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k-1}\sinh^{2k-1} x + (-1)^m \arctan(\sinh x) \qquad [m \geq 1]$$

4. $$\int \frac{dx}{\sinh^{2m+1} x \cosh x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln\tanh x$$

5. $$\int \frac{dx}{\sinh^{2m} x \cosh x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \arctan\sinh x$$

2.418

1. $$\int \frac{\cosh^p x}{\sinh^{2n} x}\,dx = -\frac{\cosh^{p+1} x}{2n-1}\left[\operatorname{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec} h^{2n-2k-1} x\right] + \frac{(-1)^n(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \cosh^p x\,dx$$

This formula is applicable for arbitrary real p. For the integral $\int \cosh^p x\,dx$, where p is a natural number, see **2.413** 2 and **2.413** 3. If p is a negative integer, we have for this integral:

2. $$\int \frac{dx}{\cosh^{2m} x} = \frac{\sinh x}{2m-1}\left\{\operatorname{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k(m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{sech}^{2m-2k-1} x\right\}$$

3. $$\int \frac{dx}{\cosh^{2m+1} x} = \frac{\sinh x}{2m}\left\{\operatorname{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m-1)(m-2)\dots(m-k)} \operatorname{sech}^{2m-2k} x\right\} + \frac{(2m-1)!!}{(2m)!!}\arctan\sinh x$$

2.419

1. $$\int \frac{\cosh^p x}{\sinh^{2n+1} x}\,dx = -\frac{\cosh^{p+1} x}{2n}\left[\operatorname{cosech}^{2n} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \operatorname{cosec} h^{2n-2k} x\right] + \frac{(-1)^n (2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\cosh^p x}{\sinh x}\,dx$$

This formula is applicable for arbitrary real p. For $n=0$ and p an integer

2. $$\int \frac{\cosh^{2m} x}{\sinh x}\,dx = \sum_{k=1}^{m} \frac{\cosh^{2k-1} x}{2k-1} + \ln\tanh\frac{x}{2}$$

3. $$\int \frac{\cosh^{2m+1} x}{\sinh x}\,dx = \sum_{k=1}^{m} \frac{\cosh^{2k} x}{2k} + \ln\sinh x = \sum_{k=1}^{m} \binom{m}{k} \frac{\sinh^{2k} x}{2k} + \ln\sinh x$$

4. $$\int \frac{dx}{\sinh x \cosh^{2m} x} = \sum_{k=1}^{m} \frac{\operatorname{sech}^{2m-2k+1} x}{2m-2k+1} + \ln\tanh\frac{x}{2}$$

5. $$\int \frac{dx}{\sinh x \cosh^{2m+1} x} = \sum_{k=1}^{m} \frac{\operatorname{sech}^{2m-2k+2} x}{2m-2k+2} + \ln\tanh x$$

2.421 In formulas **2.421** 1 and **2.421** 2, $s=1$ for m odd and $m<2n+1$; in all other cases, $s=0$.

GI (351)(11, 13)

1.[10] $$\int \frac{\sinh^{2n+1} x}{\cosh^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} (-1)^{n+k} \binom{n}{k} \frac{\cosh^{2k-m+1} x}{2k-m+1} + s(-1)^{n+\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln\cosh x$$

2. $$\int \frac{\cosh^{2n+1} x}{\sinh^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} \binom{n}{k} \frac{\sinh^{2k-m+1} x}{2k-m+1} + s\binom{n}{\frac{m-1}{2}} \ln\sinh x$$

2.422

1. $$\int \frac{dx}{\sinh^{2m} x \cosh^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1} \binom{m+n-1}{k} \tanh^{2k-2m+1} x$$

2. $$\int \frac{dx}{\sinh^{2m+1} x \cosh^{2n+1} x} = \sum_{\substack{k=0\\ k\neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \tanh^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln\tanh x$$

GI (351)(15)

2.423

1. $$\int \frac{dx}{\sinh x} = \ln\tanh\frac{x}{2} = \frac{1}{2}\ln\frac{\cosh x - 1}{\cosh x + 1}$$

2. $$\int \frac{dx}{\sinh^2 x} = -\coth x$$

3. $$\int \frac{dx}{\sinh^3 x} = -\frac{\cosh x}{2\sinh^2 x} - \frac{1}{2}\ln\tanh\frac{x}{2}$$

4. $$\int \frac{dx}{\sinh^4 x} = -\frac{\cosh x}{3\sinh^3 x} + \frac{2}{3}\coth x = -\frac{1}{3}\coth^3 x + \coth x$$

5. $$\int \frac{dx}{\sinh^5 x} = -\frac{\cosh x}{4\sinh^4 x} + \frac{3}{8}\frac{\cosh x}{\sinh^2 x} + \frac{3}{8}\ln\tanh\frac{x}{2}$$

6. $$\begin{aligned}\int \frac{dx}{\sinh^6 x} &= -\frac{\cosh x}{5\sinh^5 x} + \frac{4}{15}\coth^3 x - \frac{4}{5}\coth x \\ &= -\frac{1}{5}\coth^5 x + \frac{2}{3}\coth^3 x - \coth x\end{aligned}$$

7. $$\int \frac{dx}{\sinh^7 x} = -\frac{\cosh x}{6\sinh^2 x}\left(\frac{1}{\sinh^4 x} - \frac{5}{4\sinh^2 x} + \frac{15}{8}\right) - \frac{5}{16}\ln\tanh\frac{x}{2}$$

8. $$\int \frac{dx}{\sinh^8 x} = \coth x - \coth^3 x + \frac{3}{5}\coth^5 x - \frac{1}{7}\coth^7 x$$

9. $$\begin{aligned}\int \frac{dx}{\cosh x} &= \arctan(\sinh x) \\ &= \arcsin(\tanh x) \\ &= 2\arctan(e^x) \\ &= \operatorname{gd} x\end{aligned}$$

10. $$\int \frac{dx}{\cosh^2 x} = \tanh x$$

11. $$\int \frac{dx}{\cosh^3 x} = \frac{\sinh x}{2\cosh^2 x} + \frac{1}{2}\arctan(\sinh x)$$

12. $$\begin{aligned}\int \frac{dx}{\cosh^4 x} &= \frac{\sinh x}{3\cosh^3 x} + \frac{2}{3}\tanh x \\ &= -\frac{1}{3}\tanh^3 x + \tanh x\end{aligned}$$

13. $$\int \frac{dx}{\cosh^5 x} = \frac{\sinh x}{4\cosh^4 x} + \frac{3}{8}\frac{\sinh x}{\cosh^2 x} + \frac{3}{8}\arctan(\sinh x)$$

14. $$\begin{aligned}\int \frac{dx}{\cosh^6 x} &= \frac{\sinh x}{5\cosh^5 x} - \frac{4}{15}\tanh^3 x + \frac{4}{5}\tanh x \\ &= \frac{1}{5}\tanh^5 x - \frac{2}{3}\tanh^3 x + \tanh x\end{aligned}$$

15. $$\int \frac{dx}{\cosh^7 x} = \frac{\sinh x}{6\cosh^2 x}\left(\frac{1}{\cosh^4 x} + \frac{5}{4\cosh^2 x} + \frac{15}{8}\right) + \frac{5}{16}\arctan(\sinh x)$$

16. $$\int \frac{dx}{\cosh^8 x} = -\frac{1}{7}\tanh^7 x + \frac{3}{5}\tanh^5 x - \tanh^3 x + \tanh x$$

17. $$\int \frac{\sinh x}{\cosh x}\,dx = \ln\cosh x$$

18. $$\int \frac{\sinh^2 x}{\cosh x}\,dx = \sinh x - \arctan(\sinh x)$$

19. $$\int \frac{\sinh^3 x}{\cosh x}\,dx = \frac{1}{2}\sinh^2 x - \ln\cosh x = \frac{1}{2}\cosh^2 x - \ln\cosh x$$

20. $$\int \frac{\sinh^4 x}{\cosh x}\,dx = \frac{1}{3}\sinh^3 x - \sinh x + \arctan(\sinh x)$$

21. $$\int \frac{\sinh x}{\cosh^2 x}\,dx = -\frac{1}{\cosh x}$$

22. $$\int \frac{\sinh^2 x}{\cosh^2 x}\,dx = x - \tanh x$$

23. $$\int \frac{\sinh^3 x}{\cosh^2 x}\,dx = \cosh x + \frac{1}{\cosh x}$$

24. $$\int \frac{\sinh^4 x}{\cosh^2 x}\,dx = -\frac{3}{2}x + \frac{1}{4}\sinh 2x + \tanh x$$

25. $$\int \frac{\sinh x}{\cosh^3 x}\,dx = -\frac{1}{2\cosh^2 x} = \frac{1}{2}\tanh^2 x$$

26. $$\int \frac{\sinh^2 x}{\cosh^3 x}\,dx = -\frac{\sinh x}{2\cosh^2 x} + \frac{1}{2}\arctan(\sinh x)$$

27. $$\int \frac{\sinh^3 x}{\cosh^3 x}\,dx = -\frac{1}{2}\tanh^2 x + \ln\cosh x = \frac{1}{2\cosh^2 x} + \ln\cosh x$$

28. $$\int \frac{\sinh^4 x}{\cosh^3 x}\,dx = \frac{\sinh x}{2\cosh x} + \sinh x - \frac{3}{2}\arctan(\sinh x)$$

29. $$\int \frac{\sinh x}{\cosh^4 x}\,dx = -\frac{1}{3\cosh^3 x}$$

30. $$\int \frac{\sinh^2 x}{\cosh^4 x}\,dx = \frac{1}{3}\tanh^3 x$$

31. $$\int \frac{\sinh^3 x}{\cosh^4 x}\,dx = -\frac{1}{\cosh x} + \frac{1}{3\cosh^3 x}$$

32. $$\int \frac{\sinh^4 x}{\cosh^4 x}\,dx = -\frac{1}{3}\tanh^3 x - \tanh x + x$$

33. $$\int \frac{\cosh x}{\sinh x}\,dx = \ln\sinh x$$

34. $$\int \frac{\cosh^2 x}{\sinh x}\,dx = \cosh x + \ln\tanh\frac{x}{2}$$

35. $$\int \frac{\cosh^3 x}{\sinh x}\,dx = \frac{1}{2}\cosh^2 x + \ln\sinh x$$

36. $$\int \frac{\cosh^4 x}{\sinh x}\,dx = \frac{1}{3}\cosh^3 x + \cosh x + \ln\tanh\frac{x}{2}$$

37. $$\int \frac{\cosh x}{\sinh^2 x}\,dx = -\frac{1}{\sinh x}$$

38. $$\int \frac{\cosh^2 x}{\sinh^2 x}\,dx = x - \coth x$$

39. $$\int \frac{\cosh^3 x}{\sinh^2 x}\,dx = \sinh x - \frac{1}{\sinh x}$$

40. $$\int \frac{\cosh^4 x}{\sinh^2 x}\,dx = \frac{3}{2}x + \frac{1}{4}\sinh 2x - \coth x$$

41. $$\begin{aligned}\int \frac{\cosh x}{\sinh^3 x}\,dx &= -\frac{1}{2\sinh^2 x}\\ &= -\frac{1}{2}\coth^2 x\end{aligned}$$

42. $$\int \frac{\cosh^2 x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \ln\tanh\frac{x}{2}$$

43. $$\begin{aligned}\int \frac{\cosh^3 x}{\sinh^3 x}\,dx &= -\frac{1}{2\sinh^2 x} + \ln\sinh x\\ &= -\frac{1}{2}\coth^2 x + \ln\sinh x\end{aligned}$$

44. $$\int \frac{\cosh^4 x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \cosh x + \frac{3}{2}\ln\tanh\frac{x}{2}$$

45. $$\int \frac{\cosh x}{\sinh^4 x}\,dx = -\frac{1}{3\sinh^3 x}$$

46. $$\int \frac{\cosh^2 x}{\sinh^4 x}\,dx = -\frac{1}{3}\coth^3 x$$

47. $$\int \frac{\cosh^3 x}{\sinh^4 x}\,dx = -\frac{1}{\sinh x} - \frac{1}{3\sinh^3 x}$$

48. $$\int \frac{\cosh^4 x}{\sinh^4 x}\,dx = -\frac{1}{3}\coth^3 x - \coth x + x$$

49. $$\int \frac{dx}{\sinh x\cosh x} = \ln\tanh x$$

50. $$\int \frac{dx}{\sinh x\cosh^2 x} = \frac{1}{\cosh x} + \ln\tanh\frac{x}{2}$$

51. $$\begin{aligned}\int \frac{dx}{\sinh x\cosh^3 x} &= \frac{1}{2\cosh^2 x} + \ln\tanh x\\ &= -\frac{1}{2}\tanh^2 x + \ln\tanh x\end{aligned}$$

52. $$\int \frac{dx}{\sinh x \cosh^4 x} = \frac{1}{\cosh x} + \frac{1}{3\cosh^3 x} + \ln\tanh\frac{x}{2}$$

53. $$\int \frac{dx}{\sinh^2 x \cosh x} = -\frac{1}{\sinh x} - \arctan\sinh x$$

54. $$\int \frac{dx}{\sinh^2 x \cosh^2 x} = -2\coth 2x$$

55. $$\int \frac{dx}{\sinh^2 x \cosh^3 x} = -\frac{\sinh x}{2\cosh^2 x} - \frac{1}{\sinh x} - \frac{3}{2}\arctan\sinh x$$

56. $$\int \frac{dx}{\sinh^2 x \cosh^4 x} = \frac{1}{3\sinh x\cosh^3 x} - \frac{8}{3}\coth 2x$$

57. $$\int \frac{dx}{\sinh^3 x \cosh x} = -\frac{1}{2\sinh^2 x} - \ln\tanh x = -\frac{1}{2}\coth^2 x + \ln\coth x$$

58. $$\int \frac{dx}{\sinh^3 x \cosh^2 x} = -\frac{1}{\cosh x} - \frac{\cosh x}{2\sinh^2 x} - \frac{3}{2}\ln\tanh\frac{x}{2}$$

59. $$\int \frac{dx}{\sinh^3 x \cosh^3 x} = -\frac{2\cosh 2x}{\sinh^2 2x} - 2\ln\tanh x = \frac{1}{2}\tanh^2 x - \frac{1}{2}\coth^2 x - 2\ln\tanh x$$

60. $$\int \frac{dx}{\sinh^3 x \cosh^4 x} = -\frac{2}{\cosh x} - \frac{1}{3\cosh^2 x} - \frac{\cosh x}{2\sinh^2 x} - \frac{5}{2}\ln\tanh\frac{x}{2}$$

61. $$\int \frac{dx}{\sinh^4 x \cosh x} = \frac{1}{\sinh x} - \frac{1}{3\sinh^3 x} + \arctan\sinh x$$

62. $$\int \frac{dx}{\sinh^4 x \cosh^2 x} = -\frac{1}{3\cosh x\sinh^3 x} + \frac{8}{3}\coth 2x$$

63. $$\int \frac{dx}{\sinh^4 x \cosh^3 x} = \frac{2}{\sinh x} - \frac{1}{3\sinh^3 x} + \frac{\sinh x}{2\cosh^2 x} + \frac{5}{2}\arctan\sinh x$$

64. $$\int \frac{dx}{\sinh^4 x \cosh^4 x} = 8\coth 2x - \frac{8}{3}\coth^3 2x$$

2.424

1. $$\int \tanh^p x\,dx = -\frac{\tanh^{p-1} x}{p-1} + \int \tanh^{p-2} x\,dx \qquad [p \neq 1]$$

2. $$\int \tanh^{2n+1} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k}\binom{n}{k}\frac{1}{\cosh^{2k} x} + \ln\cosh x = -\sum_{k=1}^{n}\frac{\tanh^{2n-2k+2} x}{2n-2k+2} + \ln\cosh x$$

3. $$\int \tanh^{2n} x\,dx = -\sum_{k=1}^{n}\frac{\tanh^{2n-2k+1} x}{2n-2k+1} + x$$ GU (351)(12)

4. $$\int \coth^p x\,dx = -\frac{\coth^{p-1} x}{p-1} + \int \coth^{p-2} x\,dx \qquad [p \neq 1]$$

5. $$\int \coth^{2n+1} x\,dx = -\sum_{k=1}^{n} \frac{1}{2n}\binom{n}{k}\frac{1}{\sinh^{2k} x} + \ln\sinh x$$
$$= -\sum_{k=1}^{n} \frac{\coth^{2n-2k+2} x}{2n-2k+2} + \ln\sinh x$$

6. $$\int \coth^{2n} x\,dx = -\sum_{k=1}^{n} \frac{\coth^{2n-2k+1} x}{2n-2k+1} + x$$ GU (351)(14)

For formulas containing powers of $\tanh x$ and $\coth x$ equal to $n = 1, 2, 3, 4$, see **2.423** 17, **2.423** 22, **2.423** 27, **2.423** 32, **2.423** 33, **2.423** 38, **2.423** 43, **2.423** 48.

Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

2.425

1. $$\int \sinh(ax+b)\sinh(cx+d)\,dx = \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] - \frac{1}{2(a-c)}\sinh[(a-c)x+b-d] \qquad [a^2 \neq c^2]$$ GU (352)(2a)

2. $$\int \sinh(ax+b)\cosh(cx+d)\,dx = \frac{1}{2(a+c)}\cosh[(a+c)x+b+d] + \frac{1}{2(a-c)}\cosh[(a-c)x+b-d] \qquad [a^2 \neq c^2]$$ GU (352)(2c)

3. $$\int \cosh(ax+b)\cosh(cx+d)\,dx = \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] + \frac{1}{2(a-c)}\sinh[(a-c)x+b-d] \qquad [a^2 \neq c^2]$$ GU (352)(2b)

When $a = c$:

4. $$\int \sinh(ax+b)\sinh(ax+d)\,dx = -\frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$$ GU (352)(3a)

5. $$\int \sinh(ax+b)\cosh(ax+d)\,dx = \frac{x}{2}\sinh(b-d) + \frac{1}{4a}\cosh(2ax+b+d)$$ GU (352)(3c)

6. $$\int \cosh(ax+b)\cosh(ax+d)\,dx = \frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$$ GU (352)(3b)

2.426

1. $$\int \sinh ax \sinh bx \sinh cx\,dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} - \frac{\cosh(a-b+c)x}{4(a-b+c)} - \frac{\cosh(a+b-c)x}{4(a+b-c)}$$ GU (352)(4a)

2. $$\int \sinh ax \sinh bx \cosh cx\, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} - \frac{\sinh(-a+b+c)x}{4(-a+b+c)} - \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4b)

3. $$\int \sinh ax \cosh bx \cosh cx\, dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} + \frac{\cosh(a-b+c)x}{4(a-b+c)} + \frac{\cosh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4c)

4. $$\int \cosh ax \cosh bx \cosh cx\, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} + \frac{\sinh(-a+b+c)x}{4(-a+b+c)} + \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4d)

2.427

1. $$\int \sinh^p x \sinh ax\, dx = \frac{1}{p+a}\left\{\sinh px \cosh ax - p\int \sinh^{p-1} x \cosh(a-1)x\, dx\right\}$$

2. $$\int \sinh^p x \sinh(2n+1)x\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k+1)x - \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k)x\right] + \frac{\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma(p+1-2n)} \int \sinh^{p-2n} x \sinh x\, dx$$

[p is not a negative integer]

3. $$\int \sinh^p x \sinh 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k)x - \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k-1)x\right]$$

[p is not a negative integer] GU (352)(5)a

2.428

1. $$\int \sinh^p x \cosh ax\, dx = \frac{1}{p+a}\left\{\sinh^p x \sinh ax - p\int \sinh^{p-1} x \sinh(a-1)x\, dx\right\}$$

2. $$\int \sinh^p x \cosh(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times \left\{ \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k+1)x - \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k)x \right] + \frac{\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p+1-2n)} \int \sinh^{p-2n} x \cosh x\,dx \right\}$$

[p is not a negative integer]

3. $$\int \sinh^p x \cosh 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k)x - \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k-1)x \right] + \frac{\Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\,\Gamma(p+1-2n)} \int \sinh^{p-2n} x\,dx \right\}$$

[p is not a negative integer] GU (352)(6)a

2.429

1. $$\int \cosh^p x \sinh ax\,dx = \frac{1}{p+a} \left\{ \cosh^p x \cosh ax + p \int \cosh^{p-1} x \sinh(a-1)x\,dx \right\}$$

2. $$\int \cosh^p x \sinh(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)} \int \cosh^{p-n} x \sinh(n+1)x\,dx \right]$$

[p is not a negative integer]

3. $$\int \cosh^p x \sinh 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cosh^{p-n} x \sinh nx\,dx \right]$$

[p is not a negative integer] GU (352)(7)a

2.431

1. $$\int \cosh^p x \cosh ax\, dx = \frac{1}{p+a}\left\{\cosh^p x \sinh ax + p\int \cosh^{p-1} x \cosh(a-1)x\, dx\right\}$$

2. $$\int \cosh^p x \cosh(2n+1)x\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left[\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)}\cosh^{p-k} x \sinh(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)}\int \cosh^{p-n} x \cosh(n+1)x\, dx\right]$$
[p is not a negative integer]

3. $$\int \cosh^p x \cosh 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left[\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)}\cosh^{p-k} x \sinh(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)}\cosh^{p-n} x \cosh nx\, dx\right]$$
[p is not a negative integer] GU (352)(8)a

2.432

1. $$\int \sinh(n+1)x \sinh^{n-1} x\, dx = \frac{1}{n}\sinh^n x \sinh nx$$

2. $$\int \sinh(n+1)x \cosh^{n-1} x\, dx = \frac{1}{n}\cosh^n x \cosh nx$$

3. $$\int \cosh(n+1)x \sinh^{n-1} x\, dx = \frac{1}{n}\sinh^n x \cosh nx$$

4. $$\int \cosh(n+1)x \cosh^{n-1} x\, dx = \frac{1}{n}\cosh^n x \sinh nx$$

2.433

1. $$\int \frac{\sinh(2n+1)x}{\sinh x}\, dx = 2\sum_{k=0}^{n-1}\frac{\sinh(2n-2k)x}{2n-2k} + x$$

2. $$\int \frac{\sinh 2nx}{\sinh x}\, dx = 2\sum_{k=0}^{n-1}\frac{\sinh(2n-2k-1)x}{2n-2k-1}$$ GU (352)(5d)

3. $$\int \frac{\cosh(2n+1)x}{\sinh x}\, dx = 2\sum_{k=0}^{n-1}\frac{\cosh(2n-2k)x}{2n-2k} + \ln\sinh x$$

4. $$\int \frac{\cosh 2nx}{\sinh x}\, dx = 2\sum_{k=0}^{n-1}\frac{\cosh(2n-2k-1)x}{2n-2k-1} + \ln\tanh\frac{x}{2}$$ GU (352)(6d)

5. $$\int \frac{\sinh(2n+1)x}{\cosh x}\, dx = 2\sum_{k=0}^{n-1}(-1)^k\frac{\cosh(2n-2k)x}{2n-2k} + (-1)^n \ln\cosh x$$

6. $$\int \frac{\sinh 2nx}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\cosh(2n-2k-1)x}{2n-2k-1}$$ GU (352)(7d)

7. $$\int \frac{\cosh(2n+1)x}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\sinh(2n-2k)x}{2n-2k} + (-1)^n x$$

8. $$\int \frac{\cosh 2nx}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\sinh(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\tanh x)$$ GU (352)(8d)

9. $$\int \frac{\sinh 2x}{\sinh^n x}\,dx = -\frac{2}{(n-2)\sinh^{n-2} x}$$

For $n = 2$:

10. $$\int \frac{\sinh 2x}{\sinh^2 x}\,dx = 2\ln\sinh x$$

11. $$\int \frac{\sinh 2x\,dx}{\cosh^n x} = \frac{2}{(2-n)\cosh^{n-2} x}$$

For $n = 2$:

12. $$\int \frac{\sinh 2x}{\cosh^2 x}\,dx = 2\ln\cosh x$$

13. $$\int \frac{\cosh 2x}{\sinh x}\,dx = 2\cosh x + \ln\tanh\frac{x}{2}$$

14. $$\int \frac{\cosh 2x}{\sinh^2 x}\,dx = -\coth x + 2x$$

15. $$\int \frac{\cosh 2x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \frac{3}{2}\ln\tanh\frac{x}{2}$$

16. $$\int \frac{\cosh 2x}{\cosh x}\,dx = 2\sinh x - \arcsin(\tanh x)$$

17. $$\int \frac{\cosh 2x}{\cosh^2 x}\,dx = -\tanh x + 2x$$

18. $$\int \frac{\cosh 2x}{\cosh^3 x}\,dx = -\frac{\sinh x}{2\cosh^2 x} + \frac{3}{2}\arcsin(\tanh x)$$

19. $$\int \frac{\sinh 3x}{\sinh x}\,dx = x + \sinh 2x$$

20. $$\int \frac{\sinh 3x}{\sinh^2 x}\,dx = 3\ln\tanh\frac{x}{2} + 4\cosh x$$

21. $$\int \frac{\sinh 3x}{\sinh^3 x}\,dx = -3\coth x + 4x$$

22. $$\int \frac{\sinh 3x}{\cosh^n x}\,dx = \frac{4}{(3-n)\cosh^{n-3} x} - \frac{1}{(1-n)\cosh^{n-1} x}$$

For $n = 1$ and $n = 3$:

23. $$\int \frac{\sinh 3x}{\cosh x}\,dx = 2\sinh^2 x - \ln\cosh x$$

24. $$\int \frac{\sinh 3x}{\cosh^3 x}\,dx = \frac{1}{2\cosh^2 x} + 4\ln\cosh x$$

25. $$\int \frac{\cosh 3x}{\sinh^n x}\,dx = \frac{4}{(3-n)\sinh^{n-3} x} + \frac{1}{(1-n)\sinh^{n-1} x}$$

For $n = 1$ and $n = 3$:

26. $$\int \frac{\cosh 3x}{\sinh x}\,dx = 2\sinh^2 x + \ln\sinh x$$

27. $$\int \frac{\cosh 3x}{\sinh^3 x}\,dx = -\frac{1}{2\sinh^2 x} + 4\ln\sinh x$$

28. $$\int \frac{\cosh 3x}{\cosh x}\,dx = \sinh 2x - x$$

29. $$\int \frac{\cosh 3x}{\cosh^2 x}\,dx = 4\sinh x - 3\arcsin(\tanh x)$$

30. $$\int \frac{\cosh 3x}{\cosh^3 x}\,dx = 4x - 3\tanh x$$

2.44–2.45 Rational functions of hyperbolic functions

2.441

1. $$\int \frac{A + B\sinh x}{(a + b\sinh x)^n}\,dx = \frac{aB - bA}{(n-1)(a^2+b^2)} \cdot \frac{\cosh x}{(a + b\sinh x)^{n-1}} + \frac{1}{(n-1)(a^2+b^2)} \int \frac{(n-1)(aA+bB) + (n-2)(aB-bA)\sinh x}{(a + b\sinh x)^{n-1}}\,dx$$

For $n = 1$:

2. $$\int \frac{A + B\sinh x}{a + b\sinh x}\,dx = \frac{B}{b}x - \frac{aB - bA}{b}\int \frac{dx}{a + b\sinh x} \qquad \text{(see } \mathbf{2.441}\ 3)$$

3. $$\int \frac{dx}{a + b\sinh x} = \frac{1}{\sqrt{a^2+b^2}} \ln \frac{a\tanh\frac{x}{2} - b + \sqrt{a^2+b^2}}{a\tanh\frac{x}{2} - b - \sqrt{a^2+b^2}} = \frac{2}{\sqrt{a^2+b^2}} \operatorname{arctanh} \frac{a\tanh\frac{x}{2} - b}{\sqrt{a^2+b^2}}$$

2.442

1. $$\int \frac{A + B\cosh x}{(a + b\sinh x)^n}\,dx = -\frac{B}{(n-1)b\,(a + b\sinh x)^{n-1}} + A\int \frac{dx}{(a + b\sinh x)^n}$$

For $n = 1$:

2. $$\int \frac{A + B\cosh x}{a + b\sinh x}\,dx = \frac{B}{b}\ln(a + b\sinh x) + A\int \frac{dx}{a + b\sinh x}$$

(see **2.441** 3)

2.443

1. $$\int \frac{A+B\cosh x}{(a+b\cosh x)^n}\,dx = \frac{aB-bA}{(n-1)\,(a^2-b^2)}\cdot\frac{\sinh x}{(a+b\cosh x)^{n-1}} + \frac{1}{(n-1)\,(a^2-b^2)}\int\frac{(n-1)(aA-bB)+(n-2)(aB-bA)\cosh x}{(a+b\cosh x)^{n-1}}\,dx$$

For $n=1$:

2. $$\int \frac{A+B\cosh x}{a+b\cosh x}\,dx = \frac{B}{b}x - \frac{aB-bA}{b}\int\frac{dx}{a+b\cosh x} \qquad \text{(see } \mathbf{2.443}\ 3)$$

3. $$\int \frac{dx}{a+b\cosh x} = \frac{1}{\sqrt{b^2-a^2}}\arcsin\frac{b+a\cosh x}{a+b\cosh x} \qquad \left[b^2>a^2,\quad x<0\right]$$
$$= -\frac{1}{\sqrt{b^2-a^2}}\arcsin\frac{b+a\cosh x}{a+b\cosh x} \qquad \left[b^2>a^2,\quad x>0\right]$$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}\tanh\frac{x}{2}}{a+b-\sqrt{a^2-b^2}\tanh\frac{x}{2}} \qquad \left[a^2>b^2\right]$$

2.444

1. $$\int \frac{dx}{\cosh a+\cosh x} = \operatorname{cosech} a\left[\ln\cosh\frac{x+a}{2} - \ln\cosh\frac{x-a}{2}\right]$$
$$= 2\operatorname{cosech} a\operatorname{arctanh}\left(\tanh\frac{x}{2}\tanh\frac{a}{2}\right)$$

2.[11] $$\int \frac{dx}{\cos a+\cosh x} = 2\operatorname{cosec} a\arctan\left(\tanh\frac{x}{2}\tan\frac{a}{2}\right)$$

2.445

1. $$\int \frac{B\sinh x}{(a+b\cosh x)^n}\,dx = -\frac{B}{(n-1)b(a+b\cosh x)^{n-1}} \qquad [n\neq 1]$$

For $n=1$:

2. $$\int \frac{B\sinh x}{a+b\cosh x}\,dx = \frac{B}{b}\ln(a+b\cosh x) \qquad \text{(see } \mathbf{2.443}\ 3)$$

In evaluating definite integrals by use of formulas **2.441**–**2.443** and **2.445**, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{arcsinh}\left(-\frac{a}{b}\right)$$

in formulas **2.441** or **2.442** or over the points

$$x = \operatorname{arccosh}\left(-\frac{a}{b}\right)$$

in formulas **2.443** or **2.445**. Formulas **2.443** are not applicable for $a^2=b^2$. Instead, we may use the following formulas in these cases:

2.446

1. $$\int \frac{A+B\cosh x}{(\varepsilon+\cosh x)^n}\,dx = \frac{B\sinh x}{(1-n)(\varepsilon+\cosh x)^n} + \left(\varepsilon A + \frac{n}{n-1}B\right)\frac{(n-1)!}{(2n-1)!!}\sinh x \sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!} \times \frac{\varepsilon^h}{(\varepsilon+\cosh x)^{n-k}} \qquad [\varepsilon=\pm 1,\quad n>1]$$

For $n=1$:

2. $$\int \frac{A+B\cosh x}{\varepsilon+\cosh x}\,dx = Bx + (\varepsilon A - B)\frac{\cosh x-\varepsilon}{\sinh x} \qquad [\varepsilon=\pm 1]$$

2.447

1. $$\int \frac{\sinh x\,dx}{a\cosh x + b\sinh x} = \frac{a\ln\cosh\left(x+\operatorname{arctanh}\frac{b}{a}\right)bx}{a^2-b^2} \qquad [a>|b|]$$
$$= \frac{bx - a\ln\sinh\left(x+\operatorname{arctanh}\frac{a}{b}\right)}{b^2-a^2} \qquad [b>|a|]$$ MZ 215

For $a=b=1$:

2. $$\int \frac{\sinh x\,dx}{\cosh x+\sinh x} = \frac{x}{2}+\frac{1}{4}e^{-2x}$$

For $a=-b=1$:

3. $$\int \frac{\sinh x\,dx}{\cosh x-\sinh x} = -\frac{x}{2}+\frac{1}{4}e^{2x}$$ MZ 215

2.448

1. $$\int \frac{\cosh x\,dx}{a\cosh x + b\sinh x} = \frac{ax - b\ln\cosh\left(x+\operatorname{arctanh}\frac{b}{a}\right)}{a^2-b^2} \qquad [a>|b|]$$
$$= \frac{-ax + b\ln\sinh\left(x+\operatorname{arctanh}\frac{a}{b}\right)}{b^2-a^2} \qquad [b>|a|]$$

For $a=b=1$:

2. $$\int \frac{\cosh x\,dx}{\cosh x+\sinh x} = \frac{x}{2}-\frac{1}{4}e^{-2x}$$

For $a=-b=1$:

3. $$\int \frac{\cosh x\,dx}{\cosh x-\sinh x} = \frac{x}{2}+\frac{1}{4}e^{2x}$$ MZ 214, 215

2.449

1.[6] $$\int \frac{dx}{(a\cosh x + b\sinh x)^n} = \frac{1}{\sqrt{(a^2-b^2)^n}}\int \frac{dx}{\sinh^n\left(x+\operatorname{arctanh}\frac{b}{a}\right)} \qquad [a>|b|]$$
$$= \frac{1}{\sqrt{(b^2-a^2)^n}}\int \frac{dx}{\cosh^n\left(x+\operatorname{arctanh}\frac{a}{b}\right)} \qquad [b>|a|]$$

For $n=1$:

2. $$\int \frac{dx}{a\cosh x + b\sinh x} = \frac{1}{\sqrt{a^2-b^2}} \arctan\left|\sinh\left(x + \operatorname{arctanh}\frac{b}{a}\right)\right| \qquad [a>|b|]$$
$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\tanh\frac{x+\operatorname{arctanh}\frac{a}{b}}{2}\right| \qquad [b>|a|]$$

For $a=b=1$:

3. $$\int \frac{ax}{\cosh x + \sinh x} = -e^{-x} = \sinh x - \cosh x$$

For $a=-b=1$:

4. $$\int \frac{dx}{\cosh x - \sinh x} = e^{x} = \sinh x + \cosh x$$ MZ 214

2.451

1. $$\int \frac{A + B\cosh x + C\sinh x}{(a + b\cosh x + c\sinh x)^n}\,dx$$
$$= \frac{Bc - Cb + (Ac - Ca)\cosh x + (Ab - Ba)\sinh x}{(1-n)\,(a^2-b^2+c^2)\,(a + b\cosh x + c\sinh x)^{n-1}} + \frac{1}{(n-1)\,(a^2-b^2+c^2)}$$
$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba)\cosh x - (n-2)(Ac - Ca)\sinh x}{(a + b\cosh x + c\sinh x)^{n-1}}\,dx$$
$$\left[a^2+c^2 \neq b^2\right]$$
$$= \frac{Bc - Cb - Ca\cosh x - Ba\sinh x}{(n-1)a\,(a + b\cosh x + c\sinh x)^n} + \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2}\right](c\cosh x + b\sinh x)\,\frac{(n-1)!}{(2n-1)!!}$$
$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k}\,\frac{1}{(a + b\cosh x + c\sinh x)^{n-k}}$$
$$\left[a^2+c^2 = b^2\right]$$

2. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x + c\sinh x}\,dx = \frac{Cb - Bc}{b^2-c^2}\ln(a + b\cosh x + c\sinh x)$$
$$+ \frac{Bb - Cc}{b^2-c^2}x + \left(A - a\frac{Bb - Cc}{b^2-c^2}\right)\int \frac{dx}{a + b\cosh x + c\sinh x}$$
$$\left[b^2 \neq c^2\right] \qquad \text{(see } \mathbf{2.451}\ 4)$$

3. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x \pm b\sinh x}\,dx = \frac{C \mp B}{2a}(\cosh x \mp \sinh x) + \left[\frac{A}{a} - \frac{(B \mp C)\,b}{2a^2}\right]x$$
$$+ \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)\,b}{2a^2}\right]\ln(a + b\cosh x \pm b\sinh x)$$
$$[ab \neq 0]$$

4. $$\int \frac{dx}{a+b\cosh x+c\sinh x}$$

$$= \frac{2}{\sqrt{b^2-a^2-c^2}}\arctan\frac{(b-a)\tanh\frac{x}{2}+c}{\sqrt{b^2-a^2-c^2}} \qquad \left[b^2>a^2+c^2 \text{ and } a\neq b\right]$$

$$= \frac{1}{\sqrt{a^2-b^2+c^2}}\ln\frac{(a-b)\tanh\frac{x}{2}-c+\sqrt{a^2-b^2+c^2}}{(a-b)\tanh\frac{x}{2}-c-\sqrt{a^2-b^2+c^2}} \qquad \left[b^2<a^2+c^2 \text{ and } a\neq b\right]$$

$$= \frac{1}{c}\ln\left(a+c\tanh\frac{x}{2}\right) \qquad [a=b \text{ and } c\neq 0]$$

$$= \frac{2}{(a-b)\tanh\frac{x}{2}+c} \qquad \left[b^2=a^2+c^2\right]$$

GU (351)(18)

2.452

1. $$\int \frac{A+B\cosh x+C\sinh x}{(a_1+b_1\cosh x+c_1\sinh x)\,(a_2+b_2\cosh x+c_2\sinh x)}\,dx$$

$$= A_0\ln\frac{a_1+b_1\cosh x+c_1\sinh x}{a_2+b_2\cosh x+c_2\sinh x}+A_1\int\frac{dx}{a_1+b_1\cosh x+c_1\sinh x}+A_2\int\frac{dx}{a_2+b_2\cosh x+c_2\sinh x}$$

where GU (351)(19)

$$A_0=\frac{\begin{vmatrix}a_1 & b_1 & c_1\\ A & B & C\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2-\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2} \qquad A_1=\frac{\begin{vmatrix}a_1 & b_1 & c_1\\ \begin{vmatrix}b_1 & c_1\\ B & C\end{vmatrix} & \begin{vmatrix}c_1 & a_1\\ C & A\end{vmatrix} & \begin{vmatrix}a_1 & b_1\\ A & B\end{vmatrix}\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2-\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2},$$

$$A_2=\frac{\begin{vmatrix}a_1 & b_1 & c_1\\ \begin{vmatrix}C & B\\ c_2 & b_2\end{vmatrix} & \begin{vmatrix}C & A\\ c_2 & a_2\end{vmatrix} & \begin{vmatrix}B & A\\ b_2 & a_2\end{vmatrix}\\ a_2 & b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2-\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2}, \qquad \left[\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}^2+\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}^2\neq\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}^2\right].$$

2. $$\int\frac{A\cosh^2 x+2B\sinh x\cosh x+C\sinh^2 x}{a\cosh^2 x+2b\sinh x\cosh x+c\sinh^2 x}\,dx$$

$$= \frac{1}{4b^2-(a+c)^2}\Big\{[4Bb-(A+C)(a+c)]x$$

$$+[(A+C)b-\mathrm{B}(a+c)]\ln\left(a\cosh^2 x+2b\sinh x\cosh x+c\sinh^2 x\right)$$

$$+\left[2(A-C)b^2-2Bb(a-c)+(Ca-Ac)(a+c)\right]f(x)\Big\}$$

where GU (351)(24)

$$f(x) = \frac{1}{2\sqrt{b^2-ac}} \ln \frac{c\tanh x + b - \sqrt{b^2-ac}}{c\tanh x + b + \sqrt{b^2-ac}} \qquad [b^2 > ac]$$

$$= \frac{1}{\sqrt{ac-b^2}} \arctan \frac{c\tanh x + b}{\sqrt{ac-b^2}} \qquad [b^2 < ac]$$

$$= -\frac{1}{c\tanh x + b} \qquad [b^2 = ac]$$

2.453

1. $$\int \frac{(A + B\sinh x)\,dx}{\sinh x\,(a + b\sinh x)} = \frac{1}{a}\left[A \ln\left|\tanh\frac{x}{2}\right| + (aB - bA)\int \frac{dx}{a + b\sinh x}\right]$$

(see **2.441** 3)

2. $$\int \frac{(A + B\sinh x)\,dx}{\sinh x\,(a + b\cosh x)} = \frac{A}{a^2-b^2}\left(a \ln\left|\tanh\frac{x}{2}\right| + b\ln\left|\frac{a + b\cosh x}{\sinh x}\right|\right) + B\int \frac{dx}{a + b\cosh x}$$

(see **2.443** 3)

For $a^2 = b^2 = 1$:

3. $$\int \frac{(A + B\sinh x)\,dx}{\sinh x\,(1 + \cosh x)} = \frac{A}{2}\left(\ln\left|\tanh\frac{x}{2}\right| - \frac{1}{2}\tanh^2\frac{x}{2}\right) + B\tanh\frac{x}{2}$$

4. $$\int \frac{(A + B\sinh x)\,dx}{\sinh x\,(1 - \cosh x)} = \frac{A}{2}\left(-\ln\left|\coth\frac{x}{2}\right| + \frac{1}{2}\coth^2\frac{x}{2}\right) + B\coth\frac{x}{2}$$

2.454

1. $$\int \frac{(A + B\sinh x)\,dx}{\cosh x\,(a + b\sinh x)} = \frac{1}{a^2+b^2}\left[(Aa + Bb)\arctan(\sinh x) + (Ab - Ba)\ln\left|\frac{a + b\sinh x}{\cosh x}\right|\right]$$

2. $$\int \frac{(A + B\cosh x)\,dx}{\sinh x\,(a + b\sinh x)} = \frac{1}{a}\left(A\ln\left|\tanh\frac{x}{2}\right| + B\ln\left|\frac{\sinh x}{a + b\sinh x}\right| - Ab\int\frac{dx}{a + b\sinh x}\right)$$

(see **2.441** 3)

2.455

1. $$\int \frac{(A + B\cosh x)\,dx}{\sinh x\,(a + b\cosh x)} = \frac{1}{a^2-b^2}\left[(Aa + Bb)\ln\left|\tanh\frac{x}{2}\right| + (Ab - Ba)\ln\left|\frac{a + b\cosh x}{\sinh x}\right|\right]$$

For $a^2 = b^2 = 1$:

2. $$\int \frac{(A + B\cosh x)\,dx}{\sinh x\,(1 + \cosh x)} = \frac{A+B}{2}\ln\left|\tanh\frac{x}{2}\right| - \frac{A-B}{4}\tanh^2\frac{x}{2}$$

3. $$\int \frac{(A + B\cosh x)\,dx}{\sinh x\,(1 - \cosh x)} = \frac{A+B}{4}\coth^2\frac{x}{2} - \frac{A-B}{2}\ln\coth\frac{x}{2}$$

2.456 $$\int \frac{(A + B\cosh x)\,dx}{\cosh x\,(a + b\sinh x)} = \frac{A}{a^2+b^2}\left[a\arctan(\sinh x) + b\ln\left|\frac{a + b\sinh x}{\cosh x}\right|\right] + B\int\frac{dx}{a + b\sinh x}$$

(see **2.441** 3)

2.457

1. $$\int \frac{(A+B\cosh x)\,dx}{\cosh x\,(a+b\cosh x)} = \frac{1}{a}\left[A\arctan\sinh x-(Ab-Ba)\int\frac{dx}{a+b\cosh x}\right]$$

(see **2.443** 3)

2.458

1. $$\int \frac{dx}{a+b\sinh^2 x}$$
$$= \frac{1}{\sqrt{a(b-a)}}\arctan\left(\sqrt{\frac{b}{a}-1}\tanh x\right) \qquad \left[\frac{b}{a}>1\right]$$
$$= \frac{1}{\sqrt{a(a-b)}}\operatorname{arctanh}\left(\sqrt{1-\frac{b}{a}}\tanh x\right) \qquad \left[0<\frac{b}{a}<1 \text{ or } \frac{b}{a}<0 \text{ and } \sinh^2 x<-\frac{a}{b}\right]$$
$$= \frac{1}{\sqrt{a(a-b)}}\operatorname{arccoth}\left(\sqrt{1-\frac{b}{a}}\tanh x\right) \qquad \left[\frac{b}{a}<0 \text{ and } \sinh^2 x>-\frac{a}{b}\right]$$

MZ 195

2. $$\int \frac{dx}{a+b\cosh^2 x}$$
$$= \frac{1}{\sqrt{-a(a+b)}}\arctan\left(\sqrt{-\left(1+\frac{b}{a}\right)}\coth x\right) \qquad \left[\frac{b}{a}<-1\right]$$
$$= \frac{1}{\sqrt{a(a+b)}}\operatorname{arctanh}\left(\sqrt{1+\frac{b}{a}}\coth x\right) \qquad \left[-1<\frac{b}{a}<0 \text{ and } \cosh^2 x>-\frac{a}{b}\right]$$
$$= \frac{1}{\sqrt{a(a+b)}}\operatorname{arccoth}\left(\sqrt{1+\frac{b}{a}}\coth x\right) \qquad \left[\frac{b}{a}>0 \text{ or } -1<\frac{b}{a}<0 \text{ and } \cosh^2 x<-\frac{a}{b}\right]$$

MZ 202

For $a^2=b^2=1$:

3. $$\int \frac{dx}{1+\sinh^2 x} = \tanh x$$

4. $$\int \frac{dx}{1-\sinh^2 x} = \frac{1}{\sqrt{2}}\operatorname{arctanh}\left(\sqrt{2}\tanh x\right) \qquad \left[\sinh^2 x<1\right]$$
$$= \frac{1}{\sqrt{2}}\operatorname{arccoth}\left(\sqrt{2}\tanh x\right) \qquad \left[\sinh^2 x>1\right]$$

5. $$\int \frac{dx}{1+\cosh^2 x} = \frac{1}{\sqrt{2}}\operatorname{arccoth}\left(\sqrt{2}\coth x\right)$$

6. $$\int \frac{dx}{1-\cosh^2 x} = \coth x$$

2.459

1. $$\int \frac{dx}{\left(a+b\sinh^2 x\right)^2} = \frac{1}{2a(b-a)}\left[\frac{b\sinh x\cosh x}{a+b\sinh^2 x}+(b-2a)\int\frac{dx}{a+b\sinh^2 x}\right]$$

(see **2.458** 1) MZ 196

2. $$\int \frac{dx}{\left(a+b\cosh^2 x\right)^2} = \frac{1}{2a(a+b)}\left[-\frac{b\sinh x\cosh x}{a+b\cosh^2 x} + (2a+b)\int \frac{dx}{a+b\cosh^2 x}\right]$$

(see **2.458** 2) MZ 203

3. $$\int \frac{dx}{\left(a+b\sinh^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\tanh x\right) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\tanh x}{1+p^2\tanh^2 x}\right.$$
$$\left. + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\tanh^2 x\right)\frac{2p\tanh x}{\left(1+p^2\tanh^2 x\right)^2}\right]$$
$$\left[p^2 = \frac{b}{a} - 1 > 0\right]$$
$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\tanh x\right) + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\tanh x}{1-q^2\tanh^2 x}\right.$$
$$\left. + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\tanh^2 x\right)\frac{2q\tanh x}{\left(1-q^2\tanh^2 x\right)^2}\right]$$
$$\left[q^2 = 1 - \frac{b}{a} > 0\right]$$

MZ 196

4. $$\int \frac{dx}{\left(a+b\cosh^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\coth x\right) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\coth x}{1+p^2\coth^2 x}\right.$$
$$\left. + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\coth^2 x\right)\frac{2p\coth x}{\left(1+p^2\coth^2 x\right)^2}\right]$$
$$\left[p^2 = -1 - \frac{b}{a} > 0\right]$$
$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\varphi(x)^* + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\coth x}{1-q^2\coth^2 x}\right.$$
$$\left. + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\coth^2 x\right)\frac{2q\coth x}{\left(1-q^2\coth^2 x\right)^2}\right]$$
$$\left[q^2 = 1 + \frac{b}{a} > 0\right]$$

2.46 Algebraic functions of hyperbolic functions

2.461

1. $$\int \sqrt{\tanh x}\,dx = \operatorname{arctanh}\sqrt{\tanh x} - \arctan\sqrt{\tanh x}$$ MZ 221

*In 2.459.4, if $\frac{b}{a} < 0$ and $\cosh^2 x > -\frac{a}{b}$, then $\varphi(x) = \operatorname{arctanh}(q\coth x)$. If $\frac{b}{a} < 0$, but $\cosh^2 x < -\frac{a}{b}$, or if $\frac{b}{a} > 0$, then $\varphi(x) = \operatorname{arccoth}(q\coth x)$.

2. $$\int \sqrt{\coth x}\,dx = \operatorname{arccoth}\sqrt{\coth x} - \arctan\sqrt{\coth x}$$ MZ 222

2.462

1. $$\int \frac{\sinh x\,dx}{\sqrt{a^2+\sinh^2 x}} = \operatorname{arcsinh}\frac{\cosh x}{\sqrt{a^2-1}} = \ln\left(\cosh x + \sqrt{a^2+\sinh^2 x}\right) \qquad [a^2>1]$$
$$= \operatorname{arccosh}\frac{\cosh x}{\sqrt{1-a^2}} = \ln\left(\cosh x + \sqrt{a^2+\sinh^2 x}\right) \qquad [a^2<1]$$
$$= \ln\cosh x \qquad [a^2=1]$$

2. $$\int \frac{\sinh x\,dx}{\sqrt{a^2-\sinh^2 x}} = \arcsin\frac{\cosh x}{\sqrt{a^2+1}} \qquad [\sinh^2 x < a^2]$$

3. $$\int \frac{\sinh x\,dx}{\sqrt{\sinh^2 x-a^2}} = \operatorname{arccosh}\frac{\cosh x}{\sqrt{a^2+1}} = \ln\left(\cosh x + \sqrt{\sinh^2 x - a^2}\right)$$
$$[\sinh^2 x > a^2]$$ MZ 199

4. $$\int \frac{\cosh x\,dx}{\sqrt{a^2+\sinh^2 x}} = \operatorname{arcsinh}\frac{\sinh x}{a} = \ln\left(\sinh x + \sqrt{a^2+\sinh^2 x}\right)$$

5. $$\int \frac{\cosh x\,dx}{\sqrt{a^2-\sinh^2 x}} = \arcsin\frac{\sinh x}{a} \qquad [\sinh^2 x < a^2]$$

6. $$\int \frac{\cosh x\,dx}{\sqrt{\sinh^2 x-a^2}} = \operatorname{arccosh}\frac{\sinh x}{a} = \ln\left(\sinh x + \sqrt{\sinh^2 x - a^2}\right)$$
$$[\sinh^2 x > a^2]$$

7. $$\int \frac{\sinh x\,dx}{\sqrt{a^2+\cosh^2 x}} = \operatorname{arcsinh}\frac{\cosh x}{a} = \ln\left(\cosh x + \sqrt{a^2+\cosh^2 x}\right)$$

8. $$\int \frac{\sinh x\,dx}{\sqrt{a^2-\cosh^2 x}} = \arcsin\frac{\cosh x}{a} \qquad [\cosh^2 x < a^2]$$

9. $$\int \frac{\sinh x\,dx}{\sqrt{\cosh^2 x-a^2}} = \operatorname{arccosh}\frac{\cosh x}{a} = \ln\left(\cosh x + \sqrt{\cosh^2 x - a^2}\right)$$
$$[\cosh^2 x > a^2]$$ MZ 215, 216

10. $$\int \frac{\cosh x\,dx}{\sqrt{a^2+\cosh^2 x}} = \operatorname{arcsinh}\frac{\sinh x}{\sqrt{a^2+1}} = \ln\left(\sinh x + \sqrt{a^2+\cosh^2 x}\right)$$

11. $$\int \frac{\cosh x\,dx}{\sqrt{a^2-\cosh^2 x}} = \arcsin\frac{\sinh x}{\sqrt{a^2-1}} \qquad [\cosh^2 x < a^2]$$

12. $$\int \frac{\cosh x\,dx}{\sqrt{\cosh^2 x-a^2}} = \operatorname{arccosh}\frac{\sinh x}{\sqrt{a^2-1}} \qquad [a^2>1]$$
$$= \ln\sinh x \qquad [a^2=1]$$

MZ 206

13. $$\int \frac{\coth x\,dx}{\sqrt{a+b\sinh x}} = 2\sqrt{a}\,\operatorname{arccoth}\sqrt{1+\frac{b}{a}\sinh x} \qquad [b\sinh x>0,\quad a>0]$$
$$= 2\sqrt{a}\,\operatorname{arctanh}\sqrt{1+\frac{b}{a}\sinh x} \qquad [b\sinh x<0,\quad a>0]$$
$$= 2\sqrt{-a}\,\operatorname{arctanh}\sqrt{-\left(1+\frac{b}{a}\sinh x\right)} \qquad a<0$$

14. $$\int \frac{\tanh x\,dx}{\sqrt{a+b\cosh x}} = 2\sqrt{a}\,\operatorname{arccoth}\sqrt{1+\frac{b}{a}\cosh x} \qquad [b\cosh x>0,\quad a>0]$$
$$= 2\sqrt{a}\,\operatorname{arctanh}\sqrt{1+\frac{b}{a}\cosh x} \qquad [b\cosh x<0,\quad a>0]$$
$$= 2\sqrt{-a}\,\operatorname{arctanh}\sqrt{-\left(1+\frac{b}{a}\cosh x\right)} \qquad [a<0]$$

MZ 220, 221

2.463

1. $$\int \frac{\sinh x\sqrt{a+b\cosh x}}{p+q\cosh x}\,dx$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arccoth}\sqrt{\frac{q\,(a+b\cosh x)}{aq-bp}} \qquad \left[b\cosh x>0,\quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\cosh x)}{aq-bp}} \qquad \left[b\cosh x<0,\quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{bp-aq}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\cosh x)}{bp-aq}} \qquad \left[\frac{aq-bp}{q}<0\right]$$

MZ 220

2. $$\int \frac{\cosh x\sqrt{a+b\sinh x}}{p+q\sinh x}\,dx$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arccoth}\sqrt{\frac{q\,(a+b\sinh x)}{aq-bp}} \qquad \left[b\sinh x>0,\quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\sinh x)}{aq-bp}} \qquad \left[b\sinh x<0,\quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{bp-aq}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\sinh x)}{bp-aq}} \qquad \left[\frac{aq-bp}{q}<0\right]$$

MZ 221

2.464

1. $$\int \frac{dx}{\sqrt{k^2+k'^2\cosh^2 x}} = \int \frac{dx}{\sqrt{1+k'^2\sinh^2 x}} = F\left(\arcsin\left(\tanh x\right),k\right)$$
$$[x>0]$$ BY (295.00)(295.10)

2. $$\int \frac{dx}{\sqrt{\cosh^2 x-k^2}} = \int \frac{dx}{\sqrt{\sinh^2 x+k'^2}} = F\left(\arcsin\left(\frac{1}{\cosh x}\right),k\right)$$
$$[x>0]$$ BY (295.40)(295.30)

3. $$\int \frac{dx}{\sqrt{1-k'^2\cosh^2 x}} = F\left(\arcsin\left(\frac{\tanh x}{k}\right), k\right) \qquad \left[0 < x < \operatorname{arccosh}\frac{1}{k'}\right]$$ BY (295.20)

Notation: In **2.464** 4–**2.464** 8, we set $\alpha = \arccos\dfrac{1-\sinh 2ax}{1+\sinh 2ax}$, $r = \dfrac{1}{\sqrt{2}}$ $\qquad [ax > 0]$

4. $$\int \frac{dx}{\sqrt{\sinh 2ax}} = \frac{1}{2a}\,F(\alpha, r)$$ BY (296.50)

5. $$\int \sqrt{\sinh 2ax}\,dx = \frac{1}{2a}\left[F(\alpha,r) - 2\,E(\alpha,r)\right] + \frac{1}{a}\frac{\sqrt{\sinh 2ax\left(1+\sinh^2 2ax\right)}}{1+\sinh 2ax}$$ BY (296.53)

6. $$\int \frac{\cosh^2 2ax\,dx}{(1+\sinh 2ax)^2\sqrt{\sinh 2ax}} = \frac{1}{2a}\,E(\alpha, r)$$ BY (296.51)

7. $$\int \frac{(1-\sinh 2ax)^2\,dx}{(1+\sinh 2ax)^2\sqrt{\sinh 2ax}} = \frac{1}{2a}\left[2\,E(\alpha,r) - F(\alpha,r)\right]$$ BY (296.55)

8. $$\int \frac{\sqrt{\sinh 2ax}\,dx}{(1+\sinh 2ax)^2} = \frac{1}{4a}\left[F(\alpha,r) - E(\alpha,r)\right]$$ BY (296.54)

Notation: In **2.464** 9–**2.464** 15, we set $\alpha = \arcsin\sqrt{\dfrac{\cosh 2ax - 1}{\cosh 2ax}}$, $r = \dfrac{1}{\sqrt{2}}$ $\qquad [x \neq 0]$:

9. $$\int \frac{dx}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}}\,F(\alpha, r)$$ BY (296.00)

10. $$\int \sqrt{\cosh 2ax}\,dx = \frac{1}{a\sqrt{2}}\left[F(\alpha,r) - 2\,E(\alpha,r)\right] + \frac{\sinh 2ax}{a\sqrt{\cosh 2ax}}$$ BY (296.03)

11. $$\int \frac{dx}{\sqrt{\cosh^3 2ax}} = \frac{1}{a\sqrt{2}}\left[2\,E(\alpha,r) - F(\alpha,r)\right]$$ BY (296.04)

12. $$\int \frac{dx}{\sqrt{\cosh^5 2ax}} = \frac{1}{3\sqrt{2}a}\,F(\alpha,r) + \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}$$ BY (296.04)

13. $$\int \frac{\sinh^2 2ax\,dx}{\sqrt{\cosh 2ax}} = -\frac{\sqrt{2}}{3a}\,F(\alpha,r) + \frac{1}{3a}\sinh 2ax\sqrt{\cosh 2ax}$$ BY (296.07)

14. $$\int \frac{\tanh^2 2ax\,dx}{\sqrt{\cosh 2ax}} = \frac{\sqrt{2}}{3a}\,F(\alpha,r) - \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}$$ BY (296.05)

15. $$\int \frac{\sqrt{\cosh 2ax}\,dx}{p^2 + (1-p^2)\cosh 2ax} = \frac{1}{a\sqrt{2}}\,\Pi\left(\alpha, p^2, r\right)$$ BY (296.02)

Notation: In **2.464** 16–**2.464** 20, we set:

$$\alpha = \arccos\frac{\sqrt{a^2+b^2} - a - b\sinh x}{\sqrt{a^2+b^2} + a + b\sinh x},$$

$$r = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}} \qquad \left[a > 0, \quad b > 0, \quad x > -\operatorname{arcsinh}\frac{a}{b}\right]$$

16. $$\int \frac{dx}{\sqrt{a+b\sinh x}} = \frac{1}{\sqrt[4]{a^2+b^2}}\,F(\alpha, r)$$ BY (298.00)

17. $$\int \sqrt{a+b\sinh x}\,dx = \sqrt[4]{a^2+b^2}\,[F(\alpha,r)-2\,E(\alpha,r)] + \frac{2b\cosh x\sqrt{a+b\sinh x}}{\sqrt{a^2+b^2}+a+b\sinh x}$$ BY (298.02)

18. $$\int \frac{\sqrt{a+b\sinh x}}{\cosh^2 x}\,dx = \sqrt[4]{a^2+b^2}\,E(\alpha,r) - \frac{\sqrt{a^2+b^2}-a}{2\sqrt[4]{a^2+b^2}}\,F(\alpha,r) - \frac{a+\sqrt{a^2+b^2}}{b}\cdot\frac{\sqrt{a^2+b^2}-a-b\sinh x}{\sqrt{a^2+b^2}+a+b\sinh x}\cdot\frac{\sqrt{a+b\sinh x}}{\cosh x}$$ BY (298.03)

19. $$\int \frac{\cosh^2 x\,dx}{\left[\sqrt{a^2+b^2}+a+b\sinh x\right]^2\sqrt{a+b\sinh x}} = \frac{1}{b^2\sqrt[4]{a^2+b^2}}\,E(\alpha,r)$$ BY (298.01)

20. $$\int \frac{\sqrt{a+b\sinh x}\,dx}{\left[\sqrt{a^2+b^2}-a-b\sinh x\right]^2} = -\frac{1}{\sqrt[4]{a^2+b^2}\left(\sqrt{a^2+b^2}-a\right)}\,E(\alpha,r) + \frac{b}{\sqrt{a^2+b^2}-a}\cdot\frac{\cosh x\sqrt{a+b\sinh x}}{a^2+b^2-(a+b\sinh x)^2}$$ BY (298.04)

Notation: In **2.464** 21–**2.464** 31, we set $\alpha = \arcsin\left(\tanh\frac{x}{2}\right)$, $r = \sqrt{\frac{a-b}{a+b}}$ $[0<b<a, x>0]$:

21. $$\int \frac{dx}{\sqrt{a+b\cosh x}} = \frac{2}{\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.25)

22. $$\int \sqrt{a+b\cosh x}\,dx = 2\sqrt{a+b}\,[F(\alpha,r)-E(\alpha,r)] + 2\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.29)

23. $$\int \frac{\cosh x\,dx}{\sqrt{a+b\cosh x}} = \frac{2}{\sqrt{a+b}}\,F(\alpha,r) - \frac{2\sqrt{a+b}}{b}\,E(\alpha,r) + \frac{2}{b}\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.33)

24. $$\int \frac{\tanh^2\frac{x}{2}}{\sqrt{a+b\cosh x}}\,dx = \frac{2\sqrt{a+b}}{a-b}\,[F(\alpha,r)-E(\alpha,r)]$$ BY (297.28)

25.[11] $$\int \frac{\tanh^4\frac{x}{2}}{\sqrt{a+b\cosh x}}\,dx = \frac{2\sqrt{a+b}}{3(a-b)^2}\,[(3a+b)\,F(\alpha,r)-4a\,E(\alpha,r)] + \frac{2}{3(a-b)}\,\frac{\sinh\frac{x}{2}\sqrt{a+b\cosh x}}{\cosh^3\frac{x}{2}}$$ BY (297.28)

26. $$\int \frac{\cosh x-1}{\sqrt{a+b\cosh x}}\,dx = \frac{2}{b}\left[\left(\tanh\frac{x}{2}\right)\sqrt{a+b\cosh x} - \sqrt{a+b}\,E(\alpha,r)\right]$$ BY (297.31)

27. $$\int \frac{(\cosh x-1)^2}{\sqrt{a+b\cosh x}}\,dx = \frac{4\sqrt{a+b}}{3b^2}\,[(a+3b)\,E(\alpha,r)-b\,F(\alpha,r)] + \frac{4}{3b^2}\left[b\cosh^2\frac{x}{2}-(a+3b)\right]\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.31)

28. $$\int \frac{\sqrt{a+b\cosh x}}{\cosh x+1}\,dx = \sqrt{a+b}\,E(\alpha,r)$$ BY (297.26)

29. $$\int \frac{dx}{(\cosh x+1)\sqrt{a+b\cosh x}} = \frac{\sqrt{a+b}}{a-b}\,E(\alpha,r) - \frac{2b}{(a-b)\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.30)

30. $$\int \frac{dx}{(\cosh x+1)^2\sqrt{a+b\cosh x}} = \frac{1}{3(a-b)^2\sqrt{a+b}}\left[b(5b-a)\,F(\alpha,r) + (a-3b)(a+b)\,E(\alpha,r)\right] + \frac{1}{6(a-b)}\cdot\frac{\sinh\frac{x}{2}}{\cosh^3\frac{x}{2}}\sqrt{a+b\cosh x}$$ 297.30)

31. $$\int \frac{(1+\cosh x)\,dx}{[1+p^2+(1-p^2)\cosh x]\sqrt{a+b\cosh x}} = \frac{2}{\sqrt{a+b}}\,\Pi\left(\alpha,p^2,r\right)$$ BY (297.27)

Notation: In **2.464** 32–**2.464** 40, we set:

$$\alpha = \arcsin\sqrt{\frac{a-b\cosh x}{a-b}}$$
$$r = \sqrt{\frac{a-b}{a+b}} \qquad \left[0<b<a, \quad 0<x<\operatorname{arccosh}\frac{a}{b}\right]$$

32. $$\int \frac{dx}{\sqrt{a-b\cosh x}} = \frac{2}{\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.50)

33. $$\int \sqrt{a-b\cosh x}\,dx = 2\sqrt{a+b}\,[F(\alpha,r)-E(\alpha,r)]$$ BY (297.54)

34. $$\int \frac{\cosh x\,dx}{\sqrt{a-b\cosh x}} = \frac{2\sqrt{a+b}}{b}\,E(\alpha,r) - \frac{2}{\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.56)

35. $$\int \frac{\cosh^2 x\,dx}{\sqrt{a-b\cosh x}} = \frac{2(b-2a)}{3b\sqrt{a+b}}\,F(\alpha,r) + \frac{4a\sqrt{a+b}}{3b^2}\,E(\alpha,r) + \frac{2}{3b}\sinh x\sqrt{a-b\cosh x}$$ BY (297.56)

36. $$\int \frac{(1+\cosh x)\,dx}{\sqrt{a-b\cosh x}} = \frac{2\sqrt{a+b}}{b}\,E(\alpha,r)$$ BY (297.51)

37. $$\int \frac{dx}{\cosh x\sqrt{a-b\cosh x}} = \frac{2b}{a\sqrt{a+b}}\,\Pi\left(\alpha,\frac{a-b}{a},r\right)$$ BY (297.57)

38. $$\int \frac{dx}{(1+\cosh x)\sqrt{a-b\cosh x}} = \frac{1}{\sqrt{a+b}}\,E(\alpha,r) - \frac{1}{a+b}\tanh\frac{x}{2}\sqrt{a-b\cosh x}$$ BY (297.58)

39. $$\int \frac{dx}{(1+\cosh x)^2\sqrt{a-b\cosh x}} = \frac{1}{3\sqrt{(a+b)^3}}\,[(a+3b)\,E(\alpha,r) - b\,F(\alpha,r)] - \frac{1}{3(a+b)^2}\,\frac{\tanh\frac{x}{2}\sqrt{a-b\cosh x}}{\cosh x+1}\,[2a+4b+(a+3b)\cosh x]$$ BY (297.58)

40. $$\int \frac{dx}{(a-b-ap^2+bp^2\cosh x)\sqrt{a-b\cosh x}} = \frac{2}{(a-b)\sqrt{a+b}}\,\Pi\left(\alpha,p^2,r\right)$$ BY (297.52)

Notation: In **2.464** 41 –**2.464** 47, we set:

$$\alpha = \arcsin\sqrt{\frac{b\,(\cosh x-1)}{b\cosh x-a}},$$
$$r = \sqrt{\frac{a+b}{2b}} \qquad [0<a<b, x>0]$$

41. $$\int \frac{dx}{\sqrt{b\cosh -a}} = \sqrt{\frac{2}{b}}\, F(\alpha, r)$$ BY (297.00)

42. $$\int \sqrt{b\cosh x - a}\, dx = (b-a)\sqrt{\frac{2}{b}}\, F(\alpha, r) - 2\sqrt{2b}\, E(\alpha, r) + \frac{2b\sinh x}{\sqrt{b\cosh x - a}}$$ BY (297.05)

43. $$\int \frac{dx}{\sqrt{(b\cosh x - a)^3}} = \frac{1}{b^2 - a^2} \cdot \sqrt{\frac{2}{b}}\, [2b\, E(\alpha, r) - (b-a)\, F(\alpha, r)]$$ BY (297.06)

44. $$\int \frac{dx}{\sqrt{(b\cosh x - a)^5}} = \frac{1}{3\,(b^2 - a^2)^2}\sqrt{\frac{2}{b}}\, [(b-3a)(b-a)\, F(\alpha, r) + 8ab\, E(\alpha, r)] + \frac{2b}{3\,(b^2 - a^2)} \cdot \frac{\sinh x}{\sqrt{(b\cosh x - a)^3}}$$ BY (297.06)

45. $$\int \frac{\cosh x\, dx}{\sqrt{b\cosh x - a}} = \sqrt{\frac{2}{b}}\, [F(\alpha, r) - 2\, E(\alpha, r)] + \frac{2\sinh x}{\sqrt{b\cosh x - a}}$$ BY (297.03)

46. $$\int \frac{(\cosh x + 1)\, dx}{\sqrt{(b\cosh x - a)^3}} = \frac{2}{b-a}\sqrt{\frac{2}{b}}\, E(\alpha, r)$$ BY (297.01)

47. $$\int \frac{\sqrt{b\cosh x - a}\, dx}{p^2 b - a + b\,(1 - p^2)\cosh x} = \sqrt{\frac{2}{b}}\, \Pi\left(\alpha, p^2, r\right)$$ BY (297.02)

Notation: In **2.464** 48–**2.464** 55, we set $\alpha = \arcsin\sqrt{\dfrac{b\cosh x - a}{b\,(\cosh x - 1)}}$ and $r = \sqrt{\dfrac{2b}{a+b}}$ for $\left[0 < b < a, x > \operatorname{arccosh}\dfrac{a}{b}\right]$:

48. $$\int \frac{dx}{\sqrt{b\cosh x - a}} = \frac{2}{\sqrt{a+b}}\, F(\alpha, r)$$ BY (297.75)

49. $$\int \sqrt{b\cosh x - a}\, dx = -2\sqrt{a+b}\, E(\alpha, r) + 2\coth\frac{x}{2}\sqrt{b\cosh x - a}$$ BY (297.79)

50. $$\int \frac{\coth^2\frac{x}{2}\, dx}{\sqrt{b\cosh x - a}} = \frac{2\sqrt{a+b}}{a-b}\, E(\alpha, r)$$ BY (297.76)

51. $$\int \frac{\sqrt{b\cosh x - a}}{\cosh x - 1}\, dx = \sqrt{a+b}\, [F(\alpha, r) - E(\alpha, r)]$$ BY (297.77)

52. $$\int \frac{dx}{(\cosh x - 1)\sqrt{b\cosh x - a}} = \frac{\sqrt{a+b}}{a-b}\, E(\alpha, r) - \frac{1}{\sqrt{a+b}}\, F(\alpha, r)$$ BY (297.78)

53. $$\int \frac{dx}{(\cosh x - 1)^2\sqrt{b\cosh x - a}} = \frac{1}{3(a-b)^2\sqrt{a+b}}\left[(a-2b)(a-b)\, F(\alpha, r) + (3a-b)(a+b)\, E(\alpha, r)\right] + \frac{a+b}{6b(a-b)} \cdot \frac{\cosh\frac{x}{2}}{\sinh^3\frac{x}{2}}\sqrt{b\cosh x - a}$$ BY (297.78)

54. $$\int \frac{dx}{(\cosh x + 1)\sqrt{b\cosh x - a}} = \frac{1}{\sqrt{a+b}}\, [F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b\cosh x - a}}{(a+b)\sinh x}$$ BY (297.80)

55. $$\int \frac{dx}{(\cosh x+1)^2\sqrt{b\cosh x-a}} = \frac{1}{3\sqrt{(a+b)^3}}\Big[(a+b)\,F(\alpha,r) - (a+3b)\,E(\alpha,r)\Big] + \frac{\sqrt{b\cosh x-a}}{3(a+b)\sinh x}\left(2\frac{a+3b}{a+b}-\tanh^2\frac{x}{2}\right)$$

BY (297.80)

Notation: In **2.464** 56–**2.464** 60, we set

$$\alpha = \arccos\frac{\sqrt[4]{b^2-a^2}}{\sqrt{a\sinh x+b\cosh x}},$$
$$r = \frac{1}{\sqrt{2}} \qquad \left[0<a<b, \quad -\operatorname{arcsinh}\frac{a}{\sqrt{b^2-a^2}}<x\right]$$

56. $$\int \frac{dx}{\sqrt{a\sinh x+b\cosh x}} = \sqrt[4]{\frac{4}{b^2-a^2}}\,F(\alpha,r)$$ BY (299.00)

57. $$\int \sqrt{a\sinh x+b\cosh x}\,dx = \sqrt[4]{4\,(b^2-a^2)}\,[F(\alpha,r)-2\,E(\alpha,r)] + \frac{2\,(a\cosh x+b\sinh x)}{\sqrt{a\sinh x+b\cosh x}}$$

BY (299.02)

58. $$\int \frac{dx}{\sqrt{(a\sinh x+b\cosh x)^3}} = \sqrt[4]{\frac{4}{(b^2-a^2)^3}}\,[2\,E(\alpha,r)-F(\alpha,r)]$$ BY (299.03)

59. $$\int \frac{dx}{\sqrt{(a\sinh x+b\cosh x)^5}} = \frac{1}{3}\sqrt[4]{\frac{4}{(b^2-a^2)^5}}\,F(\alpha,r) + \frac{2}{3\,(b^2-a^2)}\cdot\frac{a\cosh x+b\sinh x}{\sqrt{(a\sinh x+b\cosh x)^3}}$$

BY (299.03)

60. $$\int \frac{\left(\sqrt{b^2-a^2}+a\sinh x+b\cosh x\right)\,dx}{\sqrt{(a\sinh x+b\cosh x)^3}} = 2\sqrt[4]{\frac{4}{b^2-a^2}}\,E(\alpha,r)$$ BY (299.01)

2.47 Combinations of hyperbolic functions and powers

2.471

1. $$\int x^r\sinh^p x\cosh^q x\,dx$$
$$= \frac{1}{(p+q)^2}\Big[(p+q)x^r\sinh^{p-1}x\cosh^{q-1}x - rx^{r-1}\sinh^p x\cosh^q x + r(r+1)\int x^{r-2}\sinh^p x\cosh^q x\,dx + rp\int x^{r-1}\sinh^{p-1}x\cosh^{q-1}x\,dx + (q-1)(p+q)\int x^r\sinh^p x\cosh^{q-2}x\,dx\Big]$$
$$= \frac{1}{(p+q)^2}\Big[(p+q)x^r\sinh^{p-1}x\cosh^{q+1}x - rx^{r-1}\sinh^p x\cosh^q x + r(r-1)\int x^{r-2}\sinh^p x\cosh^q x\,dx - rq\int x^{r-1}\sinh^{p-1}x\cosh^{q-1}x\,dx - (p-1)(p+q)\int x^r\sinh^{p-2}x\cosh^q x\,dx\Big]$$

GU (353)(1)

2. $$\int x^n \sinh^{2m} x\,dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cosh(2m-2k)x\,dx$$

3. $$\int x^n \sinh^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int x^n \sinh(2m-2k+1)x\,dx$$

4. $$\int x^n \cosh^{2m} x\,dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cosh(2m-2k)x\,dx$$

5. $$\int x^n \cosh^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \int x^n \cosh(2m-2k+1)x\,dx$$

2.472

1. $$\int x^n \sinh x\,dx = x^n \cosh x - n\int x^{n-1} \cosh x\,dx$$
$$= x^n \cosh x - nx^{n-1} \sinh x + n(n-1)\int x^{n-2} \sinh x\,dx$$

2. $$\int x^n \cosh x\,dx = x^n \sinh x - n\int x^{n-1} \sinh x\,dx$$
$$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1)\int x^{n-2} \cosh x\,dx$$

3. $$\int x^{2n} \sinh x\,dx = (2n)! \left\{ \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \cosh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!} \sinh x \right\}$$

4. $$\int x^{2n+1} \sinh x\,dx = (2n+1)! \sum_{k=0}^{n} \left\{ \frac{x^{2k+1}}{(2k+1)!} \cosh x - \frac{x^{2k}}{(2k)!} \sinh x \right\}$$

5.[11] $$\int x^{2n} \cosh x\,dx = (2n)! \left\{ \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \sinh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!} \cosh x \right\}$$

6. $$\int x^{2n+1} \cosh x\,dx = (2n+1)! \sum_{k=0}^{n} \left\{ \frac{x^{2k+1}}{(2k+1)!} \sinh x - \frac{x^{2k}}{(2k)!} \cosh x \right\}$$

7. $$\int x \sinh x\,dx = x \cosh x - \sinh x$$

8. $$\int x^2 \sinh x\,dx = \left(x^2+2\right) \cosh x - 2x \sinh x$$

9. $$\int x \cosh x\,dx = x \sinh x - \cosh x$$

10. $$\int x^2 \cosh x\,dx = \left(x^2+2\right) \sinh x - 2x \cosh x$$

2.473 **Notation**: $z_1 = a + bx$

1. $$\int z_1 \sinh kx\,dx = \frac{1}{k} z_1 \cosh kx - \frac{b}{k^2} \sinh kx$$

2. $$\int z_1 \cosh kx\,dx = \frac{1}{k} z_1 \sinh kx - \frac{b}{k^2}\cosh kx$$

3. $$\int z_1^2 \sinh kx\,dx = \frac{1}{k}\left(z_1^2 + \frac{2b^2}{k^2}\right)\cosh kx - \frac{2bz_1}{k^2}\sinh kx$$

4. $$\int z_1^2 \cosh kx\,dx = \frac{1}{k}\left(z_1^2 + \frac{2b^2}{k^2}\right)\sinh kx - \frac{2bz_1}{k^2}\cosh kx$$

5. $$\int z_1^3 \sinh kx\,dx = \frac{z_1}{k}\left(z_1^2 + \frac{6b^2}{k^2}\right)\cosh kx - \frac{3b}{k^2}\left(z_1^2 + \frac{2b^2}{k^2}\right)\sinh kx$$

6. $$\int z_1^3 \cosh kx\,dx = \frac{z_1}{k}\left(z_1^2 + \frac{6b^2}{k^2}\right)\sinh kx - \frac{3b}{k^2}\left(z_1^3 + \frac{2b^2}{k^2}\right)\cosh kx$$

7. $$\int z_1^4 \sinh kx\,dx = \frac{1}{k}\left(z_1^4 + \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cosh kx - \frac{4bz_1}{k^2}\left(z_1^2 + \frac{6b^2}{k^2}\right)\sinh kx$$

8. $$\int z_1^4 \cosh kx\,dx = \frac{1}{k}\left(z_1^4 + \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sinh kx - \frac{4bz_1}{k^2}\left(z_1^2 + \frac{6b^2}{k^2}\right)\cosh kx$$

9. $$\int z_1^5 \sinh kx\,dx = \frac{z_1}{k}\left(z_1^4 + \frac{20b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\cosh kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2}z_1^2 + 24\frac{b^4}{k^4}\right)\sinh kx$$

10. $$\int z_1^5 \cosh kx\,dx = \frac{z_1}{k}\left(z_1^4 + 20\frac{b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\sinh kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2}z_1^2 + 24\frac{b^4}{k^4}\right)\cosh kx$$

11. $$\int z_1^6 \sinh kx\,dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2}z_1^4 + 360\frac{b^4}{k^4}z_1^2 + 720\frac{b^6}{k^6}\right)\cosh kx - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\sinh kx$$

12. $$\int z_1^6 \cosh kx\,dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2}z_1^4 + 360\frac{b^4}{k^4}z_1^2 + 720\frac{b^6}{k^6}\right)\sinh kx - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2}z_1 + 120\frac{b^4}{k^4}\right)\cosh kx$$

2.474

1. $$\int x^n \sinh^2 x\,dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!}\sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cosh 2x\right\}$$

GU (353)(2b)

2. $$\int x^n \cosh^2 x\,dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!}\sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cosh 2x\right\}$$

GU (353)(3e)

3. $$\int x \sinh^2 x\,dx = \frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x - \frac{x^2}{4}$$

4. $$\int x^2 \sinh^2 x\,dx = \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\sinh 2x - \frac{x}{4}\cosh 2x - \frac{x^3}{6}$$ MZ 257

5. $$\int x\cosh^2 x\,dx = \frac{x}{4}\sinh 2x - \frac{1}{8}\cosh 2x + \frac{x^2}{4}$$

6. $$\int x^2\cosh^2 x\,dx = \frac{1}{4}\left(x^2+\frac{1}{2}\right)\sinh 2x - \frac{x}{4}\cosh 2x + \frac{x^3}{6}$$ MZ 261

7. $$\int x^n\sinh^3 x\,dx$$
$$= \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\cosh 3x}{3^{2k+1}} - 3\cosh x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\sinh 3x}{3^{2k+2}} - 3\sinh x\right)\right\}$$
GU (353)(2f)

8. $$\int x^n\cosh^3 x\,dx$$
$$= \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\sinh 3x}{3^{2k+1}} + 3\sinh x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\cosh 3x}{3^{2k+2}} + 3\cosh x\right)\right\}$$
GU (353)(3f)

9. $$\int x\sinh^3 x\,dx = \frac{3}{4}\sinh x - \frac{1}{36}\sinh 3x - \frac{3}{4}x\cosh x - \frac{x}{12}\cosh 3x$$

10. $$\int x^2\sinh^3 x\,dx = -\left(\frac{3x^2}{4}+\frac{3}{2}\right)\cosh x + \left(\frac{x^2}{12}+\frac{1}{54}\right)\cosh 3x + \frac{3x}{2}\sinh x - \frac{x}{18}\sinh 3x.$$ MZ 257

11. $$\int x\cosh^3 x\,dx = -\frac{3}{4}\cosh x - \frac{1}{36}\cosh 3x + \frac{3}{4}x\sinh x + \frac{x}{12}\sinh 3x$$

12. $$\int x^2\cosh^3 x\,dx = \left(\frac{3}{4}x^2+\frac{3}{2}\right)\sinh x + \left(\frac{x^2}{12}+\frac{1}{54}\right)\sinh 3x - \frac{3}{2}x\cosh x - \frac{x}{18}\cosh 3x$$ MZ 262

2.475

1. $$\int \frac{\sinh^q x}{x^p}\,dx = -\frac{(p-2)\sinh^q x + qx\sinh^{q-1} x\cosh x}{(p-1)(p-2)x^{p-1}}$$
$$+\frac{q(q-1)}{(p-1)(p-2)}\int\frac{\sinh^{q-2} x}{x^{p-2}}\,dx + \frac{q^2}{(p-1)(p-2)}\int\frac{\sinh^q x}{x^{p-2}}\,dx \qquad [p>2]$$
GU (353)(6a)

2. $$\int \frac{\cosh^q x}{x^p}\,dx = -\frac{(p-2)\cosh^q x + qx\cosh^{q-1} x\sinh x}{(p-1)(p-2)x^{p-1}}$$
$$-\frac{q(q-1)}{(p-1)(p-2)}\int\frac{\cosh^{q-2} x}{x^{p-2}}\,dx + \frac{q^2}{(p-1)(p-2)}\int\frac{\cosh^q x}{x^{p-2}}\,dx \qquad [p>2]$$
GU (353)(7a)

3. $$\int \frac{\sinh x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\cosh x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\sinh x\right\} + \frac{1}{(2n-1)!}\operatorname{chi}(x)$$
GU (353)(6b)

4. $$\int \frac{\sinh x}{x^{2n+1}}\,dx = -\frac{1}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\cosh x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\sinh x\right\} + \frac{1}{(2n)!}\operatorname{shi}(x)$$ GU (353)(6b)

5. $$\int \frac{\cosh x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\sinh x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\cosh x\right\} + \frac{1}{(2n-1)!}\operatorname{shi}(x)$$ GU (353)(7b)

6. $$\int \frac{\cosh x}{x^{2n+1}}\,dx = -\frac{1}{(2n)!x}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\sinh x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\cosh x\right\} + \frac{1}{(2n)!}\operatorname{chi}(x)$$ GU (353)(7b)

7. $$\int \frac{\sinh^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\operatorname{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}}\binom{2m}{m}\ln x$$ GU (353)(6c)

8. $$\int \frac{\sinh^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\operatorname{shi}(2m-2k+1)x$$ GU (353)(6d)

9. $$\int \frac{\cosh^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\operatorname{chi}(2m-2k)x + \frac{1}{2^{2m}}\binom{2m}{m}\ln x$$ GU (353)(7c)

10. $$\int \frac{\cosh^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\operatorname{chi}(2m-2k+1)x$$ GU (353)(7c)

11. $$\int \frac{\sinh^{2m} x}{x^2}\,dx = \frac{(-1)^{m-1}}{2^{2m}x}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+1}\binom{2m}{k}\left\{\frac{\cosh(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x\right\}$$

12. $$\int \frac{\sinh^{2m+1} x}{x^2}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+1}\binom{2m+1}{k} \times\left\{\frac{\sinh(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{chi}(2m-2k+1)x\right\}$$

13. $$\int \frac{\cosh^{2m} x}{x^2}\,dx = -\frac{1}{2^{2m}x}\binom{2m}{m} - \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\left\{\frac{\cosh(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x\right\}$$

14. $$\int \frac{\cosh^{2m+1} x}{x^2}\,dx = -\frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\left\{\frac{\cosh(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{shi}(2m-2k+1)x\right\}$$

2.476

1. $$\int \frac{\sinh kx}{a+bx}\,dx = \frac{1}{b}\left[\cosh\frac{ka}{b}\operatorname{shi}(u) - \sinh\frac{ka}{b}\operatorname{chi}(u)\right] = \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\operatorname{Ei}(u) - \exp\left(\frac{ka}{b}\right)\operatorname{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

2. $$\int \frac{\cosh kx}{a+bx}\,dx = \frac{1}{b}\left[\cosh\frac{ka}{b}\,\mathrm{chi}(u) - \sinh\frac{ka}{b}\,\mathrm{shi}(u)\right]$$
$$= \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\mathrm{Ei}(u) + \exp\left(\frac{ka}{b}\right)\mathrm{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

3. $$\int \frac{\sinh kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\sinh kx}{a+bx} + \frac{k}{b}\int\frac{\cosh kx}{a+bx}\,dx \qquad \text{(see } \mathbf{2.476}\ 2)$$

4. $$\int \frac{\cosh kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\cosh kx}{a+bx} + \frac{k}{b}\int\frac{\sinh kx}{a+bx}\,dx \qquad \text{(see } \mathbf{2.476}\ 1)$$

5. $$\int \frac{\sinh kx}{(a+bx)^3}\,dx = -\frac{\sinh kx}{2b(a+bx)^2} - \frac{k\cosh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int\frac{\sinh kx}{a+bx}\,dx$$
(see **2.476** 1)

6. $$\int \frac{\cosh kx}{(a+bx)^3}\,dx = -\frac{\cosh kx}{2b(a+bx)^2} - \frac{k\sinh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int\frac{\cosh kx}{a+bx}\,dx$$
(see **2.476** 2)

7. $$\int \frac{\sinh kx}{(a+bx)^4}\,dx = -\frac{\sinh kx}{3b(a+bx)^3} - \frac{k\cosh kx}{6b^2(a+bx)^2} - \frac{k^2\sinh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int\frac{\cosh kx}{a+bx}\,dx$$
(see **2.476** 2)

8. $$\int \frac{\cosh kx}{(a+bx)^4}\,dx = -\frac{\cosh kx}{3b(a+bx)^3} - \frac{k\sinh kx}{6b^2(a+bx)^2} - \frac{k^2\cosh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int\frac{\sinh kx}{a+bx}\,dx$$
(see **2.476** 1)

9. $$\int \frac{\sinh kx}{(a+bx)^5}\,dx = -\frac{\sinh kx}{4b(a+bx)^4} - \frac{k\cosh kx}{12b^2(a+bx)^3} - \frac{k^2\sinh kx}{24b^3(a+bx)^2}$$
$$-\frac{k^3\cosh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int\frac{\sinh kx}{a+bx}\,dx$$
(see **2.476** 1)

10. $$\int \frac{\cosh kx}{(a+bx)^5}\,dx = -\frac{\cosh kx}{4b(a+bx)^4} - \frac{k\sinh kx}{12b^2(a+bx)^3} - \frac{k^2\cosh kx}{24b^3(a+bx)^2}$$
$$-\frac{k^3\sinh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int\frac{\cosh kx}{a+bx}\,dx$$
(see **2.476** 2)

11. $$\int \frac{\sinh kx}{(a+bx)^6}\,dx = -\frac{\sinh kx}{5b(a+bx)^5} - \frac{k\cosh kx}{20b^2(a+bx)^4} - \frac{k^2\sinh kx}{60b^3(a+bx)^3} - \frac{k^3\cosh kx}{120b^4(a+bx)^2}$$
$$-\frac{k^4\sinh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int\frac{\cosh kx}{a+bx}\,dx$$
(see **2.476** 2)

12. $$\int \frac{\cosh kx}{(a+bx)^6}\,dx = -\frac{\cosh kx}{5b(a+bx)^5} - \frac{k\sinh kx}{20b^2(a+bx)^4} - \frac{k^2\cosh kx}{60b^3(a+bx)^3} - \frac{k^3\sinh kx}{120b^4(a+bx)^2}$$
$$-\frac{k^4\cosh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int\frac{\sinh kx}{a+bx}\,dx$$
(see **2.476** 1)

2.477

1. $$\int \frac{x^p\,dx}{\sinh^q x} = \frac{-px^{p-1}\sinh x - (q-2)x^p\cosh x}{(q-1)(q-2)\sinh^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}}{\sinh^{q-2} x}\,dx - \frac{q-2}{q-1}\int \frac{x^p\,dx}{\sinh^{q-2} x}$$
$[q > 2]$ GU (353)(8a)

2. $$\int \frac{x^p\,dx}{\cosh^q x} = \frac{px^{p-1}\cosh x + (q-2)x^p\sinh x}{(q-1)(q-2)\cosh^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\cosh^{q-2} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\cosh^{q-2} x}$$
$[q > 2]$ GU (353)(10a)

3. $$\int \frac{x^n}{\sinh x}\,dx = \sum_{k=0}^{\infty} \frac{\left(2-2^{2k}\right)B_{2k}}{(n+2k)(2k)!}x^{n+2k}$$
$[|x| < \pi, \quad n > 0]$ GU(353)(8b)

4. $$\int \frac{x^n}{\cosh x}\,dx = \sum_{k=0}^{\infty} \frac{E_{2k}x^{n+2k+1}}{(n+2k+1)(2k)!}$$
$\left[|x| < \frac{\pi}{2}, \quad n \geq 0\right]$ GU (353)(10b)

5. $$\int \frac{dx}{x^n\sinh x} = -\left[1+(-1)^n\right]\frac{2^{n-1}-1}{n!}B_n\ln x + \sum_{\substack{k=0\\ k\neq\frac{n}{2}}}^{\infty} \frac{2-2^{2k}}{(2k-n)(2k)!}B_{2k}x^{2k-n}$$
$[|x| < \pi, \quad n \geq 1]$ GU (353)(9b)

6.[11] $$\int \frac{dx}{x^n\cosh x} = \sum_{\substack{k=0\\ k\neq\frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!}x^{2k-n+1} + \frac{1}{2}\left[1+(-1)^n\right] + \frac{E_{n-1}}{(n-1)!}\ln x$$
$\left[|x| < \frac{\pi}{2}\right]$ GU (353)(11b)

7. $$\int \frac{x^n}{\sinh^2 x}\,dx = -x^n\coth x + n\sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1}$$
$[n > 1, \quad |x| < \pi]$ GU (353)(8c)

8. $$\int \frac{x^n}{\cosh^2 x}\,dx = x^n\tanh x - n\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1}$$
$\left[n > 1, \quad |x| < \frac{\pi}{2}\right]$ GU (353)(10c)

9. $$\int \frac{dx}{x^n\sinh^2 x} = -\frac{\coth x}{x^n} - \left[1-(-1)^n\right]\frac{2^n n}{(n+1)!}B_{n+1}\ln x - \frac{n}{x^{n+1}}\sum_{\substack{k=0\\ k\neq\frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!}(2x)^{2^k}$$
$[|x| < \pi]$ GU (353)(9c)

10. $$\int \frac{dx}{x^n \cosh^2 x} = \frac{\tanh x}{x^n} + [1-(-1)^n] - \frac{2n\left(2^{n+1}-1\right)n}{(n+1)!} B_{n+1} \ln x + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k\neq \frac{n+1}{2}}}^{\infty} \frac{\left(2^{2k}-1\right) B_{2k}}{(2k-n-1)(2k)!} (2x)^{2^k}$$
$$\left[|x| < \frac{\pi}{2}\right]$$ GU (353)(11c)

11. $$\int \frac{x}{\sinh^{2n} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+1)} \times \left\{ \frac{x\cosh x}{\sinh^{2n-2k+1} x} + \frac{1}{(2n-2k)\sinh^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\, dx}{\sinh^2 x}$$
(see **2.477** 17) GU (353)(8e)

12. $$\int \frac{x}{\sinh^{2n-1} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5)\ldots(2n-2k+1)}{(2n-2)(2n-4)\ldots(2n-2k)} \times \left\{ \frac{x\cosh x}{\sinh^{2n-2k} x} + \frac{1}{(2n-2k-1)\sinh^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\, dx}{\sinh x}$$
(see **2.477** 15) GU (353)(8e)

13. $$\int \frac{x}{\cosh^{2n} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+1)} \times \left\{ \frac{x\sinh x}{\cosh^{2n-2k+1} x} + \frac{1}{(2n-2k)\cosh^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\, dx}{\cosh^2 x}$$
(see **2.477** 18) GU (353)(10e)

14. $$\int \frac{x}{\cosh^{2n-1} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\ldots(2n-2k+1)}{(2n-2)(2n-4)\ldots(2n-2k)} \times \left\{ \frac{x\sinh x}{\cosh^{2n-2k} x} + \frac{1}{(2n-2k-1)\cosh^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\, dx}{\cosh x}$$
(see **2.477** 16) GU (353)(10e)

15. $$\int \frac{x\, dx}{\sinh x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1} \qquad |x| < \pi$$ GU (353)(8b)a

16. $$\int \frac{x\, dx}{\cosh x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!} \qquad |x| < \frac{\pi}{2}$$ GU (353)(10b)a

17. $$\int \frac{x\, dx}{\sinh^2 x} = -x\coth x + \ln\sinh x$$ MZ 257

18. $$\int \frac{x\, dx}{\cosh^2 x} = x\tanh x - \ln\cosh x$$ MZ 262

19. $$\int \frac{x\, dx}{\sinh^3 x} = -\frac{x\cosh x}{2\sinh^2 x} - \frac{1}{2\sinh x} - \frac{1}{2}\int \frac{x\, dx}{\sinh x}$$ (see **2.477** 15) MZ 257

20. $$\int \frac{x\,dx}{\cosh^3 x} = \frac{x\sinh x}{2\cosh^2 x} + \frac{1}{2\cosh x} + \frac{1}{2}\int \frac{x\,dx}{\cosh x} \qquad \text{(see } \mathbf{2.477}\ 16)$$ MZ 262

21. $$\int \frac{x\,dx}{\sinh^4 x} = -\frac{x\cosh x}{3\sinh^3 x} - \frac{1}{6\sinh^2 x} + \frac{2}{3}x\coth x - \frac{2}{3}\ln\sinh x$$ MZ 258

22. $$\int \frac{x\,dx}{\cosh^4 x} = \frac{x\sinh x}{3\cosh^3 x} + \frac{1}{6\cosh^2 x} + \frac{2}{3}x\tanh x - \frac{2}{3}\ln\cosh x$$ MZ 262

23. $$\int \frac{x\,dx}{\sinh^5 x} = -\frac{x\cosh x}{4\sinh^4 x} - \frac{1}{12\sinh^3 x} + \frac{3x\cosh x}{8\sinh^2 x} + \frac{3}{8\sinh x} + \frac{3}{8}\int \frac{x\,dx}{\sinh x}$$
(see **2.477** 15) MZ 258

24. $$\int \frac{x\,dx}{\cosh^5 x} = \frac{x\sinh x}{4\cosh^4 x} + \frac{1}{12\cosh^3 x} + \frac{3x\sinh x}{8\cosh^2 x} + \frac{3}{8\cosh x} + \frac{3}{8}\int \frac{x\,dx}{\cosh x}$$
(see **2.477** 16) MZ 262

2.478

1. $$\int \frac{x^n\cosh x\,dx}{(a+b\sinh x)^m} = -\frac{x^n}{(m-1)b\,(a+b\sinh x)^{m-1}} + \frac{n}{(m-1)b}\int \frac{x^{n-1}\,dx}{(a+b\sinh x)^{m-1}}$$
$[m \neq 1]$ MZ 263

2. $$\int \frac{x^n\sinh x\,dx}{(a+b\cosh x)^m} = -\frac{x^n}{(m-1)b\,(a+b\cosh x)^{m-1}} + \frac{n}{(m-1)b}\int \frac{x^{n-1}\,dx}{(a+b\cosh x)^{m-1}}$$
$[m \neq 1]$ MZ 263

3. $$\int \frac{x\,dx}{1+\cosh x} = x\tanh\frac{x}{2} - 2\ln\cosh\frac{x}{2}$$

4. $$\int \frac{x\,dx}{1-\cosh x} = x\coth\frac{x}{2} - 2\ln\sinh\frac{x}{2}$$

5. $$\int \frac{x\sinh x\,dx}{(1+\cosh x)^2} = -\frac{x}{1+\cosh x} + \tanh\frac{x}{2}$$

6. $$\int \frac{x\sinh x\,dx}{(1-\cosh x)^2} = \frac{x}{1-\cosh x} - \coth\frac{x}{2}$$ MZ 262-264

7. $$\int \frac{x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2\sin 2t}\left[L(u+t) - L(u-t) - 2\,L(t)\right]$$
$[u = \arctan(\tanh x\cot t)\,, \quad t \neq \pm n\pi]$
LO III 402

8. $$\int \frac{x\cosh x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2\sin t}\left[L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{\upsilon+t}{2}\right) + L\left(\frac{\upsilon-t}{2}\right) - 2\,L\left(\frac{t}{2}\right) - 2\,L\left(\frac{\pi-t}{2}\right)\right]$$
$\left[u = 2\arctan\left(\tanh\frac{x}{2}\cdot\cot\frac{t}{2}\right), \quad \upsilon = 2\arctan\left(\coth\frac{x}{2}\cdot\cot\frac{t}{2}\right); \quad t \neq \pm n\pi\right]$ LO III 403

2.479

1. $$\int x^p \frac{\sinh^{2m} x}{\cosh^n x}\,dx = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} \int \frac{x^p\,dx}{\cosh^{n-2k} x} \qquad \text{(see } \mathbf{4.477}\ 2)$$

2. $$\int x^p \frac{\sinh^{2m+1} x}{\cosh^n x}\,dx = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} \int x^p \frac{\sinh x}{\cosh^{n-2k} x}\,dx$$
$$[n > 1] \qquad \text{(see } \mathbf{2.479}\ 3)$$

3. $$\int x^p \frac{\sinh x}{\cosh^n x}\,dx = -\frac{x^p}{(n-1)\cosh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}\,dx}{\cosh^{n-1} x}$$
$$[n > 1] \qquad \text{(see } \mathbf{2.477}\ 2) \qquad \text{GU (353)(12)}$$

4. $$\int x^p \frac{\cosh^{2m} x}{\sinh^n x}\,dx = \sum_{k=0}^{m} \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x} \qquad \text{(see } \mathbf{2.477}\ 1)$$

5. $$\int x^p \frac{\cosh^{2m+1} x}{\sinh^n x}\,dx = \sum_{k=0}^{m} \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x}\,dx \qquad \text{(see } \mathbf{2.479}\ 6)$$

6. $$\int x^p \frac{\cosh x}{\sinh^n x}\,dx = -\frac{x^p}{(n-1)\sinh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}\,dx}{\sinh^{n-1} x}$$
$$[n > 1] \qquad \text{(see } \mathbf{2.477}\ 1)$$
GU (353)(13c)

7. $$\int x^p \tanh x\,dx = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right) B_{2k}}{(2k+p)(2k)!} x^{p+2k} \qquad \left[p > -1, \quad |x| < \frac{\pi}{2}\right] \qquad \text{GU (353)(12d)}$$

8. $$\int x^p \coth x\,dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \qquad [p \geq +1, \quad |x| < \pi] \qquad \text{GU (353)(13d)}$$

9. $$\int \frac{x \cosh x}{\sinh^2 x}\,dx = \ln \tanh \frac{x}{2} - \frac{x}{\sinh x}$$

10. $$\int \frac{x \sinh x}{\cosh^2 x}\,dx = -\frac{x}{\cosh x} + \arctan(\sinh x) \qquad \text{MZ 263}$$

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.481

1. $$\int e^{ax} \sinh(bx+c)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a \sinh(bx+c) - b \cosh(bx+c)\right]$$
$$\left[a^2 \neq b^2\right]$$

2. $$\int e^{ax} \cosh(bx+c)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a \cosh(bx+c) - b \sinh(bx+c)\right]$$
$$\left[a^2 \neq b^2\right]$$

For $a^2 = b^2$:

3. $$\int e^{ax}\sinh(ax+c)\,dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

4. $$\int e^{-ax}\sinh(ax+c)\,dx = \frac{1}{2}xe^{c} + \frac{1}{4a}e^{-(2ax+c)}$$

5. $$\int e^{ax}\cosh(ax+c)\,dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

6. $$\int e^{-ax}\cosh(ax+c)\,dx = \frac{1}{2}xe^{c} - \frac{1}{4a}e^{-(2ax+c)}$$ MZ 275-277

2.482

1. $$\int x^p e^{ax}\sinh bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx - \int x^p e^{(a-b)x}\,dx\right\}$$

$[a^2 \neq b^2]$

2. $$\int x^p e^{ax}\cosh bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx + \int x^p e^{(a-b)x}\,dx\right\}$$

$[a^2 \neq b^2]$

For $a^2 = b^2$:

3. $$\int x^p e^{ax}\sinh ax\,dx = \frac{1}{2}\int x^p e^{2ax}\,dx - \frac{x^{p+1}}{2(p+1)}$$ (see **2.321**)

4. $$\int x^p e^{-ax}\sinh ax\,dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2}\int x^p e^{-2ax}\,dx$$ (see **2.321**)

5. $$\int x^p e^{ax}\cosh ax\,dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2}\int x^p e^{2ax}\,dx$$ (see **2.321**) MZ 276, 278

2.483

1. $$\int xe^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\sinh bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\cosh bx\right]$$

$[a^2 \neq b^2]$

2. $$\int xe^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\cosh bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\sinh bx\right]$$

$[a^2 \neq b^2]$

3. $$\int x^2 e^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left\{\left[ax^2 - \frac{2\left(a^2+b^2\right)}{a^2-b^2}x + \frac{2a\left(a^2+3b^2\right)}{\left(a^2-b^2\right)^2}\right]\sinh bx - \left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b\left(3a^2+b^2\right)}{\left(a^2-b^2\right)^2}\right]\cosh x\right\}$$ $[a^2 \neq b^2]$

4. $$\int x^2 e^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left\{\left[ax^2 - \frac{2\left(a^2+b^2\right)}{a^2-b^2}x + \frac{2a\left(a^2+3b^2\right)}{\left(a^2-b^2\right)^2}\right]\cosh bx - \left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b\left(3a^2+b^2\right)}{\left(a^2-b^2\right)^2}\right]\sinh x\right\} \qquad \left[a^2 \neq b^2\right]$$

For $a^2 = b^2$:

5. $$\int xe^{ax}\sinh ax\,dx = \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right) - \frac{x^2}{4}$$

6. $$\int xe^{-ax}\sinh ax\,dx = \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right) + \frac{x^2}{4}$$ MZ 276, 278

7. $$\int xe^{ax}\cosh ax\,dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right)$$

8. $$\int xe^{-ax}\cosh ax\,dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right)$$

9. $$\int x^2 e^{ax}\sinh ax\,dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) - \frac{x^3}{6}$$

10. $$\int x^2 e^{-ax}\sinh ax\,dx = \frac{e^{-2ax}}{4a}\left(x^2 + \frac{x}{a} + \frac{1}{2a^2}\right) + \frac{x^3}{6}$$

11. $$\int x^2 e^{ax}\cosh ax\,dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right)$$

2.484

1. $$\int e^{ax}\sinh bx\frac{dx}{x} = \frac{1}{2}\left\{\mathrm{Ei}[(a+b)x] - \mathrm{Ei}[(a-b)x]\right\} \qquad \left[a^2 \neq b^2\right]$$

2. $$\int e^{ax}\cosh bx\frac{dx}{x} = \frac{1}{2}\left\{\mathrm{Ei}[(a+b)x] + \mathrm{Ei}[(a-b)x]\right\} \qquad \left[a^2 \neq b^2\right]$$

3. $$\int e^{ax}\sinh bx\frac{dx}{x^2} = -\frac{e^{ax}\sinh bx}{2x} + \frac{1}{2}\left\{(a+b)\,\mathrm{Ei}[(a+b)x] - (a-b)\,\mathrm{Ei}[(a-b)x]\right\} \qquad \left[a^2 \neq b^2\right]$$

4. $$\int e^{ax}\cosh bx\frac{dx}{x^2} = -\frac{e^{ax}\cosh bx}{2x} + \frac{1}{2}\left\{(a+b)\,\mathrm{Ei}[(a+b)x] + (a-b)\,\mathrm{Ei}[(a-b)x]\right\} \qquad \left[a^2 \neq b^2\right]$$

For $a^2 = b^2$:

5. $$\int e^{ax}\sinh ax\frac{dx}{x} = \frac{1}{2}\left[\mathrm{Ei}(2ax) - \ln x\right]$$

6. $$\int e^{-ax}\sinh ax\frac{dx}{x} = \frac{1}{2}\left[\ln x - \mathrm{Ei}(-2ax)\right]$$

7. $$\int e^{ax}\cosh ax\frac{dx}{x} = \frac{1}{2}\left[\ln x + \mathrm{Ei}(2ax)\right]$$

8. $$\int e^{ax}\sinh ax\frac{dx}{x^2}=-\frac{1}{2x}\left(e^{2ax}-1\right)+a\,\mathrm{Ei}(2ax)$$

9. $$\int e^{-ax}\sinh ax\frac{dx}{x^2}=-\frac{1}{2x}\left(1-e^{-2ax}\right)+a\,\mathrm{Ei}(-2ax)$$

10. $$\int e^{ax}\cosh ax\frac{dx}{x^2}=-\frac{1}{2x}\left(e^{2ax}+1\right)+a\,\mathrm{Ei}(2ax)$$ MZ 276, 278

2.5–2.6 Trigonometric Functions

2.50 Introduction

2.501 Integrals of the form $\int R\left(\sin x,\cos x\right)\,dx$ can always be reduced to integrals of rational functions by means of the substitution $t=\tan\frac{x}{2}$.

2.502 If $R\left(\sin x,\cos x\right)$ satisfies the relation

$$R\left(\sin x,\cos x\right)=-R\left(-\sin x,\cos x\right),$$

it is convenient to make the substitution $t=\cos x$.

2.503 If this function satisfies the relation

$$R\left(\sin x,\cos x\right)=-R\left(\sin x,-\cos x\right),$$

it is convenient to make the substitution $t=\sin x$.

2.504 If this function satisfies the relation

$$R\left(\sin x,\cos x\right)=R\left(-\sin x,-\cos x\right),$$

it is convenient to make the substitution $t=\tan x$.

2.51–2.52 Powers of trigonometric functions

2.510
$$\begin{aligned}\int\sin^p x\cos^q x\,dx&=-\frac{\sin^{p-1}x\cos^{q+1}x}{q+1}+\frac{p-1}{q+1}\int\sin^{p-2}x\cos^{q+2}x\,dx\\&=-\frac{\sin^{p-1}x\cos^{q+1}x}{p+q}+\frac{p-1}{p+q}\int\sin^{p-2}x\cos^{q}x\,dx\\&=\frac{\sin^{p+1}x\cos^{q+1}x}{p+1}+\frac{p+q+2}{p+1}\int\sin^{p+2}x\cos^{q}x\,dx\\&=\frac{\sin^{p+1}x\cos^{q-1}x}{p+1}+\frac{q-1}{p+1}\int\sin^{p+2}x\cos^{q-2}x\,dx\\&=\frac{\sin^{p+1}x\cos^{q-1}x}{p+q}+\frac{q-1}{p+q}\int\sin^{p}x\cos^{q-2}x\,dx\\&=-\frac{\sin^{p+1}x\cos^{q+1}x}{q+1}+\frac{p+q+2}{q+1}\int\sin^{p}x\cos^{q+2}x\,dx\\&=\frac{\sin^{p-1}x\cos^{q-1}x}{p+q}\left\{\sin^2x-\frac{q-1}{p+q-2}\right\}\\&\quad+\frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int\sin^{p-2}x\cos^{q-2}x\,dx\end{aligned}$$

FI II 89, TI 214

2.511

1. $$\int \sin^p x \cos^{2n} x\,dx = \frac{\sin^{p+1} x}{2n+p}\left\{\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\} + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \sin^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have:

2. $$\int \sin^{2l} x\,dx = -\frac{\cos x}{2l}\left\{\sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)}\sin^{2l-2k-1} x\right\} + \frac{(2l-1)!!}{2^l l!}x$$

(see also **2.513** 1)

TI (232)

3. $$\int \sin^{2l+1} x\,dx = -\frac{\cos x}{2l+1}\left\{\sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)}\sin^{2l-2k-2} x\right\}$$

(see also **2.513** 2) TI (233)

4. $$\int \sin^p x \cos^{2n+1} x\,dx = \frac{\sin^{p+1} x}{2n+p+1}\left\{\cos^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}$$

This formula is applicable for arbitrary real p, except for the negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.512

1. $$\int \cos^p x \sin^{2n} x\,dx = -\frac{\cos^{p+1} x}{2n+p}\left\{\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\} + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \cos^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

2. $$\int \cos^{2l} x\,dx = \frac{\sin x}{2l}\left\{\cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)}\cos^{2l-2k-1} x\right\} + \frac{(2l-1)!!}{2^l l!}x$$

(see also **2.513** 3) TI (230)

3. $$\int \cos^{2l+1} x\,dx = \frac{\sin x}{2l+1}\left\{\cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \cos^{2l-2k-2} x\right\}$$

(see also **2.513** 4) TI (231)

4. $$\int \cos^p x \sin^{2n+1} x\,dx$$
$$= -\frac{\cos^{p+1} x}{2n+p+1}\left\{\sin^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: -1, -3, $\dots$, $-(2n+1)$.

2.513

1. $$\int \sin^{2n} x\,dx = \frac{1}{2^{2n}}\binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also **2.511** 2) TI (226)

2. $$\int \sin^{2n+1} x\,dx = \frac{1}{2^{2n}}(-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$

(see also **2.511** 3) TI (227)

3. $$\int \cos^{2n} x\,dx = \frac{1}{2^{2n}}\binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also **2.512** 2) TI (224)

4. $$\int \cos^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}$$

(see also **2.512** 3) TI (225)

5. $$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x$$

6. $$\int \sin^3 x\,dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x = \frac{1}{3}\cos^3 x - \cos x$$

7. $$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$
$$= -\frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x$$

8. $$\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos 3x - \frac{1}{80}\cos 5x$$
$$= -\frac{1}{5}\sin^4 x\cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x$$

9. $$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x$$

10. $$\int \sin^7 x\,dx = -\frac{35}{64}\cos x + \frac{7}{64}\cos 3x - \frac{7}{320}\cos 5x + \frac{1}{448}\cos 7x$$
$$= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x + \frac{8}{35}\cos^3 x - \frac{24}{35}\cos x$$

11. $$\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x\cos x + \frac{1}{2}x$$

12. $$\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x$$

13. $$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = \frac{3}{8}x + \frac{3}{8}\sin x\cos x + \frac{1}{4}\sin x\cos^3 x$$

14. $$\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x = \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x\sin x$$

15. $$\int \cos^6 x\,dx = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x$$
$$= \frac{5}{16}x + \frac{5}{16}\sin x\cos x + \frac{5}{24}\sin x\cos^3 x + \frac{1}{6}\sin x\cos^5 x$$

16. $$\int \cos^7 x\,dx = \frac{35}{64}\sin x + \frac{7}{64}\sin 3x + \frac{7}{320}\sin 5x + \frac{1}{448}\sin 7x$$
$$= \frac{24}{35}\sin x - \frac{8}{35}\sin^3 x + \frac{6}{35}\sin x\cos^4 x + \frac{1}{7}\sin x\cos^6 x$$

17. $$\int \sin x\cos^2 x\,dx = -\frac{1}{4}\left(\frac{1}{3}\cos 3x + \cos x\right) = -\frac{\cos^3 x}{3}$$

18. $$\int \sin x\cos^3 x\,dx = -\frac{\cos^4 x}{4}$$

19. $$\int \sin x\cos^4 x\,dx = -\frac{\cos^5 x}{5}$$

20. $$\int \sin^2 x\cos x\,dx = -\frac{1}{4}\left(\frac{1}{3}\sin 3x - \sin x\right) = \frac{\sin^3 x}{3}$$

21. $$\int \sin^2 x\cos^2 x\,dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right)$$

22. $$\int \sin^2 x\cos^3 x\,dx = -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right)$$
$$= \frac{\sin^3 x}{5}\left(\cos^2 x + \frac{2}{3}\right) = \frac{\sin^3 x}{5}\left(\frac{5}{3} - \sin^2 x\right)$$

23. $$\int \sin^2 x\cos^4 x\,dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x$$

24. $$\int \sin^3 x\cos x\,dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{\sin^4 x}{4}$$

25. $$\int \sin^3 x\cos^2 x\,dx = \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right)$$
$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

26. $$\int \sin^3 x \cos^3 x\,dx = \frac{1}{32}\left(\frac{1}{6}\cos 6x - \frac{3}{2}\cos 2x\right)$$

27. $$\int \sin^3 x \cos^4 x\,dx = \frac{1}{7}\cos^3 x\left(-\frac{2}{5} - \frac{3}{5}\sin^2 x + \sin^4 x\right)$$

28. $$\int \sin^4 x \cos x\,dx = \frac{\sin^5 x}{5}$$

29. $$\int \sin^4 x \cos^2 x\,dx = \frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x$$

30. $$\int \sin^4 x \cos^3 x\,dx = \frac{1}{7}\sin^3 x\left(\frac{2}{5} + \frac{3}{5}\cos^2 x - \cos^4 x\right)$$

31. $$\int \sin^4 x \cos^4 x\,dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x$$

2.514 $$\int \frac{\sin^p x}{\cos^{2n} x}\,dx$$
$$= \frac{\sin^{p+1} x}{2n-1}\left\{\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\sec^{2n-2k-1} x\right\}$$
$$+\frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \sin^p x\,dx$$

This formula is applicable for arbitrary real p. For $\int \sin^p x\,dx$, where p is a natural number, see **2.511** 2, 3 and **2.513** 1, 2. If $n = 0$ and p is a negative integer, we have for this integral:

2.515

1. $$\int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1}\left\{\operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k (l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)}\operatorname{cosec}^{2l-2k-1} x\right\}$$

TI (242)

2. $$\int \frac{dx}{\sin^{2l+1} x} = -\frac{\cos x}{2l}\left\{\operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{28^k (l-1)(l-2)\dots(l-k)}\operatorname{cosec}^{2l-2k} x\right\}$$
$$+\frac{(2l-1)!!}{2^l l!}\ln\tan\frac{x}{2}$$

TI (243)

2.516

1. $$\int \frac{\sin^p x\,dx}{\cos^{2n+1} x}$$
$$= \frac{\sin^{p+1} x}{2n}\left\{\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\cdots(2n-p-2k+1)}{2^k (n-1)(n-2)\cdots(n-k)}\sec^{2n-2k} x\right\}$$
$$+\frac{(2n-p-1)(2n-p-3)\cdots(3-p)(1-p)}{2^n n!}\int \frac{\sin^p x}{\cos x}\,dx$$

This formula is applicable for arbitrary real p. For $n = 0$ and p a natural number, we have

2. $$\int \frac{\sin^{2l+1} x\,dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k} x}{2k} - \ln\cos x$$

3. $$\int \frac{\sin^{2l} x\,dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k-1} x}{2k-1} + \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

2.517

1. $$\int \frac{dx}{\sin^{2m+1} x\cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+2)\sin^{2m-2k+2} x} + \ln\tan x$$

2. $$\int \frac{dx}{\sin^{2m} x\cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+1)\sin^{2m-2k+1} x} + \ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

2.518

1. $$\int \frac{\sin^p x}{\cos^2 x}\,dx = \frac{\sin^{p-1} x}{\cos x} - (p-1)\int \sin^{p-2} x\,dx$$

2. $$\int \frac{\cos^p x\,dx}{\sin^{2n} x} = \frac{\cos^{p+1} x}{2n-1}\left\{\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x\right\} + \frac{(2n-p-2)(2n-p-4)\dots(2-p)(-p)}{(2n-1)!!}\int \cos^p x\,dx$$

This formula is applicable for arbitrary real p. For $\int \cos^p x\,dx$ where p is a natural number, see **2.512** 2, 3 and **2.513** 3, 4. If $n=0$ and p is a negative integer, we have for this integral:

2.519

1. $$\int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1}\left\{\sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k (l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)} \sec^{2l-2k-1} x\right\}$$ TI (240)

2. $$\int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l}\left\{\sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k (l-1)(l-2)\dots(l-k)} \sec^{2l-2k} x\right\} + \frac{(2l-1)!!}{2^l l!}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$ TI (241)

2.521

1. $$\int \frac{\cos^p x\,dx}{\sin^{2n+1} x} = -\frac{\cos^{p+1} x}{2n}\left\{\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \operatorname{cosec}^{2n-2k} x\right\} + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n\cdot n!}\int \frac{\cos^p x}{\sin x}\,dx$$

This formula is applicable for arbitrary real p. For $n=0$ and p a natural number, we have

2. $$\int \frac{\cos^{2l+1} x\,dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k} x}{2k} + \ln \sin x$$

3. $$\int \frac{\cos^{2l} x\,dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k-1} x}{2k-1} + \ln \tan \frac{x}{2}$$

2.522

1. $$\int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+2)\cos^{2m-2k+2} x} + \ln \tan x$$

2. $$\int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+1)\cos^{2m-2k+1} x} + \ln \tan \frac{x}{2}$$ GW (331)(15)

2.523 $$\int \frac{\cos^m x}{\sin^2 x}\,dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1)\int \cos^{m-2} x\,dx$$

2.524 In formulas **2.524** 1 and **2.524** 2, $s = 1$ for m odd and $m < 2n+1$; in other cases, $s = 0$.

1. $$\int \frac{\sin^{2n+1} x}{\cos^m x}\,dx = \sum_{\substack{k=0\\ k\neq \frac{m-1}{2}}}^{n} (-1)^{k+1} \binom{n}{k} \frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cos x$$

GU (331)(11d)

2. $$\int \frac{\cos^{2n+1} x}{\sin^m x}\,dx = \sum_{\substack{k=0\\ k\neq \frac{m-1}{2}}}^{n} (-1)^{k} \binom{n}{k} \frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \sin x$$

2.525

1. $$\int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\tan^{2k-2m+1} x}{2k-2m+1}$$ TI (267)

2. $$\int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\tan^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln \tan x$$

TI (268), GU (331)(15f)

2.526

1. $$\int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

2. $$\int \frac{dx}{\sin^2 x} = -\cot x$$

3. $$\int \frac{dx}{\sin^3 x} = -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln \tan \frac{x}{2}$$

4. $$\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3}\cot x = -\frac{1}{3}\cot^3 x - \cot x$$

5. $$\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8}\ln \tan \frac{x}{2}$$

6. $$\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\sin^5 x} - \frac{4}{15}\cot^3 x - \frac{4}{5}\cot x = -\frac{1}{5}\cot^5 x - \frac{2}{3}\cot^3 x - \cot x$$

7. $$\int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6\sin^2 x}\left(\frac{1}{\sin^4 x} + \frac{5}{4\sin^2 x} + \frac{15}{8}\right) + \frac{5}{16}\ln\tan\frac{x}{2}$$

8. $$\int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7}\cot^7 x + \frac{3}{5}\cot^5 x + \cot^3 x + \cot x\right)$$

9. $$\int \frac{dx}{\cos x} = \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \ln\cot\left(\frac{\pi}{4} - \frac{x}{2}\right) = \ln\sqrt{\frac{1+\sin x}{1-\sin x}}$$

10. $$\int \frac{dx}{\cos^2 x} = \tan x$$

11. $$\int \frac{dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

12. $$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}\tan x = \frac{1}{3}\tan^3 x + \tan x$$

13. $$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{8}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

14. $$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5\cos^5 x} + \frac{4}{15}\tan^3 x + \frac{4}{5}\tan x = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x$$

15. $$\int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\cos^6 x} + \frac{5\sin x}{24\cos^4 x} + \frac{5\sin x}{16\cos^2 x} + \frac{5}{16}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

16. $$\int \frac{dx}{\cos^8 x} = \frac{1}{7}\tan^7 x + \frac{3}{5}\tan^5 x + \tan^3 x + \tan x$$

17. $$\int \frac{\sin x}{\cos x}\,dx = -\ln\cos x$$

18. $$\int \frac{\sin^2 x}{\cos x}\,dx = -\sin x + \ln\tan\left(\frac{\pi}{4} = \frac{x}{2}\right)$$

19. $$\int \frac{\sin^3 x}{\cos x}\,dx = -\frac{\sin^2 x}{2} - \ln\cos x = \frac{1}{2}\cos^2 x - \ln\cos x$$

20. $$\int \frac{\sin^4 x}{\cos x}\,dx = -\frac{1}{3}\sin^3 x - \sin x + \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

21. $$\int \frac{\sin^2 x\,dx}{\cos^2 x} = \frac{1}{\cos x}$$

22. $$\int \frac{\sin^2 x\,dx}{\cos^2 x} = \tan x - x$$

23. $$\int \frac{\sin^3 x\,dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}$$

24. $$\int \frac{\sin^4 x\,dx}{\cos^2 x} = \tan x + \frac{1}{2}\sin x\cos x - \frac{3}{2}x$$

25. $$\int \frac{\sin x\,dx}{\cos^3 x} = \frac{1}{2\cos^2 x} = \frac{1}{2}\tan^2 x$$

26. $$\int \frac{\sin^2 x\,dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

27. $$\int \frac{\sin^3 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \ln\cos x$$

28. $$\int \frac{\sin^4 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

29. $$\int \frac{\sin x\,dx}{\cos^4 x} = \frac{1}{3\cos^3 x}$$

30. $$\int \frac{\sin^2 x\,dx}{\cos^4 x} = \frac{1}{3}\tan^3 x$$

31. $$\int \frac{\sin^3 x\,dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x}$$

32. $$\int \frac{\sin^4 x\,dx}{\cos^4 x} = \frac{1}{3}\tan^3 x - \tan x + x$$

33. $$\int \frac{\cos x\,dx}{\sin x} = \ln\sin x$$

34. $$\int \frac{\cos^2 x\,dx}{\sin x} = \cos x + \ln\tan\frac{x}{2}$$

35. $$\int \frac{\cos^3 x\,dx}{\sin x} = \frac{\cos^2 x}{2} + \ln\sin x$$

36. $$\int \frac{\cos^4 x\,dx}{\sin x} = \frac{1}{3}\cos^3 x + \cos x + \ln\tan\left(\frac{x}{2}\right)$$

37. $$\int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x}$$

38. $$\int \frac{\cos^2 x}{\sin^2 x}\,dx = -\cot x - x$$

39. $$\int \frac{\cos^3 x}{\sin^2 x}\,dx = -\sin x - \frac{1}{\sin x}$$

40. $$\int \frac{\cos^4 x}{\sin^2 x}\,dx = -\cot x - \frac{1}{2}\sin x\cos x - \frac{3}{2}x$$

41. $$\int \frac{\cos x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x}$$

42. $$\int \frac{\cos^2 x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\tan\frac{x}{2}$$

43. $$\int \frac{\cos^3 x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - \ln\sin x$$

44. $$\int \frac{\cos^4 x}{\sin^3 x}\,dx = -\frac{1}{2}\frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2}\ln\tan\frac{x}{2}$$

45. $\displaystyle\int \frac{\cos x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x}$

46. $\displaystyle\int \frac{\cos^2 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x$

47. $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}$

48. $\displaystyle\int \frac{\cos^4 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x + \cot x + x$

49. $\displaystyle\int \frac{dx}{\sin x\cos x} = \ln\tan x$

50. $\displaystyle\int \frac{dx}{\sin x\cos^2 x} = \frac{1}{\cos x} + \ln\tan\frac{x}{2}$

51. $\displaystyle\int \frac{dx}{\sin x\cos^3 x} = \frac{1}{2\cos^2 x} + \ln\tan x$

52. $\displaystyle\int \frac{dx}{\sin x\cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln\tan\frac{x}{2}$

53. $\displaystyle\int \frac{dx}{\sin^2 x\cos x} = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - \operatorname{cosec} x$

54. $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x} = -2\cot 2x$

55. $\displaystyle\int \frac{dx}{\sin^2 x\cos^3 x} = \left(\frac{1}{2\cos^2 x} - \frac{3}{2}\right)\frac{1}{\sin x} + \frac{3}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

56. $\displaystyle\int \frac{dx}{\sin^2 x\cos^4 x} = \frac{1}{3\sin x\cos^3 x} - \frac{8}{3}\cot 2x$

57. $\displaystyle\int \frac{dx}{\sin^3 x\cos x} = -\frac{1}{2\sin^2 x} + \ln\tan x$

58. $\displaystyle\int \frac{dx}{\sin^3 x\cos^2 x} = -\frac{1}{\cos x}\left(\frac{1}{2\sin^2 x} - \frac{3}{2}\right) + \frac{3}{2}\ln\tan\frac{x}{2}$

59. $\displaystyle\int \frac{dx}{\sin^3 x\cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2\ln\tan x$

60. $\displaystyle\int \frac{dx}{\sin^3 x\cos^4 x} = \frac{2}{\cos x} + \frac{1}{3\cos^3 x} - \frac{\cos x}{2\sin^2 x} + \frac{5}{2}\ln\tan\frac{x}{2}$

61. $\displaystyle\int \frac{dx}{\sin^4 x\cos x} = -\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

62. $\displaystyle\int \frac{dx}{\sin^4 x\cos^2 x} = -\frac{1}{3\cos x\sin^3 x} - \frac{8}{3}\cot 2x$

63. $\displaystyle\int \frac{dx}{\sin^4 x\cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

64. $\displaystyle\int \frac{dx}{\sin^4 x\cos^4 x} = -8\cot 2x - \frac{8}{3}\cot^3 2x$

2.527

1. $$\int \tan^p x\,dx = \frac{\tan^{p-1} x}{p-1} - \int \tan^{p-2} x\,dx \qquad [p \neq 1]$$

2. $$\int \tan^{2n+1} x\,dx = \sum_{k=1}^{n} (-1)^{n+k} \binom{n}{k} \frac{1}{2k\cos^{2k} x} - (-1)^n \ln\cos x$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}\tan^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln\cos x$$

3. $$\int \tan^{2n} x\,dx = \sum_{k=1}^{n} (-1)^{k-1} \frac{\tan^{2n-2k+1} x}{2n-2k+1} + (-1)^n x$$ GU (331)(12)

4. $$\int \cot^p x\,dx = -\frac{\cot^{p-1} x}{p-1} - \int \cot^{p-2} x\,dx \qquad [p \neq 1]$$

5. $$\int \cot^{2n+1} x\,dx = \sum_{k=1}^{n} (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k\sin^{2k} x} + (-1)^n \ln\sin x$$
$$= \sum_{k=1}^{n} (-1)^k \frac{\cot^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln\sin x$$

6. $$\int \cot^{2n} x\,dx = \sum_{k=1}^{n} (-1)^k \frac{\cot^{2n-2k+1} x}{2n-2k+1} + (-1)^n x$$ GU (331)(14)

For special formulas for $p = 1, 2, 3, 4$, see **2.526** 17, **2.526** 33, **2.526** 22, **2.526** 38, **2.526** 27, **2.526** 43, **2.526** 32, and **2.526** 48.

2.53–2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.531

1. $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$$

2. $$\int \cos(ax+b)\,dx = -\frac{1}{a}\sin(ax+b)$$

2.532

1. $$\int \sin(ax+b)\sin(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)}$$
$$\left[a^2 \neq c^2\right]$$

2.[8] $$\int \sin(ax+b)\cos(cx+d)\,dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)}$$
$$\left[a^2 \neq c^2\right]$$

3. $$\int \cos(ax+b)\cos(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)} \qquad \left[a^2 \neq c^2\right]$$

For $c = a$:

4. $$\int \sin(ax+b)\sin(ax+d)\,dx = \frac{x}{2}\cos(b-d) - \frac{\sin(2ax+b+d)}{4a}$$

5. $$\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d)}{4a}$$

6. $$\int \cos(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\cos(b-d) + \frac{\sin(2ax+b+d)}{4a}$$ GU (332)(3)

2.533

1.[8] $$\int \sin ax \cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \qquad \left[a^2 \neq b^2\right]$$

2.[8] $$\int \sin ax \sin bx \sin cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right\}$$

PE (376)

3. $$\int \sin ax \cos bx \cos cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right\}$$

PE (378)

4. $$\int \cos ax \sin bx \sin cx\,dx = \frac{1}{4}\left\{\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right\}$$

PE (379)

5. $$\int \cos ax \cos bx \cos cx\,dx = \frac{1}{4}\left\{\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c}\right\}$$

PE (377)

2.534

1. $$\int \frac{\cos px + i\sin px}{\sin nx}\,dx = -2\int \frac{z^{p+n-1}}{1-z^{2n}}\,dz \qquad [z=\cos x + i\sin x]$$ Pe (374)

2. $$\int \frac{\cos px + i\sin px}{\cos nx}\,dx = -2i\int \frac{z^{p+n-1}}{1-z^{2n}}\,dz \qquad [z=\cos x + i\sin x]$$ Pe (373)

2.535

1. $$\int \sin^p x \sin ax\,dx = \frac{1}{p+a}\left\{-\sin^p x\cos ax + p\int \sin^{p-1} x\cos(a-1)x\,dx\right\}$$ GU (332)(5a)

2. $$\int \sin^p x \sin(2n+1)x\,dx$$

$$= (2n+1)\left\{\int \sin^{p+1} x\,dx + \sum_{k=1}^{n}(-1)^k \frac{\left[(2n+1)^2-1^2\right]\left[(2n+1)^2-3^2\right]\ldots \ldots\left[(2n+1)^2-(2k-1)^2\right]}{(2k+1)!} \times \int \sin^{2k+p+1} x\,dx\right\}$$

TI (299)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\dfrac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^{k-1}\,\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)}\sin^{p-2k} x\cos(2n-2k+1)x + (-1)^k\frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)}\sin^{p-2k-1} x\sin(2n-2k)x\right] + \frac{(-1)^n\,\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p-2n+1)}\int \sin^{p-2n+1} x\,dx\right\}$$

GU (332)(5c)

3. $$\int \sin^p x \sin 2nx\, dx = 2n \left\{ \frac{\sin^{p+2} x}{p+2} \right.$$

$$\left. + \sum_{k=1}^{n-1} (-1)^k \frac{\left(4n^2 - 2^2\right)\left(4n^2 - 4^2\right) \ldots \left[4n^2 - (2k)^2\right]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x \right\}$$

TI (303)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k-1}\, \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1}\, \Gamma(p - 2k + 1)} \sin^{p-2k} x \cos(2n - 2k)x \right.$$

$$\left. - \frac{(-1)^k\, \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2}\, \Gamma(p - 2k)} \sin^{p-2k-1} x \sin(2n - 2k - 1)x \right\}$$

[p is not equal to $-2, -4, \ldots, -2n$]

GU (332)(5c)

2.536

1. $$\int \sin^p x \cos ax\, dx = \frac{1}{p+1} \left\{ \sin^p x \sin ax - p \int \sin^{p-1} x \sin(a-1)x\, dx \right\}$$ GU (332)(6a)

2. $$\int \sin^p x \cos(2n+1)x\, dx$$

$$= \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^{n} (-1)^k \frac{\left[(2n+1)^2 - 1^2\right]\left[(2n+1)^2 - 3^2\right] \ldots \left[(2n+1)^2 - (2k-1)^2\right]}{(2k)!(2k+p+1)}$$

$$\times \sin^{2k+p+1} x$$

TI (301)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k\, \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1}\, \Gamma(p - 2k + 1)} \sin^{p-2k} x \sin(2n - 2k + 1)x \right.\right.$$

$$\left. + \frac{(-1)^k\, \Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2}\, \Gamma(p - 2k)} \sin^{p-2k-1} x \cos(2n - 2k)x \right]$$

$$\left. + \frac{(-1)^n\, \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n}\, \Gamma(p - 2n + 1)} \int \sin^{p-2n} x \cos x\, dx \right\}$$

[p is not equal to $-3, -5, \ldots, -(2n+1)$]

GU (332)(6c)

3. $$\int \sin^p x \cos 2nx\, dx$$

$$= \int \sin^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2 \cdot \left(4n^2 - 2^2\right) \ldots \left[4n^2 - (2k-2)^2\right]}{(2k)!} \int \sin^{2k+p} x\, dx$$

TI (300)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p - 2k + 1)} \sin^{p-2k} x \sin(2n - 2k)x \right.\right.$$

$$\left. + \frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p - 2k)} \sin^{p-2k-1} x \cos(2n - 2k - 1)x \right]$$

$$\left. + \frac{(-1)^n \Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n} \Gamma(p - 2n + 1)} \int \sin^{p-2n} x\, dx \right\}$$

GU (332)(6c)

2.537

1. $$\int \cos^p x \sin ax\, dx = \frac{1}{p+a} \left\{ -\cos^p x \cos ax + p \int \cos^{p-1} x \sin(a-1)x\, dx \right\}$$ GU (332)(7a)

2. $$\int \cos^p x \sin(2n+1)x\, dx$$

$$= (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} \right.$$

$$\left. + \sum_{k=1}^{n} (-1)^k \frac{\left[(2n+1)^2 - 1^2\right] \left[(2n+1)^2 - 3^2\right] \ldots \left[(2n+1)^2 - (2k-1)^2\right]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x \right\}$$

TI (295)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ -\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{2k+1} \Gamma(p - 2k + 1)} \cos^{p-k} x \cos(2n - k + 1)x \right.$$

$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p - n + 1)} \int \cos^{p-n} x \sin(n+1)x\, dx \right\}$$

[p is not equal to $-3, -5, \ldots, -(2n+1)$]

GU (332)(7b)a

3. $$\int \cos^p x \sin 2nx\, dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} \right.$$

$$\left. + \sum_{k=1}^{n-1} (-1)^k \frac{\left(4n^2-2^2\right)\left(4n^2-4^2\right)\ldots\left[4n^2-(2k)^2\right]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\}$$

TI (297)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ -\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \cos(2n-k)x \right.$$

$$\left. + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \sin nx\, dx \right\}$$

$[p$ is not equal to $-2, -4, \ldots, -2n]$

GU (332)(7b)a

2.538

1. $$\int \cos^p x \cos ax\, dx = \frac{1}{p+a} \left\{ \cos^p x \sin ax + p \int \cos^{p-1} x \cos(a-1)x\, dx \right\}$$ GU (332)(8a)

2. $$\int \cos^p x \cos(2n+1)x\, dx$$

$$= (-1)^n (2n+1) \left\{ \int \cos^{p+1} x\, dx \right.$$

$$+ \sum_{k=1}^{n} (-1)^k \frac{\left[(2n+1)^2-1^2\right]\left[(2n+1)^2-3^2\right]\ldots\left[(2n+1)^2-(2k-1)^2\right]}{(2k+1)!}$$

$$\left. \times \int \cos^{2k+p+1} x\, dx \right\}$$

TI (293)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x \right.$$

$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x\, dx \right\}$$

GU (332)(8b)a

3. $$\int \cos^p x \cos 2nx\, dx$$
$$= (-1)^n \left\{ \int \cos^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2\left[4n^2-2^2\right]\ldots\left[4n^2-(2k-2)^2\right]}{(2k)!} \int \cos^{2k+p} x\, dx \right\}$$
TI (294)
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x \right.$$
$$\left. + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \cos nx\, dx \right\}$$
GU (332)(8b)a

2.539

1. $$\int \frac{\sin(2n+1)x}{\sin x}\, dx = 2\sum_{k=1}^{n} \frac{\sin 2kx}{2k} + x$$

2. $$\int \frac{\sin 2nx}{\sin x}\, dx = 2\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$$ GU (332)(5e)

3. $$\int \frac{\cos(2n+1)x}{\sin x}\, dx = 2\sum_{k=1}^{n} \frac{\cos 2kx}{2k} + \ln \sin x$$

4. $$\int \frac{\cos 2nx}{\sin x}\, dx = 2\sum_{k=1}^{n} \frac{\cos(2k-1)x}{2k-1} + \ln \tan \frac{x}{2}$$ GI (332)(6e)

5. $$\int \frac{\sin(2n+1)x}{\cos x}\, dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1} \ln \cos x$$

6. $$\int \frac{\sin 2nx}{\cos x}\, dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1}$$ GU (332)(7d)

7. $$\int \frac{\cos(2n+1)x}{\cos x}\, dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^n x$$

8. $$\int \frac{\cos 2nx}{\cos x}\, dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$ GU (332)(8d)

2.541

1. $$\int \sin(n+1)x \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \sin nx$$ BI (71)(1)a

2. $$\int \sin(n+1)x \cos^{n-1} x\, dx = -\frac{1}{n} \cos^n x \cos nx$$ BI (71)(2)a

3. $$\int \cos(n+1)x \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \cos nx$$ BI (71)(3)a

4. $$\int \cos(n+1)x \cos^{n-1} x\, dx = \frac{1}{n} \cos^n x \sin nx$$ BI (71)(4)a

5. $$\int \sin\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \cos n\left(\frac{\pi}{2} - x\right)$$ BI (71)(5)a

6. $$\int \cos\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x\, dx = -\frac{1}{n} \sin^n x \sin n\left(\frac{\pi}{2} - x\right)$$ BI (71)(6)a

2.542

1. $$\int \frac{\sin 2x}{\sin^n x}\, dx = -\frac{2}{(n-2)\sin^{n-2} x}$$

For $n = 2$:

2. $$\int \frac{\sin 2x}{\sin^2 x}\, dx = 2 \ln \sin x$$

2.543

1. $$\int \frac{\sin 2x\, dx}{\cos^n x} = \frac{2}{(n-2)\cos^{n-2} x}$$

For $n = 2$:

2. $$\int \frac{\sin 2x}{\cos^2 x}\, dx = -2 \ln \cos x$$

2.544

1. $$\int \frac{\cos 2x\, dx}{\sin x} = 2 \cos x + \ln \tan \frac{x}{2}$$

2. $$\int \frac{\cos 2x\, dx}{\sin^2 x} = -\cot x - 2x$$

3. $$\int \frac{\cos 2x\, dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \tan \frac{x}{2}$$

4. $$\int \frac{\cos 2x\, dx}{\cos x} = 2 \sin x - \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

5. $$\int \frac{\cos 2x\, dx}{\cos^2 x} = 2x - \tan x$$

6. $$\int \frac{\cos 2x\, dx}{\cos^3 x} = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

7. $$\int \frac{\sin 3x\, dx}{\sin x} = x + \sin 2x$$

8. $$\int \frac{\sin 3x}{\sin^2 x}\, dx = 3 \ln \tan \frac{x}{2} + 4 \cos x$$

9. $$\int \frac{\sin 3x}{\sin^3 x}\, dx = -3 \cot x - 4x$$

2.545

1. $$\int \frac{\sin 3x}{\cos^n x}\,dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}$$

For $n = 1$ and $n = 3$:

2. $$\int \frac{\sin 3x}{\cos x}\,dx = 2\sin^2 x + \ln\cos x$$

3. $$\int \frac{\sin 3x}{\cos^3 x}\,dx = -\frac{1}{2\cos^2 x} - 4\ln\cos x$$

2.546

1. $$\int \frac{\cos 3x}{\sin^n x}\,dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}$$

For $n = 1$ and $n = 3$:

2. $$\int \frac{\cos 3x}{\sin x}\,dx = -2\sin^2 x + \ln\sin x$$

3. $$\int \frac{\cos 3x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - 4\ln\sin x$$

2.547

1. $$\int \frac{\sin nx}{\cos^p x}\,dx = 2\int \frac{\sin(n-1)x\,dx}{\cos^{p-1} x} - \int \frac{\sin(n-2)x\,dx}{\cos^p x}$$

2. $$\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x$$

3. $$\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

4. $$\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\tan x$$

2.548

1. $$\int \frac{\sin^m x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\sum_{k=0}^{2n}(-1)^{n+k}\cos^m\left[\frac{2k+1}{2(2n+1)}\pi\right]\ln\frac{\sin\left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2}\right]}{\sin\left[\frac{k+n+1}{(2n+1)}\pi - \frac{x}{2}\right]}$$

$[m$ a natural number $\leq 2n]$ TI (378)

2. $$\int \frac{\sin^{2m} x\,dx}{\sin 2nx} = \frac{(-1)^n}{2n}\left\{\ln\cos x + \sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left(\cos^2 x - \sin^2\frac{k\pi}{2n}\right)\right\}$$

$[m$ a natural number $\leq n]$ TI (379)

3. $$\int \frac{\sin^{2m+1} x}{\sin 2nx}\,dx = \frac{(-1)^n}{2n}\left\{\ln\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left[\tan\left(\frac{n+k}{4n}\pi - \frac{x}{2}\right)\tan\left(\frac{n-k}{4n}\pi - \frac{x}{2}\right)\right]\right\}$$

$[m$ a natural number $< n]$

4. $$\int \frac{\sin^{2m} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1}\left\{\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+\sum_{k=1}^{n}(-1)^k \right.$$
$$\left. \times\ \cos^{2m}\frac{k\pi}{2n+1}\ln\left[\tan\left(\frac{2n+2k+1}{4(2n+1)}\pi-\frac{x}{2}\right)\tan\left(\frac{2n-2k+1}{2(2n+1)}\pi-\frac{x}{2}\right)\right]\right\}$$
[m a natural number $\le n$] TI (381)

5. $$\int \frac{\sin^{2m+1} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1}\left\{\ln\cos x+\sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left(\cos^2 x-\sin^2\frac{k\pi}{2n+1}\right)\right\}$$
[m a natural number $\le n$] TI (382)a

6. $$\int \frac{\sin^{m} x\,dx}{\cos 2nx} = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^{n+k}\cos^m\left[\frac{2k+1}{4n}\pi\right]\ln\frac{\sin\left[\frac{2k-2n+1}{8n}\pi+\frac{x}{2}\right]}{\sin\left[\frac{2k+2n+1}{8n}\pi-\frac{x}{2}\right]}$$
[m a natural number $< 2n$] TI (377)

7. $$\int \frac{\cos^{2m+1} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\left\{\ln\sin x+\sum_{k=1}^{n}(-1)^k\cos^{2m+1}\frac{k\pi}{2n+1}\ln\left(\sin^2 x-\sin^2\frac{k\pi}{2n+1}\right)\right\}$$
[m a natural number $\le n$] TI (376)

8. $$\int \frac{\cos^{2m} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\left\{\ln\tan\frac{x}{2}\right.$$
$$\left. +\sum_{k=1}^{n}(-1)^k\cos^{2m}\frac{k\pi}{2n+1}\ln\left[\tan\left(\frac{x}{2}+\frac{k\pi}{4n+2}\right)\tan\left(\frac{x}{2}-\frac{k\pi}{4n+2}\right)\right]\right\}$$
[m a natural number $\le n$] TI (375)

9. $$\int \frac{\cos^{2m+1} x}{\sin 2nx}\,dx = \frac{1}{2n}\left\{\ln\tan\frac{x}{2}+\sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left[\tan\left(\frac{x}{2}+\frac{k\pi}{4}\right)\tan\left(\frac{x}{2}-\frac{k\pi}{4n}\right)\right]\right\}$$
[m a natural number $< n$] TI (374)

10. $$\int \frac{\cos^{2m} x}{\sin 2nx}\,dx = \frac{1}{2n}\left\{\ln\sin x+\sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left(\sin^2 x-\sin^2\frac{k\pi}{2n}\right)\right\}$$
[m a natural number $\le n$] TI (373)

11. $$\int \frac{\cos^{m} x}{\cos nx}\,dx = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k\cos^m\frac{2k+1}{2n}\pi\ln\frac{\sin\left[\frac{2k+1}{4n}\pi+\frac{x}{2}\right]}{\sin\left[\frac{2k+1}{4n}\pi-\frac{x}{2}\right]}$$
[m is a natural number $\le n$] TI (372)

2.549

1. $$\int \sin x^2\,dx = \sqrt{\frac{\pi}{2}}\,S(x)$$

2. $$\int \cos x^2\,dx = \sqrt{\frac{\pi}{2}}\,C(x)$$

3.[11] $$\int \sin\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{ac-b^2}{a}\,S\left(\frac{ax+b}{\sqrt{a}}\right)+\sin\frac{ac-b^2}{a}\,C\left(\frac{ax+b}{\sqrt{a}}\right)\right\}$$

$[a>0]$

4.[11] $$\int \cos\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{ac-b^2}{a}\,C\left(\frac{ax+b}{\sqrt{a}}\right)-\sin\frac{ac-b^2}{a}\,S\left(\frac{ax+b}{\sqrt{a}}\right)\right\}$$

$[a>0]$

5. $$\int \sin\ln x\,dx = \frac{x}{2}\left(\sin\ln x-\cos\ln x\right)$$ PE (444)

6. $$\int \cos\ln x\,dx = \frac{x}{2}\left(\sin\ln x+\cos\ln x\right)$$ PE (445)

2.55–2.56 Rational functions of the sine and cosine

2.551

1. $$\int \frac{A+B\sin x}{(a+b\sin x)^n}\,dx = \frac{1}{(n-1)\left(a^2-b^2\right)}\left[\frac{(Ab-aB)\cos x}{(a+b\sin x)^{n-1}} + \int \frac{(Aa-Bb)(n-1)+(aB-bA)(n-2)\sin x}{(a+b\sin x)^{n-1}}\,dx\right]$$

TI (358)a

For $n=1$:

2. $$\int \frac{A+B\sin x}{a+b\sin x}\,dx = \frac{B}{b}x+\frac{Ab-aB}{b}\int\frac{dx}{a+b\sin x}$$ (see **2.551** 3) TI (342)

3. $$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} \qquad \left[a^2>b^2\right]$$
$$= \frac{1}{\sqrt{b^2-a^2}}\ln\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \qquad \left[a^2<b^2\right]$$

2.552

1. $$\int \frac{A+B\cos x}{(a+b\sin x)^n}\,dx = -\frac{B}{(n-1)b\,(a+b\sin x)^{n-1}}+A\int\frac{dx}{(a+b\sin x)^n}$$

(see **2.552** 3) TI (361)

For $n=1$:

2. $$\int \frac{A+B\cos x}{a+b\sin x}\,dx = \frac{B}{b}\ln\left(a+b\sin x\right)+A\int\frac{dx}{a+b\sin x}$$

(see **2.551** 3) TI (344)

3. $$\int \frac{dx}{(a+b\sin x)^n} = \frac{1}{(n-1)(a^2-b^2)}\left[\frac{b\cos x}{(a+b\sin x)^{n-1}} + \int \frac{(n-1)a-(n-2)b\sin x}{(a+b\sin x)^{n-1}}\,dx\right] \qquad \text{(see } \mathbf{2.551}\ 1)$$

TI (359)

2.553

1. $$\int \frac{A+B\sin x}{(a+b\cos x)^n}\,dx = \frac{B}{(n-1)b\,(a+b\cos x)^{n-1}} + A\int \frac{dx}{(a+b\cos x)^n}$$

(see **2.554** 3) TI (355)

For $n=1$:

2. $$\int \frac{A+B\sin x}{a+b\cos x}\,dx = -\frac{B}{b}\ln(a+b\cos x) + A\int \frac{dx}{a+b\cos x}$$

(see **2.553** 3*) TI (343)

3.* $$\int \frac{dx}{a+b\cos x}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\arctan\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right) \qquad \left[a^2>b^2\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}}\ln\left|\frac{(b-a)\tan\left(\frac{x}{2}\right)+\sqrt{b^2-b^a}}{(b-a)\tan\left(\frac{x}{2}\right)-\sqrt{b^2-b^a}}\right| \qquad \left[b^2>a^2\right]$$

$$= \frac{2}{\sqrt{b^2-a^2}}\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right) \qquad \left[b^2>a^2,\quad \left|(b-a)\tan\left(\frac{x}{2}\right)\right|<\sqrt{b^2-a^2}\right]$$

$$= \frac{2}{\sqrt{b^2-a^2}}\operatorname{arccoth}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right) \qquad \left[b^2>a^2,\quad \left|(b-a)\tan\left(\frac{x}{2}\right)\right|>\sqrt{b^2-a^2}\right]$$

(compare with **2.551** 3)

2.554

1. $$\int \frac{A+B\cos x}{(a+b\cos x)^n}\,dx = \frac{1}{(n-1)(a^2-b^2)}\left[\frac{(aB-Ab)\sin x}{(a+b\cos x)^{n-1}} + \int \frac{(Aa-bB)(n-1)+(n-2)(aB-bA)\cos x}{(a+b\cos x)^{n-1}}\,dx\right]$$

TI (353)

For $n = 1$:

2. $$\int \frac{A + B\cos x}{a + b\cos x}\,dx = \frac{B}{b}x + \frac{Ab - aB}{b}\int \frac{dx}{a + b\cos x} \qquad \text{(see } \mathbf{2.553}\ 3) \qquad \text{TI (341)}$$

3. $$\int \frac{dx}{(a + b\cos x)^n} = -\frac{1}{(n-1)\,(a^2 - b^2)}\left\{\frac{b\sin x}{(a + b\cos x)^{n-1}} - \int \frac{(n-1)a - (n-2)b\cos x}{(a + b\cos x)^{n-1}}\,dx\right\} \qquad \text{(see } \mathbf{2.554}\ 1)$$

TI (354)

In integrating the functions in formulas **2.551** 3 and **2.553** 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points $x = \arcsin\left(-\frac{a}{b}\right)$ in formula **2.551** 3 or over the points $x = \arccos\left(-\frac{a}{b}\right)$ in formula **2.553** 3.

2.555 Formulas **2.551** 3 and **2.553** 3 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

1. $$\int \frac{A + B\sin x}{(1 \pm \sin x)^n}\,dx = -\frac{1}{2^{n-1}}\left\{2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1} \pm (A \mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1}\right.$$

TI (361)a

2. $$\int \frac{A + B\cos x}{(1 \pm \cos x)^n}\,dx = \frac{1}{2^{n-1}}\left\{2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \pm (A \mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1}\right\}$$

TI (356)

For $n = 1$:

3.[11] $$\int \frac{A + B\sin x}{1 \pm \sin x}\,dx = \pm Bx + (B \mp A)\tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right) \qquad \text{TI (250)}$$

4. $$\int \frac{A + B\cos x}{1 \pm \cos x}\,dx = \pm Bx \pm (A \mp B)\tan\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right] \qquad \text{TI (248)}$$

2.556

1. $$\int \frac{\left(1 - a^2\right)\,dx}{1 - 2a\cos x + a^2} = 2\arctan\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right) \qquad [0 < a < 1, \quad |x| < \pi] \qquad \text{FI II 93}$$

2. $$\int \frac{(1 - a\cos x)\,dx}{1 - 2a\cos x + a^2} = \frac{x}{2} + \arctan\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right) \qquad [0 < a < 1, \quad |x| < \pi] \qquad \text{FI II 93}$$

2.557

1. $$\int \frac{dx}{(a\cos x + b\sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}}\int \frac{dx}{\sin^n\left(x + \arctan\frac{a}{b}\right)}$$

(see **2.515**) MZ 173a

2.[6] $$\int \frac{\sin x\,dx}{a\sin x + b\cos x} = \frac{ax - b\ln\sin\left(x + \arctan\frac{b}{a}\right)}{a^2+b^2}$$

3. $$\int \frac{\cos x\,dx}{a\cos x + b\sin x} = \frac{ax + b\ln\sin\left(x + \arctan\frac{a}{b}\right)}{a^2+b^2}$$ MZ 174a

4. $$\int \frac{dx}{a\cos x + b\sin x} = \frac{\ln\tan\left[\frac{1}{2}\left(x + \arctan\frac{a}{b}\right)\right]}{\sqrt{a^2+b^2}}$$

5. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = -\frac{\cot\left(x + \arctan\frac{a}{b}\right)}{a^2+b^2} = +\frac{1}{a^2+b^2}\cdot\frac{a\sin x - b\cos x}{a\cos x + b\sin x}$$ MZ 174a

2.558

1. $$\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}\,dx$$
$$= \frac{(Bc - Cb) + (Ac - Ca)\cos x - (Ab - Ba)\sin x}{(n-1)(a^2-b^2-c^2)(a + b\cos x + c\sin x)^{n-1}} + \frac{1}{(n-1)(a^2-b^2-c^2)}$$
$$\times\int \frac{(n-1)(Aa - Bb - Cc) - (n-2)\left[(Ab - Ba)\cos x - (Ac - Ca)\sin x\right]}{(a + b\cos x + c\sin x)^{n-1}}\,dx$$
$$\left[n \neq 1, \quad a^2 \neq b^2 + c^2\right]$$
$$= \frac{Cb - Bc + Ca\cos x - Ba\sin x}{(n-1)a(a + b\cos x + c\sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right)(-c\cos x + b\sin x)$$
$$\times\frac{(n-1)!}{(2n-1)!!}\sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!a^k}\cdot\frac{1}{(a + b\cos x + c\sin x)^{n-k}}$$
$$\left[n \neq 1, \quad a^2 = b^2 + c^2\right]$$

For $n = 1$:

2.[11] $$\int \frac{A + B\cos x + C\sin x}{a + b\cos x + c\sin x}\,dx = \frac{Bc - Cb}{b^2+c^2}\ln(a + b\cos x + c\sin x) + \frac{Bb + Cc}{b^2+c^2}x$$
$$+\left(A - \frac{Bb + Cc}{b^2+c^2}a\right)\int\frac{dx}{a + b\cos x + c\sin x}$$ (see **2.558** 4)

GU (331)(18)

3. $$\int \frac{dx}{(a + b\cos x + c\sin x)^n} = \int \frac{d(x-\alpha)}{[a + r\cos(x-\alpha)]^n},$$
where $b = r\cos\alpha$, $c = r\sin\alpha$ (see **2.554** 3)

4. $$\int \frac{dx}{a + b\cos x + c\sin x}$$
$$= \frac{2}{\sqrt{a^2-b^2-c^2}}\arctan\frac{(a-b)\tan\frac{x}{2} + c}{\sqrt{a^2-b^2-c^2}} \qquad \left[a^2 > b^2 + c^2\right]$$ TI (253), FI II 94
$$= \frac{1}{\sqrt{b^2+c^2-a^2}}\ln\frac{(a-b)\tan\frac{x}{2} + c - \sqrt{b^2+c^2-a^2}}{(a-b)\tan\frac{x}{2} + c + \sqrt{b^2+c^2-a^2}} \qquad \left[a^2 < b^2 + c^2\right]$$ TI (253)a
$$= \frac{1}{c}\ln\left(a + c\cdot\tan\frac{x}{2}\right) \qquad [a = b]$$
$$= \frac{-2}{c + (a-b)\tan\frac{x}{2}} \qquad \left[a^2 = b^2 + c^2\right]$$ TI (253)a

2.559

1. $$\int \frac{dx}{[a(1+\cos x)+c\sin x]^2} = \frac{1}{c^3}\left[\frac{c(a\sin x - c\cos x)}{a(1+\cos x)+c\sin x} - a\ln\left(a + c\tan\frac{x}{2}\right)\right]$$

2. $$\int \frac{A + B\cos x + C\sin x}{(a_1 + b_1\cos x + c_1\sin x)(a_2 + b_2\cos x + c_2\sin x)}\,dx$$
$$= A_0 \ln\frac{a_1 + b_1\cos x + c_1\sin x}{a_2 + b_2\cos x + c + 2\sin x} + A_1\int\frac{dx}{a_1 + b_1\cos x + c_1\sin x} + A_2\int\frac{dx}{a_2 + b_2\cos x + c_2\sin x}$$
(see **2.558 4**) GU (331)(19)

where

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad A_1 = \frac{\begin{vmatrix} \begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix} & \begin{vmatrix} A & C \\ a_1 & c_1 \end{vmatrix} & \begin{vmatrix} B & A \\ b_1 & a_1 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2\right]$$

3. $$\int \frac{A\cos^2 x + 2B\sin x\cos x + C\sin^2 x}{a\cos^2 x + 2b\sin x\cos x + c\sin^2 x}\,dx$$
$$= \frac{1}{4b^2 + (a-c)^2}\Big\{[4Bb + (A-C)(a-c)]x + [(A-C)b - \mathrm{B}(a-c)]$$
$$\times \ln\left(a\cos^2 x + 2b\sin x\cos x + c\sin^2 x\right)$$
$$+ \left[2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)\right] f(x)\Big\}$$

where GU (331)(24)

$$f(x) = \frac{1}{2\sqrt{b^2 - ac}}\ln\frac{c\tan x + b - \sqrt{b^2 - ac}}{c\tan x + b + \sqrt{b^2 - ac}} \qquad [b^2 > ac]$$
$$= \frac{1}{\sqrt{ac - b^2}}\arctan\frac{c\tan x + b}{\sqrt{ac - b^2}} \qquad [b^2 < ac]$$
$$= -\frac{1}{c\tan x + b} \qquad [b^2 = ac]$$

2.561

1. $$\int \frac{(A + B\sin x)\,dx}{\sin x\,(a + b\sin x)} = \frac{A}{a}\ln\tan\frac{x}{2} + \frac{Ba - Ab}{a}\int\frac{dx}{a + b\sin x}$$
(see **2.551** 3) TI (348)

2. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(a+b\cos x)} = \frac{A}{a^2-b^2}\left\{a\ln\tan\frac{x}{2} + b\ln\frac{a+b\cos x}{\sin x}\right\} + B\int\frac{dx}{a+b\cos x} \qquad \text{(see } \mathbf{2.553}\ 3)$$

TI (349)

For $a^2 = b^2\,(=1)$:

3. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(a+b\cos x)} = \frac{A}{2}\left\{\ln\tan\frac{x}{2} + \frac{1}{1+\cos x}\right\} + B\tan\frac{x}{2}$$

4. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(1-\cos x)} = \frac{A}{2}\left\{\ln\tan\frac{x}{2} - \frac{1}{1-\cos x}\right\} - B\cot\frac{x}{2}$$

5. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(a+b\sin x)} = \frac{1}{a^2-b^2}\left\{(Aa-Bb)\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - (Ab-aB)\ln\frac{a+b\sin x}{\cos x}\right\}$$

TI (346)

For $a^2 = b^2\,(=1)$:

6. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(1\pm\sin x)} = \frac{A\pm B}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \mp \frac{A\mp B}{2\,(1\pm\sin x)}$$

7. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(a+b\cos x)} = \frac{A}{a}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + \frac{B}{a}\ln\frac{a+b\cos x}{\cos x} - \frac{Ab}{a}\int\frac{dx}{a+b\cos x} \qquad \text{(see } \mathbf{2.553}\ 3)$$

TI (351)a

8. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(a+b\sin x)} = \frac{A}{a}\ln\tan\frac{x}{2} - \frac{B}{a}\ln\frac{a+b\sin x}{\sin x} - \frac{Ab}{a}\int\frac{dx}{a+b\sin x}$$

(see **2.551** 3) TI (352)

9. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(a+b\cos x)} = \frac{1}{a^2-b^2}\left\{(Aa-Bb)\ln\tan\frac{x}{2} + (Ab-Ba)\ln\frac{a+b\cos x}{\sin x}\right\}$$ TI (345)

For $a^2 = b^2\,(=1)$:

10. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(1\pm\cos x)} = \pm\frac{A\mp B}{2\,(1\pm\cos x)} + \frac{A\pm B}{2}\ln\tan\frac{x}{2}$$

11. $$\int \frac{(A+B\cos x)\,dx}{\cos x\,(a+b\sin x)} = \frac{A}{a^2-b^2}\left\{a\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - b\ln\frac{a+b\sin x}{\cos x}\right\} + B\int\frac{dx}{a+b\sin x}$$

(see **2.551** 3) TI (350)

For $a^2 = b^2\,(=1)$:

12. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(1\pm\sin x)} = \frac{A\pm B}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \mp \frac{A\mp B}{2\,(1\pm\sin x)}$$

13. $$\int \frac{(A+B\cos x)\,dx}{\cos x\,(a+b\cos x)} = \frac{A}{a}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + \frac{Ba-Ab}{a}\int\frac{dx}{a+b\cos x}$$

(see **2.553** 3) TI (347)

2.562

1. $$\int \frac{dx}{a+b\sin^2 x} = \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan\left(\sqrt{\frac{a+b}{a}}\tan x\right) \qquad \left[\frac{b}{a} > -1\right]$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh}\left(\sqrt{-\frac{a+b}{a}}\tan x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2 x < -\frac{a}{b}\right]$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth}\left(\sqrt{-\frac{a+b}{a}}\tan x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2 x > -\frac{a}{b}\right]$$

MZ 155

2. $$\int \frac{dx}{a+b\cos^2 x} = \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan\left(\sqrt{\frac{a+b}{a}}\cot x\right) \qquad \left[\frac{b}{a} > -1\right]$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh}\left(\sqrt{-\frac{a+b}{a}}\cot x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b}\right]$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth}\left(\sqrt{-\frac{a+b}{a}}\cot x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b}\right]$$

MZ 162

3. $$\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right)$$

4. $$\int \frac{dx}{1-\sin^2 x} = \tan x$$

5. $$\int \frac{dx}{1+\cos^2 x} = -\frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\cot x\right)$$

6. $$\int \frac{dx}{1-\cos^2 x} = -\cot x$$

2.563

1. $$\int \frac{dx}{\left(a+b\sin^2 x\right)^2} = \frac{1}{2a(a+b)}\left[(2a+b)\int \frac{dx}{a+b\sin^2 x} + \frac{b\sin x\cos x}{a+b\sin^2 x}\right]$$

(see **2.562** 1) MZ 155

2. $$\int \frac{dx}{\left(a+b\cos^2 x\right)^2} = \frac{1}{2a(a+b)}\left[(2a+b)\int \frac{dx}{a+b\cos^2 x} - \frac{b\sin x\cos x}{a+b\cos^2 x}\right]$$

(see **2.562** 2) MZ 163

3. $$\int \frac{dx}{\left(a+b\sin^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\tan x\right)\right.$$
$$\left.+\left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\tan x}{1+p^2\tan^2 x}+\left(1-\frac{2}{p^2}-\frac{1}{p^2}\tan^2 x\right)\frac{2p\tan x}{\left(1+p^2\tan^2 x\right)^2}\right]$$
$$\left[p^2 = 1+\frac{b}{a} > 0\right]$$
$$= \frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\tan x\right)\right.$$
$$\left.+\left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\tan x}{1-q^2\tan^2 x}+\left(1+\frac{2}{q^2}+\frac{1}{q^2}\tan^2 x\right)\frac{2q\tan x}{\left(1-q^2\tan^2 x\right)^2}\right]$$
$$\left[q^2 = -1-\frac{b}{a} > 0, \quad \sin^2 x < -\frac{a}{b}; \quad \text{for } \sin^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}\left(q\tan x\right) \text{ to } \operatorname{arccoth}\left(q\tan x\right)\right]$$

MZ 156

4. $$\int \frac{dx}{\left(a+b\cos^2 x\right)^3} = -\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\cot x\right)\right.$$
$$\left.+\left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\cot x}{1+p^2\cot^2 x}+\left(1-\frac{2}{p^2}-\frac{1}{p^2}\cot^2 x\right)\frac{2p\cot x}{\left(1+p^2\cot^2 x\right)^2}\right]$$
$$\left[p^2 = 1+\frac{b}{a} > 0\right]$$
$$= -\frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\cot x\right)\right.$$
$$\left.+\left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\cot x}{1-q^2\cot^2 x}+\left(1+\frac{2}{q^2}+\frac{1}{q^2}\cot^2 x\right)\frac{2p\cot x}{\left(1-q^2\cot^2 x\right)^2}\right]$$
$$\left[q^2 = -1-\frac{b}{a} > 0, \quad \cos^2 x < -\frac{a}{b}; \quad \text{for } \cos^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}\left(q\cot x\right) \text{ to } \operatorname{arccoth}\left(q\cot x\right)\right]$$

MZ 163a

2.564

1. $$\int \frac{\tan x\,dx}{1+m^2\tan^2 x} = \frac{\ln\left(\cos^2 x+m^2\sin^2 x\right)}{2\left(m^2-1\right)}$$ LA 210 (10)

2. $$\int \frac{\tan\alpha-\tan x}{\tan\alpha+\tan x}\,dx = \sin 2\alpha\ln\sin(x+\alpha)-x\cos 2\alpha$$ LA 210 (11)a

3. $$\int \frac{\tan x\,dx}{a+b\tan x} = \frac{1}{a^2+b^2}\left\{bx-a\ln\left(a\cos x+b\sin x\right)\right\}$$ PE (335)

4. $$\int \frac{dx}{a+b\tan^2 x} = \frac{1}{a-b}\left[x-\sqrt{\frac{b}{a}}\arctan\left(\sqrt{\frac{b}{a}}\tan x\right)\right]$$ PE (334)

2.57 Integrals containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$

Notation:

$$\alpha = \arcsin \sqrt{\frac{1 - \sin x}{2}}, \qquad \beta = \arcsin \sqrt{\frac{b\,(1 - \sin x)}{a + b}},$$

$$\gamma = \arcsin \sqrt{\frac{b\,(1 - \cos x)}{a + b}}, \qquad \delta = \arcsin \sqrt{\frac{(a + b)\,(1 - \cos x)}{2\,(a - b \cos x)}}, \qquad r = \sqrt{\frac{2b}{a + b}}$$

2.571

1. $$\int \frac{dx}{\sqrt{a + b \sin x}} = \frac{-2}{\sqrt{a + b}}\, F(\alpha, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
$$= -\sqrt{\frac{2}{b}}\, F\left(\beta, \frac{1}{r}\right) \qquad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.00, 288.50)

2. $$\int \frac{\sin x\, dx}{\sqrt{a + b \sin x}}$$
$$= \frac{2a}{b\sqrt{a + b}}\, F(\alpha, r) - \frac{2\sqrt{a + b}}{b}\, E(\alpha, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right] \qquad \text{BY (288.03)}$$
$$= \sqrt{\frac{2}{b}} \left\{ F\left(\beta, \frac{1}{r}\right) - 2\, E\left(\beta, \frac{1}{r}\right) \right\} \qquad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right] \qquad \text{BY (288.54)}$$

3. $$\int \frac{\sin^2 x\, dx}{\sqrt{a + b \sin x}} = \frac{4a\sqrt{a + b}}{3b^2}\, E(\alpha, r) - \frac{2\left(2a^2 + b^2\right)}{3b^2\sqrt{a + b}}\, F(\alpha, r) - \frac{2}{3b} \cos x \sqrt{a + b \sin x}$$
$$\left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
$$= \sqrt{\frac{2}{b}} \left\{ \frac{4a}{3b}\, E\left(\beta, \frac{1}{r}\right) - \frac{2a + b}{3b}\, F\left(\beta, \frac{1}{r}\right) \right\} - \frac{2}{3b} \cos x \sqrt{a + b \sin x}$$
$$\left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.03, 288.54)

4. $$\int \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{\sqrt{a + b}}\, F\left(\frac{x}{2}, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$
$$= \sqrt{\frac{2}{b}}\, F\left(\gamma, \frac{1}{r}\right) \qquad \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$
BY (289.00)

5. $$\int \frac{dx}{\sqrt{a - b \cos x}} = \frac{2}{\sqrt{a + b}}\, F(\delta, r) \qquad [a > b > 0, \quad 0 \le x \le \pi] \qquad \text{BY (291.00)}$$

6. $$\int \frac{\cos x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{b\sqrt{a+b}}\left\{(a+b)\,E\left(\frac{x}{2},r\right) - a\,F\left(\frac{x}{2},r\right)\right\}$$
$[a>b>0, \quad 0\le x\le \pi]$
BY (289.03)

$$= \sqrt{\frac{2}{b}}\left\{2\,E\left(\gamma,\frac{1}{r}\right) - F\left(\gamma,\frac{1}{r}\right)\right\}$$
$\left[b>|a|>0, \quad 0\le x<\arccos\left(-\frac{a}{b}\right)\right]$
BY (290.04)

7.[6] $$\int \frac{\cos x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{b\sqrt{a+b}}\left\{(b-a)\,\Pi\left(\delta,r^2,r\right) + a\,F(\delta,r)\right\}$$
$[a>b>0, \quad 0\le x\le \pi]$ BY (291.03)

8. $$\int \frac{\cos^2 x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\left\{\left(2a^2+b^2\right)F\left(\frac{x}{2},r\right) - 2a(a+b)\,E\left(\frac{x}{2},r\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x}$$
$[a>b>0, \quad 0\le x\le \pi]$
BY (289.03)

$$= \frac{1}{3b}\sqrt{\frac{2}{b}}\left\{(2a+b)\,F\left(\gamma,\frac{1}{r}\right) - 4a\,E\left(\gamma,\frac{1}{r}\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x}$$
$\left[b\ge|a|>0, \quad 0\le x<\arccos\left(-\frac{a}{b}\right)\right]$
BY (290.04)

9. $$\int \frac{\cos^2 x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\left\{\left(2a^2+b^2\right)F(\delta,r) - 2a(a+b)\,E(\delta,r)\right\}$$
$$+\frac{2}{3b}\sin x\frac{a+b\cos x}{\sqrt{a-b\cos x}}\,[a>b>0,]$$
BY (291.04)a

2.572 $$\int \frac{\tan^2 x\,dx}{\sqrt{a+b\sin x}}$$
$$= \frac{1}{\sqrt{a+b}}\,F(\alpha,r) + \frac{a}{(a-b)\sqrt{a+b}}\,E(\alpha,r)$$
$$-\frac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x} \qquad \left[0<b<a, -\frac{\pi}{2}<x<\frac{\pi}{2}\right]$$
$$= \sqrt{\frac{2}{b}}\left\{\frac{2a+b}{2(a+b)}\,F\left(\beta,\frac{1}{r}\right) + \frac{ab}{a^2-b^2}\,E\left(\beta,\frac{1}{r}\right)\right\}$$
$$-\frac{b-a\sin x}{(a^2-b^2)\cos x}\sqrt{a+b\sin x} \qquad \left[0<|a|<b, \quad -\arcsin\frac{a}{b}<x<\frac{\pi}{2}\right]$$
BY(288.08, 288.58)

2.573

1. $$\int \frac{1-\sin x}{1+\sin x}\cdot\frac{dx}{\sqrt{a+b\sin x}} = \frac{2}{a-b}\left\{\sqrt{a+b}\,E(\alpha,r)\right\} - \tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\sqrt{a+b\sin x}\Bigg\}$$
$\left[0<b<a, \quad -\frac{\pi}{2}\le x<\frac{\pi}{2}\right]$ BY (288.07)

2. $$\int \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{a-b} \tan \frac{x}{2} \sqrt{a+b\cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, r\right)$$
$$[a > b > 0, \quad 0 \le x < \pi]$$ BY (289.07)

2.574

1. $$\int \frac{dx}{(2-p^2+p^2\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b} \Pi\left(\alpha, p^2, r\right)$$
$$\left[0 < b < a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
BY (288.02)

2. $$\int \frac{dx}{(a+b-p^2b+p^2b\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b}\sqrt{\frac{2}{b}} \Pi\left(\beta, p^2, \frac{1}{r}\right)$$
$$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.52)

3. $$\int \frac{dx}{(2-p^2+p^2\cos x)\sqrt{a+b\cos x}} = \frac{1}{\sqrt{a+b}} \Pi\left(\frac{x}{2}, p^2, r\right)$$
$$[a > b > 0, \quad 0 \le x < \pi]$$ BY (289.02)

4. $$\int \frac{dx}{(a+b-p^2b+p^2b\cos x)\sqrt{a+b\cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}} \Pi\left(\gamma, p^2, \frac{1}{r}\right)$$
$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$
BY (290.02)

2.575

1. $$\int \frac{dx}{\sqrt{(a+b\sin x)^3}} = \frac{2b\cos x}{(a^2-b^2)\sqrt{a+b\sin x}} - \frac{2}{(a-b)\sqrt{a+b}} E(\alpha, r)$$
$$\left[0 < b < a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
BY (288.05)
$$= \sqrt{\frac{2}{b}} \left\{ \frac{2b}{b^2-a^2} E\left(\beta, \frac{1}{r}\right) - \frac{1}{a+b} F\left(\beta, \frac{1}{r}\right) \right\} + \frac{2b}{b^2-a^2} \cdot \frac{\cos x}{\sqrt{a+b\sin x}}$$
$$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.56)

2. $$\int \frac{dx}{\sqrt{(a+b\sin x)^5}} = \frac{2}{3\left(a^2-b^2\right)^2\sqrt{a+b}}\left\{\left(a^2-b^2\right)F(\alpha,r)-4a(a+b)\,E(\alpha,r)\right\}$$
$$+\frac{2b\left(5a^2-b^2+4ab\sin x\right)}{3\left(a^2-b^2\right)^2\sqrt{(a+b\sin x)^3}}\cos x$$
$$\left[0<b<a,\quad -\frac{\pi}{2}\le x<\frac{\pi}{2}\right]$$
BY (288.05)
$$= -\frac{1}{3\left(a^2-b^2\right)^2}\sqrt{\frac{2}{b}}\left\{(3a-b)(a-b)\,F\left(\beta,\frac{1}{r}\right)+8ab\,E\left(\beta,\frac{1}{r}\right)\right\}$$
$$+\frac{2b\left[a^2-b^2+4a\left(a+b\sin x\right)\right]}{3\left(a^2-b^2\right)^2\sqrt{(a+b\sin x)^3}}\cos x$$
$$\left[0<|a|<b,\quad -\arcsin\frac{a}{b}<x<\frac{\pi}{2}\right]$$
BY (288.56)

3. $$\int \frac{dx}{\sqrt{(a+b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E\left(\frac{x}{2},r\right)-\frac{2b}{a^2-b^2}\cdot\frac{\sin x}{\sqrt{a+b\cos x}}$$
$$[a>b>0,\quad 0\le x\le\pi]$$
BY (289.05)
$$= \frac{1}{a^2-b^2}\sqrt{\frac{2}{b}}\left\{(a-b)\,F\left(\gamma,\frac{1}{r}\right)+2b\,E\left(\gamma,\frac{1}{r}\right)\right\}+\frac{2b}{b^2-a^2}\cdot\frac{\sin x}{\sqrt{a+b\cos x}}$$
$$\left[b\ge|a|>0,\quad 0\le x<\arccos\left(-\frac{a}{b}\right)\right]$$
BY (290.06)

4. $$\int \frac{dx}{\sqrt{(a-b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E(\delta,r) \qquad [a>b>0,\quad 0\le x\le\pi] \qquad (291.01)$$

5. $$\int \frac{dx}{\sqrt{(a+b\cos x)^5}} = \frac{2\sqrt{a+b}}{3\left(a^2-b^2\right)^2}\left\{4a\,E\left(\frac{x}{2},r\right)-(a-b)\,F\left(\frac{x}{2},r\right)\right\}$$

$$-\frac{2b}{3\left(a^2-b^2\right)^2}\cdot\frac{5a^2-b^2+4ab\cos x}{\sqrt{(a+b\cos x)^3}}\sin x$$

$[a>b>0, \quad 0\le x\le\pi]$

BY (289.05)

$$=\frac{1}{3\left(a^2-b^2\right)^2}\sqrt{\frac{2}{b}}\left\{(a-b)(3a-b)\,F\left(\gamma,\frac{1}{r}\right)+8ab\,E\left(\gamma,\frac{1}{r}\right)\right\}$$

$$+\frac{2b\left(5a^2-b^2+4ab\cos x\right)\sin x}{3\left(a^b-b^2\right)^2\sqrt{(a+b\cos x)^3}}$$

$\left[b\ge|a|>0, \quad 0\le x<\arccos\left(-\frac{a}{b}\right)\right]$

BY (290.06)

2.576

1. $$\int\sqrt{a+b\cos x}\,dx=2\sqrt{a+b}\,E\left(\frac{x}{2},r\right)$$

$[a>b>0, \quad 0\le x\le\pi]$

BY (289.01)

$$=\sqrt{\frac{2}{b}}\left\{(a-b)\,F\left(\gamma,\frac{1}{r}\right)+2b\,E\left(\gamma,\frac{1}{r}\right)\right\}$$

$\left[b\ge|a|>0, \quad 0\le x<\arccos\left(-\frac{a}{b}\right)\right]$

BY (290.03)

2. $$\int\sqrt{a-b\cos x}\,dx=2\sqrt{a+b}\,E(\delta,r)-\frac{2b\sin x}{\sqrt{a-b\cos x}}$$ $[a>b>0, \quad 0\le x\le\pi]$ BY (291.05)

2.577

1.[3] $$\int\frac{\sqrt{a-b\cos x}}{1+p\cos x}\,dx=\frac{2(a-b)}{(1+p)\sqrt{a+b}}\,\Pi\left(\delta,\frac{2ap}{(a+b)(1+p)},r\right)$$

$[a>b>0, \quad 0\le x\le\pi, \quad p\ne-1]$

BY (291.02)

2.[3] $$\int\sqrt{\frac{a-b\cos x}{1+p\cos x}}\,dx=\frac{2(a-b)}{\sqrt{(1+p)(a+b)}}\,\Pi\left(\delta,-r^2,\sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right)$$

$[a>b>0, \quad 0\le x\le\pi, \quad p\ne-1]$

2.578 $$\int\frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}}=\frac{1}{\sqrt{b-a}}\arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}}\cos x\right)$$ $[b>a, \quad b>0]$ PE (333)

2.58–2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

1. $$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi}} = 2\int \frac{d\psi}{\sqrt{a-p+2p\cos^2\psi}} \qquad \left[\varphi = 2\psi+\alpha, \tan\alpha = \frac{c}{b}, p = \sqrt{b^2+c^2}\right]$$

2. $$\int \frac{d\varphi}{\sqrt{a+b\cos\varphi+c\sin\varphi+d\cos^2\varphi+e\sin\varphi\cos\varphi+f\sin^2\varphi}} = 2\int \frac{dx}{\sqrt{A+Bx+Cx^2-Dx^3+Ex^4}}$$
$$\left[\tan\frac{\varphi}{2} = x, A = a+b+d, B = 2c+2e, C = 2a-2d+4f, D = 2c-2e, E = a-b+d\right]$$

Forms containing $\sqrt{1-k^2\sin^2 x}$

Notation: $\Delta = \sqrt{1-k^2\sin^2 x}$, $k' = \sqrt{1-k^2}$

2.581

1. $$\int \sin^m x\cos^n x\Delta^r\,dx$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m-3}x\cos^{n+1}x\Delta^{r+2} + \left[(m+n-2)+(m+r-1)k^2\right]\right.$$
$$\left.\times\int \sin^{m-2}x\cos^n x\Delta^r\,dx - (m-3)\int \sin^{m-4}x\cos^n x\Delta^r\,dx\right\}$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m+1}x\cos^{n-3}x\Delta^{r+2} + \left[(n+r-1)k^2-(m+n-2)k'^2\right]\right.$$
$$\left.\times\int \sin^m x\cos^{n-2}x\Delta^r\,dx + (n-3)k'^2\int \sin^m x\cos^{n-4}x\Delta^r\,dx\right\}$$
$$[m+n+r\neq 0]$$

For $r=-3$ and $r=-5$:

2. $$\int \frac{\sin^m x\cos^n x}{\Delta^3}\,dx = \frac{\sin^{m-1}x\cos^{n-1}x}{k^2\Delta} - \frac{m-1}{k^2}\int \frac{\sin^{m-2}x\cos^n x}{\Delta}\,dx + \frac{n-1}{k^2}\int \frac{\sin^m x\cos^{n-2}x}{\Delta}\,dx$$

3. $$\int \frac{\sin^m x\cos^n x}{\Delta^5}\,dx = \frac{\sin^{m-1}x\cos^{n-1}x}{3k^2\Delta^3} - \frac{m-1}{3k^2}\int \frac{\sin^{m-2}x\cos^n x}{\Delta^3}\,dx + \frac{n-1}{3k^2}\int \frac{\sin^m x\cos^{n-2}x}{\Delta^3}\,dx$$

For $m=1$ or $n=1$:

4. $$\int \sin x\cos^n x\Delta^r\,dx = -\frac{\cos^{n-1}x\Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1)k'^2}{(n+r+1)k^2}\int \cos^{n-2}x\sin x\Delta^r\,dx$$

5. $$\int \sin^m x\cos x\Delta^r\,dx = -\frac{\sin^{m-1}x\Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2}\int \sin^{m-2}x\cos x\Delta^r\,dx$$

For $m=3$ or $n=3$:

6. $$\int \sin^3 x \cos^n x \Delta^r \, dx = \frac{(n+r+1)k^2 \cos^2 x - \left[(r+2)k^2 + n + 1\right]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x \Delta^{r+2} - \frac{\left[(r+2)k^2 + n + 1\right](n-1)k'^2}{(n+r+1)(n+r+3)k^4} \int \cos^{n-2} x \sin x \Delta^r \, dx$$

7. $$\int \sin^m x \cos^3 x \Delta^r \, dx = \frac{(m+r+1)k^2 \sin^2 x - \left[(r+2)k^2 - (m+1)k'^2\right]}{(m+r+1)(m+r+3)k^4} \times \sin^{m-1} x n^{m-1} x \Delta^{r+2} + \frac{\left[(r+2)k^2 - (m-1)k'^2\right](m-1)}{(m+r+1)(m+r+3)k^4} \int \sin^{m-2} x \cos x \Delta^r \, dx$$

2.582

1. $$\int \Delta^n \, dx = \frac{n-1}{n}\left(2-k^2\right) \int \Delta^{n-2} \, dx - \frac{n-2}{n}\left(1-k^2\right) \int \Delta^{n-4} \, dx + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2}$$

LA (316)(1)a

2. $$\int \frac{dx}{\Delta^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1)k'^2 \Delta^{n-1}} + \frac{n-2}{n-1} \frac{2-k^2}{k'^2} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1} \frac{1}{k'^2} \int \frac{dx}{\Delta^{n-3}}$$

LA 317(8)a

3. $$\int \frac{\sin^n x}{\Delta} \, dx = \frac{\sin^{n-3} x}{(n-1)k^2} \cos x \cdot \Delta + \frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} \, dx - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} \, dx$$

LA 316(1)a

4. $$\int \frac{\cos^n x}{\Delta} \, dx = \frac{\cos^{n-3} x}{(n-1)k^2} \sin x \cdot \Delta + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} \, dx + \frac{n-3}{n-1} \frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} \, dx$$

LA 316(2)a

5. $$\int \frac{\tan^n x}{\Delta} \, dx = \frac{\tan^{n-3} x}{(n-1)k'^2} \frac{\Delta}{\cos^2 x} - \frac{(n-2)\left(2-k^2\right)}{(n-1)k'^2} \int \frac{\tan^{n-2} x}{\Delta} \, dx - \frac{n-3}{(n-1)k'^2} \int \frac{\tan^{n-4} x}{\Delta} \, dx$$

LA 317(3)

6. $$\int \frac{\cot^n x}{\Delta} \, dx = -\frac{\cot^{n-1} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1}\left(2-k^2\right) \int \frac{\cot^{n-2} x}{\Delta} \, dx - \frac{n-3}{n-1} k'^2 \int \frac{\cot^{n-4} x}{\Delta} \, dx$$

LA 317(6)

2.583

1. $$\int \Delta \, dx = E(x,k)$$

2. $$\int \Delta \sin x\, dx = -\frac{\Delta \cos x}{2} - \frac{k'^2}{2k} \ln(k\cos x + \Delta)$$

3. $$\int \Delta \cos x\, dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x)$$

4. $$\int \Delta \sin^2 x\, dx = -\frac{\Delta}{3} \sin x \cos x + \frac{k'^2}{3k^2} F(x,k) + \frac{2k^2-1}{3k^2} E(x,k)$$

5. $$\int \Delta \sin x \cos x\, dx = -\frac{\Delta^3}{3k^2}$$

6. $$\int \Delta \cos^2 x\, dx = \frac{\Delta}{3} \sin x \cos x - \frac{k'^2}{3k^2} F(x,k) + \frac{k^2+1}{3k^2} E(x,k)$$

7. $$\int \Delta \sin^3 x\, dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k\cos x + \Delta)$$

8. $$\int \Delta \sin^2 x \cos x\, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x)$$

9. $$\int \Delta \sin x \cos^2 x\, dx = -\frac{2k^2 \cos^2 x + k'^2}{8k^2} \Delta \cos x + \frac{k'^4}{8k^3} \ln(k\cos x + \Delta)$$

10. $$\int \Delta \cos^3 x\, dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x)$$

11. $$\int \Delta \sin^4 x\, dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x - \frac{2\left(2k^4 - k^2 - 1\right)}{15k^4} F(x,k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x,k)$$

12. $$\int \Delta \sin^3 x \cos x\, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta$$

13. $$\int \Delta \sin^2 x \cos^2 x\, dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x - \frac{k'^2\left(1 + k'^2\right)}{15k^4} F(x,k) + \frac{2\left(k^4 - k^2 + 1\right)}{15k^4} E(x,k)$$

14. $$\int \Delta \sin x \cos^3 x\, dx = -\frac{3k^4 \sin^4 x - k^2\left(5k^2 + 1\right) \sin^2 x + 5k^2 - 2}{15k^4} \Delta$$

15. $$\int \Delta \cos^4 x\, dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x + \frac{2k'^2\left(k'^2 - 2k^2\right)}{15k^4} F(x,k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x,k)$$

16. $$\int \Delta \sin^5 x\, dx = \frac{-8k^4 \sin^4 x - 2k^2\left(5k^2 - 1\right) \sin^2 x - 15k^4 + 4k^2 + 3}{48k^4} \Delta \cos x + \frac{5k^6 - 3k^4 - k^2 - 1}{16k^5} \ln(k\cos x + \Delta)$$

17. $$\int \Delta \sin^4 x \cos x\, dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin(k \sin x)$$

18. $$\int \Delta \sin^3 x \cos^2 x\, dx = \frac{8k^4 \sin^4 x - 2k^2\left(k^2+1\right)\sin^2 x - 3k^4 + 2k^2 - 3}{48k^4}\Delta \cos x + \frac{k'^4\left(k^2+1\right)}{16k^5}\ln\left(k\cos x + \Delta\right)$$

19. $$\int \Delta \sin^2 x \cos^3 x\, dx = \frac{-8k^4 \sin^4 x + 2k^2\left(6k^2+1\right)\sin^2 x - 6k^2 + 3}{48k^4}\Delta \sin x + \frac{2k^2-1}{16k^5}\arcsin\left(k\sin x\right)$$

20. $$\int \Delta \sin x \cos^4 x\, dx = \frac{-8k^4 \sin^4 x + 2k^2\left(7k^2+1\right)\sin^2 x - 3k^4 - 8k^2 + 3}{48k^4}\Delta \cos x - \frac{k'^6}{16k^5}\ln\left(k\cos x + \Delta\right)$$

21. $$\int \Delta \cos^5 x\, dx = \frac{8k^4 \sin^4 x - 2k^2\left(12k^2+1\right)\sin^2 x + 24k^4 + 12k^2 - 3}{48k^4}\Delta \sin x + \frac{8k^4 - 4k^2 + 1}{16k^5}\arcsin\left(k\sin x\right)$$

22. $$\int \Delta^3\, dx = \frac{2}{3}\left(1+k'^2\right)E(x,k) - \frac{k'^2}{3}F(x,F) + \frac{k^2}{3}\Delta \sin x \cos x$$

23. $$\int \Delta^3 \sin x\, dx = \frac{2k^2\sin^2 x + 3k^2 - 5}{8}\Delta \cos x - \frac{3k'^4}{8k}\ln\left(k\cos x + \Delta\right)$$

24. $$\int \Delta^3 \cos x\, dx = \frac{-2k^2\sin^2 x + 5}{8}\Delta \sin x + \frac{3}{8k}\arcsin\left(k\sin x\right)$$

25. $$\int \Delta^3 \sin^2 x\, dx = \frac{3k^2\sin^2 x + 4k^2 - 6}{15}\Delta \sin x \cos x + \frac{k'^2\left(3-4k^2\right)}{15k^2}F(x,k) - \frac{8k^4 - 13k^2 + 3}{15k^2}E(x,k)$$

26. $$\int \Delta^3 \sin x \cos x\, dx = -\frac{\Delta^5}{5k^2}$$

27. $$\int \Delta^3 \cos^2 x\, dx = \frac{-3k^2\sin^2 x + k^2 + 5}{15}\Delta \sin x \cos x - \frac{k'^2\left(k^2+3\right)}{15k^2}F(x,k) - \frac{2k^4 - 7k^2 - 3}{15k^2}E(x,k)$$

28. $$\int \Delta^3 \sin^3 x\, dx = \frac{8k^4\sin^4 x + 2k^2\left(5k^2-7\right)\sin^2 x + 15k^4 - 22k^2 + 3}{48k^2}\Delta \cos x - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3}\ln\left(k\cos x + \Delta\right)$$

29. $$\int \Delta^3 \sin^2 x \cos x\, dx = \frac{-8k^4\sin^4 x + 14k^2\sin^2 x - 3}{48k^2}\Delta \sin x + \frac{1}{16k^3}\arcsin\left(k\sin x\right)$$

30. $$\int \Delta^3 \sin x \cos^2 x\, dx = \frac{-8k^4 \sin^4 x + 2k^2\left(k^2+7\right)\sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \times \Delta \cos x + \frac{k'^6}{16k^3} \ln\left(k\cos x + \Delta\right)$$

31. $$\int \Delta^3 \cos^3 x\, dx = \frac{8k^4 \sin^4 x - 2k^2\left(6k^2+7\right)\sin^2 x + 30k^2 + 3}{48k^2}\Delta \sin x + \frac{6k^2-1}{16k^3} \arcsin\left(k \sin x\right)$$

32. $$\int \frac{\Delta\, dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln k\left(k\cos x + \Delta\right)$$

33. $$\int \frac{\Delta\, dx}{\cos x} = \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} + k \arcsin\left(k \sin x\right)$$

34. $$\int \frac{\Delta\, dx}{\sin^2 x} = k'^2\, F(x,k) - E(x,k) - \Delta \cot x$$

35. $$\int \frac{\Delta\, dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

36. $$\int \frac{\Delta\, dx}{\cos^2 x} = F(x,k) - E(x,k) + \Delta \tan x$$

37. $$\int \frac{\sin x}{\cos x} \Delta\, dx = \int \Delta \tan x\, dx = -\Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

38. $$\int \frac{\cos x}{\sin x} \Delta\, dx = \int \Delta \cot x\, dx = \Delta + \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}$$

39. $$\int \frac{\Delta\, dx}{\sin^3 x} = -\frac{\Delta \cos x}{2\sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

40. $$\int \frac{\Delta\, dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1+k^2}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

41. $$\int \frac{\Delta\, dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

42. $$\int \frac{\Delta\, dx}{\cos^3 x} = \frac{\Delta \sin x}{2\cos^2 x} + \frac{1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

43. $$\int \frac{\Delta \sin x\, dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k \ln\left(k\cos x + \Delta\right)$$

44. $$\int \frac{\Delta \cos x\, dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k \arcsin\left(k \sin x\right)$$

45. $$\int \frac{\Delta \sin^2 x\, dx}{\cos x} = -\frac{\Delta \sin x}{2} + \frac{2k^2-1}{2k} \arcsin\left(k \sin x\right) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

46. $$\int \frac{\Delta \cos^2 x\, dx}{\sin x} = \frac{\Delta \cos x}{2} + \frac{k^2+1}{2k} \ln\left(k\cos x + \Delta\right) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

47. $$\int \frac{\Delta\, dx}{\sin^4 x} = \frac{1}{3}\left\{-\Delta \cot^3 x + \left(k^2-3\right)\Delta \cot x + 2k'^2\, F(x,k) + \left(k^2-2\right) E(x,k)\right\}$$

48. $$\int \frac{\Delta\, dx}{\sin^3 x \cos x} = -\frac{\Delta}{2\sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

49. $$\int \frac{\Delta\, dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{k'^2} \tan x - \cot x \right) \Delta + 2\, F(x, k) - \frac{1 + k'^2}{k'^2}\, E(x, k)$$

50. $$\int \frac{\Delta\, dx}{\sin x \cos^3 x} = \frac{\Delta}{2\cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

51. $$\int \frac{\Delta\, dx}{\cos^4 x} = \frac{1}{3k'^2} \left\{ \left[k'^2 \tan^2 x - \left(2k^2 - 3\right) \tan x \right] \Delta + 2k'^2\, F(x, k) + \left(k^2 - 2\right) E(x, k) \right\}$$

52. $$\int \frac{\sin x}{\cos^3 x} \Delta\, dx = \frac{\Delta}{2\cos^2 x} + \frac{k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

53. $$\int \frac{\cos x}{\sin^3 x} \Delta\, dx = -\frac{\Delta}{2\sin^2 x} + \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

54. $$\int \frac{\sin^2 x}{\cos^2 x} \Delta\, dx = \int \tan^2 x \Delta\, dx = \Delta \tan x + F(x, k) - 2\, E(x, k)$$

55. $$\int \frac{\cos^2 x}{\sin^2 x} \Delta\, dx = \int \cot^2 x \Delta\, dx = -\Delta \cot x + k'^2\, F(x, k) - 2\, E(x, k)$$

56. $$\int \frac{\sin^3 x}{\cos x} \Delta\, dx = -\frac{k^2 \sin^2 x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

57. $$\int \frac{\cos^3 x}{\sin x} \Delta\, dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$

58. $$\int \frac{\Delta\, dx}{\sin^5 x} = \frac{\left(k^2 - 3\right) \sin^2 x + 2}{8 \sin^4 x} \cos x \Delta + \frac{k'^2 \left(k^2 + 3\right)}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

59. $$\int \frac{\Delta\, dx}{\sin^4 x \cos x} = -\frac{\left(3 - k^2\right) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{k'}{2} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

60. $$\int \frac{\Delta\, dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2 - 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

61. $$\int \frac{\Delta\, dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2 - 3}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

62. $$\int \frac{\Delta\, dx}{\sin x \cos^4 x} = \frac{\left(2k^2 - 3\right) \sin^2 x - 3k^2 + 4}{3k'^2 \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

63. $$\int \frac{\Delta\, dx}{\cos^5 x} = \frac{\left(2k^2 - 3\right) \sin^2 x - 4k^2 + 5}{8k'^2 \cos^4 x} \sin x \Delta - \frac{4k^2 - 3}{16k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

64. $$\int \frac{\sin x}{\cos^4 x} \Delta\, dx = \frac{-\left(2k^2 + 1\right) k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta$$

65. $$\int \frac{\cos x}{\sin^4 x} \Delta\, dx = -\frac{\Delta^3}{3 \sin^3 x}$$

66. $$\int \frac{\sin^2 x}{\cos^3 x} \Delta\, dx = \frac{\sin x}{2\cos^2 x} \Delta + \frac{2k^2 - 1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin\left(k \sin x\right)$$

67. $$\int \frac{\cos^2 x}{\sin^3 x}\Delta\, dx = -\frac{\cos x}{2\sin^2 x}\Delta - \frac{k^2+1}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x} - k\ln(k\cos x+\Delta)$$

68. $$\int \frac{\sin^3 x}{\cos^2 x}\Delta\, dx = -\frac{\sin^2 x-3}{2\cos x}\Delta - \frac{3k^2-1}{2k}\ln(k\cos x+\Delta)$$

69. $$\int \frac{\cos^3 x}{\sin^2 x}\Delta\, dx = -\frac{\sin^2 x+2}{2\sin x}\Delta - \frac{2k^2+1}{2k}\arcsin(k\sin x)$$

70. $$\int \frac{\sin^4 x}{\cos x}\Delta\, dx = -\frac{2k^2\sin^2 x+4k^2-1}{8k^2}\Delta\sin x + \frac{8k^4-4k^2-1}{8k^3}\arcsin(k\sin x) + \frac{k'}{2}\ln\frac{\Delta+k'\sin x}{\Delta-k'\sin x}$$

71. $$\int \frac{\cos^4 x}{\sin x}\Delta\, dx = \frac{-2k^2\sin^2 x+5k^2+1}{8k^2}\Delta\cos x + \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} + \frac{3k^4+6k^2-1}{8k^3}\ln(k\cos x+\Delta)$$

2.584

1. $$\int \frac{dx}{\Delta} = F(x,k)$$

2. $$\int \frac{\sin x\, dx}{\Delta} = \frac{1}{2k}\ln\frac{\Delta-k\cos x}{\Delta+k\cos x} = -\frac{1}{k}\ln(k\cos x+\Delta)$$

3. $$\int \frac{\cos x\, dx}{\Delta} = \frac{1}{k}\arcsin(k\sin x) = \frac{1}{k}\arctan\frac{k\sin x}{\Delta}$$

4. $$\int \frac{\sin^2 x\, dx}{\Delta} = \frac{1}{k^2}F(x,k) - \frac{1}{k^2}E(x,k)$$

5. $$\int \frac{\sin x\cos x\, dx}{\Delta} = -\frac{\Delta}{k^2}$$

6. $$\int \frac{\cos^2 x\, dx}{\Delta} = \frac{1}{k^2}E(x,k) - \frac{k'^2}{k^2}F(x,k)$$

7. $$\int \frac{\sin^3 x\, dx}{\Delta} = \frac{\cos x\Delta}{2k^2} - \frac{1+k^2}{2k^3}\ln(k\cos x+\Delta)$$

8. $$\int \frac{\sin^2 x\cos x\, dx}{\Delta} = -\frac{\sin x\Delta}{2k^2} + \frac{\arcsin(k\sin x)}{2k^3}$$

9. $$\int \frac{\sin x\cos^2 x\, dx}{\Delta} = -\frac{\cos x\Delta}{2k^2} + \frac{k'^2}{2k^3}\ln(k\cos x+\Delta)$$

10. $$\int \frac{\cos^3 x\, dx}{\Delta} = \frac{\sin x\Delta}{2k^2} + \frac{2k^2-1}{2k^3}\arcsin(k\sin x)$$

11. $$\int \frac{\sin^4 x\, dx}{\Delta} = \frac{\sin x\cos x\Delta}{3k^2} + \frac{2+k^2}{3k^4}F(x,k) - \frac{2\left(1+k^2\right)}{3k^4}E(x,k)$$

12. $$\int \frac{\sin^3 x\cos x\, dx}{\Delta} = -\frac{1}{3k^4}\left(2+k^2\sin^2 x\right)\Delta$$

13. $$\int \frac{\sin^2 x \cos^2 x\, dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2-k^2}{3k^4}\, E(x,k) + \frac{2k^2-2}{3k^4}\, F(x,k)$$

14. $$\int \frac{\sin x \cos^3 x\, dx}{\Delta} = -\frac{1}{3k^4}\left(k^2 \cos^2 x - 2k'^2\right)\Delta$$

15. $$\int \frac{\cos^4 x\, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2-2}{3k^4}\, E(x,k) + \frac{3k^4-5k^2+2}{3k^4}\, F(x,k)$$

16. $$\int \frac{\sin^5 x\, dx}{\Delta} = \frac{2k^2 \sin^2 x + 3k^2 + 3}{8k^4} \cos x \Delta - \frac{3+2k^2+3k^4}{8k^5} \ln\left(k \cos x + \Delta\right)$$

17. $$\int \frac{\sin^4 x \cos x\, dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \sin x \Delta + \frac{3}{8k^5} \arcsin\left(k \sin x\right)$$

18. $$\int \frac{\sin^3 x \cos x\, dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \cos x \Delta - \frac{k^4+2k^2-3}{8k^5} \ln\left(k \cos x + \Delta\right)$$

19. $$\int \frac{\sin^2 x \cos^3 x\, dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \sin x \Delta + \frac{4k^2-3}{8k^5} \arcsin\left(k \sin x\right)$$

20. $$\int \frac{\sin x \cos^4 x\, dx}{\Delta} = \frac{3-5k^2+2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^5} \ln\left(k \cos x + \Delta\right)$$

21. $$\int \frac{\cos^5 x\, dx}{\Delta} = \frac{2k^2 \cos^2 x + 6k^2 - 3}{8k^4} \sin x \Delta + \frac{8k^4-8k^2+3}{8k^5} \arcsin\left(k \sin x\right)$$

22. $$\int \frac{\sin^6 x\, dx}{\Delta} = \frac{3k^2 \sin^2 x + 4k^2 + 4}{15k^4} \sin x \cos x \Delta + \frac{4k^4+3k^2+8}{15k^6}\, F(x,k) - \frac{8k^4+7k^2+8}{15k^6}\, E(x,k)$$

23. $$\int \frac{\sin^5 x \cos x\, dx}{\Delta} = -\frac{3k^4 \sin^4 x + 4k^2 \sin^2 x + 8}{15k^6} \Delta$$

24. $$\int \frac{\sin^4 x \cos x\, dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{k^4+7k^2-8}{15k^6}\, F(x,k) - \frac{2k^4+3k^2-8}{15k^6}\, E(x,k)$$

25. $$\int \frac{\sin^3 x \cos^3 x\, dx}{\Delta} = \frac{3k^4 \sin^4 x - \left(5k^4 - 4k^2\right) \sin^2 x - 10k^2 + 8}{15k^6} \Delta$$

26. $$\int \frac{\sin^2 x \cos^4 x\, dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{9k^4-17k^2+8}{15k^6}\, F(x,k) - \frac{3k^4-13k^2+8}{15k^6}\, E(x,k)$$

27. $$\int \frac{\sin x \cos^5 x\, dx}{\Delta} = \frac{-3k^4 \cos^4 x + 4k^2 k'^2 \cos^2 x - 8k^4 + 16k^2 - 8}{15k^6} \Delta$$

28. $$\int \frac{\cos^6 x\, dx}{\Delta} = \frac{3k^2 \cos^2 x + 8k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{15k^6-34k^4+27k^2-8}{15k^6}\, F(x,k) + \frac{23k^4-23k^2+8}{15k^6}\, E(x,k)$$

29. $$\int \frac{\sin^7 x\,dx}{\Delta} = \frac{8k^4\sin^4 x + 10k^2\left(k^2+1\right)\sin^2 x + 15k^4 + 14k^2 + 15}{48k^6}\cos x\Delta - \frac{\left(5k^4 - 2k^2 + 5\right)\left(k^2+1\right)}{16k^7}\ln\left(k\cos x + \Delta\right)$$

30. $$\int \frac{\sin^6 x\cos x\,dx}{\Delta} = -\frac{8k^4\sin^4 x + 10k^2\sin^2 x + 15}{48k^6}\sin x\Delta + \frac{5}{16k^7}\arcsin\left(k\sin x\right)$$

31. $$\int \frac{\sin^5 x\cos^2 x\,dx}{\Delta} = \frac{-8k^4\sin^4 x + 2k^2\left(k^2-5\right)\sin^2 x + 3k^4 + 4k^2 - 15}{48k^6}\cos x\Delta - \frac{k^6 + k^4 + 3k^2 - 5}{16k^7}\ln\left(k\cos x + \Delta\right)$$

32. $$\int \frac{\sin^4 x\cos^3 x\,dx}{\Delta} = \frac{8k^4\sin^4 x - 2k^2\left(6k^2-5\right)\sin^2 x - 18k^2 + 15}{48k^6}\sin x\Delta + \frac{6k^2-5}{16k^7}\arcsin\left(k\sin x\right)$$

33. $$\int \frac{\sin^3 x\cos^4 x\,dx}{\Delta} = \frac{8k^4\sin^4 x - 2k^2\left(6k^2-5\right)\sin^2 x + 3k^4 - 22k^2 + 15}{48k^6}\cos x\Delta - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7}\ln\left(k\cos x + \Delta\right)$$

34. $$\int \frac{\sin^2 x\cos^5 x\,dx}{\Delta} = \frac{-8k^4\sin^4 x + 2k^2\left(12k^2-5\right)\sin^2 x - 24k^4 + 36k^2 - 15}{48k^6}\sin x\Delta + \frac{8k^4 - 12k^2 + 5}{16k^7}\arcsin\left(k\sin x\right)$$

35. $$\int \frac{\sin x\cos^6 x\,dx}{\Delta} = \frac{-8k^4\sin^4 x + 2k^2\left(13k^2-5\right)\sin^2 x - 33k^4 + 40k^2 - 15}{48k^6}\cos x\Delta + \frac{5k'^6}{16k^7}\ln\left(k\cos x + \Delta\right)$$

36. $$\int \frac{\cos^7 x\,dx}{\Delta} = \frac{8k^4\sin^4 x - 2k^2\left(18k^2-5\right)\sin^2 x + 72k^4 - 54k^2 + 15}{48k^6}\sin x\Delta + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7}\arcsin\left(k\sin x\right)$$

37. $$\int \frac{dx}{\Delta^3} = \frac{1}{k'^2}E(x,k) - \frac{k^2}{k'^2}\frac{\sin x\cos x}{\Delta}$$

38. $$\int \frac{\sin x\,dx}{\Delta^3} = -\frac{\cos x}{k'^2\Delta}$$

39. $$\int \frac{\cos x\,dx}{\Delta^3} = \frac{\sin x}{\Delta}$$

40.[11] $$\int \frac{\sin^2 x\,dx}{\Delta^3} = \frac{1}{k'^2k^2}E(x,k) - \frac{1}{k^2}F(x,k) - \frac{1}{k'^2}\frac{\sin x\cos x}{\Delta}$$

41. $$\int \frac{\sin x\cos x\,dx}{\Delta^3} = \frac{1}{k^2\Delta}$$

42. $$\int \frac{\cos^2 x\,dx}{\Delta^3} = \frac{1}{k^2}F(x,k) - \frac{1}{k^2}E(x,k) + \frac{\sin x\cos x}{\Delta}$$

43. $$\int \frac{\sin^3 x\,dx}{\Delta^3} = -\frac{\cos x}{k^2 k'^2 \Delta} + \frac{1}{k^3} \ln\left(k\cos x + \Delta\right)$$

44. $$\int \frac{\sin^2 x \cos x\,dx}{\Delta^3} = \frac{\sin x}{k^2\Delta} - \frac{1}{k^3} \arcsin\left(k \sin x\right)$$

45. $$\int \frac{\sin x \cos^2 x\,dx}{\Delta^3} = \frac{\cos x}{k^2\Delta} - \frac{1}{k^3} \ln\left(k\cos x + \Delta\right)$$

46. $$\int \frac{\cos^3 x\,dx}{\Delta^3} = -\frac{k'^2 \sin x}{k^2\Delta} + \frac{1}{k^3} \arcsin\left(k \sin x\right)$$

47. $$\int \frac{\sin^4 x\,dx}{\Delta^3} = \frac{k'^2+1}{k'^2 k^4}\, E(x,k) - \frac{2}{k^4}\, F(x,k) - \frac{\sin x \cos x}{k^2 k'^2 \Delta}$$

48. $$\int \frac{\sin^3 x \cos x\,dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \delta}$$

49. $$\int \frac{\sin^2 x \cos^2 x\,dx}{\Delta^3} = \frac{2-k^2}{k^4}\, F(x,k) - \frac{2}{k^4}\, E(x,k) + \frac{\sin x \cos x}{k^2\Delta}$$

50. $$\int \frac{\sin x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}$$

51. $$\int \frac{\cos^4 x\,dx}{\Delta^3} = \frac{k'^2+1}{k^4}\, E(x,k) - \frac{2k'^2}{k^4}\, F(x,k) - \frac{k'^2 \sin x \cos x}{k^2\Delta}$$

52.[9] $$\int \frac{\sin^5 x\,dx}{\Delta^3} = \frac{k^2 k'^2 \sin^2 x + k^2 - 3}{2k^4 k'^2 \Delta} \cos x + \frac{k^2+3}{2k^5} \ln\left(k\cos x + \Delta\right)$$

53. $$\int \frac{\sin^4 x \cos x\,dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4\Delta} \sin x - \frac{3}{2k^5} \arcsin\left(k \sin x\right)$$

54. $$\int \frac{\sin^3 x \cos^2 x\,dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4\Delta} \cos x + \frac{k^2-3}{2k^5} \ln\left(k\cos x + \Delta\right)$$

55. $$\int \frac{\sin^2 x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4\Delta} \sin x - \frac{2k^2-3}{2k^5} \arcsin\left(k \sin x\right)$$

56. $$\int \frac{\sin x \cos^4 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4\Delta} \cos x + \frac{3k'^2}{2k^5} \ln\left(k\cos x + \Delta\right)$$

57. $$\int \frac{\cos^5 x\,dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4\Delta} \sin x + \frac{4k^2-3}{2k^5} \arcsin\left(k \sin x\right)$$

58. $$\int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2\left(k'^2+1\right) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2}\, F(x,k) + \frac{2\left(k'^2+1\right)}{3k'^4}\, E(x,k)$$

59. $$\int \frac{\sin x\,dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x$$

60. $$\int \frac{\cos x\,dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x$$

61. $$\int \frac{\sin^2 x\,dx}{\Delta^5} = \frac{k^2+1}{3k'^4k^2}\,E(x,k) - \frac{1}{3k'^2k^2}\,F(x,k) + \frac{k^2\left(k^2+1\right)\sin^2 x - 2}{3k'^4\Delta^3}\sin x\cos x$$

62. $$\int \frac{\sin x\cos x\,dx}{\Delta^5} = \frac{1}{3k^2\Delta^3}$$

63. $$\int \frac{\cos^2 x\,dx}{\Delta^5} = \frac{1}{3k^2}\,F(x,k) + \frac{2k^2-1}{3k^2k'^2}\,E(x,k) + \frac{k^2\left(2k^2-1\right)\sin^2 x - 3k^2 + 2}{2k'^2\Delta}\sin x\cos x$$

64. $$\int \frac{\sin^3 x}{\Delta^5}\,dx = \frac{\left(3k^2-1\right)\sin^2 x - 2}{3k'^4\Delta^3}\cos x$$

65. $$\int \frac{\sin^2 x\cos x}{\Delta^5}\,dx = \frac{\sin^3 x}{3\Delta^3}$$

66. $$\int \frac{\sin x\cos^2 x}{\Delta^5}\,dx = -\frac{\cos^3 x}{3k'^2\Delta^3}$$

67. $$\int \frac{\cos^3 x\,dx}{\Delta^5} = \frac{-\left(2k^2+1\right)\sin^2 x + 3}{3\Delta^3}\sin x$$

68. $$\int \frac{dx}{\Delta\sin x} = -\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}$$

69. $$\int \frac{dx}{\Delta\cos x} = -\frac{1}{2k'}\ln\frac{\Delta-k'\sin x}{\Delta+k'\sin x}$$

70. $$\int \frac{dx}{\Delta\sin^2 x} = \int \frac{1+\cot^2 x}{\Delta}\,dx = F(x,k) - E(x,k) - \Delta\cot x$$

71. $$\int \frac{dx}{\Delta\sin x\cos x} = \int (\tan x + \cot x)\,\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

72. $$\int \frac{dx}{\Delta\cos^2 x} = \int \left(1+\tan^2 x\right)\frac{dx}{\Delta} = F(x,k) - \frac{1}{k'^2}\,E(x,k) + \frac{1}{k'^2}\Delta\tan x$$

73. $$\int \frac{\sin x}{\cos x}\frac{dx}{\Delta} = \int \tan x\frac{dx}{\Delta} = \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

74. $$\int \frac{\cos x}{\sin x}\frac{dx}{\Delta} = \int \cot x\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$$

75. $$\int \frac{dx}{\Delta\sin^3 x} = -\frac{\Delta\cos x}{2\sin^2 x} - \frac{1+k^2}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x}$$

76. $$\int \frac{dx}{\Delta\sin^2 x\cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'}\ln\frac{\Delta-k'\sin x}{\Delta+k'\sin x}$$

77. $$\int \frac{dx}{\Delta\sin x\cos^2 x} = \frac{\Delta}{k'^2\cos x} + \frac{1}{2}\ln\frac{\Delta-\cos x}{\Delta+\cos x}$$

78. $$\int \frac{dx}{\Delta\cos^3 x} = \frac{\Delta\sin x}{2k'^2\cos^2 x} + \frac{2k^2-1}{4k'^3}\ln\frac{\Delta-k'\sin x}{\Delta+k'\sin x}$$

79. $$\int \frac{\sin x}{\cos^2 x}\frac{dx}{\Delta} = \frac{\Delta}{k'^2\cos x}$$

80. $$\int \frac{\cos x}{\sin^2 x}\frac{dx}{\Delta} = -\frac{\Delta}{\sin x}$$

81. $$\int \frac{\sin^2 x}{\cos x}\frac{dx}{\Delta} = \frac{1}{2k'}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x} - \frac{1}{k}\arcsin(k\sin x)$$

82. $$\int \frac{\cos^2 x}{\sin x}\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k}\ln(k\cos x + \Delta)$$

83. $$\int \frac{dx}{\Delta\sin^4 x} = \frac{1}{3}\left\{-\Delta\cot^3 x - \Delta\left(2k^2+3\right)\cot x + \left(k^2+2\right)F(x,k) - 2\left(k^2+1\right)E(x,k)\right\}$$

84. $$\int \frac{dx}{\Delta\sin^3 x\cos x} = \int\left(\tan x + 2\cot x + \cot^3 x\right)\frac{dx}{\Delta}$$
$$= -\frac{\Delta}{2\sin^2 x} + \frac{1}{2k'}\ln\frac{\Delta + k'}{\Delta - k'} - \frac{k^2+2}{4}\ln\frac{1+\Delta}{1-\Delta}$$

85. $$\int \frac{dx}{\Delta\sin^2 x\cos^2 x} = \int\left(\tan^2 x + 2 + \cot^2 x\right)\frac{dx}{\Delta}$$
$$= \left(\frac{\tan x}{k'^2} - \cot x\right)\Delta + \frac{k^2-2}{k'^2}E(x,k) + 2F(x,k)$$

86. $$\int \frac{dx}{\Delta\sin x\cos^3 x} = \int\left(\cot x + 2\tan x + \tan^3 x\right)\frac{dx}{\Delta}$$
$$= -\frac{\Delta}{2k'^2\cos^2 x} - \frac{1}{2}\ln\frac{1+\Delta}{1-\Delta} + \frac{2-3k^2}{4k'^3}\ln\frac{\Delta+k'}{\Delta-k'}$$

87. $$\int \frac{dx}{\Delta\cos^4 x} = \frac{1}{3k'^2}\left\{\Delta\tan^3 x - \frac{5k^2-3}{k'^2}\Delta\tan x - \left(3k^2-2\right)F(x,k) + \frac{2\left(2k^2-1\right)}{k'^2}E(x,k)\right\}$$

88. $$\int \frac{\sin x}{\cos^3 x}\frac{dx}{\Delta} = \int\tan x\left(1+\tan^2 x\right)\frac{dx}{\Delta} = \frac{\Delta}{2k'^2\cos^2 x} - \frac{k^2}{4k'^3}\ln\frac{\Delta+k'}{\Delta-k'}$$

89. $$\int \frac{\cos x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta}{2\sin^2 x} - \frac{k^2}{4}\ln\frac{1+\Delta}{1-\Delta}$$

90. $$\int \frac{\sin^2 x}{\cos^2 x}\frac{dx}{\Delta} = \int\frac{\tan^2 x}{\Delta}\,dx = \frac{\Delta}{k'^2}\tan x - \frac{1}{k'^2}E(x,k)$$

91. $$\int \frac{\cos^2 x}{\sin^2 x}\frac{dx}{\Delta} = \int\frac{\cot^2 x}{\Delta}\,dx = -\Delta\cot x - E(x,k)$$

92. $$\int \frac{\sin^3 x}{\cos x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

93. $$\int \frac{\cos^3 x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2}\ln\frac{1+\Delta}{1-\Delta}$$

94. $$\int \frac{dx}{\Delta\sin^5 x} = -\frac{\left[3\left(1+k^2\right)\sin^2 x + 2\right]}{8\sin^2 x}\Delta\cos x + \frac{3k^4+2k^2+3}{16}\ln\frac{\Delta+\cos x}{\Delta-\cos x}$$

95. $$\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{\left(3+2k^2\right)\sin^2 x + 1}{3\sin^3 x}\Delta - \frac{1}{2k'}\ln\frac{\Delta - k'\sin x}{\Delta + k'\sin x}$$

96. $$\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{\left(3-k^2\right)\sin^2 x - k'^2}{2k'^2 \sin^2 x \cos x}\Delta + \frac{k^2+3}{4}\ln\frac{\Delta - \cos x}{\Delta + \cos x}$$

97. $$\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{\left(3-2k^2\right)\sin^2 x - 2k'^2}{2k'^2 \sin x \cos^2 x}\Delta - \frac{4k^2-3}{4k'^3}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

98. $$\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{\left(5k^2-3\right)\sin^2 x - 6k^2 + 4}{3k'^4 \cos^3 x}\Delta - \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

99. $$\int \frac{dx}{\Delta \cos^5 x} = \frac{3\left(2k^2-1\right)\sin^2 x - 8k^2 + 5}{8k'^4 \cos^4 x}\Delta \sin x + \frac{8k^4 - 8k^2 + 3}{16k'^5}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

100. $$\int \frac{\sin x}{\cos^4 x}\frac{dx}{\Delta} = -\frac{2k^2\cos^2 x - k'^2}{2k'^4 \cos^3 x}\Delta$$

101. $$\int \frac{\cos x}{\sin^4 x}\frac{dx}{\Delta} = -\frac{2k^2\sin^2 x + 1}{3\sin^3 x}\Delta$$

102. $$\int \frac{\sin^2 x}{\cos^3 x}\frac{dx}{\Delta} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} - \frac{1}{4k'^3}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

103. $$\int \frac{\cos^3 x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta \cos x}{2\sin^2 x} + \frac{k'^2}{4}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

104. $$\int \frac{\sin^3 x}{\cos^2 x}\frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{k}\ln\left(k\cos x + \Delta\right)$$

105. $$\int \frac{\cos^3 x}{\sin^2 x}\frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k}\arcsin\left(k\sin x\right)$$

106. $$\int \frac{\sin^4 x}{\cos x}\frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{1}{2k'}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x} - \frac{2k^2+1}{2k^3}\arcsin\left(k\sin x\right)$$

107. $$\int \frac{\cos^4 x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2-1}{2k^3}\ln\left(k\cos x + \Delta\right)$$

2.585

1. $$\int \frac{(a+\sin x)^{p+3}\,dx}{\Delta}$$
$$= \frac{1}{(p+2)k^2}\Bigg[(a+\sin x)^p \cos x\Delta$$
$$+2(2p+3)ak^2\int\frac{(a+\sin x)^{p+2}\,dx}{\Delta} + (p+1)\left(1+k^2-6a^2k^2\right)\int\frac{(a+\sin x)^{p+1}\,dx}{\Delta}$$
$$-a(2p+1)\left(1+k^2-2a^2k^2\right)\int\frac{(a+b\sin x)^p\,dx}{\Delta}$$
$$-\,p\left(1-a^2\right)\left(1-a^2k^2\right)\int\frac{(a+\sin x)^{p-1}\,dx}{\Delta}\Bigg]$$
$$\left[p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm\frac{1}{k}\right]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{a+\sin x}{\Delta}\,dx = a\,F(x,k) + \frac{1}{2k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x}$$

3. $$\int \frac{(a+\sin x)^2}{\Delta}\,dx = \frac{1+k^2a^2}{k^2}\,F(x,k) - \frac{1}{k^2}\,E(x,k) + \frac{a}{k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x}$$

4.[6] $$\int \frac{dx}{(a+\sin x)\,\Delta} = \frac{1}{a}\,\Pi\left(x, \frac{1}{a^2}, k\right) - \int \frac{\sin x\,dx}{\left(a^2-\sin^2 x\right)\Delta},$$

where

5. $$\int \frac{\sin x\,dx}{\left(a^2-\sin^2 x\right)\Delta} = \frac{-1}{2\sqrt{(1-a^2)\,(1-a^2k^2)}}\ln\frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2}\cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2}\cos x}$$

2.586

1. $$\int \frac{dx}{(a+\sin x)^n\,\Delta} = \frac{1}{(n-1)\,(1-a^2)\,(1-a^2k^2)}\left[-\frac{\cos x\Delta}{(a+\sin x)^{n-1}}\right.$$
$$-(2n-3)\left(1+k^2-2a^2k^2\right)a\int\frac{dx}{(a+\sin x)^{n-1}\,\Delta}$$
$$-(n-2)\left(6a^2k^2-k^2-1\right)\int\frac{dx}{(a+\sin x)^{n-2}\,\Delta}$$
$$\left.-\,(10-4n)ak^2\int\frac{dx}{(a+\sin x)^{n-3}\,\Delta} - (n-3)k^2\int\frac{dx}{(a+\sin x)^{n-4}\,\Delta}\right]$$
$$\left[n\neq 1,\quad a\neq\pm 1,\quad a\neq\pm\frac{1}{k}\right]$$

This integral can be reduced to the integrals:

2. $$\int \frac{dx}{(a+\sin x)^2\,\Delta} = \frac{1}{(1-a^2)\,(1-a^2k^2)}\left[-\frac{\cos x\Delta}{a+\sin x} - a\left(1+k^2-2a^2k^2\right)\int\frac{dx}{(a+\sin x)\,\Delta}\right.$$
$$\left.-\,2ak^2\int\frac{(a+\sin x)\,dx}{\Delta} + k^2\int\frac{(a+\sin x)^2\,dx}{\Delta}\right]$$
(see **2.585** 2, 3, 4)

3. $$\int \frac{dx}{(a+\sin x)^3\,\Delta} = \frac{1}{2\,(1-a^2)\,(1-a^2k^2)}\left[-\frac{\cos x\Delta}{(a+\sin x)^2} - 3a\left(1+k^2-2a^2k^2\right)\int\frac{dx}{(a+\sin x)^2\,\Delta}\right.$$
$$\left.-\,\left(6a^2k^2-k^2-1\right)\int\frac{dx}{(a+\sin x)\,\Delta} + 2ak^2\,F(x,k)\right]$$
(see **2.585** 4 and **2.586** 2)

For $a=\pm 1$, we have:

4. $$\int \frac{dx}{(1\pm\sin x)^n\,\Delta} = \frac{1}{(2n-1)k'^2}\left[\mp\frac{\cos x\Delta}{(1\pm\sin x)^n} + (n-1)\left(1-5k^2\right)\int\frac{dx}{(1\pm\sin x)^{n-1}\,\Delta}\right.$$
$$\left.+\,2(2n-3)k^2\int\frac{dx}{(1\pm\sin x)^{n-2}\,\Delta} - (n-2)k^2\int\frac{dx}{(1\pm\sin x)^{n-3}\,\Delta}\right]$$
GU (241)(6a)

This integral can be reduced to the following integrals:

5. $$\int \frac{dx}{(1 \pm \sin x)\,\Delta} = \frac{\mp \cos x \Delta}{k'^2\,(1 \pm \sin x)} + F(x,k) - \frac{1}{k'^2}\,E(x,k)$$ GU (241)(6c)

6. $$\int \frac{dx}{(1 \pm \sin x)^2\,\Delta} = \frac{1}{3k'^4}\left\{\mp \frac{k'^2 \cos x \Delta}{(1 \pm \sin x)^2} \mp \frac{\left(1-5k^2\right)\cos x \Delta}{1 \pm \sin x} + \left(1-3k^2\right)k'^2\,F(x,k) - \left(1-5k^2\right)E(x,k)\right\}$$ GU (241)(6b)

For $a = \pm\dfrac{1}{k}$, we have

7. $$\int \frac{dx}{(1 \pm k \sin x)^n\,\Delta} = \frac{1}{(2n-1)k'^2}\left[\pm \frac{k \cos x \Delta}{(1 \pm k \sin x)^n} + (n-1)\left(5-k^2\right)\int \frac{dx}{(1 \pm k \sin x)^{n-1}\,\Delta} - 2(2n-3)\int \frac{dx}{(1 \pm k \sin x)^{n-2}\,\Delta} + (n-2)\int \frac{dx}{(1 \pm k \sin x)^{n-3}\,\Delta}\right]$$ GU (241)(7a)

This integral can be reduced to the following integrals:

8. $$\int \frac{dx}{(1 \pm k \sin x)\,\Delta} = \pm \frac{k \cos x \Delta}{k'^2\,(1 \pm k \sin x)} + \frac{1}{k'^2}\,E(x,k)$$ GU (241)(7b)

9. $$\int \frac{dx}{(1 \pm k \sin x)^2\,\Delta} = \frac{1}{3k'^4}\left[\pm \frac{kk'^2 \cos x \Delta}{(1 \pm k \sin x)^2} \pm \frac{k\left(5-k^2\right)\cos x \Delta}{1 \pm k \sin x} - 2k'^2\,F(x,k) + \left(5-k^2\right)E(x,k)\right]$$ GU(241)(7c)

2.587

1. $$\int \frac{(b+\cos x)^{p+3}\,dx}{\Delta} \quad \frac{1}{(p+2)k^2}\left[(b+\cos x)^p \sin x \Delta + 2(2p+3)bk^2 \int \frac{(b+\cos x)^{p+2}\,dx}{\Delta} - (p+1)\left(k'^2 - k^2 + 6b^2k^2\right)\int \frac{(b+\cos x)^{p+1}\,dx}{\Delta} + (2p+1)b\left(k'^2 - k^2 + b^2k^2\right)\int \frac{(b+\cos x)^{p}\,dx}{\Delta} + p\left(1-b^2\right)\left(k'^2 + k^2b^2\right)\int \frac{(b+\cos x)^{p-1}\,dx}{\Delta}\right]$$
$$\left[p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k}\right]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{b+\cos x}{\Delta}\,dx = b\,F(x,k) + \frac{1}{k}\arcsin(k \sin x)$$

3. $$\int \frac{(b+\cos x)^2}{\Delta}\,dx = \frac{b^2k^2 - k'^2}{k^2}\,F(x,k) + \frac{1}{k^2}\,E(x,k) + \frac{2b}{k}\arcsin(k \sin x)$$

4. $$\int \frac{dx}{(b+\cos x)\,\Delta} = \frac{b}{b^2-1}\,\Pi\left(x, \frac{1}{b^2-1}, k\right) + \int \frac{\cos x\,dx}{\left(1-b^2-\sin^2 x\right)\Delta},$$

where

5. $$\int \frac{\cos x\,dx}{\left(1-b^2-\sin^2 x\right)\Delta} = \frac{1}{2\sqrt{\left(1-b^2\right)\left(k'^2+k^2b^2\right)}} \ln \frac{\sqrt{1-b^2}\Delta + k\sqrt{k'^2+k^2b^2}\sin x}{\sqrt{1-b^2}\Delta - k\sqrt{k'^2+k^2b^2}\sin x}$$

2.588

1. $$\int \frac{dx}{(b+\cos x)^n\,\Delta} = \frac{1}{(n-1)\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2\sin x\Delta}{(b+\cos x)^{-1}}\right.$$
$$-(2n-3)\left(1-2k^2+2b^2k^2\right) b\int \frac{dx}{(b+\cos x)^{n-1}\,\Delta}$$
$$-(n-2)\left(2k^2-1-6b^2k^2\right)\int \frac{dx}{(b+\cos x)^{n-2}\,\Delta}$$
$$\left. -\,(4n-10)bk^2\int \frac{dx}{(b+\cos x)^{n-3}\,\Delta} + (n-3)k^2\int \frac{dx}{(b+\cos x)^{n-4}\,\Delta}\right]$$
$$\left[n\neq 1,\quad b\neq \pm 1,\quad b\neq \pm\frac{ik'}{k}\right]$$

This integral can be reduced to the following integrals:

2. $$\int \frac{dx}{(b+\cos x)^2\,\Delta} = \frac{1}{\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2\sin x\Delta}{b+\cos x} - \left(1-2k^2+2b^2k^2\right) b\int \frac{dx}{(b+\cos x)\,\Delta}\right.$$
$$\left. +\,2bk^2\int \frac{b+\cos x}{\Delta}dx - k^2\int \frac{(b+\cos x)^2}{\Delta}\,dx\right]$$
(see **2.587** 2, 3, 4)

3. $$\int \frac{dx}{(b+\cos x)^3\,\Delta} = \frac{1}{2\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2\sin x\Delta}{(b+\cos x)^2}\right.$$
$$-3b\left(1-2k^2+2k^2b^2\right)\int \frac{dx}{(b+\cos x)^2\,\Delta}$$
$$\left. -\,\left(2k^2-1-6b^2k^2\right)\int \frac{dx}{(b+\cos x)\,\Delta} - 2bk^2\,F(x,k)\right]$$
(see **2.588** 2 and **2.587** 4)

2.589

1. $$\int \frac{(c+\tan x)^{p+3}\,dx}{\Delta} = \frac{1}{(p+2)k'^2} \left[\frac{(c+\tan x)^p\,\Delta}{\cos^2 x} + 2(2n+3)ck'^2\int \frac{(c+\tan x)^{p+2}\,dx}{\Delta}\right.$$
$$-(p+1)\left(1+k'^2+6c^2k'^2\right)\int \frac{(c+\tan x)^{p+1}\,dx}{\Delta}$$
$$+(2p+1)c\left(1+k'^2+2c^2k'^2\right)\int \frac{(c+\tan x)^{p}\,dx}{\Delta}$$
$$\left. -\,p\left(1+c^2\right)\left(1+k'^2c^2\right)\int \frac{(c+\tan x)^{p-1}\,dx}{\Delta}\right]$$
$$[p\neq -2]$$

For $p=n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{c+\tan x}{\Delta}\,dx = c\,F(x,k) + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

3. $$\int \frac{(c+\tan x)^2}{\Delta}\,dx = \frac{1}{k'^2}\tan x\Delta + c^2\,F(x,k) - \frac{1}{k'^2}\,E(x,k) + \frac{c}{k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

4. $$\int \frac{dx}{(c+\tan x)\,\Delta} = \frac{c}{1+c^2}\,F(x,k) + \frac{1}{c\,(1+c^2)}\,\Pi\left(x, -\frac{1+c^2}{c^2}, k\right) - \int \frac{\sin x\cos x\,dx}{\left[c^2-(1+c^2)\sin^2 x\right]\Delta},$$

where

5. $$\int \frac{\sin x\cos x\,dx}{\left[c^2-(1+c^2)\sin^2 x\right]\Delta} = \frac{1}{2\sqrt{(1+c^2)\left(1+c^2k'^2\right)}}\ln\frac{\sqrt{1+c^2k'^2}+\sqrt{1+c^2}\Delta}{\sqrt{1+c^2k'^2}-\sqrt{1+c^2}\Delta}$$

2.591

1. $$\int \frac{dx}{(c+\tan x)^n\,\Delta} = \frac{1}{(n-1)\,(1+c^2)\left(1+k'^2c^2\right)}\left[-\frac{\Delta}{(c+\tan x)^{n-1}\cos^2 x} + (2n-3)c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)^{n-1}\,\Delta} - (n-2)\left(1+k'^2+6c^2k'^2\right)\int\frac{dx}{(c+\tan x)^{n-2}\,\Delta} + (4n-10)ck'^2\int\frac{dx}{(c+\tan x)^{n-3}\,\Delta} - (n-3)k'^2\int\frac{dx}{(c+\tan x)^{n-4}\,\Delta}\right]$$

This integral can be reduced to the integrals:

2. $$\int \frac{dx}{(c+\tan x)^2\,\Delta} = \frac{1}{(1+c^2)\left(1+k'^2c^2\right)}\left[\frac{-\Delta}{(c+\tan x)\cos^2 x} + c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)\,\Delta} - 2ck'^2\int\frac{c+\tan x}{\Delta}dx + k'^2\int\frac{(c+\tan x)^2}{\Delta}\,dx\right]$$

(see **2.589** 2, 3, 4)

3. $$\int \frac{dx}{(c+\tan x)^3\,\Delta} = \frac{1}{2\,(1+c^2)\left(1+k'^2c^2\right)}\left[\frac{-\Delta}{(c+\tan x)^2\cos^2 x} + 3c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)^2\,\Delta} - \left(1+k'^2+6c^2k'^2\right)\int\frac{dx}{(c+\tan x)\,\Delta} + 2ck'^2\,F(x,k)\right]$$

(see **2.591** 2 and **2.589** 4)

2.592

1. $$P_n = \int \frac{\left(a+\sin^2 x\right)^n}{\Delta}\,dx$$

The recursion formula

$$P_{n+1} = \frac{1}{(2n+3)k^2}\left\{\left(a+\sin^2 x\right)^n \sin x\cos x\Delta + (2n+2)\left(1+k^2+3ak^2\right)P_{n+1}\right.$$
$$\left. -\,(2n+1)\left[1+2a\left(1+k^2\right)+3a^2k^2\right]P_n + 2na(1+a)\left(1+k^2a\right)P_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals:

2. P_1 (see **2.584** 1 and **2.584** 4)

3. P_0 (see **2.584** 1)

4. $$P_{-1} = \int \frac{dx}{\left(a+\sin^2 x\right)\Delta} = \frac{1}{a}\,\Pi\left(x, \frac{1}{a}, k\right)$$

For $a = 0$

5. $$\int \frac{dx}{\sin^2 x\Delta}$$ (see **2.584** 70) H (124)a

6. $$T_n = \int \frac{dx}{\left(h+g\sin^2 x\right)^n\Delta}$$

can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2}\left\{\frac{-g^2\sin x\cos x\Delta}{\left(h+g\sin^2 x\right)^{n-1}} + 2(n-2)\left[g\left(1+k^2\right)+3hk^2\right]T_{n-2}\right.$$
$$\left. -\,(2n-3)\left[g^2+2hg\left(1+k^2\right)+3h^2k^2\right]T_{n-1} + 2(n-1)h(g+h)\left(g+hk^2\right)T_n\right\}$$

2.593

1. $$Q_n = \int \frac{\left(b+\cos^2 x\right)^n}{\Delta}\,dx$$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2}\left\{\left(b+\cos^2 x\right)^n \sin x\sin x\Delta - (2n+2)\left(1-2k^2-3bk^2\right)Q_{n+1}\right.$$
$$\left. +\,(2n+1)\left[k'^2+2b\left(k'^2-k^2\right)-3b^2k^2\right]n_2nb(1-b)\left(k'^2-k^2b\right)Q_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals:

2. Q_1 (see **2.584** 1 and **2.584** 6)

3. Q_0 (see **2.584** 1)

4. $Q_{-1} = \int \frac{dx}{(b+\cos^2 x)\,\Delta} = \frac{1}{b+1}\,\Pi\left(x, -\frac{1}{b+1}, k\right)$

For $b = 0$

5. $\int \frac{dx}{\cos^2 x \Delta}$ (see **2.584** 72) H (123)

2.594

1. $R_n = \int \frac{\left(c+\tan^2 x\right)^n\, dx}{\Delta}$

The recursion formula

$$R_{n+2} = \frac{1}{(2n+3)k'^2}\left\{\frac{\left(c+\tan^2 x\right)^n \tan x\Delta}{\cos^2 x} - (2n+2)\left(1+k'^2-3ck'^2\right)R_{n+1}\right.$$
$$\left. + (2n-1)\left[1-2c\left(1+k'^2\right)+3c^2k'^2\right]R_n + 2nc(1-c)\left(1-k'^2c\right)R_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals:

2. R_1 (see **2.584** 1 and **2.584** 90)

3. R_0 (see **2.584** 1)

4. $R_{-1} = \int \frac{dx}{\left(c+\tan^2 x\right)\Delta} = \frac{1}{c-1}\,F(x,k) + \frac{1}{c(1-c)}\,\Pi\left(x, \frac{1-c}{c}, k\right)$

For $c = 0$, see **2.582** 5.

2.595 Integrals of the type $\int R\left(\sin x, \cos x, \sqrt{1-p^2\sin^2 x}\right)\,dx$ for $p^2 > 1$.

Notation: $\alpha = \arcsin(p\sin x)$.

Basic formulas

1. $\int \frac{dx}{\sqrt{1-p^2\sin^2 x}} = \frac{1}{p}\,F\left(\alpha, \frac{1}{p}\right)$ $[p^2 > 1]$ BY (283.00)

2. $\int \sqrt{1-p^2\sin^2 x}\,dx = p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p}\,F\left(\alpha, \frac{1}{p}\right)$

$[p^2 > 1]$ BY (283.03)

3. $\int \frac{dx}{\left(1-r^2\sin^2 x\right)\sqrt{1-p^2\sin^2 x}} = \frac{1}{p}\,\Pi\left(\alpha, \frac{r^2}{p^2}, \frac{1}{p}\right)$ $[p^2 > 1]$ BY (283.02)

To evaluate integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-p^2\sin^2 x}\right)\,dx$ for $p^2 > 1$, we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

(1) k with p;

(2) k'^2 with $1-p^2$;

(3) $F(x,k)$ with $\dfrac{1}{p}\,F\left(\alpha,\frac{1}{p}\right)$;

(4) $E(x,k)$ with $p\,E\left(\alpha,\dfrac{1}{p}\right)-\dfrac{p^2-1}{p}\,F\left(\alpha,\dfrac{1}{p}\right)$.

For example (see **2.584** 15):

2.596

1.[10]
$$\int\frac{\cos^4 x\,dx}{\sqrt{1-p^2\sin^2 x}}=\frac{\sin x\cos x\sqrt{1-p^2\sin^2 x}}{3p^2}+\frac{4p^2-2}{3p^4}\left[p\,E\left(\alpha,\frac{1}{p}\right)\right.$$
$$\left.-\frac{p^2-1}{p}\,F\left(\alpha,\frac{1}{p}\right)\right]+\frac{2-5p^2+3p^4}{3p^4}\cdot\frac{1}{p}\,F\left(\alpha,\frac{1}{p}\right)$$
$$=\frac{\sin x\cos x\sqrt{1-p^2\sin^2 x}}{3p^2}-\frac{p^2-1}{3p^3}\,F\left(\alpha,\frac{1}{p}\right)+\frac{4p^2-2}{3p^3}\,E\left(\alpha,\frac{1}{p}\right)\qquad[p^2>1]$$

For example (see **2.583** 36):

2.
$$\int\frac{\sqrt{1-p^2\sin^2 x}}{\cos^2 x}\,dx=\tan x\sqrt{1-p^2\sin^2 x}+\frac{1}{p}\,F\left(\alpha,\frac{1}{p}\right)-\left[p\,E\left(\alpha,\frac{1}{p}\right)-\frac{p^2-1}{p}\,F\left(\alpha,\frac{1}{p}\right)\right]$$
$$=p\left[F\left(\alpha,\frac{1}{p}\right)-E\left(\alpha,\frac{1}{p}\right)\right]+\tan x\sqrt{1-p^2\sin^2 x}$$
$$[p^2>1]$$

For example (see **2.584** 37):

3.
$$\int\frac{dx}{\sqrt{\left(1-p^2\sin^2 x\right)^3}}=\frac{-1}{p^2-1}\left[p\,E\left(\alpha,\frac{1}{p}\right)-\frac{p^2-1}{p}\,F\left(\alpha,\frac{1}{p}\right)\right]-\frac{p^2}{1-p^2}\frac{\sin x\cos x}{\sqrt{1-p^2\sin^2 x}}$$
$$=\frac{p^2}{p^2-1}\frac{\sin x\cos x}{\sqrt{1-p^2\sin^2 x}}+\frac{1}{p}\,F\left(\alpha,\frac{1}{p}\right)-\frac{p}{p^2-1}\,E\left(\alpha,\frac{1}{p}\right)$$
$$[p^2>1]$$

2.597 Integrals of the form $\displaystyle\int R\left(\sin x,\cos x,\sqrt{1+p^2\sin^2 x}\right)\,dx$

Notation: $\alpha=\arcsin\left(\dfrac{\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}}\right)$

Basic formulas

1.
$$\int\frac{dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{\sqrt{1+p^2}}\,F\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)$$ BY (282.00)

2.
$$\int\sqrt{1+p^2\sin^2 x}\,dx=\sqrt{1+p^2}\,E\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)-p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}$$ BY (282.03)

3. $$\frac{\sqrt{1+p^2\sin^2 x}\,dx}{1+(p^2-r^2p^2-r^2)\sin^2 x}=\frac{1}{\sqrt{1+p^2}}\,\Pi\left(\alpha,r^2,\frac{p}{\sqrt{1+p^2}}\right)$$ BY (282.02)

4. $$\int\frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}}=-\frac{1}{p}\arcsin\left(\frac{p\cos x}{\sqrt{1+p^2}}\right)$$

5. $$\int\frac{\cos x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{p}\ln\left(p\sin x+\sqrt{1+p^2\sin^2 x}\right)$$

6. $$\int\frac{dx}{\sin x\sqrt{1+p^2\sin^2 x}}=\frac{1}{2}\ln\frac{\sqrt{1+p^2\sin^2 x}-\cos x}{\sqrt{1+p^2\sin^2 x}+\cos x}$$

7. $$\int\frac{dx}{\cos x\sqrt{1+p^2\sin^2 x}}=\frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}\sin x}$$

8. $$\int\frac{\tan x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}}$$

9. $$\int\frac{\cot x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{2}\ln\frac{1-\sqrt{1+p^2\sin^2 x}}{1+\sqrt{1+p^2\sin^2 x}}$$

2.598 To calculate integrals of the form $\int R\left(\sin x,\cos x,\sqrt{1+p^2\sin^2 x}\right)dx$, we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

(1) k^2 with $-p^2$;

(2) k'^2 with $1+p^2$;

(3) $F(x,k)$ with $\frac{1}{\sqrt{1+p^2}}\,F\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)$;

(4) $E(x,k)$ with $\sqrt{1+p^2}\,E\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)-p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}$;

(5) $\frac{1}{k}\ln(k\cos x+\Delta)$ with $\frac{1}{p}\arcsin\frac{p\cos x}{\sqrt{1+p^2}}$;

(6) $\frac{1}{k}\arcsin(k\sin x)$ with $\frac{1}{p}\ln\left(p\sin x+\sqrt{1+p^2\sin^2 x}\right)$.

For example (see **2.584** 90):

1. $$\int\frac{\tan^2 x\,dx}{\sqrt{1+p^2\sin^2 x}}=\frac{1}{(1+p^2)}\left[\tan x\sqrt{1+p^2\sin^2 x}\right.$$
$$\left.-\sqrt{1+p^2}\,E\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)+p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}\right]$$
$$=-\frac{1}{\sqrt{1+p^2}}\,E\left(\alpha,\frac{p}{\sqrt{1+p^2}}\right)+\frac{\tan x}{\sqrt{1+p^2\sin^2 x}}$$

For example (see **2.584** 37):

2. $$\int \frac{dx}{\sqrt{\left(1+p^2\sin^2 x\right)^3}} = \frac{1}{\sqrt{1+p^2}}\, E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$

2.599 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{a^2\sin^2 x - 1}\right)\, dx \qquad \left[a^2 > 1\right]$

Notation: $\alpha = \arcsin\left(\dfrac{a\cos x}{\sqrt{a^2-1}}\right)$.

Basic formulas:

1. $$\int \frac{dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{1}{a}\, F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) \qquad \left[a^2 > 1\right]$$ BY (285.00)a

2. $$\int \sqrt{a^2\sin^2 x - 1}\, dx = \frac{1}{a}\, F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) - a\, E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)$$ $$\left[a^2 > 1\right]$$ BY (285.06)a

3. $$\int \frac{dx}{\left(1 - r^2\sin^2 x\right)\sqrt{a^2\sin^2 x - 1}} = \frac{1}{a\left(r^2-1\right)}\, \Pi\left(\alpha, \frac{r^2\left(a^2-1\right)}{a^2\left(r^2-1\right)}, \frac{\sqrt{a^2-1}}{a}\right)$$ $$\left[a^2 > 1, \quad r^2 > 1\right]$$ BY (285.02)a

4. $$\int \frac{\sin x\, dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{\alpha}{a} \qquad \left[a^2 > 1\right]$$

5. $$\int \frac{\cos x\, dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{a}\ln\left(a\sin x + \sqrt{a^2\sin^2 x - 1}\right) \qquad \left[a^2 > 1\right]$$

6. $$\int \frac{dx}{\sin x\sqrt{a^2\sin^2 x - 1}} = -\arctan\frac{\cos x}{\sqrt{a^2\sin^2 x - 1}} \qquad \left[a^2 > 1\right]$$

7. $$\int \frac{dx}{\cos x\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-2}}\ln\frac{\sqrt{a^2-1}\sin x + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1}\sin x - \sqrt{a^2\sin^2 x - 1}}$$ $$\left[a^2 > 1\right]$$

8. $$\int \frac{\tan x\, dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1} + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2\sin^2 x - 1}}$$ $$\left[a^2 > 1\right]$$

9. $$\int \frac{\cot x\, dx}{\sqrt{a^2\sin^2 x - 1}} = -\arcsin\left(\frac{1}{a\sin x}\right) \qquad \left[a^2 > 1\right]$$

2.611 To calculate integrals of the type $\int R\left(\sin x, \cos x, \sqrt{a^2\sin^2 x - 1}\right)\, dx$ for $a^2 > 1$, we may use formulas **2.583** and **2.584**. In doing so, we should follow the procedure outlined below:

(1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$F(x,k) \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta}$$

$$E(x,k) \quad \text{should be replaced with} \quad \int \Delta dx$$

$$-\frac{1}{k}\ln(k\cos x+\Delta) \quad \text{should be replaced with} \quad \int \frac{\sin x\,dx}{\Delta}$$

$$\frac{1}{k}\arcsin(k\sin x) \quad \text{should be replaced with} \quad \int \frac{\cos x\,dx}{\Delta}$$

$$\frac{1}{2}\ln\frac{\Delta-\cos x}{\Delta+\cos x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta\sin x}$$

$$\frac{1}{2k'}\ln\frac{\Delta+k'\sin x}{\Delta-k'\sin x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta\cos x}$$

$$\frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'} \quad \text{should be replaced with} \quad \int \frac{\tan x}{\Delta}\,dx$$

$$\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} \quad \text{should be replaced with} \quad \int \frac{\cot x}{\Delta}\,dx$$

(2) Then, on both sides of the equations, we should replace Δ with $i\sqrt{a^2\sin^2 x-1}$, k with a and k'^2 with $1-a^2$.

(3) Both sides of the resulting equations should be multiplied by i, as a result of which only real functions $\left(a^2>1\right)$ should appear on both sides of the equations.

(4) The integrals on the right sides of the equations should be replaced with their values found from formulas **2.599**.

Examples:

1. We rewrite equation **2.584** 4 in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2\sin^2 x-1}}\,dx = \frac{1}{a^2}\int \frac{dx}{i\sqrt{a^2\sin^2 x-1}} - \frac{1}{a^2}\int i\sqrt{a^2\sin^2 x-1}\,dx,$$

from which we get

$$\int \frac{\sin^2 x\,dx}{\sqrt{a^2\sin^2 x-1}} = \frac{1}{a^2}\left\{\int \frac{dx}{\sqrt{a^2\sin^2 x-1}} + \int \sqrt{a^2\sin^2 x-1}\,dx\right\} = -\frac{1}{a}\,E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)$$

$$\left[a^2>1\right]$$

2. We rewrite equation **2.584** 58 as follows:

$$\int \frac{dx}{i^5\sqrt{\left(a^2\sin^2 x-1\right)^5}} = -\frac{2a^4\left(a^2-2\right)\sin^2 x-\left(3a^2-5\right)a^2}{3\left(1-a^2\right)^2 i^3\sqrt{\left(a^2\sin^2 x-1\right)^3}}\sin x\cos x$$
$$-\frac{1}{3\left(1-a^2\right)}\int \frac{dx}{i\sqrt{a^2\sin^2 x-1}} - \frac{2a^2-4}{3\left(1-a^2\right)^2}\int i\sqrt{a^2\sin^2 x-1}\,dx$$

from which we obtain

$$\int \frac{dx}{\sqrt{\left(a^2\sin^2 x-1\right)^5}} = \frac{2a^4\left(a^2-2\right)\sin^2 x-\left(3a^2-5\right)a^2}{3\left(1-a^2\right)^2\sqrt{\left(a^2\sin^2 x-1\right)^3}}\sin x\cos x+\frac{1}{3\left(1-a^2\right)^2 a}$$
$$\times\left\{\left(a^2-3\right)F\left(\alpha,\frac{\sqrt{a^2-1}}{a}\right)-2a^2\left(a^2-2\right)E\left(\alpha,\frac{\sqrt{a^2-1}}{a}\right)\right\}$$
$$\left[a^2>1\right]$$

3. We rewrite equation **2.584** 71 in the form

$$\int \frac{dx}{\sin x\cos x i\sqrt{a^2\sin^2 x-1}} = \int \frac{\cot x\,dx}{i\sqrt{a^2\sin^2 x-1}}+\int \frac{\tan x\,dx}{i\sqrt{a^2\sin^2 x-1}},$$

from which we obtain

$$\int \frac{dx}{\sin x\cos x\sqrt{a^2\sin^2 x-1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}-\sqrt{a^2\sin^2 x-1}}-\arcsin\left(\frac{1}{a\sin x}\right)$$
$$\left[a^2>1\right]$$

2.612 Integrals of the form $\int R\left(\sin x,\cos x,\sqrt{1-k^2\cos^2 x}\right)\,dx$.

To find integrals of the form $\int R\left(\sin x,\cos x,\sqrt{1-k^2\cos^2 x}\right)\,dx$, we make the substitution $x=\frac{\pi}{2}-y$, which yields

$$\int R\left(\sin x,\cos x,\sqrt{1-k^2\cos^2 x}\right)\,dx=-\int R\left(\cos y,\sin y,\sqrt{1-k^2\sin^2 y}\right)\,dy.$$

The integrals $\int R\left(\cos y,\sin y,\sqrt{1-k^2\sin^2 y}\right)\,dy$ are found from formulas **2.583** and **2.584**. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions $F(x,k)$ and $E(x,k)$ with the corresponding integrals, we obtain an expression of the form

$$-g\left(\cos y,\sin y\right)-A\int\frac{dy}{\sqrt{1-k^2\sin^2 y}}-B\int\sqrt{1-k^2\sin^2 y}\,dy$$

Returning now to the original variable x, we obtain

$$\int R\left(\sin x,\cos x,\sqrt{1-k^2\cos^2 x}\right)\,dx=-g\left(\sin x,\cos x\right)-A\int\frac{dx}{\sqrt{1-k^2\cos^2 x}}-B\int\sqrt{1-k^2\cos^2 x}\,dx$$

The integrals appearing in this expression are found from the formulas

1. $$\int\frac{dx}{\sqrt{1-k^2\cos^2 x}}=F\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\right),k\right)$$

2. $$\int\sqrt{1-k^2\cos^2 x}\,dx=E\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\right),k\right)-\frac{k^2\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}$$

2.613 Integrals of the form $\int R\left(\sin x,\cos x,\sqrt{1-p^2\cos^2 x}\right)\,dx \qquad [p>1]$.

To find integrals of the type $\int R\left(\sin x,\cos x,\sqrt{1-p^2\cos^2 x}\right)\,dx$, where $[p>1]$, we proceed as in section **2.612**. Here, we use the formulas

1. $$\int \frac{dx}{\sqrt{1-p^2\cos^2 x}} = -\frac{1}{p}\,F\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right) \qquad [p>1]$$

2. $$\int \sqrt{1-p^2\cos^2 x}\,dx = \frac{p^2-1}{p}\,F\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right) - p\,E\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right)$$

2.614 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right)\,dx$.

To find integrals of the type $\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right)\,dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right)\,dx = -\int R\left(\cos y, \sin y, \sqrt{1+p^2\sin^2 y}\right)\,dy.$$

To calculate the integrals $-\int R\left(\cos y, \sin y, \sqrt{1+p^2\sin^2 y}\right)\,dy$, we need to use first what was said in **2.598** and **2.612** and then, after returning to the variable x, the formulas

1. $$\int \frac{dx}{\sqrt{1+p^2\cos^2 x}} = \frac{1}{\sqrt{1+p^2}}\,F\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

2. $$\int \sqrt{1+p^2\cos^2 x}\,dx = \sqrt{1+p^2}\,E\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

2.615 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right)\,dx \qquad [a>1]$.

To find integrals of the type $\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right)\,dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right)\,dx = -\int R\left(\cos y, \sin y, \sqrt{a^2\sin^2 y - 1}\right)\,dy$$

To calculate the integrals $-\int R\left(\cos y, \sin y, \sqrt{a^2\sin^2 y - 1}\right)\,dy$, we use what was said in **2.611** and then, after returning to the variable x, we use the formulas

1. $$\int \frac{dx}{\sqrt{a^2\cos^2 x - 1}} = \frac{1}{a}\,F\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a>1]$$

2. $$\int \sqrt{a^2\cos^2 x - 1}\,dx = a\,E\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) - \frac{1}{a}\,F\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a>1]$$

2.616[11] Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-p^2\sin^2 x}, \sqrt{1-q^2\sin^2 x}\right)\,dx$.

Notation: $\alpha = \arcsin\left(\dfrac{\sqrt{1-p^2}\sin x}{\sqrt{1-p^2\sin^2 x}}\right)$.

1. $$\int \frac{dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}} = \frac{1}{\sqrt{1-p^2}}\,F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.00)

2. $$\int \frac{\tan^2 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}} = \frac{\tan x\sqrt{1-q^2\sin^2 x}}{\left(1-q^2\right)\sqrt{1-p^2\sin^2 x}} - \frac{1}{\left(1-q^2\right)\sqrt{1-p^2}}\,E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.07)

3. $$\int \frac{\tan^4 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}}$$
$$= \frac{1}{3\left(1-q^2\right)^2\left(1-p^2\right)^{\frac{3}{2}}} \times \left[2\left(2-p^2-q^2\right)E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \left(1-q^2\right)F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right]$$
$$+\frac{2p^2+q^2-3+\sin^2 x\left(4-3p^2-2q^2+p^2q^2\right)}{3\left(1-p^2\right)\left(1-q^2\right)^2}\,\frac{\sin x}{\cos^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.07)

4. $$\int \frac{\sin^2 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)^3}}$$
$$= \frac{\sqrt{1-p^2}}{\left(1-q^2\right)\left(q^2-p^2\right)}\,E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1}{\left(q^2-p^2\right)\sqrt{1-p^2}}\,F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$-\frac{\sin x\cos x}{\left(1-q^2\right)\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.06)

5. $$\int \frac{\cos^2 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)^3\left(1-q^2\sin^2 x\right)}}$$
$$= \frac{\sqrt{1-p^2}}{q^2-p^2}\,E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1-q^2}{\left(q^2-p^2\right)\sqrt{1-p^2}}\,F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.05)

6. $$\int \frac{\cos^4 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)^5\left(1-q^2\sin^2 x\right)}}$$
$$= \frac{\left(1-p^2\right)^{\frac{3}{2}}}{3\left(q^2-p^2\right)^2}\left[\frac{\left(2+p^2-3q^2\right)\left(1-q^2\right)}{\left(1-p^2\right)^2}F\left(\alpha,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right.$$
$$\left.+\,2\frac{2q^2-p^2-1}{1-p^2}E\left(\alpha,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right]+\frac{\left(1-p^2\right)\sin x\cos x\sqrt{1-q^2\sin^2 x}}{3\left(q^2-p^2\right)\sqrt{\left(1-p^2\sin^2 x\right)^3}}$$
$$\left[0<p^2<q^2<1,\quad 0<x\le\frac{\pi}{2}\right]$$ BY (284.05)

7. $$\int \frac{dx}{1-p^2\sin^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}}=\frac{1}{\sqrt{1-p^2}}E\left(\alpha,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0<p^2<q^2<1,\quad 0<x\le\frac{\pi}{2}\right]$$
BY (284.01)

8. $$\int\sqrt{\frac{1-p^2\sin^2 x}{\left(1-q^2\sin^2 x\right)^3}}\,dx=\frac{\sqrt{1-p^2}}{1-q^2}E\left(\alpha,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)-\frac{q^2-p^2}{1-q^2}\frac{\sin x\cos x}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}}$$
$$\left[0<p^2<q^2<1,\quad 0<x\le\frac{\pi}{2}\right].$$
BY (284.04)

9. $$\int\frac{dx}{1+\left(p^2r^2-p^2-r^2\right)\sin^2 x}\sqrt{\frac{1-p^2\sin^2 x}{1-q^2\sin^2 x}}=\frac{1}{\sqrt{1-p^2}}\Pi\left(\alpha,r^2,\sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0<p^2<q^2<1,\quad 0<x\le\frac{\pi}{2}\right].$$
BY (284.02)

2.617 **Notation**: $\alpha=\arcsin\sqrt{\dfrac{\sqrt{b^2+c^2}-b\sin x-c\cos x}{2\sqrt{b^2+c^2}}},\quad r=\sqrt{\dfrac{2\sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}.$

1. $$\int\frac{dx}{\sqrt{a+b\sin x+c\cos x}}$$
$$=-\frac{2}{\sqrt{a+\sqrt{b^2+c^2}}}F(\alpha,r)$$
$$\left[0<\sqrt{b^2+c^2}<a,\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\pi\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (294.00)
$$=-\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}}F(\alpha,r)$$
$$\left[0<|a|<\sqrt{b^2+c^2},\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right)\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.00)

2. $$\int \frac{\sin x\,dx}{\sqrt{a+b\sin x+c\cos x}} = -\frac{\sqrt{2b}}{\sqrt[4]{(b^2+c^2)^3}}\left\{2\,E(\alpha,r)-F(\alpha,r)\right\}+\frac{2c}{b^2+c^2}\sqrt{a+b\sin x+c\cos x}$$
$$\left[0<|a|<\sqrt{b^2+c^2},\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right)\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.05)

3. $$\int \frac{(b\cos x-c\sin x)\,dx}{\sqrt{a+b\sin x+c\cos x}} = 2\sqrt{a+b\sin x+c\cos x}$$

4. $$\int \frac{\sqrt{b^2+c^2}+b\sin x+c\cos x}{\sqrt{a+b\sin x+c\cos x}}\,dx$$
$$= -2\sqrt{a+\sqrt{b^2+c^2}}\,E(\alpha,r)+\frac{2\left(a-\sqrt{b^2+c^2}\right)}{\sqrt{a+\sqrt{b^2+c^2}}}\,F(\alpha,r)$$
$$\left[0<\sqrt{b^2+c^2}<a,\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\pi\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (294.04)
$$= -2\sqrt{2}\sqrt[4]{b^2+c^2}\,E(\alpha,r)$$
$$\left[0<|a|<\sqrt{b^2+c^2},\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right)\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.01)

5. $$\int \sqrt{a+b\sin x+c\cos x}\,dx$$
$$= -2\sqrt{a+\sqrt{b^2+c^2}}\,E(\alpha,r)$$
$$\left[0<\sqrt{b^2+c^2}<a,\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\pi\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (294.01)
$$= -2\sqrt{2}\sqrt[4]{b^2+c^2}\,E(\alpha,r)+\frac{\sqrt{2}\left(\sqrt{b^2+c^2}-a\right)}{\sqrt[4]{b^2+c^2}}\,F(\alpha,r)$$
$$\left[0<|a|<\sqrt{b^2+c^2},\quad \arcsin\frac{b}{\sqrt{b^2+c^2}}-\arccos\left(\frac{-a}{\sqrt{b^2+c^2}}\right)\le x<\arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.03)

2.618 Integrals of the form $\int R\left(\sin ax,\cos ax,\sqrt{\cos 2ax}\right)\,dx = \frac{1}{a}\int R\left(\sin t,\cos t,\sqrt{1-2\sin^2 t}\right)\,dt$ where the substitution $t=ax$ has been used.
Notation: $\alpha=\arcsin\left(\sqrt{2}\sin ax\right)$

The integrals $\int R\left(\sin ax,\cos ax,\sqrt{\cos 2ax}\right)\,dx$ are special cases of the integrals **2.595**. for $(p=2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}}\,F\left(\alpha,\frac{1}{\sqrt{2}}\right) \qquad \left[0<ax\le\frac{\pi}{4}\right]$$

2. $$\int \frac{\cos^2 ax}{\sqrt{\cos 2ax}}\,dx = \frac{1}{a\sqrt{2}}\,E\left(\alpha,\frac{1}{\sqrt{2}}\right) \qquad \left[0<ax\le\frac{\pi}{4}\right]$$

3. $$\int \frac{dx}{\cos^2 ax\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\tan x}{a}\sqrt{\cos 2ax} \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

4. $$\int \frac{dx}{\cos^4 ax\sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\left(6\cos^2 ax + 1\right)\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{4}\right]$$

5. $$\int \frac{\tan^2 ax\, dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a}\tan ax\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{2}\right]$$

6. $$\int \frac{\tan^4 ax\, dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax} \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

7. $$\int \frac{dx}{\left(1 - 2r^2\sin^2 ax\right)\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}}\, \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

8. $$\int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}} \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

9. $$\int \frac{\sin^2 ax\, dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

10. $$\int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

11. $$\int \sqrt{\cos 2ax}\, dx = \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

12. $$\int \frac{\sqrt{\cos 2ax}}{\cos^2 ax}\, dx = \frac{\sqrt{2}}{a}\left\{F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\} + \frac{1}{a}\tan ax\sqrt{\cos 2ax} \qquad \left[0 < x \le \frac{\pi}{4}\right]$$

2.619 Integrals of the form $\int R\left(\sin ax, \cos ax, \sqrt{-\cos 2ax}\right)\, dx = \frac{1}{a}\int R\left(\sin x, \cos x, \sqrt{2\sin^2 x - 1}\right)\, dx$

Notation: $\alpha = \arcsin\left(\sqrt{2}\cos ax\right)$

The integrals $\int R\left(\sin x, \cos x, \sqrt{2\sin^2 x - 1}\right)\, dx$ are special cases of the integrals **2.599** and **2.611** for $(a = 2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

2. $$\int \frac{\cos^2 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$

3. $$\int \frac{\cos^4 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}}\left[3\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{5}{2}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] - \frac{1}{12a}\sin 2ax\sqrt{-\cos 2ax}$$

4. $$\int \frac{dx}{\sin^2 ax\sqrt{-\cos 2ax}} = \frac{1}{a}\cot ax\sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

5. $$\int \frac{dx}{\sin^4 ax\sqrt{-\cos 2ax}} = \frac{2}{3a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 6\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{3a}\frac{\cos ax}{\sin^3 ax}\left(6\sin^2 ax + 1\right)\sqrt{-\cos 2ax}$$

6. $$\int \frac{\cot^2 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{a}\cot ax\sqrt{-\cos 2ax}$$

7. $$\int \frac{dx}{(1 - 2r^2\cos^2 ax)\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\,\Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right)$$

8. $$\int \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}$$

9. $$\int \frac{\cos^2 ax\, dx}{\sqrt{-\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

10. $$\int \frac{dx}{\sqrt{-\cos^5 2ax}} = -\frac{1}{3a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3 2ax}}$$

11. $$\int \sqrt{-\cos 2ax}\, dx = \frac{1}{a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$

2.621 Integrals of the form $\int R\left(\sin ax, \cos ax, \sqrt{\sin 2ax}\right)\, dx$.

Notation: $\alpha = \arcsin\sqrt{\dfrac{2\sin ax}{1 + \sin ax + \cos ax}}$.

1. $$\int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ BY (287.50)

2. $$\int \frac{\sin ax\, dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left[\frac{1+i}{2}\,\Pi\left(\alpha, \frac{1+i}{2}, \frac{1}{\sqrt{2}}\right) + \frac{1-i}{2}\,\Pi\left(\alpha, \frac{1-i}{2}, \frac{1}{\sqrt{2}}\right) + F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$ BY (287.57)

3. $$\int \frac{\sin ax\, dx}{(1 + \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$ BY (287.54)

4. $$\int \frac{\sin ax\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{\sqrt{\tan ax} - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\} \qquad \left[ax \neq \frac{\pi}{2}\right]$$ BY (287.55)

5. $$\int \frac{(1+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\,E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ BY (287.51)

6. $$\int \frac{(1+\cos ax)\,dx}{(1-\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \sqrt{\tan ax}\right\} \qquad \left[ax \neq \frac{\pi}{2}\right]$$ BY (287.56)

7. $$\int \frac{(1-\sin ax+\cos ax)\,dx}{(1+\sin ax+\cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{2\,E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\}$$ BY (287.53)

8. $$\int \frac{(1+\sin ax+\cos ax)\,dx}{[1+\cos ax+(1-2r^2)\sin ax]\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\,\Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right).$$ BY (287.52)

2.63–2.65 Products of trigonometric functions and powers

2.631

1. $$\begin{aligned}\int x^r \sin^p x \cos^q x\,dx &= \frac{1}{(p+q)^2}\Bigg[(p+q)x^r \sin^{p+1} x \cos^{q-1} x \\ &\quad + r x^{r-1}\sin^p x\cos^q x - r(r-1)\int x^{r-2}\sin^p x\cos^q x\,dx \\ &\quad - rp\int x^{r-1}\sin^{p-1}x\cos^{q-1}x\,dx + (q-1)(p+q)\int x^r\sin^p x\cos^{q-2}x\,dx\Bigg] \\ &= \frac{1}{(p+q)^2}\Bigg[-(p+q)x^r\sin^{p-1}x\cos^{q+1}x \\ &\quad + r x^{r-1}\sin^p x\cos^q x - r(r-1)\int x^{r-2}\sin^p x\cos^q x\,dx \\ &\quad + rq\int x^{r-1}\sin^{p-1}x\cos^{q-1}x\,dx + (p-1)(p+q)\int x^r\sin^{p-2}x\cos^q x\,dx\Bigg]\end{aligned}$$ GU (331)(1)

2. $$\begin{aligned}\int x^m \sin^n x\,dx &= \frac{x^{m-1}\sin^{n-1}x}{n^2}\{m\sin x - nx\cos x\} \\ &\quad + \frac{n-1}{n}\int x^m\sin^{n-2}x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin^n x\,dx\end{aligned}$$

3. $$\begin{aligned}\int x^m \cos^n x\,dx &= \frac{x^{m-1}\cos^{n-1}x}{n^2}\{m\cos x + nx\sin x\} \\ &\quad + \frac{n-1}{n}\int x^m\cos^{n-2}x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos^n x\,dx\end{aligned}$$

4. $$\int x^n \sin^{2m} x\,dx = \binom{2m}{m}\frac{x^{n+1}}{2^{2m}(n+1)} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\int x^n\cos(2m-2k)x\,dx$$
(see **2.633** 2) TI 333

5. $$\int x^n \sin^{2m+1} x\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\int x^n\sin(2m-2k+1)x\,dx$$
(see **2.633** 1) TI 333

6. $$\int x^n \cos^{2m} x\,dx = \binom{2m}{m}\frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\int x^n\cos(2m-2k)x\,dx$$
(see **2.633** 2) TI 333

7. $$\int x^n \cos^{2m+1} x\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\int x^n\cos(2m-2k+1)x\,dx$$
(see **2.633** 2) TI 333

2.632

1. $$\int x^{\mu-1}\sin\beta x\,dx = \frac{i}{2}(i\beta)^{-\mu}\gamma(\mu, i\beta x) - \frac{i}{2}(-i\beta)^{-\mu}\gamma(\mu, -i\beta x)$$
$[\operatorname{Re}\mu > -1, \quad x > 0]$ ET I 317(2)

2. $$\int x^{\mu-1}\sin ax\,dx = -\frac{1}{2a^\mu}\left\{\exp\left[\frac{\pi i}{2}(\mu-1)\right]\Gamma(\mu, -iax) + \exp\left[\frac{\pi i}{2}(1-\mu)\right]\Gamma(\mu, iax)\right\}$$
$[\operatorname{Re}\mu < 1, \quad a > 0, \quad x > 0]$ ET I 317(3)

3. $$\int x^{\mu-1}\cos\beta x\,dx = \frac{1}{2}\left\{(i\beta)^{-\mu}\gamma(\mu, i\beta x) + (-i\beta)^{-\mu}\gamma(\mu, -i\beta x)\right\}$$
$[\operatorname{Re}\mu > 0, \quad x > 0]$ ET I 319(22)

4. $$\int x^{\mu-1}\cos ax\,dx = -\frac{1}{2a^\mu}\left\{\exp\left(i\mu\frac{\pi}{2}\right)\Gamma(\mu, -iax) + \exp\left(-i\mu\frac{\pi}{2}\right)\Gamma(\mu, iax)\right\}$$
ET I 319(23)

2.633

1. $$\int x^n\sin ax\,dx = -\sum_{k=0}^{n}k!\binom{n}{k}\frac{x^{n-k}}{a^{k+1}}\cos\left(ax + \frac{1}{2}k\pi\right)$$
TI (487)

2.[8] $$\int x^n\cos ax\,dx = \sum_{k=0}^{n}k!\binom{n}{k}\frac{x^{n-k}}{a^{k+1}}\sin\left(ax + \frac{1}{2}k\pi\right)$$
TI (486)

3. $$\int x^{2n}\sin x\,dx = (2n)!\left\{\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k}}{(2n-2k)!}\cos x + \sum_{k=0}^{n-1}(-1)^k\frac{x^{2n-2k-1}}{(2n-2k-1)!}\sin x\right\}$$

4. $$\int x^{2n+1}\sin x\,dx = (2n+1)!\left\{\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + \sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right\}$$

5. $$\int x^{2n}\cos x\,dx = (2n)!\left\{\sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x + \sum_{k=0}^{n-1}(-1)^{k}\frac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right\}$$

6. $$\int x^{2n+1}\cos x\,dx = (2n+1)!\left\{\sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x + \sum_{k=0}^{n}\frac{x^{2n-2k}}{(2n-2k)!}\cos x\right\}$$

2.634

1. $$\int P_n(x)\sin mx\,dx = -\frac{\cos mx}{m}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$$

2. $$\int P_n(x)\cos mx\,dx = \frac{\sin mx}{m}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$$

In formulas **2.634**, $P_n(x)$ is any n^{th}-degree polynomial, and $P_n^{(k)}(x)$ is its kth derivative with respect to x.

2.635 **Notation**: $z_1 = a + bx$.

1. $$\int z_1\sin kx\,dx = -\frac{1}{k}z_1\cos kx + \frac{b}{k^2}\sin kx$$

2. $$\int z_1\cos kx\,dx = \frac{1}{k}z_1\sin kx + \frac{b}{k^2}\cos kx$$

3. $$\int z_1^2\sin kx\,dx = \frac{1}{k}\left(\frac{2b^2}{k^2} - z_1^2\right)\cos kx + \frac{2bz_1}{k^2}\sin kx$$

4. $$\int z_1^2\cos kx\,dx = \frac{1}{k}\left(z_1^2 - \frac{2b^2}{k^2}\right)\sin kx + \frac{2bz_1}{k^2}\cos kx$$

5. $$\int z_1^3\sin kx\,dx = \frac{z_1}{k}\left(\frac{6b^2}{k^2} - z_1^2\right)\cos kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\sin kx$$

6. $$\int z_1^3\cos kx\,dx = \frac{z_1}{k}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\cos kx$$

7. $$\int z_1^4\sin kx\,dx = -\frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx$$

8. $$\int z_1^4\cos kx\,dx = \frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\cos kx$$

9. $$\int z_1^5\sin kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx - \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx$$

10. $$\int z_1^5\cos kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx$$

11. $$\int z_1^6 \sin kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx - \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\cos kx$$

12. $$\int z_1^6 \cos kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx + \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\sin kx$$

2.636

1. $$\int x^n \sin^2 x\,dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}$$

GU (333)(2e)

2. $$\int x^n \cos^2 x\,dx = \frac{x^{n+1}}{2(n+1)} - \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}$$

GU (333)(3e)

3. $$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x$$

4. $$\int x^2 \sin^2 x\,dx = \frac{x^3}{6} - \frac{x}{4}\cos 2x - \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x$$ MZ 241

5. $$\int x\cos^2 x\,dx = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x$$

6. $$\int x^2 \cos^2 x\,dx = \frac{x^3}{6} + \frac{x}{4}\cos 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x$$ MZ 245

2.637

1.[11] $$\int x^n \sin^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\cos 3x}{3^{2k+1}} - 3\cos x\right) - \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\sin 3x}{3^{2k+2}} - 3\sin x\right)\right\}$$

GU(333)(2f)

2. $$\int x^n \cos^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\sin 3x}{3^{2k+1}}+3\sin x\right)\right.$$
$$\left.+\sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\cos 3x}{3^{2k+2}}+3\cos x\right)\right\}$$
GU(333)(3f)

3. $$\int x\sin^3 x\,dx = \frac{3}{4}\sin x - \frac{1}{36}\sin 3x - \frac{3}{4}x\cos x + \frac{x}{12}\cos 3x$$

4. $$\int x^2\sin^3 x\,dx = -\left(\frac{3}{4}x^2+\frac{3}{2}\right)\cos x + \left(\frac{x^2}{12}+\frac{1}{54}\right)\cos 3x + \frac{3}{2}x\sin x - \frac{x}{18}\sin 3x$$ MZ 241

5. $$\int x\cos^3 x\,dx = \frac{3}{4}\cos x + \frac{1}{36}\cos 3x + \frac{3}{4}x\sin x + \frac{x}{12}\sin 3x$$

6. $$\int x^2\cos^3 x\,dx = \left(\frac{3}{4}x^2-\frac{3}{2}\right)\sin x + \left(\frac{x^2}{12}-\frac{1}{54}\right)\sin 3x + \frac{3}{2}x\cos x + \frac{x}{18}\cos 3x$$ MZ 245, 246

2.638

1. $$\int\frac{\sin^q x}{x^p}\,dx = -\frac{\sin^{q-1}x\,[(p-2)\sin x + qx\cos x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)}\int\frac{\sin^q x\,dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)}\int\frac{\sin^{q-2}x\,dx}{x^{p-2}}$$
$[p\neq 1,\quad p\neq 2]$ TI (496)

2. $$\int\frac{\cos^q x}{x^p}\,dx = -\frac{\cos^{q-1}x\,[(p-2)\cos x - qx\sin x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)}\int\frac{\cos^q x\,dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)}\int\frac{\cos^{q-2}x\,dx}{x^{p-2}}$$
$[p\neq 1,\quad p\neq 2]$ TI (495)

3.[6] $$\int\frac{\sin x\,dx}{x^p} = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1}\int\frac{\cos x\,dx}{x^{p-1}} = -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)}\int\frac{\sin x\,dx}{x^{p-2}}$$
$(p>2)$ TI (492)

4.[6] $$\int\frac{\cos x\,dx}{x^p} = -\frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1}\int\frac{\sin x\,dx}{x^{p-1}} = -\frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)}\int\frac{\cos x\,dx}{x^{p-2}}$$
$(p>2)$ TI (491)

2.639

1. $$\int \frac{\sin x\,dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(-1)^k(2k+1)!}{x^{2k+1}}\cos x + \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\sin x\right\} + \frac{(-1)^{n+1}}{(2n-1)!}\operatorname{ci}(x)$$

GU (333)(6b)a

2. $$\int \frac{\sin x}{x^{2n+1}}\,dx = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\cos x + \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}}\sin x\right\} + \frac{(-1)^{n}}{(2n)!}\operatorname{si}(x)$$

GU (333)(6b)a

3. $$\int \frac{\cos x\,dx}{x^{2n}}\,dx = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\cos x - \sum_{k=0}^{n-2}\frac{(-1)^{k}(2k+1)!}{x^{2k+1}}\sin x\right\} + \frac{(-1)^{n}}{(2n-1)!}\operatorname{si}(x)$$

GU (333)(7b)

4. $$\int \frac{\cos x\,dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}}\cos x - \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\sin x\right\} + \frac{(-1)^{n}}{(2n)!}\operatorname{ci}(x)$$

GU (333)(7b)

2.641

1. $$\int \frac{\sin kx}{a+bx}\,dx = \frac{1}{b}\left[\cos\frac{ka}{b}\operatorname{si}(u) - \sin\frac{ka}{b}\operatorname{ci}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

2. $$\int \frac{\cos kx}{a+bx}\,dx = \frac{1}{b}\left[\cos\frac{ka}{b}\operatorname{ci}(u) + \sin\frac{ka}{b}\operatorname{si}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

3. $$\int \frac{\sin kx}{(a+bx)^2}\,dx = -\frac{1}{b}\frac{\sin kx}{a+bx} + \frac{k}{b}\int\frac{\cos kx}{a+bx}\,dx \qquad \text{(see } \mathbf{2.641}\ 2)$$

4. $$\int \frac{\cos kx}{(a+bx)^2}\,dx = -\frac{1}{b}\frac{\cos kx}{a+bx} - \frac{k}{b}\int\frac{\sin kx}{a+bx}\,dx \qquad \text{(see } \mathbf{2.641}\ 1)$$

5. $$\int \frac{\sin kx}{(a+bx)^3}\,dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k\cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2}\int\frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

6. $$\int \frac{\cos kx}{(a+bx)^3}\,dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k\sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2}\int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

7. $$\int \frac{\sin kx}{(a+bx)^4}\,dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k\cos kx}{6b^2(a+bx)^2} + \frac{k^2\sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3}\int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

8. $$\int \frac{\cos kx}{(a+bx)^4}\,dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k\sin kx}{6b^2(a+bx)^2} + \frac{k^2\cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

9. $$\int \frac{\sin kx}{(a+bx)^5}\,dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k\cos kx}{12b^2(a+bx)^3} + \frac{k^2\sin kx}{24b^3(a+bx)^2} + \frac{k^3\cos kx}{24b^4(a+bx)}\,\frac{k^4}{24b^4}\int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

10. $$\int \frac{\cos kx}{(a+bx)^5}\,dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k\sin kx}{12b^2(a+bx)^3} + \frac{k^2\cos kx}{24b^3(a+bx)^2} - \frac{k^3\sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

11. $$\int \frac{\sin kx}{(a+bx)^6}\,dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k\cos kx}{20b^2(a+bx)^4} + \frac{k^2\sin kx}{60b^3(a+bx)^3} + \frac{k^3\cos kx}{120b^4(a+bx)^2} - \frac{k^4\sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

12. $$\int \frac{\cos kx}{(a+bx)^6}\,dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k\sin kx}{20b^2(a+bx)^4} + \frac{k^2\cos kx}{60b^3(a+bx)^3} - \frac{k^3\sin kx}{120b^4(a+bx)^2} - \frac{k^4\cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5}\int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

2.642

1. $$\int \frac{\sin^{2m} x}{x}\,dx = \binom{2m}{m}\frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\operatorname{ci}[(2m-2k)x]$$

2. $$\int \frac{\sin^{2m+1} x}{x}\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\operatorname{si}[(2m-2k+1)x]$$

3. $$\int \frac{\cos^{2m} x}{x}\,dx = \binom{2m}{m}\frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\operatorname{ci}[(2m-2k)x]$$

4. $$\int \frac{\cos^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\operatorname{ci}[(2m-2k+1)x]$$

5. $$\int \frac{\sin^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\,\mathrm{si}[(2m-2k)x]\right\}$$

6. $$\int \frac{\sin^{2m+1} x}{x^2}\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+1}\binom{2m+1}{k} \times\left\{\frac{\sin(2m-2k+1)x}{x} - (2m-2k+1)\,\mathrm{ci}[(2m-2k+1)x]\right\}$$

7. $$\int \frac{\cos^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} - \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\,\mathrm{si}[(2m-2k)x]\right\}$$

8. $$\int \frac{\cos^{2m+1} x}{x^2} = -\frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\left\{\frac{\cos(2m-2k+1)x}{x} + (2m-2k+1)\,\mathrm{si}[(2m-2k+1)x]\right\}$$

2.643

1. $$\int \frac{x^p\,dx}{\sin^q x} = -\frac{x^{p-1}\left[p\sin x + (q-2)x\cos x\right]}{(q-1)(q-2)\sin^{q-1} x} + \frac{q-2}{q-1}\int\frac{x^p\,dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}\,dx}{\sin^{q-2} x}$$

2. $$\int \frac{x^p\,dx}{\cos^q x} = -\frac{x^{p-1}\left[p\cos x - (q-2)x\sin x\right]}{(q-1)(q-2)\cos^{q-1} x} + \frac{q-2}{q-1}\int\frac{x^p\,dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}\,dx}{\cos^{q-2} x}$$

3.[4] $$\int \frac{x^n}{\sin x}\,dx = \frac{x^n}{n} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2\left(2^{2k-1}-1\right)}{(n+2k)(2k)!}B_{2k}x^{n+2k}$$

$[|x| < \pi, \quad n > 0]$ TU (333)(8b)

4. $$\int \frac{dx}{x^n \sin x} = -\frac{1}{nx^n} - \left[1+(-1)^n\right](-1)^{\frac{n}{2}}\frac{2^{n-1}-1}{n!}B_n \ln x - \sum_{\substack{k=1\\ k\neq\frac{n}{2}}}^{\infty}(-1)^k\frac{2\left(2^{2n}-1\right)}{(2k-n)\cdot(2k)!}B_{2k}x^{2k-n}$$

$[n > 1, \quad |x| > \pi]$ GU (333)(9b)

5.[8] $$\int \frac{x^n\,dx}{\cos x} = \sum_{k=0}^{\infty}\frac{|E_{2k}|x^{n+2k+1}}{(n+2k+1)(2k)!}$$

$\left[|x| < \frac{\pi}{2}, \quad n > 0\right]$ GU (333)(10b)

6. $$\int \frac{dx}{x^n \cos x} = \frac{1}{2}\left[1-(-1)^n\right]\frac{|E_{n-1}|}{(n-1)!}\ln x + \sum_{\substack{k=0\\ k\neq\frac{n-1}{2}}}^{\infty}\frac{|E_{2k}|x^{2k-n+1}}{(2k-n+1)\cdot(2k)!}$$

$\left[|x| < \frac{\pi}{2}\right]$ GU (333)(11b)

7. $$\int \frac{x^n\,dx}{\sin^2 x} = -x^n \cot x + \frac{n}{n-1}x^{n-1} + n\sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}x^{n+2k-1}}{(n+2k-1)(2k)!}B_{2k}$$
$$[|x| < \pi, \quad n > 1]$$ GU (333)(8c)

8. $$\int \frac{dx}{x^n \sin^2 x} = -\frac{\cot x}{x^n} + \frac{n}{(n+1)x^{n+1}} - [1-(-1)^n](-1)^{\frac{n+1}{2}}\frac{2^n n}{(n+1)!}B_{n+1}\ln x$$
$$-\frac{n}{2^{n+1}}\sum_{\substack{k=1\\ k\neq\frac{n+1}{2}}}^{\infty}\frac{(-1)^k(2x)^{2k}}{(2k-n-1)(2k)!}B_{2k}$$
$$[|x| < \pi]$$ GU (333)(9c)

9. $$\int \frac{x^n\,dx}{\cos^2 x} = x^n \tan x + n\sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}\left(2^{2k}-1\right)x^{n+2k-1}}{(n+2k-1)\cdot(2k)!}B_{2k}$$
$$\left[n > 1, \quad |x| < \frac{\pi}{2}\right]$$ GU (333)(10c)

10. $$\int \frac{dx}{x^n \cos^2 x} = \frac{\tan x}{x^n} - [1-(-1)^n](-1)^{\frac{n+1}{2}}\frac{2^n n}{(n+1)!}\left(2^{n+1}-1\right)B_{n+1}\ln x$$
$$-\frac{n}{x^{n+1}}\sum_{\substack{k=1\\ k\neq\frac{n+1}{2}}}^{\infty}\frac{(-1)^k\left(2^{2k}-1\right)(2x)^{2k}}{(2k-n-1)(2k)!}B_{2k}$$
$$\left[|x| < \frac{\pi}{2}\right]$$ GU (333)(11c)

2.644

1. $$\int \frac{x\,dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1}\frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)}\frac{\sin x + (2n-2k)x\cos x}{(2n-2k+1)(2n-2k)\sin^{2n-2k+1} x}$$
$$+\frac{2^{n-1}(n-1)!}{(2n-1)!!}(\ln\sin x - x\cot x)$$

2. $$\int \frac{x\,dx}{\sin^{2n+1} x} = -\sum_{k=0}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2n(2n-2)\ldots(2n-2k+2)}\frac{\sin x + (2n-2k-1)x\cos x}{(2n-2k)(2n-2k-1)\sin^{2n-2k} x}$$
$$+\frac{(2n-1)!!}{2^n n!}\int\frac{x\,dx}{\sin x}$$
(see **2.644** 5)

3. $$\int \frac{x\,dx}{\cos^{2n} x} = \sum_{k=0}^{n-1}\frac{(2n-2)(2n-4)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)}\frac{(2n-2k)x\sin x - \cos x}{(2n-2k+1)(2n-2k)\cos^{2n-2k+1} x}$$
$$+\frac{2^{n-1}(n-1)!}{(2n-1)!!}(x\tan x + \ln\cos x)$$

4. $$\int \frac{x\,dx}{\cos^{2n+1} x} = \sum_{k=0}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2n(2n-2)\ldots(2n-2k+2)}\frac{(2n-2k+1)x\sin x - \cos x}{(2n-2k)(2n-2k-1)\cos^{2n-2k} x}$$
$$+\frac{(2n-1)!!}{2^n n!}\int\frac{x\,dx}{\cos x}$$
(see **2.644** 6)

5. $$\int \frac{x\,dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2\left(2^{2k-1}-1\right)}{(2k+1)!} B_{2k} x^{2k+1}$$

6. $$\int \frac{x\,dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}$$

7. $$\int \frac{x\,dx}{\sin^2 x} = -x\cot x + \ln \sin x$$

8. $$\int \frac{x\,dx}{\cos^2 x} = x\tan x + \ln \cos x$$

9. $$\int \frac{x\,dx}{\sin^3 x} = -\frac{\sin x + x\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{x}{\sin x}\,dx$$ (see **2.644** 5)

10. $$\int \frac{x\,dx}{\cos^3 x} = \frac{x\sin x - \cos x}{2\cos^2 x} + \frac{1}{2}\int \frac{x\,dx}{\cos x}$$ (see **2.644** 6)

11. $$\int \frac{x\,dx}{\sin^4 x} = -\frac{x\cos x}{3\sin^3 x} - \frac{1}{6\sin^2 x} - \frac{2}{3}x\cot x + \frac{2}{3}\ln(\sin x)$$

12. $$\int \frac{x\,dx}{\cos^4 x} = \frac{x\sin x}{3\cos^3 x} - \frac{1}{6\cos^2 x} + \frac{2}{3}x\tan x - \frac{2}{3}\ln(\cos x)$$

13. $$\int \frac{x\,dx}{\sin^5 x} = -\frac{x\cos x}{4\sin^4 x} - \frac{1}{12\sin^3 x} - \frac{3x\cos x}{8\sin^2 x} - \frac{3}{8\sin x} + \frac{3}{8}\int \frac{x\,dx}{\sin x}$$

(see **2.644** 5)

14. $$\int \frac{x\,dx}{\cos^5 x} = \frac{x\sin x}{4\cos^4 x} - \frac{1}{12\cos^3 x} + \frac{3x\sin x}{8\cos^2 x} - \frac{3}{8\cos x} + \frac{3}{8}\int \frac{x\,dx}{\cos x}$$

(see **2.644** 6)

2.645

1. $$\int x^p \frac{\sin^{2m} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\cos^{n-2k} x}$$ (see **2.643** 2)

2. $$\int x^p \frac{\sin^{2m+1} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x}\,dx$$ (see **2.645** 3)

3. $$\int x^p \frac{\sin x\,dx}{\cos^n x} = \frac{x^p}{(n-1)\cos^{n-1} x} - \frac{p}{n-1}\int \frac{x^{p-1}}{\cos^{n-1} x}\,dx$$

$[n > 1]$ (see **2.643** 2) GU (333)(12)

4. $$\int x^p \frac{\cos^{2m} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\sin^{n-2k} x}$$ (see **2.643** 1)

5. $$\int x^p \frac{\cos^{2m+1} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x}\,dx$$ (see **2.645** 6)

6. $$\int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1)\sin^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}\,dx}{\sin^{n-1} x}$$ $[n > 1]$ (see **2.643** 1) GU (333)(13)

7. $$\int \frac{x\cos x}{\sin^2 x}\,dx = -\frac{x}{\sin x} + \ln\tan\frac{x}{2}$$

8. $$\int \frac{x\sin x}{\cos^2 x}\,dx = \frac{x}{\cos x} - \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

2.646

1. $$\int x^p \tan x\,dx = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k}\left(2^{2k-1}-1\right)}{(p+2k)\cdot(2k)!}B_{2k}x^{p+2k}$$

$\left[p \geq -1, \quad |x| < \frac{\pi}{2}\right]$ GU (333)(12d)

2. $$\int x^p \cot x\,dx = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(p+2k)(2k)!}x^{p+2k}$$ $[p \geq 1, \quad |x| < \pi]$ GU (333)(13d)

3. $$\int x^p \tan^2 x\,dx = x\tan x + \ln\cos x - \frac{x^2}{2}$$

4. $$\int x\cot^2 x\,dx = -x\cot x + \ln\sin x - \frac{x^2}{2}$$

2.647

1. $$\int \frac{x^n\cos x\,dx}{(a+b\sin x)^m} = -\frac{x^n}{(m-1)b\,(a+b\sin x)^{m-1}} + \frac{n}{(m-1)b}\int\frac{x^{n-1}\,dx}{(a+b\sin x)^{m-1}}$$

$[m \neq 1]$ MZ 247

2. $$\int \frac{x^n\sin x\,dx}{(a+b\cos x)^m} = \frac{x^n}{(m-1)b\,(a+b\cos x)^{m-1}} - \frac{n}{(m-1)b}\int\frac{x^{n-1}\,dx}{(a+b\cos x)^{m-1}}$$

$[m \neq 1]$ MZ 247

3. $$\int \frac{x\,dx}{1+\sin x} = -x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + 2\ln\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)$$ PE (329)

4. $$\int \frac{x\,dx}{1-\sin x} = x\cot\left(\frac{\pi}{4}-\frac{x}{2}\right) + 2\ln\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)$$ PE (330)

5. $$\int \frac{x\,dx}{1+\cos x} = x\tan\frac{x}{2} + 2\ln\cos\frac{x}{2}$$ PE (331)

6. $$\int \frac{x\,dx}{1-\cos x} = -x\cot\frac{x}{2} + 2\ln\cos\frac{x}{2}$$ PE (332)

7. $$\int \frac{x\cos x}{(1+\sin x)^2}\,dx = -\frac{x}{1+\sin x} + \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)$$

8. $$\int \frac{x\cos x}{(1-\sin x)^2}\,dx = \frac{x}{1-\sin x} + \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

9. $$\int \frac{x\sin x}{(1+\cos x)^2}\,dx = \frac{x}{1+\cos x} - \tan\frac{x}{2}$$

10. $$\int \frac{x\sin x}{(1-\cos x)^2}\,dx = -\frac{x}{1-\cos x} - \cot\frac{x}{2}$$ MZ 247a

2.648

1. $$\int \frac{x+\sin x}{1+\cos x}\,dx = x\tan\frac{x}{2}$$

2. $$\int \frac{x-\sin x}{1-\cos x}\,dx = -x\cot\frac{x}{2}$$ GU (333)(16)

2.649 $$\int \frac{x^2\,dx}{\left[(ax-b)\sin x+(a+bx)\cos x\right]^2} = \frac{x\sin x+\cos x}{b\left[(ax-b)\sin x+(a+bx)\cos x\right]}$$ GU (333)(17)

2.651 $$\int \frac{dx}{\left[a+(ax+b)\tan x\right]^2} = \frac{\tan x}{a\left[a+(ax+b)\tan x\right]}$$ GU (333)(18)

2.652 $$\int \frac{x\,dx}{\cos(x+t)\cos(x-t)} = \operatorname{cosec} 2t\left\{x\ln\frac{\cos(x-t)}{\cos(x+t)} - L(x+t)+L(x-t)\right\}$$
$$\left[t\neq n\pi;\quad |x|<\left|\frac{\pi}{2}-|t_0|\right|\right],$$

where t_0 is the value of the argument t, which is reduced by multiples of the argument π to lie in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. LO III 288

2.653

1. $$\int \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{2\pi}\,S\left(\sqrt{x}\right)$$ (cf. **8.251** 21)

2. $$\int \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{2\pi}\,C\left(\sqrt{x}\right)$$ (cf. **8.251** 3)

2.654 **Notation**: $\Delta=\sqrt{1-k^2\sin^2 x}$, $\quad k'=\sqrt{1-k^2}$:

1. $$\int \frac{x\sin x\cos x}{\Delta}\,dx = -\frac{x\Delta}{k^2}+\frac{1}{k^2}\,E(x,k)$$

2. $$\int \frac{x\sin^3 x\cos x}{\Delta}\,dx = -\frac{k'^2}{9k^4}\,F(x,k)+\frac{2k^2+5}{9k^4}\,E(x,k)-\frac{1}{9k^4}\left[3\left(3-\Delta^2\right)x+k^2\sin x\cos x\right]\Delta$$

3. $$\int \frac{x\sin x\cos^3 x}{\Delta}\,dx = -\frac{k'^2}{9k^4}\,F(x,k)+\frac{7k^2-5}{9k^4}\,E(x,k)-\frac{1}{9k^4}\left[3\left(\Delta^2-3k'^2\right)x-k^2\sin x\cos x\right]\Delta$$

4. $$\int \frac{x\sin x\,dx}{\Delta^3}\,dx = -\frac{x\cos x}{k'^2\Delta}+\frac{1}{kk'^2}\arcsin\left(k\sin x\right)$$

5. $$\int \frac{x\cos x\,dx}{\Delta^3} = \frac{x\sin x}{\Delta}+\frac{1}{k}\ln\left(k\cos x+\Delta\right)$$

6. $$\int \frac{x\sin x\cos x\,dx}{\Delta^3} = \frac{x}{k^2\Delta}-\frac{1}{k^2}\,F(x,k)$$

7. $$\int \frac{x\sin^3 x\cos x\,dx}{\Delta^3} = x\frac{2-k^2\sin^2 x}{k^4\Delta}-\frac{1}{k^4}\left[E(x,k)+F(x,k)\right]$$

8. $$\int \frac{x\sin x\cos^3 x\,dx}{\Delta^3} = x\frac{k^2\sin^2 x+k^2-2}{k^4\Delta}+\frac{k'^2}{k^4}\,F(x,k)+\frac{1}{k^4}\,E(x,k)$$

2.655 Integrals containing $\sin x^2$ and $\cos x^2$

In integrals containing $\sin x^2$ and $\cos x^2$, it is expedient to make the substitution $x^2 = u$.

1. $$\int x^p \sin x^2\,dx = -\frac{x^{p-1}}{2}\cos x^2 + \frac{p-1}{2}\int x^{p-2}\cos x^2\,dx$$

2. $$\int x^p \cos x^2\,dx = \frac{x^{p-1}}{2}\sin x^2 - \frac{p-1}{2}\int x^{p-2}\sin x^2\,dx$$

3. $$\int x^n \sin x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^k\left[\frac{x^{n-4k+3}\cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1}\sin x^2}{2^{2k}(n-4k+1)!!}\right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\sin x^2\,dx\right\}$$
$$\left[r = \left\lfloor\frac{n}{4}\right\rfloor\right]$$ GU (336)(4a)

4. $$\int x^n \cos x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^{k-1}\left[\frac{x^{n-4k+3}\sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1}\cos x^2}{2^{2k}(n-4k+1)!!}\right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\cos x^2\,dx\right\}$$
$$\left[r = \left\lfloor\frac{n}{4}\right\rfloor\right]$$ GU (336)(5a)

5. $$\int x\sin x^2\,dx = -\frac{\cos^2 x}{2}$$

6. $$\int x\cos x^2\,dx = -\frac{\sin^2 x}{2}$$

7. $$\int x^2\sin x^2\,dx = -\frac{x}{2}\cos x^2 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,C(x)$$

8. $$\int x^2\cos x^2\,dx = \frac{x}{2}\sin x^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,S(x)$$

9. $$\int x^3\sin x^2\,dx = -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2$$

10. $$\int x^3\cos x^2\,dx = \frac{x^2}{2}\sin x^2 + \frac{1}{2}\cos x^2$$

2.66 Combinations of trigonometric functions and exponentials

2.661 $$\int e^{ax}\sin^p x\cos^q x\,dx = \frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^p x\cos^{q-1}x\,[a\cos x+(p+q)\sin x]$$

$$-\,pa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\,dx+(q-1)(p+q)\int e^{ax}\sin^p x\cos^{q-2}x\,dx\Big\}$$

TI (523)

$$=\frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1}x\cos^q x\,[a\sin x-(p+q)\cos x]$$

$$+\,qa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\,dx+(p-1)(p+q)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\Big\}$$

TI (524)

$$=\frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1}x\cos^{q-1}x\,\big[a\sin x\cos x+q\sin^2 x-p\cos^2 x\big]$$

$$+\,q(q-1)\int e^{ax}\sin^p x\cos^{q-2}x\,dx+p(p-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\Big\}$$

TI (525)

$$=\frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1}x\cos^{q-1}x\,\big(a\sin x\cos x+q\sin^2 x-p\cos^2 x\big)$$

$$+q(q-1)\int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx$$

$$-\,(q-p)(p+q-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\Big\}$$

TI (526)

$$=\frac{1}{a^2+(p+q)^2}\Big[e^{ax}\sin^{p-1}x\cos^{q-1}x\,\big(a\sin x\cos x+q\sin^2 x-p\cos^2 x\big)$$

$$+p(p-1)\int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx$$

$$+\,(q-p)(p+q-1)\int e^{ax}\sin^p x\cos^{q-2}x\,dx\Big]$$

GU (334)(1a)

For $p=m$ and $q=n$ even integers, the integral $\int e^{ax}\sin^m x\cos^n x\,dx$ can be reduced by means of these formulas to the integral $\int e^{ax}\,dx$. However, when only m or only n is even, they can be reduced to

integrals of the form $\int e^{ax}\cos^n x\,dx$ or $\int e^{ax}\sin^m x\,dx$, respectively.

2.662

1. $$\int e^{ax}\sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)\,e^{ax}\sin^{n-1} bx + n(n-1)b^2\int e^{ax}\sin^{n-2} bx\,dx\right]$$

2. $$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1} bx + n(n-1)b^2\int e^{ax}\cos^{n-2} bx\,dx\right]$$

3. $$\int e^{ax}\sin^{2m} bx\,dx$$
$$= \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\sin^{2m-2k-1} bx}{(2m-2k)!\,[a^2+(2m)^2b^2]\,[a^2+(2m-2)^2b^2]\cdots[a^2+(2m-2k)^2b^2]} \times[a\sin bx-(2m-2k)b\cos bx] + \frac{(2m)!b^{2m}e^{ax}}{[a^2+(2m)^2b^2]\,[a^2+(2m-2)^2b^2]\cdots[a^2+4b^2]\,a}$$
$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}\,(a\cos 2bkx + 2bk\sin 2bkx)$$

4. $$\int e^{ax}\sin^{2m+1} bx\,dx$$
$$= \sum_{k=0}^{m}\frac{(2m+1)!b^{2k}e^{ax}\sin^{2m-2k} bx\,[a\sin bx-(2m-2k+1)b\cos bx]}{(2m-2k+1)!\,[a^2+(2m+1)^2b^2]\,[a^2+(2m-1)^2b^2]\cdots[a^2+(2m-2k+1)^2b^2]}$$
$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\frac{(-1)^k}{a^2+(2k+1)^2b^2}\binom{2m+1}{m-k}[a\sin(2k+1)bx-(2k+1)b\cos(2k+1)bx]$$

5.[8] $$\int e^{ax}\cos^{2m} bx\,dx = \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\cos^{2m-2k-1} bx\,[a\cos bx+(2m-2k)b\sin bx]}{(2m-2k)!\,[a^2+(2m)^2b^2]\,[a^2+(2m-2)^2b^2]\cdots[a^2+(2m-2k)^2b^2]} + \frac{(2m)!b^{2m}e^{ax}}{[a^2+(2m)^2b^2]\,[a^2+(2m-2)^2b^2]\cdots[a^2+4b^2]\,a}$$
$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}\,[a\cos 2kbx + 2kb\sin 2kbx]$$

6. $$\int e^{ax}\cos^{2m+1} bx\,dx$$
$$= \sum_{k=0}^{m}\frac{(2m+1)!b^{2k}e^{ax}\cos^{2m-2k} bx}{(2m-2k+1)!\,[a^2+(2m-1)^2b^2]\cdots[a^2+(2m-2k+1)^2b^2]}$$
$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{1}{a^2+(2k+1)^2b^2}\,[a\cos(2k+1)bx+(2k+1)b\sin(2k+1)bx]$$

2.663

1. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\,(a\sin bx - b\cos bx)}{a^2+b^2}$$

2. $$\int e^{ax}\sin^2 bx\,dx = \frac{e^{ax}\sin bx\,(a\sin bx - 2b\cos bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)\,a}$$
$$= \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right)$$

3. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\,(a\cos bx + b\sin bx)}{a^2+b^2}$$

4. $$\int e^{ax}\cos^2 bx\,dx = \frac{e^{ax}\cos bx\,(a\cos bx + 2b\sin bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)\,a}$$
$$= \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right)$$

2.664

1. $$\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}}{2}\left[\frac{a\sin(b+c)x - (b+c)\cos(b+c)x}{a^2+(b+c)^2} + \frac{a\sin(b-c)x - (b-c)\cos(b-c)x}{a^2+(b-c)^2}\right]$$

GU (334)(6b)

2. $$\int e^{ax}\sin^2 bx\cos cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\cos cx + c\sin cx}{a^2+c^2} - \frac{a\cos(2b+c)x + (2b+c)\sin(2b+c)x}{a^2+(2b+c)^2} - \frac{a\cos(2b-c)x + (2b-c)\sin(2b-c)x}{a^2+(2b-c)^2}\right]$$

GU (334)(6c)

3. $$\int e^{ax}\sin bx\cos^2 cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\sin bx - b\cos bx}{a^2+b^2} + \frac{a\sin(b+2c)x - (b+2c)\cos(b+2c)x}{a^2+(b+2c)^2} + \frac{a\sin(b-2c)x - (b-2c)\cos(b-2c)x}{a^2+(b-2c)^2}\right]$$

GU (334)(6d)

2.665

1. $$\int \frac{e^{ax}\,dx}{\sin^p bx} = -\frac{e^{ax}\,[a\sin bx + (p-2)b\cos bx]}{(p-1)(p-2)b^2\sin^{p-1} bx} + \frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int \frac{e^{ax}\,dx}{\sin^{p-2} bx}$$ TI (530)a

2. $$\int \frac{e^{ax}\,dx}{\cos^p bx} = -\frac{e^{ax}\,[a\cos bx - (p-2)b\sin bx]}{(p-1)(p-2)b^2\cos^{p-1} bx} + \frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int \frac{e^{ax}\,dx}{\cos^{p-2} bx}$$ TI (529)a

By successive applications of formulas **2.665** for p a natural number, we obtain integrals of the form $\int \frac{e^{ax}\,dx}{\sin bx}$, $\int \frac{e^{ax}\,dx}{\sin^2 bx}$, $\int \frac{e^{ax}\,dx}{\cos bx}$, $\int \frac{e^{ax}\,dx}{\cos^2 bx}$, which are not expressible in terms of a finite combination of elementary functions.

2.666

1. $$\int e^{ax}\tan^p x\,dx = \frac{e^{ax}}{p-1}\tan^{p-1} x - \frac{a}{p-1}\int e^{ax}\tan^{p-1} x\,dx - \int e^{ax}\tan^{p-2} x\,dx$$ TI (527)

2. $$\int e^{ax}\cot^p x\,dx = -\frac{e^{ax}\cot^{p-1} x}{p-1} + \frac{a}{p-1}\int e^{ax}\cot^{p-1} x\,dx - \int e^{ax}\cot^{p-2} x\,dx$$ TI (528)

3. $$\int e^{ax}\tan x\,dx = \frac{e^{ax}\tan x}{a} - \frac{1}{a}\int \frac{e^{ax}\,dx}{\cos^2 x}$$ (see remark following **2.665**)

4. $\displaystyle\int e^{ax}\tan^2 x\,dx = \frac{e^{ax}}{a}\left(a\tan x - 1\right) - a\int e^{ax}\tan x\,dx$ (see **2.666** 3) TI 355

5. $\displaystyle\int e^{ax}\cot x\,dx = \frac{e^{ax}\cot x}{a} + \frac{1}{a}\int\frac{e^{ax}\,dx}{\sin^2 x}$ (see remark following **2.665**)

6. $\displaystyle\int e^{ax}\cot^2 x\,dx = -\frac{e^{ax}}{a}\left(a\cot x + 1\right) + a\int e^{ax}\cot x\,dx$

(see **2.666** 5)

Integrals of type $\int R\left(x, e^{ax}, \sin bx, \cos cx\right)\,dx$

Notation: $\sin t = -\dfrac{b}{\sqrt{a^2+b^2}}$; $\quad \cos t = \dfrac{a}{\sqrt{a^2+b^2}}$.

2.667

1. $$\int x^p e^{ax}\sin bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}\left(a\sin bx - b\cos bx\right)\,dx$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}}\sin(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax}\sin(bx+t)\,dx$$

2. $$\int x^p e^{ax}\cos bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}\left(a\cos bx + b\sin bx\right)\,dx$$
$$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}}\cos(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax}\cos(bx+t)\,dx$$

3. $$\int x^n e^{ax}\sin bx\,dx = e^{ax}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}n!x^{n-k+1}}{(n-k+1)!\left(a^2+b^2\right)^{k/2}}\sin(bx+kt)$$

4. $$\int x^n e^{ax}\cos bx\,dx = e^{ax}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}n!x^{n-k+1}}{(n-k+1)!\left(a^2+b^2\right)^{k/2}}\cos(bx+kt)$$

5. $$\int xe^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\sin bx - \left(bx - \frac{2ab}{a^2+b^2}\right)\cos bx\right]$$

6. $$\int xe^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\cos bx + \left(bx - \frac{2ab}{a^2+b^2}\right)\sin bx\right]$$

7. $$\int x^2 e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left\{\left[ax^2 - \frac{2\left(a^2-b^2\right)}{a^2+b^2}x + \frac{2a\left(a^2-3b^2\right)}{\left(a^2+b^2\right)^2}\right]\sin bx\right.$$
$$\left. - \left[bx^2 - \frac{4ab}{a^2+b^2}x + \frac{2b\left(3a^2-b^2\right)}{\left(a^2+b^2\right)^2}\right]\cos bx\right\}$$

8. $$\int x^2 e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} \left\{ \left[ax^2 - \frac{2\left(a^2-b^2\right)}{a^2+b^2} x + \frac{2a\left(a^2-3b^2\right)}{\left(a^2+b^2\right)^2} \right] \cos bx + \left[bx^2 - \frac{4ab}{a^2+b^2} x + \frac{2b\left(3a^2-b^2\right)}{\left(a^2+b^2\right)^2} \right] \sin bx \right\}$$

GU (335), MZ 274-275

2.67 Combinations of trigonometric and hyperbolic functions

2.671

1. $$\int \sinh(ax+b)\sin(cx+d)\, dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d)$$

2. $$\int \sinh(ax+b)\cos(cx+d)\, dx = \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d)$$

3. $$\int \cosh(ax+b)\sin(cx+d)\, dx = \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d)$$

4. $$\int \cosh(ax+b)\cos(cx+d)\, dx = \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d)$$

GU (354)(1)

2.672

1. $$\int \sinh x \sin x\, dx = \frac{1}{2}\left(\cosh x \sin x - \sinh x \cos x\right)$$

2. $$\int \sinh x \cos x\, dx = \frac{1}{2}\left(\cosh x \cos x + \sinh x \sin x\right)$$

3. $$\int \cosh x \sin x\, dx = \frac{1}{2}\left(\sinh x \sin x - \cosh x \cos x\right)$$

4. $$\int \cosh x \cos x\, dx = \frac{1}{2}\left(\sinh x \cos x + \cosh x \sin x\right)$$

2.673

1. $$\int \sinh^{2m}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{(-1)^{m+n}}{2^{2m+2n-1}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k)c}\binom{2n}{k}\sin[(2n-2k)(cx+d)]$$
$$+\frac{(-1)^n}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(3a)

2. $$\int \sinh^{2m}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{m+n}}{2^{2m+2n-2}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k-1)c}\binom{2n-1}{k}\cos[(2n-2k-1)(cx+d)]$$
$$+\frac{(-1)^{n-1}}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(3b)

3. $$\int \sinh^{2m-1}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\cosh[(2m-2j-1)(ax+d)]$$
$$+\frac{(-1)^n}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(3c)

4. $$\int \sinh^{2m-1}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}}{2^{2m-2n-4}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(3d)

5. $$\int \sinh^{2m}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x+\frac{\binom{2n}{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m}{j}}{(2m-2j)a}\sinh[(2m-2j)(ax+b)]$$
$$+\frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{\binom{2n}{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)]$$
$$+\frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(4a)

6. $$\int \sinh^{2m}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\sin[(2n-2k-2)(cx+d)]$$
$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^j\binom{2m}{j}\binom{2-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(4a)

7. $$\int \sinh^{2m-1}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)]$$
$$+ \frac{1}{2^{2m-2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(4b)

8. $$\int \sinh^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(4b)

9. $$\int \cosh^{2m}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{(-1)^n\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)]$$
$$+ \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)]$$
$$+ \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(5a)

10. $$\int \cosh^{2m-1}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)]$$
$$+\frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(5a)

11. $$\int \cosh^{2m}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1}\binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)]$$
$$+\frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(5b)

12. $$\int \cosh^{2m-1}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(5b)

13. $$\int \cosh^{2m}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{\binom{2m}{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{\binom{2}{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)]$$
$$+\frac{\binom{2n}{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{\binom{2m}{j}}{(2m-2j)a}\sinh[(2m-2j)(ax+b)]$$
$$+\frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(6)

14. $$\int \cosh^{2m-1}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{\binom{2m-1}{j}}{(2m-2j-1)a}\sinh[(2m-2j-1)(ax+b)]$$
$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(6)

15. $$\int \cosh^{2m}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\sin[(2n-2k-1)(cx+d)]$$
$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(6)

16. $$\int \cosh^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{1}{2^{2m+2n-4}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(6)

2.674

1. $$\int e^{ax}\sinh bx\sin cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\sin cx - c\cos cx\right] - \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\sin cx - c\cos cx\right]$$

2. $$\int e^{ax}\sinh bx\cos cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\cos cx + c\sin cx\right] - \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\cos cx + c\sin cx\right]$$

3. $$\int e^{ax}\cosh bx\sin cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\sin cx - c\cos cx\right] + \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\sin cx - c\cos cx\right]$$

4. $$\int e^{ax}\cosh bx\cos cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\cos cx + c\sin cx\right] + \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\cos cx + c\sin cx\right]$$

MZ 379

2.7 Logarithms and Inverse-Hyperbolic Functions

2.71 The logarithm

2.711 $$\int \ln^m x\,dx = x\ln^m x - m\int \ln^{m-1} x\,dx$$
$$= \frac{x}{m+1}\sum_{k=0}^{m}(-1)^k(m+1)m(m-1)\cdots(m-k+1)\ln^{m-k} x$$
$$(m>0)$$

TI (603)

2.72–2.73 Combinations of logarithms and algebraic functions

2.721

1. $$\int x^n \ln^m x\,dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x\,dx \qquad \text{(see } \mathbf{2.722}\text{)}$$

For $n=-1$

2. $$\int \frac{\ln^m x\,dx}{x} = \frac{\ln^{m+1} x}{m+1}$$

For $n=-1$ and $m=-1$

3. $$\int \frac{dx}{x\ln x} = \ln(\ln x)$$

2.722 $$\int x^n \ln^m x\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}(-1)^k(m+1)m(m-1)\cdots(m-k+1)\frac{\ln^{m-k} x}{(n+1)^{k+1}}$$ TI (604)

2.723

1. $$\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right]$$ TI 375

2. $$\int x^n \ln^2 x\,dx = x^{n+1}\left[\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right]$$ TI 375

3. $$\int x^n \ln^3 x\,dx = x^{n+1}\left[\frac{\ln^3 x}{n+1} - \frac{3\ln^2 x}{(n+1)^2} + \frac{6\ln x}{(n+1)^3} - \frac{6}{(n+1)^4}\right]$$

2.724

1. $$\int \frac{x^n\,dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\ln x)^{m-1}}$$

For $m=1$

2. $$\int \frac{x^n\,dx}{\ln x} = \operatorname{li}\left(x^{n+1}\right)$$

2.725

1. $$\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1}\ln x - \int \frac{(a+bx)^{m+1}\,dx}{x}\right]$$ TI 374

2. $$\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1} - a^{m+1}\right]\ln x - \sum_{k=0}^{m}\frac{\binom{m}{k}a^{m-k}b^k x^{k+1}}{(k+1)^2}$$

For $m=-1$, see **2.727** 2.

2.726

1. $$\int (a+bx)\ln x\,dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b}\right]\ln x - \left(ax + \frac{1}{4}bx^2\right)$$

2. $$\int (a+bx)^2 \ln x\,dx = \frac{1}{3b}\left[(a+bx)^3 - a^3\right]\ln x - \left(a^2x + \frac{abx^2}{2} + \frac{b^2x^3}{9}\right)$$

3. $$\int (a+bx)^3 \ln x\,dx = \frac{1}{4b}\left[(a+bx)^4 - a^4\right]\ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4\right)$$

2.727

1.[8] $$\int \frac{\ln x\,dx}{(a+bx)^m} = \frac{1}{b(m-1)}\left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right]$$ TI 376

For $m=1$

2.[8] $$\int \frac{\ln x\,dx}{a+bx} = \frac{1}{b}\ln x \ln(a+bx) - \frac{1}{b}\int \frac{\ln(a+bx)\,dx}{x}$$ (see **2.728** 2)

3. $$\int \frac{\ln x\,dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab}\ln\frac{x}{a+bx}$$

4. $$\int \frac{\ln x\,dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b}\ln\frac{x}{a+bx}$$

5. $$\int \frac{\ln x\,dx}{\sqrt{a+bx}} = \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} - 2\sqrt{a}\ln\left[\frac{(a+bx)^{1/2} - a^{1/2}}{x^{1/2}}\right]\right\} \qquad [a>0]$$
$$= \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a}\arctan\sqrt{\frac{a+bx}{-a}}\right\} \qquad [a<0]$$

2.728

1. $$\int x^m \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1}\ln(a+bx) - b\int \frac{x^{m+1}\,dx}{a+bx}\right]$$

2.[9] $$\int \frac{\ln(a+bx)}{x} = \ln a \ln x + \frac{bx}{a}\,\Phi\left(-\frac{bx}{a}, 2, 1\right) \qquad [a>0]$$

2.729

1. $$\int x^m \ln(a+bx)\,dx = \frac{1}{m+1}\left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}}\right]\ln(a+bx) + \frac{1}{m+1}\sum_{k=1}^{m+1}\frac{(-1)^k x^{m-k+2}a^{k-1}}{(m-k+2)b^{k-1}}$$

2. $$\int x\ln(a+bx)\,dx = \frac{1}{2}\left[x^2 - \frac{a^2}{b^2}\right]\ln(a+bx) - \frac{1}{2}\left[\frac{x^2}{2} - \frac{ax}{b}\right]$$

3. $$\int x^2\ln(a+bx)\,dx = \frac{1}{3}\left[x^3 + \frac{a^3}{b^3}\right]\ln(a+bx) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right]$$

4. $$\int x^3\ln(a+bx)\,dx = \frac{1}{4}\left[x^4 - \frac{a^4}{b^4}\right]\ln(a+bx) - \frac{1}{4}\left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3}\right]$$

2.731 $$\int x^{2n}\ln\left(x^2+a^2\right)\,dx = \frac{1}{2n+1}\left\{x^{2n+1}\ln\left(x^2+a^2\right) + (-1)^n 2a^{2n+1}\arctan\frac{x}{a} - 2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2k+1}a^{2n-2k}x^{2k+1}\right\}$$

2.732[7] $$\int x^{2n+1}\ln\left(x^2+a^2\right)\,dx=\frac{1}{2n+2}\left\{\left(x^{2n+2}+(-1)^n a^{2n+2}\right)\ln\left(x^2+a^2\right)+\sum_{k=1}^{n+1}\frac{(-1)^{n-k}}{k}a^{2n-2k+2}x^{2k}\right\}$$

2.733

1. $$\int \ln\left(x^2+a^2\right)\,dx=x\ln\left(x^2+a^2\right)-2x+2a\arctan\frac{x}{a}$$ DW

2. $$\int x\ln\left(x^2+a^2\right)\,dx=\frac{1}{2}\left[\left(x^2+a^2\right)\ln\left(x^2+a^2\right)-x^2\right]$$ DW

3. $$\int x^2\ln\left(x^2+a^2\right)\,dx=\frac{1}{3}\left[x^3\ln\left(x^2+a^2\right)-\frac{2}{3}x^3+2a^2x-2a^3\arctan\frac{x}{a}\right]$$ DW

4. $$\int x^3\ln\left(x^2+a^2\right)\,dx=\frac{1}{4}\left[\left(x^4-a^4\right)\ln\left(x^2+a^2\right)-\frac{x^4}{2}+a^2x^2\right]$$ DW

5. $$\int x^4\ln\left(x^2+a^2\right)\,dx=\frac{1}{5}\left[x^5\ln\left(x^2+a^2\right)-\frac{2}{5}x^5+\frac{2}{3}a^2x^3-2a^4x+2a^5\arctan\frac{x}{a}\right]$$ DW

2.734 $$\int x^{2n}\ln\left|x^2-a^2\right|dx=\frac{1}{2n+1}\left\{x^{2n+1}\ln\left|x^2-a^2\right|+a^{2n+1}\ln\left|\frac{x+a}{x-a}\right|-2\sum_{k=0}^{n}\frac{1}{2k+1}a^{2n-2k}x^{2k+1}\right\}$$

2.735 $$\int x^{2n+1}\ln\left|x^2-a^2\right|dx=\frac{1}{2n+2}\left\{\left(x^{2n+2}-a^{2n+2}\right)\ln\left|x^2-a^2\right|-\sum_{k=1}^{n+1}\frac{1}{k}a^{2n-2k+2}x^{2k}\right\}$$

2.736

1. $$\int \ln\left|x^2-a^2\right|dx=x\ln\left|x^2-a^2\right|-2x+a\ln\left|\frac{x+a}{x-a}\right|$$ DW

2. $$\int x\ln\left|x^2-a^2\right|dx=\frac{1}{2}\left\{\left(x^2-a^2\right)\ln\left|x^2-a^2\right|-x^2\right\}$$ DW

3. $$\int x^2\ln\left|x^2-a^2\right|dx=\frac{1}{3}\left\{x^3\ln\left|x^2-a^2\right|-\frac{2}{3}x^3-2a^2x+a^3\ln\left|\frac{x+a}{x-a}\right|\right\}$$ DW

4. $$\int x^3\ln\left|x^2-a^2\right|dx=\frac{1}{4}\left\{\left(x^4-a^4\right)\ln\left|x^2-a^2\right|-\frac{x^4}{2}-a^2x^2\right\}$$ DW

5. $$\int x^4\ln\left|x^2-a^2\right|dx=\frac{1}{5}\left\{x^5\ln\left|x^2-a^2\right|-\frac{2}{5}x^5-\frac{2}{3}a^2x^3-2a^4x+a^5\ln\left|\frac{x+a}{x-a}\right|\right\}$$ DW

2.74 Inverse hyperbolic functions

2.741

1. $$\int \operatorname{arcsinh}\frac{x}{a}\,dx=x\operatorname{arcsinh}\frac{x}{a}-\sqrt{x^2+a^2}$$ DW

2. $$\int \operatorname{arccosh}\frac{x}{a}\,dx = x\operatorname{arccosh}\frac{x}{a} - \sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} > 0\right]$$ DW
$$= x\operatorname{arccosh}\frac{x}{a} + \sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} < 0\right]$$ DW

3. $$\int \operatorname{arctanh}\frac{x}{a}\,dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln\left(a^2-x^2\right)$$ DW

4. $$\int \operatorname{arccoth}\frac{x}{a}\,dx = x\operatorname{arccoth}\frac{x}{a} + \frac{a}{2}\ln\left(x^2-a^2\right)$$ DW

2.742

1. $$\int x\operatorname{arcsinh}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\operatorname{arcsinh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}$$ DW

2. $$\int x\operatorname{arccosh}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{arccosh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} > 0\right]$$
$$= \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{arccosh}\frac{x}{a} + \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} < 0\right]$$
DW

2.8 Inverse Trigonometric Functions

2.81 Arcsines and arccosines

2.811 $$\int \left(\arcsin\frac{x}{a}\right)^n dx = x\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cdot(2k)!\left(\arcsin\frac{x}{a}\right)^{n-2k}$$
$$+\sqrt{a^2-x^2}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\binom{n}{2k-1}\cdot(2k-1)!\left(\arcsin\frac{x}{a}\right)^{n-2k+1}$$

2.812 $$\int \left(\arccos\frac{x}{a}\right)^n dx = x\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cdot(2k)!\left(\arccos\frac{x}{a}\right)^{n-2k}$$
$$+\sqrt{a^2-x^2}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k}\binom{n}{2k-1}\cdot(2k-1)!\left(\arccos\frac{x}{a}\right)^{n-2k+1}$$

2.813

1.[11] $$\int \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[x\arcsin\frac{x}{|a|} + \sqrt{a^2-x^2}\right]$$

2.[9] $$\int \left(\arcsin\frac{x}{a}\right)^2 dx = x\left(\arcsin\frac{x}{|a|}\right)^2 + 2\sqrt{a^2-x^2}\arcsin\frac{x}{|a|} - 2x$$

3. $$\int \left(\arcsin\frac{x}{a}\right)^3 dx = \operatorname{sign}(a)\left[x\left(\arcsin\frac{x}{|a|}\right)^3 + 3\sqrt{a^2-x^2}\left(\arcsin\frac{x}{|a|}\right)^2\right.$$
$$\left. - 6x\arcsin\frac{x}{|a|} - 6\sqrt{a^2-x^2}\right]$$

2.814

1. $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}$$

2. $$\int \left(\arccos\frac{x}{a}\right)^2 dx = x\left(\arccos\frac{x}{a}\right)^2 - 2\sqrt{a^2-x^2}\arccos\frac{x}{a} - 2x$$

3. $$\int \left(\arccos\frac{x}{a}\right)^3 dx = x\left(\arccos\frac{x}{a}\right)^3 - 3\sqrt{a^2-x^2}\left(\arccos\frac{x}{a}\right)^2 - 6x\arccos\frac{x}{a} + 6\sqrt{a^2-x^2}$$

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.821

1. $$\int \operatorname{arccosec}\frac{x}{a}\,dx = \int \arcsin\frac{a}{x}\,dx = x\arcsin\frac{x}{2} + a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[0<\arcsin\frac{a}{x}<\frac{\pi}{2}\right]$$
$$= x\arcsin\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[-\frac{\pi}{2}<\arcsin\frac{a}{x}<0\right]$$
DW

2. $$\int \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = x\arccos\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[0<\arccos\frac{a}{x}<\frac{\pi}{2}\right]$$
$$= x\arccos\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[-\frac{\pi}{2}<\arccos\frac{a}{x}<0\right]$$
DW

2.822

1.[8] $$\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln\left(a^2+x^2\right)$$ DW

2. $$\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} - \frac{a}{2}\ln\left(a^2+x^2\right)$$ DW

3.[9] $$\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a} - \frac{ax}{2}$$

4.[9] $$\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a}$$

5.[9] $$\int x^2\arctan\frac{x}{a}\,dx = \frac{1}{3}x^3\arctan\frac{x}{a} + \frac{1}{6}a^3\ln\left(x^2+a^2\right) - \frac{ax^2}{6}$$

6.[9] $$\int x^2\operatorname{arccot}\frac{x}{a}\,dx = -\frac{1}{3}x^3\arctan\frac{x}{a} - \frac{1}{6}a^3\ln\left(x^2+a^2\right) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

2.83 Combinations of arcsine or arccosine and algebraic functions

2.831 $$\int x^n\arcsin\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arcsin\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}$$ (see **2.263** 1, **2.264**, **2.27**)

2.832 $$\int x^n\arccos\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arccos\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}$$ (see **2.263** 1, **2.264**, **2.27**)

1. For $n=-1$, these integrals (that is, $\int\frac{\arcsin x}{x}\,dx$ and $\int\frac{\arccos x}{x}\,dx$) cannot be expressed as a finite combination of elementary functions.

2. $$\int \frac{\arccos x}{x}\,dx = -\frac{\pi}{2}\ln\frac{1}{x} - \int \frac{\arcsin x}{x}\,dx$$

2.833[9]

1. $$\int x\arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{|a|}+\frac{x}{4}\sqrt{a^2-x^2}\right]$$

2. $$\int x\arccos\frac{x}{a}\,dx = \frac{\pi x^2}{4} - \operatorname{sign}(a)\left[\frac{1}{4}\left(2x^2-a^2\right)\arcsin\frac{x}{|a|}+\frac{x}{4}\sqrt{a^2-x^2}\right]$$

3. $$\int x^2\arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|}+\frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2}\right]$$

4. $$\int x^2\arccos\frac{x}{a}\,dx = \frac{\pi x^3}{6} - \operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|}+\frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2}\right]$$

5. $$\int x^3\arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^4}{4}-\frac{3a^4}{32}\right)\arcsin\frac{x}{|a|}+\frac{1}{32}x\left(2x^2+3a^2\right)\sqrt{a^2-x^2}\right]$$

6. $$\int x^3\arccos\frac{x}{a}\,dx = \frac{\pi x^4}{8} - \operatorname{sign}(a)\left[\frac{\left(8x^4-3a^4\right)}{32}\arcsin\frac{x}{|a|}+\frac{1}{32}x\left(2x^2+3a^2\right)\sqrt{a^2-x^2}\right]$$

2.834

1. $$\int \frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}$$

2. $$\int \frac{1}{x^2}\arccos\frac{x}{a}\,dx = -\frac{1}{x}\arccos\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}$$

2.835 $$\int \frac{\arcsin x}{(a+bx)^2}\,dx = -\frac{\arcsin x}{b(a+bx)} - \frac{2}{b\sqrt{a^2-b^2}}\arctan\sqrt{\frac{(a-b)(1-x)}{(a+b)(1+x)}} \qquad \left[a^2>b^2\right]$$

$$= -\frac{\arcsin x}{b(a+bx)} - \frac{1}{b\sqrt{b^2-a^2}}\ln\frac{\sqrt{(a+b)(1+x)}+\sqrt{(b-a)(1-x)}}{\sqrt{(a+b)(1+x)}-\sqrt{(b-a)(1-x)}} \qquad \left[a^2<b^2\right]$$

2.836[8] $$\int \frac{x\arcsin x}{\left(1+cx^2\right)^2}\,dx = -\frac{\arcsin x}{2c\left(1+cx^2\right)} + \frac{1}{2c\sqrt{c+1}}\arctan\frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \qquad [c>-1]$$

$$= -\frac{\arcsin x}{2c\left(1+cx^2\right)} + \frac{1}{4c\sqrt{-(c+1)}}\ln\frac{\sqrt{1-x^2}+x\sqrt{-(c+1)}}{\sqrt{1-x^2}-x\sqrt{-(c+1)}} \qquad [c<-1]$$

2.837

1. $$\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = x - \sqrt{1-x^2}\arcsin x$$

2. $$\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^2}{4} - \frac{x}{2}\sqrt{1-x^2}\arcsin x + \frac{1}{4}\left(\arcsin x\right)^2$$

3. $$\int \frac{x^3\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3}\left(x^2+2\right)\sqrt{1-x^2}\arcsin x$$

2.838

1. $$\int \frac{\arcsin x}{\sqrt{\left(1-x^2\right)^3}}\,dx = \frac{x\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\left(1-x^2\right)$$

2. $$\int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}}\,dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\frac{1-x}{1+x}$$

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.841

1. $$\int x \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arccos\frac{a}{x} - a\sqrt{x^2-a^2}\right\} \qquad \left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{2}\left\{x^2 \arccos\frac{a}{x} + a\sqrt{x^2-a^2}\right\} \qquad \left[\frac{\pi}{2} < \arccos\frac{a}{x} < \pi\right]$$
DW

2. $$\int x^2 \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = \frac{1}{3}\left\{x^3 \arccos\frac{a}{x} - \frac{a}{2}x\sqrt{x^2-a^2} - \frac{a^3}{2}\ln\left(x+\sqrt{x^2-a^2}\right)\right\}$$
$$\left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{3}\left\{x^3 \arccos\frac{a}{x} + \frac{a}{2}x\sqrt{x^2-a^2} + \frac{a^3}{2}\ln\left(x+\sqrt{x^2-z^2}\right)\right\}$$
$$\left[\frac{\pi}{2} < \arccos\frac{a}{x} < \pi\right]$$
DW

3. $$\int x \operatorname{arccosec}\frac{x}{a}\,dx = \int \arcsin\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arcsin\frac{a}{x} + a\sqrt{x^2-a^2}\right\} \qquad \left[0 < \arcsin\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{2}\left\{x^2 \arcsin\frac{a}{x} - a\sqrt{x^2-a^2}\right\} \qquad \left[-\frac{\pi}{2} < \arcsin\frac{a}{x} < 0\right]$$
DW

2.85 Combinations of the arctangent and arccotangent with algebraic functions

2.851 $$\int x^n \arctan\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arctan\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}$$

2.852

1. $$\int x^n \operatorname{arccot}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arccot}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}$$
For $n=-1$

2. $\displaystyle\int \frac{\arctan x}{x}\,dx$ cannot be expressed as a finite combination of elementary functions.

3. $$\int \frac{\operatorname{arccot} x}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\arctan x}{x}\,dx$$

2.853

1. $$\int x \arctan\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a} - \frac{ax}{2}$$

2. $$\int x \operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}$$

3.[9] $$\int x^2 \arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} + \frac{a^3}{6}\ln\left(x^2+a^2\right) - \frac{ax^2}{6}$$

4.[9] $$\int x^2 \operatorname{arccot}\frac{x}{a}\,dx = -\frac{x^3}{3}\arctan\frac{x}{a} - \frac{a^3}{6}\ln\left(x^2+a^2\right) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

2.854 $$\int \frac{1}{x^2}\arctan\frac{x}{a}\,dx = -\frac{1}{x}\arctan\frac{x}{a} - \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}$$

2.855 $$\int \frac{\arctan x}{(\alpha+\beta x)^2}\,dx = \frac{1}{\alpha^2+\beta^2}\left\{\ln\frac{\alpha+\beta x}{\sqrt{1+x^2}} - \frac{\beta-\alpha x}{\alpha+\beta x}\arctan x\right\}$$

2.856

1. $$\int \frac{x\arctan x}{1+x^2}\,dx = \frac{1}{2}\arctan x\ln\left(1+x^2\right) - \frac{1}{2}\int\frac{\ln\left(1+x^2\right)\,dx}{1+x^2}$$ TI (689)

2. $$\int \frac{x^2\arctan x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) - \frac{1}{2}\left(\arctan x\right)^2$$ TI (405)

3. $$\int \frac{x^3\arctan x}{1+x^2}\,dx = -\frac{1}{2}x + \frac{1}{2}\left(1+x^2\right)\arctan x - \int\frac{x\arctan x}{1+x^2}\,dx$$

(see **2.8511**)

4. $$\int \frac{x^4\arctan x}{1+x^2}\,dx = -\frac{1}{6}x^2 + \frac{2}{3}\ln\left(1+x^2\right) + \left(\frac{x^3}{3} - x\right)\arctan x + \frac{1}{2}\left(\arctan x\right)^2$$

2.857 $$\int \frac{\arctan x\,dx}{\left(1+x^2\right)^{n+1}} = \left[\sum_{k=1}^{n}\frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!}\frac{x}{\left(1+x^2\right)^{n-k+1}} + \frac{1}{2}\frac{(2n-1)!!}{(2)!!}\arctan x\right]\arctan x$$
$$+\frac{1}{2}\sum_{k=1}^{n}\frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)}\frac{1}{\left(1+x^2\right)^{n-k+1}}$$

2.858 $$\int \frac{x\arctan x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\arctan x + \sqrt{2}\arctan\frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x$$

2.859 $$\int \frac{\arctan x}{\sqrt{\left(a+bx^2\right)^3}}\,dx = \frac{x\arctan x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}}\arctan\sqrt{\frac{a+bx^2}{b-a}} \qquad [a<b]$$
$$= \frac{x\arctan x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}}\ln\frac{\sqrt{a+bx^2}-\sqrt{a-b}}{\sqrt{a+bx^2}+\sqrt{a-b}} \qquad [a>b]$$

This page intentionally left blank

3–4 Definite Integrals of Elementary Functions

3.0 Introduction

3.01 Theorems of a general nature

3.011 Suppose that $f(x)$ is integrable† over the largest of the intervals $(p,q),(p,r),(r,q)$. Then (depending on the relative positions of the points p, q, and r) it is also integrable over the other two intervals, and we have

$$\int_p^q f(x)\,dx = \int_p^r f(x)\,dx + \int_r^q f(x)\,dx. \qquad \text{FI II 126}$$

3.012 *The first mean-value theorem.* Suppose (1) that $f(x)$ is continuous and that $g(x)$ is integrable over the interval (p,q), (2) that $m \le f(x) \le M$, and (3) that $g(x)$ does not change sign anywhere in the interval (p,q). Then, there exists at least one point ξ (with $p \le \xi \le q$) such that

$$\int_p^q f(x)g(x)\,dx = f(\xi)\int_p^q g(x)\,dx. \qquad \text{FI II 132}$$

3.013 *The second mean-value theorem.* If $f(x)$ is monotonic and non-negative throughout the interval (p,q), where $p <$q, and if $g(x)$ is integrable over that interval, then there exists at least one point ξ (with $p \le \xi \le q$) such that

1. $$\int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx$$

Under the conditions of Theorem **3.013** 1, if $f(x)$ is nondecreasing, then

2. $$\int_p^q f(x)g(x)\,dx = f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q]\,.$$

If $f(x)$ is monotonic in the interval (p,q), where $p <$q, and if $g(x)$ is integrable over that interval, then

*We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

†A function $f(x)$ is said to be integrable over the interval (p,q), if the integral $\int_p^q f(x)\,dx$ exists. Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

3. $$\int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx + f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q]\,,$$

or

4. $$\int_p^q f(x)g(x)\,dx = A\int_p^\xi g(x)\,dx + B\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q]\,,$$

where A and B are any two numbers satisfying the conditions

$$\begin{array}{llll} A \ge f(p+0) & \text{and} & B \le f(q-0) & [\text{if } f \text{ decreases}], \\ A \le f(p+0) & \text{and} & B \ge f(q-0) & [\text{if } f \text{ increases}]. \end{array}$$

In particular,

5. $$\int_p^q f(x)g(x)\,dx = f(p+0)\int_p^\xi g(x)\,dx + f(q-0)\int_\xi^q g(x)\,dx$$ FI II 138

3.02 Change of variable in a definite integral

3.020 $$\int_\alpha^\beta f(x)\,dx = \int_\varphi^\psi f[g(t)]g'(t)\,dt; \qquad x = g(t).$$

This formula is valid under the following conditions:

1. $f(x)$ is continuous on some interval $A \le x \le B$ containing the original limits of integration α and β.
2. The equalities $\alpha = g(\varphi)$ and $\beta = g(\psi)$ hold.
3. $g(t)$ and its derivative $g'(t)$ are continuous on the interval $\varphi \le t \le \psi$.
4. As t varies from φ to ψ, the function $g(t)$ always varies in the same direction from $g(\varphi) = \alpha$ to $g(\psi) = \beta$.*

3.021 The integral $\int_\alpha^\beta f(x)\,dx$ can be transformed into another integral with given limits φ and ψ by means of the linear substitution

$$x = \frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}\ :$$

1. $$\int_\alpha^\beta f(x)\,dx = \frac{\beta-\alpha}{\psi-\varphi}\int_\varphi^\psi f\left(\frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}\right)dt$$

In particular, for $\varphi = 0$ and $\psi = 1$,

*If this last condition is not satisfied, the interval $\varphi \le t \le \psi$ should be partitioned into subintervals throughout each of which the condition is satisfied:

$$\int_\alpha^\beta f(x)\,dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t)\,dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t)\,dt + \cdots + \int_{\varphi_{n-1}}^{\psi} f[g(t)]g'(t)\,dt.$$

2. $$\int_\alpha^\beta f(x)\,dx = (\beta-\alpha)\int_0^1 f((\beta-\alpha)t+\alpha)\,dt$$

For $\varphi = 0$ and $\psi = \infty$,

3. $$\int_\alpha^\beta f(x)\,dx = (\beta-\alpha)\int_0^\infty f\left(\frac{\alpha+\beta t}{1+t}\right)\frac{dt}{(1+t)^2}$$

3.022 The following formulas also hold:

1. $$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(\alpha+\beta-x)\,dx$$

2. $$\int_0^\beta f(x)\,dx = \int_0^\beta f(\beta-x)\,dx$$

3. $$\int_{-\alpha}^\alpha f(x)\,dx = \int_{-\alpha}^\alpha f(-x)\,dx$$

3.03 General formulas

3.031

1. Suppose that a function $f(x)$ is integrable over the interval $(-p,p)$ and satisfies the relation $f(-x) = f(x)$ on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^p f(x)\,dx = 2\int_0^p f(x)\,dx.$$ FI II 159

2. Suppose that $f(x)$ is a function that is integrable on the interval $(-p,p)$ and satisfies the relation $f(-x) = -f(x)$ on that interval. (A function satisfying the latter condition is called an *odd* function). Then,

$$\int_{-p}^p f(x)\,dx = 0.$$ FI II 159

3.032

1. $$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx,$$

where $f(x)$ is a function that is integrable on the interval $(0,1)$. FI II 159

2. $$\int_0^{2\pi} f(p\cos x + q\sin x)\,dx = 2\int_0^\pi f\left(\sqrt{p^2+q^2}\cos x\right)dx,$$

where $f(x)$ is integrable on the interval $\left(-\sqrt{p^2+q^2}, \sqrt{p^2+q^2}\right)$. FI II 160

3. $$\int_0^{\frac{\pi}{2}} f(\sin 2x)\cos x\,dx = \int_0^{\frac{\pi}{2}} f\left(\cos^2 x\right)\cos x\,dx,$$

where $f(x)$ is integrable on the interval $(0,1)$. FI II 161

3.033

1. If $f(x+\pi) = f(x)$ and $f(-x) = f(x)$, then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\,dx \qquad \text{LO V 277(3)}$$

2. If $f(x+\pi) = -f(x)$ and $f(-x) = f(x)$, then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx \qquad \text{LO V 279(4)}$$

In formulas **3.033**, it is assumed that the integrals in the left members of the formulas exist.

3.034 $\displaystyle\int_0^\infty \frac{f(px)-f(qx)}{x}\,dx = [f(0)-f(+\infty)]\ln\frac{q}{p},$

if $f(x)$ is continuous for $x \geq 0$ and if there exists a finite limit $f(+\infty) = \lim\limits_{x\to+\infty} f(x)$. FI II 633

3.035

1. $$\int_0^\pi \frac{f\left(\alpha+e^{xi}\right)+f\left(\alpha+e^{-xi}\right)}{1+2p\cos x+p^2}\,dx = \frac{2\pi}{1-p^2}f(\alpha+p) \qquad [|p|<1] \qquad \text{LA 230(16)}$$

2. $$\int_0^\pi \frac{1-p\cos x}{1-2p\cos x+p^2}\left\{f\left(\alpha+e^{xi}\right)+f\left(\alpha+e^{-xi}\right)\right\}dx = \pi\left\{f(\alpha+p)+f(\alpha)\right\}$$
$$[|p|<1] \qquad \text{BE 169}$$

3. $$\int_0^\pi \frac{f\left(\alpha+e^{-xi}\right)-f\left(\alpha+e^{xi}\right)}{1-2p\cos x+p^2}\sin x\,dx = \frac{\pi}{\pi}\left\{f(\alpha+p)-f(\alpha)\right\}$$
$$[|p|<1] \qquad \text{BE 169}$$

In formulas **3.035**, it is assumed that the function f is analytic in the closed unit circle with its center at the point α.

3.036

1.[11] $$\int_0^\pi f\left(\frac{\sin^2 x}{1+2p\cos x+p^2}\right)dx = \int_0^\pi f\left(\sin^2 x\right)dx \qquad \left[p^2<1\right]$$
$$= \int_0^\pi f\left(\frac{\sin^2 x}{p^2}\right)dx \qquad \left[p^2\geq 1\right]$$
LA 228(6)

2. $$\int_0^\pi F^{(n)}(\cos x)\sin^{2n} x\,dx = (2n-1)!!\int_0^\pi F(\cos x)\cos nx\,dx \qquad \text{B 174}$$

3.037 If f is analytic in the circle of radius r and if

$$f\left[r(\cos x+i\sin x)\right] = f_1(r,x)+if_2(r,x),$$

then

1. $$\int_0^\infty \frac{f_1(r,x)}{p^2+x^2}\,dx = \frac{\pi}{2p}f\left(re^{-p}\right) \qquad \text{LA 230(19)}$$

2. $$\int_0^\infty f_2(r,x)\frac{x\,dx}{p^2+x^2} = \frac{\pi}{2}\left[f\left(re^{-p}\right)-f(0)\right] \qquad \text{LA 230(20)}$$

3. $$\int_0^\infty \frac{f_2(r,x)}{x}\,dx = \frac{\pi}{2}[f(r)-f(0)] \qquad \text{LA 230(21)}$$

4. $$\int_0^\infty \frac{f_2(r,x)}{x\left(p^2+x^2\right)}\,dx = \frac{\pi}{2p^2}\left[f(r) - f\left(re^{-p}\right)\right]$$ LA 230(22)

3.038 $$\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{1+x^2}}\,F\left(qx + p\sqrt{1+x^2}\right) = \int_{-\infty}^{\infty} F\left(p\cosh x + q\sinh x\right)\sinh x\,dx$$
$$= 2q\int_0^\infty F'\left(\operatorname{sign} p\cdot\sqrt{p^2-q^2}\cosh x\right)\sinh^2 x\,dx$$

[If F is a function with a continuous derivative in the interval $(-\infty,\infty)$, all these integrals converge.]

3.04 Improper integrals

3.041 Suppose that a function $f(x)$ is defined on an interval $(p,+\infty)$ and that it is integrable over an arbitrary finite subinterval of the form (p,P). Then, by definition

$$\int_p^{+\infty} f(x)\,dx = \lim_{P\to+\infty}\int_p^P f(x)\,dx,$$

if this limit exists. If it does exist, we say that the integral $\displaystyle\int_p^{+\infty} f(x)\,dx$ exists or that it converges. Otherwise, we say that the integral diverges.

3.042 Suppose that a function $f(x)$ is bounded and integrable in an arbitrary interval $(p,q-\eta)$ (for $0<\eta<q-p$) but is unbounded in every interval $(q-\eta,q)$ to the left of the point q. The point q is then called a *singular point*. Then, by definition,

$$\int_p^q f(x)\,dx = \lim_{\eta\to 0}\int_p^{q-\eta} f(x)\,dx,$$

if this limit exists. In this case, we say that the integral $\displaystyle\int_p^q f(x)\,dx$ *exists* or that it *converges*.

3.043 If not only the integral of $f(x)$ but also the integral of $|f(x)|$ exists, we say that the integral of $f(x)$ converges *absolutely*.

3.044 The integral $\displaystyle\int_p^{+\infty} f(x)\,dx$ converges absolutely if there exists a number $\alpha>1$ such that the limit

$$\lim_{x\to+\infty}\{x^\alpha|f(x)|\}$$

exists. On the other hand, if

$$\lim_{x\to+\infty}\{x|f(x)|\} = L > 0,$$

the integral $\displaystyle\int_p^{+\infty} |f(x)|\,dx$ diverges.

3.045 Suppose that the upper limit q of the integral $\displaystyle\int_p^q f(x)\,dx$ is a singular point. Then, this integral converges absolutely if there exists a number $\alpha<1$ such that the limit

$$\lim_{x\to q}\left[(q-x)^\alpha|f(x)|\right]$$

exists. On the other hand, if

$$\lim_{x\to q}\left[(q-x)|f(x)|\right] = L > 0,$$

the integral $\displaystyle\int_p^q f(x)\,dx$ diverges.

3.046 Suppose that the functions $f(x)$ and $g(x)$ are defined on the interval $(p, +\infty)$, that $f(x)$ is integrable over every finite interval of the form (p, P), that the integral

$$\int_p^P f(x)\,dx$$

is a bounded function of P, that $g(x)$ is monotonic, and that $g(x) \to 0$ as $x \to +\infty$. Then, the integral

$$\int_p^{+\infty} f(x)g(x)\,dx$$

converges. FI II 577

3.05 The principal values of improper integrals

3.051 Suppose that a function $f(x)$ has a singular point r somewhere inside the interval (p, q), that $f(x)$ is defined at r, and that $f(x)$ is integrable over every portion of this interval that does not contain the point r. Then, by definition

$$\int_p^q f(x)\,dx = \lim_{\substack{\eta\to 0\\ \eta'\to 0}} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta'}^q f(x)\,dx \right\}.$$

Here, the limit must exist for *independent* modes of approach of η and η' to zero. If this limit does not exist but the limit

$$\lim_{\eta\to 0} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta}^q f(x)\,dx \right\}$$

does exist, we say that this latter limit is the *principal value* of the improper integral $\int_p^q f(x)\,dx$, and we say that the integral $\displaystyle\int_p^q f(x)\,dx$ exists in the sense of principal values. FI II 603

3.052 Suppose that the function $f(x)$ is continuous over the interval (p, q) and vanishes at only one point r inside this interval. Suppose that the first derivative $f'(x)$ exists in a neighborhood of the point r. Suppose that $f'(r) \neq 0$ and that the second derivative $f''(r)$ exists at the point r itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$ FI II 605

diverges, but exists in the sense of principal values.

3.053 A divergent integral of a positive function cannot exist in the sense of principal values.

3.054 Suppose that the function $f(x)$ has no singular points in the interval $(-\infty, +\infty)$. Then, by definition

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{P\to -\infty\\ Q\to +\infty}} \int_P^Q f(x)\,dx.$$

Here, the limit must exist for independent approach of P and Q to $\pm\infty$. If this limit does not exist but the limit

$$\lim_{P\to +\infty} \int_{-P}^{+P} f(x)\,dx$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,dx.$$ FI II 607

3.055 The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense). FI II 607

3.1–3.2 Power and Algebraic Functions

3.11 Rational functions

1. $$\int_{-\infty}^{\infty} \frac{p+qx}{r^2+2rx\cos\lambda+x^2}\,dx = \frac{\pi}{r\sin\lambda}\,(p-qr\cos\lambda) \qquad \text{(principal value)}$$

(see also **3.194** 8 and **3.252** 1 and 2) BI (22)(14)

3.112[11] Integrals of the form $\displaystyle\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)}$, where

$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots b_{n-1},$$
$$h_n(x) = a_0x^n + a_1x^{n-1} + \cdots a_n$$

[All roots of $h_n(x)$ lie in the upper half-plane.]

1. $$\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0}\frac{M_n}{\Delta_n},$$ JE

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}, \qquad M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}.$$

2. $$\int_{-\infty}^{\infty} \frac{g_1(x)\,dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0a_1}$$ JE

3.[8] $$\int_{-\infty}^{\infty} \frac{g_2(x)\,dx}{h_2(x)h_2(-x)} = \pi i\frac{-b_0+\frac{a_0b_1}{a_2}}{a_0a_1}$$

4.[11] $$\int_{-\infty}^{\infty} \frac{g_3(x)\,dx}{h_3(x)h_3(-x)} = \pi i\frac{-a_2b_0+a_0b_1-\dfrac{a_0a_1b_2}{a_3}}{a_0\left(a_0a_3-a_1a_2\right)}$$ JE

5. $$\int_{-\infty}^{\infty} \frac{g_4(x)\,dx}{h_4(x)h_4(-x)} = \pi i\frac{b_0\left(-a_1a_4+a_2a_3\right)-a_0a_3b_1+a_0a_1b_2+\dfrac{a_0b_3}{a_4}\left(a_0a_3-a_1a_2\right)}{a_0\left(a_0a_3^2+a_1^2a_4-a_1a_2a_3\right)}$$ JE

6. $$\int_{-\infty}^{\infty} \frac{g_5(x)\,dx}{h_5(x)h_5(-x)} = \pi i\frac{M_5}{a_0\Delta_5},$$

where

$$M_5 = b_0\left(-a_0a_4a_5+a_1a_4^2+a_2^2a_5-a_2a_3a_4\right)+a_0b_1\left(-a_2a_5+a_3a_4\right)$$
$$+a_0b_2\left(a_0a_5-a_1a_4\right)+a_0b_3\left(-a_0a_3+a_1a_2\right)+\frac{a_0b_4}{a_5}\left(-a_0a_1a_5+a_0a_3^2+a_1^2a_4-a_1a_2a_3\right),$$

$$\Delta_5 = a_0^2a_5^2-2a_0a_1a_4a_5-a_0a_2a_3a_5+a_0a_3^2a_4+a_1^2a_4^2+a_1a_2^2a_5-a_1a_2a_3a_4$$ JE

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.121

1. $$\int_0^1 \frac{1}{1-2x\cos\lambda+x^2}\frac{dx}{\sqrt{x}} = 2\operatorname{cosec}\lambda\sum_{k=1}^{\infty}\frac{\sin k\lambda}{2k-1}$$ BI (10)(17)

2. $$\int_0^1 \frac{1}{q-px}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \qquad [0<p<q]$$ BI (10)(9)

3. $$\int_0^1 \frac{dx}{1-2rx+r^2}\sqrt{\frac{1\mp x}{1\pm x}} = \pm\frac{\pi}{4r}\mp\frac{1}{r}\frac{1\mp r}{1\pm r}\arctan\frac{1+r}{1-r}$$ LI (14)(5, 16)

3.13–3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

Notation: In **3.131**–**3.137** we set: $\alpha = \arcsin\sqrt{\dfrac{a-c}{a-u}}$, $\beta = \arcsin\sqrt{\dfrac{c-u}{b-u}}$,

$$\gamma = \arcsin\sqrt{\frac{u-c}{b-c}}, \qquad \delta = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\kappa = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \lambda = \arcsin\sqrt{\frac{a-u}{a-b}},$$

$$\mu = \arcsin\sqrt{\frac{u-a}{u-b}}, \qquad \nu = \arcsin\sqrt{\frac{a-c}{u-c}}, \qquad p = \sqrt{\frac{a-b}{a-c}}, \qquad q = \sqrt{\frac{b-c}{a-c}}.$$

3.131

1. $$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\alpha,p) \qquad [a>b>c\geq u]$$ BY (231.00)

2. $$\int_{u}^{c} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\beta,p) \qquad [a>b>c>u]$$ BY (232.00)

3. $$\int_{c}^{u} \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\gamma,q) \qquad [a>b\geq u>c]$$ BY (233.00)

4. $$\int_{u}^{b} \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\delta,q) \qquad [a>b>u\geq c]$$ BY (234.00)

5. $$\int_{b}^{u} \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\kappa,p) \qquad [a\geq u>b>c]$$ BY (235.00)

6. $$\int_{u}^{a} \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\lambda,p) \qquad [a>u\geq b>c]$$ BY (236.00)

7. $$\int_{a}^{u} \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\mu,q) \qquad [u>a>b>c]$$ BY (237.00)

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}\,F(\nu,q) \qquad [u \ge a > b > c]$$ BY (238.00)

3.132

1. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}\,[c\,F(\beta,p) + (a-c)\,E(\beta,p)] - 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.19)

2. $$\int_c^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}}\,F(\gamma,q) - 2\sqrt{a-c}\,E(\gamma,q)$$
$[a > b \ge u > c]$ BY (233.17)

3. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}\left[(b-a)\,\Pi\left(\delta,q^2,q\right) + a\,F(\delta,q)\right]$$
$[a > b > u \ge c]$ BY (234.16)

4. $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}\left[(b-c)\,\Pi\left(\kappa,p^2,p\right) + c\,F\left(\kappa,p\right)\right]$$
$[a \ge u > b > c]$ BY (235.16)

5. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}}\,F(\lambda,p) + 2\sqrt{a-c}\,E(\lambda,p)$$
$[a > u \ge b > c]$ BY (236.16)

6. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}}\left[a(a-b)\,\Pi(\mu,1,q) + b^2\,F(\mu,q)\right]$$
$[u > a > b > c]$ BY (237.16)

3.133

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}\,[F(\alpha,p) - E(\alpha,p)]$$
$[a > b > c \ge u]$ BY (231.08)

2. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}}\,[F(\beta,p) - E(\beta,p)] + \frac{2}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.13)

3. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}\,E(\gamma,q) - \frac{2}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \ge u > c]$ BY (233.09)

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}\,E(\delta,q) \qquad [a > b > u \ge c]$$ BY (234.05)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}\,[F\left(\kappa,p\right) - E\left(\kappa,p\right)] + \frac{2}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.04)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}}\,E(\nu,q) + \frac{2}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\alpha,p) - \frac{2}{(a-b)\sqrt{a-c}}\,F(\alpha,p) - \frac{2}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\beta,p) - \frac{2}{(a-b)\sqrt{a-c}}\,F(\beta,p)$$
$[a > b > c > u]$ BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}}\,F(\gamma,q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\gamma,q) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a > b > u > c]$ BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}\,F(\lambda,p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\lambda,p) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a > u > b > c]$ BY (236.09)

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\mu,q) - \frac{2}{(b-c)\sqrt{a-c}}\,F(\mu,q)$$
$[u > a > b > c]$ BY (237.12)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\nu,q) - \frac{2}{(b-c)\sqrt{a-c}}\,F(\nu,q) - \frac{2}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}}\,E(\alpha,p) + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\delta,q) - E(\delta,q)] + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.04)

15. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,E(\kappa,p)$$
$[a \geq u > b > c]$ BY (235.01)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,E(\lambda,p) - \frac{2}{(b-c)(a-c)}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\mu,q) - E(\mu,q)] + \frac{2}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.13)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\nu,q) - E(\nu,q)]$$
$[u \geq a > b > c]$ BY (238.03)

3.134

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}}$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\alpha,p) - 2(2a-b-c)\,E(\alpha,p)]$$
$$+\frac{2}{3(a-c)(a-b)}\sqrt{\frac{(c-u)(b-u)}{(a-u)^3}}$$
$[a > b > c \geq u]$ BY (231.08)

2. $$\int_u^c \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\beta,p) - 2(2a-b-c)\,E(\beta,p)]$$
$$+\frac{2\left[4a^2-3ab-2ac+bc-u(3a-2b-c)\right]}{3(a-b)(a-c)^2}\sqrt{\frac{c-u}{(a-u)^3(b-u)}}$$
$[a > b > c > u]$ BY (232.13)

3. $$\int_c^u \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^3\sqrt{(a-c)^3}}\,[2(2a-b-c)\,E(\gamma,q) - (a-b)\,F(\gamma,q)]$$
$$-\frac{2\left[5a^2-3ab-3ac+bc-2u(2a-b-c)\right]}{3(a-b)^2(a-c)^2}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$
$[a > b \geq u > c]$ BY (233.09)

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[2(2a-b-c)\,E(\delta,q) - (a-b)\,F(\delta,q)]$$
$$-\frac{2}{3(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$
$[a > b > u \geq c]$ BY (234.05)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}}$$
$$= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\kappa,p) - 2(2a-b-c)\,E(\kappa,p)]$$
$$+\frac{2\left[4a^2-2ab-3ac+bc-u(3a-b-2c)\right]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(a-u)^3(u-c)}}$$
$[a > u > b > c]$ BY (235.04)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\left[2(2a-b-c)\,E(\nu,q)-(a-b)\,F(\nu,q)\right] + \frac{2\left[4a^2-2ab-3ac+bc+u(b+2c-3a)\right]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(u-a)^3(u-c)}}$$
$[u>a>b>c]$ BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times\left[2(a-c)(a+c-2b)\,E(\alpha,p)+(b-c)(3b-a-2c)\,F(\alpha,p)\right] - \frac{2\left[3ab-ac+2bc-4b^2-u(2a-3b+c)\right]}{3(a-b)(b-c)^2}\sqrt{\frac{c-u}{(a-u)(b-u)^3}}$$
$[a>b>c\geq u]$ BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times\left[(b-c)(3b-a-2c)\,F(\beta,p)+2(a-c)(a-2b+c)\,E(\beta,p)\right] + \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(a-u)(c-u)}{(b-u)^3}}$$
$[a>b>c>u]$ BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times\left[(a-b)(2a-3b+c)\,F(\gamma,q)+2(a-c)(2b-a-c)\,E(\gamma,q)\right] + \frac{2\left[3ab+3bc-ac-5b^2-2u(a-2b+c)\right]}{3(a-b)^2(b-c)^2}\sqrt{\frac{(a-u)(u-c)}{(b-u)^3}}$$
$[a>b>u>c]$ BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times\left[(b-c)(3b-2c-a)\,F(\lambda,p)+2(a-c)(a+c-2b)\,E(\lambda,p)\right] + \frac{2\left[3ab+3bc-ac-5b^2+2u(2b-a-c)\right]}{3(a-b)^2(b-c)^2}\sqrt{\frac{(a-u)(u-c)}{(u-b)^3}}$$
$[a>u>b>c]$ BY (236.09)

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times\left[(a-b)(2a+c-3b)\,F(\mu,q)+2(a-c)(2b-a-c)\,E(\mu,q)\right] + \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(u-a)(u-c)}{(u-b)^3}}$$
$[u>a>b>c]$ BY (237.12)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times [(a-b)(2a+c-3b)\,F(\nu,q) + 2(a-c)(2b-c-a)\,E(\nu,q)] - \frac{2\left[3bc+2ab-ac-4b^2+u(3b-a-2c)\right]}{3(a-b)^2(b-c)}\sqrt{\frac{u-a}{(u-b)^3(u-c)}}$$
$[u \geq a > b > c]$ BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E(\alpha,p) - (b-c)\,F(\alpha,p)] + \frac{2\left[ab-3ac-2bc+4c^2+u(2a+b-3c)\right]}{3(a-c)(b-c)^2}\sqrt{\frac{b-u}{(a-u)(c-u)^3}}$$
$[a > b > c > u]$ By (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\delta,q) - 2(a+b-2c)\,E(\delta,q)] + \frac{2\left[ab-3ac-2bc+4c^2+u(2a+b-3c)\right]}{3(b-c)^2(a-c)}\sqrt{\frac{b-u}{(a-u)(u-c)^3}}$$
$[a > b > u > c]$ BY (234.04)

15. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E\,(\kappa,p) - (b-c)\,F\,(\kappa,p)] + \frac{2}{3(a-c)(b-c)}\sqrt{\frac{(a-u)(u-b)}{(u-c)^3}}$$
$[a \geq u > b > c]$ BY (235.20)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E(\lambda,p) - (b-c)\,F(\lambda,p)] - \frac{2\left[ab-3ac-3bc+5c^2+2u(a+b-2c)\right]}{3(b-c)^2(a-c)^2}\sqrt{\frac{(a-u)(u-b)}{(u-c)^3}}$$
$[a > u \geq b > c]$ BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\mu,q) - 2(a+b-2c)\,E(\mu,q)] + \frac{2\left[4c^2-ab-2ac-bc+u(3a+2b-5c)\right]}{3(b-c)(a-c)^2}\sqrt{\frac{u-a}{(u-b)(u-c)^3}}$$
$[u > a > b > c]$ BY (237.13)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\nu,q) - 2(a+b-2c)\,E(\nu,q)] + \frac{2}{3(a-c)(b-c)}\sqrt{\frac{(u-a)(u-b)}{(u-c)^3}}$$
$[u \geq a > b > c]$ BY (238.03)

3.135

1.[6] $$\int_{-\infty}^{u}\frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}}=\frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)\,F(\alpha,p)-(2a-b-c)\,E(\alpha,p)\right]+\frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}}$$
$[a>b>c>u]$ BY (231.13)

2. $$\int_{u}^{a}\frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}}=\frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)\,F(\lambda,p)-2(2a-b-c)\,E(\lambda,p)\right]+\frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)}\sqrt{\frac{a-u}{(u-b)(u-c)}}$$
$[a>u>b>c]$ BY (236.15)

3. $$\int_{a}^{u}\frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}}=\frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(2a-b-c)\,E(\mu,q)-2(a-b)\,F(\mu,q)\right]+\frac{2}{(a-c)(b-c)}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u>a>b>c]$ BY (236.14)

4. $$\int_{u}^{\infty}\frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}}=\frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(2a-b-c)\,E(\nu,q)-2(a-b)\,F(\nu,q)\right]-\frac{2}{(a-b)(b-c)}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u\geq a>b>c]$ BY (238.13)

5. $$\int_{-\infty}^{u}\frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}}=\frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(2b-a-c)\,E(\alpha,p)-(b-c)\,F(\alpha,p)\right]+\frac{2}{(b-c)(a-c)}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a>b>c>u]$ BY(231.12)

6. $$\int_{u}^{b}\frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}}=\frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}}\left[(a-b)\,F(\delta,q)+(2b-a-c)\,E(\delta,q)\right]+\frac{2}{(b-c)(a-c)}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a>b>u>c]$ BY (234.03)

7. $$\int_{b}^{u}\frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}}=\frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(b-c)\,F(\kappa,p)-(2b-a-c)\,E(\kappa,p)\right]+\frac{2}{(a-b)(a-c)}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a>u>b>c]$ BY (235.15)

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(a+c-2b)\,E(\nu,q)-(a-b)\,F(\nu,q)\right] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u>a>b>c]$ BY (238.14)

9. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(b-c)(a-b)^2\sqrt{a-c}}\left[(a+b-2c)\,E(\alpha,p)-2(b-c)\,F(\alpha,p)\right] - \frac{2}{(a-b)(b-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a>b>c\geq u]$ BY (231.11)

10. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a+b-2c)\,E(\beta,p)-2(b-c)\,F(\beta,p)\right] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a>b>c>u]$ BY (232.15)

11. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a-b)\,F(\gamma,q)-(a+b-2c)\,E(\gamma,q)\right] + \frac{2\left[a^2+b^2-ac-bc-u(a+b-2c)\right]}{(a-b)^2(b-c)(a-c)}\sqrt{\frac{u-c}{(a-u)(b-u)}}$$
$[a>b>u>c]$ BY (233.11)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a-b)\,F(\nu,q)-(a+b-2c)\,E(\nu,q)\right] + \frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}}$$
$[u>a>b>c]$ BY (238.15)

3.136

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times\left[(b-c)(a+b-2c)\,F(\alpha,p)-2\left(c^2+a^2+b^2-ab-ac-bc\right)E(\alpha,p)\right] + \frac{2[c(a-c)+b(a-b)-u(2a-c-b)]}{(a-b)(a-c)(b-c)^2\sqrt{(a-u)(b-u)(c-u)}}$$
$[a>b>c>u]$ BY (231.14)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times \left[(a-b)(2a-b-c)\,F(\nu,q) - 2\left(a^2+b^2+c^2-ab-ac-bc\right)E(\nu,q)\right] + \frac{2[u(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (238.16)

3.137

1.[6] $$\int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-r)\sqrt{a-c}}\left[\Pi\left(\alpha, \frac{a-r}{a-c}, p\right) - F(\alpha,p)\right]$$
$[a > b > c \geq u]$ BY (231.15)

2. $$\int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \times \Pi\left(\beta, \frac{r-b}{r-c}, p\right) + \frac{2}{(r-b)\sqrt{a-c}}\,F(\beta,p)$$
$[a > b > c > u, \quad r \neq 0]$ BY (232.17)

3. $$\int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}}\,\Pi\left(\gamma, \frac{b-c}{r-c}, q\right)$$
$[a > b \geq u > c, \quad r \neq c]$ BY (233.02)

4. $$\int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-a)(r-b)\sqrt{a-c}} \times \left[(b-a)\,\Pi\left(\delta, q^2\frac{r-a}{r-b}, q\right) + (r-b)\,F(\delta,q)\right]$$
$[a > b > u \geq c, \quad r \neq b]$ BY (234.18)

5. $$\int_b^u \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(c-r)(b-r)\sqrt{a-c}} \times \left[(c-b)\,\Pi\left(\kappa, p^2\frac{c-r}{b-r}, p\right) + (b-r)\,F(\kappa,p)\right]$$
$[a \geq u > b > c, \quad r \neq b]$ BY (235.17)

6.[8] $$\int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}}\,\Pi\left(\lambda, \frac{a-b}{a-r}, p\right)$$
$[a > u \geq b > c, \quad r \neq a]$ BY (236.02)

7. $$\int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \times \left[(b-a)\,\Pi\left(\mu, \frac{b-r}{a-b}, q\right) + (a-p)\,F(\mu,q)\right]$$
$[u > a > b > c, \quad r \neq a]$ BY (237.17)

8. $$\int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}}\left[\Pi\left(\nu, \frac{r-c}{a-c}, q\right) - F(\nu, q)\right]$$
$[u \geq a > b > c]$ BY (238.06)

3.138

1. $$\int_0^u \frac{dx}{\sqrt{x(1-x)\,(1-k^2x)}} = 2\,F\left(\arcsin\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE (532), JA

2. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)\left(k'^2 + k^2x\right)}} = 2\,F\left(\arccos\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE(533)

3. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)\left(x - k'^2\right)}} = 2\,F\left(\arcsin\frac{\sqrt{1-u}}{k}, k\right)$$ $[0 < u < 1]$ PE (534)

4. $$\int_0^u \frac{dx}{\sqrt{x(1+x)\left(1 + k'^2x\right)}} = 2\,F\left(\arctan\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE (535)

5. $$\int_0^u \frac{dx}{\sqrt{x\left[1 + x^2 + 2\left(k'^2 - k^2\right)x\right]}} = F\left(2\arctan\sqrt{u}, k\right)$$
$[0 < u < 1]$ JA

6. $$\int_u^1 \frac{dx}{\sqrt{x\left[k'^2\left(1+x^2\right) + 2\left(1+k^2\right)x\right]}} = F\left(\frac{\pi}{2} - 2\arctan\sqrt{u}, k\right)$$
$[0 < u < 1]$ JA

7. $$\int_a^u \frac{dx}{\sqrt{(x-\alpha)\left[(x-m)^2 + n^2\right]}} = \frac{1}{\sqrt{p}}\,F\left(2\arctan\sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right)$$
$[\alpha < u]$,

8. $$\int_u^a \frac{dx}{\sqrt{(\alpha-x)\left[(x-m)^2 + n^2\right]}} = \frac{1}{\sqrt{p}}\,F\left(2\,\mathrm{arccot}\sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right)$$
$[u < \alpha]$,

where $p = \sqrt{(m-\alpha)^2 + n^2}$.

3.139 **Notation** $\alpha = \arccos\dfrac{1-\sqrt{3}-u}{1+\sqrt{3}-u}$, $\beta = \arccos\dfrac{\sqrt{3}-1+u}{\sqrt{3}+1-u}$,
$\gamma = \arccos\dfrac{\sqrt{3}+1-u}{\sqrt{3}-1+u}$, $\delta = \arccos\dfrac{u-1-\sqrt{3}}{u-1+\sqrt{3}}$.

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,F\left(\alpha, \sin 75^\circ\right)$$ H 66 (285)

2. $$\int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,F\left(\beta, \sin 75^\circ\right)$$ H 65 (284)

3. $$\int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\, F\left(\gamma, \sin 15^\circ\right)$$ H 65 (283)

4. $$\int_u^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\, F\left(\delta, \sin 15^\circ\right)$$ H 65 (282)

5. $$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}} \left\{\Gamma\left(\frac{1}{3}\right)\right\}^3$$ MO 9

6. $$\int_0^1 \frac{x\,dx}{\sqrt{1-x^3}} = \frac{1}{\pi}\frac{\sqrt{3}}{\sqrt[3]{4}} \left\{\Gamma\left(\frac{2}{3}\right)\right\}^3$$ MO 9

7. $$\int_u^1 \sqrt{1-x^3}\,dx = \frac{1}{5}\left\{\sqrt[4]{27}\, F\left(\beta, \sin 75^\circ\right) - 2u\sqrt{1-u^3}\right\}$$ BY (244.01)

8. $$\int_u^1 \frac{x\,dx}{\sqrt{1-x^3}} = \left(3^{-\frac{1}{4}} - 3^{\frac{1}{4}}\right) F\left(\beta, \sin 75^\circ\right) + 2\sqrt[4]{3}\, E\left(\beta, \sin 75^\circ\right) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}$$ BY (244.05)

9. $$\int_u^1 \frac{x^m\,dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1}\int_u^1 \frac{x^{m-3}\,dx}{\sqrt{1-x^3}}$$ BY (244.07)

10. $$\int_1^u \frac{x\,dx}{\sqrt{x^3-1}} = \left(3^{-\frac{1}{4}} + 3^{\frac{1}{4}}\right) F\left(\gamma, \sin 15^\circ\right) - 2\sqrt[4]{3}\, E\left(\gamma, \sin 15^\circ\right) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}$$ BY (240.05)

11. $$\int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}}\left[F\left(\alpha, \sin 75^\circ\right) - 2\,E\left(\alpha, \sin 75^\circ\right)\right] + \frac{2}{\sqrt{3}}\frac{\sqrt{1+u+u^2}}{\left(1+\sqrt{3}-u\right)\sqrt{1-u}}$$
$[u \neq 1]$ BY (246.06)

12. $$\int_u^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}}\left[F\left(\delta, \sin 15^\circ\right) - 2\,E\left(\delta, \sin 15^\circ\right)\right] + \frac{2}{\sqrt{3}}\frac{\sqrt{1+u+u^2}}{\left(u-1+\sqrt{3}\right)\sqrt{u-1}}$$
$[u \neq 1]$ BY (242.03)

13. $$\int_{-\infty}^u \frac{(1-x)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}\left[F\left(\alpha, \sin 75^\circ\right) - E\left(\alpha, \sin 75^\circ\right)\right]$$ BY (246.07)

14. $$\int_u^1 \frac{(1-x)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}\left[F\left(\beta, \sin 75^\circ\right) - E\left(\beta, \sin 75^\circ\right)\right]$$ BY (244.04)

15. $$\int_1^u \frac{(x-1)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2\left(\sqrt{3}-2\right)}{\sqrt{3}}\frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}}\, E\left(\gamma, \sin 15^\circ\right)$$ BY (240.08)

16. $$\int_u^\infty \frac{(x-1)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2\left(2-\sqrt{3}\right)}{\sqrt{3}}\frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}}\, E\left(\delta, \sin 15^\circ\right)$$ BY (242.07)

17. $$\int_{-\infty}^u \frac{(1-x)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[\frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E\left(\alpha, \sin 75^\circ\right)\right]$$ BY (246.08)

18. $$\int_1^u \frac{(x-1)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[F\left(\gamma, \sin 15^\circ\right) - E\left(\gamma, \sin 15^\circ\right)\right]$$ BY (240.04)

19. $$\int_u^\infty \frac{(x-1)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[F\left(\delta,\sin 15^\circ\right)-E\left(\delta,\sin 15^\circ\right)\right]$$ BY (242.05)

20. $$\int_{-\infty}^u \frac{\left(x^2+x+1\right)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,E\left(\alpha,\sin 75^\circ\right)$$ BY (246.01)

21. $$\int_u^1 \frac{\left(x^2+x+1\right)\,dx}{\left(x-1+\sqrt{3}\right)^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,E\left(\beta,\sin 75^\circ\right)$$ BY (244.02)

22. $$\int_1^u \frac{\left(x^2+x+1\right)\,dx}{\left(\sqrt{3}+x-1\right)^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,E\left(\gamma,\sin 15^\circ\right)$$ BY (240.01)

23. $$\int_u^\infty \frac{\left(x^2+x+1\right)\,dx}{\left(x-1+\sqrt{3}\right)^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,E\left(\delta,\sin 15^\circ\right)$$ BY (242.01)

24. $$\int_1^u \frac{(x-1)\,dx}{\left(x^2+x+1\right)\sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}}\,E\left(\gamma,\sin 15^\circ\right)-\frac{2+\sqrt{3}}{\sqrt[4]{27}}\,F\left(\gamma,\sin 15^\circ\right) -\frac{2-\sqrt{3}}{\sqrt{3}}\,\frac{2(u-1)\left(\sqrt{3}+1-u\right)}{\left(\sqrt{3}-1+u\right)\sqrt{u^3-1}}$$ BY (240.09)

25. $$\int_{-\infty}^u \frac{\left(1+\sqrt{3}-x\right)^2\,dx}{\left[\left(1+\sqrt{3}-x\right)^2-4\sqrt{3}p^2(1-x)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\alpha,p^2,\sin 75^\circ\right)$$ BY (246.02)

26. $$\int_u^1 \frac{\left(1+\sqrt{3}-x\right)^2\,dx}{\left[\left(1+\sqrt{3}-x\right)^2-4\sqrt{3}p^2(1-x)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\beta,p^2,\sin 75^\circ\right)$$ BY (244.03)

27. $$\int_1^u \frac{\left(1-\sqrt{3}-x\right)^2\,dx}{\left[\left(1-\sqrt{3}-x\right)^2-4\sqrt{3}p^2(x-1)\right]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\gamma,p^2,\sin 15^\circ\right)$$ BY (240.02)

28. $$\int_u^\infty \frac{\left(1-\sqrt{3}-x\right)^2\,dx}{\left[\left(1-\sqrt{3}-x\right)^2-4\sqrt{3}p^2(x-1)\right]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\delta,p^2,\sin 15^\circ\right)$$ BY (242.02)

3.141 **Notation**: In **3.141** and **3.142** we set:

$$\alpha=\arcsin\sqrt{\frac{a-c}{a-u}},\qquad \beta=\arcsin\sqrt{\frac{c-u}{b-u}},\qquad \gamma=\arcsin\sqrt{\frac{u-c}{b-c}}$$

$$\delta=\arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},\qquad \kappa=\arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}},\qquad \lambda=\arcsin\sqrt{\frac{a-u}{a-b}},$$

$$\mu=\arcsin\sqrt{\frac{u-a}{u-b}},\qquad \nu=\arcsin\sqrt{\frac{a-c}{u-c}},\qquad p=\sqrt{\frac{a-b}{a-c}},\qquad q=\sqrt{\frac{b-c}{a-c}}.$$

1. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}}\,dx = 2\sqrt{a-c}\,[F(\beta,p)-E(\beta,p)] + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a>b>c>u]$ BY (232.06)

2. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\gamma,q)$$
$[a>b\geq u>c]$ BY (233.01)

3. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\delta,q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a>b>u\geq c]$ BY (234.06)

4. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}\,[F(\kappa,p)-E(\kappa,p)] + 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a\geq u>b>c]$ BY (235.07)

5. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}\,[F(\lambda,p)-E(\lambda,p)]$$
$[a>u\geq b>c]$ BY (236.04)

6. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)}}\,dx = -2\sqrt{a-c}\,E(\mu,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u>a>b>c]$ BY (237.03)

7. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}}\,dx = \frac{2(b-c)}{\sqrt{a-c}}\,F(\beta,p) - 2\sqrt{a-c}\,E(\beta,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a>b>c>u]$ BY (232.07)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\gamma,q) - \frac{2(a-b)}{\sqrt{a-c}}\,F(\gamma,q)$$
$[a>b\geq u>c]$ BY (233.04)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\delta,q) - \frac{2(a-b)}{\sqrt{a-c}}\,F(\delta,q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a>b>u\geq c]$ BY (234.07)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\kappa,p) - \frac{2(b-c)}{\sqrt{a-c}}\,F(\kappa,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a\geq u>b>c]$ BY (235.06)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\lambda,p) - \frac{2(b-c)}{\sqrt{a-c}}\,F(\lambda,p)$$
$[a>u\geq b>c]$ BY (236.03)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}}\,dx = \frac{2(a-b)}{\sqrt{a-c}}\,F(\mu,q) - 2\sqrt{a-c}\,E(\mu,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u>a>b>c]$ BY (237.04)

13. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}}\,dx = -2\sqrt{a-c}\,E(\beta,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a>b>c>u]$ BY (232.08)

14. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}\,[F(\gamma,q)-E(\gamma,q)]$$
$[a>b\geq u>c]$ BY (233.03)

15. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}\,[F(\delta,q)-E(\delta,q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a>b>u\geq c]$ BY (234.08)

16. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\kappa,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a\geq u>b>c]$ BY (235.07)

17. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\lambda,p)$$
$[a>u\geq b>c]$ BY (236.01)

18. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}}\,dx = 2\sqrt{a-c}\,[F(\mu,q)-E(\mu,q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u>a>b>c]$ BY (237.05)

19. $$\int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\beta,p)-(b-c)\,F(\beta,p)] + \frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a>b>c>u]$ BY (232.11)

20. $$\int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\gamma,q)-2(a-b)\,F(\gamma,q)] - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a>b\geq u>c]$ BY (233.06)

21.[11] $$\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-a)\,F(\delta,q)+(2a-b-c)\,E(\delta,q)] + \frac{2}{3}(b+c-a-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a>b>u\geq c]$ BY (234.11)

22. $$\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\kappa,p)-(b-c)\,F(\kappa,p)] + \frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a\geq u>b>c]$ BY (235.10)

23.[11] $$\int_u^a \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\lambda,p)-(b-c)\,F(\lambda,p)] + \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.07)

24. $$\int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(a-b)\,F(\mu,q)+(b+c-2a)\,E(\mu,q)] + \frac{2}{3}(u+2a-2b-c)\sqrt{\frac{(u-a)(u-b)}{u-c}}$$
$[u > a > b > c]$ BY (237.08)

25. $$\int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)\,E(\beta,p)-(b-c)\,F(\beta,p)] + \frac{2}{3}(a+c-b-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.10)

26. $$\int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)\,E(\gamma,q)+(a-b)\,F(\gamma,q)] - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a > b \geq u > c]$ BY (233.05)

27. $$\int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a-b)\,F(\delta,q)+(2b-a-c)\,E(\delta,q)] + \frac{2}{3}(2a+c-2b-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.10)

28. $$\int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(b-c)\,F(\kappa,p)+(a+c-2b)\,E(\kappa,p)] + \frac{2}{3}(2b-a-2c+u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.11)

29. $$\int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+c-2b)\,E(\lambda,p)+(b-c)\,F(\lambda,p)] - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.06)

30.[11] $$\int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+c-2b)\,E(\mu,q)-(a-b)\,F(\mu,q)] + \frac{2}{3}(u+b-a-c)\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.06)

31. $$\int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-c)\,F(\beta,p) + (2c-a-b)\,E(\beta,p)] + \frac{2}{3}(a+2b-2c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.09)

32. $$\int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\gamma,q) - (a-b)\,F(\gamma,q)] + \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a > b \geq u > c]$ BY (233.07)

33. $$\int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\delta,q) - (a-b)\,F(\delta,q)] + \frac{2}{3}(2c-2a-b+u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.09)

34. $$\int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E\,(\kappa,p) - 2(b-c)\,F\,(\kappa,p)] + \frac{2}{3}(u+c-a-b)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.09)

35. $$\int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\lambda,p) - 2(b-c)\,F(\lambda,p)] - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.05)

36. $$\int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\mu,q) - (a-b)\,F(\mu,q)] + \frac{2}{3}(u+2c-a-2b)\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.07)

3.142

1. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,F(\alpha,p) - \frac{2\sqrt{a-c}}{b-c}\,E(\alpha,p) + \frac{2(a-c)}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.05)

2. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}}\,dx = 2\frac{a-b}{(b-c)\sqrt{a-c}}\,F(\delta,q) - \frac{2\sqrt{a-c}}{b-c}\,E(\delta,q) + 2\frac{a-c}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.13)

3. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E\,(\kappa,p) - \frac{2}{\sqrt{a-c}}\,F\,(\kappa,p)$$
$[a \geq u > b > c]$ BY (235.12)

4. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\lambda,p) - \frac{2}{\sqrt{a-c}}\,F(\lambda,p) - \frac{2}{b-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.12)

5. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\mu,q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}}\,F(\mu,q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.10)

6. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\nu,q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}}\,F(\nu,q)$$
$[u \geq a > b > c]$ BY (238.09)

7. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\alpha,p) - 2\frac{a-b}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.03)

8. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\beta,p) \qquad [a > b > c > u]$$ BY (232.01)

9. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\gamma,q) - E(\gamma,q)] + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a > b > u > c]$ BY (233.15)

10. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{c-b}\,E(\lambda,p) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a > u > b > c]$ BY (236.11)

11. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\mu,q) - E(\mu,q)]$$
$[u > a > b > c]$ BY (237.09)

12. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\nu,q) - E(\nu,q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.10)

13. $$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\alpha,p) \qquad [a > b > c \geq u]$$ BY (231.01)

14. $$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\beta,p) - \frac{2(a-b)}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.05)

15. $$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\gamma,q) - E(\gamma,q)] + \frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \geq u > c]$ BY (233.13)

16. $$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\delta,q) - E(\delta,q)] \qquad [a > b > u \geq c]$$ BY (234.15)

17. $$\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}}\,dx = -\frac{2}{\sqrt{a-c}}\,E\left(\kappa,p\right) + 2\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.08)

18. $$\int_u^\infty \sqrt{\frac{x-b}{(x-a)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\left[F(\nu,q) - E(\nu,q)\right] + 2\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.07)

19. $$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}\left[F(\alpha,p) - E(\alpha,p)\right] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.04)

20. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}}\,dx = -\frac{2}{\sqrt{a-c}}\,E(\delta,q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.14)

21. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\left[F\left(\kappa,p\right) - E\left(\kappa,p\right)\right]$$
$[a \geq u > b > c]$ BY (235.03)

22. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\left[F(\lambda,p) - E(\lambda,p)\right] + \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.14)

23. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\mu,q) - 2\frac{b-c}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.11)

24. $$\int_u^\infty \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\nu,q) \qquad [u \geq a > b > c]$$ BY (238.01)

25. $$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\alpha,p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\alpha,p)$$
$[a > b > c \geq u]$ BY (231.07)

26. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\beta,p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\beta,p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.03)

27. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\gamma,q) - \frac{2}{\sqrt{a-c}}\,F(\gamma,q) - \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \geq u > c]$ BY (233.14)

28. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\delta,q) - \frac{2}{\sqrt{a-c}}\,F(\delta,q)$$
$[a > b > u \geq c]$ BY (234.20)

29. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\kappa,p) - \frac{2\sqrt{a-c}}{a-b}\,E(\kappa,p) + 2\frac{a-c}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.13)

30. $$\int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}}\,dx = \frac{2}{\sqrt{a-c}}\,F(\nu,q) - \frac{2\sqrt{a-c}}{a-b}\,E(\nu,q) + \frac{2(a-c)}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.08)

31. $$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\alpha,p) - E(\alpha,p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.06)

32. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\beta,p) - E(\beta,p)]$$
$[a > b > c > u]$ BY (232.04)

33. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}}\,dx = -\frac{2\sqrt{a-c}}{a-b}\,E(\gamma,q) + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a > b > u > c]$ BY (233.16)

34. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\lambda,p) - E(\lambda,p)] + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a > u > b > c]$ BY (236.13)

35. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\mu,q)$$
$[u > a > b > c]$ BY (237.01)

36. $$\int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\nu,q) - 2\frac{b-c}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.11)

3.143

1.[6] $$\int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\,F\left(\arctan\frac{\left(1+\sqrt{2}\right)(1-u)}{(1+u)}, 2\sqrt[4]{2}\left(\sqrt{2}-1\right)\right)$$
H 66 (286)

2. $$\int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\,F\left(\arccos\frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2}\right)$$
H 66 (287)

3.144 **Notation**: $\alpha = \arcsin\dfrac{1}{\sqrt{u^2-u+1}}$.

1. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\left(x^2-x+1\right)}} = F\left(\alpha, \frac{\sqrt{3}}{2}\right)$$
$[u \geq 1]$ BY (261.50)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3\,(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)\,(u^2-u+1)}} - 4\,E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$
$[u>1]$ BY (261.54)

3. $$\int_u^\infty \frac{(2x-1)^2\,dx}{\sqrt{x^3(x-1)^3\,(x^2-x+1)}} = 4\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u(u-1)\,(u^2-u-1)}}\right]$$
$[u>1]$ BY (261.56)

4. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\,(x^2-x+1)^3}} = \frac{4}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right]$$
$[u\geq 1]$ BY (261.52)

5. $$\int_u^\infty \frac{(2x-1)^2\,dx}{\sqrt{x(x-1)\,(x^2-x+1)^3}} = 4\,E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$
$[u>1]$ BY (261.51)

6. $$\int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}}\,dx = \frac{4}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right)$$
$[u>1]$ BY (261.53)

7. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right] + \frac{1}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$[u>1]$ BY (261.57)

8. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$[u>1]$ BY (261.58)

9. $$\int_u^\infty \frac{dx}{(2x-1)^2\sqrt{x(x-1)\,(x^2-x+1)}} = \frac{4}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$[u>1]$ BY (261.55)

10. $$\int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5\,(x^2-x+1)}} = \frac{40}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{4}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)\left(9u^2-9u-1\right)}{3\sqrt{u^3(u-1)^3\,(u^2-u+1)}}$$
$[u>1]$ BY (261.54)

11. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\,(x^2-x+1)^5}} = \frac{44}{27}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{56}{27}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}}$$
$[u>1]$ BY (261.52)

12. $$\int_u^\infty \frac{dx}{(2x-1)^4\sqrt{x(x-1)\,(x^2-x+1)}} = \frac{16}{27}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{27}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{8\left(5u^2-5u+2\right)}{9(2u-1)^3}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$$[u>1]$$ BY (261.55)

3.145

1. $$\int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)\,[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}}\,F\left(2\arctan\sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$$[\beta<\alpha<u]$$

2. $$\int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)\,[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}}\,F\left(2\,\mathrm{arccot}\sqrt{\frac{q(\alpha-u)}{p(u-\beta)}}, \frac{1}{2}\sqrt{\frac{-(p-q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$$[\beta<u<\alpha]$$

3. $$\int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)\,[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}}\,F\left(2\arctan\sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$$[u<\beta<\alpha]$$

where $(m-\alpha)^2+n^2=p^2$, and $(m-\beta)^2+n^2=q^2$.*

4. Set

$$(m_1-m)^2+(n_1+n)^2=p^2, \qquad (m_1-m)^2+(n_1-n)^2=p_1^2,$$
$$\cot\alpha=\sqrt{\frac{(p+p_1)^2-4n^2}{4n^2-(p-p_1)^2}};$$

then

$$\int_{m-n\tan\alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2]\left[(xm_1)^2+n_1^2\right]}} = \frac{2}{p+p_1}\,F\left(\alpha+\arctan\frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right)$$
$$[m-n\tan\alpha<u<m+n\cot\alpha]$$

3.146

1. $$\int_0^1 \frac{1}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}+\frac{1}{4}\sqrt{2}\,\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (13)(6)

2. $$\int_0^1 \frac{x^2}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}$$ BI (13)(7)

*Formulas **3.145** are not valid for $\alpha+\beta=2m$. In this case, we make the substitution $x-m=z$, which leads to one of the formulas in **3.152**.

3. $$\int_0^1 \frac{x^4}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2}\,\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (13)(8)

3.147 **Notation**: In **3.147**–**3.151** we set: $\alpha = \arcsin\sqrt{\dfrac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \kappa = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

1. $$\int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\alpha, q)$$
$[a > b > c > d > u]$ BY (251.00)

2. $$\int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\beta, r)$$
$[a > b > c \geq u > d]$ BY (254.00)

3. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\gamma, r)$$
$[a > b > c > u \geq d]$ BY (253.00)

4. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\delta, q)$$
$[a > b \geq u > c > d]$ BY (254.00)

5. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\kappa, q)$$
$[a > b > u \geq c > d]$ BY (255.00)

6. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\lambda, r)$$
$[a \geq u > b > c > d]$ BY (256.00)

7.[11] $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\mu, r)$$
$[a > u \geq b > c > d]$ BY (257.00)

8. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\, F(\nu, q)$$
$[u > a > b > c > d]$ BY (258.00)

3.148

1.[8] $$\int_u^d \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-c)\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + c\,F(\alpha, q)\right\}$$
$[a > b > c > d > u]$ BY (251.03)

2. $$\int_d^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-a)\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right) + a\,F(\beta, r)\right\}$$
$[a > b > c \geq u > d]$ BY (252.11)

3. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(c-b)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + b\,F(\gamma, r)\right\}$$
$[a > b > c > u \geq d]$ BY (253.11)

4. $$\int_c^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(c-d)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + d\,F(\delta, q)\right\}$$
$[a > b \geq u > c > d]$ BY (254.10)

5. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-a)\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) + a\,F(\kappa, q)\right\}$$
$[a > b > u \geq c > d]$ BY (255.17)

6.[8] $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-c)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + c\,F(\lambda, r)\right\}$$
$[a \geq u > b > c > d]$ BY (256.11)

7. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-d)\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right) + d\,F(\mu, r)\right\}$$
$[a > u \geq b > c > d]$ BY (257.11)

8. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-b)\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right) + b\,F(\nu, q)\right\}$$
$[u > a > b > c > d]$ BY (258.11)

3.149

1. $$\int_u^d \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(d-x)}}$$
$$= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{(c-d)\,\Pi\left(\alpha, \frac{c(a-d)}{d(a-c)}, q\right) + d\,F(\alpha, q)\right\}$$
$[a > b > c > d > u]$ BY (251.04)

2. $$\int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}}\left\{(a-d)\,\Pi\left(\beta, \frac{a(d-c)}{d(a-c)}, r\right) + d\,F(\beta, r)\right\}$$
$[a > b > c \geq u > d]$ BY (252.12)

3. $$\int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{bc\sqrt{(a-c)(b-d)}}\left\{(b-c)\,\Pi\left(\gamma, \frac{b(c-d)}{c(b-d)}, r\right) + c\,F(\gamma, r)\right\}$$
$[a > b > c > u \geq d]$ BY (253.12)

4. $$\int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{cd\sqrt{(a-c)(b-d)}}\left\{(d-c)\,\Pi\left(\delta, \frac{d(b-c)}{c(b-d)}, q\right) + c\,F(\delta, q)\right\}$$
$[a > b \geq u > c > d]$ BY (254.11)

5. $$\int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{(a-b)\,\Pi\left(\kappa, \frac{a(b-c)}{b(a-c)}, q\right) + b\,F\left(\kappa, q\right)\right\}$$
$[a > b > u \geq c > d]$ BY (255.18)

6. $$\int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{(c-b)\,\Pi\left(\lambda, \frac{c(a-b)}{b(a-c)}, r\right) + b\,F(\lambda, r)\right\}$$
$[a \geq u > b > c > d]$ BY (256.12)

7. $$\int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{(d-a)\,\Pi\left(\mu, \frac{d(b-a)}{a(b-d)}, r\right) + a\,F(\mu, r)\right\}$$
$[a > u \geq b > c > d]$ BY (257.12)

8. $$\int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{ab\sqrt{(a-c)(b-d)}}\left\{(b-a)\,\Pi\left(\nu, \frac{b(a-d)}{a(b-d)}, q\right) + a\,F(\nu, q)\right\}$$
$[u > a > b > c > d]$ BY (258.12)

3.151

1. $$\int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(d-c)\,\Pi\left(\alpha, \frac{(a-d)(p-c)}{(a-c)(p-d)}, q\right) + (p-d)\,F(\alpha, q)\right]$$
$[a > b > c > d > u, \quad p \neq d]$ BY (251.39)

2. $$\int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(d-a)\,\Pi\left(\beta, \frac{(d-c)(p-a)}{(a-c)(p-d)}, r\right) + (p-d)\,F(\beta, r)\right]$$
$[a > b > c \geq u > d, \quad p \neq d]$ BY (252.39)

3. $$\int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \times \left[(c-b)\,\Pi\left(\gamma, \frac{(c-d)(p-b)}{(b-d)(p-c)}, r\right) + (p-c)\,F(\gamma, r)\right]$$
$[a > b > c > u \geq d, \quad p \neq c]$ BY (253.39)

4. $$\int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(c-d)\,\Pi\left(\delta, \frac{(b-c)(p-d)}{(b-d)(p-c)}, q\right) + (p-c)\,F(\delta, q)\right]$$
$[a > b \geq u > c > d, \quad p \neq c]$ BY (254.39)

5. $$\int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \left[(b-a)\,\Pi\left(\kappa, \frac{(b-c)(p-a)}{(a-c)(p-b)}, q\right) + (p-b)\,F(\kappa, q)\right]$$
$[a > b > u \geq c > d, \quad p \neq b]$ BY (255.38)

6. $$\int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \times \left[(b-c)\,\Pi\left(\lambda, \frac{(a-b)(p-c)}{(a-c)(p-b)}, r\right) + (p-b)\,F(\lambda, r)\right]$$
$[a \geq u > b > c > d, \quad p \neq b]$ BY (256.39)

7. $$\int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(a-d)\,\Pi\left(\mu, \frac{(b-a)(p-d)}{(b-d)(p-a)}, r\right) + (p-a)\,F(\mu, r)\right]$$
$[a > u \geq b > c > d, \quad p \neq a]$ BY (257.39)

8. $$\int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \left[(a-b)\,\Pi\left(\nu, \frac{(a-d)(p-b)}{(b-d)(p-a)}, q\right) + (p-a)\,F(\nu, q)\right]$$
$[u > a > b > c > d, \quad p \neq a]$ BY (258.39)

3.152 **Notation**: In **3.152**–**3.163** we set: $\alpha = \arctan\dfrac{u}{b}, \qquad \beta = \operatorname{arccot}\dfrac{u}{a}$

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \qquad \delta = \arccos\frac{u}{b}, \qquad \varepsilon = \arccos\frac{b}{u}, \qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
$$\eta = \arcsin\frac{u}{b}, \qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \qquad \kappa = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \qquad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \qquad \nu = \arcsin\frac{a}{u}, \qquad q = \frac{\sqrt{a^2-b^2}}{a},$$
$$r = \frac{b}{\sqrt{a^2+b^2}}, \qquad s = \frac{a}{\sqrt{a^2+b^2}}, \qquad t = \frac{b}{a}.$$

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)}} = \frac{1}{a}\,F(\alpha, q)$$ $[a > b > 0]$ H 62(258), BY (221.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)}} = \frac{1}{a}\,F(\beta, q)$$ $[a > b > 0]$ H 63 (259), BY (222.00)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)\,(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,F(\gamma, r)$$ $[b \geq u > 0]$ H 63 (260)

4. $$\int_u^b \frac{dx}{\sqrt{(x^2+a^2)\,(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,F(\delta, r)$$ $[b > u \geq 0]$ H 63 (261), BY (213.00)

5. $$\int_b^u \frac{dx}{\sqrt{(x^2+a^2)\,(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,F(\varepsilon, s)$$ $[u > b > 0]$ H 63 (262), BY (211.00)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)\,(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,F(\xi, s)$$ $[u > b > 0]$ H 63 (263), BY (212.00)

7. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)}} = \frac{1}{a}\,F(\eta,t) \qquad [a>b\geq u>0]$$ H 63 (264), BY (219.00)

8. $$\int_u^b \frac{dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)}} = \frac{1}{a}\,F(\zeta,t) \qquad [a>b>u\geq 0]$$ H 63 (265), BY (220.00)

9. $$\int_b^u \frac{dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)}} = \frac{1}{a}\,F(\kappa,q) \qquad [a\geq u>b>0]$$ H 63 (266), BY (217.00)

10. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)}} = \frac{1}{a}\,F(\lambda,q) \qquad [a>u\geq b>0]$$ H 63 (257), BY (218.00)

11. $$\int_a^u \frac{dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)}} = \frac{1}{a}\,F(\mu,t) \qquad [u>a>b>0]$$ H 63 (268), BY (216.00)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)}} = \frac{1}{a}\,F(\nu,t) \qquad [u\geq a>b>0]$$ H 64(269), BY (215.00)

3.153

1. $$\int_0^u \frac{x^2\,dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)}} = u\sqrt{\frac{a^2+u^2}{b^2+u^2}} - a\,E(\alpha,q) \qquad [u>0,\quad a>b]$$ BY (221.09)

2. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)\,(b^2-x^2)}} = \sqrt{a^2+b^2}\,E(\gamma,r) - \frac{a^2}{\sqrt{a^2+b^2}}\,F(\gamma,r) - u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b\geq u>0]$ BY (214.05)

3. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2+x^2)\,(b^2-x^2)}} = \sqrt{a^2+b^2}\,E(\delta,r) - \frac{a^2}{\sqrt{a^2+b^2}}\,F(\delta,r)$$
$[b>u\geq 0]$ BY (213.06)

4. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)\,(x^2-b^2)}} = \frac{b^2}{\sqrt{a^2+b^2}}\,F(\varepsilon,s) - \sqrt{a^2+b^2}\,E(\varepsilon,s) + \frac{1}{u}\sqrt{(u^2+a^2)\,(u^2-b^2)}$$
$[u>b>0]$ BY (211.09)

5. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)}} = a\,\{F(\eta,t)-E(\eta,t)\} \qquad [a>b\geq u>0]$$ BY (219.05)

6. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)}} = a\,\{F(\zeta,t)-E(\zeta,t)\} + u\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a>b>u\geq 0]$ BY (220.06)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)}} = a\,E(\kappa,q) - \frac{1}{u}\sqrt{(a^2-u^2)\,(u^2-b^2)}$$
$[a\geq u>b>0]$ BY (217.05)

8. $$\int_u^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)}} = a\,E(\lambda,q) \qquad [a>u\geq b>0]$$ BY (218.06)

9.[6] $$\int_a^u \frac{x^2\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = a\left\{F(\mu,t)-E(\mu,t)\right\}+u\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u>a>b>0]$ BY (216.06)

10. $$\int_0^1 \frac{x^2\,dx}{\sqrt{(1+x^2)(1+k^2x^2)}} = \frac{1}{k^2}\left\{\sqrt{\frac{1+k^2}{2}}-E\left(\frac{\pi}{4},\sqrt{1-k^2}\right)\right\}$$
BI (14)(9)

3.154

1. $$\int_0^u \frac{x^4\,dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E(\alpha,q)-b^2F(\alpha,q)\right\}+\frac{u}{3}\left(u^2-2a^2-b^2\right)\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a>b,\quad u>0]$ BY (221.09)

2. $$\int_0^u \frac{x^4\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2a^2-b^2\right)a^2F(\gamma,r)-2\left(a^4-b^4\right)E(\gamma,r)\right\}$$
$$-\frac{u}{3}\left(2b^2-a^2+u^2\right)\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[a\geq u>0]$ BY (214.05)

3. $$\int_u^b \frac{x^4\,dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2a^2-b^2\right)a^2F(\delta,r)-2\left(a^4-b^4\right)E(\delta,r)\right\}$$
$$+\frac{u}{3}\sqrt{(a^2+u^2)(b^2-u^2)}$$
$[b>u\geq 0]$ BY (213.06)

4. $$\int_b^u \frac{x^4\,dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2b^2-a^2\right)b^2F(\varepsilon,s)+2\left(a^4-b^4\right)E(\varepsilon,s)\right\}$$
$$+\frac{2b^2-2a^2+u^2}{3u}\sqrt{(u^2+a^2)(u^2-b^2)}$$
$[u>b>0]$ BY (211.09)

5. $$\int_0^u \frac{x^4\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\eta,t)-2\left(a^2+b^2\right)E(\eta,t)\right\}+\frac{u}{3}\sqrt{(a^2-u^2)(b^2-u^2)}$$
$[a>b\geq u>0]$ BY (219.05)

6. $$\int_u^b \frac{x^4\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\zeta,t)-2\left(a^2+b^2\right)E(\zeta,t)\right\}$$
$$+\frac{u}{3}\left(u^2+a^2+2b^2\right)\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a>b>u\geq 0]$ BY (220.06)

7. $$\int_b^u \frac{x^4\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E(\kappa,q)-b^2F(\kappa,q)\right\}$$
$$-\frac{u^2+2a^2+2b^2}{3u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a\geq u>b>0]$ BY (217.05)

8. $$\int_u^a \frac{x^4\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E(\lambda,q)-b^2F(\lambda,q)\right\}+\frac{u}{3}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a>u\geq b>0]$ BY (218.06)

9. $$\int_a^u \frac{x^4\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\mu,t) - 2\left(a^2+b^2\right)E(\mu,t)\right\} + \frac{u}{3}\left(u^2+2a^2+b^2\right)\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.06)

3.155

1. $$\int_u^a \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\lambda,q) - 2b^2F(\lambda,q)\right\} - \frac{u}{3}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a > u \geq b > 0]$ BY (218.11)

2. $$\int_a^u \sqrt{(x^2-a^2)(x^2-b^2)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\mu,t) - \left(a^2-b^2\right)F(\mu,t)\right\} + \frac{u}{3}\left(u^2-a^2-2b^2\right)\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.10)

3. $$\int_0^u \sqrt{(x^2+a^2)(x^2+b^2)}\,dx = \frac{a}{3}\left\{2b^2F(\alpha,q) - \left(a^2+b^2\right)E(\alpha,q)\right\} + \frac{u}{3}\left(u^2+a^2+2b^2\right)\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a > b, \quad u > 0]$ BY (221.08)

4. $$\int_0^u \sqrt{(a^2+x^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{a^2F(\gamma,r) - \left(a^2-b^2\right)E(\gamma,r)\right\} + \frac{u}{3}\left(u^2+2a^2-b^2\right)\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[a \geq u > 0]$ BY (214.12)

5.[9] $$\int_u^b \sqrt{(a^2+x^2)(b^2-x^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{a^2F(\delta,r) + \left(b^2-a^2\right)E(\delta,r)\right\} + \frac{u}{3}\sqrt{(a^2+u^2)(b^2-u^2)}$$
$[b > u \geq 0]$ BY (213.13)

6. $$\int_b^u \sqrt{(a^2+x^2)(x^2-b^2)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{\left(b^2-a^2\right)E(\varepsilon,s) - b^2F(\varepsilon,s)\right\} + \frac{u^2+a^2-b^2}{3u}\sqrt{(a^2+u^2)(u^2-b^2)}$$
$[u > b > 0]$ BY (211.08)

7. $$\int_0^u \sqrt{(a^2-x^2)(b^2-x^2)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\eta,t) - \left(a^2-b^2\right)F(\eta,t)\right\} + \frac{u}{3}\sqrt{(a^2-u^2)(b^2-u^2)}$$
$[a > b \geq u > 0]$ BY (219.11)

8. $$\int_u^b \sqrt{(a^2-x^2)(b^2-x^2)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\zeta,t) - \left(a^2-b^2\right)F(\zeta,t)\right\} + \frac{u}{3}\left(u^2-2a^2-b^2\right)\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a > b > u \geq 0]$ BY (220.05)

9. $$\int_b^u \sqrt{(a^2-x^2)(x^2-b^2)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\kappa,q)-2b^2F(\kappa,q)\right\} + \frac{u^2-a^2-b^2}{3u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a \geq u > b > 0]$ BY (217.09)

3.156

1.[6] $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{ub^2}\sqrt{\frac{b^2+u^2}{a^2+u^2}} - \frac{1}{ab^2}E(\beta,q)$$
$[a \geq b, \quad u > 0]$ BY (222.04)

2. $$\int_u^b \frac{dx}{x^2\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{a^2F(\delta,r)-\left(a^2+b^2\right)E(\delta,r)\right\} + \frac{1}{a^2b^2u}\sqrt{(a^2+u^2)(b^2-u^2)}$$
$[b > u > 0]$ BY (213.09)

3. $$\int_b^u \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{\left(a^2+b^2\right)E(\varepsilon,s)-b^2F(\varepsilon,s)\right\}$$
$[u > b > 0]$ BY (211.11)

4. $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{\left(a^2+b^2\right)E(\xi,s)-b^2F(\xi,s)\right\} - \frac{1}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$
$[u \geq b > 0]$ BY (212.06)

5. $$\int_u^b \frac{dx}{x^2\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ab^2}\left\{F(\zeta,t)-E(\zeta,t)\right\} + \frac{1}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a > b > u > 0]$ BY (220.09)

6. $$\int_b^u \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\kappa,q)$$
$[a \geq u > b > 0]$ BY (217.01)

7. $$\int_u^a \frac{dx}{x^2\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ab^2}E(\lambda,q) - \frac{1}{a^2b^2u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a > u \geq b > 0]$ BY (218.12)

8. $$\int_a^u \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\left\{F(\mu,t)-E(\mu,t)\right\} + \frac{1}{a^2u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.09)

9. $$\int_u^\infty \frac{dx}{x^2\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ab^2}\left\{F(\nu,t)-E(\nu,t)\right\}$$
$[u \geq a > b > 0]$ BY (215.07)

3.157

1. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p+b^2)}\left\{\frac{b^2}{p}\Pi\left(\alpha, \frac{p+b^2}{p}, q\right) + F(\alpha, q)\right\}$$
$[p \neq 0]$ BY (221.13)

2. $$\int_u^\infty \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)}\left\{\Pi\left(\beta, \frac{a^2+p}{a^2}, q\right) - F(\beta, q)\right\}$$
BY (222.11)

3. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{p(p+a^2)\sqrt{a^2+b^2}}\left\{a^2\Pi\left(\gamma, \frac{b^2(p+a^2)}{p(a^2+b^2)}, r\right) + pF(\gamma, r)\right\}$$
$[b \geq u > 0, \quad p \neq 0]$ BY (214.13)a

4. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{(p-b^2)\sqrt{a^2+b^2}}\Pi\left(\delta, \frac{b^2}{b^2-p}, r\right)$$
$[b > u \geq 0, \quad p \neq b^2]$ BY (213.02)

5. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{p(p-b^2)\sqrt{a^2+b^2}}\left\{b^2\Pi\left(\varepsilon, \frac{p}{p-b^2}, s\right) + (p-b^2)F(\varepsilon, s)\right\}$$
$[u > b > 0, \quad p \neq b^2]$ BY (211.14)

6. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{(a^2+p)\sqrt{a^2+b^2}}\left\{\Pi\left(\xi, \frac{a^2+p}{a^2+b^2}, s\right) - F(\xi, s)\right\}$$
$[u \geq b > 0]$ BY (212.12)

7. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ap}\Pi\left(\eta, \frac{b^2}{p}, t\right) \qquad [a > b \geq u > 0; \quad p \neq b]$$
BY (219.02)

8. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times \left\{(b^2-a^2)\Pi\left(\zeta, \frac{b^2(p-a^2)}{a^2(p-b^2)}, t\right) + (p-b^2)F(\zeta, t)\right\}$$
$[a > b > u \geq 0; \quad p \neq b^2]$ BY (220.13)

9. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)}\left\{b^2\Pi\left(\kappa, \frac{p(a^2-b^2)}{a^2(p-b^2)}, q\right) + (p-b^2)F(\kappa, q)\right\}$$
$[a \geq u > b > 0; \quad p \neq b^2]$ BY (217.12)

10. $$\int_u^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)}\Pi\left(\lambda, \frac{a^2-b^2}{a^2-p}, q\right)$$
$[a > u \geq b > 0; \quad p \neq a^2]$ BY (218.02)

11. $$\int_a^u \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a(p-a^2)(p-b^2)}\left\{(a^2-b^2)\Pi\left(\mu, \frac{p-b^2}{p-a^2}, t\right) + (p-a^2)F(\mu, t)\right\}$$
$[u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2]$ BY (216.12)

12. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap}\left\{\Pi\left(\nu,\frac{p}{a^2},t\right)-F(\nu,t)\right\}$$

$[u \geq a > b > 0; \quad p \neq 0]$ BY (215.12)

3.158

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left\{a^2\,E(\alpha,q)-b^2\,F(\alpha,q)\right\}$$

$[a > b; \quad u > 0]$ BY (221.05)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left\{a^2\,E(\beta,q)-b^2\,F(\beta,q)\right\}-\frac{u}{b^2\sqrt{(a^2+u^2)(b^2+u^2)}}$$

$[a > b, \quad u \geq 0]$ BY (222.05)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\left\{F(\alpha,q)-E(\alpha,q)\right\}+\frac{u}{a^2\sqrt{(u^2+a^2)(u^2+b^2)}}$$

$[a > b; \quad u > 0]$ BY (221.06)

4. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\left\{F(\beta,q)-E(\beta,q)\right\}$$

$[a > b, \quad u \geq 0]$ BY (222.03)

5. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\,E(\gamma,r) \qquad [b \geq u > 0]$$ BY (214.01)a

6. $$\int_u^b \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\,E(\delta,r)-\frac{u}{a^2(a^2+b^2)}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$

$[b > u \geq 0]$ BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\left\{F(\varepsilon,s)-E(\varepsilon,s)\right\}+\frac{1}{(a^2+b^2)\,u}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$

$[u > b > 0]$ BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\left\{F(\xi,s)-E(\xi,s)\right\}$$

$[u \geq b > 0]$ BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)^3}} = \frac{1}{b^2\sqrt{a^2+b^2}}\left\{F(\gamma,r)-E(\gamma,r)\right\}+\frac{u}{b^2\sqrt{(a^2+u^2)(b^2-u^2)}}$$

$[b > u > 0]$ BY (214.10)

10. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^3}} = \frac{u}{b^2\sqrt{(a^2+u^2)(u^2-b^2)}}-\frac{1}{b^2\sqrt{a^2+b^2}}\,E(\xi,s)$$

$[u \geq b > 0]$ BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)^3\,(b^2-x^2)}} = \frac{1}{a^2\,(a^2-b^2)}\left\{a\,E(\eta,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$

$[a > b \geq u > 0]$ BY (219.07)

12. $$\int_u^b \frac{dx}{\sqrt{(a^2-x^2)^3\,(b^2-x^2)}} = \frac{1}{a\,(a^2-b^2)}\,E(\zeta,t)$$

$[a > b > u \geq 0]$ BY (220.10)

13. $$\int_b^u \frac{dx}{\sqrt{(a^2-x^2)^3\,(x^2-b^2)}} = \frac{1}{a\,(a^2-b^2)}\left\{F\,(\kappa,q) - E\,(\kappa,q) + \frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}}\right\}$$

$[a > u > b > 0]$ BY (217.10)

14. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)^3\,(x^2-b^2)}} = \frac{1}{a\,(b^2-a^2)}\left\{E(\nu,t) - \frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}}\right\}$$

$[u > a > b > 0]$ BY (215.04)

15. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)^3}} = \frac{1}{ab^2}\,F(\eta,t) - \frac{1}{b^2\,(a^2-b^2)}\left\{a\,E(\eta,t) - u\sqrt{\frac{a^2-u^2}{b^2-u^2}}\right\}$$

$[a > b > u > 0]$ BY (219.06)

16. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)^3}} = \frac{1}{ab^2\,(a^2-b^2)}\left\{b^2\,F(\lambda,q) - a^2\,E(\lambda,q) + au\sqrt{\frac{a^2-u^2}{u^2-b^2}}\right\}$$

$[a > u > b > 0]$ BY (218.04)

17. $$\int_a^u \frac{dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)^3}} = \frac{a}{b^2\,(a^2-b^2)}\,E(\mu,t) - \frac{1}{ab^2}\,F(\mu,t)$$

$[u > a > b > 0]$ BY (216.11)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)^3}} = \frac{1}{b^2\,(a^2-b^2)}\left\{a\,E(\nu,t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\} - \frac{1}{ab^2}\,F(\nu,t)$$

$[u \geq a > b > 0]$ BY (215.06)

3.159

1. $$\int_0^u \frac{x^2\,dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)^3}} = \frac{a}{a^2-b^2}\,\{F(\alpha,q) - E(\alpha,q)\}$$

$[a > b, \quad u > 0]$ BY (221.12)

2. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)^3}} = \frac{a}{a^2-b^2}\,\{F(\beta,q) - E(\beta,q)\} + \frac{u}{\sqrt{(a^2+u^2)\,(b^2+u^2)}}$$

$[a > b, \quad u \geq 0]$ BY (222.10)

3. $$\int_0^u \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3\,(x^2+b^2)}} = \frac{1}{a\,(a^2-b^2)}\left\{a^2\,E(\alpha,q) - b^2\,F(\alpha,q)\right\} - \frac{u}{\sqrt{(a^2+u^2)\,(b^2+u^2)}}$$

$[a>b, \quad u>0]$ BY (221.11)

4. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2+a^2)^3\,(x^2+b^2)}} = \frac{1}{a\,(a^2-b^2)}\left\{a^2\,E(\beta,q) - b^2\,F(\beta,q)\right\}$$

$[a>b, \quad u\geq 0]$ BY (222.07)

5. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3\,(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\gamma,r) - E(\gamma,r)\right\}$$

$[b\geq u>0]$ BY (214.04)

6. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3\,(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\delta,r) - E(\delta,r)\right\} + \frac{u}{a^2+b^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$

$[b>u\geq 0]$ BY (213.07)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3\,(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,E(\varepsilon,s) - \frac{a^2}{u\,(a^2+b^2)}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$

$[u>b>0]$ BY (211.13)

8. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3\,(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,E(\xi,s) \qquad [u\geq b>0]$$ BY (212.01)

9. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2+x^2)\,(b^2-x^2)^3}} = \frac{u}{\sqrt{(a^2+u^2)\,(b^2-u^2)}} - \frac{1}{\sqrt{a^2+b^2}}\,E(\gamma,r)$$

$[b>u>0]$ BY (214.07)

10. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(a^2+x^2)\,(x^2-b^2)^3}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\xi,s) - E(\xi,s)\right\} + \frac{u}{\sqrt{(a^2+u^2)\,(u^2-b^2)}}$$

$[u>b>0]$ BY (212.10)

11. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3\,(b^2-x^2)}} = \frac{1}{a^2-b^2}\left\{a\,E(\eta,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} - \frac{1}{a}\,F(\eta,t)$$

$[a>b\geq u>0]$ BY (219.04)

12. $$\int_u^b \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3\,(b^2-x^2)}} = \frac{a}{a^2-b^2}\,E(\zeta,t) - \frac{1}{a}\,F(\zeta,t)$$

$[a>b>u\geq 0]$ BY (220.08)

13. $$\int_b^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3\,(x^2-b^2)}} = \frac{1}{a\,(a^2-b^2)}\left\{b^2\,F(\kappa,q) - a^2\,E(\kappa,q) + \frac{a^3}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}}\right\}$$

$[a>u>b>0]$ BY (217.06)

14. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3\,(x^2-b^2)}} = \frac{a}{a^2-b^2}\left\{\frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} - E(\nu,t)\right\} + \frac{1}{a}\,F(\nu,t)$$
$[u > a > b > 0]$ BY (215.09)

15. $$\int_0^u \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(b^2-x^2)^3}} = \frac{1}{a^2-b^2}\left\{u\sqrt{\frac{a^2-u^2}{b^2-u^2}} - a\,E(\eta,t)\right\}$$
$[a > b > u > 0]$ BY (219.12)

16. $$\int_u^a \frac{x^2\,dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left\{a\,F(\lambda,q) - a\,E(\lambda,q) + u\sqrt{\frac{a^2-u^2}{u^2-b^2}}\right\}$$
$[a > u > b > 0]$ BY (218.07)

17. $$\int_a^u \frac{x^2\,dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)^3}} = \frac{a}{a^2-b^2}\,E(\mu,t)$$
$[u > a > b > 0]$ BY (216.01)

18. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^2-a^2)\,(x^2-b^2)^3}} = \frac{1}{a^2-b^2}\left\{a\,E(\nu,t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\}$$
$[u \geq a > b > 0]$ BY (215.11)

3.161

1. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2+a^2)\,(x^2+b^2)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E(\beta,q) - b^2\,F(\beta,q)\right\} + \frac{a^2b^2-u^2\left(2a^2+b^2\right)}{3a^2b^4u^3}$$
$[a > b, \quad u > 0]$ BY (222.04)

2. $$\int_u^b \frac{dx}{x^4\sqrt{(x^2+a^2)\,(b^2-x^2)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\left\{a^2\left(2a^2-b^2\right)F(\delta,r) - 2\left(a^4-b^4\right)E(\delta,r)\right\} + \frac{a^2b^2+2u^2\left(a^2-b^2\right)}{3a^4b^4u^3}\sqrt{(b^2-u^2)\,(a^2+u^2)}$$
$[b > u > 0]$ BY (213.09)

3. $$\int_b^u \frac{dx}{x^4\sqrt{(x^2+a^2)\,(x^2-b^2)}} = \frac{2b^2-a^2}{3a^4b^2\sqrt{a^2+b^2}}\,F(\varepsilon,s) + \frac{2}{3}\frac{\left(a^2-b^2\right)\sqrt{a^2+b^2}}{a^4b^4}\,E(\varepsilon,s) + \frac{1}{3a^2b^2u^3}\sqrt{(u^2+a^2)\,(u^2-b^2)}$$
$[u > b > 0]$ BY (211.11)

4. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2+a^2)\,(x^2-b^2)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\left\{2\left(a^4-b^4\right)E(\xi,s) + b^2\left(2b^2-a^2\right)F(\xi,s)\right\} - \frac{a^2b^2+u^2\left(2a^2-b^2\right)}{3a^2b^4u^3}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$
$[u \geq b > 0]$ BY (212.06)

5. $$\int_u^b \frac{dx}{x^4\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{3a^3b^4}\left\{\left\{\left(2a^2+b^2\right)F(\zeta,t)-2\left(a^2+b^2\right)E(\zeta,t)\right\} + \frac{\left[\left(2a^2+b^2\right)u^2+a^2b^2\right]a}{u^3}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a>b>u>0]$ BY (220.09)

6. $$\int_b^u \frac{dx}{x^4\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E(\kappa,q)-b^2F(\kappa,q)\right\} + \frac{1}{3a^2b^2u^3}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a\geq u>b>0]$ BY (217.14)

7. $$\int_u^a \frac{dx}{x^4\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E(\lambda,q)-b^2F(\lambda,q) - \frac{2\left(a^2+b^2\right)u^2+a^2b^2}{au^3}\sqrt{(a^2-u^2)(u^2-b^2)}\right\}$$
$[a>u\geq b>0]$ BY (218.12)

8. $$\int_a^u \frac{dx}{x^4\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{\left\{\left(2a^2+b^2\right)F(\mu,t)-2\left(a^2+b^2\right)E(\mu,t)\right\} + \frac{\left[\left(a^2+2b^2\right)u^2+a^2b^2\right]b^2}{au^3}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\}$$
$[u>a>b>0]$ BY (216.09)

9. $$\int_u^\infty \frac{dx}{x^4\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{3a^3b^4}\left\{\left(2a^2+b^2\right)F(\nu,t)-2\left(a^2+b^2\right)E(\nu,t) + \frac{ab^2}{u^3}\sqrt{(u^2-a^2)(u^2-b^2)}\right\}$$
$[u\geq a>b>0]$ BY (215.07)

3.162

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^5(x^2+b^2)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(3a^2-b^2\right)F(\alpha,q)-2\left(2a^2-b^2\right)E(\alpha,q)\right\} + \frac{u\left[a^2\left(4a^2-3b^2\right)+u^2\left(3a^2-2b^2\right)\right]}{3a^4\left(a^2-b^2\right)\sqrt{\left(u^2+a^2\right)^3\left(u^2+b^2\right)}}$$
$[a>b,\quad u>0]$ BY (221.06)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)^5\,(x^2+b^2)}} = \frac{1}{3a^3\,(a^2-b^2)^2}\left\{\left(3a^2-b^2\right)F(\beta,q)-2\left(2a^2-b^2\right)E(\beta,q)\right\} + \frac{u}{3a^2\,(a^2-b^2)}\sqrt{\frac{u^2+b^2}{(a^2+u^2)^3}}$$
$[a>b, \quad u\geq 0]$ BY (222.03)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)^5}} = \frac{3b^2-a^2}{3ab^2\,(a^2-b^2)^2}F(\alpha,q)+\frac{a\left(2a^2-4b^2\right)}{3b^4\,(a^2-b^2)^2}E(\alpha,q) + \frac{u}{3b^2\,(a^2-b^2)}\sqrt{\frac{u^2+a^2}{(u^2+b^2)^3}}$$
$[a>b, \quad u>0]$ BY (221.05)

4. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)\,(x^2+b^2)^5}} = \frac{1}{3ab^4\,(a^2-b^2)^2}\left\{2a^2\left(a^2-2b^2\right)E(\beta,q)+b^2\left(3b^2-a^2\right)F(\beta,q)\right\} - \frac{u\left[b^2\left(3a^2-4b^2\right)+u^2\left(2a^2-3b^2\right)\right]}{3b^4\,(a^2-b^2)\sqrt{(u^2+a^2)\,(u^2+b^2)^3}}$$
$[a>b, \quad u\geq 0]$ BY (222.05)

5. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)^5\,(b^2-x^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{2\left(b^2+2a^2\right)E(\gamma,r)-a^2F(\gamma,r)\right\} + \frac{u}{3a^2\,(a^2+b^2)}\sqrt{\frac{b^2-u^2}{(a^2+u^2)^3}}$$
$[b\geq u>0]$ BY (214.15)

6. $$\int_u^b \frac{dx}{\sqrt{(a^2+x^2)^5\,(b^2-x^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{\left(4a^2+2b^2\right)E(\delta,r)-a^2F(\delta,r)\right\} - \frac{u\left[a^2\left(5a^2+3b^2\right)+u^2\left(4a^2+2b^2\right)\right]}{3a^4\,(a^2+b^2)^2}\sqrt{\frac{b^2-u^2}{(a^2+u^2)^3}}$$
$[b>u>0]$ BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{(a^2+x^3)^5\,(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{\left(3a^2+2b^2\right)F(\varepsilon,s)-\left(4a^2+2b^2\right)E(\varepsilon,s)\right\} + \frac{\left(3a^2+b^2\right)u^2+2\left(2a^2+b^2\right)a^2}{3a^2\,(a^2+b^2)^2\,u}\sqrt{\frac{u^2-b^2}{(u^2+a^2)^3}}$$
$[u>b>0]$ BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^5\,(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}}\left\{\left(3a^2+2b^2\right)F(\xi,s)-\left(4a^2+2b^2\right)E(\xi,s)\right\} + \frac{u}{3a^2\,(a^2+b^2)}\sqrt{\frac{u^2-b^2}{(a^2+u^2)^3}}$$
$[u>b>0]$ BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)\left(b^2-x^2\right)^5}} = \frac{1}{3b^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(2a^2+3b^2\right)F(\gamma,r)-\left(2a^2+4b^2\right)E(\gamma,r)\right\} + \frac{u\left[\left(3a^3+4b^2\right)b^2-\left(2a^2+3b^2\right)u^2\right]}{3b^4\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)^3}}$$
$[b>u>0]$ BY (214.10)

10. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)\left(x^2-b^2\right)^5}} = \frac{1}{3b^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(2a^2+4b^2\right)E(\xi,s)-b^2F(\xi,s)\right\} + \frac{u\left[\left(3a^2+4b^2\right)b^2-\left(2a^2+3b^2\right)u^2\right]}{3b^4\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)^3}}$$
$[u>b>0]$ BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)\left(b^2-x^2\right)^5}} = \frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\eta,t)+\frac{2a\left(2b^2-a^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\eta,t) + \frac{u\left[\left(3a^2-5b^2\right)b^2-2\left(a^2-2b^2\right)u^2\right]}{3b^4\left(a^2-b^2\right)^2\left(b^2-u^2\right)}\sqrt{\frac{a^2-u^2}{b^2-u^2}}$$
$[a>b>a>0]$ BY (219.06)

12. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)\left(x^2-b^2\right)^5}} = \frac{3b^2-a^2}{3ab^2\left(a^2-b^2\right)^2}F(\lambda,q)+\frac{2a\left(a^2-2b^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\lambda,q) + \frac{u\left[2\left(2b^2-a^2\right)u^2+\left(3a^2-5b^2\right)b^2\right]}{3b^4\left(a^2-b^2\right)^2\left(u^2-b^2\right)}\sqrt{\frac{a^2-u^2}{u^2-b^2}}$$
$[a>u>b>0]$ BY (218.04)

13. $$\int_a^u \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^5}} = \frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\mu,t)+\frac{2a\left(2b^2-a^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\mu,t) + \frac{u}{3b^2\left(a^2-b^2\right)\left(u^2-b^2\right)}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u>a>b>0]$ BY (216.11)

14. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^5}} = \frac{\left(4b^2-2a^2\right)a}{3b^4\left(a^2-b^2\right)^2}E(\nu,t)+\frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\nu,t) - \frac{\left(3b^2-a^2\right)u^2-\left(4b^2-2a^2\right)b^2}{3b^2u\left(a^2-b^2\right)^2\left(u^2-b^2\right)}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u\geq a>b>0]$ BY (215.06)

15. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(b^2-x^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(4a^2-2b^2\right)E(\eta,t)-\left(a^2-b^2\right)F(\eta,t) - \frac{u\left[\left(5a^2-3b^2\right)a^2-\left(4a^2-2b^2\right)u^2\right]}{a\left(a^2-u^2\right)}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a>b\geq u>0]$ BY (219.07)

16. $$\int_u^b \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(b^2-x^2\right)}} = \frac{2\left(2a^2-b^2\right)}{3a^3\left(a^2-b^2\right)^2} E\left(\zeta,r\right) - \frac{1}{3a^3\left(a^2-b^2\right)} F(\zeta,t) + \frac{u}{3a^2\left(a^2-b^2\right)\left(a^2-u^2\right)}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a>b>u\geq 0]$ BY (220.10)

17. $$\int_b^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(x^2-b^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(3a^2-b^2\right)F\left(\kappa,q\right)-\left(4a^2-2b^2\right)E\left(\kappa,q\right)\right\} + \frac{2\left(2a^2-b^2\right)a^2+\left(b^2-3a^2\right)u^2}{3a^2u\left(a^2-b^2\right)^2\left(a^2-u^2\right)}\sqrt{\frac{u^2-b^2}{a^2-u^2}},$$
$[a>u>b>0]$ BY (217.10)

18. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)^5\left(x^2-b^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(4a^2-2b^2\right)E(\nu,t)-\left(a^2-b^2\right)F(\nu,t)\right\} + \frac{\left(4a^2-2b^2\right)a^2+\left(b^2-3a^2\right)u^2}{3a^2u\left(a^2-b^2\right)^2\left(u^2-a^2\right)}\sqrt{\frac{u^2-b^2}{u^2-a^2}}$$
$[u>a>b>0]$ BY (215.04)

3.163

1. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)^2}\left\{\left(a^2+b^2\right)E(\alpha,q)-2b^2F(\alpha,q)\right\} - \frac{u}{a^2\left(a^2-b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$
$[a>b,\quad u>0]$ BY (221.07)

2. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)^2}\left\{\left(a^2+b^2\right)E(\beta,q)-2b^2F(\beta,q)\right\} - \frac{u}{b^2\left(a^2-b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$
$[a>b,\quad u\geq 0]$ BY (222.12)

3. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(b^3-x^2\right)^3}} = \frac{1}{a^2b^2\sqrt{\left(a^2+b^2\right)^3}}\left\{a^2F(\gamma,r)-\left(a^2-b^2\right)E(\gamma,r)\right\} + \frac{u}{b^2\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}}$$
$[b>u>0]$ BY (214.15)

4. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2-b^2\right)^3}} = \frac{b^2-a^2}{a^2b^2\sqrt{\left(a^2+b^2\right)^3}}E(\xi,s) - \frac{1}{a^2\sqrt{\left(a^2+b^2\right)^3}}F(\xi,s) + \frac{u}{b^2\left(a^2+b^2\right)\sqrt{\left(u^2+a^2\right)\left(u^2-b^2\right)}}$$
$[u>b>0]$ BY (212.05)

5. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)}F(\eta,t) - \frac{a^2+b^2}{ab^2\left(a^2-b^2\right)^2}E(\eta,t) + \frac{\left[a^4+b^4-\left(a^2+b^2\right)u^2\right]u}{a^2b^2\left(a^2-b^2\right)^2\sqrt{\left(a^2-u^2\right)\left(b^2-u^2\right)}}$$
$[a>b>u>0]$ BY (279.08)

6. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)^3\left(x^2-b^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)}F(\nu,t) - \frac{a^2+b^2}{ab^2\left(a^2-b^2\right)^2}E(\nu,t) + \frac{1}{u\left(a^2-b^2\right)\sqrt{\left(u^2-a^2\right)\left(u^2-b^2\right)}}$$
$[u>a>b>0]$ BY (215.10)

3.164 **Notation**: $\alpha = \arccos\dfrac{u^2-\rho\overline{\rho}}{u^2+\rho\overline{\rho}}, \qquad r = \dfrac{1}{2}\sqrt{-\dfrac{\left(\rho-\overline{\rho}\right)^2}{\rho\overline{\rho}}}.$

1. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = \frac{1}{\sqrt{\rho\overline{\rho}}}F(\alpha,r)$$ BY (225.00)

2. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2-\rho\overline{\rho}\right)^2\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = \frac{2u\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}{\left(\rho+\overline{\rho}\right)^2\left(u^4-\rho^2\overline{\rho}^2\right)} - \frac{1}{\left(\rho+\overline{\rho}\right)^2\sqrt{\rho\overline{\rho}}}E(\alpha,r)$$
BY (225.03)

3. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2+\rho\overline{\rho}\right)^2\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = -\frac{1}{\left(\rho-\overline{\rho}\right)^2\sqrt{\rho\overline{\rho}}}\left[F(\alpha,r)-E(\alpha,r)\right]$$ BY (225.07)

4. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2+\rho^2\right)^3\left(x^2+\overline{\rho}^2\right)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{\left(\rho^2-\overline{\rho}^2\right)^2}E(\alpha,r) + \frac{1}{\left(\rho-\overline{\rho}\right)^2\sqrt{\rho\overline{\rho}}}F(\alpha,r) - \frac{2u\left(u^2-\rho\overline{\rho}\right)}{\left(\rho+\overline{\rho}\right)^2\left(u^2+\rho\overline{\rho}\right)\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}$$
BY (225.05)

5. $$\int_u^\infty \frac{\left(x^2-\rho\overline{\rho}\right)^2\,dx}{\sqrt{\left(x^2+\rho^2\right)^3\left(x^2+\overline{\rho}^2\right)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{\left(\rho-\overline{\rho}\right)^2}\left[F(\alpha,r)-E(\alpha,r)\right] + \frac{2u\left(u^2-\rho\overline{\rho}\right)}{\left(u^2+\rho\overline{\rho}\right)\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}$$
BY (225.06)

6. $$\int_u^\infty \frac{\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}}{\left(x^2+\rho\overline{\rho}\right)^2}\,dx = \frac{1}{\sqrt{\rho\overline{\rho}}}E(\alpha,r)$$ BY(225.01)

7. $$\int_u^\infty \frac{\left(x^2-\varrho\overline{\varrho}\right)^2\,dx}{\left(x^2+\varrho\overline{\varrho}\right)^2\sqrt{\left(x^2+\varrho^2\right)\left(x^2+\overline{\varrho}^2\right)}} = -\frac{4\sqrt{\varrho\overline{\varrho}}}{\left(\varrho-\overline{\varrho}\right)^2}E(\alpha,r) + \frac{\left(\varrho+\overline{\varrho}\right)^2}{\left(\varrho-\overline{\varrho}\right)^2\sqrt{\varrho\overline{\varrho}}}F(\alpha,r)$$ BY (225.08)

8. $$\int_u^\infty \frac{\left(x^2+\varrho\overline{\varrho}\right)^2\,dx}{\left[\left(x^2+\varrho\overline{\varrho}\right)^2-4p^2\varrho\overline{\varrho}x^2\right]\sqrt{\left(x^2+\varrho^2\right)\left(x^2+\overline{\varrho}^2\right)}} = \frac{1}{\sqrt{\varrho\overline{\varrho}}}\Pi\left(\alpha,p^2,r\right)$$ BY (225.02)

3.165 **Notation**: $\alpha = \arccos \dfrac{u^2 - a^2}{u^2 + a^2}, \quad r = \dfrac{\sqrt{a^2 - b^2}}{a\sqrt{2}}.$

1. $$\int_u^a \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} \times F\left[\arctan\left(\frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a - u}{a + u}\right), \frac{2\sqrt{a\sqrt{2}\,(a^2 - b^2)}}{a\sqrt{2} + \sqrt{a^2 - b^2}}\right]$$
$[a > b, \quad a > u \geq 0]$ BY (264.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} F(\alpha, r)$$
$[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$ BY (263.00, 266.00)

3. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a^3} [F(\alpha, r) - 2\,E(\alpha, r)] + \frac{\sqrt{u^4 + 2b^2u^2 + a^4}}{a^2u\,(u^2 + a^2)}$$
$[a > b > 0, \quad u > 0]$ BY (263.06)

4. $$\int_u^\infty \frac{x^2\,dx}{(x^2 + a^2)^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{4a\,(a^2 - b^2)} [F(\alpha, r) - E(\alpha, r)]$$
$[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$ BY (263.03, 266.05)

5. $$\int_u^\infty \frac{x^2\,dx}{(x^2 - a^2)^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{u\sqrt{u^4 + 2b^2u^2 + a^4}}{2\,(a^2 + b^2)\,(u^4 - a^4)} - \frac{1}{4a\,(a^2 + b^2)} E(\alpha, r)$$
$[a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0]$ BY (263.05, 266.02)

6. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{2\,(a^4 - b^4)} E(\alpha, r) - \frac{1}{4a\,(a^2 - b^2)} F(\alpha, r) - \frac{u\left(u^2 - a^2\right)}{2\,(a^2 + b^2)\,(u^2 + a^2)\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$[a^2 > b^2 > -\infty, a^2 > 0, \quad u \geq 0]$ BY (263.08, 266.03)

7. $$\int_u^\infty \frac{\left(x^2 - a^2\right)^2\,dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 - b^2} [F(\alpha, r) - E(\alpha, r)] + \frac{u^2 - a^2}{u^2 + a^2} \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$[|b^2| < a^2, \quad u \geq 0]$ BY (266.08)

8. $$\int_u^\infty \frac{\left(x^2 + a^2\right)^2\,dx}{\sqrt{(x^2 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\alpha, r) - \frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{u^2 - a^2}{u^2 + a^2} \cdot \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$[|b^2| < a^2, \quad u \geq 0]$ BY (266.06)a

9. $$\int_u^\infty \frac{\left(x^2 - a^2\right)^2\,dx}{(x^2 + a^2)^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{a}{a^2 - b^2} E(\alpha, r) - \frac{a^2 + b^2}{2a\,(a^2 - b^2)} F(\alpha, r)$$
$[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$ BY (263.04, 266.07)

10. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{\left(x^2+a^2\right)^2}\,dx = \frac{1}{2a}\,E(\alpha,r) \qquad \left[a^2>b^2>-\infty,\quad a^2>0,\quad u\ge 0\right]$$
BY (263.01, 266.01)

11. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{\left(x^2-a^2\right)^2}\,dx = \frac{1}{2a}\left[F(\alpha,r)-E(\alpha,r)\right]+\frac{u}{u^4-a^4}\sqrt{u^4+2b^2u^2+a^4}$$
$[a>b>0,\quad u>a]$ BY (263)

12. $$\int_u^\infty \frac{\left(x^2+a^2\right)^2\,dx}{\left[\left(x^2+a^2\right)^2-4a^2p^2x^2\right]\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}\,\Pi\left(\alpha,p^2,r\right)$$
$[a>b>0,\quad u\ge 0]$ BY (263.02)

3.166 **Notation**: $\alpha=\arccos\dfrac{u^2-1}{u^2+1},\quad \beta=\arctan\left\{\left(1+\sqrt{2}\right)\dfrac{1-u}{1+u}\right\},$

$$\gamma=\arccos u,\quad \delta=\arccos\frac{1}{u},\quad \varepsilon=\arccos\frac{1-u^2}{1+u^2},$$
$$r=\frac{\sqrt{2}}{2},\quad q=2\sqrt{3\sqrt{2}-4}=2\sqrt[4]{2}\left(\sqrt{2}-1\right)\approx 0.985171$$

1. $$\int_u^\infty \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}\,F(\alpha,r) \qquad [u\ge 0]$$
H (287), BY (263.50)

2. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4+1}} = \frac{1}{2}\left[F(\alpha,r)-2\,E(\alpha,r)\right]+\frac{\sqrt{u^4+1}}{u\left(u^2+1\right)}$$
$[u>0]$ BY (263.57)

3. $$\int_u^\infty \frac{x^2\,dx}{\left(x^4+1\right)\sqrt{x^4+1}} = \frac{1}{2}\,E(\alpha,r)-\frac{1}{4}\,F(\alpha,r)-\frac{u\left(u^2-1\right)}{2\left(u^2+1\right)\sqrt{u^4+1}}$$
$[u\ge 0]$ BY (263.59)

4. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2+1\right)^2\sqrt{x^4+1}} = \frac{1}{4}\left[F(\alpha,r)-E(\alpha,r)\right] \qquad [u\ge 0]$$
BY (263.53)

5. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2-1\right)^2\sqrt{x^4+1}} = \frac{u\sqrt{u^4+1}}{2\left(u^4-1\right)}-\frac{1}{4}\,E(\alpha,r) \qquad [u>1]$$
BY (263.55)

6. $$\int_u^\infty \frac{\sqrt{x^4+1}}{\left(x^2-1\right)^2}\,dx = \frac{1}{2}\left[F(\alpha,r)-E(\alpha,r)\right]+\frac{u\sqrt{u^4+1}}{u^4-1}$$
$[u>1]$ BY (263.58)

7. $$\int_u^\infty \frac{\left(x^2-1\right)^2\,dx}{\left(x^2+1\right)^2\sqrt{x^4+1}} = E(\alpha,r)-\frac{1}{2}\,F(\alpha,r) \qquad [u\ge 0]$$
BY (263.54)

8. $$\int_u^\infty \frac{\sqrt{x^4+1}\,dx}{\left(x^2+1\right)^2} = \frac{1}{2}\,E(\alpha,r) \qquad [u\ge 0]$$
BY (263.51)

9. $$\int_u^\infty \frac{\left(x^2+1\right)^2\,dx}{\left[\left(x^2+1\right)^2-4p^2x^2\right]\sqrt{x^4+1}} = \frac{1}{2}\,\Pi\left(\alpha,p^2,r\right) \qquad [u\geq 0]$$ BY (263.52)

10. $$\int_0^u \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}\,F(\varepsilon,r)$$ H 66(288)

11. $$\int_u^1 \frac{dx}{\sqrt{x^4+1}} = \left(2-\sqrt{2}\right)F(\beta,q) \qquad [0\leq u<1]$$ BY (264.50)

12. $$\int_u^1 \frac{\left(x^2+x\sqrt{2}+1\right)\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \left(2+\sqrt{2}\right)E(\beta,q) \qquad [0\leq u<1]$$ BY (264.51)

13. $$\int_u^1 \frac{(1-x)^2\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \frac{1}{\sqrt{2}}\left[F(\beta,q)-E(\beta,q)\right]$$
$[0\leq u<1]$ BY (264.55)

14. $$\int_u^1 \frac{(1+x)^2\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \frac{3\sqrt{2}+4}{2}\,E(\beta,q)-\frac{3\sqrt{2}-4}{2}\,F(\beta,q)$$
$[0\leq u<1]$ BY (264.56)

15. $$\int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}}\,F(\gamma,r) \qquad [u<1]$$ H 66 (290), BY (259.75)

16. $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2$$

17. $$\int_1^u \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}\,F(\delta,r) \qquad [u>1]$$ H 66 (289), BY (260.75)

18.[8] $$\int_u^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \sqrt{2}\,E(\gamma,r)-\frac{1}{\sqrt{2}}\,F(\gamma,r) \qquad [u<1]$$
$$= \frac{1}{\sqrt{2\pi}}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2 \qquad [u=0]$$
BY (259.76)

19. $$\int_1^u \frac{x^2\,dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}\,F(\delta,r)-\sqrt{2}\,E(\delta,r)+\frac{1}{u}\sqrt{u^4-1} \qquad [u>1]$$ BY (260.77)

20. $$\int_u^1 \frac{x^4\,dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}}\,F(\gamma,r)+\frac{u}{3}\sqrt{1-u^4} \qquad [u<1]$$ BY (259.76)

21.[3] $$\int_1^u \frac{x^4\,dx}{\sqrt{x^4-1}} = \frac{1}{3\sqrt{2}}\,F(\delta,r)+\frac{1}{3}u\sqrt{u^4-1} \qquad [u>1]$$ BY (260.77)

22. $$\int_0^u \frac{dx}{\sqrt{x\left(1+x^3\right)}} = \frac{1}{\sqrt[4]{3}}\,F\left(\arccos\frac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u},\frac{\sqrt{2+\sqrt{3}}}{2}\right)$$
$[u>0]$ BY (260.50)

23. $$\int_0^u \frac{dx}{\sqrt{x\left(1-x^3\right)}} = \frac{1}{\sqrt[4]{3}}\,F\left(\arccos\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u},\frac{\sqrt{2-\sqrt{3}}}{2}\right)$$
$[1\geq u>0]$ BY (259.50)

3.167 **Notation**: In **3.167** and **3.168** we set: $\alpha = \arcsin\sqrt{\dfrac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \kappa = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

1. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) - F(\alpha, q)\right\}$$
$[a > b > c > d > u]$ BY (251.05)

2. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - F(\beta, r)\right\}$$
$[a > b > c \geq u > d]$ BY (252.14)

3. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(c-b)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (b-d)\,F(\gamma, r)\right\}$$
$[a > b > c > u \geq d]$ BY (253.14)

4. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right)$$
$[a > b \geq u > c > d]$ BY (254.02)

5. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-a)\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) + (a-d)\,F(\kappa, q)\right\}$$
$[a > b > u \geq c > d]$ BY (255.20)

6. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-c)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (c-d)\,F(\lambda, r)\right\}$$
$[a \geq u > b > c > d]$ BY (256.13)

7. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right)$$
$[a > u \geq b > c > d]$ BY (257.02)

8. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(a-b)\,\Pi\left(\nu,\frac{a-d}{b-d},q\right)+(b-d)\,F(\nu,q)\right\}$$
$[u>a>b>c>d]$ BY (258.14)

9. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\alpha,\frac{a-d}{a-c},q\right)$$
$[a>b>c>d>u]$ BY (251.02)

10. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\beta,\frac{d-c}{a-c},r\right)-(a-c)\,F(\beta,r)\right]$$
$[a>b>c\geq u>d]$ BY (252.13)

11. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\gamma,\frac{c-d}{b-d},r\right)-F(\gamma,r)\right]$$
$[a>b>c>u\geq d]$ BY (253.13)

12. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\delta,\frac{b-c}{b-d},q\right)-F(\delta,q)\right]$$
$[a>b\geq u>c>d]$ BY (254.12)

13. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-a)\,\Pi\left(\kappa,\frac{b-c}{a-c},q\right)+(a-c)\,F\left(\kappa,q\right)\right]$$
$[a>b>u\geq c>d]$ BY (259.19)

14. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\lambda,\frac{a-b}{a-c},r\right)$$
$[a\geq u>b>c>d]$ BY (256.02)

15. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\mu,\frac{b-a}{b-d},r\right)+(d-c)\,F(\mu,r)\right]$$
$[a>u\geq b>c>d]$ BY (257.13)

16. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-b)\,\Pi\left(\nu,\frac{a-d}{b-d},q\right)+(b-c)\,F(\nu,q)\right]$$
$[u>a>b>c>d]$ BY (258.13)

17. $$\int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\alpha,\frac{a-d}{a-c},q\right)+(b-c)\,F(\alpha,q)\right]$$
$[a>b>c>d>u]$ BY (251.07)

18. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\beta,\frac{d-c}{a-c},r\right)-(a-b)\,F(\beta,r)\right]$$
$[a>b>c\geq u>d]$ BY (252.15)

19. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right)$$
$[a > b > c > u \geq d]$ BY (253.02)

20. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (b-d)\,F(\delta, q)\right]$$
$[a > b \geq u > c > d]$ BY (254.14)

21. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) - F(\kappa, q)\right]$$
$[a > b > u \geq c > d]$ BY (255.21)

22. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r)\right]$$
$[a \geq u > b > c > d]$ BY (256.15)

23.[8] $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right) - (b-d)\,F(\mu, r)\right]$$
$[a > u \geq b > c > d]$ BY (257.15)

24. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right)$$
$[u > a > b > c > d]$ BY (258.02)

25. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (a-c)\,F(\alpha, q)\right]$$
$[a > b > c > d > u]$ BY (251.06)

26. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right)$$
$[a > b > c \geq u > d]$ BY (252.02)

27. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-c)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (a-b)\,F(\gamma, r)\right]$$
$[a > b > c > u \geq d]$ BY (253.15)

28. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (a-d)\,F(\delta, q)\right]$$
$[a > b \geq u > c > d]$ BY (254.13)

29. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right)$$
$[a > b > u \geq c > d]$ BY (255.02)

30. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-b)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (a-c)\,F(\lambda, r)\right]$$
$[a \geq u > b > c > d]$ BY (256.14)

31. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\mu, \frac{b-a}{b-d}, r\right) - F(\mu, r)\right]$$
$[a > u \geq b > c > d]$ BY (257.14)

32. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\nu, \frac{a-d}{b-d}, q\right) - F(\nu, q)\right]$$
$[u > a > b > c > d]$ BY (258.15)

3.168

1. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{d-a}\left[\sqrt{\frac{a-c}{b-d}}\,E(\gamma, r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}\right]$$
$[a > b > c > u > d]$ BY (253.06)

2. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\delta, q) - E(\delta, q)]$$
$[a > b \geq u > c > d]$ BY (254.04)

3. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b > u \geq c > d]$ BY (255.09)

4. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\left[\sqrt{\frac{a-c}{b-d}}\,E(\lambda, r) - \frac{c-d}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}\right]$$
$[a \geq u > b > c > d]$ BY (256.06)

5. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,E(\mu, r)$$
$[a > u \geq b > c > d]$ BY (257.01)

6. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\nu, q) - E(\nu, q)] + \frac{2}{a-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$
$[u > a > b > c > d]$ BY (258.10)

7. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(b-c)(a-d)\,F(\gamma, r) - (a-c)(b-d)\,E(\gamma, r)] + \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$
$[a > b > c > u > d]$ BY (253.03)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \left[(a-c)(b-d)\,E(\delta,q) - (a-b)(c-d)\,F(\delta,q)\right]$$
$$[a > b \geq u > c > d]$$ BY (254.15)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \left[(a-c)(b-d)\,E(\kappa,q) - (a-b)(c-d)\,F(\kappa,q)\right] - \frac{2}{c-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$$[a > b > u \geq c > d]$$ BY (255.06)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \left[(a-c)(b-d)\,E(\lambda,r) - (a-d)(b-c)\,F(\lambda,r)\right] - \frac{2}{a-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$$[a \geq u > b > c > d]$$ BY (256.03)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}\,E(\mu,r) - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\mu,r)$$
$$[a > u \geq b > c > d]$$ BY (257.09)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}}\,dx = \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\nu,q) + 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}\,E(\nu,q)$$
$$[u > a > b > c > d]$$ BY (258.09)

13. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\gamma,r) - E(\gamma,r)] + \frac{2}{c-d}\sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$
$$[a > b > c > u > d]$$ BY (253.04)

14. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\delta,q)$$
$$[a > b \geq u > c > d]$$ BY (254.01)

15. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\kappa,q) - \frac{2(a-d)}{(b-d)(c-d)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b > u \geq c > d]$ BY (255.08)

16. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\lambda,r) - E(\lambda,r)] + \frac{2}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a \geq u > b > c > d]$ BY (256.05)

17. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\mu,r) - E(\mu,r)]$$
$[a > u \geq b > c > d]$ BY (257.06)

18. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{-2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\nu,q) + \frac{2}{c-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$
$[u > a > b > c > d]$ BY (258.05)

19. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\alpha,q) - E(\alpha,q)]$$
$[a > b > c > d > u]$ BY (251.01)

20. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{-2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\beta,r) + \frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a > b > c \geq u > d]$ BY (252.06)

21. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\kappa,q) - E(\kappa,q)] + \frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a > b > u > c > d]$ BY (255.05)

22. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\lambda,r)$$
$[a \geq u > b > c > d]$ BY (256.01)

23. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\mu,r) - \frac{2(c-d)}{(a-c)(b-c)}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a > u \geq b > c > d]$ BY (257.06)

24. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\nu,q) - E(\nu,q)] + \frac{2}{a-c}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u > a > b > c > d]$ BY (258.06)

25. $$\int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\alpha,q)$$
$[a > b > c > d > u]$ BY (251.01)

26. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\beta,r)-E(\beta,r)] + \frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a>b>c>u>d]$ BY (252.03)

27. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{d-c}\sqrt{\frac{b-d}{a-c}}\,E(\kappa,q) + \frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a>b>u>c>d]$ BY (255.03)

28. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\lambda,r)-E(\lambda,r)]$$
$[a\geq u>b>c>d]$ BY (256.08)

29. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\mu,r)-E(\mu,r)] + \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a>u\geq b>c>d]$ BY (257.03)

30. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\nu,q) - \frac{2(b-c)}{(a-c)(c-d)}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u>a>b>c>d]$ BY (258.03)

31. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\alpha,q) - \frac{a-b}{b-c}\,\frac{2}{\sqrt{(a-c)(b-d)}}\,F(\alpha,q)$$
$[a>b>c>d>u]$ BY (251.08)

32. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}}\,dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\beta,r) - 2\frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\beta,r)$$
$$+2\frac{a-c}{(b-c)(c-d)}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a>b>c>u>d]$ BY (252.04)

33. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}}\,dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}}\,F(\kappa,q) - 2\sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}}\,E(\kappa,q)$$
$$+\frac{2(a-c)}{(b-c)(c-d)}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a>b>u>c>d]$ BY (255.04)

34. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\lambda,r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\lambda,r)$$
$[a\geq u>b>c>d]$ BY (256.09)

35. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\mu,r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\mu,r)$$
$$-\frac{2}{b-c}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a>u\geq b>c>d]$ BY (257.04)

36. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\nu,q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\nu,q) - \frac{2}{c-d}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u > a > b > c > d]$ BY (258.04)

37. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\alpha,q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\alpha,q) - \frac{2}{a-b}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.11)

38. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\beta,r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\beta,r) + \frac{2}{b-c}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c \geq u > d]$ BY (252.07)

39. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\gamma,r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\gamma,r)$$
$[a > b > c > u \geq d]$ BY (253.07)

40. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}}\,dx = \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\delta,q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\delta,q) + \frac{2(b-d)}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a > b > u > c > d]$ BY (254.05)

41. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}}\,dx = \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\mu,r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\mu,r) + \frac{2(b-d)}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a > u > b > c > d]$ BY (257.07)

42. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\nu,q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\nu,q)$$
$[u > a > b > c > d]$ BY (258.07)

43. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\alpha,q) - \frac{2(b-c)}{(a-b)(b-d)}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a > b > c > d > u]$

44. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\beta,r)-E(\beta,r)] + \frac{2}{b-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a>b>c\geq u>d]$ BY (252.10)

45. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\gamma,r)-E(\gamma,r)]$$
$[a>b>c>u\geq d]$ BY (254.08)

46. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{b-a}\sqrt{\frac{a-c}{b-d}}\,E(\delta,q) + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a>b\geq u>c>d]$ BY (254.08)

47. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\mu,r)-E\,(\mu,r)] + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a>u\geq b>c>d]$ BY (257.10)

48. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\nu,q)$$
$[u>a>b>c>d]$ BY (258.01)

49. $$\int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\alpha,q)-E\,(\alpha,q)] + \frac{2}{b-d}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a>b>c>d>u]$ BY (251.12)

50. $$\int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\beta,r) - \frac{2(a-b)}{(b-c)(b-d)}\sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}}$$
$[a>b>c\geq u>d]$ BY (252.09)

51. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\gamma,r)$$
$[a>b>c>u\geq d]$ BY (253.01)

52. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\delta,q)-E(\delta,q)] + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a>b>u>c>d]$ BY (254.06)

53. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{c-b}\sqrt{\frac{a-c}{b-d}}\,E(\mu,r) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a>u>b>c>d]$ BY (257.08)

54. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\nu,q)-E(\nu,q)]$$
$[u>a>b>c>d]$ BY (258.08)

55. $$\int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{b-a}\sqrt{\frac{b-d}{a-c}}\,E(\alpha,q) + \frac{2}{a-b}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a>b>c>d>u]$ BY (251.09)

56. $$\int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\beta,q)-E(\beta,q)]$$
$[a>b>c\geq u>d]$ BY (252.05)

57. $$\int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\gamma,r)-E(\gamma,r)] + \frac{2}{a-c}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a>b>c>u\geq d]$ BY (253.05)

58. $$\int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\delta,q) - \frac{2(a-d)}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a>b\geq u>c>d]$ BY (254.03)

59. $$\int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\kappa,q)$$
$[a>b>u\geq c>d]$ BY (255.01)

60. $$\int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\lambda,r)-E(\lambda,r)] + \frac{2}{a-b}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a>u>b>c>d]$ BY (256.10)

61. $$\int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}}\,dx = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\alpha,q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\alpha,q) + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a>b>c>d>u]$ BY (251.15)

62. $$\int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\beta,r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\beta,r)$$
$[a>b>c\geq u>d]$ BY (252.08)

63. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\gamma,r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\gamma,r) - \frac{2}{a-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a>b>c>u\geq d]$ BY (253.10)

64. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\delta,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\delta,q) - \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a>b\geq u>c>d]$ BY (254.09)

65. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\kappa,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\kappa,q)$$
$[a>b>u\geq c>d]$ BY (255.10)

66. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\lambda,r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\lambda,r) + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a>u>b>c>d]$ BY (256.07)

67. $$\int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\alpha,q)-E(\alpha,q)] + \frac{2}{a-d}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a>b>c>d>u]$ BY (251.13)

68. $$\int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\beta,r)$$
$[a>b>c\geq u>d]$ BY (252.01)

69. $$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\gamma,r) - \frac{2(a-b)}{(a-c)(a-d)}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a>b>c>u\geq d]$ BY (253.08)

70. $$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\delta,q)-E(\delta,q)] + \frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a>b\geq u>c>d]$ BY (254.07)

71. $$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\kappa,q)-E(\kappa,q)]$$
$[a>b>u\geq c>d]$ BY (255.07)

72. $$\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{-2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\lambda,r) + \frac{2}{a-d}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a\geq u>b>c>d]$ BY (256.04)

3.169 **Notation**: In **3.169**–**3.172**, we set: $\alpha = \arctan\frac{u}{b}$, $\beta = \arctan\frac{a}{u}$,

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}},\qquad \delta = \arccos\frac{u}{b},\qquad \varepsilon = \arccos\frac{b}{u},\qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
$$\eta = \arcsin\frac{u}{b},\qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}},\qquad \kappa = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}},\qquad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}},\qquad \nu = \arcsin\frac{a}{u},\qquad q = \frac{\sqrt{a^2-b^2}}{a},$$
$$r = \frac{b}{\sqrt{a^2+b^2}},\qquad s = \frac{a}{\sqrt{a^2+b^2}}\qquad t = \frac{b}{a}.$$

1. $$\int_0^u \sqrt{\frac{x^2+a^2}{x^2+b^2}}\,dx = a\left\{F(\alpha,q)-E(\alpha,q)\right\}+u\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a>b, \quad u>0]$ BY (221.03)

2.[6] $$\int_0^u \sqrt{\frac{x^2+b^2}{x^2+a^2}}\,dx = \frac{b^2}{a}\,F(\alpha,q)-a\,E(\alpha,q)+u\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a>b, \quad u>0]$ BY (221.04)

3. $$\int_0^u \sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\gamma,r)-u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b\geq u>0]$ BY (214.11)

4. $$\int_u^b \sqrt{\frac{a^2+x^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\delta,r)$$
$[b>u\geq 0]$ BY (213.01), ZH 64 (273)

5. $$\int_b^u \sqrt{\frac{a^2+x^2}{x^2-b^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\varepsilon,s)-E(\varepsilon,s)\right\}+\frac{1}{u}\sqrt{(u^2+a^2)(u^2-b^2)}$$
$[u>b>0]$ BY (211.03)

6. $$\int_0^u \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\gamma,r)-E(\gamma,r)\right\}+u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b\geq u>0]$ BY (214.03)

7. $$\int_u^b \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\delta,r)-E(\delta,r)\right\}$$
$[b>u\geq 0]$ BY (213.03)

8. $$\int_b^u \sqrt{\frac{x^2-b^2}{a^2+x^2}}\,dx = \frac{1}{u}\sqrt{(a^2+u^2)(u^2-b^2)}-\sqrt{a^2+b^2}\,E(\varepsilon,s)$$
$[u>b>0]$ BY (211.04)

9. $$\int_0^u \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = a\,E(\eta,t)-\frac{a^2-b^2}{a}\,F(\eta,t)$$
$[a>b\geq u>0]$ BY (219.03)

10. $$\int_u^b \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = a\,E(\zeta,t)-\frac{a^2-b^2}{a}\,F(\zeta,t)-u\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a>b>u\geq 0]$ BY (220.04)

11. $$\int_b^u \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = a\,E(\kappa,q)-\frac{b^2}{a}\,F(\kappa,q)-\frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a\geq u>b>0]$ BY (217.04)

12. $$\int_u^a \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = a\,E(\lambda,q)-\frac{b^2}{a}\,F(\lambda,q)$$
$[a>u\geq b>0]$ BY (218.03)

13. $$\int_a^u \sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{a^2-b^2}{a}\,F(\mu,t)-a\,E(\mu,t)+\mu\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u>a>b>0]$ BY (216.03)

14. $$\int_0^u \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\,E(\eta,t)$$
$[a>b\geq u>0]$ H 64 (276), BY (219.01)

15. $$\int_u^b \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\left\{E(\zeta,t) - \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} \qquad [a>b>u\geq 0]$$ BY (220.03)

16. $$\int_b^u \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\kappa,q) - E(\kappa,q)\right\} + \frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$$[a\geq u>b>0]$$ BY (217.03)

17. $$\int_u^a \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\lambda,q) - E(\lambda,q)\right\} \qquad [a>u\geq b>0]$$ BY (218.09)

18. $$\int_a^u \sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = u\sqrt{\frac{u^2-a^2}{u^2-b^2}} - a\,E(\mu,t) \qquad [u>a>b>0]$$ BY (216.04)

3.171

1. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\,E(\varepsilon,s) \qquad [u>b>0]$$ BY (211.01), ZH 64 (274)

2. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\,E(\xi,s) - \frac{a^2}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$
$$[u\geq b>0]$$ BY (212.09)

3. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}} = \frac{a^2-b^2}{ab^2}\,F(\zeta,t) - \frac{a}{b^2}\,E(\zeta,t) + \frac{a^2}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$$[a>b>u>0]$$ BY (220.12)

4. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\kappa,q) - \frac{1}{a}\,F(\kappa,q) \qquad [a\geq u>b>0]$$ BY (217.11)

5. $$\int_u^a \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\lambda,q) - \frac{1}{a}f(\lambda,q) - \frac{\sqrt{(a^2-u^2)(u^2-b^2)}}{b^2u}$$
$$[a>u\geq b>0]$$ BY (218.10)

6. $$\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\mu,t) - \frac{a^2-b^2}{ab^2}\,F(\mu,t) - \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$$[u>a>b>0]$$ BY (216.08)

7. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+a^2}{x^2+b^2}} = \frac{1}{a}\,F(\beta,q) - \frac{a}{b^2}\,E(\beta,q) + \frac{a^2}{b^2u}\sqrt{\frac{b^2+u^2}{a^2+u^2}}$$
$$[a>b,\quad u>0]$$ BY (222.08)

8. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+b^2}{x^2+a^2}} = \frac{1}{a}\left\{F(\beta,q) - E(\beta,q)\right\} + \frac{1}{u}\sqrt{\frac{b^2+u^2}{a^2+u^2}}$$
$$[a>b,\quad u>0]$$ BY (222.09)

9. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2+x^2}} = \frac{\sqrt{(b^2-u^2)(a^2+u^2)}}{a^2u} - \frac{\sqrt{a^2+b^2}}{a^2}\,E(\delta,r)$$
$$[b>u>0]$$ BY (213.10)

10. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\left\{F(\varepsilon,s)-E(\varepsilon,s)\right\} \qquad [a>b>0]$$ BY (211.07)

11. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\left\{F(\xi,s)-E(\xi,s)\right\} + \frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$
$$[u\geq b>0]$$ BY (212.11)

12. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{b^2-x^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\left\{F(\delta,r)-E(\delta,r)\right\} + \frac{\sqrt{(b^2-u^2)(a^2+u^2)}}{b^2u}$$
$$[b>u>0]$$ BY (213.05)

13. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}E(\nu,t) - \frac{a^2-b^2}{ab^2}F(\nu,t) \qquad [u\geq a>b>0]$$ BY (215.08)

14. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2-x^2}} = \frac{1}{u}\sqrt{\frac{b^2-u^2}{a^2-u^2}} - \frac{1}{a}E(\zeta,t) \qquad [a>b>u>0]$$ BY (220.11)

15. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2-x^2}} = \frac{1}{a}\left\{F(\kappa,q)-E(\kappa,q)\right\} \qquad [a\geq u>b>0]$$ BY (217.08)

16. $$\int_u^a \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{u^2-x^2}} = \frac{1}{a}\left\{F(\lambda,q)-E(\lambda,q)\right\} + \frac{\sqrt{(a^2-u^2)(u^2-b^2)}}{a^2u}$$
$$[a>u\geq b>0]$$ BY (218.08)

17. $$\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\mu,t) - \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}} \qquad [u>a>b>0]$$ BY (216.07)

18. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\nu,t) \qquad [u\geq a>b>0]$$
BY (215.01), ZH 65 (281)

3.172

1. $$\int_0^u \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\alpha,q) - \frac{a^2-b^2}{a^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$
$$[a>b,\quad u>0]$$ BY (221.10)

2. $$\int_u^\infty \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\beta,q) \qquad [a>b,\quad u\geq 0]$$ H 64 (271)

3. $$\int_0^u \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\alpha,q) \qquad [a>b,\quad u>0]$$ H 64 (270)

4. $$\int_u^\infty \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\beta,q) - \frac{a^2-b^2}{b^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$
$$[a>b,\quad u\geq 0]$$ BY (222.06)

5. $$\int_0^u \sqrt{\frac{b^2-x^2}{(a^2+x^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\gamma,r) - \frac{1}{\sqrt{a^2+b^2}}F(\gamma,r)$$
$$[b\geq u>0]$$ BY (214.08)

6. $$\int_u^b \sqrt{\frac{b^2-x^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\delta,r) - \frac{1}{\sqrt{a^2+b^2}}\,F(\delta,r) - \frac{u}{a^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b > u \geq 0]$ BY (213.04)

7. $$\int_b^u \sqrt{\frac{x^2-b^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\varepsilon,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}\,F(\varepsilon,s) - \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$
$[u > b > 0]$ BY (211.06)

8. $$\int_u^\infty \sqrt{\frac{x^2-b^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\xi,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}\,F(\xi,s)$$
$[u \geq b > 0]$ BY (212.08)

9. $$\int_0^u \sqrt{\frac{x^2+a^2}{\left(b^2-x^2\right)^3}}\,dx = \frac{a^2}{b^2\sqrt{a^2+b^2}}\,F(\gamma,r) - \frac{\sqrt{a^2+b^2}}{b^2}\,E(\gamma,r) + \frac{\left(a^2+b^2\right)u}{b^2\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}}$$
$[b > u > 0]$ BY (214.09)

10. $$\int_u^\infty \sqrt{\frac{x^2+a^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{1}{\sqrt{a^2+b^2}}\,F(\xi,s) - \frac{\sqrt{a^2+b^2}}{b^2}\,E(\xi,s) + \frac{\left(a^2+b^2\right)u}{b^2\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)}}$$
$[u > b > 0]$ BY (212.07)

11. $$\int_0^u \sqrt{\frac{b^2-x^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{a}\left\{F(\eta,t) - E(\eta,t) + \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a > b \geq u > 0]$ BY (219.09)

12. $$\int_u^b \sqrt{\frac{b^2-x^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{a}\left\{F(\zeta,t) - E(\zeta,t)\right\}$$
$[a > b > u \geq 0]$ BY (220.07)

13. $$\int_b^u \sqrt{\frac{x^2-b^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}} - \frac{1}{a}\,E(\kappa,q)$$
$[a > u > b > 0]$ BY (217.07)

14. $$\int_u^\infty \sqrt{\frac{x^2-b^2}{\left(x^2-a^2\right)^3}}\,dx = \frac{1}{a}\left[F(\nu,t) - E(\nu,t)\right] + \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}}$$
$[u > a > b > 0]$ BY (215.05)

15. $$\int_0^u \sqrt{\frac{a^2-x^2}{\left(b^2-x^2\right)^3}}\,dx = \frac{a}{b^2}\left[F(\eta,t) - E(\eta,t)\right] + \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{b^2-u^2}}$$
$[a > b > u > 0]$ BY (219.10)

16. $$\int_u^a \sqrt{\frac{a^2-x^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{u^2-b^2}} - \frac{a}{b^2}\,E(\lambda,q)$$
$[a > u > b > 0]$ BY (218.05)

17. $$\int_a^u \sqrt{\frac{x^2-a^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{a}{b^2}\left[F(\mu,t) - E(\mu,t)\right]$$
$[u > a > b > 0]$ BY (216.05)

18. $$\int_u^\infty \sqrt{\frac{x^2-a^2}{(x^2-b^2)^3}}\,dx = \frac{a}{b^2}\left[F(\nu,t)-E(\nu,t)\right]+\frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u \ge a > b > 0]$ BY (215.03)

3.173

1. $$\int_u^1 \frac{dx}{x^2}\sqrt{\frac{x^2+1}{1-x^2}} = \sqrt{2}\left[F\left(\arccos u, \frac{\sqrt{2}}{2}\right) - E\left(\arccos u, \frac{\sqrt{2}}{2}\right)\right] + \frac{\sqrt{1-u^4}}{u}$$
$[u < 1]$ BY (259.77)

2. $$\int_1^u \frac{dx}{x^2}\sqrt{\frac{x^2+1}{x^2-1}} = \sqrt{2}\,E\left(\arccos\frac{1}{u}, \frac{\sqrt{2}}{2}\right)$$
$[u > 1]$ BY (260.76)

3.174 **Notation**: In **3.174** and **3.175**, we take: $\alpha = \arccos\dfrac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u}$,

$$\beta = \arccos\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u}, \qquad p = \frac{\sqrt{2+\sqrt{3}}}{2}, \qquad q = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

1. $$\int_0^u \frac{dx}{\left[1+\left(1+\sqrt{3}\right)x\right]^2}\sqrt{\frac{1-x+x^2}{x(1+x)}} = \frac{1}{\sqrt[4]{3}}\,E(\alpha,p)$$
$[u > 0]$ BY (260.51)

2. $$\int_0^u \frac{dx}{\left[1+\left(\sqrt{3}-1\right)x\right]^2}\sqrt{\frac{1+x+x^2}{x(1-x)}} = \frac{1}{\sqrt[4]{3}}\,E(\beta,q)$$
$[1 \ge u > 0]$ BY (259.51)

3. $$\int_0^u \frac{dx}{1-x+x^2}\sqrt{\frac{x(1+x)}{1-x+x^2}} \quad \frac{1}{\sqrt[4]{27}}\,E(\alpha,p) + -\frac{2-\sqrt{3}}{\sqrt[4]{27}}\,F(\alpha,p) - \frac{2\left(2+\sqrt{3}\right)}{\sqrt{3}}\,\frac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u} \times\sqrt{\frac{u(1+u)}{1-u+u^2}}$$
$[u > 0]$ BY (260.54)

4. $$\int_0^u \frac{dx}{1+x+x^2}\sqrt{\frac{x(1-x)}{1+x+x^2}} \quad \frac{4}{\sqrt[4]{27}}\,E(\beta,q) - \frac{2+\sqrt{3}}{\sqrt[4]{27}}\,F(\beta,q) - \frac{2\left(2-\sqrt{3}\right)}{\sqrt{3}}\,\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u} \times\sqrt{\frac{u(1-u)}{1+u+u^2}}$$
$[1 \ge u > 0]$ BY (259.55)

3.175

1. $$\int_0^u \frac{dx}{1+x}\sqrt{\frac{x}{1+x^3}} = \frac{1}{\sqrt[4]{27}}\left[F(\alpha,p)-2\,E(\alpha,p)\right] + \frac{2}{\sqrt{3}}\,\frac{\sqrt{u\left(1-u+u^2\right)}}{\sqrt{1+u}\left[1+\left(1+\sqrt{3}\right)u\right]}$$
$[u > 0]$ BY (260.55)

2. $$\int_0^u \frac{dx}{1-x}\sqrt{\frac{x}{1-x^3}} = \frac{1}{\sqrt[4]{27}}\left[F(\beta,q)-2\,E(\beta,q)\right] + \frac{2}{\sqrt{3}}\,\frac{\sqrt{u\left(1+u+u^2\right)}}{\sqrt{1-u}\left[1+\left(\sqrt{3}-1\right)u\right]}$$
$[0 < u < 1]$ BY (259.52)

3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

3.181

1. $$\int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b}\left\{2\left[\boldsymbol{E}\left(\frac{1}{\sqrt{2}}\right) + E\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right] - \left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]\right\}$$
$[a \geq u > b]$ BY (271.05)

2. $$\int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} \sqrt{\frac{a-b}{2}} F\left[\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right)\right] + \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}$$
$[u > a > b]$ BY (272.05)

3.182

1. $$\int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}}\left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]$$
$[a \geq u > b]$ BY (271.01)

2. $$\int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}}\, F\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right)$$
$[u > a > b]$ BY (272.00)

3.183 **Notation**: In **3.183**–**3.186** we set:

$$\alpha = \arccos\frac{1}{\sqrt[4]{u^2+1}}, \qquad \beta = \arccos\sqrt[4]{1-u^2}, \qquad \gamma = \arccos\frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

1. $$\int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}}$$
$[u > 0]$ BY (273.55)

2. $$\int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2}\left[2\,E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right)\right]$$
$[0 < u \leq 1]$ BY (271.55)

3. $$\int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\gamma, \frac{1}{\sqrt{2}}\right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}}$$
$[u > 1]$ BY (272.55)

3.184

1. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5}\left[2\,E\left(\beta,\frac{1}{\sqrt{2}}\right) - F\left(\beta,\frac{1}{\sqrt{2}}\right)\right] - \frac{2u}{5}\sqrt[4]{(1-u^2)^3}$$
$[0 < u \le 1]$ BY (271.59)

2. $$\int_1^u \frac{dx}{x^2\sqrt[4]{x^2-1}} = E\left(\gamma,\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\,F\left(\gamma,\frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}\cdot\frac{\sqrt{u^2-1}}{u}$$
$[u > 1]$ BY (272.54)

3.185

1. $$\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2}\,F\left(\alpha,\frac{1}{\sqrt{2}}\right)$$
$[u > 0]$ BY (273.50)

2. $$\int_0^u \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2}\,F\left(\beta,\frac{1}{\sqrt{2}}\right)$$
$[0 < u \le 1]$ BY (271.51)

3. $$\int_1^u \frac{dx}{\sqrt[4]{(x^2-1)^3}} = F\left(\gamma,\frac{1}{\sqrt{2}}\right)$$
$[u > 1]$ BY (272.50)

4. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3}\,F\left(\beta,\frac{1}{\sqrt{2}}\right) - \frac{2}{3}u\sqrt[4]{1-u^2}$$
$[0 < u \le 1]$ BY (271.54)

5. $$\int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2}\,E\left(\alpha,\frac{1}{\sqrt{2}}\right) - \sqrt{2}\,F\left(\alpha,\frac{1}{\sqrt{2}}\right)$$
$[u > 0]$ BY (273.54)

6. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2}\left[F\left(\alpha,\frac{1}{\sqrt{2}}\right) - 2\,E\left(\alpha,\frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}}$$
$[u > 0]$ BY (273.56)

7. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{(x^2+1)^7}} = \frac{1}{3\sqrt{2}}\,F\left(\alpha,\frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{(u^2+1)^3}}$$
$[u > 0]$ BY (273.53)

3.186

1. $$\int_0^u \frac{1+\sqrt{x^2+1}}{(x^2+1)\sqrt[4]{x^2+1}}\,dx = 2\sqrt{2}\,E\left(\alpha,\frac{1}{\sqrt{2}}\right)$$
$[u > 0]$ BY (273.51)

2. $$\int_0^u \frac{dx}{\left(1+\sqrt{1-x^2}\right)\sqrt[4]{1-x^2}} = \sqrt{2}\left[F\left(\beta,\frac{1}{\sqrt{2}}\right) - E\left(\beta,\frac{1}{\sqrt{2}}\right)\right] + \frac{u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}}$$
$[0 < u \le 1]$ BY (271.58)

3. $$\int_1^u \frac{dx}{\left(x^2+2\sqrt{x^2-1}\right)\sqrt[4]{x^2-1}} = \frac{1}{2}\left[F\left(\gamma,\frac{1}{\sqrt{2}}\right) - E\left(\gamma,\frac{1}{\sqrt{2}}\right)\right]$$
$[u > 1]$ BY (272.53)

4. $$\int_0^u \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2}\left[2\,E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right)\right] - \frac{2u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}}$$
$[0 < u \le 1]$ BY (271.57)

5. $$\int_1^u \frac{x^2\,dx}{\left(x^2+2\sqrt{x^2-1}\right)\sqrt[4]{(x^2-1)^3}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right)$$
$[u > 1]$ BY (272.51)

3.19–3.23 Combinations of powers of x and powers of binomials of the form $(\alpha+\beta x)$

3.191

1. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\,dx = u^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 185(7)

2. $$\int_u^\infty x^{-\nu}(x-u)^{\mu-1}\,dx = u^{\mu-\nu}\,\mathrm{B}(\nu-\mu,\mu)$$
$[\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ ET II 201(6)

3. $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\,dx = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 774(1)

3.192

1. $$\int_0^1 \frac{x^p\,dx}{(1-x)^p} = p\pi \operatorname{cosec} p\pi$$
$[p^2 < 1]$ BI (3)(4)

2. $$\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\pi \operatorname{cosec} p\pi$$
$[-1 < p < 0]$ BI (3)(5)

3. $$\int_0^1 \frac{(1-x)^p}{x^{p+1}}\,dx = -\pi \operatorname{cosec} p\pi$$
$[-1 < p < 0]$ BI (4)(6)

4. $$\int_1^\infty (x-1)^{p-\frac{1}{2}}\,\frac{dx}{x} = \pi \sec p\pi$$
$\left[-\frac{1}{2} < p < \frac{1}{2}\right]$ BI (23)(7)

3.193 $$\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n!\,n^{\nu+n}}{\nu(\nu+1)(\nu+2)\ldots(\nu+n)}$$
$[\operatorname{Re}\nu > 0]$ EH I 2

3.194

1. $$\int_0^u \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu}\,{}_2F_1(\nu,\mu;1+\mu;-\beta u)$$
$[|\arg(1+\beta u)| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 310(20)

2.[6] $$\int_u^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu(\nu-\mu)}\,{}_2F_1\left(\nu,\nu-\mu;\quad \nu-\mu+1;\quad -\frac{1}{\beta u}\right)$$
$[\operatorname{Re}\nu > \operatorname{Re}\mu]$ ET I 310(21)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \beta^{-\mu}\,\mathrm{B}(\mu,\nu-\mu)$$
$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ FI II 775a, ET I 310(19)

4.[11] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu}\binom{\mu-1}{n}\operatorname{cosec}(\mu\pi) \qquad [|\arg\beta|<\pi, \quad 0<\operatorname{Re}\mu<n+1]$$

ET I 308(6)

5. $$\int_0^u \frac{x^{\mu-1}\,dx}{1+\beta x} = \frac{u^\mu}{\mu}\,{}_2F_1(1,\mu;1+\mu;-\beta u) \qquad [|\arg(1+u\beta)|<\pi, \quad \operatorname{Re}\mu>0]$$

ET I 308(5)

6. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^2} = \frac{(1-\mu)\pi}{\beta^\mu}\operatorname{cosec}\mu\pi \qquad [0<\operatorname{Re}\mu<2]$$

BI (16)(4)

7. $$\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+\frac12}} = 2^{m+1}m!\frac{(2n-2m-3)!!}{(2n-1)!!}\frac{a^{m-n+\frac12}}{b^{m+1}}$$

$$\left[m<n-\tfrac12, \quad a>0, \quad b>0\right]$$

BI (21)(2)

8. $$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^m} = 2^{-n}\sum_{k=0}^\infty \binom{m-n-1}{k}\frac{(-2)^{-k}}{n+k}$$

BI (3)(1)

3.195[11] $$\int_0^\infty \frac{(1+x)^{p-1}}{(a+x)^{p+1}}\,dx = \frac{1-a^{-p}}{p(a-1)} \qquad [p\neq 0, \quad a>0, \quad a\neq 1]$$

$$= \frac{\ln a}{a-1} \qquad [p=0, \quad a>0, \quad a\neq 1]$$

$$= 1 \qquad [a=1]$$

LI (19)(6)

3.196

1. $$\int_0^u (x+\beta)^\nu (u-x)^{\mu-1}\,dx = \frac{\beta^\nu u^\mu}{\mu}\,{}_2F_1\left(1,-\nu;1+\mu;-\frac{u}{\beta}\right)$$

$$\left[\left|\arg\frac{u}{\beta}\right|<\pi\right]$$

ET II 185(8)

2. $$\int_u^\infty (x+\beta)^{-\nu}(x-u)^{\mu-1}\,dx = (u+\beta)^{\mu-\nu}\,\mathrm{B}(\nu-\mu,\mu)$$

$$\left[\left|\arg\frac{u}{\beta}\right|<\pi, \quad \operatorname{Re}\nu>\operatorname{Re}\mu>0\right]$$

ET II 201(7)

3. $$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\,dx = (b-a)^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu) \qquad [b>a, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$

EH I 10(13)

4. $$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b}\operatorname{cosec}\nu\pi\left(\frac{b}{b-a}\right)^\nu \qquad [a<b, \quad b>0, \quad 0<\nu<1]$$

LI (23)(5)

5. $$\int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b}\operatorname{cosec}\nu\pi\left(\frac{b}{a-b}\right)^\nu \qquad [a>b>0, \quad 0<\nu<1]$$

LI (24)(10)

3.197

1. $$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho}\,dx=\beta^{-\mu}\gamma^{\nu-\varrho}\,\mathrm{B}(\nu,\mu-\nu+\varrho)\;{}_2F_1\left(\mu,\nu;\mu+\varrho;1-\frac{\gamma}{\beta}\right)$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\mu>\operatorname{Re}(\nu-\varrho)]$ ET II 233(9)

2.[11] $$\int_u^\infty x^{-\lambda}(x+\beta)^{\nu}(x-u)^{\mu-1}\,dx=u^{-\lambda}(\beta+u)^{\mu+\nu}\,\mathrm{B}(\lambda-\mu-\nu,\mu)\;{}_2F_1\left(\lambda,\mu;\lambda-\mu;-\frac{\beta}{u}\right)$$
$\left[\left|\arg\frac{u}{\beta}\right|<\pi \text{ or } \left|\frac{\beta}{u}\right|<1,\quad 0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-\nu)\right]$ ET II 201(8)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}\,dx=\mathrm{B}(\lambda,\mu)\;{}_2F_1(\nu,\lambda;\lambda+\mu;\beta)$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\mu>0,\quad |\beta|<1]$ WH

4. $$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu}\,dx=(1+a)^{-\mu}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad a>-1]$
BI(5)4, EH I 10(11)

5. $$\int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+\alpha x)^{\mu}\,dx=\mathrm{B}(\lambda,-\mu-\nu-\lambda)\;{}_2F_1(-\mu,\lambda;-\mu-\nu;1-\alpha)$$
$[|\arg\alpha|<\pi,\quad -\operatorname{Re}(\mu+\nu)>\operatorname{Re}\lambda>0]$
EH I 60(12), ET I 310(23)

6. $$\int_1^\infty x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda}\,dx=\alpha^{-\lambda}\,\mathrm{B}(\mu,\nu-\mu)\;{}_2F_1\left(\nu,\mu;\lambda;\alpha^{-1}\right)$$
$[1+\operatorname{Re}\nu>\operatorname{Re}\lambda>\operatorname{Re}\mu,\quad |\arg(\alpha-1)|<\pi]$ EH I 115(6)

7. $$\int_0^\infty x^{\mu-\frac12}(x+a)^{-\mu}(x+b)^{-\mu}\,dx=\sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2\mu}\frac{\Gamma\left(\mu-\frac12\right)}{\Gamma(\mu)}$$
$[\operatorname{Re}\mu>0]$ BI 19(5)

8. $$\int_0^u x^{\nu-1}(x+\alpha)^{\lambda}(u-x)^{\mu-1}\,dx=\alpha^{\lambda}u^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)\;{}_2F_1\left(-\lambda,\nu;\mu+\nu;-\frac{u}{\alpha}\right)$$
$\left[\left|\arg\left(\frac{u}{\alpha}\right)\right|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0\right]$
ET II 186(9)

9. $$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu}\,dx=\mathrm{B}(\mu-\lambda,\lambda)\;{}_2F_1(\nu,\mu-\lambda;\mu;1-\beta)$$
$[\operatorname{Re}\mu>\operatorname{Re}\lambda>0]$ EH I 205

10. $$\int_0^1\frac{x^{q-1}\,dx}{(1-x)^q(1+px)}=\frac{\pi}{(1+p)^q}\operatorname{cosec}q\pi$$
$[0<q<1,\quad p>-1]$ BI (5)(1)

11. $$\int_0^1\frac{x^{p-\frac12}\,dx}{(1-x)^p(1+qx)^p}=\frac{2\,\Gamma\left(p+\frac12\right)\Gamma(1-p)}{\sqrt{\pi}}\cos^{2p}\left(\arctan\sqrt{q}\right)\frac{\sin\left[(2p-1)\arctan\left(\sqrt{q}\right)\right]}{(2p-1)\sin\left[\arctan\left(\sqrt{q}\right)\right]}$$
$\left[-\frac12<p<1,\quad q>0\right]$ BI (11)(1)

12. $$\int_0^1 \frac{x^{p-\frac{1}{2}}\,dx}{(1-x)^p(1-qx)^p} = \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(1-p)}{\sqrt{\pi}}\,\frac{\left(1-\sqrt{q}\right)^{1-2p}-\left(1+\sqrt{q}\right)^{1-2p}}{(2p-1)\sqrt{q}}$$
$\left[-\frac{1}{2}<p<1, \quad 0<q<1\right]$ BI (11)(2)

3.198 $$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}[ax+b(1-x)+c]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}\,\mathrm{B}(\mu,\nu)$$
$[a\geq 0, \quad b\geq 0, \quad c>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ FI II 787

3.199 $$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\,dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad c<a<b]$
EH I 10(14)

3.211 $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\varrho}(1-vx)^{-\sigma}\,dx = \mathrm{B}(\mu,\lambda)\,F_1\left((\lambda,\varrho,\sigma,\lambda+\mu;u,v)\right)$$
$[\operatorname{Re}\lambda>0, \quad \operatorname{Re}\mu>0]$ EH I 231(5)

3.212 $$\int_0^\infty \left[(1+ax)^{-p}+(1+bx)^{-p}\right]x^{q-1}\,dx = 2(ab)^{-\frac{q}{2}}\,\mathrm{B}(q,p-q)\cos\left\{q\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$
$[p>q>0]$ BI (19)(9)

3.213 $$\int_0^\infty \left[(1+ax)^{-p}-(1+bx)^{-p}\right]x^{q-1}\,dx = -2i(ab)^{-\frac{q}{2}}\,\mathrm{B}(q,p-q)\sin\left\{q\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$
$[p>q>0]$ BI (19)(10)

3.214 $$\int_0^1 \left[(1+x)^{\mu-1}(1-x)^{\nu-1}+(1+x)^{\nu-1}(1-x)^{\mu-1}\right]\,dx = 2^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$
LI(1)(15), EH I 10(10)

3.215 $$\int_0^1 \left\{a^\mu x^{\mu-1}(1-ax)^{\nu-1}+(1-a)^\nu x^{\nu-1}[1-(1-a)x]^{\mu-1}\right\}\,dx = \mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad |a|<1]$
BI (1)(16)

3.216

1. $$\int_0^1 \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ FI II 775

2. $$\int_1^\infty \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ FI II 775

3.217 $$\int_0^\infty \left\{\frac{b^p x^{p-1}}{(1+bx)^p}-\frac{(1+bx)^{p-1}}{b^{p-1}x^p}\right\}\,dx = \pi\cot p\pi$$ $[0<p<1, \quad b>0]$ BI(18)(13)

3.218 $$\int_0^\infty \frac{x^{2p-1}-(a+x)^{2p-1}}{(a+x)^p x^p}\,dx = \pi\cot p\pi$$ $[p<1]$ (cf. **3.217**) BI (18)(7)

3.219 $$\int_0^\infty \left\{\frac{x^\nu}{(x+1)^{\nu+1}}-\frac{x^\mu}{(x+1)^{\mu+1}}\right\}\,dx = \psi(\mu+1)-\psi(\nu+1)$$
$[\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1]$ BI (19)(13)

3.221

1. $$\int_a^\infty \frac{(x-a)^{p-1}}{x-b}\,dx = \pi(a-b)^{p-1}\operatorname{cosec} p\pi$$ $[a>b, \quad 0<p<1]$ LI (24)(8)

2. $$\int_{-\infty}^{a} \frac{(a-x)^{p-1}}{x-b}\,dx = -\pi(b-a)^{p-1}\operatorname{cosec} p\pi \qquad [a<b, \quad 0<p<1]$$ LI (24)(8)

3.222

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x} = \beta(\mu) \qquad [\operatorname{Re}\mu>0]$$ WH

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = \pi\operatorname{cosec}(\mu\pi)a^{\mu-1} \qquad \text{for } a>0$$ FI II 718, FI II 737

$$= -\pi\cot(\mu\pi)(-a)^{\mu-1} \qquad \text{for } a<0$$ BI(18)(2), ET II 249(28)

$$[0<\operatorname{Re}\mu<1]$$

3.223

1. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta}\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\operatorname{cosec}(\mu\pi)$$
$$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(7)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta}\left[\beta^{\mu-1}\operatorname{cosec}(\mu\pi)+\alpha^{\mu-1}\cot(\mu\pi)\right]$$
$$[|\arg\beta|<\pi, \quad \alpha>0, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(8)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)} = \pi\cot(\mu\pi)\frac{a^{\mu-1}-b^{\mu-1}}{b-a} \qquad [a>b>0, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(9)

3.224 $$\int_0^\infty \frac{(x+\beta)x^{\mu-1}\,dx}{(x+\gamma)(x+\delta)} = \pi\operatorname{cosec}(\mu\pi)\left\{\frac{\gamma-\beta}{\gamma-\delta}\gamma^{\mu-1}+\frac{\delta-\beta}{\delta-\gamma}\delta^{\mu-1}\right\}$$
$$[|\arg\gamma|<\pi, \quad |\arg\delta|<\pi, \quad 0<\operatorname{Re}\mu<1]$$ ET I 309(10)

3.225

1. $$\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx = (1-p)\pi\operatorname{cosec} p\pi \qquad [-1<p<1]$$ BI (23)(8)

2. $$\int_1^\infty \frac{(x-1)^{1-p}}{x^3}\,dx = \frac{1}{2}p(1-p)\pi\operatorname{cosec} p\pi \qquad [0<p<1]$$ BI (23)(1)

3. $$\int_0^\infty \frac{x^p\,dx}{(1+x)^3} = \frac{\pi}{2}p(1-p)\operatorname{cosec} p\pi \qquad [-1<p<2]$$ BI (16)(5)

3.226

1. $$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = 2\frac{(2n)!!}{(2n+1)!!}$$ BI (8)(1)

2. $$\int_0^1 \frac{x^{n-\frac{1}{2}}\,dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!}\pi.$$ BI (8)(2)

3.227

1. $$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x}\,dx = \beta^{1-\mu}\gamma^{\nu-1}\,\mathrm{B}(\nu,\mu-\nu)\;{}_2F_1\left(\mu-1,\nu;\mu;1-\frac{\gamma}{\beta}\right)$$
$$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu]$$ ET II 217(9)

2. $$\int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = \pi\gamma^{-\varrho}(\beta-\gamma)^{-\sigma}\operatorname{cosec}(\varrho\pi)\,I_{1-\gamma/\beta}(\sigma,\varrho)$$
$$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad -\operatorname{Re}\sigma<\operatorname{Re}\varrho<1]$$ ET II 217(10)

3.228

1. $$\begin{aligned}\int_a^b \frac{(x-a)^{\nu}(b-x)^{-\nu}}{x-c}\,dx &= \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{a-c}{b-c}\right)^{\nu}\right] && \text{for } c<a\\ &= \pi\operatorname{cosec}(\nu\pi)\left[1-\cos(\nu\pi)\left(\frac{c-a}{b-c}\right)^{\nu}\right] && \text{for } a<c<b\\ &= \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{c-a}{c-b}\right)^{\nu}\right] && \text{for } c>b\end{aligned}$$
$$[|\operatorname{Re}\nu|<1]$$ ET II 250(31)

2. $$\begin{aligned}\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c}\,dx &= \frac{\pi\operatorname{cosec}(\nu\pi)}{b-c}\left|\frac{a-c}{b-c}\right|^{\nu-1} && \text{for } c<a \text{ or } c>b;\\ &= -\frac{\pi(c-a)^{\nu-1}}{(b-c)^{\nu}}\cot(\nu\pi) && \text{for } a<c<b\end{aligned}$$
$$[0<\operatorname{Re}\nu<1]$$ ET II 250(32)

3. $$\begin{aligned}&\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x-c}\,dx\\ &\quad= \frac{(b-a)^{\mu+\nu-1}}{b-c}\,\mathrm{B}(\mu,\nu)\;{}_2F_1\left(1,\mu;\mu+\nu;\frac{b-a}{b-c}\right) && \text{for } c<a \text{ or } c>b;\\ &\quad= \pi(c-a)^{\nu-1}(b-c)^{\mu-1}\cot\mu\pi-(b-a)^{\mu+\nu-2}\,\mathrm{B}(\mu-1,\nu)\\ &\qquad\times{}_2F_1\left(2-\mu-\nu,1;2-\mu;\frac{b-c}{b-a}\right) && \text{for } a<c<b\end{aligned}$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad \mu+\nu\neq 1,\quad \mu\neq 1,2,\ldots]$$ ET II 250(33)

4. $$\int_0^1 \frac{(1-x)^{\nu-1}x^{-\nu}}{a-bx}\,dx = \frac{\pi(a-b)^{\nu-1}}{a^{\nu}}\operatorname{cosec}(\nu\pi) \qquad [0<\operatorname{Re}\nu<1,\quad 0<b<a]$$ BI (5)(8)

5. $$\begin{aligned}\int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c}\,dx &= a^{1-\mu}(-c)^{\nu-1}\,\mathrm{B}(\mu-\nu,\nu)\;{}_2F_1\left(\mu-1,\nu;\mu;1+\frac{c}{a}\right) && \text{for } c<0;\\ &= \pi c^{\nu-1}(a+c)^{1-\mu}\cot[(\mu-\nu)\pi]-\frac{a^{1-\mu-\nu}}{a+c}\,\mathrm{B}(\mu-\nu-1,\nu)\\ &\quad\times{}_2F_1\left(2-\mu,1;2-\mu+\nu;\frac{a}{a+c}\right) && \text{for } c>0\end{aligned}$$
$$[a>0,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu]$$ ET II 251(34)

6. $$\int_0^\infty x^{\nu-1}\frac{(\gamma+x)^{-n}}{x+\beta}\,dx=\frac{\pi}{\sin\pi\nu}\frac{\beta^{\nu-1}}{(\gamma-\beta)^n}\left[1-\left(\frac{\gamma}{\beta}\right)^{\nu-1}\sum_{j=0}^{n-1}\frac{(1-\nu)_j}{j!}\left(\frac{\gamma-\beta}{\gamma}\right)^j\right]$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\nu<n]$ AS 256 (6.1.22)

3.229 $$\int_0^1\frac{x^{\mu-1}\,dx}{(1-x)^\mu(1+ax)(1+bx)}=\frac{\pi\operatorname{cosec}\mu\pi}{a-b}\left[\frac{a}{(1+a)^\mu}-\frac{b}{(1+b)^\mu}\right]$$
$[0<\operatorname{Re}\mu<1]$ BI (5)(7)

3.231

1. $$\int_0^1\frac{x^{p-1}-x^{-p}}{1-x}\,dx=\pi\cot p\pi$$ $[p^2<1]$ BI (4)(4)

2.[11] $$\int_0^1\frac{x^{p-1}+x^{-p}}{1+x}\,dx=\pi\operatorname{cosec}p\pi$$ $[p^2<1]$ BI (4)(1)

3. $$\int_0^1\frac{x^{p}-x^{-p}}{x-1}\,dx=\frac{1}{p}-\pi\cot p\pi$$ $[p^2<1]$ BI (4)(3)

4. $$\int_0^1\frac{x^{p}-x^{-p}}{1+x}\,dx=\frac{1}{p}-\pi\operatorname{cosec}p\pi$$ $[p^2<1]$ BI (4)(2)

5. $$\int_0^1\frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx=\psi(\nu)-\psi(\mu)$$ $[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$
FI II 815, BI(4)(5)

6. $$\int_0^\infty\frac{x^{p-1}-x^{q-1}}{1-x}\,dx=\pi\left(\cot p\pi-\cot q\pi\right)$$ $[p>0,\quad q>0]$ FI II 718

3.232 $$\int_0^\infty\frac{(c+ax)^{-\mu}-(c+bx)^{-\mu}}{x}\,dx=c^{-\mu}\ln\frac{b}{a}$$ $[\operatorname{Re}\mu>-1;\quad a>0;\quad b>0;\quad c>0]$
BI (18)(14)

3.233 $$\int_0^\infty\left\{\frac{1}{1+x}-(1+x)^{-\nu}\right\}\frac{dx}{x}=\psi(\nu)+\boldsymbol{C}$$ $[\operatorname{Re}\nu>0]$ EH I 17, WH

3.234

1.[11] $$\int_0^1\left(\frac{x^{q-1}}{1-ax}-\frac{x^{-q}}{a-x}\right)dx=\pi a^{-q}\cot q\pi$$ $[0<q<1,\quad a>0]$ BI (5)(11)

2. $$\int_0^1\left(\frac{x^{q-1}}{1+ax}+\frac{x^{-q}}{a+x}\right)dx=\pi a^{-q}\operatorname{cosec}q\pi$$ $[0<q<1,\quad a>0]$ BI (5)(10)

3.235 $$\int_0^\infty\frac{(1+x)^\mu-1}{(1+x)^\nu}\frac{dx}{x}=\psi(\nu)-\psi(\nu-\mu)$$ $[\operatorname{Re}\nu>\operatorname{Re}\mu>0]$ BI (18)(5)

3.236[10] $$\int_0^1\frac{x^{\frac{\mu}{2}}\,dx}{\left[(1-x)\left(1-a^2x\right)\right]^{\frac{\mu+1}{2}}}=\frac{(1-a)^{-\mu}-(1+a)^{-\mu}}{2a\mu\sqrt{\pi}}\Gamma\left(1+\frac{\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)$$
$[-2<\mu<1,\quad |a|<1]$ BI (12)(32)

3.237 $$\sum_{n=0}^\infty(-1)^{n+1}\int_n^{n+1}\frac{dx}{x+u}=\ln\frac{u\left[\Gamma\left(\frac{u}{2}\right)\right]^2}{2\left[\Gamma\left(\frac{u+1}{2}\right)\right]^2}$$ $[|\arg u|<\pi]$ ET II 216(1)

3.238

1. $$\int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u}\,dx = -\pi\cot\frac{\nu\pi}{2}|u|^{\nu-1}\operatorname{sign} u \qquad [0<\operatorname{Re}\nu<1 \quad u \text{ real}, \quad u\neq 0]$$
ET II 249(29)

2. $$\int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u}\operatorname{sign} x\,dx = \pi\tan\frac{\nu\pi}{2}|u|^{\nu-1} \qquad [0<\operatorname{Re}\nu<1 \quad u \text{ real}, \quad u\neq 0]$$
ET II 249(30)

3. $$\int_a^b \frac{(b-x)^{\mu-1}(x-a)^{\nu-1}}{|x-u|^{\mu+\nu}}\,dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^{\mu}|b-u|^{\nu}}\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad 0<u<a<b \text{ and } 0<a<b<u]$ MO 7

3.24–3.27 Powers of x, of binomials of the form $\alpha+\beta x^p$ and of polynomials in x

3.241

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x^p} = \frac{1}{p}\,\beta\left(\frac{\mu}{p}\right) \qquad [\operatorname{Re}\mu>0, \quad p>0]$$
WH, BI (2)(13)

2. $$\int_0^{\infty} \frac{x^{\mu-1}\,dx}{1+x^{\nu}} = \frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu} = \frac{1}{\nu}\,\mathrm{B}\left(\frac{\mu}{\nu},\frac{\nu-\mu}{\nu}\right) \qquad [\operatorname{Re}\nu>\operatorname{Re}\mu>0]$$
ET I 309(15)a, BI (17)(10)

3.[11] $$\mathrm{PV}\int_0^{\infty} \frac{x^{p-1}\,dx}{1-x^q} = \frac{\pi}{q}\cot\frac{p\pi}{q} \qquad [p<q]$$
BI (17)(11)

4.[11] $$\int_0^{\infty} \frac{x^{\mu-1}\,dx}{(p+qx^{\nu})^{n+1}} = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu}\frac{\Gamma\left(\frac{\mu}{\nu}\right)\Gamma\left(1+n-\frac{\mu}{\nu}\right)}{\Gamma(1+n)}$$
$\left[0<\frac{\mu}{\nu}<n+1, \quad p\neq 0, \quad q\neq 0\right]$
BI (17)(22)a

5. $$\int_0^{\infty} \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2}\operatorname{cosec}\frac{(p-q)\pi}{q} \qquad [p<2q]$$
BI (17)(18)

6.[10] $$G(x) = \int_a^b \operatorname{sign}\left[\frac{x}{c}-\left(\frac{b-u}{b-a}\right)^p\right]du = (b-a)F\left[\left(\frac{x}{c}\right)^{1/p}\right]$$
where
$$F(x) = \int_0^1 \operatorname{sign}(x-t)\,dt = \begin{cases} -1 & x\le 0 \\ 2x-1 & 0<x<1 \\ 1 & x\ge 1 \end{cases}$$

3.242

1. $$\int_{-\infty}^{\infty} \frac{x^{2m}\,dx}{x^{4n}+2x^{2n}\cos t+1} = \frac{\pi}{n}\sin\left[\frac{(2n-2m-1)}{2n}t\right]\operatorname{cosec} t\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$
$[m<n, \quad t^2<\pi^2]$ FI II 642

2.[11] $$\int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^c \left(\frac{x^2+1}{x^b+1}\right)\frac{dx}{x^2} = 2^{-1/2-c}(1+a)^{1/2-c}\,\mathrm{B}\left(c-\frac{1}{2},\frac{1}{2}\right)$$

3.243[11] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+x^{2\nu})\,(1+x^{3\nu})}$$
$$= \frac{\pi}{48\nu}\left[8\operatorname{cosec}(2\rho)+12\operatorname{cosec}(3\rho)-8\operatorname{cosec}\left(2\rho-\frac{4\pi}{3}\right)+8\operatorname{cosec}\left(2\rho-\frac{2\pi}{3}\right)\right.$$
$$\left.-3\operatorname{cosec}\left(\rho-\frac{\pi}{6}\right)\operatorname{cosec}\left(\rho+\frac{\pi}{6}\right)\sec(\rho)\right]$$
where $\rho=\dfrac{\mu\pi}{6\nu}$, $[0<\operatorname{Re}\mu<5\operatorname{Re}\nu]$ ET I 312(34)

3.244

1. $$\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p>0]$ BI (2)(14)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^q}\,dx = \frac{\pi}{q}\cot\frac{p\pi}{q}$$ $[q>p>0]$ BI (2)(16)

3. $$\int_0^1 \frac{x^{\nu-1}-x^{\mu-1}}{1-x^\nu}\,dx = \frac{1}{\nu}\left[\boldsymbol{C}+\psi\left(\frac{\mu}{\nu}\right)\right]$$ $[\operatorname{Re}\mu>\operatorname{Re}\nu>0]$ BI (2)(17)

4. $$\int_{-\infty}^\infty \frac{x^{2m}-x^{2n}}{1-x^{2l}}\,dx = \frac{\pi}{l}\left[\cot\left(\frac{2m+1}{2l}\pi\right)-\cot\left(\frac{2n+1}{2l}\pi\right)\right]$$
$[m<l,\quad n<l]$ FI II 640

3.245 $$\int_0^\infty \left[x^{\nu-\mu}-x^\nu(1+x)^{-\mu}\right]\,dx = \frac{\nu}{\nu-\mu+1}\,\mathrm{B}(\nu,\mu-\nu)$$
$[\operatorname{Re}\mu>\operatorname{Re}\nu>0]$ BI (16)(13)

3.246 $$\int_0^\infty \frac{1-x^q}{1-x^r}x^{p-1}\,dx = \frac{\pi}{r}\sin\frac{q\pi}{r}\operatorname{cosec}\frac{p\pi}{r}\operatorname{cosec}\frac{(p+q)\pi}{r}$$
$[p+q<r,\quad p>0]$
ET I 331(33), BI (17)(12)

Integrals of the form $\int f\left(x^p\pm x^{-p},x^q\pm x^{-q},\ldots\right)\dfrac{dx}{x}$ can be transformed by the substitution $x=e^t$ or $x=e^{-t}$. For example, instead of $\int_0^1\left(x^{1+p}+x^{1-p}\right)^{-1}dx$, we should seek to evaluate $\int_0^\infty \operatorname{sech} px\,dx$ and, instead of $\int_0^1 \dfrac{x^{n-m-1}+x^{n+m-1}}{1+2x^n\cos a+x^{2n}}\,dx$, we should seek to evaluate $\int_0^\infty \cosh mx\,(\cosh nx-\cos a)^{-1}\,dx$ (see **3.514** 2).

3.247

1.[11] $$\int_0^1 \frac{x^{\alpha-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = (n-1)!\sum_{k=0}^\infty \frac{\xi^k}{(\alpha+kb)(\alpha+kb+1)\ldots(\alpha+kb+n-1)}$$
$[b>0,\quad |\xi|<1]$ AD (6704)

2. $$\int_0^\infty \frac{(1-x^p)\,x^{\nu-1}}{1-x^{np}}\,dx = \frac{\pi}{np}\sin\left(\frac{\pi}{n}\right)\operatorname{cosec}\frac{(p+\nu)\pi}{np}\operatorname{cosec}\frac{\pi\nu}{np}$$
$[0<\operatorname{Re}\nu<(n-1)p]$ ET I 311(33)

3.248

1. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu}\,\mathrm{B}\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right) \qquad [\operatorname{Re}\nu > \operatorname{Re}2\mu > 0]$$ BI (21)(9)

2. $$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}$$ BI (8)(14)

3. $$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$ BI (8)(13)

4.[3] $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$

6.* $$\int_{-\infty}^\infty \frac{dx}{\left(1+x^2\right)^2\sqrt{b+ax^2}} = \begin{cases} \dfrac{2}{\sqrt{b-a}}\arctan\left(\sqrt{\dfrac{b}{a}-1}\right) & \text{if } a<b \\ \dfrac{2}{\sqrt{a}} & \text{if } a=b \\ \dfrac{1}{\sqrt{a-b}}\ln\left(\dfrac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a>b \end{cases}$$

3.249

1.[0] $$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2\cdot(2n-2)!!}\frac{\pi}{a^{2n-1}}$$ FI II 743

2.[9] $$\int_0^a \left(a^2-x^2\right)^{n-\frac{1}{2}}\,dx = a^{2n}\frac{(2n-1)!!}{2(2n)!!}\pi.$$ FI II 156

3. $$\int_{-1}^1 \frac{\left(1-x^2\right)^n\,dx}{(a-x)^{n+1}} = 2^{n+1}\,Q_n(a)$$ EH II 181(31)

4. $$\int_0^1 \frac{x^\mu\,dx}{1+x^2} = \frac{1}{2}\,\beta\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1]$$ BI (2)(7)

5. $$\int_0^1 \left(1-x^2\right)^{\mu-1}\,dx = 2^{2\mu-2}\,\mathrm{B}(\mu,\mu) = \tfrac{1}{2}\,\mathrm{B}\left(\tfrac{1}{2},\mu\right) \qquad [\operatorname{Re}\mu > 0]$$ FI II 784

6. $$\int_0^1 \left(1-\sqrt{x}\right)^{p-1}\,dx = \frac{2}{p(p+1)} \qquad [p>0]$$ BI (7)(7)

7. $$\int_0^1 \left(1-x^\mu\right)^{-\frac{1}{\nu}}\,dx = \frac{1}{\mu}\,\mathrm{B}\left(\frac{1}{\mu}, 1-\frac{1}{\nu}\right) \qquad [\operatorname{Re}\mu > 0, \quad |\nu| > 1]$$

8.[11] $$\int_{-\infty}^\infty \left(1+\frac{x^2}{n-1}\right)^{-n/2}\,dx = \frac{\sqrt{\pi(n-1)}}{\Gamma\left(\frac{n}{2}\right)}\,\Gamma\left(\frac{n-1}{2}\right) \qquad [n>1]$$

3.251

1. $$\int_0^1 x^{\mu-1}\left(1-x^\lambda\right)^{\nu-1}\,dx = \frac{1}{\lambda}\,\mathrm{B}\left(\frac{\mu}{\lambda},\nu\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \lambda > 0]$$ FI II 787

2. $$\int_0^\infty x^{\mu-1}\left(1+x^2\right)^{\nu-1}\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right) \qquad \left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\left(\nu+\tfrac{1}{2}\mu\right) < 1\right]$$

3. $$\int_1^\infty x^{\mu-1}\left(x^p-1\right)^{\nu-1}\,dx=\frac{1}{p}\,\mathrm{B}\left(1-\nu-\frac{\mu}{p},\nu\right)$$ $[p>0,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\mu<p-p\operatorname{Re}\nu]$ ET I 311(32)

4. $$\int_0^\infty \frac{x^{2m}\,dx}{\left(ax^2+c\right)^n}=\frac{(2m-1)!!(2n-2m-3)!!\pi}{2\cdot(2n-2)!!a^m c^{n-m-1}\sqrt{ac}}$$ $[a>0,\quad c>0,\quad n>m+1]$ GU (141)(8a)

5. $$\int_0^\infty \frac{x^{2m+1}\,dx}{\left(ax^2+c\right)^n}=\frac{m!(n-m-2)!}{2(n-1)!a^{m+1}c^{n-m-1}}$$ $[ac>0,\quad n>m+1\geq 1]$ GU (141)(8b)

6. $$\int_0^\infty \frac{x^{\mu+1}}{\left(1+x^2\right)^2}\,dx=\frac{\mu\pi}{4\sin\frac{\mu\pi}{2}}$$ $[-2<\operatorname{Re}\mu<2]$ WH

7. $$\int_0^1 \frac{x^\mu\,dx}{\left(1+x^2\right)^2}=-\frac{1}{4}+\frac{\mu-1}{4}\,\beta\left(\frac{\mu-1}{2}\right)$$ $[\operatorname{Re}\mu>1]$ LI (3)(11)

8. $$\int_0^1 x^{q+p-1}\left(1-x^q\right)^{-\frac{p}{q}}\,dx=\frac{p\pi}{q^2}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p]$ BI (9)(22)

9. $$\int_0^1 x^{\frac{q}{p}-1}\left(1-x^q\right)^{-\frac{1}{p}}\,dx=\frac{\pi}{q}\operatorname{cosec}\frac{\pi}{p}$$ $[p>1,\quad q>0]$ BI (9)(23)a

10. $$\int_0^1 x^{p-1}\left(1-x^q\right)^{-\frac{p}{q}}\,dx=\frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p>0]$ BI (9)(20)

11. $$\int_0^\infty x^{\mu-1}\left(1+\beta x^p\right)^{-\nu}\,dx=\frac{1}{p}\beta^{-\frac{\mu}{p}}\,\mathrm{B}\left(\frac{\mu}{p},\nu-\frac{\mu}{p}\right)$$ $[|\arg\beta|<\pi,\quad p>0,\quad 0<\operatorname{Re}\mu<p\operatorname{Re}\nu]$ BI (17)(20), EH I 10(16)

3.252

1. $$\int_0^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n}=\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{1}{\sqrt{ac-b^2}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right]$$ $\left[a>0,\quad ac>b^2\right]$ GW (131)(4)

2. $$\int_{-\infty}^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n}=\frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!\left(ac-b^2\right)^{n-\frac{1}{2}}}$$ $\left[a>0,\quad ac>b^2\right]$ GW (131)(5)

3. $$\int_0^\infty \frac{dx}{\left(ax^2+2bx+c\right)^{n+\frac{3}{2}}}=\frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left\{\frac{1}{\sqrt{c}\left(\sqrt{ac}+b\right)}\right\}$$ $\left[a\geq 0,\quad c>0,\quad b>-\sqrt{ac}\right]$ GW (213)(4)

4. $$\int_0^\infty \frac{x\,dx}{(ax^2+2bx+c)^n}$$
$$= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2\,(ac-b^2)} - \frac{b}{2\,(ac-b^2)^{\frac{3}{2}}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right\} \quad \text{for } ac>b^2;$$
$$= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2\,(ac-b^2)} + \frac{b}{4\,(b^2-ac)^{\frac{3}{2}}}\ln\frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}}\right\} \quad \text{for } b^2>ac>0;$$
$$= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad \text{for } ac=b^2$$
$[a>0, \quad b>0, \quad n\geq 2]$ GW (141)(5)

5. $$\int_{-\infty}^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = -\frac{(2n-3)!!\pi b a^{n-2}}{(2n-2)!!\,(ac-b^2)^{\frac{(2n-1)}{2}}} \qquad \left[ac>b^2, \quad a>0, \quad n\geq 2\right]$$
GW (141)(6)

6. $$\int_{-\infty}^\infty \frac{x^m\,dx}{(ax^2+2bx+c)^n} = \frac{(-1)^m\pi a^{n-m-1}b^m}{(2n-2)!!\,(ac-b^2)^{n-\frac{1}{2}}} \times \sum_{k=0}^{[m/2]}\binom{m}{2k}(2k-1)!!(2n-2k-3)!!\left(\frac{ac-b^2}{b^2}\right)^k$$
$\left[ac>b^2, \quad 0\leq m\leq 2n-2\right]$ GW (141)(17)

7. $$\int_0^\infty \frac{x^n\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{c}\,(\sqrt{ac}+b)^{n+1}}$$
$\left[a\geq 0, \quad c>0, \quad b>-\sqrt{ac}\right]$
GW (213)(5a)

8. $$\int_0^\infty \frac{x^{n+1}\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{a}\,(\sqrt{ac}+b)^{n+1}}$$
$\left[a>0, \quad c\geq 0, \quad b>-\sqrt{ac}\right]$
GW (213)(5b)

9. $$\int_0^\infty \frac{x^{n+\frac{1}{2}}\,dx}{(ax^2+2bx+c)^{n+1}} = \frac{(2n-1)!!\pi}{2^{2n+\frac{1}{2}}\,(b+\sqrt{ac})^{n+\frac{1}{2}}\,n!\sqrt{a}}$$
$\left[a>0, \quad c>0, \quad b+\sqrt{ac}>0\right]$
LI (21)(19)

10.[6] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+2x\cos t+x^2)^\nu} = 2^{\nu-\frac{1}{2}}\,(\sin t)^{\frac{1}{2}-\nu}\,t\,\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{B}(\mu,2\nu-\mu)\,P^{\frac{1}{2}-\nu}_{\mu-\nu-\frac{1}{2}}(\cos t)$$
$[0<t<\pi, \quad 0<\operatorname{Re}\mu<\operatorname{Re}2\nu]$
ET I 310(22)

11. $$\int_0^\infty \left(1+2\beta x+x^2\right)^{\mu-\frac{1}{2}} x^{-\nu-1}\,dx = 2^{-\mu}\left(\beta^2-1\right)^{\frac{\mu}{2}}\Gamma(1-\mu)\,\mathrm{B}(\nu-2\mu+1,-\nu)\,P^{\mu}_{\nu-\mu}(\beta)$$
$$[\operatorname{Re}\nu<0,\quad \operatorname{Re}(2\mu-\nu)<1,\quad |\arg(\beta\pm1)|<\pi]$$
EH I 160(33)
$$= -\pi\operatorname{cosec}\nu\pi\; C_\nu^{\frac{1}{2}-\mu}(\beta)$$
$$\left[-2<\operatorname{Re}\left(\tfrac{1}{2}-\mu\right)<\operatorname{Re}\nu<0,\quad |\arg(\beta\pm1)|<\pi\right]$$
EH I 178(24)

12. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x^2+2ax\cos t+a^2} = -\pi a^{\mu-2}\operatorname{cosec} t\operatorname{cosec}(\mu\pi)\sin[(\mu-1)t]$$
$$[a>0,\quad 0<|t|<\pi,\quad 0<\operatorname{Re}\mu<2]$$
FI II 738, BI(20)(3)

13. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\left(x^2+2ax\cos t+a^2\right)^2} = \frac{\pi a^{\mu-4}}{2}\operatorname{cosec}\mu\pi\operatorname{cosec}^3 t$$
$$\times\{(\mu-1)\sin t\cos[(\mu-2)t]-\sin[(\mu-1)t]\}$$
$$[a>0,\quad 0<|t|<\pi,\quad 0<\operatorname{Re}\mu<4]$$ LI(20)(8)a, ET I 309(13)

14. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+2x\cos t+x^2}} = \pi\operatorname{cosec}(\mu\pi)\,P_{\mu-1}(\cos t)\qquad [-\pi<t<\pi,\quad 0<\operatorname{Re}\mu<1]$$
ET I 310(17)

3.253 $$\int_{-1}^1 \frac{(1+x)^{2\mu-1}(1-x)^{2\nu-1}}{(1+x^2)^{\mu+\nu}}\,dx = 2^{\mu+\nu-2}\,\mathrm{B}(\mu,\nu)\qquad [\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ FI II 787

3.254

1. $$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}\left(x^2+\beta^2\right)^\nu\,dx$$
$$= \beta^{2\nu}u^{\lambda+\mu-1}\,\mathrm{B}(\lambda,\mu)\;{}_3F_2\left(-\nu,\frac{\lambda}{2},\frac{\lambda+1}{2};\frac{\lambda+\mu}{2},\frac{\lambda+\mu+1}{2};\frac{-u^2}{\beta^2}\right)$$
$$\left[\operatorname{Re}\left(\frac{u}{\beta}\right)>0,\quad \lambda>0,\quad \operatorname{Re}\mu>0\right]$$ ET II 186(10)

2.[6] $$\int_u^\infty \left(x^{-\lambda}(x-u)^{\mu-1}\left(x^2+\beta^2\right)\right)^\nu\,dx$$
$$= u^{\mu-\lambda+2\nu}\frac{\Gamma(\mu)\,\Gamma(\lambda-\mu-2\nu)}{\Gamma(\lambda-2\nu)}$$
$$\times\;{}_3F_2\left(-\nu,\frac{\lambda-\mu}{2}-\nu,\frac{1+\lambda-\mu}{2}-\nu;\frac{\lambda}{2}-\nu,\frac{1+\lambda}{2}-\nu;-\frac{\beta^2}{u^2}\right)$$
$$\left[|u|>|\beta|\text{ and }\operatorname{Re}\left(\frac{\beta}{u}\right)>0,\quad 0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-2\nu)\right]$$ ET II 202(9)

3.255 $$\int_0^1 \frac{x^{\mu+\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}}{(c+2bx-ax^2)^{\mu+1}}\,dx = \frac{\sqrt{\pi}}{\left\{a+\left(\sqrt{c+2b-a}+\sqrt{c}\right)^2\right\}^{\mu+\frac{1}{2}}\sqrt{c+2b-a}}\,\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)}$$
$$\left[a+\left(\sqrt{c+2b-a}+\sqrt{c}\right)^2>0,\quad c+2b-a>0,\quad \operatorname{Re}\mu>-\tfrac{1}{2}\right]$$
BI (14)(2)

3.256

1. $$\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\cos\left(\frac{q-p}{4}\pi\right)\sec\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\frac{q}{2}\right)$$
$[p>0, \quad q>0, \quad p+q<2]$ BI (8)(25)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\sin\left(\frac{q-p}{4}\pi\right)\operatorname{cosec}\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\frac{q}{2}\right)$$
$[p>0, \quad q>0, \quad p+q<2]$ BI (8)(26)

3.257[9] $$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx$$
$$= \frac{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}{2ac^{p+\frac{1}{2}}\,\Gamma(p+1)} \qquad \left[a>0, \quad b<0, \quad c>0, \quad p>-\tfrac{1}{2}\right]$$ BI (20)(4)
$$= \frac{1}{2}\frac{\mathrm{B}\left(p+\frac{1}{2},\frac{1}{2}\right)}{a\,(4ab+x)^{p+\frac{1}{2}}} \qquad \left[a>0, \quad b>0, \quad c>-4ab, \quad p>-\tfrac{1}{2}\right]$$

3.258

1. $$\int_b^\infty \left(x-\sqrt{x^2-a^2}\right)^n dx = \frac{a^2}{2(n-1)}\left(b-\sqrt{b^2-a^2}\right)^{n-1} - \frac{1}{2(n+1)}\left(b-\sqrt{b^2-a^2}\right)^{n+1}$$
$[0<a\le b, \quad n\ge 2]$ GW (215)(5)

2. $$\int_b^\infty \left(\sqrt{x^2+1}-x\right)^n dx = \frac{\left(\sqrt{b^2+1}-b\right)^{n-1}}{2(n-1)} + \frac{\left(\sqrt{b^2+1}-b\right)^{n+1}}{2(n+1)}$$
$[n\ge 2]$ GW (214)(7)

3. $$\int_0^\infty \left(\sqrt{x^2+a^2}-x\right)^n dx = \frac{na^{n+1}}{n^2-1}$$ $[n\ge 2]$ GW (214)(6a)

4. $$\int_0^\infty \frac{dx}{\left(x+\sqrt{x^2+a^2}\right)^n} = \frac{n}{a^{n-1}\left(n^2-1\right)}$$ $[n\ge 2]$ GW (214)(5a)

5. $$\int_0^\infty x^m\left(\sqrt{x^2+a^2}-x\right)^n dx = \frac{n\cdot m!a^{m+n+1}}{(n-m-1)(n-m+1)\ldots(m+n+1)}$$
$[a>0, \quad 0\le m\le n-2]$ GW (214)(6)

6. $$\int_0^\infty \frac{x^m\,dx}{\left(x+\sqrt{x^2+a^2}\right)^n} = \frac{n\cdot m!}{(n-m-1)(n-m+1)\ldots(m+n+1)a^{n-m-1}}$$
$[a>0, \quad 0\le m\le n-2]$ GW (214)(5)

7. $$\int_a^\infty (x-a)^m\left(x-\sqrt{x^2-a^2}\right)^n dx = \frac{n\cdot(n-m-2)!(2m+1)!a^{m+n+1}}{2^m(n+m+1)!}$$
$[a>0, \quad n\ge m+2]$ GH (215)(6)

3.259

1.[6] $$\int_0^1 x^{p-1}(1-x)^{n-1}\left(1+bx^m\right)^l\,dx=(n-1)!\sum_{k=0}^{\infty}\binom{l}{k}\frac{b^k\,\Gamma(p+km)}{\Gamma(p+n+km)}$$
$[|b|<1 \text{ unless } l=0,1,2,\ldots;\quad p,n,p+ml>0]$ BI (1)(14)

2.[11] $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\left(x^m+\beta^m\right)^\lambda\,dx$$
$$=\beta^{m\lambda}u^{\mu+\nu-1}\,\mathrm{B}\,(\mu,\nu)$$
$$\times\ {}_{m+1}F_m\left(-\lambda,\frac{\nu}{m},\frac{\nu+1}{m},\ldots,\frac{\nu+m-1}{m};\frac{\mu+\nu}{m},\frac{\mu+\nu+1}{m},\ldots,\frac{\mu+\nu+m-1}{m};\frac{-u^m}{\beta^m}\right)$$
$\left[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad \left|\arg\left(\frac{u}{\beta}\right)\right|<\frac{\pi}{m}\right]$ ET II 186(11)

3.[11] $$\int_0^\infty x^{\lambda-1}\left(1+\alpha x^p\right)^{-\mu}\left(1+\beta x^p\right)^{-\nu}\,dx=\frac{1}{p}\alpha^{-\lambda/p}\,\mathrm{B}\left(\frac{\lambda}{p},\mu+\nu-\frac{\lambda}{p}\right)\ {}_2F_1\left(\nu,\frac{\lambda}{p};\mu+\nu;1-\frac{\beta}{\alpha}\right)$$
$[|\arg\alpha|<\pi,\quad |\arg\beta|<\pi,\quad p>0,\quad 0<\operatorname{Re}\lambda<2\operatorname{Re}(\mu+\nu)]$ ET I 312(35)

3.261

1.[11] $$\mathrm{PV}\int_0^1\frac{(1-x\cos t)\,x^{\mu-1}\,dx}{1-2x\cos t+x^2}=\sum_{k=0}^{\infty}\frac{\cos kt}{\mu+k}$$
$[\operatorname{Re}\mu>0,\quad t\neq 2n\pi]$ BI (6)(9)

2. $$\int_0^1\frac{\left(x^\nu+x^{-\nu}\right)\,dx}{1+2x\cos t+x^2}=\frac{\pi\sin\nu t}{\sin t\sin\nu\pi}$$
$\left[\nu^2<1,\quad t\neq(2n+1)\pi\right]$ BI (6)(8)

3. $$\int_0^1\frac{\left(x^{1+p}+x^{1-p}\right)\,dx}{\left(1+2x\cos t+x^2\right)^2}=\frac{\pi\left(p\sin t\cos pt-\cos t\sin pt\right)}{2\sin^3 t\sin p\pi}$$
$\left[p^2<1,\quad t\neq(2n+1)\pi\right]$ BI (6)(18)

4. $$\int_0^1\frac{x^{\mu-1}}{1+2ax\cos t+a^2x^2}\cdot\frac{dx}{(1-x)^\mu}=\frac{\pi\operatorname{cosec}t\operatorname{cosec}\mu\pi}{\left(1+2a\cos t+a^2\right)^{\frac{\mu}{2}}}\sin\left(t-\mu\arctan\frac{a\sin t}{1+a\cos t}\right)$$
$[a>0,\quad 0<\operatorname{Re}\mu<1]$ BI (6)(21)

3.262 $$\int_0^\infty\frac{x^{-p}\,dx}{1+x^3}=\frac{\pi}{3}\operatorname{cosec}\frac{(1-p)\pi}{3}$$
$[-2<p<1]$ LI (18)(3)

3.263 $$\int_0^\infty\frac{x^\nu\,dx}{(x+\gamma)\left(x^2+\beta^2\right)}=\frac{\pi}{2\left(\beta^2+\gamma^2\right)}\left[\gamma\beta^{\nu-1}\sec\frac{\nu\pi}{2}+\beta^\nu\operatorname{cosec}\frac{\nu\pi}{2}-2\gamma^\nu\operatorname{cosec}(\nu\pi)\right]$$
$[\operatorname{Re}\beta>0,\quad |\arg\gamma|<\pi,\quad -1<\operatorname{Re}\nu<2,\quad \nu\neq 0]$ ET II 216(7)

3.264

1. $$\int_0^\infty\frac{x^{p-1}\,dx}{\left(a^2+x^2\right)\left(b^2-x^2\right)}=\frac{\pi}{2}\frac{a^{p-2}+b^{p-2}\cos\frac{p\pi}{2}}{a^2+b^2}\operatorname{cosec}\frac{p\pi}{2}$$
$[0<p<4,\quad a>0,\quad b>0]$
BI (19)(14)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x^2)\,(\gamma+x^2)} = \frac{\pi}{2}\frac{\gamma^{\frac{\mu}{2}-1}-\beta^{\frac{\mu}{2}-1}}{\beta-\gamma}\operatorname{cosec}\frac{\mu\pi}{2}$$
$$= \frac{\pi}{2(\gamma-\beta)}\left(\frac{1}{\sqrt{\beta}}-\frac{1}{\sqrt{\gamma}}\right) \qquad \left[\mu=\tfrac{1}{2}\right]$$
$$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad 0<\operatorname{Re}\mu<4]$$ ET I 309(4)

3. $$\int_0^\infty \frac{dx}{(b+x^2)\,(a+b+x^2)^2} = \frac{\pi}{2}\left(\frac{1}{a^2b^{1/2}}-\frac{1}{2a\,(a+b)^{3/2}}-\frac{1}{a^2\,(a+b)^{1/2}}\right)$$ MC

4. $$\int_0^\infty \frac{dx}{(b+x^2)\,(a+b+x^2)^3} = \frac{\pi}{4}\left(\frac{2}{a^3b^{1/2}}-\frac{3}{4a\,(a+b)^{5/2}}-\frac{1}{a^2\,(a+b)^{3/2}}-\frac{2}{a^3\,(a+b)^{1/2}}\right)$$

5. $$\int_0^\infty \frac{dx}{(b+x^2)\,(a+b+x^2)^4} = \frac{\pi}{4}\left(\frac{2}{a^4b^{1/2}}-\frac{5}{8a\,(a+b)^{7/2}}-\frac{3}{4a^2\,(a+b)^{5/2}}\right.$$
$$\left.-\frac{1}{a^3\,(a+b)^{3/2}}-\frac{2}{a^4\,(a+b)^{1/2}}\right)$$

6. $$\int_0^\infty \frac{dx}{(b+x^2)\,(a+b+x^2)^n}$$
$$= \frac{\pi}{2}\frac{1}{a^nb^{1/2}}-\frac{1}{2a\,(a+b)^{n-1/2}}\,\mathrm{B}\left(n-\frac{1}{2},\frac{1}{2}\right)\ {}_2F_1\left(1-n,1;\ \frac{3}{2}-n;\ \frac{a+b}{a}\right)$$
AS 263 (6.6.3.2)
$$= \frac{\pi}{2}\frac{1}{a^nb^{1/2}}-\frac{\pi}{2a^n\,(a+b)^{n-1/2}}\sum_{j=0}^{n-1}\frac{\left(\frac{1}{2}\right)_j}{j!}\left(\frac{a}{a+b}\right)^j$$
$$[n>0, \quad a+b>0]$$

7. $$\int_0^\infty \frac{x^2\,dx}{(x^2+\alpha^2)\,(x^2+\beta^2)\,(x^2+\gamma^2)} = \frac{\pi}{2\alpha\,(\beta^2-\gamma^2)}\left[\frac{\beta}{\beta+\alpha}-\frac{\gamma}{\gamma+\alpha}\right] = \frac{\pi}{2(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)}$$

3.265 $$\int_0^1 \frac{1-x^{\mu-1}}{1-x}\,dx = \psi(\mu)+\boldsymbol{C} \qquad [\operatorname{Re}\mu>0]$$ FI II 796, WH, ET I 16(13)
$$= \psi(1-\mu)+\boldsymbol{C}-\pi\cot(\mu\pi) \qquad [\operatorname{Re}\mu>0]$$ EH I 16(15)a

3.266 $$\int_0^\infty \frac{(x^\nu-a^\nu)\ dx}{(x-a)(\beta+x)} = \frac{\pi}{a+\beta}\left\{\beta^\nu\operatorname{cosec}(\nu\pi)-a^\nu\cot(\nu\pi)-\frac{a^\nu}{\pi}\ln\frac{\beta}{a}\right\}$$
$$[|\arg\beta|<\pi, \quad |\operatorname{Re}\nu|<1, \quad \nu\neq 0]$$
ET II 216(8)

3.267

1. $$\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n+1)}$$ BI (9)(6)

2. $$\int_0^1 \frac{x^{3n-1}\,dx}{\sqrt[3]{1-x^3}} = \frac{(n-1)!\,\Gamma\left(\frac{2}{3}\right)}{3\,\Gamma\left(n+\frac{2}{3}\right)}$$ BI (9)(7)

3.* $$\int_0^1 \frac{x^{3n-2}\,dx}{\sqrt[3]{1-x^3}} = \frac{\Gamma\left(n-\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{3\,\Gamma\left(n+\frac{1}{3}\right)}$$

3.268

1. $$\int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p}\right) dx = \ln p$$ BI (5)(14)

2. $$\int_0^1 \frac{1-x^\mu}{1-x} x^{\nu-1}\,dx = \psi(\mu+\nu) - \psi(\nu) \qquad [\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0]$$ BI (2)(3)

3. $$\int_0^1 \left[\frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}}\right] dx = n\boldsymbol{C} + \sum_{k=1}^{n} \psi\left(\mu + \frac{n-k}{n}\right)$$
$$[\operatorname{Re}\mu > 0]$$ BI (13)(10)

3.269

1. $$\int_0^1 \frac{x^p - x^{-p}}{1-x^2} x\,dx = \frac{\pi}{2}\cot\frac{p\pi}{2} - \frac{1}{p} \qquad [p^2 < 1]$$ BI (4)(12)

2. $$\int_0^1 \frac{x^p - x^{-p}}{1+x^2} x\,dx = \frac{1}{p} - \frac{\pi}{2}\operatorname{cosec}\frac{p\pi}{2} \qquad [p^2 < 1]$$ BI (4)(8)

3. $$\int_0^1 \frac{x^\mu - x^\nu}{1-x^2}\,dx = \frac{1}{2}\psi\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\psi\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$$ BI (2)(9)

3.271

1. $$\int_0^\infty \frac{x^p - x^q}{x-1}\frac{dx}{x+a} = \frac{\pi}{1+a}\left(\frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi}\right)$$
$$[p^2 < 1, \quad q^2 < 1, \quad a > 0]$$ BI (19)(2)

2. $$\int_0^\infty \frac{x^p - a^p}{x-a}\frac{x^p - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{\frac{a^{2p}-1}{\sin(2p\pi)} - \frac{1}{\pi}a^p \ln a\right\}$$
$$\left[p^2 < \frac{1}{4}\right]$$ BI (19)(3)

3. $$\int_0^\infty \frac{x^p - a^p}{x-a}\frac{x^{-p} - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{2\left(a^p - 1\right)\cot p\pi - \frac{1}{\pi}\left(a^p + 1\right)\ln a\right\}$$
$$[p^2 < 1]$$ BI (18)(9)

4. $$\int_0^\infty \frac{x^p - a^p}{x-a}\frac{1-x^{-p}}{1-x} x^q\,dx = \frac{\pi}{a-1}\left\{\frac{a^{p+q}-1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]}\right\}\frac{\sin p\pi}{\sin q\pi}$$
$$\left[(p+q)^2 < 1, \quad (p-q)^2 < 1\right]$$ BI (19)(4)

5. $$\int_0^\infty \left(\frac{x^p - x^{-p}}{1-x}\right)^2 dx = 2\left(1 - 2p\pi\cot 2p\pi\right) \qquad \left[0 < p^2 < \tfrac{1}{4}\right]$$ BI (16)(3)

3.272

1. $$\int_0^1 \frac{x^{n-1} + x^{n-\frac{1}{2}} - 2x^{2n-1}}{1-x}\,dx = 2\ln 2$$ BI (8)(8)

2. $$\int_0^1 \frac{x^{n-1}+x^{n-\frac{2}{3}}+x^{n-\frac{1}{3}}-3x^{3n-1}}{1-x}\,dx = 3\ln 3$$ BI (8)(9)

3.273

1. $$\int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1}x^{n+1}\sin nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\sin kt}{\Gamma(p+k)}$$
$[p>0]$ BI (6)(13)

2. $$\int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1}x^{n+1}\cos nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\cos kt}{\Gamma(p+k)}$$
$[p>0]$ BI (6)(14)

3. $$\int_0^1 x\frac{\sin t - x^n\sin[(n+1)t] + x^{n+1}\sin nt}{1-2x\cos t + x^2}\,dx = \sum_{k=1}^{n}\frac{\sin kt}{k+1}$$ BI (6)(12)

4. $$\int_0^1 \frac{1 - x\cos t - x^{n+1}\cos[(n+1)t] + x^{n+2}\cos nt}{1-2x\cos t + x^2}\,dx = \sum_{k=0}^{n}\frac{\cos kt}{k+1}$$ BI (6)(11)

3.274

1. $$\int_0^\infty \frac{x^{\mu-1}(1-x)}{1-x^n}\,dx = \frac{\pi}{n}\sin\frac{\pi}{n}\operatorname{cosec}\frac{\mu\pi}{n}\operatorname{cosec}\frac{(\mu+1)\pi}{n}$$
$[0<\operatorname{Re}\mu<n-1]$ BI (20)(13)

2. $$\int_0^1 \frac{1-x^n}{(1+x)^{n+1}}\frac{dx}{1-x} = \frac{1}{2^{n+1}}\sum_{k=1}^{n}\frac{2^k}{k}$$ BI (5)(3)

3. $$\int_0^\infty \frac{x^q-1}{x^p-x^{-p}}\frac{dx}{x} = \frac{\pi}{2p}\tan\frac{q\pi}{2p}$$ $[p>q]$ BI (18)(6)

3.275

1. $$\int_0^1 \left(\frac{x^{n-1}}{1-x^{1/p}} - \frac{px^{np-1}}{1-x}\right)dx = p\ln p$$ $[p>0]$ BI (13)(9)

2. $$\int_0^1 \left(\frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x}\right)dx = \boldsymbol{C} + \frac{1}{n}\sum_{k=1}^{n}\psi\left(m+\frac{n-k}{n}\right)$$ BI (5)(13)

3. $$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q}\right)dx = \ln q$$ $[q>0]$ BI (5)(12)

4. $$\int_0^\infty \left(\frac{1}{1+x^{2^n}} - \frac{1}{1+x^{2^m}}\right)\frac{dx}{x} = 0.$$ BI (18)(17)

3.276

1.[10] $$\int_0^\infty \frac{\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx}{x^2} = \frac{1}{2|b|}\frac{\mathrm{B}\left(p+\frac{1}{2},\frac{1}{2}\right)}{(2a(b+|b|)+c)^{p+\frac{1}{2}}}$$
$\left[a>0,\quad c>-4ac,\quad p>-\frac{1}{2}\right]$

2.[10] $$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx = \frac{\mathrm{B}\left(p+\frac{1}{2},\frac{1}{2}\right)}{(4ab+c)^{p+\frac{1}{2}}}$$

$$\left[a>0, \quad b>0, \quad c>-4ac, \quad p>-\tfrac{1}{2}\right]$$

3.277

1.[11] $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{1+x^2}+\beta\right]^\nu}{\sqrt{1+x^2}}\,dx = 2^{\frac{\mu}{2}-1}\left(\beta^2-1\right)^{\frac{\nu}{2}+\frac{\mu}{4}}\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)\,P^{\nu+\frac{\mu}{2}}_{\frac{\mu}{2}-1}(\beta)$$

$$[\operatorname{Re}\beta>-1, \quad 0<\operatorname{Re}\mu<1-\operatorname{Re}\nu]$$

ET I 310(25)

2. $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{\beta^2+x^2}+x\right]^\nu}{\sqrt{\beta^2+x^2}}\,dx = \frac{\beta^{\mu+\nu-1}}{2^\mu}\mathrm{B}\left(\mu,\frac{1-\mu-\nu}{2}\right)$$

$$[\operatorname{Re}\beta>0, \quad 0<\operatorname{Re}\mu<1-\operatorname{Re}\nu]$$

ET I 311(28)

3. $$\int_0^\infty \frac{x^{\mu-1}\left[\cos t\pm i\sin t\sqrt{1+x^2}\right]^\nu}{\sqrt{1+x^2}}\,dx = 2^{\frac{\mu-1}{2}}\sin^{\frac{1-\mu}{2}} t\,\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}$$

$$\times\left[\pi^{-\frac{1}{2}}\,Q^{\frac{\mu+1}{2}}_{-\frac{\mu+1}{2}-\nu}(\cos t)\mp\frac{i}{2}\pi^{\frac{1}{2}}\,P^{-\frac{\mu+1}{2}-\nu}_{\frac{\mu-1}{2}}(\cos t)\right]$$

$$[\operatorname{Re}\mu>0]$$

ET I 311 (27)

4. $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{(\beta^2-1)(x^2+1)}+\beta\right]^\nu}{\sqrt{x^2+1}}\,dx$$

$$=\frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}}e^{-\frac{1}{2}i\pi(\mu-1)}\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}\left(\beta^2-1\right)\frac{1-\mu}{4}\,Q^{\frac{\mu-1}{2}}_{-\frac{\mu+1}{2}-\nu}(\beta)$$

$$[\operatorname{Re}\beta>1, \quad \operatorname{Re}\nu<0, \quad \operatorname{Re}\mu<1-\operatorname{Re}\nu]$$

ET I 311(26)

5. $$\int_u^\infty \frac{(x-u)^{\mu-1}\left(\sqrt{x+1}-\sqrt{x-1}\right)^{2\nu}}{\sqrt{x^2-1}}\,dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}}e^{\left(\mu-\frac{1}{2}\right)\pi i}\left(u^2-1\right)^{\frac{2\mu-1}{4}}\,Q^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(u)$$

$$[|\arg(u-1)|<\pi, \quad 0<\operatorname{Re}\mu<1+\operatorname{Re}\nu]$$

ET II 202(10)

6. $$\int_1^\infty \frac{x^{\mu-1}\left[\left(x-\sqrt{x^2-1}\right)^\nu+\left(x-\sqrt{x^2-1}\right)^{-\nu}\right]}{\sqrt{x^2-1}}\,dx = 2^{-\mu}\,\mathrm{B}\left(\frac{1-\mu+\nu}{2},\frac{1-\mu-\nu}{2}\right)$$

$$[\operatorname{Re}\mu<1+\operatorname{Re}\nu]$$

ET I 311(29)

7. $$\int_0^u \frac{(u-x)^{\mu-1}\left[\left(\sqrt{x+2}+\sqrt{x}\right)^{2\nu}+\left(\sqrt{x+2}-\sqrt{x}\right)^{2\nu}\right]}{\sqrt{x(x+2)}}\,dx = 2\frac{2\mu+1}{2}\sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}}\,P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(u+1)$$

$$[|\arg u|<\pi, \quad \operatorname{Re}\mu>0]$$

ET II 186(12)

3.278[8]

1. $$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^q \frac{dx}{1-x^2} = 0 \qquad [pq>1]$$

3.3–3.4 Exponential Functions

3.31 Exponential functions

3.310[11] $\displaystyle\int_0^\infty e^{-px}\,dx = \frac{1}{p}$ $[\operatorname{Re} p > 0]$

3.311

1. $\displaystyle\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}$ LO III 284a

2. $\displaystyle\int_0^\infty \frac{e^{-\mu x}}{1+e^{-x}}\,dx = \beta(\mu)$ $[\operatorname{Re}\mu > 0]$ EH I 20(3), ET I 144(7)

3.[11] $\displaystyle\int_{-\infty}^\infty \frac{e^{-px}}{1+e^{-qx}}\,dx = \frac{\pi}{|q|}\operatorname{cosec}\frac{p\pi}{q}$

$[q > p > 0 \text{ or } 0 > p > q]$ (cf. **3.241** 2) BI (28)(7)

4. $\displaystyle\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \sum_{k=0}^\infty \frac{a^k}{q+kp}$ $[0 < a < 1]$ BI (27)(7)

5. $\displaystyle\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \psi(\nu) + \boldsymbol{C} + \pi\cot(\pi\nu)$ $[\operatorname{Re}\nu < 1]$ (cf. **3.265**) EH I 16(16)

6. $\displaystyle\int_0^\infty \frac{e^{-x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) + \boldsymbol{C}$ $[\operatorname{Re}\nu > 0]$ WH, EH I 16(14)

7. $\displaystyle\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) - \psi(\mu)$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ (cf. **3.231** 5)

BI (27)(8)

8. $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = \pi b^{\mu-1}\cot(\mu\pi)$ $[b > 0, \quad 0 < \operatorname{Re}\mu < 1]$ ET I 120(14)a

9. $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b+e^{-x}} = \pi b^{\mu-1}\operatorname{cosec}(\mu\pi)$ $[|\arg b| < \pi, \quad 0 < \operatorname{Re}\mu < 1]$

ET I 120(15)a

10.[11] $\displaystyle\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\frac{p\pi}{p+q}$ $[p > 0, \quad q > 0]$ GW (311)(16c)

11. $\displaystyle\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\,dx = \frac{1}{r-s}\left[\psi\left(\frac{r-q}{r-s}\right) - \psi\left(\frac{r-p}{r-s}\right)\right]$

$[r > s, r > p, r > q]$ GW (311)(16)

12. $\displaystyle\int_0^\infty \frac{a^x-b^x}{c^x-d^x}\,dx = \frac{1}{\ln\frac{c}{d}}\left[\psi\left(\frac{\ln\frac{c}{b}}{\ln\frac{c}{d}}\right) - \psi\left(\frac{\ln\frac{c}{a}}{\ln\frac{c}{d}}\right)\right]$ $[c > a > 0, \quad b > 0, \quad d > 0]$

GW (311)(16a)

13.* $\displaystyle\int_0^\infty \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\operatorname{cosec}\left(\frac{\pi p}{p+q}\right)$

3.312

1. $$\int_0^\infty \left(1-e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x}\,dx = \beta\,\mathrm{B}(\beta\mu,\nu) \qquad [\operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\mu>0]$$

LI(25)(13), EH I 11(24)

2. $$\int_0^\infty \left(1-e^{-x}\right)^{-1}\left(1-e^{-\alpha x}\right)\left(1-e^{-\beta x}\right)e^{-px}\,dx = \psi(p+\alpha)+\psi(p+\beta)-\psi(p+\alpha+\beta)-\psi(p)$$

$[\operatorname{Re}p>0,\quad \operatorname{Re}p>-\operatorname{Re}\alpha,\quad \operatorname{Re}p>-\operatorname{Re}\beta,\quad \operatorname{Re}p>-\operatorname{Re}(\alpha+\beta)]$ ET I 145(15)

3.[11] $$\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\left(1-\beta e^{-x}\right)^{-\varrho} e^{-\mu x}\,dx = \mathrm{B}(\mu,\nu)\ {}_2F_1(\varrho,\mu;\mu+\nu;\beta)$$

$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad |\arg(1-\beta)|<\pi]$ EH I 116(15)

3.313

1.[7] $$\mathrm{PV}\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{1-e^{-x}} = \pi\cot\pi\mu \qquad [0<\operatorname{Re}\mu<1]$$

2.[7] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{(1+e^{-x})^\nu} = \mathrm{B}(\mu,\nu-\mu) \qquad [0<\operatorname{Re}\mu<\operatorname{Re}\nu]$$

3.314 $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(e^{\beta/\gamma}+e^{-x/\gamma}\right)^\nu} = \gamma\exp\left[\beta\left(\mu-\frac{\nu}{\gamma}\right)\right]\mathrm{B}(\gamma\mu,\nu-\gamma\mu)$$

$\left[\operatorname{Re}\left(\frac{\nu}{\gamma}\right)>\operatorname{Re}\mu>0,\quad |\operatorname{Im}\beta|<\pi\operatorname{Re}\gamma\right]$ ET I 120(21)

3.315

1. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(e^\beta+e^{-x}\right)^\nu\left(e^\gamma+e^{-x}\right)^\varrho} = \exp[\gamma(\mu-\varrho)-\beta\nu]\,\mathrm{B}(\mu,\nu+\varrho-\mu)\ {}_2F_1\left(\nu,\mu;\nu+\varrho;1-e^{\nu-\beta}\right)$$

$[|\operatorname{Im}\beta|<\pi,\quad |\operatorname{Im}\gamma|<\pi,\quad 0<\operatorname{Re}\mu<\operatorname{Re}(\nu+\varrho)]$ ET I 121(22)

2. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{(\beta+e^{-x})(\gamma+e^{-x})} = \frac{\pi\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)}{\gamma-\beta}\operatorname{cosec}(\mu\pi)$$

$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad \beta\neq\gamma,\quad 0<\operatorname{Re}\mu<2]$ ET I 120(18)

3.316 $$\int_{-\infty}^\infty \frac{(1+e^{-x})^\nu-1}{(1+e^{-x})^\mu}\,dx = \psi(\mu)-\psi(\mu-\nu) \qquad [\operatorname{Re}\mu>\operatorname{Re}\nu>0] \qquad (\text{cf. } \mathbf{3.235})$$

BI (28)(8)

3.317

1. $$\int_{-\infty}^\infty \left(\frac{1}{1+e^{-x}}-\frac{1}{(1+e^{-x})^\mu}\right)dx = \boldsymbol{C}+\psi(\mu) \qquad [\operatorname{Re}\mu>0] \qquad (\text{cf. } \mathbf{3.233})$$ BI (28)(10)

2. $$\int_{-\infty}^\infty \left(\frac{1}{(1+e^{-x})^\nu}-\frac{1}{(1+e^{-x})^\mu}\right)dx = \psi(\mu)-\psi(\nu) \qquad [\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0] \qquad (\text{cf. } \mathbf{3.219})$$

BI (28)(11)

3.318

1. $$\int_0^\infty \frac{\left[\beta+\sqrt{1-e^{-x}}\right]^{-\nu}+\left[\beta-\sqrt{1-e^{-x}}\right]^{-\nu}}{\sqrt{1-e^{-x}}}e^{-\mu x}\,dx = \frac{2^{\mu+1}e^{(\mu-\nu)\pi i}\left(\beta^2-1\right)^{(\mu-\nu)/2}\Gamma(\mu)\,Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)}$$
$[\operatorname{Re}\mu>0]$ ET I 145(18)

2.[7] $$\int_u^\infty \frac{1}{\sqrt{1-e^{-2x}}}\left(e^{-u}\sqrt{1-e^{-2x}}-e^{-x}\sqrt{1-e^{-2u}}\right)^\nu e^{-\mu x}\,dx = \frac{2^{-\frac{1}{2}(\mu+\nu)}\sqrt{\pi}e^{-\frac{u}{2}(\mu+\nu)}\Gamma(\mu)\Gamma(\nu+1)\,P_{-\frac{1}{2}(\mu-\nu)}^{-\frac{1}{2}(\mu+\nu)}\left(\sqrt{1-e^{-2u}}\right)}{\Gamma[(\mu+\nu+1)/2]}$$
$[u>0,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>-1]$ ET I 145(19)

3.32–3.34 Exponentials of more complicated arguments

3.321

1.[11] $$\frac{\sqrt{\pi}}{2}\,\Phi(u)=\frac{\sqrt{\pi}}{2}\operatorname{erf}(u)=\int_0^u e^{-x^2}\,dx=\sum_{k=0}^\infty\frac{(-1)^k u^{2k+1}}{k!(2k+1)} = e^{-u^2}\sum_{k=0}^\infty\frac{2^k u^{2k+1}}{(2k+1)!!}$$
(cf. **8.25**) AD 6.700

2. $$\int_0^u e^{-q^2x^2}\,dx=\frac{\sqrt{\pi}}{2q}\,\Phi(qu)$$ $[q>0]$

3. $$\int_0^\infty e^{-q^2x^2}\,dx=\frac{\sqrt{\pi}}{2q}$$ $[q>0]$ FI II 624

4.* $$\int_0^u xe^{-q^2x^2}\,dx=\frac{1}{2q^2}\left[1-e^{-q^2u^2}\right]$$

5.* $$\int_0^u x^2e^{-q^2x^2}\,dx=\frac{1}{2q^3}\left[\frac{\sqrt{\pi}}{2}\,\Phi(qu)-que^{-q^2u^2}\right]$$

6.* $$\int_0^u x^3e^{-q^2x^2}\,dx=\frac{1}{2q^4}\left[1-\left(1+q^2u^2\right)e^{-q^2u^2}\right]$$

7.* $$\int_0^u x^4e^{-q^2x^2}\,dx=\frac{1}{2q^5}\left[\frac{3\sqrt{\pi}}{4}\,\Phi(qu)-\left(\frac{3}{2}+q^2u^2\right)que^{-q^2u^2}\right]$$

3.322

1.[11] $$\int_u^\infty \exp\left(-\frac{x^2}{4\beta}-\gamma x\right)dx=\sqrt{\pi\beta}e^{\beta\gamma^2}\left[1-\Phi\left(\gamma\sqrt{\beta}+\frac{u}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0]$ ET I 146(21)

2. $$\int_0^\infty \exp\left(-\frac{x^2}{4\beta}-\gamma x\right)dx=\sqrt{\pi\beta}\exp\left(\beta\gamma^2\right)\left[1-\Phi\left(\gamma\sqrt{\beta}\right)\right]$$
$[\operatorname{Re}\beta>0]$ NT 27(1)a

3.11 $\mathrm{PV}\int_0^\infty e^{\pm i\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}e^{\pm\pi i/4}$ $[\lambda > 0]$ PBM 343 (2.3.15(2))

3.323

1.11 $\int_1^\infty \exp\left(-qx - x^2\right)\,dx = \frac{\sqrt{\pi}}{2}e^{q^2/4}\left[1 - \Phi\left(1 + \frac{1}{2}q\right)\right]$ BI (29)(4)

2.10 $\int_{-\infty}^\infty \exp\left(-p^2x^2 \pm qx\right)\,dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p}$ $\left[\operatorname{Re} p^2 > 0\right]$ BI (28)(1)

3.11 $\int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)\,dx = 2^{-\frac{3}{2}}\frac{\gamma}{\beta}e^{\frac{\gamma^4}{2\beta^2}}\,K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right)$

$\left[|\arg\beta| < \frac{\pi}{4},\quad |\arg\gamma| < \frac{\pi}{4}\right]$

ET I 147(34)a

3.324

1. $\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right)\,dx = \sqrt{\frac{\beta}{\gamma}}\,K_1\left(\sqrt{\beta\gamma}\right)$ $[\operatorname{Re}\beta \geq 0,\quad \operatorname{Re}\gamma > 0]$ ET I 146(25)

2.11 $\int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right]\,dx = \frac{1}{n}\,\Gamma\left(\frac{1}{2n}\right)$ $[b \geq 0]$

3.325 $\int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{ab}\right)$ $[a > 0,\quad b > 0]$ FI II 644

3.326

1.8 $\int_0^\infty \exp\left(-x^\mu\right)\,dx = \frac{1}{\mu}\,\Gamma\left(\frac{1}{\mu}\right)$ $[\operatorname{Re}\mu > 0]$ BI (26)(4)

2.10 $\int_0^\infty x^m\exp\left(-\beta x^n\right)\,dx = \frac{\Gamma(\gamma)}{n\beta^\gamma}\qquad \gamma = \frac{m+1}{n}$ $[\operatorname{Re}\beta > 0,\quad \operatorname{Re} m > 0,\quad \operatorname{Re} n > 0]$

3.* $\int_0^\infty (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n},\beta(-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n},\beta(-b)^n\right)}{n\beta^{1/n}}$

$[\operatorname{Re} n > 0,\quad \operatorname{Re}\beta > 0,\quad |\arg b| < \pi]$

4.* $\int_0^u (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n},\beta(-b)^n\right) - \Gamma\left(\frac{2}{n},\beta(u-b)^n\right)}{n\beta^{2/n}}$

$-(a-b)\frac{\Gamma\left(\frac{1}{n},\beta(-b)^n\right) - \Gamma\left(\frac{1}{n},\beta(u-b)^n\right)}{n\beta^{1/n}}$

$[\operatorname{Re} n > 0,\quad \operatorname{Re}\beta > 0,\quad |\arg b| < \pi,\quad |\arg(u-b)| < \pi]$

5.* $\int_u^\infty (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n},\beta(-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n},\beta(u-b)^n\right)}{n\beta^{1/n}}$

$[\operatorname{Re} n > 0,\quad \operatorname{Re}\beta > 0,\quad |\arg(u-b)| < \pi]$

Exponentials of exponentials

3.327 $\displaystyle\int_0^\infty \exp\left(-ae^{nx}\right)\,dx = -\frac{1}{n}\,\mathrm{Ei}(-a)$ $\quad[n \geq 1, \quad \mathrm{Re}\,a \geq 0, \quad a \neq 0]$ LI (26)(5)

3.328 $\displaystyle\int_{-\infty}^\infty \exp\left(-e^x\right) e^{\mu x}\,dx = \Gamma(\mu)$ $\quad[\mathrm{Re}\,\mu > 0]$ NH 145(14)

3.329 $\displaystyle\int_0^\infty \left[\frac{a\exp\left(-ce^{ax}\right)}{1-e^{-ax}} - \frac{b\exp\left(-ce^{bx}\right)}{1-e^{-bx}}\right] dx = e^{-c}\ln\frac{b}{a}$ $\quad[a > 0, \quad b > 0, \quad c > 0]$ BI (27)(12)

3.331

1. $\displaystyle\int_0^\infty \exp\left(-\beta e^{-x} - \mu x\right)\,dx = \beta^{-\mu}\,\gamma(\mu,\beta)$ $\quad[\mathrm{Re}\,\mu > 0]$ ET I 147(36)

2. $\displaystyle\int_0^\infty \exp\left(-\beta e^{x} - \mu x\right)\,dx = \beta^{\mu}\,\Gamma(-\mu,\beta)$ $\quad[\mathrm{Re}\,\beta > 0]$ ET I 147(37)

3.[11] $\displaystyle\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\exp\left(\beta e^{-x} - \mu x\right)\,dx = \mathrm{B}(\mu,\nu)\beta^{-\frac{\mu+\nu}{2}}e^{\frac{\beta}{2}}\,M_{\frac{\nu-\mu}{2},\frac{\nu+\mu-1}{2}}(\beta)$

$[\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0]$ ET I 147(38)

4. $\displaystyle\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\exp\left(-\beta e^{x} - \mu x\right)\,dx = \Gamma(\nu)\beta^{\frac{\mu-1}{2}}e^{-\frac{\beta}{2}}\,W_{\frac{1-\mu-2\nu}{2},\frac{-\mu}{2}}(\beta)$

$[\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\nu > 0]$ ET I 147(39)

3.332 $\displaystyle\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\left(1-\lambda e^{-x}\right)^{-\varrho}\exp\left(\beta e^{-x} - \mu x\right)\,dx = \mathrm{B}(\mu,\nu)\,\Phi_1(\mu,\varrho,\nu,\lambda,\beta)$

$[\mathrm{Re}\,\mu > 0, \quad \mathrm{Re}\,\nu > 0, \quad |\arg(1-\lambda)| < \pi]$ ET I 147(40)

3.333

1.[3] $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)-1} = \Gamma(\mu)\,\zeta(\mu)$ $\quad[\mathrm{Re}\,\mu > 1]$ ET I 121(24)

2.[3] $\displaystyle\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)+1} = \left(1-2^{1-\mu}\right)\Gamma(\mu)\,\zeta(\mu)$ $\quad[\mathrm{Re}\,\mu > 0, \quad \mu \neq 1]$

$= \ln 2$ $\quad[\mu = 1]$

ET I 121(25)

3.* $\displaystyle\int_0^\infty \left(\frac{\tanh(x)}{x^3} - \frac{1}{x^2\cosh^2(x)}\right) dx = \frac{7\,\zeta(3)}{\pi^2}$

3.334[11] $\displaystyle\int_0^\infty (e^x-1)^{\nu-1}\exp\left[-\frac{\beta}{e^x-1} - \mu x\right] dx = \Gamma(\mu-\nu+1)e^{\frac{\beta}{2}}\beta^{\frac{\nu-1}{2}}\,W_{\frac{\nu-2\mu-1}{2},-\frac{\nu}{2}}(\beta)$

$[\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > \mathrm{Re}\,\nu - 1]$

ET I 137(41)

Exponentials of hyperbolic functions

3.335 $\displaystyle\int_0^\infty \left(e^{\nu x} + e^{-\nu x}\cos\nu\pi\right)\exp\left(-\beta\sinh x\right)\,dx = -\pi\left[\mathbf{E}_\nu(\beta) + Y_\nu(\beta)\right]$

$[\mathrm{Re}\,\beta > 0]$ EH II 35(34)

3.336

1. $$\int_0^\infty \exp\left(-\nu x - \beta \sinh x\right)\, dx = \pi \operatorname{cosec} \nu\pi \left[\mathbf{J}_\nu(\beta) - J_\nu(\beta)\right]$$
$$\left[|\arg\beta| < \frac{\pi}{2} \text{ and } |\arg\beta| = \frac{\pi}{2} \text{ for } \operatorname{Re}\nu > 0; \quad \nu \text{ is not an integer}\right]$$ WA 341(2)

2. $$\int_0^\infty \exp\left(nx - \beta \sinh x\right)\, dx = \frac{1}{2}\left[S_n(\beta) - \pi\, \mathbf{E}_n(\beta) - \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0; \quad n = 0, 1, 2, \ldots]$$ WA 342(6)

3. $$\int_0^\infty \exp\left(-nx - \beta \sinh x\right)\, dx = \frac{1}{2}(-1)^{n+1}\left[S_n(\beta) + \pi\, \mathbf{E}_n(\beta) + \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0; \quad n = 0, 1, 2, \ldots]$$
EH II 84(47)

3.337

1. $$\int_{-\infty}^\infty \exp\left(-\alpha x - \beta \cosh x\right)\, dx = 2\, K_\alpha(\beta) \qquad \left[|\arg\beta| < \frac{\pi}{2}\right]$$ WA 201(7)

2. $$\int_{-\infty}^\infty \exp\left(-\nu x + i\beta \cosh x\right)\, dx = i\pi e^{\frac{i\nu\pi}{2}}\, H_\nu^{(1)}(\beta) \qquad [0 < \arg z < \pi]$$ EH II 21(27)

3. $$\int_{-\infty}^\infty \exp\left(-\nu x - i\beta \cosh x\right)\, dx = -i\pi e^{-\frac{i\nu\pi}{2}}\, H_\nu^{(2)}(\beta) \qquad [-\pi < \arg z < 0]$$ EH II 21(30)

Exponentials of trigonometric functions and logarithms

3.338

1. $$\int_0^\pi \left\{\exp i\left[(\nu-1)x - \beta \sin x\right] - \exp i\left[(\nu+1)x - \beta \sin x\right]\right\}\, dx = 2\pi\left[\mathbf{J}'_\nu(\beta) + i\, \mathbf{E}'_\nu(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ EH II 36

2. $$\int_0^\pi \exp\left[\pm i\left(\nu x - \beta \sin x\right)\right]\, dx = \pi\left[\mathbf{J}_\nu(\beta) \pm i\, \mathbf{E}_\nu(\beta)\right] \qquad [\operatorname{Re}\beta > 0]$$ EH II 35(32)

3.[10] $$\int_0^\infty \exp\left[-\gamma\left(x - \beta \sin x\right)\right]\, dx = \frac{1}{\gamma} + 2\sum_{k=1}^\infty \frac{\gamma\, J_k(k\beta)}{\gamma^2 + k^2} \qquad [\operatorname{Re}\gamma > 0]$$ WA 619(4)

4.[6] $$\int_{-\pi}^\pi \frac{\exp\left[\dfrac{a + b\sin x + c\cos x}{1 + p\sin x + q\cos x}\right]}{1 + p\sin x + q\cos x}\, dx = \frac{2\pi}{\sqrt{1 - p^2 - q^2}} e^{-\alpha}\, I_0(\beta),$$
$$\text{with } \alpha = \frac{bp + cq - a}{1 - p^2 - q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2 - b^2 - c^2}{1 - p^2 - q^2}}; \qquad \left[p^2 + q^2 < 1\right]$$

5.* $$\int_0^{\pi/4} \exp\left[-\sum_{n=1}^\infty \frac{\tan^{2n} x}{n + \frac{1}{2}}\right]\, dx = \ln\sqrt{2}$$

3.339[6] $\displaystyle\int_0^{\pi} \exp(z\cos x)\,dx = \pi\, I_0(z)$ BI (277)(2)a

3.341 $\displaystyle\int_0^{\frac{\pi}{2}} \exp(-p\tan x)\,dx = \operatorname{ci}(p)\sin p - \operatorname{si}(p)\cos(p)$ $[p>0]$ BI (271)(2)a

3.342[11] $\displaystyle\int_0^{1} \exp(-px\ln x)\,dx = \int_0^{1} x^{-px}\,dx = \sum_{k=1}^{\infty}\frac{p^{k-1}}{k^k}$ BI (29)(1)

3.35 Combinations of exponentials and rational functions

3.351

1.[8] $\displaystyle\int_0^{u} x^n e^{-\mu x}\,dx = \frac{n!}{\mu^{n+1}} - e^{-u\mu}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1}\,\gamma(n+1,\mu u)$

$[u>0,\quad \operatorname{Re}\mu>0, n=0,1,2,\ldots]$

ET I 134(5)

2.[11] $\displaystyle\int_u^{\infty} x^n e^{-\mu x}\,dx = e^{-u\mu}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1}\,\Gamma(n+1,\mu u)$

$[u>0,\quad \operatorname{Re}\mu>0, n=0,1,2,\ldots]$

ET I 33(4)

3. $\displaystyle\int_0^{\infty} x^n e^{-\mu x}\,dx = n!\mu^{-n-1}$ $[\operatorname{Re}\mu>0]$ ET I 133(3)

4. $\displaystyle\int_u^{\infty}\frac{e^{-px}\,dx}{x^{n+1}} = (-1)^{n+1}\frac{p^n\operatorname{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n}\sum_{k=0}^{n-1}\frac{(-1)^k p^k u^k}{n(n-1)\ldots(n-k)}$

$[p>0]$ NT 21(3)

5. $\displaystyle\int_1^{\infty}\frac{e^{-\mu x}\,dx}{x} = -\operatorname{Ei}(-\mu)$ $[\operatorname{Re}\mu>0]$ BI (104)(10)

6. $\displaystyle\int_{-\infty}^{u}\frac{e^x}{x}\,dx = \operatorname{li}(e^u) = \operatorname{Ei}(u)$ $[u<0]$

7.[9] $\displaystyle\int_0^{u} x e^{-\mu x}\,dx = \frac{1}{\mu^2} - \frac{1}{\mu^2}e^{-\mu u}(1+\mu u)$ $[u>0]$

8.[11] $\displaystyle\int_0^{u} x^2 e^{-\mu x}\,dx = \frac{2}{\mu^3} - \frac{1}{\mu^3}e^{-\mu u}\left(2+2\mu u+\mu^2u^2\right)$ $[u>0]$

9.[7] $\displaystyle\int_0^{u} x^3 e^{-\mu x}\,dx = \frac{6}{\mu^4} - \frac{1}{\mu^4}e^{-\mu u}\left(6+6\mu u+3\mu^2u^2+\mu^3u^3\right)$

$[u>0]$

3.352

1. $\displaystyle\int_0^{u}\frac{e^{-\mu x}\,dx}{x+\beta} = e^{\mu\beta}\left[\operatorname{Ei}(-\mu u-\mu\beta) - \operatorname{Ei}(-\mu\beta)\right]$ $[u\geq 0,\quad |\arg\beta|<\pi]$ ET II 217(12)

2. $\displaystyle\int_u^{\infty}\frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu}\operatorname{Ei}(-\mu u-\mu\beta)$ $[u\geq 0,\quad |\arg(u+\beta)|<\pi,\quad \operatorname{Re}\mu>0]$

ET I 134(6), JA

3. $$\int_u^v \frac{e^{-\mu x}\,dx}{x+\alpha} = e^{\alpha\mu}\left\{\mathrm{Ei}[-(\alpha+v)\mu] - \mathrm{Ei}[-(\alpha+u)\mu]\right\} \qquad [-\alpha < n, \text{ and } -\alpha > v, \quad \mathrm{Re}\,\mu > 0]$$
ET I 134 (7)

4. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu}\,\mathrm{Ei}(-\mu\beta) \qquad [|\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0]$$
ET II 217(11)

5.[7] $$\int_u^\infty \frac{e^{-px}\,dx}{a-x} = e^{-pa}\,\mathrm{Ei}(pa-pu)$$
$[p > 0, \quad a < u;$ for $a > u,$ one should replace $\mathrm{Ei}(pa-pu)$ in this formula with $\overline{\mathrm{Ei}}(pa-pu)]$
ET II 251(37)

6.[8] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a-x} = e^{-\mu a}\,\mathrm{Ei}(a\mu)$$
$[a < 0, \quad \mathrm{Re}\,\mu > 0]$ BI (91)(4)

7. $$\int_{-\infty}^\infty \frac{e^{ipx}\,dx}{x-a} = i\pi e^{iap} \qquad [p > 0]$$
ET II 251(38)

3.353

1. $$\int_u^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = e^{-u\mu}\sum_{k=1}^{n-1}\frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\,\mathrm{Ei}[-(u+\beta)\mu]$$
$[n \geq 2, \quad |\arg(u+\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0]$
ET I 134(10)

2.[7] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(k-1)!(-\mu)^{n-k-1}\beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu)$$
$[n \geq 2, \quad |\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0]$
ET I 134(9), BI (92)(2)

3. $$\int_0^\infty \frac{e^{-px}\,dx}{(a+x)^2} = pe^{\alpha p}\,\mathrm{Ei}(-ap) + \frac{1}{a} \qquad [p > 0, \quad a > 0]$$
LI (281)(28), LI (281)(29)

4. $$\int_0^1 \frac{xe^x}{(1+x)^2}\,dx = \frac{e}{2} - 1.$$
BI (80)(6)

5.[7] $$\int_0^\infty \frac{x^n e^{-\mu x}}{x+\beta}\,dx = (-1)^{n-1}\beta^n e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu) + \sum_{k=1}^{n}(k-1)!(-\beta)^{n-k}\mu^{-k}$$
$[|\arg\beta| < \pi, \quad \mathrm{Re}\,\mu > 0]$
BI (91)(3)a, LET I 135(11)

3.354

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2+x^2} = \frac{1}{\beta}\left[\mathrm{ci}(\beta\mu)\sin\beta\mu - \mathrm{si}(\beta\mu)\cos\beta\mu\right] \qquad [\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > 0]$$
BI (91)(7)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2+x^2} = -\,\mathrm{ci}(\beta\mu)\cos\beta\mu - \mathrm{si}(\beta\mu)\sin\beta\mu \qquad [\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > 0]$$
BI (91)(8)

3.[7] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2-x^2} = \frac{1}{2\beta}\left[e^{-\beta\mu}\,\mathrm{Ei}(\beta\mu) - e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu)\right] \qquad [|\arg(\pm\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0]$$
BI (91)(14)

4. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2-x^2} = \frac{1}{2}\left[e^{-\beta\mu}\,\mathrm{Ei}(\beta\mu) + e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu)\right]$$
$\left[|\arg(\pm\beta)| < \pi, \quad \mathrm{Re}\,\mu > 0;\ \text{for } \beta > 0 \text{ one should replace } \mathrm{Ei}(\beta\mu) \text{ in this formula with } \overline{\mathrm{Ei}}(\beta\mu)\right]$

BI (91)(15)

5.[8] $$\int_{-\infty}^\infty \frac{e^{-ipx}\,dx}{a^2+x^2} = \frac{\pi}{a}e^{-|ap|} \qquad [a \neq 0, \quad p \text{ real}]$$ ET I 118(1)a

3.355

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\left(\beta^2+x^2\right)^2} = \frac{1}{2\beta^3}\left\{\mathrm{ci}(\beta\mu)\sin\beta\mu - \mathrm{si}(\beta\mu)\cos\beta\mu\right\} - \beta\mu\left[\mathrm{ci}(\beta\mu)\cos\beta\mu + \mathrm{si}(\beta\mu)\sin\beta\mu\right]$$

LI (92)(6)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\left(\beta^2+x^2\right)^2} = \frac{1}{2\beta^2}\left\{-\beta\mu\left[\mathrm{ci}(\beta\mu)\sin\beta\mu - \mathrm{si}(\beta\mu)\cos\beta\mu\right]\right\}$$
$[\mathrm{Re}\,\beta > 0, \quad \mathrm{Re}\,\mu > 0]$ BI (92)(7)

3.[3] $$\int_0^\infty \frac{e^{-px}\,dx}{\left(a^2-x^2\right)^2} = \frac{1}{4a^3}\left[(ap-1)e^{ap}\,\mathrm{Ei}(-ap) + (1+ap)e^{-ap}\,\mathrm{Ei}(ap)\right]$$
$\left[\mathrm{Im}\left(a^2\right) > 0, \quad p > 0\right]$ BI (92)(8)

4.[3] $$\int_0^\infty \frac{xe^{-px}\,dx}{\left(a^2-x^2\right)^2} = \frac{1}{4a^2}\left\{-2 + ap\left[e^{-ap}\,\mathrm{Ei}(ap) - e^{ap}\,\mathrm{Ei}(-ap)\right]\right\}$$
$\left[\mathrm{Im}\left(a^2\right) > 0, \quad p > 0\right]$ LI (92)(9)

3.356

1. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2+x^2}\,dx = (-1)^{n-1}a^{2n}\left[\mathrm{ci}(ap)\cos ap + \mathrm{si}(ap)\sin ap\right] + \frac{1}{p^{2n}}\sum_{k=1}^n (2n-2k+1)!\left(-a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(12)

2. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2+x^2}\,dx = (-1)^n a^{2n-1}\left[\mathrm{ci}(ap)\sin ap - \mathrm{si}(ap)\cos ap\right] + \frac{1}{p^{2n-1}}\sum_{k=1}^n (2n-2k)!\left(-a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(11)

3. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2-x^2}\,dx = \frac{1}{2}a^{2n}\left[e^{ap}\,\mathrm{Ei}(-ap) + e^{-ap}\,\mathrm{Ei}(ap)\right] - \frac{1}{p^{2n}}\sum_{k=1}^n (2n-2k+1)!\left(a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(17)

4. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2-x^2}\,dx = \frac{1}{2}a^{2n-1}\left[e^{-ap}\,\mathrm{Ei}(ap) - e^{ap}\,\mathrm{Ei}(-ap)\right] - \frac{1}{p^{2n-1}}\sum_{k=1}^n (2n-2k)!\left(a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(16)

3.357

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a^2}\left\{\operatorname{ci}(a\mu)\,(\sin a\mu+\cos a\mu) + \operatorname{si}(a\mu)\,(\sin a\mu-\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(18)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a}\left\{\operatorname{ci}(a\mu)\,(\sin a\mu-\cos a\mu) - \operatorname{si}(a\mu)\,(\sin a\mu+\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(19)

3. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2}\left\{-\operatorname{ci}(a\mu)\,(\sin a\mu+\cos a\mu) - \operatorname{si}(a\mu)\,(\sin a\mu-\cos a\mu) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(20)

4. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a^2}\left\{\operatorname{ci}(a\mu)\,(\sin a\mu-\cos a\mu) - \operatorname{si}(a\mu)\,(\sin a\mu+\cos a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(21)

5. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a}\left\{-\operatorname{ci}(a\mu)\,(\sin a\mu+\cos a\mu) - \operatorname{si}(a\mu)\,(\sin a\mu-\cos a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(22)

6. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2}\left\{\operatorname{ci}(a\mu)\,(\cos a\mu-\sin a\mu) + \operatorname{si}(a\mu)\,(\cos a\mu+\sin a\mu) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(23)

3.358

1. $$\int_0^\infty \frac{e^{-px}}{a^4-x^4}\,dx = \frac{1}{4a^3}\left\{e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) + 2\operatorname{ci}(ap)\sin ap - 2\operatorname{si}(ap)\cos ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(18)

2. $$\int_0^\infty \frac{xe^{-px}\,dx}{a^4-x^4} = \frac{1}{4a^2}\left\{e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) - 2\operatorname{ci}(ap)\cos ap - 2\operatorname{si}(ap)\sin ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(19)

3. $$\int_0^\infty \frac{x^2e^{-px}\,dx}{a^4-x^4} = \frac{1}{4a}\left\{e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) - 2\operatorname{ci}(ap)\sin ap + 2\operatorname{si}(ap)\cos ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(20)

4. $$\int_0^\infty \frac{x^3e^{-px}\,dx}{a^4-x^4} = \frac{1}{4}\left\{e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) + 2\operatorname{ci}(ap)\cos ap + 2\operatorname{si}(ap)\sin ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(21)

5. $$\int_0^\infty \frac{x^{4n}e^{-px}}{a^4-x^4}\,dx = \frac{1}{4}a^{4n-3}\left[e^{-ap}\,\mathrm{Ei}(ap) - e^{ap}\,\mathrm{Ei}(-ap) + 2\,\mathrm{ci}(ap)\sin ap - 2\,\mathrm{si}(ap)\cos ap\right] - \frac{1}{p^{4n-3}}\sum_{k=1}^{n}(4n-4k)!\left(a^4p^4\right)^{k-1}$$
$[p>0, \quad a>0]$ BI (91)(22)

6. $$\int_0^\infty \frac{x^{4n+1}e^{-px}}{a^4-x^4}\,dx = \frac{1}{4}a^{4n-2}\left[e^{ap}\,\mathrm{Ei}(-ap) + e^{-ap}\,\mathrm{Ei}(ap) - 2\,\mathrm{ci}(ap)\cos ap - 2\,\mathrm{si}(ap)\sin ap\right] - \frac{1}{p^{4n-2}}\sum_{k=1}^{n}(4n-4k+1)!\left(a^4p^4\right)^{k-1}$$
$[p>0, \quad a>0]$ BI (91)(23)

7. $$\int_0^\infty \frac{x^{4n+2}e^{-px}}{a^4-x^4}\,dx = \frac{1}{4}a^{4n-1}\left[e^{-ap}\,\mathrm{Ei}(ap) - e^{ap}\,\mathrm{Ei}(-ap) - 2\,\mathrm{ci}(ap)\sin ap + 2\,\mathrm{si}(ap)\cos ap\right] - \frac{1}{p^{4n-1}}\sum_{k=1}^{n}(4n-4k+2)!\left(a^4p^4\right)^{k-1}$$
$[p>0, \quad a>0]$ BI (91)(24)

8. $$\int_0^\infty \frac{x^{4n+3}e^{-px}}{a^4-x^4}\,dx = \frac{1}{4}a^{4n}\left[e^{ap}\,\mathrm{Ei}(-ap) + e^{-ap}\,\mathrm{Ei}(ap) + 2\,\mathrm{ci}(ap)\cos ap + 2\,\mathrm{si}(ap)\sin ap\right] - \frac{1}{p^{4n}}\sum_{k=1}^{n}(4n-4k+3)!\left(a^4p^4\right)^{k-1}$$
$[p>0, \quad a>0]$ BI (91)(25)

3.359 $$\int_{-\infty}^\infty \frac{(i-x)^n}{(i+x)^n}\frac{e^{-ipx}}{i+x^2}\,dx = (-1)^{n-1}2\pi p e^{-p}\,L_{n-1}(2p) \qquad \text{for } p>0;$$
$$= 0 \qquad \text{for } p<0.$$
ET I 118(2)

3.36–3.37 Combinations of exponentials and algebraic functions

3.361

1.[8] $$\int_0^u \frac{e^{-qx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{q}}\,\Phi\left(\sqrt{qu}\right)$$ $[q>0]$

2.[8] $$\int_0^\infty \frac{e^{-qx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{q}}$$ $[q>0]$ BI(98)(10)

3.[8] $$\int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}}\,dx = e^q\sqrt{\frac{\pi}{q}}$$ $[q>0]$ BI (104)(16)

3.362

1. $$\int_1^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x-1}} = \sqrt{\frac{\pi}{\mu}}e^{-\mu}$$ $[\mathrm{Re}\,\mu>0]$ BI (104)(11)a

2. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{\mu}}e^{\beta\mu}\left[1-\Phi\left(\sqrt{\beta\mu}\right)\right]$$ $[\mathrm{Re}\,\mu>0, \quad |\arg\beta|<\pi]$ ET I 135(18)

3.363

1. $$\int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x}\,dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u}\,[1-\Phi(\sqrt{u\mu})]$$ $[\operatorname{Re}\mu > 0]$ ET I 136(23)

2. $$\int_u^\infty \frac{e^{-\mu x}\,dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}\,[1-\Phi(\sqrt{u\mu})]$$ $[u>0, \quad \operatorname{Re}\mu \geq 0]$ ET I 136(26)

3.364

1. $$\int_0^2 \frac{e^{-px}\,dx}{\sqrt{x(2-x)}} = \pi e^{-p}\,I_0(p)$$ $[p>0]$ GW (312)(7a)

2. $$\int_{-1}^1 \frac{e^{2x}\,dx}{\sqrt{1-x^2}} = \pi\,I_0(2)$$ BI (277)(2)a

3. $$\int_0^\infty \frac{e^{-px}\,dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}}\,K_0\left(\frac{ap}{2}\right)$$ $[a>0, \quad p>0]$ GW (312)(8a)

3.365

1. $$\int_0^u \frac{xe^{-\mu x}\,dx}{\sqrt{u^2-x^2}} = \frac{\pi u}{2}\,[\mathbf{L}_1(\mu u) - I_1(\mu u)] + u$$ $[u>0, \quad \operatorname{Re}\mu > 0]$ ET I 136(28)

2. $$\int_u^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2-u^2}} = u\,K_1(u\mu)$$ $[u>0, \quad \operatorname{Re}\mu > 0]$ ET I 136(29)

3.366

1. $$\int_0^{2u} \frac{(u-x)e^{-\mu x}\,dx}{\sqrt{2ux-x^2}} = \pi u e^{-u\mu}\,I_1(u\mu)$$ $[\operatorname{Re}\mu > 0]$ ET I 136(31)

2. $$\int_0^\infty \frac{(x+\beta)e^{-\mu x}\,dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu}\,K_1(\beta\mu)$$ $[\operatorname{Re}\mu > 0, \quad |\arg\beta| < \pi]$ ET I 136(30)

3. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2}\,[\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \beta$$ $\left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 136(27)

3.367 $$\int_0^\infty \frac{e^{-\mu x}\,dx}{(1+\cos t + x)\sqrt{x^2+2x}} = \frac{\exp\left(2\mu\cos^2\frac{t}{2}\right)}{\sin t}\left(t - \sin t\int_0^u K_0(v)e^{-v\cos t}\,dv\right)$$ $[\operatorname{Re}\mu > 0]$ ET I 136(33)

3.368 $$\int_0^\infty \frac{e^{-\mu x}\,dx}{x+\sqrt{x^2+\beta^2}} = \frac{\pi}{2\beta\mu}\,[\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)] - \frac{1}{\beta^2\mu^2}$$ $\left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 136(32)

3.369[11] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu}e^{a\mu}\,(1-\Phi(\sqrt{a\mu}))$$ $[|\arg a| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 135(20)

3.371[11] $$\int_0^\infty x^{n-\frac{1}{2}}e^{-\mu x}\,dx = \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\mu^{-n-\frac{1}{2}}$$
$$= \sqrt{\pi}2^{-n}\mu^{-n-1/2}(2n-1)!!$$ $[n \geq 0]$

$[\operatorname{Re}\mu > 0]$ ET I 135(17)

3.372 $$\int_0^\infty x^{n-\frac{1}{2}}(2+x)^{n-\frac{1}{2}}e^{-px}\,dx = \frac{(2n-1)!!}{p^n}e^p\,K_n(p) \qquad [p>0, \quad n=0,1,2,\ldots]$$ GW (312)(8)

3.373 $$\int_0^\infty \left[\left(x+\sqrt{x^2+\beta^2}\right)^n + \left(x-\sqrt{x^2+\beta^2}\right)^n\right] e^{-\mu x}\,dx = 2\beta^{n+1}\,O_n(\beta\mu)$$
$[\operatorname{Re}\mu>0]$ WA 05(1)

3.374

1. $$\int_0^\infty \frac{\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \frac{1}{2}\left[S_n(\mu) - \pi\,\mathbf{E}_n(\mu) - \pi\,Y_n(\mu)\right]$$
$[\operatorname{Re}\mu>0]$ ET I 37(35)

2. $$\int_0^\infty \frac{\left(x-\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = -\frac{1}{2}\left[S_n(\mu) + \pi\,\mathbf{E}_n(\mu) + \pi\,Y_n(\mu)\right]$$
$[\operatorname{Re}\mu>0]$ ET I 137(36)

3.38–3.39 Combinations of exponentials and arbitrary powers

3.381

1. $$\int_0^u x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\,\gamma(\nu,\mu u) \qquad [\operatorname{Re}\nu>0]$$ EH I 266(22), EH II 133(1)

2. $$\int_0^u x^{p-1}e^{-x}\,dx = \sum_{k=0}^\infty (-1)^k\frac{u^{p+k}}{k!(p+k)} = e^{-u}\sum_{k=0}^\infty \frac{u^{p+k}}{p(p+1)\ldots(p+k)}$$
AD 6.705

3.[8] $$\int_u^\infty x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\,\Gamma(\nu,\mu u) \qquad [u>0, \quad \operatorname{Re}\mu>0]$$
EH I 256(21), EH II 133(2)

4. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\mu^\nu}\,\Gamma(\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ FI II 779

5. $$\int_0^\infty x^{\nu-1}e^{-(p+iq)x}\,dx = \Gamma(\nu)\left(p^2+q^2\right)^{-\frac{\nu}{2}}\exp\left(-i\nu\arctan\frac{q}{p}\right)$$
$[p>0, \quad \operatorname{Re}\nu>0 \text{ and } p=0, \quad 0<\operatorname{Re}\nu<1]$ EH I 12(32)

6. $$\int_u^\infty \frac{e^{-x}}{x^\nu}\,dx = u^{-\frac{\nu}{2}}e^{-\frac{u}{2}}\,W_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(u) \qquad [u>0]$$ WH

7. $$\int_0^\infty x^{k-1}e^{i\mu x}\,dx = \frac{\Gamma(k)}{(-i\mu)^k} \qquad [0<\operatorname{Re}(k)<1, \quad \mu\neq 0]$$
GH2 62 (313.14)

8.* $$\int_0^u x^m e^{-\beta x^n}\,dx = \frac{\gamma\left(v,\beta u^n\right)}{n\beta^v} \qquad v=\frac{m+1}{n} \qquad [u>0, \quad \operatorname{Re}v>0, \quad \operatorname{Re}n>0, \quad \operatorname{Re}\beta>0]$$

9.* $$\int_u^\infty x^m e^{-\beta x^n}\,dx = \frac{\Gamma\left(v,\beta u^n\right)}{n\beta^v} \qquad v=\frac{m+1}{n} \qquad [u>0, \quad \operatorname{Re}v>0, \quad \operatorname{Re}n>0, \quad \operatorname{Re}\beta>0]$$

10.* $$\int_0^\infty x^m e^{-\beta x^n}\,dx = \frac{\gamma\left(v,\beta u^n\right)+\Gamma\left(v,\beta u^n\right)}{n\beta^v}$$

$$v=\frac{m+1}{n} \qquad [u>0, \quad \operatorname{Re} v>0, \quad \operatorname{Re} n>0, \quad \operatorname{Re}\beta>0] \qquad \text{See also 3.326 1}$$

11.* $$\int_{-\infty}^\infty x^{2m} e^{-\beta x^{2n}}\,dx = 2\int_0^\infty x^{2m} e^{-\beta x^{2n}}\,dx = \frac{2\left(\gamma\left(v,\beta u^n\right)+\Gamma\left(v,\beta u^n\right)\right)}{n\beta^v} = \frac{\Gamma(v)}{n\beta^v}$$

$$v=\frac{2m+1}{2n} \qquad [u>0, \quad \operatorname{Re} v>0, \quad \operatorname{Re} n>0, \quad \operatorname{Re}\beta>0]$$

3.382

1.[6] $$\int_0^u (u-x)^\nu e^{-\mu x}\,dx = (-\mu)^{-\nu-1} e^{-u\mu}\,\gamma(\nu+1,-u\mu) \qquad [\operatorname{Re}\nu>-1, \quad u>0]$$ ET I 137(6)

2. $$\int_u^\infty (x-u)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{-u\mu}\,\Gamma(\nu+1) \qquad [u>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>0]$$ ET I 137(5), ET II 202(11)

3. $$\int_0^\infty (1+x)^{-\nu} e^{-\mu x}\,dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}}\, W_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(\mu) \qquad [\operatorname{Re}\mu>0]$$ WH

4. $$\int_0^\infty (x+\beta)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{\beta\mu}\,\Gamma(\nu+1,\beta\mu) \qquad [|\arg\beta|<\pi, \quad \operatorname{Re}\mu>0]$$ ET I 137(4), ET II 233(10)

5. $$\int_0^u (a+x)^{\mu-1} e^{-x}\,dx = e^a[\gamma(\mu,a+u)-\gamma(\mu,a)] \qquad [\operatorname{Re}\mu>0]$$ EH II 139

6. $$\int_{-\infty}^\infty (\beta+ix)^{-\nu} e^{-ipx}\,dx = 0 \qquad [\text{for } p>0]$$

$$= \frac{2\pi(-p)^{\nu-1}e^{\beta p}}{\Gamma(\nu)} \qquad [\text{for } p<0]$$

$$[\operatorname{Re}\nu>0, \quad \operatorname{Re}\beta>0]$$ ET I 118(4)

7. $$\int_{-\infty}^\infty (\beta-ix)^{-\nu} e^{-ipx}\,dx = \frac{2\pi p^{\nu-1}e^{-\beta p}}{\Gamma(\nu)} \qquad [\text{for } p>0]$$

$$= 0 \qquad [\text{for } p<0]$$

$$[\operatorname{Re}\nu>0, \quad \operatorname{Re}\beta>0]$$ ET I 118(3)

3.383

1.[11] $$\int_0^u x^{\nu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \mathrm{B}(\mu,\nu)u^{\mu+\nu-1}\ {}_1F_1(\nu;\mu+\nu;\beta u)$$

$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ ET II 187(14)

2.[11] $$\int_0^u x^{\mu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \sqrt{\pi}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right)\Gamma(\mu)\, I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

$$[\operatorname{Re}\mu>0]$$ ET II 187(13)

3. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1} e^{-\beta x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right) K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\beta u>0]$$ ET II 202(12)

4.[11] $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x}\,dx = \beta^{-\frac{\mu+\nu}{2}}u^{\frac{\mu+\nu-2}{2}}\,\Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right)W_{\frac{\nu-\mu}{2},\frac{1-\mu-\nu}{2}}(\beta u)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta u>0]$ ET II 202(13)

5.[11] $$\int_0^\infty e^{-px}x^{q-1}(1+ax)^{-\nu}\,dx$$
$$= \frac{\pi^2}{p^q\,\Gamma(\nu)\sin[\pi(q-\nu)]}\left[\left(\frac{p}{a}\right)^\nu\frac{L_{-\nu}^{\nu-q}\left(\frac{p}{a}\right)}{\sin(\pi\nu)\,\Gamma(1-q)}-\left(\frac{p}{a}\right)^q\frac{L_{-q}^{q-\nu}\left(\frac{p}{a}\right)}{\sin(\pi q)\,\Gamma(1-\nu)}\right]\qquad[\nu\neq\pm1,\pm2,\ldots]$$
$$=\frac{\Gamma(q)}{p^q}\qquad[\nu=0]$$
$[\operatorname{Re}q>0,\quad \operatorname{Re}p>0,\quad \operatorname{Re}a>0]$

6. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu+\frac12}e^{-\mu x}\,dx = 2^{\nu-\frac12}\,\Gamma(\nu)\mu^{-\frac12}e^{\frac{\beta\mu}{2}}\,D_{1-2\nu}\left(\sqrt{2\beta\mu}\right)$$
$[|\arg\beta|<\pi,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\mu\geq0,\quad \mu\neq0]$ ET I 39(20), EH II 119(2)a

7. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu-\frac12}e^{-\mu x}\,dx = 2^{\nu}\,\Gamma(\nu)\beta^{-\frac12}e^{\frac{\beta\mu}{2}}\,D_{-2\nu}\left(\sqrt{2\beta\mu}\right)$$
$[|\arg\beta|<\pi,\quad \operatorname{Re}\nu>0,\quad \operatorname{Re}\mu\geq0]$
ET I 139(21), EH II 119(1)a

8. $$\int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-\frac12}e^{\frac{\beta\mu}{2}}\,\Gamma(\nu)\,K_{\frac12-\nu}\left(\frac{\beta\mu}{2}\right)$$
$[|\arg\beta|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$
ET II 233(11), EH II 19(16)a, EH II 82(22)a

9. $$\int_u^\infty\frac{(x-u)^\nu e^{-\mu x}}{x}\,dx = u^\nu\,\Gamma(\nu+1)\,\Gamma(-\nu,u\mu)\qquad[u>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>0]$$
ET I 138(8)

10. $$\int_0^\infty\frac{x^{\nu-1}e^{-\mu x}}{x+\beta}\,dx = \beta^{\nu-1}e^{\beta\mu}\,\Gamma(\nu)\,\Gamma(1-\nu,\beta\mu)\qquad[|\arg\beta|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$
EH II 137(3)

3.384

1. $$\int_{-1}^1(1-x)^{\nu-1}(1+x)^{\mu-1}e^{-ipx}\,dx = 2^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)e^{ip}\;{}_1F_1(\mu;\nu+\mu;-2ip)$$
$[\operatorname{Re}\nu>0,\quad \operatorname{Re}\mu>0]$ ET I 119(13)

2. $$\int_u^v(x-u)^{2\mu-1}(v-x)^{2\nu-1}e^{-px}\,dx$$
$$=\mathrm{B}(2\mu,2\nu)(v-u)^{\mu+\nu-1}p^{-\mu-\nu}\exp\left(-p\frac{u+v}{2}\right)M_{\mu-\nu,\mu+\nu-\frac12}(vp-up)$$
$[v>u>0,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ ET I 139(23)

3. $$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\varrho-1}e^{-\mu x}\,dx$$
$$= \frac{(u+\beta)^{\nu+\varrho-1}}{\mu^{\nu+\varrho}}\exp\left[\frac{(\beta-u)\mu}{2}\right]\Gamma(2\varrho)\,W_{\nu-\varrho,\nu+\varrho-\frac{1}{2}}(u\mu+\beta\mu)$$
$[u>0,\quad |\arg(\beta+u)|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\varrho>0]$ ET I 139(22)

4. $$\int_u^\infty (x+\beta)^{\nu}(x-u)^{-\nu}e^{-\mu x}\,dx = \frac{1}{\mu}\nu\pi\operatorname{cosec}(\nu\pi)e^{-\frac{(\beta+u)\mu}{2}}\,\mathrm{k}_{2\nu}\left[\frac{(\beta+u)\mu}{2}\right]$$
$[\nu\neq 0,\quad u>0,\quad |\arg(u+\beta)|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu<1]$ ET I 139(17)

5. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu+\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{\mu}}2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\,D_{1-2\nu}\left(2\sqrt{u\mu}\right)$$
$[u>0,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$
ET I 139(18)

6. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu-\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{u}}2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\,D_{-2\nu}\left(2\sqrt{u\mu}\right)$$
$[u>0,\quad \operatorname{Re}\mu\geq 0,\quad \operatorname{Re}\nu>0]$
ET I 139(19)

7.[6] $$\int_{-\infty}^\infty (\beta-ix)^{-\mu}\,(\gamma-ix)^{-\nu}e^{-ipx}\,dx = \frac{2\pi e^{-\beta p}p^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\;{}_1F_1(\nu;\mu+\nu;(\beta-\gamma)p) \qquad [\text{for } p>0]$$
$$= 0 \qquad [\text{for } p<0]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad \operatorname{Re}(\mu+\nu)>1]$ ET I 119(10)

8.[6] $$\int_{-\infty}^\infty (\beta+ix)^{-\mu}(\gamma+ix)^{-\nu}e^{-ipx}\,dx = 0 \qquad [\text{for } p>0]$$
$$= \frac{2\pi e^{\gamma p}(-p)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\;{}_1F_1[\mu;\mu+\nu;(\beta-\gamma)p] \qquad [\text{for } p<0]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad \operatorname{Re}(\mu+\nu)>1]$ ET I 19(11)

Text

9.[6] $$\int_{-\infty}^\infty (\beta+ix)^{-2\mu}(\gamma-ix)^{-2\nu}e^{-ipx}\,dx$$
$$= 2\pi(\beta+\gamma)^{-\mu-\nu}\frac{p^{\mu+\nu-1}}{\Gamma(2\nu)}\exp\left(\frac{\beta-\gamma}{2}p\right)W_{\nu-\mu,\frac{1}{2}-\nu-\mu}(\beta p+\gamma p) \qquad [\text{for } p>0]$$
$$= 2\pi(\beta+\gamma)^{-\mu-\nu}\frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)}\exp\left(\frac{\beta-\gamma}{2}p\right)W_{\mu-\nu,\frac{1}{2}-\nu-\mu}(-\beta p-\gamma p) \qquad [\text{for } p<0]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad \operatorname{Re}(\mu+\nu)>\frac{1}{2}]$ ET I 19(12)

3.385[11] $$\int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\varrho}e^{-\mu x}\,dx = \mathrm{B}(\nu,\lambda)\Phi_1(\nu,\varrho,\lambda+\nu,-\mu,\beta)$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\nu>0,\quad |\arg(1-\beta)|<\pi]$ ET I 39(24)

3.386

1. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^{n} (\beta_k + ix)^{\nu_k} e^{-ipx}\, dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p} \beta_0^{\nu_0} \prod_{k=1}^{n} (\beta_0 + \beta_k)^{\nu_k}$$
$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n} \operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x, \quad p > 0\right]$$ ET I 118(8)

2. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^{n} (\beta_k + ix)^{\nu_k} e^{-ipx}\, dx}{\beta_0 + ix} = 0$$
$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n} \operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x, \quad p > 0\right]$$ ET I 119(9)

3.387

1.[6] $$\int_{-1}^{1} \left(1 - x^2\right)^{\nu-1} e^{-\mu x}\, dx = \sqrt{\pi} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, I_{\nu-\frac{1}{2}}(\mu)$$
$$\left[\operatorname{Re}\nu > 0, \quad |\arg\mu| < \frac{\pi}{2}\right]$$ WA 172(2)a

2.[6] $$\int_{-1}^{1} \left(1 - x^2\right)^{\nu-1} e^{i\mu x}\, dx = \sqrt{\pi} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, J_{\nu-\frac{1}{2}}(\mu)$$
$$[\operatorname{Re}\nu > 0]$$ WA 25(3), WA 48(4)a

3. $$\int_{1}^{\infty} \left(x^2 - 1\right)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, K_{\nu-\frac{1}{2}}(\mu)$$
$$\left[|\arg\mu| < \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right]$$ WA 190(4)a

4. $$\int_{1}^{\infty} \left(x^2 - 1\right)^{\nu-1} e^{i\mu x}\, dx$$
$$= i\frac{\sqrt{\pi}}{2} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, H^{(1)}_{\frac{1}{2}-\nu}(\mu) \qquad [\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(28)a
$$= -i\frac{\sqrt{\pi}}{2} \left(-\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, H^{(2)}_{\frac{1}{2}-\nu}(-\mu) \qquad [\operatorname{Im}\mu < 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(29)a

5. $$\int_{0}^{u} \left(u^2 - x^2\right)^{\nu-1} e^{\mu x}\, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[I_{\nu-\frac{1}{2}}(u\mu) + \mathbf{L}_{\nu-\frac{1}{2}}(u\mu)\right]$$
$$[u > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 188(20)a

6. $$\int_{u}^{\infty} \left(x^2 - u^2\right)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, K_{\nu-\frac{1}{2}}(u\mu)$$
$$[u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
ET II 203(17)a

7.[11] $$\int_0^\infty \left(x^2+u^2\right)^{\nu-1} e^{-\mu x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\left[\mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - Y_{\nu-\frac{1}{2}}(u\mu)\right]$$
$[|\arg u|<\pi, \quad \operatorname{Re}\mu>0]$ ET I 138(10)

3.388

1. $$\int_0^{2u} \left(2ux-x^2\right)^{\nu-1} e^{-\mu x}\,dx = \sqrt{\pi}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} e^{-u\mu}\,\Gamma(\nu)\,I_{\nu-\frac{1}{2}}(u\mu)$$
$[u>0, \quad \operatorname{Re}\nu>0]$ ET I 138(14)

2. $$\int_0^\infty \left(2\beta x+x^2\right)^{\nu-1} e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\beta\mu}\,\Gamma(\nu)\,K_{\nu-\frac{1}{2}}(\beta\mu)$$
$[|\arg\beta|<\pi, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0]$ ET I 138(13)

3. $$\int_0^\infty \left(x^2+ix\right)^{\nu-1} e^{-\mu x}\,dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\,\Gamma(\nu)\,H^{(2)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET I 138(15)

4. $$\int_0^\infty \left(x^2-ix\right)^{\nu-1} e^{-\mu x}\,dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\,\Gamma(\nu)\,H^{(1)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET I 138(16)

3.389

1. $$\int_0^u x^{2\nu-1}\left(u^2-x^2\right)^{\varrho-1} e^{\mu x}\,dx = \frac{1}{2}\,\mathrm{B}(\nu,\varrho)u^{2\nu+2\varrho-2}\;{}_1F_2\left(\nu;\frac{1}{2},\nu+\varrho;\frac{\mu^2u^2}{4}\right)$$
$$+\frac{\mu}{2}\,\mathrm{B}\left(\nu+\frac{1}{2},\varrho\right)u^{2\nu+2\varrho-1}\;{}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\nu+\varrho+\frac{1}{2};\frac{\mu^2u^2}{4}\right)$$
$[\operatorname{Re}\varrho>0, \quad \operatorname{Re}\nu>0]$ ET II 188(21)

2.[7] $$\int_0^\infty x^{2\nu-1}\left(u^2+x^2\right)^{\varrho-1} e^{-\mu x}\,dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi}\,\Gamma(1-\varrho)}\,G^{31}_{13}\left(\frac{\mu^2u^2}{4}\,\middle|\,\begin{matrix}1-\nu\\1-\varrho-\nu,0,\frac{1}{2}\end{matrix}\right)$$
$\left[|\arg u|<\frac{\pi}{2}, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0\right]$ ET II 234(15)a

3.[7] $$\int_0^u x\left(u^2-x^2\right)^{\nu-1} e^{\mu x}\,dx = \frac{u^{2\nu}}{2\nu}+\frac{\sqrt{\pi}}{2}\left(\frac{\mu}{2}\right)^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}}\,\Gamma(\nu)\left[I_{\nu+\frac{1}{2}}(\mu u)+\mathbf{L}_{\nu+\frac{1}{2}}(\mu u)\right]$$
$[\operatorname{Re}\nu>0]$ ET II 188(19)a

4. $$\int_u^\infty x\left(x^2-u^2\right)^{\nu-1} e^{-\mu x}\,dx = 2^{\nu-\frac{1}{2}}\left(\sqrt{\pi}\right)^{-1}\mu^{\frac{1}{2}-\nu}u^{\nu+\frac{1}{2}}\,\Gamma(\nu)\,K_{\nu+\frac{1}{2}}(u\mu)$$
$[\operatorname{Re}(u\mu)>0]$ ET II 203(16)a

5. $$\int_{-\infty}^\infty \frac{(ix)^{-\nu}e^{-ipx}\,dx}{\beta^2+x^2} = \pi\beta^{-\nu-1}e^{-|p|\beta}$$
$\left[|\nu|<1, \quad \operatorname{Re}\beta>0, \quad \arg ix = \frac{\pi}{2}\operatorname{sign}x\right]$ ET I 118(5)

6. $$\int_0^\infty \frac{x^\nu e^{-\mu x}}{\beta^2+x^2}\,dx = \frac{1}{2}\,\Gamma(\nu)\beta^{\nu-1}\left[\exp\left(i\mu\beta + i\frac{(\nu-1)\pi}{2}\right)\right.$$
$$\left.\times\,\Gamma(1-\nu, i\beta\mu) + \exp\left(-i\beta\mu - i\frac{(\nu-1)\pi}{2}\right)\Gamma\left(1-\nu, -i\beta\mu\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1]$ ET II 218(22)

7. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1+x^2} = \pi\operatorname{cosec}(\nu\pi)V_\nu(2\mu, 0)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 138(9)

8. $$\int_{-\infty}^\infty \frac{(\beta+ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{-p\gamma}$$
$[\operatorname{Re}\nu > -1, \quad p > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ ET I 118(6)

9.[6] $$\int_{-\infty}^\infty \frac{(\beta-ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{\gamma p}$$
$[p < 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > -1]$ ET I 118(7)

3.391 $$\int_0^\infty \left[\left(\sqrt{x+2\beta}+\sqrt{x}\right)^{2\nu} - \left(\sqrt{x+2\beta}-\sqrt{x}\right)^{2\nu}\right]e^{-\mu x}\,dx = 2^{\nu+1}\frac{\nu}{\mu}\beta^\nu e^{\beta\mu}\,K_\nu(\beta\mu)$$
$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 140(30)

3.392

1. $$\int_0^\infty \left(x+\sqrt{1+x^2}\right)^\nu e^{-\mu x}\,dx = \frac{1}{\mu}\,S_{1,\nu}(\mu) + \frac{\nu}{\mu}\,S_{0,\nu}(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(25)

2. $$\int_0^\infty \left(\sqrt{1+x^2}-x\right)^\nu e^{-\mu x}\,dx = \frac{1}{\mu}\,S_{1,\nu}(\mu) - \frac{\nu}{\mu}\,S_{0,\nu}(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(26)

3. $$\int_0^\infty \frac{\left(x+\sqrt{1+x^2}\right)^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \pi\operatorname{cosec}\nu\pi\,[\mathbf{J}_{-\nu}(\mu) - J_{-\nu}(\mu)]$$
$[\operatorname{Re}\mu > 0]$ ET I 140(27), EH II 35(33)

4. $$\int_0^\infty \frac{\left(\sqrt{1+x^2}-x\right)^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = S_{0,\nu}(\mu) - \nu\,S_{-1,\nu}(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(28)

3.393 $$\int_0^\infty \frac{\left(x+\sqrt{x^2+4\beta^2}\right)^{2\nu}}{\sqrt{x^3+4\beta^2x}}e^{-\mu x}\,dx$$
$$= \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2}\beta^{2\nu}}\left[J_{\nu+1/4}(\beta\mu)\,Y_{\nu-1/4}(\beta\mu) - J_{\nu-1/4}(\beta\mu)\,Y_{\nu+1/4}(\beta\mu)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ ET I 140(33)

3.394 $$\int_0^\infty \frac{\left(1+\sqrt{1+x^2}\right)^{\nu+1/2}}{x^{\nu+1}\sqrt{1+x^2}}e^{-\mu x}\,dx = \sqrt{2}\,\Gamma(-\nu)\,D_\nu\left(\sqrt{2i\mu}\right)D_\nu\left(\sqrt{-2i\mu}\right)$$
$[\operatorname{Re}\mu \geq 0, \quad \operatorname{Re}\nu < 0]$ ET I 140(32)

3.395

1. $$\int_1^\infty \frac{\left(\sqrt{x^2-1}+x\right)^\nu + \left(\sqrt{x^2-1}+x\right)^{-\nu}}{\sqrt{x^2-1}} e^{-\mu x}\,dx = 2\,K_\nu(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(29)

2. $$\int_1^\infty \frac{\left(x+\sqrt{x^2-1}\right)^{2\nu} + \left(x-\sqrt{x^2-1}\right)^{2\nu}}{\sqrt{x\,(x^2-1)}} e^{-\mu x}\,dx = \sqrt{\frac{2\mu}{\pi}}\,K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(34)

3. $$\int_0^\infty \frac{\left(x+\sqrt{x^2+1}\right)^\nu + \cos\nu\pi\left(x+\sqrt{x^2+1}\right)^{-\nu}}{\sqrt{x^2+1}} e^{-\mu x}\,dx = -\pi\left[\mathbf{E}_\nu(\mu) + Y_\nu(\mu)\right]$$
$[\operatorname{Re}\mu > 0]$ EH II 35(34)

3.41–3.44 Combinations of rational functions of powers and exponentials

3.411

1. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}-1} = \frac{1}{\mu^\nu}\,\Gamma(\nu)\,\zeta(\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 1]$ FI II 792a

2. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}$$
$[n = 1, 2, \ldots]$ FI II 721a

3. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}+1} = \frac{1}{\mu^\nu}\left(1-2^{1-\nu}\right)\Gamma(\nu)\,\zeta(\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 792a, WH

4. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = \left(1-2^{1-2n}\right)\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n}$$
$[n = 1, 2, \ldots]$ BI(83)(2), EH I 39(25)

5. $$\int_0^{\ln 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12}$$
BI (104)(5)

6.[8] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-\beta e^{-x}}\,dx = \Gamma(\nu)\sum_{n=0}^\infty (\mu+n)^{-\nu}\beta^n = \Gamma(\nu)\,\Phi(\beta,\nu,\mu)$$
$[\operatorname{Re}\mu > 0$ and either $|\beta| \le 1, \quad \beta \ne 1, \quad \operatorname{Re}\nu > 0$; or $\beta = 1, \quad \operatorname{Re}\nu > 1]$ EH I 27(3)

7.[11] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-e^{-\beta x}}\,dx = \frac{1}{\beta^\nu}\,\Gamma(\nu)\,\zeta\left(\nu, \frac{\mu}{\beta}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 1]$ ET I 144(10)

8. $$\int_0^\infty \frac{x^{n-1}e^{-px}}{1+e^x}\,dx = (n-1)!\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(p+k)^n}$$
$[p > -1; \quad n = 1, 2, \ldots]$ BI (83)(9)

9. $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \frac{\pi^2}{6} - 1$$
(cf. **4.231** 3) BI (82)(1)

10. $$\int_0^\infty \frac{xe^{-2x}\,dx}{e^{-x}+1} = 1 - \frac{\pi^2}{12}$$
(cf. **4.251** 6) BI (82)(2)

11. $\displaystyle\int_0^\infty \frac{xe^{-3x}}{e^{-x}+1}\,dx = \frac{\pi^2}{12} - \frac{3}{4}$ (cf. **4.251** 5) BI (82)(3)

12.[11] $\displaystyle\int_0^\infty \frac{xe^{-(2n-1)x}}{1+e^x}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2}$ (cf. **4.251** 6) BI (82)(5)

13.[11] $\displaystyle\int_0^\infty \frac{xe^{-2nx}}{1+e^x}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}$ (cf. **4.251** 5) BI (82)(4)

14.[7] $\displaystyle\int_0^\infty \frac{x^2e^{-nx}}{1-e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{1}{k^3} = 2\left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3}\right)$ $[n = 1, 2, \ldots]$ (cf. **4.261** 12)
BI (82)(9)

15.[7] $\displaystyle\int_0^\infty \frac{x^2e^{-nx}}{1+e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1}\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^{n-1} \frac{(-1)^k}{k^3}\right)$
$[n = 1, 2, \ldots]$ (cf. **4.261** 11)
LI (82)(10)

16. $\displaystyle\int_{-\infty}^\infty \frac{x^2e^{-\mu x}}{1+e^{-x}}\,dx = \pi^3 \csc^3 \mu\pi \left(2 - \sin^2 \mu\pi\right)$ $[0 < \operatorname{Re}\mu < 1]$ ET I 120(17)a

17. $\displaystyle\int_0^\infty \frac{x^3e^{-nx}}{1-e^{-x}}\,dx = \frac{\pi^4}{15} - 6\sum_{k=1}^{n-1} \frac{1}{k^4}$ (cf. **4.262** 5) BI (82)(12)

18.[11] $\displaystyle\int_0^\infty \frac{x^3e^{-nx}}{1+e^{-x}}\,dx = 6\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^4} = (-1)^{n+1}\left(\frac{7}{120}\pi^4 + 6\sum_{k=1}^{n-1} \frac{(-1)^k}{k^4}\right)$
(cf. **4.262** 4) LI (82)(13)

19.[9] $\displaystyle\int_0^\infty e^{-px}\left(e^{-x}-1\right)^n \frac{dx}{x} = -\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(p+n-k)$ LI (89)(10)

20.[9] $\displaystyle\int_0^\infty e^{-px}\left(e^{-x}-1\right)^n \frac{dx}{x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(p+n-k)\ln(p+n-k)$ LI (89)(15)

21. $\displaystyle\int_0^\infty x^{n-1}\frac{1-e^{-mx}}{1-e^x}\,dx = (n-1)!\sum_{k=1}^{m} \frac{1}{k^n}$ (cf. **4.272** 11) LI (83)(8)

22.[7] $\displaystyle\int_0^\infty \frac{x^{p-1}}{e^{rx}-q}\,dx = \frac{1}{qr^p}\,\Gamma(p)\sum_{k=1}^{\infty} \frac{q^k}{k^p} = \Gamma(p)r^{-p}\,\Phi(q,p,1)$
$[p > 0, \quad r > 0, \quad -1 < q < 1]$
BI (83)(5)

23. $\displaystyle\int_{-\infty}^\infty \frac{xe^{\mu x}\,dx}{\beta+e^x} = \pi\beta^{\mu-1}\operatorname{cosec}(\mu\pi)\left[\ln\beta - \pi\cot(\mu\pi)\right]$ $[|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 1]$
BI (101)(5), ET I 120(16)a

24. $\displaystyle\int_{-\infty}^\infty \frac{xe^{\mu x}}{e^{\nu x}-1}\,dx = \left(\frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu}\right)^2$ $[\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ (cf. **4.254** 2)
LI (101)(3)

25. $$\int_0^\infty x\frac{1+e^{-x}}{e^x-1}\,dx=\frac{\pi^2}{3}-1$$ (cf. **4.231** 4) BI (82)(6)

26. $$\int_0^\infty x\frac{1-e^{-x}}{1+e^{-3x}}e^{-x}\,dx=\frac{2\pi^2}{27}$$ LI (82)(7)a

27. $$\int_0^\infty \frac{1-e^{-\mu x}}{1+e^x}\frac{dx}{x}=\ln\left[\frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\sqrt{\pi}\right]$$ $[\operatorname{Re}\mu>-1]$ BI (93)(4)

28. $$\int_0^\infty \frac{e^{-\nu x}-e^{-\mu x}}{e^{-x}+1}\frac{dx}{x}=\ln\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}$$ $[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ BI (93)(6)

29. $$\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1+e^{rx}}\frac{dx}{x}=\ln\left[\tan\frac{p\pi}{2r}\cot\frac{q\pi}{2r}\right]$$ $[|r|>|p|,\quad |r|>|q|,\quad rp>0,\quad rq>0]$ BI (103)(3)

30. $$\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1-e^{rx}}\frac{dx}{x}=\ln\left[\sin\frac{p\pi}{r}\operatorname{cosec}\frac{q\pi}{r}\right]$$ $[|r|>|p|,\quad |r|>|q|,\quad rp>0,\quad rq>0]$ BI (103)(4)

31. $$\int_0^\infty \frac{e^{-qx}+e^{(q-p)x}}{1-e^{-px}}x\,dx=\left(\frac{\pi}{p}\operatorname{cosec}\frac{q\pi}{p}\right)^2$$ $[0<q<p]$ BI (82)(8)

32. $$\int_0^\infty \frac{e^{-px}-e^{(p-q)x}}{e^{-qx}+1}\frac{dx}{x}=\ln\cot\frac{p\pi}{2q}$$ $[0<p<q]$ BI (93)(7)

3.412 $$\int_0^\infty\left\{\frac{a+be^{-px}}{ce^{px}+g+he^{-px}}-\frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}}\right\}\frac{dx}{x}=\frac{a+b}{c+g+h}\ln\frac{p}{q}$$
$[p>0,\quad q>0]$ BI (96)(7)

3.413

1. $$\int_0^\infty \frac{\left(1-e^{-\beta x}\right)\left(1-e^{-\gamma x}\right)e^{-\mu x}}{1-e^{-x}}\frac{dx}{x}=\ln\frac{\Gamma(\mu)\,\Gamma(\beta+\gamma+\mu)}{\Gamma(\mu+\beta)\,\Gamma(\mu+\gamma)}$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\mu>-\operatorname{Re}\beta,\quad \operatorname{Re}\mu>-\operatorname{Re}\gamma,\quad \operatorname{Re}\mu>-\operatorname{Re}(\beta+\gamma)]$ (cf. **4.267** 25) BI (93)(13)

2. $$\int_0^\infty \frac{\left\{1-e^{(q-p)x}\right\}^2}{e^{qx}-e^{(q-2p)x}}\frac{dx}{x}=\ln\operatorname{cosec}\frac{q\pi}{2p}$$ $[0<q<p]$ BI (95)(6)

3. $$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-x}}\frac{1+e^{-(2n+1)x}}{x}\,dx$$
$$=\ln\left\{\frac{q(q+2)(q+4)\cdots(q+2n)(p+1)(p+3)\cdots(p+2n-1)}{p(p+2)(p+4)\cdots(p+2n)(q+1)(q+3)\cdots(q+2n-1)}\right\}$$
$[\operatorname{Re}p>-2n,\quad \operatorname{Re}q>-2n]$ (cf. **4.267** 14) BI (93)(11)

3.414 $$\int_0^\infty \frac{\left(1-e^{-\beta x}\right)\left(1-e^{-\gamma x}\right)\left(1-e^{-\delta x}\right)e^{-\mu x}}{1-e^{-x}}\frac{dx}{x}=\ln\frac{\Gamma(\mu)\,\Gamma(\mu+\beta+\gamma)\,\Gamma(\mu+\beta+\delta)\,\Gamma(\mu+\gamma+\delta)}{\Gamma(\mu+\beta)\,\Gamma(\mu+\gamma)\,\Gamma(\mu+\delta)\,\Gamma(\mu+\beta+\gamma+\delta)}$$
$[2\operatorname{Re}\mu>|\operatorname{Re}\beta|+|\operatorname{Re}\gamma|+|\operatorname{Re}\delta|]$ (cf. **4.267** 31) BI (93)(14), ET I 145(17)

3.415

1. $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)\,(e^{\mu x}-1)} = \frac{1}{2}\left[\ln\left(\frac{\beta\mu}{2\pi}\right) - \frac{\pi}{\beta\mu} - \psi\left(\frac{\beta\mu}{2\pi}\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ BI (97)(20), EH I 18(27)

2.[11] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2\,(e^{2\pi x}-1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta}\,\psi'(\beta) \sim \frac{1}{4\beta^4}\sum_{k=0}^\infty \frac{|B_{2k+2}|}{\beta^{2k}}$$
[asymptotic expansion for $\operatorname{Re}\beta > 0$] BI(97)(22), EH I 22(12)

3.[11] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)\,(e^{\mu x}+1)} = \frac{1}{2}\left[\psi\left(\frac{\beta\mu}{2\pi}+\frac{1}{2}\right) - \ln\left(\frac{\beta\mu}{2\pi}\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$

4.[8] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2\,(e^{2\pi x}+1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta}\,\psi'\left(\beta+\frac{1}{2}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$

3.416

1. $$\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i}\,\frac{dx}{e^{2\pi x}-1} = \frac{1}{2}\,\frac{2n-1}{2n+1}$$
$[n = 1, 2, \ldots]$ BI (88)(4)

2. $$\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i}\,\frac{dx}{e^{\pi x}+1} = \frac{1}{2n+1}$$
$[n = 1, 2, \ldots]$ BI (87)(1)

3.[8] $$\int_0^\infty \frac{(1+ix)^{2n-1}-(1-ix)^{2n-1}}{i}\,\frac{dx}{e^{\pi x}+1} = \frac{1}{2n}\left[1 - 2^{2^n} B_{2n}\right]$$
$[n = 1, 2, \ldots]$ BI (87)(2)

3.417

1. $$\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x+b^2e^{-x}} = \frac{\pi}{2ab}\ln\frac{b}{a}$$
$[ab > 0]$ (cf. **4.231** 8) BI (101)(1)

2. $$\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x-b^2e^{-x}} = \frac{\pi^2}{4ab}$$
(cf. **4.231** 10) LI (101)(2)

3.418

1.[6] $$\int_0^\infty \frac{x\,dx}{e^x+e^{-x}-1} = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right] = 1.1719536193\ldots$$
LI (88)(1)

2.[6] $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x+e^{-x}-1} = \frac{1}{6}\left[\psi'\left(\frac{1}{3}\right) - \frac{5}{6}\pi^2\right] = 0.3118211319\ldots$$
LI (88)(2)

3. $$\int_0^{\ln 2} \frac{x\,dx}{e^x+2e^{-x}-2} = \frac{\pi}{8}\ln 2$$
BI (104)(7)

3.419

1. $$\int_{-\infty}^{\infty}\frac{x\,dx}{(\beta+e^x)\,(1+e^{-x})}=\frac{(\ln\beta)^2}{2(\beta-1)}\qquad[|\arg\beta|<\pi]\qquad(\text{cf. }\mathbf{4.232}\ 2)$$
BI (101)(16)

2. $$\int_{-\infty}^{\infty}\frac{x\,dx}{(\beta+e^x)\,(1-e^{-x})}=\frac{\pi^2+(\ln\beta)^2}{2(\beta+1)}\qquad[|\arg\beta|<\pi]\qquad(\text{cf. }\mathbf{4.232}\ 3)$$
BI (101)(17)

3. $$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(\beta+e^x)\,(1-e^{-x})}=\frac{\left[\pi^2+(\ln\beta)^2\right]\ln\beta}{3(\beta+1)}\qquad[|\arg\beta|<\pi]\qquad(\text{cf. }\mathbf{4.261}\ 4)$$
BI (102)(6)

4. $$\int_{-\infty}^{\infty}\frac{x^3\,dx}{(\beta+e^x)\,(1-e^{-x})}=\frac{\left[\pi^2+(\ln\beta)^2\right]^2}{4(\beta+1)}\qquad[|\arg\beta|<\pi]\qquad(\text{cf. }\mathbf{4.262}\ 3)$$
BI (102)(9)

5. $$\int_{-\infty}^{\infty}\frac{x^4\,dx}{(\beta+e^x)\,(1-e^{-x})}=\frac{\left[\pi^2+(\ln\beta)^2\right]^2}{15(\beta+1)}\left[7\pi^2+3\,(\ln\beta)^2\right]\ln\beta$$
(cf. **4.263** 1) BI (102)(10)

6.[11] $$\int_{-\infty}^{\infty}\frac{x^5\,dx}{(\beta+e^x)\,(1-e^{-x})}=\frac{\left[\pi^2+(\ln\beta)^2\right]^2}{6(\beta+1)}\left[3\pi^2+(\ln\beta)^2\right]$$
(cf. **4.264** 3) BI (102)(11)

7. $$\int_{-\infty}^{\infty}\frac{(x-\ln\beta)\,x\,dx}{(\beta-e^x)\,(1-e^{-x})}=\frac{-\left[4\pi^2+(\ln\beta)^2\right]\ln\beta}{6(\beta-1)}\qquad[|\arg\beta|<\pi]\qquad(\text{cf. }\mathbf{4.257}\ 4)$$
BI (102)(7)

3.421

1. $$\int_0^{\infty}\left(e^{-\nu x}-1\right)^n\left(e^{-\rho x}-1\right)^m e^{-\mu x}\frac{dx}{x^2}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{l=0}^{m}(-1)^l\binom{m}{l}$$
$$\times\{(m-l)\rho+(n-k)\nu+\mu\}\ln\left[(m-l)\rho+(n-k)\nu+\mu\right]$$
$$[\operatorname{Re}\nu>0,\quad\operatorname{Re}\mu>0,\quad\operatorname{Re}\rho>0]$$ BI (89)(17)

2. $$\int_0^{\infty}\left(1-e^{-\nu x}\right)^n\left(1-e^{-\rho x}\right)e^{-x}\frac{dx}{x^3}=\frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\rho+k\nu+1)^2$$
$$\times\ln(\rho+k\nu+1)+\frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(k\nu+1)^2\ln(k\nu+1)$$
$$[n\geq 2,\quad\operatorname{Re}\nu>0,\quad\operatorname{Re}\rho>0]$$ BI (89)(31)

3. $$\int_{-\infty}^{\infty} \frac{xe^{-\mu x}\,dx}{(\beta+e^{-x})(\gamma+e^{-x})} = \frac{\pi\left(\beta^{\mu-1}\ln\beta-\gamma^{\mu-1}\ln\gamma\right)}{(\beta-\gamma)\sin\mu\pi} + \frac{\pi^2\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\cos\mu\pi}{(\gamma-\beta)\sin^2\mu\pi}$$
$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad \beta\neq\gamma. \quad 0<\operatorname{Re}\mu<2]$ ET I 120(19)

4. $$\int_0^{\infty} \left(e^{-px}-e^{-qx}\right)\left(e^{-rx}-e^{-sx}\right)e^{-x}\frac{dx}{x} = \ln\frac{(p+s+1)(q+r+1)}{(p+r+1)(q+s+1)}$$
$[p+s>-1, \quad p+r>-1, \quad q>p]$ (cf. **4.267** 24) BI (89)(11)

5. $$\int_0^{\infty} \left(1-e^{-px}\right)\left(1-e^{-qx}\right)\left(1-e^{-rx}\right)e^{-x}\frac{dx}{x}$$
$$= (p+q+1)\ln(p+q+1)$$
$$+(p+r+1)\ln(p+r+1)+(q+r+1)\ln(q+r+1)$$
$$-(p+1)\ln(p+1)-(q+1)\ln(q+1)-(r+1)\ln(r+1)$$
$$-(p+q+r)\ln(p+q+r)$$
$[p>0, \quad q>0, \quad r>0]$ (cf. **4.268** 3) BI (89)(14)

3.422 $$\int_{-\infty}^{\infty} \frac{x(x-a)e^{\mu x}\,dx}{(\beta-e^{x})(1-e^{-x})} = \frac{-\pi^2}{e^a-1}\operatorname{cosec}^2\mu\pi\left[(e^{\alpha\mu}+1)\ln\mu-2\pi\cot\mu\pi\,(e^{\alpha\mu}-1)\right]$$
$[a>0, \quad |\arg\beta|<\pi, \quad |\operatorname{Re}\mu|<1]$ (cf. **4.257** 5) BI (102)(8)a

3.423

1. $$\int_0^{\infty} \frac{x^{\nu-1}}{(e^x-1)^2}\,dx = \Gamma(\nu)\left[\zeta(\nu-1)-\zeta(\nu)\right]$$
$[\operatorname{Re}\nu>2]$ ET I 313(10)

2.[6] $$\int_0^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(e^x-1)^2}\,dx = \Gamma(\nu)\left[\zeta(\nu-1,\mu+2)-(\mu+1)\,\zeta(\nu,\mu+2)\right]$$
$[\operatorname{Re}\mu>-2, \quad \operatorname{Re}\nu>2]$ ET I 313(11)

3.[8] $$\int_0^{\infty} \frac{x^q e^{-px}\,dx}{(1-ae^{-px})^2} = \frac{\Gamma(q+1)}{ap^{q+1}}\sum_{k=1}^{\infty}\frac{a^k}{k^q}$$
$[a<1, \quad q>-1, \quad p>0]$ BI (85)(13)

4.[7] $$\int_0^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(1-\beta e^{-x})^2}\,dx = \Gamma(\nu)\left[\Phi(\beta;\nu-1;\mu)-(\mu-1)\,\Phi(\beta;\nu;\mu)\right]$$
$[\operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0, \quad |\arg(1-\beta)|<\pi]$ (cf. **9.550**) ET I 313(12)

5. $$\int_{-\infty}^{\infty} \frac{xe^{x}\,dx}{(\beta+e^{x})^2} = \frac{1}{\beta}\ln\beta$$
$[|\arg\beta|<\pi]$ (cf. **4.231** 5)
BI (101)(10)

6.* $$\int_0^{t} x^5\frac{e^{-x}}{(1-e^{-x})^2}\,dx = 120\,\zeta(5)-\sum_{k=1}^{\infty}\frac{e^{-kt}}{k^5}\left(y^5+5y^4+20y^3+60y^2+120y+120\right)$$
$$= 120\,\zeta(5)-\frac{t^5e^{-t/2}}{2\sinh(t/2)}-5\sum_{k=1}^{\infty}\frac{e^{-kt}}{k^5}\left(y^4+4y^3+12y^2+24y+24\right)$$
$y=kt$

3.424

1.[7] $$\int_0^\infty \frac{(1+a)e^x - a}{(1-e^x)^2} e^{-ax} x^n \, dx = n!\, \zeta(n,a) \qquad [a > -1, \quad n = 1, 2, \ldots]$$ BI (85)(15)

2. $$\int_0^\infty \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n \, dx = n! \sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n} \qquad [a > -1, \quad n = 1, 2, \ldots]$$ BI (85)(14)

3. $$\int_{-\infty}^\infty \frac{a^2 e^x + b^2 e^{-x}}{(a^2 e^x - b^2 e^{-x})^2} x^2 \, dx = \frac{\pi^2}{2ab} \qquad [ab > 0]$$ BI (102)(3)a

4. $$\int_{-\infty}^\infty \frac{a^2 e^x - b^2 e^{-x}}{(a^2 e^x + b^2 e^{-x})^2} x^2 \, dx = \frac{\pi}{ab} \ln \frac{b}{a} \qquad [ab > 0]$$ BI (102)(1)

5. $$\int_0^\infty \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} x^2 \, dx = \frac{2}{3}\pi^2 - 2$$ BI (85)(7)

3.425

1.[7] $$\int_{-\infty}^\infty \frac{x e^x \, dx}{(a^2 + b^2 e^{2x})^n} = \frac{\sqrt{\pi}\, \Gamma\left(n - \frac{1}{2}\right)}{4 a^{2n-1} b\, \Gamma(n)} \left[2 \ln \frac{a}{2b} - \boldsymbol{C} - \psi\left(n - \frac{1}{2}\right) \right]$$
$[ab > 0, \quad n > 0]$
BI(101)(13), LI(101)(13)

2.[7] $$\int_{-\infty}^\infty \frac{\left(a^2 e^x - e^{-x}\right) x^2 \, dx}{(a^2 e^x + e^{-x})^{p+1}} = -\frac{1}{a^{p+1}} \mathrm{B}\left(\frac{p}{2}, \frac{p}{2}\right) \ln a \qquad [a > 0, \quad p > 0]$$ BI (102)(5)

3.426

1. $$\int_{-\infty}^\infty \frac{\left(e^x - a e^{-x}\right) x^2 \, dx}{(a + e^x)^2 \left(1 + e^{-x}\right)^2} = \frac{(\ln a)^2}{a - 1}$$ BI (102)(12)

2. $$\int_{-\infty}^\infty \frac{\left(e^x - a e^{-x}\right) x^2 \, dx}{(a + e^x)^2 \left(1 - e^{-x}\right)^2} = \frac{\pi^2 + (\ln a)^2}{a + 1}$$ BI (102)(13)

3.427

1. $$\int_0^\infty \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x} - 1} \right) dx = \psi(\mu) \qquad [\operatorname{Re} \mu > 0] \qquad (\text{cf. } \mathbf{4.281}\ 4)$$ WH

2.[7] $$\int_0^\infty \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} \, dx = \boldsymbol{C} \qquad (\text{cf. } \mathbf{4.281}\ 1)$$ BI (94)(1)

3. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{1 + e^{-x}} \right) \frac{e^{-2x}}{x} \, dx = \frac{1}{2} \ln \frac{\pi}{4}$$ BI (94)(5)

4. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-\mu x}}{x} \, dx = \ln \Gamma(\mu) - \left(\mu - \frac{1}{2} \right) \ln \mu + \mu - \frac{1}{2} \ln(2\pi)$$
$[\operatorname{Re} \mu > 0]$ WH

5. $$\int_0^\infty \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi$$ BI (94)(6)

6. $$\int_0^\infty \left(\frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu \right) \frac{e^{-x}}{x} \, dx = -\ln \Gamma(\mu) - \ln \sin(\pi \mu) + \ln \pi$$
$[\operatorname{Re} \mu < 1]$ EH I 21(6)

7. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu x}}{x}\right) dx = \ln\mu - \psi(\nu) \qquad \text{(cf. } \mathbf{4.281}\ 5) \qquad \text{BI (94)(3)}$$

8. $$\int_0^\infty \left(\frac{n}{x} - \frac{e^{-\mu x}}{1-e^{-x/n}}\right) e^{-x}\, dx = n\,\psi(n\mu+n) - n\ln n$$
$$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots] \qquad \text{BI (94)(4)}$$

9. $$\int_0^\infty \left(\mu - \frac{1-e^{-\mu x}}{1-e^{-x}}\right) \frac{e^{-x}}{x}\, dx = \ln\Gamma(\mu+1) \qquad [\operatorname{Re}\mu > -1] \qquad \text{WH}$$

10. $$\int_0^\infty \left(\nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1}\right) \frac{dx}{x} = \ln\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)}$$
$$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > 0] \qquad \text{BI (94)(8)}$$

11. $$\int_0^\infty \left[(1-e^x)^{-1} + x^{-1} - 1\right] e^{-xz}\, dx = \psi(z) - \ln z \qquad [\operatorname{Re} z > 0] \qquad \text{EH I 18(24)}$$

3.428

1. $$\int_0^\infty \left(\nu e^{-\mu x} - \frac{1}{\mu}e^{-x} - \frac{1}{\mu}\frac{e^{-1} - e^{-\mu\nu x}}{1-e^{-x}}\right) \frac{dx}{x} = \frac{1}{\mu}\ln\Gamma(\mu\nu) - \nu\ln\mu$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad \text{BI (94)(18)}$$

2. $$\int_0^\infty \left(\frac{n-1}{2} + \frac{n-1}{1-e^{-x}} + \frac{e^{(1-\mu)x}}{1-e^{x/n}} + \frac{e^{-n\mu x}}{1-e^{-x}}\right) e^{-x}\frac{dx}{x} = \frac{n-1}{2}\ln 2\pi - \left(n\mu + \frac{1}{2}\right)\ln n$$
$$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots] \qquad \text{BI (94)(14)}$$

3. $$\int_0^\infty \left(n\mu - \frac{n-1}{2} - \frac{n}{1-e^{-x}} - \frac{e^{(1-\mu)x}}{1-e^{x/n}}\right) \frac{e^{-x}}{x}\, dx = \sum_{k=0}^{n-1} \ln\Gamma\left(\mu - \frac{k}{n} + 1\right)$$
$$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots] \qquad \text{BI (94)(13)}$$

4. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1-e^x} - \frac{e^{-\mu\nu x}}{1-e^{\mu x}} - \frac{e^x}{1-e^x} + \frac{e^{\mu x}}{1-e^{\mu x}}\right) \frac{dx}{x} = \nu\ln\mu$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad \text{LI (94)(15)}$$

5. $$\int_0^\infty \left[\frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1-e^{-\mu x}} + \left(a\mu - \frac{\mu+1}{2}\right)e^{-\mu x} + (1-a\mu)e^{-x}\right] \frac{dx}{x}$$
$$= \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - a\mu\right)\ln\mu$$
$$[\operatorname{Re}\mu > 0] \qquad \text{BI (94)(16)}$$

6. $$\int_0^\infty \left[\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu\nu x}}{1-e^{-\mu x}} - \frac{(\mu-1)e^{-\mu x}}{1-e^{-\mu x}} - \frac{\mu-1}{2}e^{-\mu}x\right] \frac{dx}{x} = \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - \mu\nu\right)\ln\mu$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad \text{(cf. } \mathbf{4.267}\ 37) \qquad \text{BI (94)(17)}$$

7. $$\int_0^\infty \left[1 - e^{-x} - \frac{(1-e^{-\nu x})(1-e^{-\mu x})}{1-e^{-x}}\right] \frac{dx}{x} = \ln\mathrm{B}(\mu,\nu)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0] \qquad \text{BI (94)(12)}$$

3.429 $$\int_0^\infty \left[e^{-x} - (1+x)^{-\mu}\right] \frac{dx}{x} = \psi(\mu) \qquad [\operatorname{Re}\mu > 0] \qquad \text{NH 184(7)}$$

3.431

1. $$\int_0^\infty \left(e^{-\mu x} - 1 + \mu x - \frac{1}{2}\mu^2 x^2\right) x^{\nu-1}\,dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu}\,\Gamma(\nu+3)$$
$[\operatorname{Re}\mu > 0, \quad -2 > \operatorname{Re}\nu > -3]$ LI (90)(5)

2. $$\int_0^\infty \left[x^{-1} - \frac{1}{2}x^{-2}(x+2)\left(1-e^{-x}\right)\right] e^{-px}\,dx = -1 + \left(p+\frac{1}{2}\right)\ln\left(1+\frac{1}{p}\right)$$
$[\operatorname{Re}p > 0]$ ET I 144(6)

3.432

1. $$\int_0^\infty x^{\nu-1}e^{-mx}\left(e^{-x}-1\right)^n\,dx = \Gamma(\nu)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(n+m-k)^\nu}$$
$[n = 0, 1, \ldots, \operatorname{Re}\nu > 0]$ LI (90)(10)

2. $$\int_0^\infty \left[x^{\nu-1}e^{-x} - e^{-\mu x}\left(1-e^{-x}\right)^{\nu-1}\right]\,dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ LI (81)(14)

3.433 $$\int_0^\infty x^{p-1}\left[e^{-x} + \sum_{k=1}^{n}(-1)^k\frac{x^{k-1}}{(k-1)!}\right]\,dx = \Gamma(p) \qquad [-n < p < -n+1, \quad n = 0, 1, \ldots]$$
FI II 805

3.434

1. $$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}}\,dx = \frac{\mu^\rho - \nu^\rho}{\rho}\,\Gamma(1-\rho) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\rho < 1]$$
BI (90)(6)

2. $$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x}\,dx = \ln\frac{\nu}{\mu} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
FI II 634

3.435

1. $$\int_0^\infty \left\{(x+1)e^{-x} - e^{-\frac{x}{2}}\right\}\frac{dx}{x} = 1 - \ln 2$$
LI (89)(19)

2.[11] $$\int_0^\infty \frac{1-e^{-\mu x}}{x(x+\beta)}\,dx = \frac{1}{\beta}\left[\ln(\beta\mu) + \boldsymbol{C} - e^{\beta\mu}\operatorname{Ei}(-\beta\mu)\right] \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$$
ET II 217 (18)

3. $$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} = \boldsymbol{C}$$
FI II 7 95, 802

4. $$\int_0^\infty \left(e^{-\mu x} - \frac{1}{1+ax}\right)\frac{dx}{x} = \ln\frac{a}{\mu} - \boldsymbol{C} \qquad [a > 0, \quad \operatorname{Re}\mu > 0]$$
BI (92)(10)

3.436 $$\int_0^\infty \left\{\frac{e^{-npx} - e^{-nqx}}{n} - \frac{e^{-mpx} - e^{-mqx}}{m}\right\}\frac{dx}{x^2} = (q-p)\ln\frac{m}{n} \qquad [p > 0, \quad q > 0]$$
BI (89)(28)

3.437 $$\int_0^\infty \left\{pe^{-x} - \frac{1-e^{-px}}{x}\right\}\frac{dx}{x} = p\ln p - p \qquad [p > 0]$$
BI (89)(24)

3.438

1. $$\int_0^\infty \left\{\left(\frac{1}{2}+\frac{1}{x}\right)e^{-x}-\frac{1}{x}e^{-\frac{x}{2}}\right\}\frac{dx}{x}=\frac{\ln 2-1}{2}$$ BI (89)(19)

2.[7] $$\int_0^\infty \left\{\frac{p^2}{6}e^{-x}-\frac{p^2}{2x}-\frac{p}{x^2}-\frac{1-e^{-px}}{x^3}\right\}\frac{dx}{x}=\frac{p^2}{6}\ln p-\frac{11}{36}p^3$$
$[p>0]$ BI (89)(33)

3. $$\int_0^\infty \left(e^{-x}-e^{-2x}-\frac{1}{x}e^{-2x}\right)\frac{dx}{x}=1-\ln 2$$ BI (89)(25)

4. $$\int_0^\infty \left\{\left(p-\frac{1}{2}\right)e^{-x}+\frac{x+2}{2x}\left(e^{-px}-e^{-\frac{x}{2}}\right)\right\}\frac{dx}{x}=\left(p-\frac{1}{2}\right)(\ln p-1)$$
$[p>0]$ BI (89)(22)

3.439 $$\int_0^\infty \left\{(p-q)e^{-rx}+\frac{1}{mx}\left(e^{-mpx}-e^{-mqx}\right)\right\}\frac{dx}{x}=p\ln p-q\ln q-(p-q)\left(1+\ln\frac{r}{m}\right)$$
$[p>0,\quad q>0,\quad r>0]$ LI(89)(26), LI(89)(27)

3.441 $$\int_0^\infty \left\{(p-r)e^{-qx}+(r-q)e^{-px}+(q-p)e^{-rx}\right\}\frac{dx}{x^2}=(r-q)p\ln p+(p-r)q\ln q+(q-p)r\ln r$$
$[p>0,\quad q>0,\quad r>0]$ (cf. **4.268** 6) BI (89)(18)

3.442

1. $$\int_0^\infty \left\{1-\frac{x+2}{2x}\left(1-e^{-x}\right)\right\}e^{-qx}\frac{dx}{x}=-1+\left(q+\frac{1}{2}\right)\ln\frac{q+1}{q}$$
$[q>0]$ BI (89)(23)

2. $$\int_0^\infty \left(\frac{e^{-x}-1}{x}+\frac{1}{1+x}\right)\frac{dx}{x}=\boldsymbol{C}-1$$ BI (92)(16)

3. $$\int_0^\infty \left(e^{-px}-\frac{1}{1+a^2x^2}\right)\frac{dx}{x}=-\boldsymbol{C}+\ln\frac{a}{p}\qquad [p>0]$$ BI (92)(11)

3.443

1. $$\int_0^\infty \left\{\frac{e^{-x}p^2}{2}-\frac{p}{x}+\frac{1-e^{-px}}{x^2}\right\}\frac{dx}{x}=\frac{p^2}{2}\ln p-\frac{3}{4}p^2\qquad [p>0]$$ BI (89)(32)

2. $$\int_0^\infty \frac{\left(1-e^{-px}\right)^n e^{-qx}}{x^3}\,dx=\frac{1}{2}\sum_{k=2}^{n}(-1)^{k-1}\binom{n}{k}(q+kp)^2\ln(q+kp)$$
$[n>2,\quad q>0,\quad pn+q>0]$ (cf. **4.268** 4) BI (89)(30)

3. $$\int_0^\infty \left(1-e^{-px}\right)^2 e^{-qx}\frac{dx}{x^2}=(2p+q)\ln(2p+q)-2(p+q)\ln(p+q)+q\ln q$$
$[q>0,\quad 2p>-q]$ (cf. **4.268** 2)
BI (89)(13)

3.45 Combinations of powers and algebraic functions of exponentials

3.451

1. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{3}\left(\frac{4}{3}-\ln 2\right)$$ BI (99)(1)

2. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-2x}}\,dx = \frac{\pi}{4}\left(\frac{1}{2}+\ln 2\right)$$ (cf. **4.241** 9) BI (99)(2)

3.452

1. $$\int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = 2\pi\ln 2$$ FI II 643a,BI(99)(4)

2. $$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = 4\pi\left\{(\ln 2)^2+\frac{\pi^2}{12}\right\}$$ BI (99)(5)

3. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = \frac{\pi}{2}\left[2\ln 2-1\right]$$ BI (99)(6)

4. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1-\ln 2$$ BI (99)(8)

5. $$\int_0^\infty \frac{xe^{-2x}\,dx}{\sqrt{e^x-1}} = \frac{3}{4}\pi\left(\ln 2-\frac{7}{12}\right)$$ BI (99)(7)

3.453

1. $$\int_0^\infty \frac{xe^x}{a^2e^x-(a^2-b^2)}\frac{dx}{\sqrt{e^x-1}} = \frac{2\pi}{ab}\ln\left(1+\frac{b}{a}\right)$$ $[ab>0]$ (cf. **4.298** 17) BI (99)(16)

2. $$\int_0^\infty \frac{xe^x\,dx}{\left[a^2e^x-(a^2+b^2)\right]\sqrt{e^x-1}} = \frac{2\pi}{ab}\arctan\frac{b}{a}$$ $[ab>0]$ (cf. **4.298** 18) BI (99)(17)

3.454

1.[11] $$\int_0^\infty \frac{xe^{-2nx}\,dx}{\sqrt{e^{2x}-1}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}\left\{\ln 2+\sum_{k=1}^{2n}\frac{(-1)^k}{k}\right\}$$ LI (99)(10)

2. $$\int_0^\infty \frac{xe^{-(2n-1)x}\,dx}{\sqrt{e^{2x}-1}} = -\frac{(2n-2)!!}{(2n-1)!!}\left\{\ln 2+\sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right\}$$ LI (99)(9)

3.455

1. $$\int_0^\infty \frac{x^2e^x\,dx}{\sqrt{(e^x-1)^3}} = 8\pi\ln 2$$ BI (99)(11)

2. $$\int_0^\infty \frac{x^3e^x\,dx}{\sqrt{(e^x-1)^3}} = 24\pi\left[(\ln 2)^2+\frac{\pi^2}{12}\right]$$ BI (99)(12)

3.456

1. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{e^{3x}-1}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3+\frac{\pi}{3\sqrt{3}}\right]$$ BI (99)(13)

2. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{(e^{3x}-1)^2}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3 - \frac{\pi}{3\sqrt{3}}\right]$$ (cf. **4.244** 3) BI (99)(14)

3.457

1. $$\int_0^\infty xe^{-x}\left(1-e^{-2x}\right)^{n-1/2}\,dx = \frac{(2n-1)!!}{4\cdot(2n)!!}\pi\left[\boldsymbol{C}+\psi(n+1)+2\ln 2\right]$$ (cf. **4.241** 5) BI (99)(3)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{(a+e^x)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}}\left[\ln(4a)-3\boldsymbol{C}-2\,\psi(2n)-\psi(n)\right]$$ BI (101)(12)

3. $$\int_{-\infty}^\infty \frac{x\,dx}{(a^2e^x+e^{-x})^\mu} = -\frac{1}{2a^\mu}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\mu}{2}\right)\ln a \qquad [a>0,\quad \operatorname{Re}\mu>0]$$ BI (101)(14)

3.458

1.[7] $$\int_0^{\ln 2} xe^x\,(e^x-1)^{p-1}\,dx = \frac{1}{p}\left[\ln 2+\sum_{k=0}^\infty \frac{(-1)^{k-1}}{p+k+1}\right]$$ BI (104)(4)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{(a+e^x)^{\nu+1}} = \frac{1}{\nu a^\nu}\left[\ln a-\boldsymbol{C}-\psi(\nu)\right] \qquad [a>0]$$
$$= \frac{1}{\nu a^\nu}\left[\ln a-\sum_{k=1}^{\nu-1}\frac{1}{k}\right] \qquad [a>0,\quad \nu=1,2,\ldots]$$
BI (101)(11)

3.46–3.48 Combinations of exponentials of more complicated arguments and powers

3.461

1. $$\int_u^\infty \frac{e^{-p^2x^2}}{x^{2n}}\,dx = \frac{(-1)^n 2^{n-1}p^{2n-1}\sqrt{\pi}}{(2n-1)!!}\left[1-\Phi(pu)\right] + \frac{e^{-p^2u^2}}{2u^{2n-1}}\sum_{k=0}^{n-1}\frac{(-1)^k 2^{k+1}(pu)^{2k}}{(2n-1)(2n-3)\cdots(2n-2k-1)}$$
$[p>0]$ NT 21(4)

2. $$\int_0^\infty x^{2n}e^{-px^2}\,dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \qquad [p>0,\quad n=0,1,\ldots]$$ FI II 743

3. $$\int_0^\infty x^{2n+1}e^{-px^2}\,dx = \frac{n!}{2p^{n+1}} \qquad [p>0]$$ BI (81)(7)

4. $$\int_{-\infty}^\infty (x+ai)^{2n}e^{-x^2}\,dx = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\sum_{k=0}^n(-1)^k\frac{(2a)^{2k}n!}{(2k)!(n-k)!}$$ BI (100)(12)

5.[11] $$\int_u^\infty e^{-\mu x^2}\frac{dx}{x^2} = \frac{1}{u}e^{-\mu u^2}-\sqrt{\mu\pi}\left[1-\Phi\left(u\sqrt{\mu}\right)\right] \qquad \left[|\arg\mu|<\frac{\pi}{2},\quad u>0\right]$$ ET I 135(19)a

6.* $$\int_0^\infty \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = b\,K_1(ab) \qquad [\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$$

7.* $$\int_0^\infty x^2 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{2b}{a^2}\,K_1(ab) + \frac{b^2}{a}\,K_0(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

8.* $$\int_0^\infty x^4 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{12b^2}{a^3}\,K_2(ab) + \frac{3b^3}{a^2}\,K_1(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

9.* $$\int_0^\infty x^6 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{90b^3}{a^4}\,K_3(ab) + \frac{15b^4}{a^3}\,K_2(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

3.462

1. $$\int_0^\infty x^{\nu-1} e^{-\beta x^2-\gamma x}\,dx = (2\beta)^{-\nu/2}\,\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$ EH II 119(3)a, ET I 313(13)

2.[8] $$\int_{-\infty}^\infty x^n e^{-px^2+2qx}\,dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\,\frac{d^{n-1}}{dq^{n-1}}\left(qe^{q^2/p}\right)$$ $[p > 0]$ BI (100)(8)

$$= n!e^{q^2/p}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{(n-2k)!(k)!}\left(\frac{p}{4q^2}\right)^k$$ $[p > 0]$ LI (100)(8)

3.[11] $$\int_{-\infty}^\infty (ix)^\nu e^{-\beta^2x^2-iqx}\,dx = 2^{-\frac{\nu}{2}}\sqrt{\pi}\beta^{-\nu-1}\exp\left(-\frac{q^2}{8\beta^2}\right) D_\nu\left(\frac{q}{\beta\sqrt{2}}\right)$$

$\left[\operatorname{Re}\beta^2 > 0, \quad \operatorname{Re}\nu > -1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x\right]$ ET I 121(23)

4. $$\int_{-\infty}^\infty x^n \exp\left[-(x-\beta)^2\right]\,dx = (2i)^{-n}\sqrt{\pi}\,H_n\,(i\beta)$$ EH II 195(31)

5.[11] $$\int_0^\infty xe^{-\mu x^2-2\nu x}\,dx = \frac{1}{2\mu} - \frac{\nu}{2\mu}\sqrt{\frac{\pi}{\mu}}e^{\frac{\nu^2}{\mu}}\left[1-\operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

$\left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 146(31)a

6. $$\int_{-\infty}^\infty xe^{-px^2+2qx}\,dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{p}\right)$$ $[\operatorname{Re} p > 0]$ BI (100)(7)

7.[11] $$\int_0^\infty x^2e^{-\mu x^2-2\nu x}\,dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{2\nu^2+\mu}{4}e^{\frac{\nu^2}{\mu}}\left[1-\operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

$\left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 146(32)

8. $$\int_{-\infty}^\infty x^2e^{-\mu x^2+2\nu x}\,dx = \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}}\left(1+2\frac{\nu^2}{\mu}\right)e^{\frac{\nu^2}{\mu}}$$ $[|\arg\nu| < \pi, \quad \operatorname{Re}\mu > 0]$ BI (100)(8)a

9.* $$\int_0^\infty e^{-\beta x^n \pm a}\,dx = \frac{e^{\pm a}}{n\beta^{1/n}}\,\Gamma\left(\frac{1}{n}\right)$$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re} n > 0]$

10.* $\displaystyle\int_0^\infty (x-a)e^{-\beta(x-a)}\,dx = e^{a\beta}\frac{(1-a\beta)}{\beta^2}$ $\quad[\operatorname{Re}\beta>0]$

11.* $\displaystyle\int_0^\infty (x-a)e^{-\beta(x+a)}\,dx = e^{-a\beta}\frac{(1-a\beta)}{\beta^2}$ $\quad[\operatorname{Re}\beta>0]$

12.* $\displaystyle\int_0^\infty (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\,\Gamma\left(m+1,\pm\frac{pb}{a}\right)$ $\quad\left[p>0,\quad \left|\arg\left(\frac{b}{a}\right)\right|<\pi\right]$

13.* $\displaystyle\int_u^\infty (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\,\Gamma\left(m+1,pu\pm\frac{pb}{a}\right)$

$\displaystyle\left[p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

14.* $\displaystyle\int_0^u (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\left[\Gamma\left(m+1,\pm\frac{pb}{a}\right)-\Gamma\left(m+1,pu\pm\frac{pb}{a}\right)\right]$

$\displaystyle\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

15.* $\displaystyle\int_0^\infty \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\,\Gamma\left(-n+1,\pm\frac{pb}{a}\right)$ $\quad\left[p>0,\quad \left|\arg\left(\frac{b}{a}\right)\right|<\pi\right]$

16.* $\displaystyle\int_u^\infty \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\,\Gamma\left(-n+1,pu\pm\frac{pb}{a}\right)$

$\displaystyle\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

17.* $\displaystyle\int_0^u \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\left[\Gamma\left(-n+1,\pm\frac{pb}{a}\right)-\Gamma\left(-n+1,pu\pm\frac{pb}{a}\right)\right]$

$\displaystyle\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

18.* $\displaystyle\int_0^\infty \left(\frac{x-a}{b}\right)^j \exp\left(-\beta\left(\frac{x-a}{b}\right)^k\right)dx = \frac{b\,\Gamma\left(\frac{j+1}{k},\beta\left(-\frac{a}{b}\right)^k\right)}{k\beta^{(j+1)/k}}$

$\displaystyle\left[\arg\left(-\frac{a}{b}\right)>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}k>0\right]$

19.* $\displaystyle\int_u^\infty \frac{e^{-\beta x^n}}{x^m}\,dx = \frac{\Gamma(z,\beta u^n)}{n\beta^z}\qquad z=\frac{1-m}{n}$ $\quad[u>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}n>0,\quad \operatorname{Re}z>0]$

20.* $\displaystyle\int_0^\infty \frac{\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = K_0(ab)$ $\quad[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

21.* $\displaystyle\int_0^\infty \frac{x^2\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{b}{a}\,K_1(ab)$ $\quad[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

22.* $\displaystyle\int_0^\infty \frac{x^4\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{3b^2}{a^2}\,K_1(ab)$ $\quad[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

23.* $\displaystyle\int_0^\infty \frac{x^6\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{15b^3}{a^3}\,K_3(ab)$ $\quad[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

24.* $$\int_0^\infty \frac{x^{2n}\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = (2n-1)!!\left(\frac{b}{a}\right)^n K_n(ab)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0]$

25.* $$\int_0^\infty \frac{\exp\left(-px^2\right)}{\sqrt{a^2+x^2}}\,dx = \frac{1}{2}\exp\left(\frac{a^2p}{2}\right)K_0\left(\frac{a^2p}{2}\right) \qquad [\operatorname{Re} a>0, \quad \operatorname{Re} b>0]$$

3.463 $$\int_0^\infty \left(e^{-x^2}-e^{-x}\right)\frac{dx}{x} = \frac{1}{2}\boldsymbol{C}$$ BI (89)(5)

3.464 $$\int_0^\infty \left(e^{-\mu x^2}-e^{-\nu x^2}\right)\frac{dx}{x^2} = \sqrt{\pi}\left(\sqrt{\nu}-\sqrt{\mu}\right) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ FI II 645

3.465 $$\int_0^\infty \left(1+2\beta x^2\right)e^{-\mu x^2}\,dx = \frac{\mu+\beta}{2}\sqrt{\frac{\pi}{\mu^3}} \qquad [\operatorname{Re}\mu>0]$$ ET I 136(24)a

3.466

1. $$\int_0^\infty \frac{e^{-\mu^2x^2}}{x^2+\beta^2}\,dx = [1-\Phi(\beta\mu)]\,\frac{\pi}{2\beta}e^{\beta^2\mu^2} \qquad \left[\operatorname{Re}\beta>0, \quad |\arg\mu|<\frac{\pi}{4}\right]$$ NT 19(13)

2. $$\int_0^\infty \frac{x^2e^{-\mu^2x^2}}{x^2+\beta^2}\,dx = \frac{\sqrt{\pi}}{2\mu}-\frac{\pi\beta}{2}e^{\mu^2\beta^2}\,[1-\Phi(\beta\mu)] \qquad \left[\operatorname{Re}\beta>0, \quad |\arg\mu|<\frac{\pi}{4}\right]$$ ET II 217(16)

3. $$\int_0^1 \frac{e^{x^2}-1}{x^2}\,dx = \sum_{k=1}^\infty \frac{1}{k!(2k-1)}$$ FI II 683

3.467 $$\int_0^\infty \left(e^{-x^2}-\frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}$$ BI (92)(12)

3.468

1. $$\int_{u\sqrt{2}}^\infty \frac{e^{-x^2}}{\sqrt{x^2-u^2}}\,\frac{dx}{x} = \frac{\pi}{4u}\,[1-\Phi(u)]^2 \qquad [u>0]$$ NT 33(17)

2. $$\int_0^\infty \frac{xe^{-\mu x^2}\,dx}{\sqrt{a^2+x^2}} = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{a^2\mu}\,[1-\Phi\left(a\sqrt{\mu}\right)] \qquad [\operatorname{Re}\mu>0, \quad a>0]$$ NT 19(11)

3.469

1. $$\int_0^\infty e^{-\mu x^4-2\nu x^2}\,dx = \frac{1}{4}\sqrt{\frac{2\nu}{\mu}}\exp\left(\frac{\nu^2}{2\mu}\right)K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right) \qquad [\operatorname{Re}\mu\geq 0]$$ ET I 146(23)

2. $$\int_0^\infty \left(e^{-x^4}-e^{-x}\right)\frac{dx}{x} = \frac{3}{4}\boldsymbol{C}$$ BI (89)(7)

3. $$\int_0^\infty \left(e^{-x^4}-e^{-x^2}\right)\frac{dx}{x} = \frac{1}{4}\boldsymbol{C}$$ BI (89)(6)

3.471

1. $$\int_0^u \exp\left(-\frac{\beta}{x}\right)\frac{dx}{x^2} = \frac{1}{\beta}\exp\left(-\frac{\beta}{u}\right)$$ ET II 188(22)

2. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{\nu-1}{2}}u^{\frac{2\mu+\nu-1}{2}}\exp\left(-\frac{\beta}{2u}\right)\Gamma(\mu)\,W_{\frac{1-2\mu-\nu}{2},\frac{\nu}{2}}\left(\frac{\beta}{u}\right)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\beta>0, \quad u>0]$
ET II 187(18)

3. $$\int_0^u x^{-\mu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx=\beta^{-\mu}u^{\mu-1}\,\Gamma(\mu)\exp\left(-\frac{\beta}{u}\right)$$
$[\operatorname{Re}\mu>0,\quad u>0]$ ET II 187(16)

4. $$\int_0^u x^{-2\mu}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx=\frac{1}{\sqrt{\pi u}}\beta^{\frac12-\mu}e^{-\frac{\beta}{2u}}\,\Gamma(\mu)\,K_{\mu-\frac12}\left(\frac{\beta}{2u}\right)$$
$[u>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>0]$
ET II 187(17)

5. $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx=\mathrm{B}(1-\mu-\nu,\mu)u^{\mu+\nu-1}\;{}_1F_1\left(1-\mu-\nu;1-\nu;\frac{\beta}{u}\right)$$
$[0<\operatorname{Re}\mu<\operatorname{Re}(1-\nu),\quad u>0]$
ET II 203(15)

6. $$\int_u^\infty x^{-2\mu}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx=\sqrt{\frac{\pi}{u}}\beta^{\frac12-\mu}\,\Gamma(\mu)\exp\left(\frac{\beta}{2u}\right)I_{\mu-\frac12}\left(\frac{\beta}{2u}\right)$$
$[\operatorname{Re}\mu>0,\quad u>0]$ ET II 202(14)

7. $$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}}\,dx=\beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\,\Gamma(1-\mu-\nu)e^{\frac{\beta}{2\gamma}}\;W_{\frac{\nu-1}{2}+\mu,-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right)$$
$[|\arg\gamma|<\pi,\quad \operatorname{Re}(1-\mu)>\operatorname{Re}\nu>0]$
ET II 234(13)a

8. $$\int_0^u x^{-2\mu}\left(u^2-x^2\right)^{\mu-1}e^{-\frac{\beta}{x}}\,dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^{\mu-\frac12}u^{\mu-\frac32}\,\Gamma(\mu)\,K_{\mu-\frac12}\left(\frac{\beta}{u}\right)$$
$[\operatorname{Re}\beta>0,\quad u>0,\quad \operatorname{Re}\mu>0]$
ET II 188(23)a

9. $$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}\,dx=2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu\left(2\sqrt{\beta\gamma}\right)$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$
ET II 82(23)a, LET I 146(29)

10. $$\int_0^\infty x^{\nu-1}\exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right]dx=2\beta^\nu e^{\frac{i\nu\pi}{2}}\,K_{-\nu}(\beta\mu)$$
$\left[\operatorname{Im}\mu>0,\quad \operatorname{Im}\left(\beta^2\mu\right)<0;\ \text{note that}\ K_{-\nu}\equiv K_\nu\right]$ EH II 82(24)

11. $$\int_0^\infty x^{\nu-1}\exp\left[\frac{i\mu}{2}\left(x+\frac{\beta^2}{x}\right)\right]dx=i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}}\,H^{(1)}_{-\nu}(\beta\mu)$$
$\left[\operatorname{Im}\mu>0,\quad \operatorname{Im}\left(\beta^2\mu\right)>0\right]$
EH II 21(33)

12. $$\int_0^\infty x^{\nu-1}\exp\left(-x-\frac{\mu^2}{4x}\right)dx=2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu)$$
$\left[|\arg\mu|<\frac{\pi}{2},\operatorname{Re}\mu^2>0;\ \text{note that}\ K_{-\nu}\equiv K_\nu\right]$ WA 203(15)

13. $$\int_0^\infty \frac{x^{\nu-1}e^{-\frac{\beta}{x}}}{x+\gamma}\,dx=\gamma^{\nu-1}e^{\frac{\beta}{\gamma}}\,\Gamma(1-\nu)\,\Gamma\left(\nu,\frac{\beta}{\gamma}\right)$$
$[|\arg\gamma|<\pi,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\nu<1]$
ET II 218(19)

14. $\displaystyle\int_0^1 \frac{\exp\left(1-\frac{1}{x}\right)-x^\nu}{x(1-x)}\,dx=\psi(\nu)$ $[\operatorname{Re}\nu>0]$ BI (80)(7)

15. $\displaystyle\int_0^\infty x^{-\frac{1}{2}}e^{-\gamma x-\beta/x}\,dx=\sqrt{\frac{\pi}{\gamma}}e^{-2\sqrt{\beta\gamma}}$ $[\operatorname{Re}\beta\geq 0,\quad \operatorname{Re}\gamma>0]$ ET 245 (5.6.1)

16. $\displaystyle\int_0^\infty x^{n-\frac{1}{2}}e^{-px-q/x}\,dx=(-1)^n\sqrt{\pi}\frac{\partial^n}{\partial p^n}\left(p^{-1/2}e^{-2\sqrt{pq}}\right)$

$[\operatorname{Re}p>0,\quad \operatorname{Re}q>0]$

PBM 344 (2.3.16(2))

3.472

1. $\displaystyle\int_0^\infty \left(\exp\left(-\frac{a}{x^2}\right)-1\right)e^{-\mu x^2}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left[\exp\left(-2\sqrt{a\mu}\right)-1\right]$

$[\operatorname{Re}\mu>0,\quad \operatorname{Re}a>0]$ ET I 146(30)

2. $\displaystyle\int_0^\infty x^2\exp\left(-\frac{a}{x^2}-\mu x^2\right)\,dx=\frac{1}{4}\sqrt{\frac{\pi}{\mu^3}}\left(1+2\sqrt{a\mu}\right)\exp\left(-2\sqrt{a\mu}\right)$

$[\operatorname{Re}\mu>0,\quad \operatorname{Re}a>0]$ ET I 146(26)

3. $\displaystyle\int_0^\infty \exp\left(-\frac{a}{x^2}-\mu x^2\right)\frac{dx}{x^2}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{a\mu}\right)$ $[\operatorname{Re}\mu>0,\quad a>0]$ ET I 146(28)a

4. $\displaystyle\int_0^\infty \exp\left[-\frac{1}{2a}\left(x^2+\frac{1}{x^2}\right)\right]\frac{dx}{x^4}=\sqrt{\frac{a\pi}{2}}(1+a)e^{-1/a}$ $[a>0]$ BI (98)(14)

5. $\displaystyle\int_0^\infty x^{-n-1/2}e^{-px-q/x}\,dx=(-1)^n\sqrt{\frac{\pi}{p}}\frac{\partial^n}{\partial q^n}e^{-2\sqrt{pq}}$ $[\operatorname{Re}p>0,\quad \operatorname{Re}q>0]$

PBM 344 (2.3.16(3))

3.473 $\displaystyle\int_0^\infty \exp\left(-x^n\right)x^{(m+1/2)n-1}\,dx=\frac{(2m-1)!!}{2^m n}\sqrt{\pi}$ BI (98)(6)

3.474

1. $\displaystyle\int_0^1 \left\{\frac{n\exp\left(1-x^{-n}\right)}{1-x^n}-\frac{x^{np}}{1-x}\right\}\frac{dx}{x}=\frac{1}{n}\sum_{k=1}^n\psi\left(p+\frac{k-1}{n}\right)$

$[p>0]$ BI (80)(8)

2. $\displaystyle\int_0^1 \left\{\frac{n\exp\left(1-x^{-n}\right)}{1-x^n}-\frac{\exp\left(1-\frac{1}{x}\right)}{1-x}\right\}\frac{dx}{x}=-\ln n$ BI (80)(9)

3.475

1.[7] $\displaystyle\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-\frac{1}{1+x^{2^{n+1}}}\right\}\frac{dx}{x}=-\frac{1}{2^n}\boldsymbol{C}$ $[n\in\mathbb{Z}]$ BI (92)(14)

2. $\displaystyle\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-\frac{1}{1+x^2}\right\}\frac{dx}{x}=-2^{-n}\boldsymbol{C}$ BI (92)(13)

3. $\displaystyle\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-e^{-x}\right\}\frac{dx}{x}=\left(1-2^{-n}\right)\boldsymbol{C}$ BI (89)(8)

3.476

1. $$\int_0^\infty \left[\exp\left(-\nu x^p\right) - \exp\left(-\mu x^p\right)\right] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ BI (89)(3)

2. $$\int_0^\infty \left[\exp\left(-x^p\right) - \exp\left(-x^q\right)\right] \frac{dx}{x} = \frac{p-q}{pq} \boldsymbol{C} \qquad [p > 0, \quad q > 0]$$ BI (89)(9)

3.477

1.[10] $$\int_{-\infty}^\infty \frac{e^{-a|x|}}{x-u}\, dx = e^{-au}\, \gamma(0, -au) - e^{au}\, \gamma(0, au) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Im} u \neq 0, \quad \arg u \neq 0]$$ MC

2.[8] $$\int_{-\infty}^\infty \frac{\operatorname{sign} x \exp\left(-a|x|\right)}{x-u}\, dx = -\left[\exp\left(a|u|\right) \operatorname{Ei}\left(-a|u|\right) - \exp\left(-a|u|\right) \operatorname{Ei}\left(a|u|\right)\right]$$
$$[a > 0]$$ ET II 251(36)

3.478

1. $$\int_0^\infty x^{\nu-1} \exp\left(-\mu x^p\right)\, dx = \frac{1}{p} \mu^{-\frac{\nu}{p}}\, \Gamma\left(\frac{\nu}{p}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad p > 0]$$
BI(81)(8)a, ET I 313(15, 16)

2. $$\int_0^\infty x^{\nu-1} \left[1 - \exp\left(-\mu x^p\right)\right]\, dx = -\frac{1}{|p|} \mu^{-\frac{\nu}{p}}\, \Gamma\left(\frac{\nu}{p}\right)$$
$$[\operatorname{Re}\mu > 0 \text{ and } -p < \operatorname{Re}\nu < 0 \text{ for } p > 0, \quad 0 < \operatorname{Re}\nu < -p \text{ for } p < 0]$$ ET I 313(18, 19)

3.[11] $$\int_0^u x^{\nu-1} (u-x)^{\mu-1} \exp\left(\beta x^n\right)\, dx = \mathrm{B}(\mu, \nu) u^{\mu+\nu-1}\, {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \ldots, \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \ldots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad n = 2, 3, \ldots]$$ ET II 187(15)

4. $$\int_0^\infty x^{\nu-1} \exp\left(-\beta x^p - \gamma x^{-p}\right)\, dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\beta\gamma}\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$$ ET I 313(17)

3.479

1. $$\int_0^\infty \frac{x^{\nu-1} \exp\left(-\beta\sqrt{1+x}\right)}{\sqrt{1+x}}\, dx = \frac{2}{\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu} \Gamma(\nu)\, K_{\frac{1}{2}-\nu}(\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$$ ET I 313(14)

2.[11] $$\int_0^\infty \frac{x^{\nu-1} \exp\left(i\mu\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\, dx = i\frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2}\right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) H^{(1)}_{\frac{1-\nu}{2}}(\mu)$$
$$[\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(30)

3.481

1. $$\int_{-\infty}^\infty x e^x \exp\left(-\mu e^x\right)\, dx = -\frac{1}{\mu} \left(\boldsymbol{C} + \ln \mu\right) \qquad [\operatorname{Re}\mu > 0]$$ BI (100)(13)

2. $$\int_{-\infty}^{\infty} xe^x \exp\left(-\mu e^{2x}\right)\, dx = -\frac{1}{4}\left[\boldsymbol{C} + \ln(4\mu)\right]\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0]$$ BI (100)(14)

3.482

1.[3] $$\int_0^{\infty} \exp\left(nx - \beta\sinh x\right)\, dx = \frac{1}{2}\left[S_n(\beta) - \pi\,\mathbf{E}_n(\beta) - \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(11)

2. $$\int_0^{\infty} \exp\left(-nx - \beta\sinh x\right)\, dx = (-1)^{n+1}\frac{1}{2}\left[S_n(\beta) + \pi\,\mathbf{E}_n(\beta) + \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(12)

3. $$\int_0^{\infty} \exp\left(-\nu x - \beta\sinh x\right)\, dx = \frac{\pi}{\sin\nu\pi}\left[\mathbf{J}_\nu(\beta) - J_\nu(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(13)

3.483 $$\int_{-\infty}^{\infty} \frac{\exp\left(\nu\operatorname{arcsinh} x - iax\right)}{\sqrt{1+x^2}}\, dx = \begin{cases} 2\exp\left(-\dfrac{i\nu\pi}{2}\right) K_\nu(a) & \text{for } a > 0, \\ 2\exp\left(\dfrac{i\nu\pi}{2}\right) K_\nu(-a) & \text{for } a < 0 \end{cases} \qquad [|\operatorname{Re}\nu| < 1]$$
ET I 122(32)

3.484 $$\int_0^{\infty} \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px}\right] \frac{dx}{x} = (e^a - 1)\ln\frac{q}{p} \qquad [p > 0, \quad q > 0]$$ BI (89)(34)

3.485 $$\int_0^{\pi/2} \exp\left(-\tan^2 x\right)\, dx = \frac{\pi e}{2}\left[1 - \Phi(1)\right]$$

3.486[6] $$\int_0^1 x^{-x}\, dx = \int_0^1 e^{-x\ln x}\, dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627\ldots$$ FI II 483

3.487

1.* $$\int_0^{\pi/4} \exp\left[-\sum_{k=0}^{\infty}\left(\frac{\tan^{2k+1} x}{k + \frac{1}{2}}\right)\right] dx = \ln 2$$

3.5 Hyperbolic Functions

3.51 Hyperbolic functions

3.511

1. $$\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a} \qquad [a > 0]$$

2. $$\int_0^{\infty} \frac{\sinh ax}{\sinh bx}\, dx = \frac{\pi}{2b}\tan\frac{a\pi}{2b} \qquad [b > |a|]$$ BI (27)(10)a

3. $$\int_0^{\infty} \frac{\sinh ax}{\cosh bx}\, dx = \frac{\pi}{2b}\sec\frac{a\pi}{2b} - \frac{1}{b}\,\beta\left(\frac{a+b}{2b}\right) \qquad [b > |a|]$$ GW (351)(3b)

4. $$\int_0^{\infty} \frac{\cosh ax}{\cosh bx}\, dx = \frac{\pi}{2b}\sec\frac{a\pi}{2b} \qquad [b > |a|]$$ BI (4)(14)a

5. $$\int_0^\infty \frac{\sinh ax \cosh bx}{\sinh cx}\,dx = \frac{\pi}{2c}\frac{\sin\frac{a\pi}{c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}} \qquad [c>|a|+|b|]$$ BI (27)(11)

6. $$\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh cx}\,dx = \frac{\pi}{c}\frac{\cos\frac{a\pi}{2c}\cos\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}} \qquad [c>|a|+|b|]$$ BI (27)(5)a

7. $$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh cx}\,dx = \frac{\pi}{c}\frac{\sin\frac{a\pi}{2c}\sin\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}} \qquad [c>|a|+|b|]$$ BI (27)(6)a

8.[11] $$\int_0^\infty \frac{dx}{\cosh^2 x} = 1$$ BI (98)(25)

9. $$\int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x}\,dx = 1 - a\pi\cot a\pi \qquad \left[a^2<1\right]$$ BI (16)(3)a

10. $$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh^2 bx}\,dx = \frac{a\pi}{2b^2}\sec\frac{a\pi}{2b} \qquad [b>|a|]$$ BI (27)(16)a

3.512

1. $$\int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax}\,dx = \frac{4^{\nu-1}}{a}\,\mathrm{B}\left(\nu+\frac{\beta}{a},\nu-\frac{\beta}{a}\right) \qquad [\mathrm{Re}\,(\nu\pm\beta)>0,\quad a>0,\quad \beta>0]$$ LI(27)(17)a, EH I 11(26)

2. $$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu+1}{2},\frac{\nu-\mu}{2}\right) \qquad [\mathrm{Re}\,\mu>-1,\quad \mathrm{Re}(\mu-\nu)<0]$$ EH I 11(23)

3.513

1. $$\int_0^\infty \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \qquad [ab\neq 0]$$ GW (351)(8)

2. $$\int_0^\infty \frac{dx}{a+b\cosh x} = \frac{2}{\sqrt{b^2-a^2}}\arctan\frac{\sqrt{b^2-a^2}}{a+b} \qquad \left[b^2>a^2\right]$$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \qquad \left[b^2<a^2\right]$$
GW (351)(7)

3. $$\int_0^\infty \frac{dx}{a\sinh x+b\cosh x} = \frac{2}{\sqrt{b^2-a^2}}\arctan\frac{\sqrt{b^2-a^2}}{a+b} \qquad \left[b^2>a^2\right]$$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}} \qquad \left[a^2>b^2\right]$$
GW (351)(9)

4. $$\int_0^\infty \frac{dx}{a+b\cosh x+c\sinh x} = \frac{2}{\sqrt{b^2-a^2-c^2}}\left[\arctan\frac{\sqrt{b^2-a^2-c^2}}{a+b+c}+\epsilon\pi\right]$$

$$\left[\text{when } b^2>a^2+c^2;\ \text{and } \begin{cases}\epsilon=0 & \text{for } (b-a)(a+b+c)>0\\ |\epsilon|=1 & \text{for } (b-a)(a+b+c)<0\\ \epsilon=1 & \text{for } a<b+c\\ \epsilon=-1 & \text{for } a>b+c\end{cases}\right]$$

$$= \frac{1}{\sqrt{a^2-b^2+c^2}}\ln\frac{a+b+c+\sqrt{a^2-b^2+c^2}}{a+b+c-\sqrt{a^2-b^2+c^2}} \qquad \left[b^2<a^2+c^2,\quad a^2\neq b^2\right]$$

$$= \frac{1}{c}\ln\frac{a+c}{a} \qquad [a=b\neq 0,\quad c\neq 0]$$

$$= \frac{2(a-b)}{c(a-b-c)} \qquad \left[b^2=a^2+c^2,\quad c(a-b-c)<0\right]$$

GW (351)(6)

3.514

1. $$\int_0^\infty \frac{dx}{\cosh ax+\cos t} = \frac{t}{a}\operatorname{cosec} t \qquad [0<t<\pi,\quad a>0]$$ BI (27)(22)a

2. $$\int_0^\infty \frac{\cosh ax-\cos t_1}{\cosh bx-\cos t_2}\,dx = \frac{\pi}{b}\frac{\sin\frac{a(\pi t_2)}{b}}{\sin t_2\sin\frac{a}{b}\pi} - \frac{\pi t_2}{b\sin t_2}\cos t_1$$

$$[0<|a|<b,\quad 0<t_2<\pi]$$ BI (6)(20)a

3. $$\int_0^\infty \frac{\cosh ax\,dx}{(\cosh x+\cos t)^2} = \frac{\pi\left(-\cos t\sin at+a\sin t\cos at\right)}{\sin^3 t\sin a\pi}$$

$$\left[0<a^2<1,\quad 0<t<\pi\right]$$ BI (6)(18)a

4. $$\int_0^\infty \frac{\sinh ax\sinh bx}{(\cosh ax+\cos t)^2}\,dx = \frac{b\pi}{a^2}\operatorname{cosec} t\operatorname{cosec}\frac{b\pi}{a}\sin\frac{bt}{a} \qquad [0<|b|<a,\quad 0<t<\pi]$$ BI (27)(27)a

3.515 $$\int_{-\infty}^\infty \left(1-\frac{\sqrt{2}\cosh x}{\sqrt{\cosh 2x}}\right)dx = -\ln 2$$ BI (21)(12)a

3.516

1. $$\int_0^\infty \frac{dx}{\left(z+\sqrt{z^2-1}\cosh x\right)^\mu} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\left(z+\sqrt{z^2-1}\cosh x\right)^\mu} = Q_{\mu-1}(z)$$

$$[\operatorname{Re}\mu>-1]$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of z in the z-plane cut from -1 to $+1$ provided $\mu<0$. If $\mu>0$, this formula ceases to be valid for points at which the denominator vanishes. CO, WH

1. $$\int_0^\infty \frac{dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{n+1}} = Q_n(\beta)$$ EH II 181(32)

2. $$\int_0^\infty \frac{\cosh \gamma x\, dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{\nu+1}} = \frac{e^{-i\gamma\pi}\,\Gamma(\nu-\gamma+1)\,Q_\nu^\gamma(\beta)}{\Gamma(\nu+1)}$$
$[\operatorname{Re}(\nu\pm\gamma)>-1, \quad \nu\neq -1,-2,-3,\ldots]$ EH I 157(12)

3. $$\int_0^\infty \frac{\sinh^{2\mu} x\, dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}\,\Gamma(\nu-2\mu+1)\,\Gamma\left(\mu+\frac12\right)}{\sqrt{\pi}\,(\beta^2-1)^{\frac{\mu}{2}}\,\Gamma(\nu+1)}\,Q_{\nu-\mu}^\mu(\beta)$$
$[\operatorname{Re}(\nu-2\mu+1)>0, \quad \operatorname{Re}(\nu+1)>0]$ EH I 155(2)

3.517

1. $$\int_0^\infty \frac{\cosh\left(\gamma+\frac12\right)x\, dx}{(\beta+\cosh x)^{\nu+\frac12}} = \sqrt{\frac{\pi}{2}}\left(\beta^2-1\right)^{-\frac{\nu}{2}}\frac{\Gamma(\nu+\gamma+1)\,\Gamma(\nu-\gamma)\,P_\gamma^{-\nu}(\beta)}{\Gamma\left(\nu+\frac12\right)}$$
$[\operatorname{Re}(\nu-\gamma)>0, \quad \operatorname{Re}(\nu+\gamma+1)>0]$ EH I 156(11)

2. $$\int_0^a \frac{\cosh\left(\gamma+\frac12\right)x\, dx}{(\cosh a-\cosh x)^{\nu+\frac12}} = \sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac12-\nu\right)}{\sinh^\nu a}\,P_\gamma^\nu(\cosh a)$$
$\left[\operatorname{Re}\nu<\frac12, \quad a>0\right]$ EH I 156(8)

3.518

1. $$\int_0^\infty \frac{\sinh^{2\mu} x\, dx}{(\cosh a+\sinh a\cosh x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi}\sinh^\mu a}\frac{\Gamma(\nu-2\mu+1)\,\Gamma\left(\mu+\frac12\right)}{\Gamma(\nu+1)}\,Q_{\nu-\mu}^\mu(\cosh a)$$
$[\operatorname{Re}(\nu+1)>0, \quad \operatorname{Re}(\nu-2\mu+1)>0, \quad a>0]$ EH I 155(3)a

2.[10] $$\int_0^\infty \frac{\sinh^{2\mu+1} x\, dx}{(\beta+\cosh x)^{\nu+1}} = 2^\mu\left(\beta^2-1\right)^{\frac{\mu-\nu}{2}}\Gamma(\nu-2\mu)\,\Gamma(\mu+1)\,P_\mu^{\mu-\nu}(\beta)$$
$[\operatorname{Re}(\nu-\mu)>\operatorname{Re}\mu>-1, \quad \beta$ does not lie on the ray $(-\infty,+1)$ of the real axis$]$ EH I 155(1)

3. $$\int_0^\infty \frac{\sinh^{2\mu-1} x\cosh x\, dx}{\left(1+a\sinh^2 x\right)^\nu} = \frac12 a^{-\mu}\,\mathrm{B}(\mu,\nu-\mu)$$
$[\operatorname{Re}\nu>\operatorname{Re}\mu>0, \quad a>0]$ EH I 11(22)

4.[7] $$\int_0^\infty \frac{\sinh^{\mu-1} x\,(\cosh x+1)^{\nu-1}\, dx}{(\beta+\cosh x)^\varrho} = 2^{\mu+\nu-\rho}\,\mathrm{B}\left(\frac12\mu,\varrho+2-\mu-\nu\right)$$
$$\times\,{}_2F_1\left(\varrho,\varrho+2-\mu-\nu;2-\frac12\mu-\nu;\frac12-\frac12\beta\right)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}(\varrho-\mu-\nu)>-2, \quad |\arg(1+\beta)|<\pi]$ EH I 115(11)

5.[6] $$\int_0^\infty \frac{\sinh^{\mu-1} x\,(\cosh x-1)^{\nu-1}\, dx}{(\beta+\cosh x)^\varrho} = 2^{-(2-\mu-\nu+\varrho)}\,{}_2F_1\left(\varrho,2-\mu-\nu+\varrho;1+\varrho-\frac{\mu}{2};\frac{1-\beta}{2}\right)$$
$$\times\,\mathrm{B}\left(2-\mu-\nu+\varrho,-1+\nu+\frac{\mu}{2}\right)$$
$[\beta\notin(-\infty,-1), \quad \operatorname{Re}(2+\varrho)\operatorname{Re}(\mu+\nu), \quad \operatorname{Re}(2\nu+\mu)>2]$ EH I 115(10)

6.[7] $$\int_0^\infty \frac{\sinh^{\mu-1} x \cosh^{\nu-1} x}{\left(\cosh^2 x-\beta\right)^{\varrho}}\,dx = {}_2F_1\left(\varrho, 1+\varrho-\frac{\mu+\nu}{2}; 1+\varrho-\frac{\nu}{2}; \beta\right) 2\,\mathrm{B}\left(\frac{\mu}{2}, 1+\varrho-\frac{\mu+\nu}{2}\right)$$
$[\beta \notin (1,\infty)\,, \quad \operatorname{Re}\mu > 0, \quad 2\operatorname{Re}(1+\varrho) > \operatorname{Re}(\mu+\nu)]$ EH I 115(9)

3.519 $$\int_0^{\pi/2} \frac{\sinh\left[(r-p)\right]\tan x}{\sinh\left(r\tan x\right)}\,dx = \pi\sum_{k=1}^{\infty}\frac{1}{k\pi+r}\sin\frac{pk\pi}{r} \qquad \left[p^2<r^2\right]$$ BI (274)(13)

3.52–3.53 Combinations of hyperbolic functions and algebraic functions

3.521

1. $$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2} \qquad [a>0]$$ GW (352)(2b)

2. $$\int_0^\infty \frac{x\,dx}{\cosh x} = 2\boldsymbol{G} = \pi\ln 2 - 4\,L\left(\frac{\pi}{4}\right) = 1.831931188\ldots$$ LI III 225(103a), BI(84)(1)a

3. $$\int_1^\infty \frac{dx}{x\sinh ax} = -2\sum_{k=0}^{\infty}\operatorname{Ei}[-(2k+1)a] \qquad [a>0]$$ LI (104)(14)

4. $$\int_1^\infty \frac{dx}{x\cosh ax} = 2\sum_{k=0}^{\infty}(-1)^{k+1}\operatorname{Ei}[-(2k+1)a] \qquad [a>0]$$ LI (104)(13)

3.522

1. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\sinh ax} = \frac{\pi}{2ab} + \pi\sum_{k=1}^{\infty}\frac{(-1)^k}{ab+k\pi} \qquad [a>0, \quad b>0]$$

2. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\sinh \pi x} = \frac{1}{2b} - \beta(b+1) \qquad [b>0]$$ BI(97)(16), GW(352)(8)

3. $$\int_0^\infty \frac{dx}{(b^2+x^2)\cosh ax} = \frac{2\pi}{b}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2ab+(2k-1)\pi} \qquad [a>0, \quad b>0]$$ BI (97)(5)

4. $$\int_0^\infty \frac{dx}{(b^2+x^2)\cosh \pi x} = \frac{1}{b}\,\beta\left(b+\frac{1}{2}\right) \qquad [b>0]$$ BI (97)(4)

5. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \pi x} = \ln 2 - \frac{1}{2}$$ BI (97)(7)

6. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh \pi x} = 2 - \frac{\pi}{2}$$ BI (97)(1)

7. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \frac{\pi x}{2}} = \frac{\pi}{2} - 1$$ BI (97)(8)

8. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh \frac{\pi x}{2}} = \ln 2$$ BI (97)(2)

9. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi + 2\ln\left(\sqrt{2}+1\right)\right] - 2$$ BI (97)(9)

10. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi - 2\ln\left(\sqrt{2}+1\right)\right]$$ BI (97)(3)

3.523

1. $$\int_0^\infty \frac{x^{\beta-1}}{\sinh ax}\,dx = \frac{2^\beta - 1}{2^{\beta-1}a^\beta}\,\Gamma(\beta)\,\zeta(\beta)$$ $[\operatorname{Re}\beta > 1, \quad a > 0]$ WH

2. $$\int_0^\infty \frac{x^{2n-1}}{\sinh ax}\,dx = \frac{2^{2n}-1}{2n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$ $[a > 0, \quad n = 1, 2, \ldots]$ WH, GW(352)(2a)

3. $$\int_0^\infty \frac{x^{\beta-1}}{\cosh ax}\,dx = \frac{2}{(2a)^\beta}\,\Gamma(\beta)\,\Phi\left(-1,\beta,\frac{1}{2}\right) = \frac{2}{(2a)^\beta}\,\Gamma(\beta)\sum_{k=0}^\infty(-1)^k\left(\frac{2}{2k+1}\right)^\beta$$ $[\operatorname{Re}\beta > 0, \quad a > 0]$ EH I 35, ET I 322(1)

4. $$\int_0^\infty \frac{x^{2n}}{\cosh ax}\,dx = \left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|$$ $[a > 0]$ BI(84)(12)a, GW(352)(1a)

5. $$\int_0^\infty \frac{x^2\,dx}{\cosh x} = \frac{\pi^3}{8}$$ (cf. **4.261** 6) BI (84)(3)

6. $$\int_0^\infty \frac{x^3\,dx}{\sinh x} = \frac{\pi^4}{8}$$ (cf. **4.262** 1 and 2) BI (84)(5)

7. $$\int_0^\infty \frac{x^4\,dx}{\cosh x} = \frac{5}{32}\pi^5$$ BI (84)(7)

8. $$\int_0^\infty \frac{x^5}{\sinh x}\,dx = \frac{\pi^6}{4}$$ BI (84)(8)

9. $$\int_0^\infty \frac{x^6}{\cosh x}\,dx = \frac{61}{128}\pi^7$$ BI (84)(9)

10. $$\int_0^\infty \frac{x^7}{\sinh x}\,dx = \frac{17}{16}\pi^8$$ BI (84)(10)

11. $$\int_0^\infty \frac{x^{1/2}\,dx}{\cosh x} = \sqrt{\pi}\sum_{k=0}^\infty(-1)^k\frac{1}{(2k+1)^{3/2}}$$ BI (98)(7)a

12. $$\int_0^\infty \frac{dx}{x^{1/2}\cosh x} = 2\sqrt{\pi}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{1/2}}$$ BI (98)(25)a

3.524

1. $$\int_0^\infty x^{\mu-1}\frac{\sinh\beta x}{\sinh\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\zeta\left[\mu,\frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right] - \zeta\left[\mu,\frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]\right\}$$ $[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > -1]$ ET I 323(10)

2.[11] $$\int_0^\infty x^{2m}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m}}{da^{2m}}\left(\tan\frac{a\pi}{2b}\right)$$ $[b > |a|]$ BI (112)(20)a

3. $$\int_0^\infty \frac{\sinh ax}{\sinh bx}\frac{dx}{x^p} = \Gamma(1-p)\sum_{k=0}^{\infty}\left\{\frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}}\right\}$$
$[b > |a|, \quad p < 1]$ BI (131)(2)a

4.[11] $$\int_0^\infty x^{2m+1}\frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m+1}}{da^{2m+1}}\left(\sec\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (112)(18)a

5. $$\int_0^\infty x^{\mu-1}\frac{\cosh\beta x}{\sinh\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\zeta\left[\mu,\frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right] + \zeta\left[\mu,\frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]\right\}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > 1]$
ET I 323(12)

6. $$\int_0^\infty x^{2m}\frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m}}{da^{2m}}\left(\sec\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI(112)(17)

7. $$\int_0^\infty \frac{\cosh ax}{\cosh bx}\cdot\frac{dx}{x^p} = \Gamma(1-p)\sum_{k=0}^{\infty}(-1)^k\left\{\frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}}\right\}$$
$[b > |a|, \quad p < 1]$ BI(131)(1)a

8. $$\int_0^\infty x^{2m+1}\frac{\cosh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m+1}}{da^{2m+1}}\left(\tan\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (112)(19)a

9.[8] $$\int_0^\infty x^2\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi^3}{4b^3}\sin\frac{a\pi}{2b}\sec^3\frac{a\pi}{2b}$$
$[b > |a|]$ BI (84)(18)

10. $$\int_0^\infty x^4\frac{\sinh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^5\cdot\sin\frac{a\pi}{2b}\cdot\left(2+\sin^2\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(17)a

11. $$\int_0^\infty x^6\frac{\sinh ax}{\sinh bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^7\sin\frac{a\pi}{2b}\left(45-30\cos^2\frac{a\pi}{2b}+2\cos^4\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(21)a

12. $$\int_0^\infty x\frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi^2}{4b^2}\sin\frac{a\pi}{2b}\sec^2\frac{a\pi}{2b}$$
$[b > |a|]$ BI (84)(15)a

13. $$\int_0^\infty x^3\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4\sin\frac{a\pi}{2b}\cdot\left(6-\cos^2\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(14)a

14. $$\int_0^\infty x^5\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^6\sin\frac{a\pi}{2b}\left(120-60\cos^2\frac{a\pi}{2b}+\cos^4\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(18)a

15. $$\int_0^\infty x^7\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8\sin\frac{a\pi}{2b}\left(5040-4200\cos^2\frac{a\pi}{2b}+546\cos^4\frac{a\pi}{2b}-\cos^6\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(22)a

16. $$\int_0^\infty x\frac{\cosh ax}{\sinh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^2$$
$[b > |a|]$ BI (84)(16)a

17. $$\int_0^\infty x^3 \frac{\cosh ax}{\sinh bx}\,dx = 2\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4\left(1+2\sin^2\frac{a\pi}{2b}\right) \qquad [b>|a|]$$ BI (82)(15)a

18. $$\int_0^\infty x^5 \frac{\cosh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^6\left(15-15\cos^2\frac{a\pi}{2b}+2\cos^4\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(19)a

19. $$\int_0^\infty x^7 \frac{\cosh ax}{\sinh bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8\left(315-420\cos^2\frac{a\pi}{2b}+126\cos^4\frac{a\pi}{2b}-4\cos^6\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI(82)(23)a

20. $$\int_0^\infty x^2 \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi^3}{8b^3}\left(2\sec^3\frac{a\pi}{2b}-\sec\frac{a\pi}{2b}\right) \qquad [b>|a|]$$ BI (84)(17)a

21. $$\int_0^\infty x^4 \frac{\cosh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^5\left(24-20\cos^2\frac{a\pi}{2b}+\cos^4\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(16)a

22. $$\int_0^\infty x^6 \frac{\cosh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^7\left(720-840\cos^2\frac{a\pi}{2b}+182\cos^4\frac{a\pi}{2b}-\cos^6\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(20)a

23. $$\int_0^\infty \frac{\sinh ax}{\cosh bx}\cdot\frac{dx}{x} = \ln\tan\left(\frac{a\pi}{4b}+\frac{\pi}{4}\right) \qquad [b>|a|]$$ BI (95)(3)a

3.525

1. $$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cdot\frac{dx}{1+x^2} = -\frac{a}{2}\cos a+\frac{1}{2}\sin a\ln\left[2\left(1+\cos a\right)\right]$$
$[\pi\geq|a|]$ BI (97)(10)a

2. $$\int_0^\infty \frac{\sinh ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\sin a+\frac{1}{2}\cos a\ln\frac{1-\sin a}{1+\sin a} \qquad [\pi\geq 2|a|]$$ BI (97)(11)a

3. $$\int_0^\infty \frac{\cosh ax}{\sinh \pi x}\cdot\frac{x\,dx}{1+x^2} = \frac{1}{2}\left(a\sin a-1\right)+\frac{1}{2}\cos a\ln\left[2\left(1+\cos a\right)\right]$$
$[\pi>|a|]$ BI (97)(12)a

4. $$\int_0^\infty \frac{\cosh ax}{\sinh \frac{\pi}{2}x}\cdot\frac{x\,dx}{1+x^2} = \frac{\pi}{2}\cos a-1+\frac{1}{2}\sin a\ln\frac{1+\sin a}{1-\sin a}$$
$\left[\frac{\pi}{2}>|a|\right]$ BI (97)(13)a

5. $$\int_0^\infty \frac{\sinh ax}{\cosh \pi x}\cdot\frac{x\,dx}{1+x^2} = -2\sin\frac{a}{2}+\frac{\pi}{2}\sin a-\cos a\ln\tan\frac{a+\pi}{4}$$
$[\pi>|a|]$ GW (352)(12)

6. $$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\cdot\frac{dx}{1+x^2} = 2\cos\frac{a}{2}-\frac{\pi}{2}\cos a-\sin a\ln\tan\frac{a+\pi}{4}$$
$[\pi>|a|]$ GW (352)(11)

7. $$\int_0^\infty \frac{\sinh ax}{\sinh bx}\cdot\frac{dx}{c^2+x^2}=\frac{\pi}{c}\sum_{k=1}^\infty \frac{\sin\frac{k(b-a)}{b}\pi}{bc+k\pi}$$ $[b\ge|a|]$ BI (97)(18)

8. $$\int_0^\infty \frac{\cosh ax}{\sinh bx}\cdot\frac{x\,dx}{c^2+x^2}=\frac{\pi}{2bc}+\pi\sum_{k=1}^\infty \frac{\cos\frac{k(b-a)}{b}\pi}{bc+k\pi}$$ $[b>|a|]$ BI (97)(19)

3.526

1. $$\int_0^\infty \frac{\sinh ax\cosh bx}{\cosh cx}\cdot\frac{dx}{x}=\frac{1}{2}\ln\left\{\tan\frac{(a+b+c)\pi}{4c}\cot\frac{(b+c-a)\pi}{4c}\right\}$$ $[c>|a|+|b|]$ BI (93)(10)a

2. $$\int_0^\infty \frac{\sinh^2 ax}{\sinh bx}\cdot\frac{dx}{x}=\frac{1}{2}\ln\sec\frac{a}{b}\pi$$ $[b>|2a|]$ BI (95)(5)a

3. $$\int_0^\infty \frac{x^{\mu-1}}{\sinh\beta x\cosh\gamma x}\,dx=\frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\Phi\left[-1,\mu,\frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]+\Phi\left[-1,\mu,\frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right]\right\}$$ $[\operatorname{Re}\gamma>|\operatorname{Re}\beta|,\quad \operatorname{Re}\mu>0]$ ET I 323(11)

3.527

1. $$\int_0^\infty \frac{x^{\mu-1}}{\sinh^2 ax}\,dx=\frac{4}{(2a)^\mu}\Gamma(\mu)\,\zeta(\mu-1)$$ $[\operatorname{Re}a>0,\quad \operatorname{Re}\mu>2]$ BI (86)(7)a

2. $$\int_0^\infty \frac{x^{2m}}{\sinh^2 ax}\,dx=\frac{\pi^{2m}}{a^{2m+1}}|B_{2m}|$$ $[a>0,\quad m=1,2,\ldots]$ BI(86)(5)a

3.[6] $$\int_0^\infty \frac{x^{\mu-1}}{\cosh^2 ax}\,dx=\frac{4}{(2a)^\mu}\left(1-2^{2-\mu}\right)\Gamma(\mu)\,\zeta(\mu-1)$$ $[\operatorname{Re}a>0,\quad \operatorname{Re}\mu>0,\quad \mu\ne 2]$
$$=\frac{1}{a^2}\ln 2$$ $[\operatorname{Re}a>0,\quad \mu=2]$
BI (86)(6)a

4. $$\int_0^\infty \frac{x\,dx}{\cosh^2 ax}=\frac{\ln 2}{a^2}$$ $[a\ne 0]$ LO III 396

5. $$\int_0^\infty \frac{x^{2m}}{\cosh^2 ax}\,dx=\frac{\left(2^{2m}-2\right)\pi^{2m}}{(2a)^{2m}a}|B_{2m}|$$ $[a>0,\quad m=1,2,\ldots]$ BI(86)(2)a

6. $$\int_0^\infty x^{\mu-1}\frac{\sinh ax}{\cosh^2 ax}\,dx=\frac{2\,\Gamma(\mu)}{a^\mu}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{\mu-1}}$$ $[\operatorname{Re}\mu>1,\quad a>0]$ BI (86)(15)a

7. $$\int_0^\infty \frac{x\sinh ax}{\cosh^2 ax}\,dx=\frac{\pi}{2a^2}$$ $[a>0]$ BI (86)(8)a

8. $$\int_0^\infty x^{2m+1}\frac{\sinh ax}{\cosh^2 ax}\,dx=\frac{2m+1}{a}\left(\frac{\pi}{2a}\right)^{2m+1}|E_{2m}|$$ $[a>0,\quad m=0,1,\ldots]$ BI (86)(12)a

9. $$\int_0^\infty x^{2m+1}\frac{\cosh ax}{\sinh^2 ax}\,dx=\frac{2^{2m+1}-1}{a^2(2a)^{2m}}(2m+1)!\,\zeta(2m+1)$$ $[a\ne 0,\quad m=1,2,\ldots]$ BI (86)(13)a

10.[11] $\displaystyle\int_0^\infty x^{2m}\frac{\cosh ax}{\sinh^2 ax}\,dx=\frac{2^{2m}-1}{a}\left(\frac{\pi}{a}\right)^{2m}|B_{2m}|$ $[a>0,\quad m=1,2,\ldots]$ BI (86)(14)a

11.[8] $\displaystyle\int_0^\infty \frac{x\sinh ax}{\cosh^{2\mu+1} ax}\,dx=\frac{\sqrt{\pi}}{4\mu a^2}\frac{\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{2}\right)}$ $[\mu>0,\quad a>0]$ LI (86)(9)

12. $\displaystyle\int_{-\infty}^\infty \frac{x^2\,dx}{\sinh^2 x}=\frac{\pi^2}{3}$ BI (102)(2)a

13. $\displaystyle\int_0^\infty x^2\frac{\cosh ax}{\sinh^2 ax}\,dx=\frac{\pi^2}{2a^3}$ $[a>0]$ BI (86)(11)a

14.[11] $\displaystyle\int_0^\infty x^2\frac{\sinh x}{\cosh^2 x}\,dx=4\boldsymbol{G}$ $[a\neq 0]$ BI (86)(10)a

15.[10] $\displaystyle\int_0^\infty \frac{\tanh\frac{x}{2}\,dx}{\cosh x}=\ln 2$ BI (93)(17)a

16.* $\displaystyle\int_0^\infty x^{\mu-1}\frac{\cosh ax}{\sinh^2 ax}=\frac{2\,\Gamma(\mu)\,\zeta(\mu-1)}{a^\mu}\left(1-2^{1-\mu}\right)$

3.528

1. $\displaystyle\int_0^\infty \frac{(1+xi)^{2n-1}-(1-xi)^{2n-1}}{i\sinh\frac{\pi x}{2}}\,dx=2$ BI (87)(8)

2. $\displaystyle\int_0^\infty \frac{(1+xi)^{2n}-(1-xi)^{2n}}{i\sinh\frac{\pi x}{2}}\,dx=(-1)^{n+1}2|E_{2n}|+2$ $[n=0,1,\ldots]$ BI (87)(7)

3.529

1. $\displaystyle\int_0^\infty \left(\frac{1}{\sinh x}-\frac{1}{x}\right)\frac{dx}{x}=-\ln 2$ BI (94)(10)a

2. $\displaystyle\int_0^\infty \frac{\cosh ax-1}{\sinh bx}\cdot\frac{dx}{x}=-\ln\cos\frac{a\pi}{2b}$ $[b>|a|]$ GW (352)(66)

3. $\displaystyle\int_0^\infty \left(\frac{a}{\sinh ax}-\frac{b}{\sinh bx}\right)\frac{dx}{x}=(b-a)\ln 2$ BI (94)(11)a

3.531

1.[7] $\displaystyle\int_0^\infty \frac{x\,dx}{2\cosh x-1}=\frac{4}{\sqrt{3}}\left[\frac{\pi}{3}\ln 2-L\left(\frac{\pi}{3}\right)\right]=1.1719536193\ldots$

[see **8.26** for $L(x)$] LI (88)(1)

2.[10] $\displaystyle\int_0^\infty \frac{x\,dx}{\cosh 2x+\cos 2t}=\frac{t\ln 2-L(t)}{\sin 2t}$ LO III 402

3. $\displaystyle\int_0^\infty \frac{x^2\,dx}{\cosh x+\cos t}=\frac{t}{3}\cdot\frac{\pi^2-t^2}{\sin t}$ $[0<t<\pi]$ BI (88)(3)a

4. $\displaystyle\int_0^\infty \frac{x^4\,dx}{\cosh x+\cos t}=\frac{t}{15}\frac{\left(\pi^2-t^2\right)\left(7\pi^2-3t^2\right)}{\sin t}$ $[0<t<\pi]$ BI (88)(4)a

5.[3] $$\int_0^\infty \frac{x^{2m}\,dx}{\cosh x-\cos 2a\pi} = 2(2m)!\operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \qquad \left[0<a<1, \quad a\neq \tfrac{1}{2}\right]$$
$$= 2\left(2^{2m-1}-1\right)\pi^{2m}|B_{2m}| \qquad \left[a=\tfrac{1}{2}\right]$$
BI (88)(5)a

6.[3] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh x-\cos t}$$
$$= \frac{i\,\Gamma(\mu)}{\sin t}\left[e^{-it}\,\Phi\left(e^{-it},\mu,1\right)-e^{it}\,\Phi\left(e^{it},\mu,1\right)\right] \qquad [\operatorname{Re}\mu>0, \quad 0<t<2\pi, \quad t\neq\pi]$$ ET I 323(5)
$$= \left(2-2^{3-\mu}\right)\Gamma(\mu)\,\zeta(\mu-1) \qquad [\mu\neq 2, \quad t=\pi]$$
$$= 2\ln 2 \qquad [\mu=2, \quad t=\pi]$$

7. $$\int_0^\infty \frac{x^\mu\,dx}{\cosh x+\cos t} = \frac{2\,\Gamma(\mu+1)}{\sin t}\sum_{k=1}^\infty (-1)^{k-1}\frac{\sin kt}{k^{\mu+1}} \qquad [\mu>-1, \quad 0<t<\pi]$$ BII (96)(14)a

8. $$\int_0^u \frac{x\,dx}{\cosh 2x-\cos 2t} = \frac{1}{2}\operatorname{cosec} 2t\,[L(\theta+t)-L(\theta-t)-2\,L(t)]$$
$$[\theta=\arctan(\tanh u\cot t), \quad t\neq n\pi]$$
LO III 402

3.532

1.[11] $$\int_0^\infty \frac{x^n\,dx}{a\cosh x+b\sinh x} = \frac{2n!}{a+b}\sum_{k=0}^\infty \frac{1}{(2k+1)^{n+1}}\left(\frac{b-a}{b+a}\right)^k$$
$$[a>0, \quad b>0, \quad n>-1]$$ GW (352)(5)

2. $$\int_0^u \frac{x\cosh x\,dx}{\cosh 2x-\cos 2t} = \frac{1}{2}\operatorname{cosec} t\left\{L\left(\frac{\theta+t}{2}\right)-L\left(\frac{\theta-t}{2}\right)+L\left(\pi-\frac{\psi+t}{2}\right)\right.$$
$$\left.+\,L\left(\frac{\psi-t}{2}\right)-2\,L\left(\frac{t}{2}\right)-2\,L\left(\frac{\pi-t}{2}\right)\right\}$$
$$\left[\tan\frac{\theta}{2}=\tanh\frac{u}{2}\cot\frac{t}{2}, \quad \tan\frac{\psi}{2}=\coth\frac{u}{2}\cot\frac{t}{2}; \quad t\neq n\pi\right]$$ LO III 288a

3.533

1. $$\int_0^\infty \frac{x\cosh x\,dx}{\cosh 2x-\cos 2t} = \operatorname{cosec} t\left[\frac{\pi}{2}\ln 2-L\left(\frac{t}{2}\right)-L\left(\frac{(\pi-t)}{2}\right)\right]$$
$$[t\neq m\pi]$$ LO III 403

2.[6] $$\int_0^\infty x\frac{\sinh ax\,dx}{(\cosh ax-\cos t)^2} = \frac{\pi-t}{a^2}\operatorname{cosec} t \qquad [a>0, \quad 0<t<\pi] \qquad (\text{cf. } \mathbf{3.514}1)$$
BI (88)(11)a

3. $$\int_0^\infty x^3\frac{\sinh x\,dx}{(\cosh x+\cos t)^2} = \frac{t\left(\pi^2-t^2\right)}{\sin t} \qquad [0<t<\pi] \qquad (\text{cf. } \mathbf{3.531}\ 3)$$
BI (88)(13)

4.[11] $$\int_0^\infty x^{2m+1}\frac{\sinh x\,dx}{(\cosh x-\cos 2a\pi)^2}=2(2m+1)!\operatorname{cosec}2a\pi\sum_{k=1}^\infty\frac{\sin 2ka\pi}{k^{2m+1}}\qquad\left[0<a<1,\quad a\neq\tfrac12\right]$$
$$=2(2m+1)\left(2^{2m-1}-1\right)\pi^{2m}|B_{2m}|\qquad\left[a=\tfrac12\right]$$
BI (88)(14)

3.534

1. $$\int_0^1\sqrt{1-x^2}\cosh ax\,dx=\frac{\pi}{2a}\,I_1(a)$$ WA 94(9)

2. $$\int_0^1\frac{\cosh ax}{\sqrt{1-x^2}}\,dx=\frac{\pi}{2}\,I_0(a)$$ WA 94(9)

3.535 $$\int_0^1\frac{x}{\sqrt{\cosh 2a-\cosh 2ax}}\cdot\frac{dx}{\sinh ax}=\frac{\pi}{2\sqrt{2}a^2}\cdot\frac{\arcsin(\tanh a)}{\sinh a}\qquad[a>0]$$ BI (80)(11)

3.536

1.[11] $$\int_0^\infty\frac{x^2}{\cosh^2 x}\,dx=\frac{\pi^2}{12}$$ BI (98)(7)

2. $$\int_0^\infty\frac{x^2\tanh x^2\,dx}{\cosh^2 x}=\frac{\sqrt{\pi}}{2}\sum_{k=0}^\infty\frac{(-1)^k}{\sqrt{2k+1}}$$ BI (98)(8)

3. $$\int_0^\infty\sinh(\nu\operatorname{arcsinh}x)\frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx=\frac{\sin\frac{\mu\pi}{2}\sin\frac{\nu\pi}{2}}{2^\mu\pi}\Gamma(\mu)\Gamma\left(\frac{1-\mu-\nu}{2}\right)\times\Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$[-1<\operatorname{Re}\mu<1-|\operatorname{Re}\nu|]$ ET I 324(14)

4. $$\int_0^\infty\cosh(\nu\operatorname{arccosh}x)\frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx=\frac{\cos\frac{\mu\pi}{2}\cos\frac{\nu\pi}{2}}{2^\mu\pi}\Gamma(\mu)\Gamma\left(\frac{1-\mu-\nu}{2}\right)\times\Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$[0<\operatorname{Re}\mu<1-|\operatorname{Re}\nu|]$ ET I 324(15)

3.54 Combinations of hyperbolic functions and exponentials

3.541

1. $$\int_0^\infty e^{-\mu x}\sinh^\nu\beta x\,dx=\frac{1}{2^{\nu+1}\beta}\,\mathrm{B}\left(\frac{\mu}{2\beta}-\frac{\nu}{2},\nu+1\right)\qquad[\operatorname{Re}\beta>0,\quad\operatorname{Re}\nu>-1,\operatorname{Re}\mu>\operatorname{Re}\beta\nu]$$
EH I 11(25), ET I 163(5)

2. $$\int_0^\infty e^{-\mu x}\frac{\sinh\beta x}{\sinh bx}\,dx=\frac{1}{2b}\left[\psi\left(\frac12+\frac{\mu+\beta}{2b}\right)-\psi\left(\frac12+\frac{\mu-\beta}{2b}\right)\right]$$
$[\operatorname{Re}(\mu+b\pm\beta)>0]$ EH I 16(14)a

3. $$\int_{-\infty}^\infty e^{-\mu x}\frac{\sinh\mu x}{\sinh\beta x}\,dx=\frac{\pi}{2\beta}\tan\frac{\mu\pi}{\beta}\qquad[\operatorname{Re}\beta>2|\operatorname{Re}\mu|]$$ BI (18)(6)

4. $$\int_0^\infty e^{-x}\frac{\sinh ax}{\sinh x}\,dx=\frac{1}{a}-\frac{\pi}{2}\cot\frac{a\pi}{2}\qquad[0<a<2]$$ BI (4)(3)

5. $$\int_0^\infty\frac{e^{-px}\,dx}{(\cosh px)^{2q+1}}=\frac{2^{2q-2}}{p}\,\mathrm{B}(q,q)-\frac{1}{2qp}\qquad[p>0,\quad q>0]$$ LI (27)(19)

6. $\displaystyle\int_0^\infty e^{-\mu x}\frac{dx}{\cosh x}=\beta\left(\frac{\mu+1}{2}\right)$ $[\operatorname{Re}\mu>-1]$ ET I 163(7)

7. $\displaystyle\int_0^\infty e^{-\mu x}\tanh x\,dx=\beta\left(\frac{\mu}{2}\right)-\frac{1}{\mu}$ $[\operatorname{Re}\mu>0]$ ET I 163(9)

8. $\displaystyle\int_0^\infty \frac{e^{-\mu x}}{\cosh^2 x}\,dx=\mu\,\beta\left(\frac{\mu}{2}\right)-1$ $[\operatorname{Re}\mu>0]$ ET I 163(8)

9. $\displaystyle\int_0^\infty e^{-\mu x}\frac{\sinh\mu x}{\cosh^2\mu x}\,dx=\frac{1}{\mu}\,(1-\ln 2)$ $[\operatorname{Re}\mu>0]$ LI (27)(15)

10. $\displaystyle\int_0^\infty e^{-qx}\frac{\sinh px}{\sinh qx}\,dx=\frac{1}{p}-\frac{\pi}{2q}\cot\frac{p\pi}{2q}$ $[0<p<2q]$ BI (27)(9)a

3.542

1. $\displaystyle\int_0^\infty e^{-\mu x}\left(\cosh\beta x-1\right)^\nu\,dx=\frac{1}{2^\nu\beta}\,\mathrm{B}\left(\frac{\mu}{\beta}-\nu,2\nu+1\right)$

$\left[\operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-\frac{1}{2},\quad \operatorname{Re}\mu>\operatorname{Re}\beta\nu\right]$ ET I 163(6)

2. $\displaystyle\int_0^\infty e^{-\mu x}\left(\cosh x-\cosh u\right)^{\nu-1}\,dx=-i\sqrt{\frac{2}{\pi}}e^{i\pi\nu}\,\Gamma(\nu)\sinh^{\nu-\frac{1}{2}u}\,Q^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh u)$

$[\operatorname{Re}\nu>0,\quad \operatorname{Re}\mu>\operatorname{Re}\nu-1]$

EH I 155(4), ET I 164(23)

3.543

1. $\displaystyle\int_{-\infty}^\infty\frac{e^{-ibx}\,dx}{\sinh x+\sinh t}=-\frac{i\pi e^{itb}}{\sinh\pi b\cosh t}\left(\cosh\pi b-e^{-2itb}\right)$

$[t>0]$ ET I 121(30)

2. $\displaystyle\int_0^\infty\frac{e^{-\mu x}}{\cosh x-\cos t}\,dx=2\operatorname{cosec}t\sum_{k=1}^\infty\frac{\sin kt}{\mu+k}$ $[\operatorname{Re}\mu>-1,\quad t\neq 2n\pi]$ BI (6)(10)a

3. $\displaystyle\int_0^\infty\frac{1-e^{-x}\cos t}{\cosh x-\cos t}e^{-(\mu-1)x}\,dx=2\sum_{k=0}^\infty\frac{\cos kt}{\mu+k}$ $[\operatorname{Re}\mu>0,\quad t\neq 2n\pi]$ BI (6)(9)a

4. $\displaystyle\int_0^\infty\frac{e^{px}-\cos t}{\left(\cosh px+\cos t\right)^2}\,dx=\frac{1}{p}\left(t\operatorname{cosec}t+\frac{1}{1+\cos t}\right)$ $[p>0]$ BI (27)(26)a

3.544 $\displaystyle\int_u^\infty\frac{\exp\left[-\left(n+\frac{1}{2}\right)x\right]}{\sqrt{2\left(\cosh x-\cosh u\right)}}\,dx=Q_n\left(\cosh u\right),\qquad [u>0]$ EH II 181(33)

3.545

1. $\displaystyle\int_0^\infty\frac{\sinh ax}{e^{px}+1}\,dx=\frac{\pi}{2p}\operatorname{cosec}\frac{a\pi}{p}-\frac{1}{2a}$ $[p>a,\quad p>0]$ BI (27)(3)

2. $\displaystyle\int_0^\infty\frac{\sinh ax}{e^{px}-1}\,dx=\frac{1}{2a}-\frac{\pi}{2p}\cot\frac{a\pi}{p}$ $[p>a,\quad p>0]$ BI (27)(9)

3.546

1. $$\int_0^\infty e^{-\beta x^2}\sinh ax\,dx=\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\beta}}\exp\frac{a^2}{4\beta}\,\Phi\left(\frac{a}{2\sqrt{\beta}}\right)\qquad[\operatorname{Re}\beta>0]$$ ET I166(38)a

2. $$\int_0^\infty e^{-\beta x^2}\cosh ax\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\frac{a^2}{4\beta}\qquad[\operatorname{Re}\beta>0]$$ FI II 720a

3. $$\int_0^\infty e^{-\beta x^2}\sinh^2 ax\,dx=\frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left(\exp\frac{a^2}{\beta}-1\right)\qquad[\operatorname{Re}\beta>0]$$ ET I 166(40)

4. $$\int_0^\infty e^{-\beta x^2}\cosh^2 ax\,dx=\frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left(\exp\frac{a^2}{\beta}+1\right)\qquad[\operatorname{Re}\beta>0]$$ ET I 166(41)

3.547

1. $$\int_0^\infty \exp(-\beta\sinh x)\sinh\gamma x\,dx=\frac{\pi}{2}\cot\frac{\gamma\pi}{2}\left[J_\gamma(\beta)-\mathbf{J}_\gamma(\beta)\right]-\frac{\pi}{2}\left[\mathbf{E}_\gamma(\beta)+Y_\gamma(\beta)\right]=\gamma\,S_{-1,\gamma}(\beta)$$
$[\operatorname{Re}\beta>0]$ WA 341(5), ET I 168(14)a

2. $$\int_0^\infty \exp(-\beta\cosh x)\sinh\gamma x\sinh x\,dx=\frac{\gamma}{\beta}\,K_\gamma(\beta)$$

3. $$\int_0^\infty \exp(-\beta\sinh x)\cosh\gamma x\,dx=\frac{\pi}{2}\tan\frac{\pi\gamma}{2}\left[\mathbf{J}_\gamma(\beta)-J_\gamma(\beta)\right]-\frac{\pi}{2}\left[\mathbf{E}_\gamma(\beta)+Y_\gamma(\beta)\right]=S_{0,\gamma}(\beta)$$
$[\operatorname{Re}\beta>0,\quad \gamma \text{ not an integer}]$
ET I 168(16)a, WA 341(4), EH II 84(50)

4. $$\int_0^\infty \exp(-\beta\cosh x)\cosh\gamma x\,dx=K_\gamma(\beta)\qquad[\operatorname{Re}\beta>0]$$ ET I 168(16)a, WA 201(5)

5. $$\int_0^\infty \exp(-\beta\sinh x)\sinh\gamma x\cosh x\,dx=\frac{\gamma}{\beta}\,S_{0,\gamma}(\beta)\qquad[\operatorname{Re}\beta>0]$$ ET I 168(7), EH II 85(51)

6. $$\int_0^\infty \exp(-\beta\sinh x)\sinh[(2n+1)x]\cosh x\,dx=O_{2n+1}(\beta)$$
$[\operatorname{Re}\beta>0]$ ET I 167(5)

7. $$\int_0^\infty \exp(-\beta\sinh x)\cosh\gamma x\cosh x\,dx=\frac{1}{\beta}\,S_{1,\gamma}(\beta)\qquad[\operatorname{Re}\beta>0]$$

8. $$\int_0^\infty \exp(-\beta\sinh x)\cosh 2nx\cosh x\,dx=O_{2n}(\beta)\qquad[\operatorname{Re}\beta>0]$$ ET I 168(6)

9. $$\int_0^\infty \exp(-\beta\cosh x)\sinh^{2\nu}x\,dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)K_\nu(\beta)$$
$\left[\operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ EH II 82(20)

10.[11] $$\int_0^\infty \exp\left[-2\left(\beta\coth x+\mu x\right)\right]\sinh^{2\nu}x\,dx=\frac{1}{2}\beta^\nu\,\Gamma(\mu-\nu)\,W_{-\mu,\nu-\frac{1}{2}}(4\beta)$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>\operatorname{Re}\nu]$

11. $$\int_0^\infty \exp\left(-\frac{\beta^2}{2}\sinh x\right)\sinh^{\nu-1}x\cosh^\nu x\,dx=-\pi\,D_\nu\left(\beta e^{i\pi/4}\right)D_\nu\left(\beta e^{-i\pi/4}\right)$$
$\left[\operatorname{Re}\nu>0,\quad |\arg\beta|\le\frac{\pi}{4}\right]$ EH II 120(10)

12. $$\int_0^\infty \frac{\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\nu+\frac14}(\beta)\,J_{\nu-\frac14}(\beta) + Y_{\nu+\frac14}(\beta)\,Y_{\nu-\frac14}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ EH I 169(20)

13. $$\int_0^\infty \frac{\exp(-2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\nu+\frac14}(\beta)\,Y_{\nu-\frac14}(\beta) - J_{\nu-\frac14}(\beta)\,Y_{\nu+\frac14}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(21)

14. $$\int_0^\infty \frac{\exp(-2\beta\sinh x)\sinh 2\nu x}{\sqrt{\sinh x}}\,dx = \frac{1}{4i}\sqrt{\frac{\pi^3\beta}{2}}\left\{e^{\nu\pi i}\,H^{(1)}_{\frac12+\nu}(\beta)\,H^{(2)}_{\frac12-\nu}(\beta) - e^{-\nu\pi i}\,H^{(1)}_{\frac12-\nu}(\beta)\,H^{(2)}_{\frac12+\nu}(\beta)\right\}$$
$[\operatorname{Re}\beta > 0]$ ET I 170(24)

15. $$\int_0^\infty \frac{\exp(-2\beta\sinh x)\cosh 2\nu x}{\sqrt{\sinh x}}\,dx = \frac{1}{4}\sqrt{\frac{\pi^3\beta}{2}}\left\{e^{\nu\pi i}\,H^{(1)}_{\frac12+\nu}(\beta)\,H^{(2)}_{\frac12-\nu}(\beta) + e^{-\nu\pi i}\,H^{(1)}_{\frac12-\nu}(\beta)\,H^{(2)}_{\frac12+\nu}(\beta)\right\}$$
$[\operatorname{Re}\beta > 0]$ ET I 170(25)

16. $$\int_0^\infty \frac{\exp(-2\beta\cosh x)\cosh 2\nu x}{\sqrt{\cosh x}}\,dx = \sqrt{\frac{\beta}{\pi}}\,K_{\nu+\frac14}(\beta)\,K_{\nu-\frac14}(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 170(26)

17.[8] $$\int_0^\infty \frac{\exp[-2\beta(\cosh x - 1)]\cosh 2\nu x}{\sqrt{\cosh x}}\,dx = \sqrt{\frac{\beta}{\pi}}\cdot e^{2\beta}\,K_{\nu+\frac14}(\beta)\,K_{\nu-\frac14}(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 170(27)

18. $$\int_0^\infty \frac{\cos\left[\left(\nu+\frac14\right)\pi\right]\exp(-2\nu x - 2\beta\sinh x) + \sin\left[\left(\nu+\frac14\right)\pi\right]\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\frac14+\nu}(\beta)\,J_{\frac14-\nu}(\beta) + Y_{\frac14+\nu}(\beta)\,Y_{\frac14-\nu}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(22)

19. $$\int_0^\infty \frac{\sin\left[\left(\nu+\frac14\right)\pi\right]\exp(-2\nu x - 2\beta\sinh x) - \cos\left[\left(\nu+\frac14\right)\pi\right]\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\frac14+\nu}(\beta)\,Y_{\frac14-\nu}(\beta) - J_{\frac14-\nu}(\beta)\,Y_{\frac14+\nu}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(23)

20. $$\int_0^\infty \frac{\exp[-\beta(\cosh x - 1)]\cosh\nu x\sinh x}{\sqrt{\cosh x\,(\cosh x - 1)}}\,dx = e^{\beta}\,K_\nu(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 169(19)

3.548

1. $$\int_0^\infty e^{-\mu x^4}\sinh ax^2\,dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\exp\left(\frac{a^2}{8\mu}\right)I_{\frac14}\left(\frac{a^2}{8\mu}\right)$$
$[\operatorname{Re}\mu > 0,\quad a \ge 0]$ ET I 166(42)

2. $$\int_0^\infty e^{-\mu x^4}\cosh ax^2\,dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\exp\left(\frac{a^2}{8\mu}\right)I_{-\frac14}\left(\frac{a^2}{8\mu}\right)$$
$[\operatorname{Re}\mu > 0,\quad a > 0]$ ET I 166(43)

3.549

1. $$\int_0^\infty e^{-\beta x}\sinh\left[(2n+1)\operatorname{arcsinh} x\right]\,dx = O_{2n+1}(\beta) \qquad [\operatorname{Re}\beta>0] \qquad (\text{cf. } \mathbf{3.547}\ 6)$$
ET I 167(5)

2. $$\int_0^\infty e^{-\beta x}\cosh\left(2n\operatorname{arcsinh} x\right)\,dx = O_{2n}(\beta) \qquad [\operatorname{Re}\beta>0] \qquad (\text{cf. } \mathbf{3.547}\ 8)$$
ET I 168(6)

3. $$\int_0^\infty e^{-\beta x}\sinh\left(\nu\operatorname{arcsinh} x\right)\,dx = \frac{\nu}{\beta}\,S_{0,\nu}(\beta) \qquad [\operatorname{Re}\beta>0] \qquad (\text{cf. } \mathbf{3.547}5)$$
ET I 168(7)

4. $$\int_0^\infty e^{-\beta x}\cosh\left(\nu\operatorname{arcsinh} x\right)\,dx = \frac{1}{\beta}\,S_{1,\nu}(\beta) \qquad [\operatorname{Re}\beta>0] \qquad (\text{cf. } \mathbf{3.547}\ 7)$$

A number of other integrals containing hyperbolic functions and exponentials, depending on $\operatorname{arcsinh} x$ or $\operatorname{arccosh} x$, can be found by first making the substitution $x=\sinh t$ or $x=\cosh t$.

3.55–3.56 Combinations of hyperbolic functions, exponentials, and powers

3.551

1. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\sinh\gamma x\,dx = \frac{1}{2}\,\Gamma(\mu)\left[(\beta-\gamma)^{-\mu}-(\beta+\gamma)^{-\mu}\right]$$
$[\operatorname{Re}\beta>-1, \quad \operatorname{Re}\beta>|\operatorname{Re}\gamma|]$
ET I 164(18)

2. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\cosh\gamma x\,dx = \frac{1}{2}\,\Gamma(\mu)\left[(\beta-\gamma)^{-\mu}+(\beta+\gamma)^{-\mu}\right]$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\beta>|\operatorname{Re}\gamma|]$
ET I 164(19)

3. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\coth x\,dx = \Gamma(\mu)\left[2^{1-\mu}\,\zeta\left(\mu,\frac{\beta}{2}\right)-\beta^{-\mu}\right]$$
$[\operatorname{Re}\mu>1, \quad \operatorname{Re}\beta>0]$ ET I 164(21)

4. $$\int_0^\infty x^n e^{-(p+mq)x}\sinh^m qx\,dx = 2^{-m}n!\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(p+2kq)^{n+1}}$$
$[p>0, \quad q>0, \quad m<p+qm]$
LI (81)(4)

5.[11] $$\int_0^1 \frac{e^{-\beta x}}{x}\sinh\gamma x\,dx = \frac{1}{2}\left[\ln\frac{\beta+\gamma}{\beta-\gamma}+\operatorname{Ei}(\gamma-\beta)-\operatorname{Ei}(-\gamma-\beta)\right]$$
$[\beta>\gamma]$ BI (80)(4)

6. $$\int_0^\infty \frac{e^{-\beta x}}{x}\sinh\gamma x\,dx = \frac{1}{2}\ln\frac{\beta+\gamma}{\beta-\gamma} \qquad [\operatorname{Re}\beta>|\operatorname{Re}\gamma|]$$
ET I 163(12)

7. $$\int_1^\infty \frac{e^{-\beta x}}{x}\cosh\gamma x\,dx = \frac{1}{2}\left[-\operatorname{Ei}(\gamma-\beta)-\operatorname{Ei}(-\gamma-\beta)\right] \qquad [\operatorname{Re}\beta>|\operatorname{Re}\gamma|]$$
ET I 164(15)

8.[6] $$\int_0^\infty xe^{-x}\coth x\,dx = \frac{\pi^2}{4}-1$$
BI (82)(6)

9. $$\int_0^\infty e^{-\beta x}\tanh x\,\frac{dx}{x}=\ln\frac{\beta}{4}+2\ln\frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4}+\frac{1}{2}\right)}$$ $[\operatorname{Re}\beta>0]$ ET I 164(16)

10.[6] $$\int_0^\infty xe^{-x}\coth(x/2)\,dx=\frac{\pi^2}{3}-1$$

3.552

1. $$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\sinh x}\,dx=2^{1-\mu}\,\Gamma(\mu)\,\zeta\left[\mu,\frac{1}{2}(\beta+1)\right]$$ $[\operatorname{Re}\mu>1,\quad \operatorname{Re}\beta>-1]$ ET I 164(20)

2. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\sinh ax}\,dx=\frac{1}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m}$$ $[a>0,\quad m=1,2,\ldots]$ EH I 38(24)a

3. $$\int_0^\infty \frac{x^{\mu-1}e^{-x}}{\cosh x}\,dx=2^{1-\mu}\left(1-2^{1-\mu}\right)\Gamma(\mu)\,\zeta(\mu)$$ $[\operatorname{Re}\mu>0,\quad \mu\neq 1]$
$$=\ln 2$$ $[\text{if } \mu=1]$
EH I 32(5)

4. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\cosh ax}\,dx=\frac{1-2^{1-2m}}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m}$$ $[a>0,\quad m=1,2,\ldots]$ EH I 39(25)a

5. $$\int_0^\infty \frac{x^2e^{-2nx}}{\sinh x}\,dx=4\sum_{k=n}^{\infty}\frac{1}{(2k+1)^3}$$ $[n=0,1,2,\ldots]$ (cf. **4.261** 13)
BI(84)(4)

6.[11] $$\int_0^\infty \frac{x^3e^{-2nx}}{\sinh x}\,dx=\frac{\pi^4}{8}-12\sum_{k=1}^{n}\frac{1}{(2k-1)^4}$$ $[n=0,1,\ldots]$ (cf. **4.262** 6)
BI (84)(6)

3.553

1. $$\int_0^\infty \frac{\sinh^2 ax}{\sinh x}\,\frac{e^{-x}\,dx}{x}=\frac{1}{2}\ln\left(a\pi\operatorname{cosec}a\pi\right)$$ $[a<1]$ BI (95)(7)

2.[11] $$\int_0^\infty \frac{\sinh^2\frac{x}{2}}{\cosh x}\cdot\frac{e^{-x}\,dx}{x}=\frac{1}{2}\ln\frac{4}{\pi}$$ (cf. **4.267** 2) BI (95)(4)

3.554

1.[11] $$\int_0^\infty e^{-\beta x}\left(1-\operatorname{sech}x\right)\frac{dx}{x}=2\ln\frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)}-\ln\frac{\beta}{4}$$ $[\operatorname{Re}\beta>0]$ ET I 164(17)

2. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x}-\operatorname{cosech}x\right)dx=\psi\left(\frac{\beta+1}{2}\right)-\ln\frac{\beta}{2}$$ $[\operatorname{Re}\beta>0]$ ET I 163(10)

3. $$\int_0^\infty\left[\frac{\sinh\left(\frac{1}{2}-\beta\right)x}{\sinh\frac{x}{2}}-(1-2\beta)e^{-x}\right]\frac{dx}{x}=2\ln\Gamma(\beta)-\ln\pi+\ln\left(\sin\pi\beta\right)$$
$[0<\operatorname{Re}\beta<1]$ EH I 21(7)

4. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x}-\coth x\right)dx=\psi\left(\frac{\beta}{2}\right)-\ln\frac{\beta}{2}+\frac{1}{\beta}\qquad[\operatorname{Re}\beta>0]$$ ET I 163(11)

5. $$\int_0^\infty\left\{-\frac{\sinh qx}{\sinh\frac{x}{2}}+2qe^{-x}\right\}\frac{dx}{x}=2\ln\Gamma\left(q+\frac{1}{2}\right)+\ln\cos\pi q-\ln\pi$$
$$\left[q^2<\tfrac{1}{2}\right]$$ WH

6. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\left(\coth x-1\right)dx=2^{1-\mu}\Gamma(\mu)\,\zeta\left(\mu,\frac{\beta}{2}+1\right)$$
$$[\operatorname{Re}\beta>0;\quad\operatorname{Re}\mu>1]$$ ET I 164(22)

3.555

1. $$\int_0^\infty\frac{\sinh^2 ax}{1-e^{px}}\cdot\frac{dx}{x}=\frac{1}{4}\ln\left(\frac{p}{2a\pi}\sin\frac{2a\pi}{p}\right)\qquad[0<2|a|<p]\qquad(\text{cf. }\mathbf{3.545}\ 2)$$
BI (93)(15)

2. $$\int_0^\infty\frac{\sinh^2 ax}{e^x+1}\cdot\frac{dx}{x}=-\frac{1}{4}\ln\left(a\pi\cot a\pi\right)\qquad\left[a<\tfrac{1}{2}\right]\qquad(\text{cf. }\mathbf{3.545}\ 1)$$ BI (93)(9)

3.556

1. $$\int_{-\infty}^\infty x\frac{1-e^{px}}{\sinh x}\,dx=-\frac{\pi^2}{2}\tan^2\frac{p\pi}{2}\qquad[p<1]\qquad(\text{cf. }\mathbf{4.255}\ 3)$$ BI (101)(4)

2. $$\int_0^\infty\frac{1-e^{-px}}{\sinh x}\cdot\frac{1-e^{-(p+1)x}}{x}\,dx=2p\ln 2\qquad[p>-1]$$ BI (95)(8)

3.557

1. $$\int_0^\infty\frac{e^{-px}-e^{-qx}}{\cosh x-\cos\frac{m}{n}\pi}\cdot\frac{dx}{x}$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q+k}{2n}\right)\Gamma\left(\frac{p+k}{2n}\right)}{\Gamma\left(\frac{n+p+k}{2n}\right)\Gamma\left(\frac{q+k}{2n}\right)}\qquad[m+n\text{ odd}]$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q-k}{n}\right)\Gamma\left(\frac{p+k}{n}\right)}{\Gamma\left(\frac{n+p-k}{n}\right)\Gamma\left(\frac{q+k}{n}\right)}\qquad[m+n\text{ even}]$$
$$[p>-1,\quad q>-1]$$ BI (96)(1)

2. $$\int_0^\infty\frac{\left(1-e^{-x}\right)^2}{\cosh x+\cos\frac{m}{n}\pi}\cdot\frac{dx}{x}$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n+k+1}{2n}\right)\right]^2\Gamma\left(\frac{k+2}{2n}\right)\Gamma\left(\frac{k}{2n}\right)}{\left[\Gamma\left(\frac{k+1}{2n}\right)\right]^2\Gamma\left(\frac{n+k}{2n}\right)\Gamma\left(\frac{n+k+2}{2n}\right)}\qquad[m+n\text{ odd}]$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n-k+1}{n}\right)\right]^2\Gamma\left(\frac{k+2}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\left[\Gamma\left(\frac{k+1}{n}\right)\right]^2\Gamma\left(\frac{n-k}{n}\right)\Gamma\left(\frac{n-k+2}{n}\right)}\qquad[m+n\text{ even}]$$
BI (96)(2)

3. $$\int_0^\infty \left[e^{-x}\tan\frac{m}{2n}\pi - \frac{e^{-px}\sin\frac{m}{n}\pi}{\cosh x+\cos\frac{m}{n}\pi}\right]\cdot\frac{dx}{x}$$
$$= \tan\left(\frac{m}{2n}\pi\right)\ln(2n) + 2\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n+k}{2n}\right)}{\Gamma\left(\frac{p+k}{2n}\right)} \qquad [m+n \text{ odd}]$$
$$= \tan\left(\frac{m}{2n}\pi\right)\ln n + 2\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n-k}{n}\right)}{\Gamma\left(\frac{p+k}{n}\right)} \qquad [m+n \text{ even}]$$
BI (96)(3)

4. $$\int_0^\infty \frac{1+e^{-x}}{\cosh x+\cos a}\cdot\frac{dx}{x^{1-p}} = 2\sec\frac{a}{2}\,\Gamma(p)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\left(k-\frac{1}{2}\right)a}{k^p}$$
$[p>0]$ LI (96)(5)

5. $$\int_0^\infty \frac{x^q e^{-\frac{x}{2}}\cosh\frac{x}{2}}{\cosh x+\cos\lambda}\,dx = \frac{\Gamma(q+1)}{\cos\frac{\lambda}{2}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\left(k-\frac{1}{2}\right)\lambda}{k^{q+1}}$$
$[q>-1]$ LI (96)(5)a

6. $$\int_0^\infty x\frac{e^{-x}-\cos a}{\cosh x-\cos a}\,dx = |a|\pi - \frac{a^2}{2} - \frac{\pi^2}{3}$$ BI (88)(8)

7. $$\int_0^\infty x^{2m+1}\frac{e^{-x}-\cos a\pi}{\cosh x-\cos a\pi}\,dx = 2\cdot(2m+1)!\sum_{k=1}^{\infty}\frac{\cos ka\pi}{k^{2m+2}}$$ BI (88)(6)

3.558

1. $$\int_0^\infty x\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = \frac{2n\pi^2}{3} - 4\sum_{k=1}^{n-1}\frac{n-k}{k^2}$$ BI (85)(3)

2. $$\int_0^\infty x\frac{1-(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = \frac{n\pi^2}{3} + 4\sum_{k=1}^{n-1}(-1)^k\frac{n-k}{k^2}$$ LI (85)(1)

3. $$\int_0^\infty x^2\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = 8n\,\zeta(3) - 8\sum_{k=1}^{n-1}\frac{n-k}{k^3}$$ BI (85)(5)

4. $$\int_0^\infty x^2 e^x\frac{1-e^{-2nx}}{\sinh^2 x}\,dx = 8n\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} - 8\sum_{k=1}^{n-1}\frac{n-k}{(2k-1)^3}$$ LI (85)(6)

5. $$\int_0^\infty x^2\frac{1+(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = 6n\,\zeta(3) - 8\sum_{k=1}^{n-1}\frac{n-k}{k^3}$$ LI (85)(4)

6. $$\int_0^\infty x^3\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = \frac{4}{15}n\pi^4 - 24\sum_{k=1}^{n-1}\frac{n-k}{k^4}$$ BI (85)(9)

7. $$\int_0^\infty x^3\frac{1+(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = \frac{7}{30}n\pi^4 + 24\sum_{k=1}^{n-1}(-1)^k\frac{n-k}{k^4}$$ BI (85)(8)

3.559 $$\int_0^\infty e^{-x}\left[a-\frac{1}{2}+\frac{(1-e^{-x})(1-ax)-xe^{-x}}{4\sinh^2\frac{x}{2}}e^{(2-a)x}\right]\frac{dx}{x}=a-\frac{1}{2}+\ln\Gamma(a)-\frac{1}{2}\ln(2\pi)\qquad [a>0]$$
BI (96)(6)

3.561 $$\int_0^\infty \frac{e^{-2x}\tanh\frac{x}{2}}{x\cosh x}\,dx=2\ln\frac{\pi}{2\sqrt{2}}$$ BI (93)(18)

3.562

1. $$\int_0^\infty x^{2\mu-1}e^{-\beta x^2}\sinh\gamma x\,dx=\frac{1}{2}\Gamma(2\mu)(2\beta)^{-\mu}\exp\left(\frac{\gamma^2}{8\beta}\right)\left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right)-D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right]$$
$[\operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}\beta>0]$ ET I 166(44)

2. $$\int_0^\infty x^{2\mu-1}e^{-\beta x^2}\cosh\gamma x\,dx=\frac{1}{2}\Gamma(2\mu)(2\beta)^{-\mu}\exp\left(\frac{\gamma^2}{8\beta}\right)\left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right)+D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right]$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0]$ ET I 166(45)

3. $$\int_0^\infty xe^{-\beta x^2}\sinh\gamma x\,dx=\frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\qquad [\operatorname{Re}\beta>0]$$ BI(81)(12)a,ET I 165(34)

4. $$\int_0^\infty xe^{-\beta x^2}\cosh\gamma x\,dx=\frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right)+\frac{1}{2\beta}$$
$[\operatorname{Re}\beta>0]$ ET I 166(35)

5. $$\int_0^\infty x^2e^{-\beta x^2}\sinh\gamma x\,dx=\frac{\sqrt{\pi}\left(2\beta+\gamma^2\right)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right)+\frac{\gamma}{4\beta^2}$$
$[\operatorname{Re}\beta>0]$ ET I 166(36)

6. $$\int_0^\infty x^2e^{-\beta x^2}\cosh\gamma x\,dx=\frac{\sqrt{\pi}\left(2\beta+\gamma^2\right)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\qquad [\operatorname{Re}\beta>0]$$ ET I 166(37)

3.6–4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

1. $$\int_0^{2\pi}(1-\cos x)^n\sin nx\,dx=0$$ BI (68)(10)

2. $$\int_0^{2\pi}(1-\cos x)^n\cos nx\,dx=(-1)^n\frac{\pi}{2^{n-1}}$$ BI (68)(11)

3. $$\int_0^{\pi}(\cos t+i\sin t\cos x)^n\,dx=\int_0^{\pi}(\cos t+i\sin t\cos x)^{-n-1}\,dx=\pi\,P_n(\cos t)$$ EH I 158(23)a

3.612

1.[6] $$\int_0^\pi \frac{\sin nx \cos mx}{\sin x}\,dx = 0 \quad \text{for } n \le m;$$
$$= \pi \quad \text{for } n > m, \quad \text{if } m+n \text{ is odd and positive}$$
$$= 0 \quad \text{for } n > m, \quad \text{if } m+n \text{ is even}$$

LI (64)(3)

2. $$\int_0^\pi \frac{\sin nx}{\sin x}\,dx = 0 \qquad \text{for } n \text{ even}$$
$$= \pi \qquad \text{for } n \text{ odd}$$

BI (64)(1, 2)

3. $$\int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx = \frac{\pi}{2}$$ FI II 145

4. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin x}\,dx = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1}\right)$$ GW (332)(21b)

5. $$\int_0^{\pi} \frac{\sin 2nx}{\cos x}\,dx = 2\int_0^{\pi/2} \frac{\sin 2nx}{\cos x}\,dx = (-1)^{n-1}4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}\right)$$ GW (332)(22a)

6. $$\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x}\,dx = (-1)^n\pi$$ GW (332)(22b)

7. $$\int_0^{\pi/2} \frac{\sin 2nx \cos x}{\sin x}\,dx = \frac{\pi}{2}$$ LI (45)(17)

3.613

1.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1 + a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad \left[a^2 < 1, \quad n \ge 0\right]$$ BI (64)(12)

2.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1 - 2a\cos x + a^2} = \frac{\pi a^n}{1-a^2} \qquad \left[a^2 < 1, \quad n \ge 0\right]$$
$$= \frac{\pi}{(a^2-1)\,a^n} \qquad \left[a^2 > 1, \quad n \ge 0\right]$$

BI (65)(3)

3. $$\int_0^\pi \frac{\sin nx \sin x\,dx}{1 - 2a\cos x + a^2} = \frac{\pi}{2}a^{n-1} \qquad \left[a^2 < 1, \quad n \ge 1\right]$$
$$= \frac{\pi}{2a^{n+1}} \qquad \left[a^2 > 1, \quad n \ge 1\right]$$

BI(65)(4), GW(332)(34a)

4.[10] $$\int_0^\pi \frac{\cos nx \cos x\,dx}{1-2a\cos x+a^2} = \frac{\pi}{2}\cdot\frac{1+a^2}{1-a^2}a^{n-1} \qquad [a^2<1,\quad n\ge 1]$$
$$= \frac{\pi}{2a^{n+1}}\cdot\frac{a^2+1}{a^2-1} \qquad [a^2>1,\quad n\ge 1]$$
$$= \frac{\pi a}{1-a^2} \qquad [n=0,\quad a^2<1]$$
$$= \frac{\pi}{a\,(a^2-1)} \qquad [n=0,\quad a^2>1]$$

BI(65)(5), GW(332)(34b)

5. $$\int_0^\pi \frac{\cos(2n-1)x\,dx}{1-2a\cos 2x+a^2} = \int_0^\pi \frac{\cos 2nx\cos x\,dx}{1-2a\cos 2x+a^2} = 0 \qquad [a^2\ne 1]$$ BI (65)(9, 10)

6. $$\int_0^\pi \frac{\cos(2n-1)x\cos 2x\,dx}{1-2a\cos 2x+a^2} = 0 \qquad [a^2\ne 1]$$ BI (65)(12)

7. $$\int_0^\pi \frac{\sin 2nx\sin x\,dx}{1-2a\cos 2x+a^2} = \int_0^\pi \frac{\sin(2n-1)x\sin 2x\,dx}{1-2a\cos 2x+a^2} = 0$$
$$[a^2\ne 1]$$ BI (65)(6, 7)

8. $$\int_0^\pi \frac{\sin(2n-1)x\sin x\,dx}{1-2a\cos 2x+a^2} = \frac{\pi}{2}\cdot\frac{a^{n-1}}{1+a} \qquad [a^2<1]$$
$$= \frac{\pi}{2}\cdot\frac{1}{(1+a)a^n} \qquad [a^2>1]$$

BI (65)(8)

9. $$\int_0^\pi \frac{\cos(2n-1)x\cos x\,dx}{1-2a\cos 2x+a^2} = \frac{\pi}{2}\cdot\frac{a^{n-1}}{1-a} \qquad [a^2<1]$$
$$= \frac{\pi}{2}\cdot\frac{1}{(a-1)a^n} \qquad [a^2>1]$$

BI (65)(11)

10. $$\int_0^\pi \frac{\sin nx - a\sin(n-1)x}{1-2a\cos x+a^2}\sin mx\,dx = 0 \qquad \text{for } m<n$$
$$= \frac{\pi}{2}a^{m-n} \qquad \text{for } m\ge n$$
$$[a^2<1]$$ LI (65)(13)

11.[6] $$\int_0^\pi \frac{\cos nx - a\cos(n-1)x}{1-2a\cos x+a^2}\cos mx\,dx = \frac{\pi}{2}\left(a^{|m|-n}-1\right)$$
$$[a^2<1]$$ BI (65)(14)

12. $$\int_0^\pi \frac{\sin nx - a\sin[(n+1)x]}{1-2a\cos x+a^2}\,dx = 0 \qquad [a^2<1]$$ BI (68)(13)

13. $$\int_0^\pi \frac{\cos nx - a\cos[(n+1)x]}{1-2a\cos x+a^2}\,dx = \pi a^n \qquad [a^2<1]$$ BI (68)(14)

3.614[7]
$$\int_0^\pi \frac{\sin x}{a^2-2ab\cos x+b^2}\cdot\frac{\sin px\cdot\,dx}{1-2a^p\cos px+a^{2p}}$$
$$=\frac{\pi b^{p-1}}{2a^{p+1}\left(1-b^p\right)}\qquad\left[0<b\le a\le 1,\quad p=1,2,3,\ldots\right]$$
$$=\frac{\pi a^{p-1}}{2b\left(b^p-a^{2p}\right)}\qquad\left[0<a\le 1,\quad a^2<b,\quad p=1,2,3,\ldots\right]$$
BI (66)(9)

3.615

1. $$\int_0^{\pi/2}\frac{\cos 2nx\,dx}{1-a^2\sin^2 x}=\frac{(-1)^n\pi}{2\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{a}\right)^{2n}\qquad\left[a^2<1\right]$$ BI (47)(27)

2. $$\int_0^\pi\frac{\cos x\sin 2nx\,dx}{1+(a+b\sin x)^2}=-\frac{\pi}{b}\sin\left\{2n\arctan\sqrt{\frac{s}{2}}\right\}\tan^{2n}\left(\frac{1}{2}\arccos\sqrt{\frac{s}{2a^2}}\right)$$

3. $$\int_0^\pi\frac{\cos x\cos(2n+1)x\,dx}{1+(a+b\sin x)^2}=\frac{\pi}{b}\cos\left\{(2n+1)\arctan\sqrt{\frac{s}{2}}\right\}\tan^{2n+1}\left(\frac{1}{2}\arccos\sqrt{\frac{s}{2a^2}}\right)$$
$$\text{where } s=-\left(1+b^2-a^2\right)+\sqrt{\left(1+b^2-a^2\right)^2+4a^2}$$ BI (65)(21, 22)

3.616

1. $$\int_0^\pi\left(1-2a\cos x+a^2\right)^n\,dx=\pi\sum_{k=0}^n\binom{n}{k}^2a^{2k}$$ BI (63)(1)

2.[10] $$\int_0^\pi\frac{dx}{\left(1-2a\cos x+a^2\right)^n}=\frac{1}{2}\int_0^{2\pi}\frac{dx}{\left(1-2a\cos x+a^2\right)^n}$$
$$=\frac{\pi}{\left(1-a^2\right)^n}\sum_{k=0}^{n-1}\frac{(n+k-1)!}{(k!)^2(n-k-1)!}\left(\frac{a^2}{1-a^2}\right)^k\qquad\left[a^2<1\right]$$
$$=\frac{\pi}{\left(a^2-1\right)^n}\sum_{k=0}^{n-1}\frac{(n+k-1)!}{(k!)^2(n-k-1)!}\frac{1}{\left(a^2-1\right)^k}\qquad\left[a^2>1\right]$$
BI (331)(63)

3. $$\int_0^\pi\left(1-2a\cos x+a^2\right)^n\cos nx\,dx=(-1)^n\pi a^n$$ BI (63)(2)

4. $$\int_0^\pi\left(1-2a\cos x+a^2\right)^n\cos mx\,dx$$
$$=\frac{1}{2}\int_0^{2\pi}\left(1-2a\cos x+a^2\right)^n\cos mx\,dx$$
$$=0\qquad[n<m]$$
$$=\pi(-a)^m\left(1+a^2\right)^{n-m}\sum_{k=0}^{[(n-m)/2]}\binom{n}{k}\binom{n-k}{m+k}\left(\frac{a}{1+a^2}\right)^{2k}\qquad[n\ge m]$$
GW (332)(35a)

5. $$\int_0^{2\pi}\frac{\sin nx\,dx}{\left(1-2a\cos 2x+a^2\right)^m}=0$$ GW (332)(32a)

6. $$\int_0^\pi \frac{\sin x\,dx}{(1-2a\cos 2x+a^2)^m} = \frac{1}{2(m-1)a}\left[\frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}}\right] \qquad [a \neq 0, \quad \pm 1]$$

GW (332)(32c)

7. $$\int_0^\pi \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m} = \frac{1}{2}\int_0^{2\pi} \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m}$$
$$= \frac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(\frac{1-a^2}{a^2}\right)^k \qquad [a^2 < 1]$$
$$= \frac{\pi}{a^n\,(a^2-1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(a^2-1\right)^k \qquad [a^2 > 1]$$

GW (332)(31)

8. $$\int_0^{\pi/2} \frac{\cos 2nx\,dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}} = \binom{2n}{n}\frac{\left(b^2-a^2\right)^n}{(2ab)^{2n+1}}\pi$$

$[a > 0, \quad b > 0]$ GW (332)(30b)

3.617[10] $$\int_0^\pi \frac{dx}{(1-2a\cos x+a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n\left(\frac{2\sqrt{|a|}}{|1+a|}\right), \qquad |a| \neq 1$$

with
$$F_n(k) = \int_0^{\pi/2} \frac{dx}{\left(1-k^2\sin^2 x\right)^{n+1/2}}$$
where the $F_n(k)$ satisfies the recurrence relation
$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1}\frac{dF_n(k)}{dk}, \qquad n = 0, 1, 2, \ldots$$
and
$$F_0(k) = \boldsymbol{K}(k) \equiv \int_0^{\pi/2} \frac{dx}{\left(1-k^2\sin^2 x\right)^{1/2}}$$
is the complete elliptic integral of the first kind.
Introducing the complete elliptic integral of the second kind
$$\boldsymbol{E}(k) = \int_0^{\pi/2} \left(1-k^2\sin^2 x\right)^{1/2}\,dx$$
the derivatives
$$\frac{d\,\boldsymbol{K}(k)}{dk} = \frac{\boldsymbol{E}(k)}{k\,(1-k^2)} - \frac{\boldsymbol{K}(k)}{k}, \qquad \frac{d\,\boldsymbol{E}(k)}{dk} = \frac{\boldsymbol{E}(k)-\boldsymbol{K}(k)}{k}$$
combined with the recurrence relation lead to

$$F_1(k) = F_0\,(k) + k\frac{d\,F_0\,(k)}{dk}$$
$$= \boldsymbol{K}(k) + \frac{\boldsymbol{E}(k)}{1-k^2} - \boldsymbol{K}(k) = \frac{\boldsymbol{E}(k)}{1-k^2},$$
$$F_2(k) = \frac{\boldsymbol{E}(k)}{1-k^2} + \frac{k}{3}\frac{d}{dk}\left(\frac{\boldsymbol{E}(k)}{1-k^2}\right)$$
$$= \frac{1}{3\,(1-k^2)}\left[\left(\frac{4-2k^2}{1-k^2}\right)\boldsymbol{E}(k) - \boldsymbol{K}(k)\right]$$

3.62 Powers of trigonometric functions

3.621

1. $$\int_0^{\pi/2} \sin^{\mu-1} x\,dx = \int_0^{\pi/2} \cos^{\mu-1} x\,dx = 2^{\mu-2}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\mu}{2}\right)$$ FI II 789

2. $$\int_0^{\pi/2} \sin^{3/2} x\,dx = \int_0^{\pi/2} \cos^{3/2} x\,dx = \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

3. $$\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2} \cos^{2m} x\,dx = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$$ FI II 151

4. $$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \int_0^{\pi/2} \cos^{2m+1} x\,dx = \frac{(2m)!!}{(2m+1)!!}$$ FI II 151

5. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\nu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
LO V 113(50), LO V 122, FI II 788

6.* $$\int_0^{\pi/2} \sqrt{\sin x}\,dx = \sqrt{\frac{2}{\pi}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2$$

7.* $$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{2\sqrt{2\pi}}$$

3.622

1. $$\int_0^{\pi/2} \tan^{\pm\mu} x\,dx = \frac{\pi}{2}\sec\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1]$$ BI (42)(1)

2. $$\int_0^{\pi/4} \tan^{\mu} x\,dx = \frac{1}{2}\,\beta\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(1)

3. $$\int_0^{\pi/4} \tan^{2n} x\,dx = (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}$$ BI (34)(2)

4.[11] $$\int_0^{\pi/4} \tan^{2n+1} x\,dx = (-1)^n\frac{\ln 2}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}$$ BI (34)(3)

3.623

1. $$\int_0^{\pi/2} \tan^{\mu-1} x \cos^{2\nu-2} x\,dx = \int_0^{\pi/2} \cot^{\mu-1} x \sin^{2\nu-2} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2},\nu-\frac{\mu}{2}\right)$$
$[0 < \operatorname{Re}\mu < 2\operatorname{Re}\nu]$ BI(42)(6), BI(45)(22)

2.[6] $$\int_0^{\pi/4} \tan^{\mu} x \sin^2 x\,dx = \frac{1+\mu}{4}\,\beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(4)

3.[6] $$\int_0^{\pi/4} \tan^{\mu} x \cos^2 x\,dx = \frac{1-\mu}{4}\,\beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(5)

3.624

1. $$\int_0^{\pi/4} \frac{\sin^p x}{\cos^{p+2} x}\,dx = \frac{1}{p+1} \qquad [p > -1]$$ GW (331)(34b)

2.[3] $$\int_0^{\pi/2} \frac{\sin^{\mu-\frac{1}{2}} x}{\cos^{2\mu-1} x}\,dx = \int_0^{\pi/2} \frac{\cos^{\mu-\frac{1}{2}} x}{\sin^{2\mu-1} x}\,dx = \frac{1}{2}\left\{\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}\right)\Gamma(1-\mu)}{\Gamma\left(\frac{5}{4}-\frac{\mu}{2}\right)}\right\}$$
$$\left[-\tfrac{1}{2} < \operatorname{Re}\mu < 1\right]$$ LI (55)(12)

3.[11] $$\int_0^{\pi/4} \frac{\cos^{n-\frac{1}{2}}(2x)}{\cos^{2n+1}(x)}\,dx = \pi\frac{(2n)!!}{2^{2n+1}\,(n!)^2}$$ BI (38)(3)

4.[8] $$\int_0^{\pi/4} \frac{\cos^{\mu} 2x}{\cos^{2(\mu+1)} x}\,dx = 2^{2\mu}\,\mathrm{B}(\mu+1,\mu+1) \qquad [\operatorname{Re}\mu > -1]$$ BI (35)(1)

5. $$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^{\mu} 2x}\,dx = 2^{1-2\mu}\,\mathrm{B}(2\mu-1,1-\mu) = \frac{\Gamma\left(\mu-\frac{1}{2}\right)\Gamma(1-\mu)}{2\sqrt{\pi}}$$
$$\left[\tfrac{1}{2} < \operatorname{Re}\mu < 1\right]$$ BI (35)(4)

6.[6] $$\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{a\pi}{2} - \frac{1}{2}\sin\pi a\,[2a\,\beta(a)-1]\,, \qquad [a > 0]$$

3.625

1. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x\cos^p 2x}{\cos^{2p+2n+1} x}\,dx = \frac{(n-1)!}{2}\cdot\frac{\Gamma(p+1)}{\Gamma(p+n+1)}$$
$$= \frac{(n-1)!}{2(p+n)(p+n-1)\cdots(p+1)} = \frac{1}{2}\,\mathrm{B}(n,p+1)$$
$$[p > -1] \qquad (\text{cf. } \mathbf{3.251}\ 1)$$ BI (35)(2)

2. $$\int_0^{\pi/4} \frac{\sin^{2n} x\cos^p 2x}{\cos^{2p+2n+2} x}\,dx = \tfrac{1}{2}\,\mathrm{B}\left(n+\tfrac{1}{2},p+1\right) \qquad [p > -1] \qquad (\text{cf. } \mathbf{3.251}\ 1)$$ BI (35)(3)

3. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x\cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x}\,dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!}$$ BI (38)(6)

4.[8] $$\int_0^{\pi/4} \frac{\sin^{2n} x\cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x}\,dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!}\cdot\frac{\pi}{2}$$ BI (38)(7)

3.626

1. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x}\sqrt{\cos 2x}\,dx = \frac{(2n-2)!!}{(2n+1)!!} \qquad (\text{cf. } \mathbf{3.251}\ 1)$$ BI (38)(4)

2. $$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x}\sqrt{\cos 2x}\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2} \qquad (\text{cf. } \mathbf{3.251}\ 1)$$ BI (38)(5)

3.627 $$\int_0^{\pi/2} \frac{\tan^{\mu} x}{\cos^{\mu} x}\,dx = \int_0^{\pi/2} \frac{\cot^{\mu} x}{\sin^{\mu} x}\,dx = \frac{\Gamma(\mu)\,\Gamma\left(\frac{1}{2}-\mu\right)}{2^{\mu}\sqrt{\pi}}\sin\frac{\mu\pi}{2}$$
$$\left[-1 < \operatorname{Re}\mu < \tfrac{1}{2}\right]$$ BI (55)(12)a

3.628[11] $$\int_0^{\frac{\pi}{2}} \sec^{2p} x\sin^{2p-1} x\,dx = \frac{1}{2\sqrt{\pi}}\,\Gamma(p)\,\Gamma\left(\tfrac{1}{2}-p\right) \qquad \left[0 < p < \tfrac{1}{2}\right]$$ WA 691

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.631

1. $$\int_0^\pi \sin^{\nu-1} x \sin ax\, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1}\nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$ $[\operatorname{Re}\nu > 0]$ LO V 121(67a), WA 337a

2.[7] $$\int_0^{\pi/2} 2\sin^{\nu-2} x \sin \nu x\, dx = \frac{1}{1-\nu}\cos\frac{\nu\pi}{2}$$ $[\operatorname{Re}\nu > 1]$ GW(332)(16d), FI I 152

3.[6] $$\int_0^\pi \sin^{\nu} x \sin \nu x\, dx = 2^{-\nu}\pi \sin\frac{\nu\pi}{2}$$ $[\operatorname{Re}\nu > -1]$ LO V 121(69)

4. $$\int_0^\pi \sin^n x \sin 2mx\, dx = 0$$ GW (332)(11a)

5. $$\int_0^\pi \sin^{2n} x \sin(2m+1)x\, dx = \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x\, dx$$
$$= \frac{(-1)^m 2^{n+1} n!(2n-1)!!}{(2n-2m-1)!!(2m+2n+1)!!} \qquad [m \le n]^*$$
$$= \frac{(-1)^n 2^{n+1} n!(2m-2n-1)!!(2n-1)!!}{(2m+2n+1)!!} \qquad [m \ge n]^*$$
GW (332)(11b)

6. $$\int_0^\pi \sin^{2n+1} x \sin(2m+1)x\, dx = 2\int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x\, dx$$
$$= \frac{(-1)^m \pi}{2^{2n+1}}\binom{2n+1}{n-m} \qquad [n \ge m]$$
$$= 0 \qquad [n < m]$$
BI(40)(12), GW(332)(11c)

7. $$\int_0^\pi \sin^n x \cos(2m+1)x\, dx = 0$$ GW (332)(12a)

8. $$\int_0^\pi \sin^{\nu-1} x \cos ax\, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1}\nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$ $[\operatorname{Re}\nu > 0]$ LO V 121(68)a, WA 337a

9. $$\int_0^{\pi/2} \cos^{\nu-1} x \cos ax\, dx = \frac{\pi}{2^{\nu}\nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$ $[\operatorname{Re}\nu > 0]$ GW (332)(9c)

10. $$\int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x\, dx = \frac{1}{\nu-1}\sin\frac{\nu\pi}{2}$$ $[\operatorname{Re}\nu > 1]$ GW(332)(16b), FI II 15 2

*In 3.631.5, for $m = n$ we should set $(2n-2m-1)!! = 1$

11. $\displaystyle\int_0^\pi \sin^\nu x \cos \nu x\, dx = \frac{\pi}{2^\nu} \cos \frac{\nu\pi}{2}$ $\quad [\operatorname{Re} \nu > -1]$ LO V 121(70)a

12. $\displaystyle\int_0^\pi \sin^{2n} x \cos 2mx\, dx = 2\int_0^{\pi/2} \sin^{2n} x \cos 2mx\, dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi$ $\quad [n \geq m]$

$\displaystyle = 0$ $\quad [n < m]$

BI(40)(16), GW(332)(12b)

13.[7] $\displaystyle\int_0^\pi \sin^{2n+1} x \cos 2mx\, dx$

$$= 2\int_0^{\pi/2} \sin^{2n+1} x \cos 2mx\, dx = \frac{(-1)^m 2^{n+1} n!(2n+1)!!}{(2m-2n-3)!!(2m+2n+1)!!} \quad [n \geq m-1]$$

$$= \frac{(-1)^{n+1} 2^{n+1} n!(2m-2n+3)!!(2n+1)!!}{(2m+2n+1)!!} \quad [n < m-1]$$

GW (332)(12c)

14. $\displaystyle\int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x\, dx = \frac{1}{\nu - 1}$ $\quad [\operatorname{Re} \nu > 1]$ GW(332)(16c), FI II 152

15. $\displaystyle\int_0^\pi \cos^m x \sin nx\, dx = \left[1 - (-1)^{m+n}\right] \int_0^{\pi/2} \cos^m x \sin nx\, dx$

$$= \left[1 - (-1)^{m+n}\right] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s \frac{m!(n-m-2)!!}{(m+n)!!} \right\}$$

$$\left[r = \begin{cases} m & \text{if } m \leq n \\ n & \text{if } m \geq n \end{cases} \quad s = \begin{cases} 2 & \text{if } n-m = 4l+2 > 0 \\ 1 & \text{if } n-m = 2l+1 > 0 \\ 0 & \text{if } n-m = 4l \text{ or } n-m < 0 \end{cases} \right]$$ GW (332)(13a)

16. $\displaystyle\int_0^{\pi/2} \cos^n x \sin nx\, dx = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k}$ FI II 153

17.[11] $\displaystyle\int_0^\pi \cos^n x \sin mx\, dx = \begin{cases} \left[1 + (-1)^{m+n}\right] \dfrac{\pi}{2^{n+1}} \dbinom{n}{k} & \text{if } m \leq n \text{ and } n - m = 2k \\ 0 & \text{otherwise} \end{cases}$

GW (332)(15a)

18.[6] $\displaystyle\int_0^\pi \cos^m x \cos ax\, dx = \frac{(-1)^m \sin a\pi}{2^m (m+a)}\, {}_2F_1\left(-m, -\frac{a+m}{2}; 1 - \frac{a+m}{2}; -1\right)$

$[a \neq 0, \pm 1, \pm 2, \ldots]$ WA 313

19. $\displaystyle\int_0^{\pi/2} \cos^{\nu-2} x \cos \nu x\, dx = 0$ $\quad [\operatorname{Re} \nu > 1]$ GW(332)(16a), FI II 152

20.[10] $\displaystyle\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$ $\quad [\operatorname{Re} n > -1]$ LO V 122(78), FI II 153

3.632

1. $\displaystyle\int_0^\pi \sin^{p-1} x \cos\left[a\left(\frac{\pi}{2} - x\right)\right] dx = 2^{p-1} \frac{\Gamma\left(\frac{p-a}{2}\right) \Gamma\left(\frac{p+a}{2}\right)}{\Gamma(p-a)\,\Gamma(p+a)} \Gamma(p)$

$[p^2 < a^2]$ BI (62)(11)

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin\left[a\left(x+\frac{\pi}{2}\right)\right]\,dx = \frac{\pi \sin\frac{a\pi}{2}}{2^{\nu-1}\nu\,\mathrm{B}\left(\frac{\nu+a+1}{2},\frac{\nu-a+1}{2}\right)}$$
$[\operatorname{Re}\nu > 0]$ WA 337a

3.[10] $$\int_0^{\pi/2} \cos^p x \sin[(p+2n)x]\,dx = (-1)^{n-1}\sum_{k=0}^{n-1}\frac{(-1)^k 2^k}{p+k+1}\binom{n-1}{k}$$
$[n > 0]$ LI (41)(12)

4. $$\int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)]\,dx = \left[1-(-1)^{n+m}\right] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)]\,dx$$
$$= \frac{\left[1-(-1)^{n+m}\right]\pi\cos ma}{2^{n-1}n\,\mathrm{B}\left(\frac{n+m+1}{2},\frac{n-m+1}{2}\right)}$$
$[n \geq m]$ LO V 123(80), LO V 139(94a)

5. $$\int_0^{\pi/2} \cos^{p+q-2} x \cos[(p-q)x]\,dx = \frac{\pi}{2^{p+q-1}(p+q-1)\,\mathrm{B}(p,q)}$$
$[p+q > 1]$ WH

3.633

1. $$\int_0^{\pi/2} \cos^{p-1} x \sin ax \sin x\,dx = \frac{a\pi}{2^{p+1}p(p+1)\,\mathrm{B}\left(\frac{p+a}{2}+1,\frac{p-a}{2}+1\right)}$$
LO V 150(110)

2. $$\int_0^{\pi/2} \cos^n x \sin nx \sin 2mx\,dx = \int_0^{\pi/2} \cos^n x \cos nx \cos 2mx\,dx = \frac{\pi}{2^{n+2}}\binom{n}{m}$$
BI (42)(19, 20)

3. $$\int_0^{\pi/2} \cos^{n-1} x \cos[(n+1)x]\cos 2mx\,dx = \frac{\pi}{2^{n+1}}\binom{n-1}{m-1}$$
$[n > m-1]$ BI (42)(21)

4. $$\int_0^{\pi/2} \cos^{p+q} x \cos px \cos qx\,dx = \frac{\pi}{2^{p+q+2}}\left[1+\frac{1}{(p+q+1)\,\mathrm{B}(p+1,q+1)}\right]$$
$[p+q > -1]$ GW (332)(10c)

5.[6] $$\int_0^{\pi/2} \cos^{p+q} x \sin px \sin qx\,dx = \frac{\pi}{2^{p+q+2}}\sum_{k=1}^{\infty}\binom{p}{k}\binom{q}{k} = \frac{\pi}{2^{p+q+2}}\left[\frac{\Gamma(p+q+1)}{\Gamma(p+1)\,\Gamma(q+1)}-1\right]$$
$[p+q > -1]$ BI (42)(16)

3.634

1. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x\,dx = \sin\frac{\mu\pi}{2}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
BI(42)(23), FI II 814a

2. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu+\nu)x\, dx = \cos\frac{\mu\pi}{2}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI(42)(24), FI II 814a

3. $$\int_0^{\pi/2} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x\, dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n!\,\Gamma(p)}$$
$[p > -n]$ BI (42)(15)

3.635

1. $$\int_0^{\pi/4} \cos^{\mu-1} 2x \tan x\, dx = \frac{1}{4}\left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu}{2}\right)\right]$$
$[\operatorname{Re}\mu > 0]$ BI (34)(7)

2.[7] $$\int_0^{\pi/2} \cos^{p+2n} x \sin px \tan x\, dx = \frac{\pi}{2^{p+2n+1}\,\Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!}$$
$$= \frac{p\pi}{2^{p+2+n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1)\,\Gamma(p+n+1)}$$
$[p > -2n]$ BI (42)(22)

3. $$\int_0^{\pi/2} \cos^{n-1} x \sin[(n+1)x] \cot x\, dx = \frac{\pi}{2}$$
BI (45)(18)

3.636

1. $$\int_0^{\pi/2} \tan^{\pm\mu} x \sin 2x\, dx = \frac{\mu\pi}{2} \operatorname{cosec}\frac{\mu\pi}{2}$$
$[0 < \operatorname{Re}\mu < 2]$ BI (45)(20)a

2. $$\int_0^{\pi/2} \tan^{\pm\mu} x \cos 2x\, dx = \mp\frac{\mu\pi}{2} \sec\frac{\mu\pi}{2}$$
$[|\operatorname{Re}\mu| < 1]$ BI (45)(21)

3.[11] $$\int_0^{\pi/2} \frac{\tan^{2\mu} x}{\cos x}\, dx = \int_0^{\pi/2} \frac{\cot^{2\mu} x}{\sin x}\, dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(-\mu)}{2\sqrt{\pi}}$$
$\left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right]$ (cf. **3.251** 1) BI (45)(13, 14)

3.637

1. $$\int_0^{\pi/2} \tan^p x \sin^{q-2} x \sin qx\, dx = -\cos\frac{(p+q)\pi}{2}\,\mathrm{B}(p+q-1, 1-p)$$
$[p+q > 1 > p]$ GW (332)(15d)

2. $$\int_0^{\pi/2} \tan^p x \sin^{q-2} x \cos qx\, dx = \sin\frac{(p+q)\pi}{2}\,\mathrm{B}(p+q-1, 1-p)$$
$[p+q > 1 > p]$ GW (332)(15b)

3. $$\int_0^{\pi/2} \cot^p x \cos^{q-2} x \sin qx\, dx = \cos\frac{p\pi}{2}\,\mathrm{B}(p+q-1, 1-p)$$
$[p+q > 1 > p]$ GW (332)(15c)

4. $$\int_0^{\pi/2} \cot^p x \cos^{q-2} x \cos qx\, dx = \sin\frac{p\pi}{2}\, \mathrm{B}(p+q-1, 1-p)$$
$[p+q>1>p]$ GW (332)(15a)

3.638

1. $$\int_0^{\pi/4} \frac{\sin^{2\mu} x\, dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2}\sec\mu\pi$$
$\left[|\mathrm{Re}\,\mu| < \frac{1}{2}\right]$ (cf. **3.192** 2)
BI (38)(8)

2. $$\int_0^{\pi/4} \frac{\sin^{\mu-\frac{1}{2}} 2x\, dx}{\cos^{\mu} 2x \cos x} = \frac{2}{2\mu-1}\cdot\frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(1-\mu)}{\sqrt{\pi}}\sin\left(\frac{2\mu-1}{4}\pi\right)$$
$\left[-\frac{1}{2} < \mathrm{Re}\,\mu < 1\right]$ BI (38)(17)

3. $$\int_0^{\pi/2} \frac{\cos^{p-1} x \sin px}{\sin x}\, dx = \frac{\pi}{2}$$
$[p>0]$ GW(332)(17), BI(45)(5)

3.64–3.65 Powers and rational functions of trigonometric functions

3.641

1. $$\int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{-p} x}{a\cos x + b\sin x}\, dx = \int_0^{\pi/2} \frac{\sin^{-p} x \cos^{p-1} x}{a\sin x + b\cos x}\, dx = \frac{\pi\,\mathrm{cosec}\, p\pi}{a^{1-p}b^p}$$
$[ab>0, \quad 0<p<1]$ GW (331)(62)

2. $$\int_0^{\pi/2} \frac{\sin^{1-p} x \cos^{p} x}{(\sin x+\cos x)^3}\, dx = \int_0^{\pi/2} \frac{\sin^{p} x \cos^{1-p} x}{(\sin x+\cos x)^3}\, dx = \frac{(1-p)p}{2}\pi\,\mathrm{cosec}\, p\pi$$
$[-1<p<2]$ BI(48)(5)

3.642

1. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{\mu+\nu}} = \frac{1}{2a^{2\mu}b^{2\nu}}\,\mathrm{B}(\mu,\nu)$$
$[\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>0]$ BI (48)(28)

2. $$\int_0^{\pi/2} \frac{\sin^{n-1} x \cos^{n-1} x\, dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n}} = \frac{\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)}{2(ab)^n}$$
$[ab>0]$ GW (331)(59a)

3. $$\int_0^{\pi/2} \frac{\sin^{2n} x\, dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}} = \frac{1}{2}\int_0^{\pi} \frac{\sin^{2n} x\, dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}}$$
$$= \int_0^{\pi/2} \frac{\cos^{2n} x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{n+1}} = \frac{1}{2}\int_0^{\pi} \frac{\cos^{2n} x\, dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!ab^{2n+1}}$$
$[ab>0]$ GW (331)(58)

4. $$\int_0^{\pi/2} \frac{\cos^{p+2n} x \cos px\, dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}} = \pi\sum_{k=0}^{n}\binom{2n-k}{n}\binom{p+k-1}{k}\frac{b^{p-1}}{(2a)^{2n-k+1}(a+b)^{p+k}}$$
$[a>0, \quad b>0, \quad p>-2n-1]$
GW (332)(30)

3.643

1. $$\int_0^{\pi/2} \frac{\cos^p x \cos px\, dx}{1-2a\cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \qquad \left[a^2 < 1, \quad p > -1\right]$$ GW (332)(33c)

2. $$\int_0^{\pi/2} \frac{\sin^{2n} x \cos^\mu x \cos \beta x}{(1-2a\cos 2x + a^2)^m}\, dx = \frac{(-1)^n \pi (1-a)^{2n-2m+1}}{2^{2m-\beta-1}(1+a)^{2m+\beta+1}} \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \binom{\beta}{k}\binom{2n}{l} \times \binom{2m-k-l-2}{m-1(-2)^l}(a-1)^k$$
$$\left[a^2 < 1, \quad \beta = 2m-2n-\mu-2, \quad \mu > -1\right]$$ GW (332)(33)

3.644

1. $$\int_0^\pi \frac{\sin^m x}{p+q\cos x}\, dx = 2^{m-2}\frac{p}{q^2}\sum_{\nu=1}^{k}\left(\frac{p^2-q^2}{-4q^2}\right)^{\nu-1} \mathrm{B}\left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2}\right) + \left(\frac{p^2-q^2}{-q^2}\right)^k A$$
$$\text{where } A = \begin{cases} \dfrac{\pi p}{q^2}\left(1-\sqrt{1-\dfrac{q^2}{p^2}}\right) & \text{if } m = 2k+2 \\ \dfrac{1}{q}\ln\dfrac{p+q}{p-q} & \text{if } m = 2k+1 \end{cases} \qquad \left[k \ge 1, \quad q \ne 0, \quad p^2 - q^2 \ge 0\right]$$

2. $$\int_0^\pi \frac{\sin^m x}{1+\cos x}\, dx = 2^{m-1}\,\mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \ge 2]$$

3. $$\int_0^\pi \frac{\sin^m x}{1-\cos x}\, dx = 2^{m-1}\,\mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \ge 2]$$

4. $$\int_0^\pi \frac{\sin^2 x}{p+q\cos x}\, dx = \frac{p\pi}{q^2}\left(1-\sqrt{1-\frac{q^2}{p^2}}\right)$$

5. $$\int_0^\pi \frac{\sin^3 x}{p+q\cos x}\, dx = 2\frac{p}{q^2} + \frac{1}{q}\left(1-\frac{p^2}{q^2}\right)\ln\frac{p+q}{p-q}$$

3.645 $$\int_0^\pi \frac{\cos^n x\, dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n}(-1)^k\frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!}\left(\frac{a+b}{a-b}\right)^k$$
$$\left[a^2 > b^2\right]$$ LI (64)(16)

3.646

1. $$\int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1-2a\cos 2x + a^2}\, dx = \frac{\pi}{4a}\left[\left(\frac{1+a}{2}\right)^n - \frac{1}{2^n}\right] \qquad \left[a^2 < 1\right]$$ BI (50)(6)

2. $$\int_0^{\pi/2} \frac{1-a\cos 2nx}{1-2a\cos 2nx + a^2}\cos^m x \cos mx\, dx = \frac{\pi}{2^{m+2}}\sum_{k=1}^{\infty}\binom{m}{kn}a^k + \frac{\pi}{2^{m+1}}$$
$$\left[a^2 < 1\right]$$ LI (50)(7)

3.647 $$\int_0^{\pi/2} \frac{\cos^p x \cos px\, dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2b}\cdot\frac{a^{p-1}}{(a+b)^p} \qquad [p > -1, \quad a > 0, \quad b > 0]$$ BI (47)(20)

3.648

1. $$\int_0^{\pi/4} \frac{\tan^l x\,dx}{1+\cos\frac{m}{n}\pi\sin 2x}$$
$$= \frac{1}{2n}\operatorname{cosec}\frac{m}{n}\pi\sum_{k=0}^{n-1}(-1)^{k-1}\sin\frac{km}{n}\pi\left[\psi\left(\frac{n+l+k}{2n}\right)-\psi\left(\frac{l+k}{2n}\right)\right] \qquad [m+n \text{ is odd}]$$
$$= \frac{1}{n}\operatorname{cosec}\frac{m}{n}\pi\sum_{k=0}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km}{n}\pi\left[\psi\left(\frac{n+l-k}{n}\right)-\psi\left(\frac{l+k}{n}\right)\right] \qquad [m+n \text{ is even}]$$
$[l$ is a natural number$]$ BI (36)(5)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,dx}{1+\cos t\sin 2x} = \pi\operatorname{cosec} t\sin\mu t\operatorname{cosec}(\mu\pi) \qquad \left[|\operatorname{Re}\mu|<1,\quad t^2<\pi^2\right]$$ BI (47)(4)

3.649

1. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\sin 2x\,dx}{1\mp 2a\cos 2x+a^2} = \frac{\pi}{4a}\operatorname{cosec}\frac{\mu\pi}{2}\left[1-\left(\frac{1-a}{1+a}\right)^{\mu}\right] \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{4a}\operatorname{cosec}\frac{\mu\pi}{2}\left[1+\left(\frac{a-1}{a+1}\right)^{\mu}\right] \qquad \left[a^2>1\right]$$
$[-2<\operatorname{Re}\mu<1]$ BI (50)(3)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,(1\mp a\cos 2x)}{1\mp 2a\cos 2x+a^2}\,dx = \frac{\pi}{4}\sec\frac{\mu\pi}{2}\left[1+\left(\frac{1-a}{1+a}\right)^{\mu}\right] \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{4}\sec\frac{\mu\pi}{2}\left[1-\left(\frac{a-1}{a+1}\right)^{\mu}\right] \qquad \left[a^2>1\right]$$
$[|\operatorname{Re}\mu|<1]$ BI (50)(4)

3.651

1. $$\int_0^{\pi/4} \frac{\tan^{\mu} x\,dx}{1+\sin x\cos x} = \frac{1}{3}\left[\psi\left(\frac{\mu+2}{3}\right)-\psi\left(\frac{\mu+1}{3}\right)\right] \qquad [\operatorname{Re}\mu>-1]$$ BI (36)(3)

2. $$\int_0^{\pi/4} \frac{\tan^{\mu} x\,dx}{1-\sin x\cos x} = \frac{1}{3}\left[\beta\left(\frac{\mu+2}{3}\right)+\beta\left(\frac{\mu+1}{3}\right)\right] \qquad [\operatorname{Re}\mu>-1]$$ BI (36)(4)a

3.652

1. $$\int_0^{\pi/2} \frac{\tan^{\mu} x\,dx}{(\sin x+\cos x)\sin x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\,dx}{(\sin x+\cos x)\cos x} = \pi\operatorname{cosec}\mu\pi$$
$[0<\operatorname{Re}\mu<1]$ BI (49)(1)

2. $$\int_0^{\pi/2} \frac{\tan^{\mu} x\,dx}{(\sin x-\cos x)\sin x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\,dx}{(\cos x-\sin x)\cos x} = -\pi\cot\mu\pi$$
$[0<\operatorname{Re}\mu<1]$ BI (49)(2)

3. $$\int_0^{\pi/2} \frac{\cot^{\mu+\frac{1}{2}x}\,dx}{(\sin x+\cos x)\cos x} = \int_0^{\pi/2} \frac{\tan^{\mu-\frac{1}{2}x}\,dx}{(\sin x+\cos x)\cos x} = \pi\sec\mu\pi$$
$\left[|\operatorname{Re}\mu|<\frac{1}{2}\right]$ BI (61)(1, 2)

3.653

1. $$\int_0^{\pi/2} \frac{\tan^{1-2\mu} x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi/2} \frac{\cot^{1-2\mu} x\,dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2a^{2\mu}b^{2-2\mu}\sin\mu\pi}$$
$[0 < \operatorname{Re}\mu < 1]$ GW (331)(59b)

2.[11] $$\int_0^{\pi/2} \frac{\tan^{\mu} x\,dx}{1 - a\sin^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\,dx}{1 - a\cos^2 x} = \frac{\pi\sec\frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}}$$
$[|\operatorname{Re}\mu| < 1, \quad a < 1]$ BI (49)(6)

3. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,dx}{1 - \cos^2 t\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} t\sec\frac{\mu\pi}{2}\cos\left[\left(\frac{\pi}{2} - t\right)\mu\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(49)(7), BI(47)(21)

4. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\sin 2x}{1 - \cos^2 t\sin^2 2x}\,dx = \pi\operatorname{cosec} 2t\operatorname{cosec}\frac{\mu\pi}{2}\sin\left[\left(\frac{\pi}{2} - t\right)\mu\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI (47)(22)a

5. $$\int_0^{\pi/2} \frac{\tan^{\mu} x\sin^2 x\,dx}{1 - \cos^2 t\sin^2 2x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\cos^2 x\,dx}{1 - \cos^2 t\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} 2t\sec\frac{\mu\pi}{2}\cos\left[\frac{\mu\pi}{2} - (\mu+1)t\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(47)(23)a, BI(49)(10)

6. $$\int_0^{\pi/2} \frac{\tan^{\mu} x\cos^2 x\,dx}{1 - \cos^2 t\sin^2 2x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\sin^2 x\,dx}{1 - \cos^2 t\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} 2t\sec\frac{\mu\pi}{2}\cos\left[\frac{\mu\pi}{2} - (\mu-1)t\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(47)(24)a, BI(49)(9)

3.654

1. $$\int_0^{\pi/2} \frac{\tan^{\mu+1} x\cos^2 x\,dx}{(1 + \cos t\sin 2x)^2} = \int_0^{\pi/2} \frac{\cot^{\mu+1} x\sin^2 x\,dx}{(1 + \cos t\sin 2x)^2} = \frac{\pi\left(\mu\sin t\cos\mu t - \cos t\sin\mu t\right)}{2\sin\mu\pi\sin^3 t}$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(48)(3), BI(49)(22)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,dx}{(\sin x + \cos x)^2} = \frac{\mu\pi}{\sin\mu\pi}$$
$[0 < \operatorname{Re}\mu < 1]$ BI (56)(9)a

3. $$\int_0^{\pi/2} \frac{\tan^{\pm(\mu-1)x}\,dx}{\cos^2 x - \sin^2 x} = \pm\frac{\pi}{2}\cot\frac{\mu\pi}{2}$$
$[0 < \operatorname{Re}\mu < 2]$ BI (45)(27, 29)

3.655 $$\int_0^{\pi/2} \frac{\tan^{2\mu-1} x\,dx}{1 - 2a\left(\cos t_1\sin^2 x + \cos t_2\cos^2 x\right) + a^2} = \int_0^{\pi/2} \frac{\cot^{2\mu-1} x\,dx}{1 - 2a\left(\cos t_1\cos^2 x + \cos t_2\sin^2 x\right) + a^2}$$
$$= \frac{\pi\operatorname{cosec}\mu\pi}{\left(1 - 2a\cos t_2 + a^2\right)^{\mu}\left(1 - 2a\cos t_1 + a^2\right)1-\mu}$$
$\left[0 < \operatorname{Re}\mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2\right]$ BI (50)(18)

3.656

1. $$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin^2 x\cos^2 x} = \frac{1}{12}\left\{-\psi\left(\frac{\mu+1}{6}\right) - \psi\left(\frac{\mu+2}{6}\right) + \psi\left(\frac{\mu+4}{6}\right) + \psi\left(\frac{\mu+5}{6}\right) + 2\,\psi\left(\frac{\mu+2}{3}\right) - 2\,\psi\left(\frac{\mu+1}{3}\right)\right\}$$
$[\operatorname{Re}\mu > -1]$ (cf. **3.651** 1 and 2) LI (36)(10)

2. $$\int_0^{\pi/2} \frac{\tan^{\mu-1} x\cos^2 x\,dx}{1-\sin^2 x\cos^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu-1} x\sin^2 x\,dx}{1-\sin^2 x\cos^2 x} = \frac{\pi}{4\sqrt{3}}\operatorname{cosec}\frac{\mu\pi}{6}\operatorname{cosec}\left(\frac{2+\mu}{6}\pi\right)$$
$[0 < \operatorname{Re}\mu < 4]$ LI (47)(26)

3.66 Forms containing powers of linear functions of trigonometric functions

3.661

1. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n+1}\,dx = 0$$ BI (68)(9)

2. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot 2\pi\left(a^2+b^2\right)^n$$ BI (68)(8)

3. $$\int_0^{\pi} (a+b\cos x)^n\,dx = \frac{1}{2}\int_0^{2\pi} (a+b\cos x)^n\,dx = \pi\left(a^2-b^2\right)^{\frac{n}{2}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right)$$
$$= \frac{\pi}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k}\left(a^2-b^2\right)^k$$
$\left[a^2 > b^2\right]$ GW (332)(37a)

4. $$\int_0^{\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{(a^2-b^2)^{\frac{n+1}{2}}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right)$$
$$= \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n} \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!}\cdot\left(\frac{a+b}{a-b}\right)^k$$
$[a > |b|]$ GW(332)(38), LI(64)(14)

3.662

1. $$\int_0^{\pi/2} (\sec x - 1)^\mu \sin x\,dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cos x\,dx = \mu\pi\operatorname{cosec}\mu\pi$$
$[|\operatorname{Re}\mu| < 1]$ BI (55)(13)

2. $$\int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \sin 2x\,dx = (1-\mu)\mu\pi\operatorname{cosec}\mu\pi \qquad [-1 < \operatorname{Re}\mu < 2]$$ BI (48)(7)

3. $$\int_0^{\pi/2} (\sec x - 1)^\mu \tan x\,dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cot x\,dx = -\pi\operatorname{cosec}\mu\pi$$
$[-1 < \operatorname{Re}\mu < 0]$ BI (46)(4,6)

4. $$\int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2}\operatorname{cosec}\mu\pi \qquad [-1 < \operatorname{Re}\mu < 0]$$ BI (38)(22)a

5. $$\int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \qquad [|\operatorname{Re}\mu| < 1]$$ BI (38)(11)a

3.663

1. $$\int_0^u (\cos x - \cos u)^{\nu-\frac{1}{2}} \cos ax\, dx = \sqrt{\frac{\pi}{2}} \sin^\nu u\, \Gamma\left(\nu + \frac{1}{2}\right) P^{-\nu}_{a-\frac{1}{2}}(\cos u)$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}; \quad a > 0, \quad 0 < u < \pi\right]$$
EH I 159(27), ET I 22(28)

2. $$\int_0^u (\cos x - \cos u)^{\nu-1} \cos[(\nu+\beta)x]\, dx = \frac{\sqrt{\pi}\,\Gamma(\beta+1)\,\Gamma(\nu)\,\Gamma(2\nu)\sin^{2\nu-1} u}{2^\nu\,\Gamma(\beta+2\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)} C^\nu_\beta(\cos u)$$
$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > -1, \quad 0 < u < \pi]$$
EH I 178(23)

3.664

1. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^q dx = \pi\, P_q(z)$$
$$\left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right]$$ SM 482

2. $$\int_0^\pi \frac{dx}{\left(z + \sqrt{z^2-1}\cos x\right)^q} = \pi\, P_{q-1}(z)$$
$$\left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right]$$ WH

3. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^q \cos nx\, dx = \frac{\pi}{(q+1)(q+2)\cdots(q+n)} P^n_q(z)$$
$$\Bigg[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2},$$
$$z \text{ lies outside the interval } (-1,1) \text{ of the real axis}\Bigg]$$
WH, SM 483(15)

4. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^\mu \sin^{2\nu-1} x\, dx$$
$$= \frac{2^{2\nu-1}\,\Gamma(\mu+1)\,[\Gamma(\nu)]^2}{\Gamma(2\nu+\mu)} C^\nu_\mu(z)$$
$$= \frac{\sqrt{\pi}\,\Gamma(\nu)\,\Gamma(2\nu)\,\Gamma(\mu+1)}{\Gamma(2\nu+\mu)\,\Gamma\left(\nu+\frac{1}{2}\right)} C^\nu_\mu(z) = 2^\nu \sqrt{\frac{\pi}{2}} \left(z^2-1\right)^{\frac{1}{4}-\frac{\nu}{2}} \Gamma(\nu)\, P^{\frac{1}{2}-\nu}_{\mu+\nu-\frac{1}{2}}(z)$$
$$[\operatorname{Re}\nu > 0]$$ EH I 155(6)a, EH I 178(22)

5. $$\int_0^{2\pi} \left[\beta + \sqrt{\beta^2-1}\cos(a-x)\right]^\nu \left(\gamma + \sqrt{\gamma^2-1}\cos x\right)^{\nu-1} dx$$
$$= 2\pi\, P_\nu\left(\beta\gamma - \sqrt{\beta^2-1}\sqrt{\gamma^2-1}\cos a\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$$ EH I 157(18)

3.665

1. $$\int_0^\pi \frac{\sin^{\mu-1} x\,dx}{(a+b\cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2-b^2)^\mu}}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\mu}{2}\right) \qquad [\operatorname{Re}\mu>0,\quad 0<b<a]$$ FI II 790a

2. $$\int_0^\pi \frac{\sin^{2\mu-1} x\,dx}{(1+2a\cos x+a^2)^\nu} = \mathrm{B}\left(\mu,\tfrac12\right) F\left(\nu,\nu-\mu+\tfrac12;\mu+\tfrac12;a^2\right)$$
$$[\operatorname{Re}\mu>0,\quad |a|<1]$$ EH I 81(9)

3.666

1. $$\int_0^\pi (\beta+\cos x)^{\mu-\nu-\frac12}\sin^{2\nu} x\,dx = \frac{2^{\nu+\frac12}e^{-i\mu\pi}\left(\beta^2-1\right)^{\frac{\mu}{2}}\Gamma\left(\nu+\frac12\right)Q^\mu_{\nu-\frac12}(\beta)}{\Gamma\left(\nu+\mu+\frac12\right)}$$
$$\left[\operatorname{Re}\left(\nu+\mu+\tfrac12\right)>0,\quad \operatorname{Re}\nu>-\tfrac12\right]$$ EH I 155(5)a

2.[6] $$\int_0^\pi (\cosh\beta+\sinh\beta\cos x)^{\mu+\nu}\sin^{-2\nu} x\,dx = \frac{\sqrt{\pi}}{2^\nu}\sinh^\nu(\beta)\,\Gamma\left(\tfrac12-\nu\right)P^\nu_\mu(\cosh\beta)$$
$$\left[\operatorname{Re}\nu<\tfrac12\right]$$ EH I 156(7)

3. $$\int_0^\pi (\cos t+i\sin t\cos x)^\mu\sin^{2\nu-1} x\,dx = 2^{\nu-\frac12}\sqrt{\pi}\sin^{\frac12-\nu} t\,\Gamma(\nu)\,P^{\frac12-\nu}_{\mu+\nu-\frac12}(\cos t)$$
$$\left[\operatorname{Re}\nu>0,\quad t^2<\pi^2\right]$$ EH I 158(23)

4. $$\int_0^{2\pi}[\cos t+i\sin t\cos(a-x)]^\nu\cos mx\,dx = \frac{i^{3m}2\pi\,\Gamma(\nu+1)}{\Gamma(\nu+m+1)}\cos ma\,P^m_\nu(\cos t)$$
$$\left[0<t<\frac{\pi}{2}\right]$$ EH I 159(25)

5.[10] $$\int_0^{2\pi}[\cos t+i\sin t\cos(a-x)]^\nu\sin mx\,dx = \frac{i^{3m}2\pi\,\Gamma(\nu+1)}{\Gamma(\nu+m+1)}\sin ma\,P^m_\nu(\cos t)$$
$$\left[0<t<\frac{\pi}{2}\right]$$ EH I 159(26)

3.667

1. $$\int_0^{\pi/4}\frac{\sin^{\mu-1}2x\,dx}{(\cos x+\sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}}\frac{\Gamma(\mu)}{\Gamma\left(\mu+\frac12\right)} \qquad [\operatorname{Re}\mu>0]$$ BI (37)(1)

2. $$\int_0^{\pi/4}\frac{\sin^\mu x\,dx}{(\cos x-\sin x)^{\mu+1}\cos x} = -\pi\operatorname{cosec}\mu\pi \qquad [-1<\operatorname{Re}\mu<0] \qquad (\text{cf. } \mathbf{3.192}\ 2)$$
BI (37)(16)

3. $$\int_0^{\pi/4}\frac{(\cos x-\sin x)^\mu}{\sin^\mu x\sin 2x}\,dx = -\frac{\pi}{2}\operatorname{cosec}\mu\pi \qquad [-1<\operatorname{Re}\mu<0]$$ BI (35)(27)

4. $$\int_0^{\pi/4}\frac{\sin^\mu x\,dx}{(\cos x-\sin x)^\mu\sin 2x} = \frac{\pi}{2}\operatorname{cosec}\mu\pi \qquad [0<\operatorname{Re}\mu<1]$$ LI (37)(20)a

5. $$\int_0^{\pi/4}\frac{\sin^\mu x\,dx}{(\cos x-\sin x)^\mu\cos^2 x} = \mu\pi\operatorname{cosec}\mu\pi \qquad [|\operatorname{Re}\mu|<1]$$ BI (37)(17)

6. $$\int_0^{\pi/4} \frac{\sin^\mu x\,dx}{(\cos x - \sin x)^{\mu-1}\cos^3 x} = \frac{1-\mu}{2}\mu\pi \operatorname{cosec}\mu\pi \qquad [|\operatorname{Re}\mu| < 1]$$ BI(35)(24), BI(37)(18)

7. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(\sin x + \cos x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ BI (48)(8)

3.668

1. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)^{\cos 2t} dx = \frac{\pi}{2\sin(\pi\cos^2 t)}$$ FI II 788

2. $$\int_u^v \frac{(\cos u - \cos x)^{\mu-1}}{(\cos x - \cos v)^\mu} \cdot \frac{\sin x\,dx}{1 - 2a\cos x + a^2} = \frac{\left(1 - 2a\cos u + a^2\right)^{\mu-1}}{(1 - 2a\cos v + a^2)^\mu} \cdot \frac{\pi}{\sin\mu\pi}$$
$$\left[0 < \operatorname{Re}\mu < 1, \quad a^2 < 1\right]$$ BI (73)(2)

3.669 $$\int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{q-p-1} x\,dx}{(a\cos x + b\sin x)^q} = \int_0^{\pi/2} \frac{\sin^{q-p-1} x \cos^{p-1} x}{(a\sin x + b\cos x)^q}\,dx = \frac{\mathrm{B}(p, q-p)}{a^{q-p}b^p}$$
$$[q > p > 0, \quad ab > 0]$$ BI (331)(9)

3.670

1. $$\int_0^\pi \sqrt{a \pm b\cos x}\,dx = \int_{-\pi/2}^{\pi/2} \sqrt{a \pm b\cos x}\,dx = 2\sqrt{a+b}\,\boldsymbol{K}\left(\sqrt{\frac{2b}{a+b}}\right)$$
$$[a > b > 0]$$

2.* $$\int_0^\pi \frac{dx}{\sqrt{a \pm b\cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b\sin x}} = \frac{2}{\sqrt{a+b}}\,\boldsymbol{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$
$$[a > b > 0]$$

3.67 Square roots of expressions containing trigonometric functions

3.671

1. $$\int_0^{\pi/2} \sin^\alpha x \cos^\beta x \sqrt{1 - k^2\sin^2 x}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$
$$[\alpha > -1, \quad \beta > -1, \quad |k| < 1]$$
GW (331)(93)

2. $$\int_0^{\pi/2} \frac{\sin^\alpha x \cos^\beta x}{\sqrt{1 - k^2\sin^2 x}}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$
$$[\alpha > -1, \quad \beta > -1, \quad |k| < 1]$$
GW (331)(92)

3. $$\int_0^\pi \frac{\sin^{2n} x\,dx}{\sqrt{1 - k^2\sin^2 x}} = \frac{\pi}{2^n}\sum_{j=0}^\infty \frac{(2j-1)!!\,(n+2j-1)!!}{2^{2j}j!(n+j)!}k^{2j} \qquad \left[k^2 < 1\right]$$
$$= \frac{(2n-1)!!\pi}{2^n\sqrt{1-k^2}}\sum_{j=0}^\infty \frac{\left[(2j-1)!!\right]^2}{2^{2j}j!(n+j)!}\left(\frac{k^2}{k^2-1}\right)^j \qquad \left[k^2 < \tfrac{1}{2}\right]$$
LI (67)(2)

4.* $$\int_0^{\pi} \sqrt{a+b\cos x}\,dx = \int_{-\pi/2}^{\pi/2} \sqrt{a+b\sin x}\,dx = 2\sqrt{a+b}\,\boldsymbol{E}\left(\sqrt{\frac{2b}{a+b}}\right) \qquad [a>b]$$

5.* $$\int_0^{\pi} \frac{dx}{\sqrt{a\pm b\cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a\pm b\sin x}} = \frac{2}{a+b}\boldsymbol{K}\left(\sqrt{\frac{2b}{a+b}}\right) \qquad [a>b]$$

3.672

1. $$\int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x}\cdot\frac{dx}{\sqrt{\cos x\,(\cos x-\sin x)}} = 2\cdot\frac{(2n)!!}{(2n+1)!!}$$ BI (39)(5)

2. $$\int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x}\cdot\frac{dx}{\sqrt{\sin x\,(\cos x-\sin x)}} = \frac{(2n-1)!!}{(2n)!!}\pi$$ BI (39)(6)

3.673 $$\int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x-\sin u}} = \sqrt{2}\boldsymbol{K}\left(\sin\frac{\pi-2u}{4}\right)$$ BI (74)(11)

3.674

1.[8] $$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-(p^2/2)\,(1-\cos 2x)}} = \boldsymbol{K}(p), \qquad [1>p>0]$$ BI (67)(5)

2. $$\int_0^{\pi} \frac{\sin x\,dx}{\sqrt{1-2p\cos x+p^2}} = 2 \qquad [p^2\le 1]$$
$$= \frac{2}{p} \qquad [p^2\ge 1]$$
BI (67)(6)

3.[8] $$\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1-2p\cos x+p^2}} = \frac{1}{p}\left[\frac{1+p^2}{1+p}\boldsymbol{K}\left(\frac{2\sqrt{p}}{1+p}\right)-(1+p)\boldsymbol{E}\left(\frac{2\sqrt{p}}{1+p}\right)\right] \qquad [p^2<1]$$ BI (67)(7)

3.675

1. $$\int_u^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x\,dx}{\sqrt{2\,(\cos u-\cos x)}} = \frac{\pi}{2}\,P_n\,(\cos u)$$ WH

2. $$\int_0^{u} \frac{\cos\left(n+\frac{1}{2}\right)x\,dx}{\sqrt{2\,(\cos x-\cos u)}} = \frac{\pi}{2}\,P_n\,(\cos u)$$ FI II 684, WH

3.676

1. $$\int_0^{\pi/2} \frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{p}\arctan p$$ BI (60)(5)

2. $$\int_0^{\pi/2} \tan^2 x\sqrt{1-p^2\sin^2 x}\,dx = \infty$$ BI (53)(8)

3. $$\int_0^{\pi/2} \frac{dx}{\sqrt{p^2\cos^2 x + q^2 \sin^2 x}} = \frac{1}{p}\boldsymbol{K}\left(\frac{\sqrt{p^2-q^2}}{p}\right) \qquad [0<q<p]$$ FI II 165

3.677

1. $$\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sqrt{1+\sin^2 x}} = \sqrt{2}\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (60)(2)

2. $$\int_0^{\pi/2} \frac{\cos^2 x\,dx}{\sqrt{1+\sin^2 x}} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right]$$ BI (60)(3)

3.678

1. $$\int_0^{\pi/4} \left(\sec^{1/2} 2x - 1\right)\frac{dx}{\tan x} = \ln 2$$ BI (38)(23)

2. $$\int_0^{\pi/4} \frac{\tan^2 x\,dx}{\sqrt{1-k^2\sin^2 2x}} = \sqrt{1-k^2} - \boldsymbol{E}(k) + \frac{1}{2}\boldsymbol{K}(k)$$ BI (39)(2)

3. $$\int_0^{u} \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}}\,dx = \frac{\pi}{2}(1-\cos u) \qquad \left[u^2 < \frac{\pi^2}{4}\right]$$ LI (74)(6)

4. $$\int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x}\sqrt{\operatorname{cosec} x}\,dx = \frac{(2n-1)!!}{(2n)!!}\pi$$ BI (38)(24)

5. $$\int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x}\tan^m x\sqrt{\operatorname{cosec} x}\,dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!}\pi$$ BI (38)(25)

3.679

1. $$\int_0^{\pi/2} \frac{\cos^2 x}{1-\cos^2\beta\cos^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{\sin\beta\cos\beta\sqrt{1-k'^2\sin^2\beta}}\left\{\frac{\pi}{2} - \boldsymbol{K}E(\beta,k') - \boldsymbol{E}F(\beta,k') + \boldsymbol{K}F(\beta,k')\right\}^*$$ MO 138

2. $$\int_0^{\pi/2} \frac{\sin^2 x}{1-\left(1-k'^2\sin^2\beta\right)\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k'^2\sin\beta\cos\beta\sqrt{1-k'^2\sin^2\beta}}\left\{\frac{\pi}{2} - \boldsymbol{K}E(\beta,k') - \boldsymbol{E}F(\beta,k') + \boldsymbol{K}F(\beta,k')\right\}^*$$ MO 138

3. $$\int_0^{\pi/2} \frac{\sin^2 x}{1-k^2\sin^2\beta\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\boldsymbol{K}E(\beta,k) - \boldsymbol{E}F(\beta,k)}{k^2\sin\beta\cos\beta\sqrt{1-k^2\sin^2\beta}}$$ MO 138

*In 3.679, $k' = \sqrt{1-k^2}$.

3.68 Various forms of powers of trigonometric functions

3.681

1. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x\, dx}{\left(1-k^2\sin^2 x\right)^{\varrho}} = \frac{1}{2}\,\mathrm{B}(\mu,\nu)\, F\left(\varrho,\mu;\mu+\nu;k^2\right)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ EH I 115(7)

2. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x\, dx}{\left(1-k^2\sin^2 x\right)^{\mu+\nu}} = \frac{\mathrm{B}(\mu,\nu)}{2\,(1-k^2)^{\mu}}$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ EH I 10(20)

3. $$\int_0^{\pi/2} \frac{\sin^{\mu} x\, dx}{\cos^{\mu-3} x\left(1-k^2\sin^2 x\right)^{\frac{\mu}{2}-1}} = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(2-\frac{\mu}{2}\right)}{k^3\sqrt{\pi(\mu-1)(\mu-3)(\mu-5)}}\left\{\frac{1+(\mu-3)k+k^2}{(1+k)^{\mu-3}} - \frac{1-(\mu-3)k+k^2}{(1-k)^{\mu-3}}\right\}$$
$[-1<\operatorname{Re}\mu<4]$ BI (54)(10)

4.[8] $$\int_0^{\pi/2} \frac{\sin^{\mu+1} x\, dx}{\cos^{\mu} x\left(1-k^2\sin^2 x\right)^{\frac{\mu+1}{2}}} = \frac{(1-k)^{-\mu}-(1+k)^{-\mu}}{2k\mu\sqrt{\pi}}\,\Gamma\left(1+\frac{\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)$$
$[-2<\operatorname{Re}\mu<1]$ BI (61)(5)

3.682 $$\int_0^{\pi/2} \frac{\sin^{\mu} x \cos^{\nu} x}{(a-b\cos^2 x)^{\varrho}}\, dx = \frac{1}{2a^{\varrho}}\,\mathrm{B}\left(\frac{\mu+1}{2},\frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2},\varrho;\frac{\mu+\nu}{2}+1;\frac{b}{a}\right)$$
$[\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1, \quad a>|b|\geq 0]$ GW (331)(64)

3.683

1. $$\int_0^{\pi/4} (\sin^n 2x-1)\tan\left(\frac{\pi}{4}+x\right)\, dx = \int_0^{\pi/4} (\cos^n 2x-1)\cot x\, dx = -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} = -\frac{1}{2}\left[\boldsymbol{C}+\psi(n+1)\right]$$
$[n\geq 0]$ BI(34)(8), BI(35)(11)

2. $$\int_0^{\pi/4} (\sin^{\mu} 2x-1)\operatorname{cosec}^{\mu} 2x\tan\left(\frac{\pi}{4}+x\right)\, dx = \int_0^{\pi/4} (\cos^{\mu} 2x-1)\sec^{\mu} 2x\cot x\, dx = \frac{1}{2}\left[\boldsymbol{C}+\psi(1-\mu)\right]$$
$[\operatorname{Re}\mu<1]$ BI (35)(20)

3. $$\int_0^{\frac{\pi}{4}} \left(\sin^{2\mu} 2x-1\right)\operatorname{cosec}^{\mu} 2x\tan\left(\frac{\pi}{4}+x\right)\, dx = \int_0^{\pi/4} \left(\cos^{2\mu} 2x-1\right)\sec^{\mu} 2x\cot x\, dx = -\frac{1}{2\mu}+\frac{\pi}{2}\cot\mu\pi$$
BI (35)(21)

4. $$\int_0^{\pi/4} (1-\sec^{\mu} 2x)\cot x\, dx = \int_0^{\pi/4} (1-\operatorname{cosec}^{\mu} 2x)\tan\left(\frac{\pi}{4}+x\right)\, dx = \frac{1}{2}\left[\boldsymbol{C}+\psi(1-\mu)\right]$$
$[\operatorname{Re}\mu<1]$ BI (35)(13)

3.684 $$\int_0^{\pi/4} \frac{(\cot^\mu x - 1)\, dx}{(\cos x - \sin x)\sin x} = \int_0^{\pi/2} \frac{(\tan^\mu x - 1)\, dx}{(\sin x - \cos x)\cos x} = -\boldsymbol{C} - \psi(1-\mu) \qquad [\operatorname{Re}\mu < 1]$$

BI (37)(9)

3.685

1. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x - \sin^{\nu-1} 2x\right) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} \left(\cos^{\mu-1} 2x - \cos^{\nu-1} 2x\right) \cot x\, dx$$
$$= \frac{1}{2}\left[\psi(\nu) - \psi(\mu)\right]$$
$[\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0]$ BI(34)(9), BI(35)(12)

2. $$\int_0^{\pi/2} \left(\sin^{\mu-1} x - \sin^{\nu-1} x\right) \frac{dx}{\cos x} = \int_0^{\pi/2} \left(\cos^{\mu-1} x - \cos^{\nu-1} x\right) \frac{dx}{\sin x} = \frac{1}{2}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\mu}{2}\right)\right]$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (46)(2)

3. $$\int_0^{\pi/2} (\sin^\mu x - \operatorname{cosec}^\mu x) \frac{dx}{\cos x} = \int_0^{\pi/2} (\cos^\mu x - \sec^\mu x) \frac{dx}{\sin x} = -\frac{\pi}{2} \tan \frac{\mu\pi}{2}$$
$[|\operatorname{Re}\mu| < 1]$ BI (46)(1, 3)

4. $$\int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \cot\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \tan x\, dx$$
$$= \frac{1}{2\mu} - \frac{\pi}{2} \operatorname{cosec} \mu\pi$$
$[|\operatorname{Re}\mu| < 1]$ BI (35)(19, 22)

5. $$\int_0^{\pi/4} (\sin^\mu 2x - \operatorname{cosec}^\mu 2x) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} (\cos^\mu 2x - \sec^\mu 2x) \cot x\, dx$$
$$= -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi$$
$[|\operatorname{Re}\mu| < 1]$ BI (35)(14)

6. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x + \operatorname{cosec}^\mu 2x\right) \cot\left(\frac{\pi}{4} + x\right) dx$$
$$= \int_0^{\pi/4} \left(\cos^{\mu-1} 2x + \sec^\mu 2x\right) \tan x\, dx = \frac{\pi}{4} \operatorname{cosec} \mu\pi$$
$[0 < \operatorname{Re}\mu < 1]$ BI (35)(18, 8)

7. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x - \operatorname{cosec}^\mu 2x\right) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} \left(\cos^{\mu-1} 2x - \sec^\mu 2x\right) \cot x\, dx = \frac{\pi}{2} \cot \mu\pi$$
$[0 < \operatorname{Re}\mu < 1]$ BI(35)(7), LI(34)(10)

3.686 $$\int_0^{\pi/2} \frac{\tan x\, dx}{\cos^\mu x + \sec^\mu x} = \int_0^{\pi/2} \frac{\cot x\, dx}{\sin^\mu x + \operatorname{cosec}^\mu x} = \frac{\pi}{4\mu}$$ BI(47)(28), BI(49)(14)

3.687

1. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x}\, dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x}\, dx = \frac{\cos\left(\frac{\nu-\mu}{4}\pi\right)}{2\cos\left(\frac{\nu+\mu}{4}\pi\right)} \mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 2]$
BI (46)(7)

2. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x}\,dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x}\,dx = \frac{\sin\left(\frac{\nu-\mu}{4}\pi\right)}{2\sin\left(\frac{\nu+\mu}{4}\pi\right)}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\nu}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 4]$
BI(46)(8)

3. $$\int_0^{\pi/2} \frac{\sin^{\mu} x + \sin^{\nu} x}{\sin^{\mu+\nu} x + 1}\cot x\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x + \cos^{\nu} x}{\cos^{\mu+\nu} x + 1}\tan x\,dx = \frac{\pi}{\mu+\nu}\sec\left(\frac{\mu-\nu}{\mu+\nu}\cdot\frac{\pi}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
BI (49)(15)a, BI (47)(29)

4. $$\int_0^{\pi/2} \frac{\sin^{\mu} x - \sin^{\nu} x}{\sin^{\mu+\nu} x - 1}\cot x\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x - \cos^{\nu} x}{\cos^{\mu+\nu} x - 1}\tan x\,dx = \frac{\pi}{\mu+\nu}\tan\left(\frac{\mu-\nu}{\mu+\nu}\cdot\frac{\pi}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
BI(149)(16)a, BI(47)(30)

5. $$\int_0^{\pi/2} \frac{\cos^{\mu} x + \sec^{\mu} x}{\cos^{\nu} x + \sec^{\nu} x}\tan x\,dx = \frac{\pi}{2\nu}\sec\left(\frac{\mu}{\nu}\cdot\frac{\pi}{2}\right) \qquad [|\operatorname{Re}\nu| > |\operatorname{Re}\mu|]$$
BI (49)(12)

6. $$\int_0^{\pi/2} \frac{\cos^{\mu} x - \sec^{\mu} x}{\cos^{\nu} x - \sec^{\nu} x}\tan x\,dx = \frac{\pi}{2\nu}\tan\left(\frac{\mu}{\nu}\cdot\frac{\pi}{2}\right) \qquad [|\operatorname{Re}\nu| > |\operatorname{Re}\mu|]$$
BI (49)(13)

3.688

1. $$\int_0^{\pi/4} \frac{\tan^{\nu} x - \tan^{\mu} x}{\cos x - \sin x}\cdot\frac{dx}{\sin x} = \psi(\mu) - \psi(\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
BI (37)(10)

2. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \tan^{1-\mu} x}{\cos x - \sin x}\cdot\frac{dx}{\sin x} = \pi\cot\mu\pi \qquad [0 < \operatorname{Re}\mu < 1]$$
BI (37)(11)

3. $$\int_0^{\pi/4} (\tan^{\mu} x + \cot^{\mu} x)\,dx = \frac{\pi}{2}\sec\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1]$$
BI (35)(9)

4. $$\int_0^{\pi/4} (\tan^{\mu} x - \cot^{\mu} x)\tan x\,dx = \frac{1}{\mu} - \frac{\pi}{2}\operatorname{cosec}\frac{\mu\pi}{2} \qquad [0 < \operatorname{Re}\mu < 2]$$
BI (35)(15)

5. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu-1} x}{\cos 2x}\,dx = \frac{\pi}{2}\cot\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 2]$$
BI (35)(10)

6. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{\cos 2x}\tan x\,dx = -\frac{1}{\mu} + \frac{\pi}{2}\cot\frac{\mu\pi}{2} \qquad [-2 < \operatorname{Re}\mu < 0]$$
BI (35)(23)

7. $$\int_0^{\pi/4} \frac{\tan^{\mu} x + \cot^{\mu} x}{1 + \cos t\sin 2x}\,dx = \pi\operatorname{cosec} t\operatorname{cosec}\mu\pi\sin\mu t \qquad [t \neq n\pi, \quad |\operatorname{Re}\mu| < 1]$$
BI (36)(6)

8. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x + \cot^{\mu} x}{(\sin x + \cos x)\cos x}\,dx = \pi\operatorname{cosec}\mu\pi \qquad [0 < \operatorname{Re}\mu < 1]$$
BI (37)(3)

9. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\sin x + \cos x)\cos x}\,dx = -\pi\operatorname{cosec}\mu\pi + \frac{1}{\mu} \qquad [0 < \operatorname{Re}\mu < 1]$$
BI (37)(4)

10. $$\int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\mu} x}{(\cos x - \sin x)\cos x}\,dx = \psi(1-\mu) - \psi(1+\nu) \qquad [\operatorname{Re}\mu < 1, \quad \operatorname{Re}\nu > -1]$$
BI (37)(5)

11. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu} x}{(\cos x - \sin x)\cos x}\, dx = \pi \cot \mu\pi \qquad [0 < \operatorname{Re}\mu < 1]$$ BI (37)(7)

12. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\cos x - \sin x)\cos x}\, dx = \pi \cot \mu\pi - \frac{1}{\mu} \qquad [0 < \operatorname{Re}\mu < 1]$$ BI (37)(8)

13. $$\int_0^{\pi/4} \frac{1}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \qquad [\operatorname{Re}\mu \neq 0]$$ BI (37)(12)

14. $$\int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\tan x} = \int_0^{\pi/2} \frac{1}{(\tan^{\mu} x + \cot^{\mu} x)^{\nu}} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1}\mu} \frac{\Gamma(\nu)}{\Gamma\left(\nu + \frac{1}{2}\right)}$$
$[\nu > 0]$ BI(49)(25), BI(49)(26)

15. $$\int_0^{\pi/4} (\tan^{\mu} x - \cot^{\mu} x)(\tan^{\nu} x - \cot^{\nu} x)\, dx = \frac{2\pi \sin\frac{\mu\pi}{2} \sin\frac{\nu\pi}{2}}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(17)

16. $$\int_0^{\pi/4} (\tan^{\mu} x + \cot^{\mu} x)(\tan^{\nu} x + \cot^{\nu} x)\, dx = \frac{2\pi \cos\frac{\mu\pi}{2} \cos\frac{\nu\pi}{2}}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(16)

17. $$\int_0^{\pi/4} \frac{(\tan^{\mu} x - \cot^{\mu} x)(\tan^{\nu} x + \cot^{\nu} x)}{\cos 2x}\, dx = -\pi \frac{\sin\mu\pi}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(25)

18. $$\int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\nu} x}{\tan^{\mu} x - \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \tan\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1]$$ BI (37)(14)

19. $$\int_0^{\pi/4} \frac{\tan^{\nu} x + \cot^{\nu} x}{\tan^{\mu} x + \cot^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \sec\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1]$$ BI (37)(13)

20. $$\int_0^{\pi/2} \frac{(1+\tan x)^{\nu} - 1}{(1+\tan x)^{\mu+\nu}} \frac{dx}{\sin x \cos x} = \psi(\mu+\nu) - \psi(\mu) \qquad [\mu > 0, \quad \nu > 0]$$ BI (49)(29)

3.689

1. $$\int_0^{\pi/2} \frac{(\sin^{\mu} x + \operatorname{cosec}^{\mu} x)\cot x\, dx}{\sin^{\nu} x - 2\cos t + \operatorname{cosec}^{\nu} x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec}\frac{\mu\pi}{\nu} \sin\frac{\mu t}{\nu}$$
$[\mu < \nu]$ LI (50)(14)

2. $$\int_0^{\pi/2} \frac{\sin^{\mu} x - 2\cos t_1 + \operatorname{cosec}^{\mu} x}{\sin^{\nu} x + 2\cos t_2 + \operatorname{cosec}^{\nu} x} \cdot \cot x\, dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec}\frac{\mu\pi}{\nu} \sin\frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1$$
$[(\nu > \mu > 0)$ or $(\nu < \mu < 0)$ or $(\mu > 0,\ \nu < 0,$ and $\mu + \nu < 0)$ or $(\mu < 0,\ \nu > 0,$ and $\mu + \nu > 0)]$
BI (50)(15)

3.69–3.71 Trigonometric functions of more complicated arguments

3.691

1. $$\int_0^\infty \sin\left(ax^2\right)\,dx = \int_0^\infty \cos ax^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \qquad [a>0]$$ FI II 743a, ET I 64(7)a

2. $$\int_0^1 \sin\left(ax^2\right)\,dx = \sqrt{\frac{\pi}{2a}}\,S\left(\sqrt{a}\right) \qquad [a>0]$$

3. $$\int_0^1 \cos\left(ax^2\right)\,dx = \sqrt{\frac{\pi}{2a}}\,C\left(\sqrt{a}\right) \qquad [a>0]$$ ET I 8(5)a

4. $$\int_0^\infty \sin\left(ax^2\right)\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\,C\left(\frac{b}{\sqrt{a}}\right)+\sin\frac{b^2}{a}\,S\left(\frac{b}{\sqrt{a}}\right)\right\}$$
$$[a>0,\quad b>0]$$ ET I 82(1)a

5. $$\int_0^\infty \sin\left(ax^2\right)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}-\sin\frac{b^2}{a}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(\frac{b^2}{a}+\frac{\pi}{4}\right)$$
$$[a>0,\quad b>0]$$
ET I 82(18), BI(70)(13) GW(334)(5a)

6. $$\int_0^\infty \cos ax^2\sin 2bx\,dx = \sqrt{\frac{\pi}{2a}}\left\{\sin\frac{b^2}{a}\,C\left(\frac{b}{\sqrt{a}}\right)-\cos\frac{b^2}{a}\,S\left(\frac{b}{\sqrt{a}}\right)\right\}$$
$$[a>0,\quad b>0]$$ ET I 83(3)a

7. $$\int_0^\infty \cos ax^2\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}+\sin\frac{b^2}{a}\right\} \qquad [a>0,\quad b>0]$$
GW(334)(5a), BI(70)(14), ET I 24(7)

8. $$\int_0^\infty (\cos ax+\sin ax)\sin\left(b^2x^2\right)\,dx$$
$$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1+2\,C\left(\frac{a}{2b}\right)\right)\cos\left(\frac{a^2}{4b^2}\right)-\left(1-2\,S\left(\frac{a}{2b}\right)\right)\sin\left(\frac{a^2}{4b^2}\right)\right\}$$
$$[a>0,\quad b>0]$$ ET I 85(22)

9. $$\int_0^\infty (\cos ax+\sin ax)\cos\left(b^2x^2\right)\,dx$$
$$= \frac{1}{2b}\sqrt{\frac{\pi}{2}}\left\{\left(1+2\,C\left(\frac{a}{2b}\right)\right)\sin\left(\frac{a^2}{4b^2}\right)+\left(1-2\,S\left(\frac{a}{2b}\right)\right)\cos\left(\frac{a^2}{4b^2}\right)\right\}$$
$$[a>0,\quad b>0]$$ ET I 25(21)

10. $$\int_0^\infty \sin\left(a^2x^2\right)\sin 2bx\sin 2cx\,dx = \frac{\sqrt{\pi}}{2a}\sin\frac{2bc}{a^2}\cos\left(\frac{b^2+c^2}{a^2}-\frac{\pi}{4}\right)$$
$$[a>0,\quad b>0,\quad c>0]$$ ET I 84(15)

11. $$\int_0^\infty \sin\left(a^2x^2\right)\cos 2bx\cos 2cx\,dx = \frac{\sqrt{\pi}}{2a}\cos\frac{2bc}{a^2}\cos\left(\frac{b^2+c^2}{a^2}+\frac{\pi}{4}\right)$$
$$[a>0,\quad b>0,\quad c>0]$$ ET I 84(21)

12. $$\int_0^\infty \cos\left(a^2x^2\right)\sin 2bx\sin 2cx\,dx=\frac{\sqrt{\pi}}{2a}\sin\frac{2bc}{a^2}\sin\left(\frac{b^2+c^2}{a^2}-\frac{\pi}{4}\right)$$
$[a>0,\quad b>0,\quad c>0]$ ET I 25(19)

13. $$\int_0^\infty \sin\left(ax^2\right)\cos\left(bx^2\right)\,dx=\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{a-b}}\right)\qquad[a>b>0]$$
$$=\frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{b+a}}-\frac{1}{\sqrt{b-a}}\right)\qquad[b>a>0]$$
BI (177)(21)

14. $$\int_0^\infty \left(\sin^2 ax^2-\sin^2 bx^2\right)\,dx=\frac{1}{8}\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right)\qquad[a>0,\quad b>0]$$ BI (178)(1)

15. $$\int_0^\infty \left(\cos^2 ax^2-\sin^2 bx^2\right)\,dx=\frac{1}{8}\left(\sqrt{\frac{\pi}{b}}+\sqrt{\frac{\pi}{a}}\right)\qquad[a>0,\quad b>0]$$ BI (178)(3)

16. $$\int_0^\infty \left(\cos^2 ax^2-\cos^2 bx^2\right)\,dx=\frac{1}{8}\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right)\qquad[a>0,\quad b>0]$$ BI (178)(5)

17. $$\int_0^\infty \left(\sin^4 ax^2-\sin^4 bx^2\right)x=\frac{1}{64}\left(8-\sqrt{2}\right)\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right)$$
$[a>0,\quad b>0]$ BI (178)(2)

18. $$\int_0^\infty \left(\cos^4 ax^2-\sin^4 bx^2\right)\,dx=\frac{1}{8}\left(\sqrt{\frac{\pi}{a}}+\sqrt{\frac{\pi}{b}}\right)+\frac{1}{32}\left(\sqrt{\frac{\pi}{2a}}-\sqrt{\frac{\pi}{2b}}\right)$$
$[a>0,\quad b>0]$ BI (178)(4)

19. $$\int_0^\infty \left(\cos^4 ax^2-\cos^4 bx^2\right)\,dx=\frac{1}{64}\left(8+\sqrt{2}\right)\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right)$$
$[a>0,\quad b>0]$ BI (178)(6)

20. $$\int_0^\infty \sin^{2n} ax^2\,dx=\int_0^\infty \cos^{2n} ax^2\,dx=\infty$$ BI (177)(5, 6)

21. $$\int_0^\infty \sin^{2n+1}\left(ax^2\right)\,dx=\frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^{n+k}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}$$
$[a>0]$ BI (70)(9)

22. $$\int_0^\infty \cos^{2n+1}\left(ax^2\right)\,dx=\frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}$$
$[a>0]$ BI(177)(7)a, BI(70)(10)

3.692

1. $$\int_0^\infty \left[\sin\left(a-x^2\right)+\cos\left(a-x^2\right)\right]\,dx=\sqrt{\frac{\pi}{a}}\sin a$$ GW(333)(30c), BI(178)(7)a

2. $$\int_0^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos ax\,dx=\sqrt{\frac{\pi}{2}}\cos\left(\frac{a^2}{2}-\frac{\pi}{8}\right)\qquad[a>0]$$ ET I 24(8)

3. $$\int_0^\infty \sin\left[a\left(1-x^2\right)\right]\cos bx\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right)$$
$[a>0]$ ET I 23(2)

4. $$\int_0^\infty \cos\left[a\left(1-x^2\right)\right]\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sin\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right)$$
$[a>0]$ ET I 24(10)

5. $$\int_0^\infty \sin\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \int_0^\infty \cos\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$
$[a>0]$ BI (70)(19, 20)

6.[8] $$\int_{-\infty}^\infty \left[\cos\sqrt{x^2-1}-\cos\sqrt{x^2+1}\right]\,dx = \sum_{n=0}^\infty \frac{\pi}{\left\{2^{4n+1}\left[(2n)!\right]^2\left(n+\frac{1}{2}\right)\right\}}$$

3.693

1. $$\int_0^\infty \sin\left(ax^2+2bx\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)-\sin\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\right\}$$
$[a>0]$ BI (70)(3)

2. $$\int_0^\infty \cos\left(ax^2+2bx\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)+\sin\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\right\}$$
$[a>0]$ BI (70)(4)

3.694

1. $$\int_0^\infty \sin\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c+\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c-\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}$$
$[a>0]$ GW (334)(4a)

2. $$\int_0^\infty \cos\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c-\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c+\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$[a>0]$ GW (334)(4b)

3.695

1. $$\int_0^\infty \sin\left(a^3x^3\right)\sin(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)-\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a>0,\quad b>0]$ ET I 83(5)

2. $$\int_0^\infty \cos\left(a^3x^3\right)\cos(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a>0,\quad b>0]$ ET I 24(11)

3.696

1. $$\int_0^\infty \sin\left(ax^4\right)\sin\left(bx^2\right)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a>0,\quad b>0]$ ET I 83(2)

2. $$\int_0^\infty \sin\left(ax^4\right)\cos\left(bx^2\right)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a>0,\quad b>0]$ ET I 84(19)

3. $$\int_0^\infty \cos\left(ax^4\right)\sin\left(bx^2\right)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a>0,\quad b>0]$ ET I 83(4), ET I 25(24)

4. $$\int_0^\infty \cos\left(ax^4\right)\cos\left(bx^2\right)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a>0,\quad b>0]$ ET I 25(25)

3.697 $$\int_0^\infty \sin\left(\frac{a^2}{x}\right)\sin(bx)\,dx = \frac{a\pi}{2\sqrt{b}}J_1\left(2a\sqrt{b}\right) \qquad [a>0,\quad b>0]$$ ET I 83(6)

3.698

1. $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\sin\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab-\cos 2ab+e^{-2ab}\right]$$
$[a>0,\quad b>0]$ ET I 83(9)

2.[8] $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\cos\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab+\cos 2ab-e^{-2ab}\right]$$ ET I 24(13)

3. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\sin\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab+\cos 2ab+e^{-2ab}\right]$$
$[a>0,\quad b>0]$ ET I 84(12)

4. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\cos\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\cos 2ab-\sin 2ab+e^{-2ab}\right]$$
$[a>0,\quad b>0]$ ET I 24(14)

3.699

1. $$\int_0^\infty \sin\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}\left(\cos 2ab+\sin 2ab\right) \qquad [a>0,\quad b>0]$$ BI (70)(27)

2. $$\int_0^\infty \cos\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}\left(\cos 2ab-\sin 2ab\right) \qquad [a>0,\quad b>0]$$ BI (70)(28)

3. $$\int_0^\infty \sin\left(a^2x^2-2ab+\frac{b^2}{x^2}\right)\,dx = \int_0^\infty \cos\left(a^2x^2-2ab+\frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}$$
$[a>0,\quad b>0]$
BI(179)(11, 12)a, ET I 83(6)

4. $\displaystyle\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}e^{-2ab}$ $[a>0,\quad b>0]$ GW (334)(9b)a

5. $\displaystyle\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}e^{-2ab}$ $[a>0,\quad b>0]$ GW (334)(9b)a

3.711 $\displaystyle\int_0^u \sin\left(a\sqrt{u^2-x^2}\right)\cos bx\,dx = \frac{\pi a u}{2\sqrt{a^2+b^2}}\,J_1\left(u\sqrt{a^2+b^2}\right)$ $[a>0,\quad b>0,\quad u>0]$

ET I 27(37)

3.712

1. $\displaystyle\int_0^\infty \sin\left(ax^p\right)\,dx = \frac{\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}}{pa^{\frac{1}{p}}}$ $[a>0,\quad p>1]$ EH I 13(40)

2. $\displaystyle\int_0^\infty \cos\left(ax^p\right)\,dx = \frac{\Gamma\left(\frac{1}{p}\right)\cos\frac{\pi}{2p}}{pa^{\frac{1}{p}}}$ $[a>0,\quad p>1]$ EH I 13(39)

3.713

1. $\displaystyle\int_0^\infty \sin\left(ax^p+bx^q\right)\,dx = \frac{1}{p}\sum_{k=0}^\infty \frac{(-b)^k}{k!}a^{-\frac{kq+1}{p}}\,\Gamma\left(\frac{kq+1}{p}\right)\sin\left[\frac{k(q-p)+1}{2p}\pi\right]$

$[a>0,\quad b>0,\quad p>0,\quad q>0]$

BI (70)(7)

2. $\displaystyle\int_0^\infty \cos\left(ax^p+bx^q\right)\,dx = \frac{1}{p}\sum_{k=0}^\infty \frac{(-b)^k}{k!}a^{-(kq+1)/p}\,\Gamma\left(\frac{kq+1}{p}\right)\cos\left[\frac{k(q-p)+1}{2p}\pi\right]$

$[a>0,\quad b>0,\quad p>0,\quad q>0]$

BI (70)(8)

3.714

1. $\displaystyle\int_0^\infty \cos\left(z\sinh x\right)\,dx = K_0(z)$ $[\operatorname{Re} z>0]$ WA 202(14)

2. $\displaystyle\int_0^\infty \sin\left(z\cosh x\right)\,dx = \frac{\pi}{2}\,J_0(z)$ $[\operatorname{Re} z>0]$ MO 36

3. $\displaystyle\int_0^\infty \cos\left(z\cosh x\right)\,dx = -\frac{\pi}{2}\,Y_0(z)$ $[\operatorname{Re} z>0]$ MO 37

4. $\displaystyle\int_0^\infty \cos\left(z\sinh x\right)\cosh\mu x\,dx = \cos\frac{\mu\pi}{2}\,K_\mu(z)$ $[\operatorname{Re} z>0,\quad |\operatorname{Re}\mu|<1]$ WA 202(13)

5. $\displaystyle\int_0^\pi \cos\left(z\cosh x\right)\sin^{2\mu}x\,dx = \sqrt{\pi}\left(\frac{2}{z}\right)^\mu\Gamma\left(\mu+\frac{1}{2}\right)I_\mu(z)$

$\left[\operatorname{Re} z>0,\quad \operatorname{Re}\mu>-\frac{1}{2}\right]$ WH

3.715

1. $\displaystyle\int_0^\pi \sin\left(z\sin x\right)\sin ax\,dx = \sin a\pi\,s_{0,a}(z) = \sin a\pi\sum_{k=1}^\infty \frac{(-1)^{k-1}z^{2k-1}}{(1^2-a^2)(3^2-a^2)\ldots[(2k-1)^2-a^2]}$

$[a>0]$ WA 338(13)

2. $$\int_0^{\pi} \sin(z\sin x)\sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(z\sin x)\sin nx\,dx$$
$$= [1-(-1)^n]\int_0^{\pi/2} \sin(z\sin x)\sin nx\,dx = [1-(-1)^n]\,\frac{\pi}{2}\,J_n(z)$$
$[n=0,\pm1,\pm2,\ldots]$ WA 30(6), GW(334)(153a)

3. $$\int_0^{\pi/2} \sin(z\sin x)\sin 2x\,dx = \frac{2}{z^2}(\sin z - z\cos z)$$ LI (43)(14)

4. $$\int_0^{\pi} \sin(z\sin x)\cos ax\,dx = (1+\cos a\pi)\,s_{0,a}(z)$$
$$= (1+\cos a\pi)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k-1}}{(1^2-a^2)(3^2-a^2)\ldots[(2k-1)^2-a^2]}$$
$[a>0]$ WA 338(14)

5. $$\int_0^{\pi} \sin(z\sin x)\cos[(2n+1)x]\,dx = 0$$ GW (334)(53b)

6. $$\int_0^{\pi} \cos(z\sin x)\sin ax\,dx = -a(1-\cos a\pi)\,s_{-1,a}(z)$$
$$= -a(1-\cos a\pi)\left\{-\frac{1}{a^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\ldots[(2k)^2-a^2]}\right\}$$
$[a>0]$ WA 338(12)

7. $$\int_0^{\pi} \cos(z\sin x)\sin 2nx\,dx = 0$$ GW (334)(54a)

8. $$\int_0^{\pi} \cos(z\sin x)\cos ax\,dx = -a\sin a\pi\,s_{-1,a}(z)$$
$$= -a\sin a\pi\left\{-\frac{1}{a^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\ldots[(2k)^2-a^2]}\right\}$$
$[a>0]$ WA 338(11)

9. $$\int_0^{\pi} \cos(z\sin x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(z\sin x)\cos nx\,dx$$
$$= [1+(-1)^n]\int_0^{\pi/2} \cos(z\sin x)\cos nx\,dx = [1+(-1)^n]\,\frac{\pi}{2}\,J_n(z)$$
GW (334)(54b)

10.[8] $$\int_0^{\pi/2} \cos(z\sin x)\cos^{2n}x\,dx = \frac{\pi}{2}\frac{(2n-1)!!}{z^n}\,J_n(z)$$ $[n=0,1,2,\ldots]$ FI II 486, WA 35a

11. $$\int_0^{\pi/2} \sin(z\cos x)\sin 2x\,dx = \frac{2}{z^2}(\sin z - z\cos z)$$ LI (43)(15)

12.[8] $$\int_0^{\pi/2} \sin(z\cos x)\cos ax\,dx = \cos\frac{a\pi}{2}\,s_{0,a}(z) = \frac{\pi}{4}\operatorname{cosec}\frac{a\pi}{2}\,[\mathbf{J}_a(z)-\mathbf{J}_{-a}(z)]$$
$$= -\frac{\pi}{4}\sec\frac{a\pi}{4}\,[\mathbf{E}_a(z)+\mathbf{E}_{-a}(z)]$$
$$= \cos\frac{a\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k-1}}{(1^2-a^2)(3^2-a^2)\ldots[(2k-1)^2-a^2]}$$
$[a>0]$ WA 339

13. $$\int_0^{\pi}\sin(z\cos x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(z\cos x)\cos nx\,dx = \pi\sin\frac{n\pi}{2}\,J_n(z)$$ GW (334)(55b)

14. $$\int_0^{\pi/2}\sin(z\cos x)\cos[(2n+1)x]\,dx = (-1)^n\frac{\pi}{2}\,J_{2n+1}(z)$$ WA 30(8)

15.[11] $$\int_0^{\pi/2}\sin(a\cos x)\tan x\,dx = \operatorname{si}(a)+\frac{\pi}{2}$$ $[a>0]$ BI (43)(17)

16. $$\int_0^{\pi/2}\sin(z\cos x)\sin^{2\nu}x\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{z}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(z)$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ WA 358(1)

17.[7] $$\int_0^{\pi/2}\cos(z\cos x)\cos ax\,dx = -a\sin\frac{a\pi}{2}\,s_{-1,a}(z)$$
$$= \frac{\pi}{4}\sec\frac{a\pi}{2}\,[\mathbf{J}_a(z)+\mathbf{J}_{-a}(z)] = \frac{\pi}{4}\operatorname{cosec}\frac{a\pi}{2}\,[\mathbf{E}_a(z)-\mathbf{E}_{-a}(z)]$$
$$= -a\sin\frac{a\pi}{2}\left\{-\frac{1}{a^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\ldots[(2k)^2-a^2]}\right\}$$
$[a>0]$ WA 339

18. $$\int_0^{\pi}\cos(z\cos x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(z\cos x)\cos nx\,dx = \pi\cos\frac{n\pi}{2}\,J_n(z)$$ GW (334)(56b)

19. $$\int_0^{\pi/2}\cos(z\cos x)\cos 2nx\,dx = (-1)^n\cdot\frac{\pi}{2}\,J_{2n}(z)$$ WA 30(9)

20. $$\int_0^{\pi/2}\cos(z\cos x)\sin^{2\nu}x\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{z}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(z)$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ WA 35, WH

21. $$\int_0^{\pi}\cos(z\cos x)\sin^{2\mu}x\,dx = \sqrt{\pi}\left(\frac{2}{z}\right)^{\mu}\Gamma\left(\mu+\frac{1}{2}\right)J_\mu(z)$$
$\left[\operatorname{Re}\mu>-\frac{1}{2}\right]$ WH

3.716

1. $$\int_0^{\pi/2}\sin(a\tan x)\,dx = \frac{1}{2}\left[e^{-a}\overline{\operatorname{Ei}}(a)-e^{a}\operatorname{Ei}(-a)\right]$$ $[a>0]$ (cf. **3.723** 1) BI (43)(1)

2. $$\int_0^{\pi/2}\cos(a\tan x)\,dx = \frac{\pi}{2}e^{-a}$$ $[a\geq 0]$ BI (43)(2)

3. $\displaystyle\int_0^{\pi/2} \sin(a\tan x)\sin 2x\,dx = \frac{a\pi}{2}e^{-a}$ $[a \geq 0]$ BI (43)(7)

4. $\displaystyle\int_0^{\pi/2} \cos(a\tan x)\sin^2 x\,dx = \frac{1-a}{4}\pi e^{-a}$ $[a \geq 0]$ BI (43)(8)

5. $\displaystyle\int_0^{\pi/2} \cos(a\tan x)\cos^2 x\,dx = \frac{1+a}{4}\pi e^{-a}$ $[a \geq 0]$ BI (43)(9)

6. $\displaystyle\int_0^{\pi/2} \sin(a\tan x)\tan x\,dx = \frac{\pi}{2}e^{-a}$ $[a > 0]$ BI (43)(5)

7. $\displaystyle\int_0^{\pi/2} \cos(a\tan x)\tan x\,dx = -\frac{1}{2}\left[e^{-a}\overline{\mathrm{Ei}}(a) + e^{a}\,\mathrm{Ei}(-a)\right]$

$[a > 0]$ (cf. **3.723** 5) BI (43)(6)

8. $\displaystyle\int_0^{\pi/2} \sin(a\tan x)\sin^2 x\tan x\,dx = \frac{2-a}{4}\pi e^{-a}$ $[a > 0]$ BI (43)(11)

9. $\displaystyle\int_0^{\pi/2} \sin^2(a\tan x)\,dx = \frac{\pi}{4}\left(1 - e^{-2a}\right)$ $[a \geq 0]$ (cf. **3.742** 1) BI (43)(3)

10. $\displaystyle\int_0^{\pi/2} \cos^2(a\tan x)\,dx = \frac{\pi}{4}\left(1 + e^{-2a}\right)$ $[a \geq 0]$ (cf. **3.742** 3) BI (43)(4)

11. $\displaystyle\int_0^{\pi/2} \sin^2(a\tan x)\cot^2 x\,dx = \frac{\pi}{4}\left(e^{-2a} + 2a - 1\right)$ $[a \geq 0]$ BI (43)(19)

12. $\displaystyle\int_0^{\pi/2} \left[1 - \sec^2 x\cos(\tan x)\right]\frac{dx}{\tan x} = \boldsymbol{C}$ BI (51)(14)

13. $\displaystyle\int_0^{\pi/2} \sin(a\cot x)\sin 2x\,dx = \frac{a\pi}{2}e^{-a}$ $[a \geq 0]$ (cf. **3.716** 3)

In general, formulas **3.716** remain valid if we replace $\tan x$ in the argument of the sine or cosine with $\cot x$ if we also replace $\sin x$ with $\cos x$, $\cos x$ with $\sin x$, hence $\tan x$ with $\cot x$, $\cot x$ with $\tan x$, $\sec x$ with $\operatorname{cosec} x$, and $\operatorname{cosec} x$ with $\sec x$ in the factors. Analogously,

3.717 $\displaystyle\int_0^{\pi/2} \sin(a\operatorname{cosec} x)\sin(a\cot x)\frac{dx}{\cos x} = \int_0^{\pi/2} \sin(a\sec x)\sin(a\tan x)\frac{dx}{\sin x} = \frac{\pi}{2}\sin a$ $[a \geq 0]$

BI (52)(11, 12)

3.718

1. $\displaystyle\int_0^{\pi/2} \sin\left(\frac{\pi}{2}p - a\tan x\right)\tan^{p-1} x\,dx = \int_0^{\pi/2} \cos\left(\frac{\pi}{2}p - a\tan x\right)\tan^{p} x\,dx = \frac{\pi}{2}e^{-a}$

$\left[p^2 < 1, \quad p \neq 0, \quad a \geq 0\right]$ BI (44)(5, 6)

2. $\displaystyle\int_0^{\pi/2} \sin(a\tan x - \nu x)\sin^{\nu-2} x\,dx = 0$ $[\operatorname{Re}\nu > 0, \quad a > 0]$ NH 157(15)

3. $\displaystyle\int_0^{\pi/2} \sin(n\tan x + \nu x)\frac{\cos^{\nu-1} x}{\sin x}\,dx = \frac{\pi}{2}$ $[\operatorname{Re}\nu > 0]$ BI (51)(15)

4. $$\int_0^{\pi/2} \cos(a\tan x - \nu x)\cos^{\nu-2} x\,dx = \frac{\pi e^{-a} a^{\nu-1}}{\Gamma(\nu)} \qquad [\operatorname{Re}\nu > 1, \quad a > 0]$$
LO V 153(112), NT 157(14)

5. $$\int_0^{\pi/2} \cos(a\tan x + \nu x)\cos^{\nu} x\,dx = 2^{-\nu-1}\pi e^{-a} \qquad [\operatorname{Re}\nu > -1, \quad a \geq 0]$$
BI (44)(4)

6. $$\int_0^{\pi/2} \cos(a\tan x - \gamma x)\cos^{\nu} x\,dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2},-\frac{\nu+1}{2}}(2a)}{\Gamma\left(1+\frac{\gamma+\nu}{2}\right)}$$
$$\left[a > 0, \quad \operatorname{Re}\nu > -1, \quad \frac{\nu+\gamma}{2} \neq -1, -2, \ldots\right]$$
EH I 274(13)a

7. $$\int_0^{\pi/2} \frac{\sin nx - \sin(nx - a\tan x)}{\sin x}\cos^{n-1} x\,dx = \begin{cases} \pi/2 & [n=0, \quad a>0], \\ \pi(1-e^{-a}) & [n=1, \quad a \geq 0] \end{cases}$$
LO V 153(114)

3.719

1.[6] $$\int_0^{\pi} \sin(\nu x - z\sin x)\,dx = \pi\,\mathbf{E}_\nu(z)$$
WA 336(2)

2. $$\int_0^{\pi} \cos(nx - z\sin x)\,dx = \pi\,J_n(z)$$
WH

3. $$\int_0^{\pi} \cos(\nu x - z\sin x)\,dx = \pi\,\mathbf{J}_\nu(z)$$
WA 336(1)

3.72–3.74 Combinations of trigonometric and rational functions

3.721

1. $$\int_0^{\infty} \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}\operatorname{sign} a$$
FI II 645

2. $$\int_1^{\infty} \frac{\sin(ax)}{x}\,dx = -\operatorname{si}(a)$$
BI 203(1)

3.[8] $$\int_1^{\infty} \frac{\cos(ax)}{x}\,dx = -\operatorname{ci}(a)$$
BI 203(5)

3.722

1. $$\int_0^{\infty} \frac{\sin(ax)}{x+\beta}\,dx = \operatorname{ci}(a\beta)\sin(a\beta) - \cos(a\beta)\operatorname{si}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0]$$
BI(16)(1), FI II 646a

2.[11] $$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x+\beta}\,dx = \pi e^{ia\beta} \qquad [a > 0, \quad \operatorname{Im}\beta > 0]$$

3. $$\int_0^{\infty} \frac{\cos(ax)}{x+\beta}\,dx = -\sin(a\beta)\operatorname{si}(a\beta) - \cos(a\beta)\operatorname{ci}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0]$$
ET I 8(7), BI(160)(2)

4.[8] $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x+\beta}\,dx = -i\pi e^{ia\beta}$ $[a>0, \quad \operatorname{Im}\beta>0]$

5.[10] $\displaystyle\int_{0}^{\infty} \frac{\sin(ax)}{\beta-x}\,dx = \sin(\beta a)\operatorname{ci}(\beta a) - \cos(\beta a)\left[\operatorname{si}(\beta a)+\pi\right]$

$[a>0, \quad \beta$ not real and positive$]$

FI II 646, BI(161)(1)

6.[8] $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(ax)}{\beta-x}\,dx = -\pi e^{ia\beta}$ $[a>0, \quad \operatorname{Im}\beta>0]$

7.[10] $\displaystyle\int_{0}^{\infty} \frac{\cos(ax)}{\beta-x}\,dx = -\cos(a\beta)\operatorname{ci}(a\beta) + \sin(a\beta)\left[\operatorname{si}(a\beta)+\pi\right]$

$[a>0, \quad \beta$ not real and positive$]$

ET I 8(8), BI(161)(2)a

8.[11] $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{\beta-x}\,dx = -i\pi e^{ia\beta}$ $[a>0, \quad \operatorname{Im}\beta>0]$

3.723

1.[11] $\displaystyle\int_{0}^{\infty} \frac{\sin(ax)}{\beta^2+x^2}\,dx = \frac{1}{2\beta}\left[e^{-a\beta}\overline{\operatorname{Ei}}(a\beta) - e^{a\beta}\operatorname{Ei}(-a\beta)\right]$ $[a>0, \quad \beta>0]$ ET I 65(14), BI(160)(3)

2. $\displaystyle\int_{0}^{\infty} \frac{\cos(ax)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}e^{-a\beta}$ $[a\geq 0, \quad \operatorname{Re}\beta>0]$

FI II 741, 750, ET I 8(11), WH

3. $\displaystyle\int_{0}^{\infty} \frac{x\sin(ax)}{\beta^2+x^2}\,dx = \frac{\pi}{2}e^{-a\beta}$ $[a>0, \quad \operatorname{Re}\beta>0]$

FI II 741, 750, ET I 65(15), WH

4. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{\beta^2+x^2}\,dx = \pi e^{-a\beta}$ $[a>0, \quad \operatorname{Re}\beta>0]$ BI (202)(10)

5.[11] $\displaystyle\int_{0}^{\infty} \frac{x\cos(ax)}{\beta^2+x^2}\,dx = -\frac{1}{2}\left[e^{-a\beta}\overline{\operatorname{Ei}}(a\beta) + e^{a\beta}\operatorname{Ei}(-a\beta)\right]$ $[a>0, \quad \beta>0]$ BI (160)(6)

6. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin[a(b-x)]}{c^2+x^2}\,dx = \frac{\pi}{c}e^{-ac}\sin(ab)$ $[a>0, \quad b>0, \quad c>0]$ LI (202)(9)

7. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos[a(b-x)]}{c^2+x^2}\,dx = \frac{\pi}{c}e^{-ac}\cos(ab)$ $[a>0, \quad b>0, \quad c>0]$ LI (202)(11)a

8. $\displaystyle\int_{0}^{\infty} \frac{\sin(ax)}{\beta^2-x^2}\,dx = \frac{1}{\beta}\left[\sin(a\beta)\operatorname{ci}(a\beta) - \cos(a\beta)\left(\operatorname{si}(a\beta)+\frac{\pi}{2}\right)\right]$

$[|\arg\beta|<\pi, \quad a>0]$ BI (161)(3)

9. $\displaystyle\int_{0}^{\infty} \frac{\cos(ax)}{b^2-x^2}\,dx = \frac{\pi}{2b}\sin(ab)$ $[a>0, \quad b>0]$ BI(161)(5), ET I 9(15)

10. $\displaystyle\int_{0}^{\infty} \frac{x\sin(ax)}{b^2-x^2}\,dx = -\frac{\pi}{2}\cos(ab)$ $[a>0]$ FI II 647, ET II 252(45)

11. $\displaystyle\int_{0}^{\infty} \frac{x\cos(ax)}{\beta^2+x^2}\,dx = \cos(a\beta)\operatorname{ci}(a\beta) + \sin(a\beta)\left[\operatorname{si}(a\beta)+\frac{\pi}{2}\right]$

$[|\arg\beta|<\pi, \quad a>0]$ BI (161)(6)

12. $$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x-b)}\,dx = \pi\frac{\cos(ab)-1}{b} \qquad [a>0, \quad b>0]$$ ET II 252(44)

3.724

1. $$\int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2}\sin(ax)\,dx = \left(\frac{cq-b}{\sqrt{p-q^2}}\sin(aq)+c\cos(aq)\right)\pi e^{-a\sqrt{p-q^2}}$$
$$\left[a>0, \quad p>q^2\right]$$ BI (202)(12)

2. $$\int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2}\cos(ax)\,dx = \left(\frac{b-cq}{\sqrt{p-q^2}}\cos(aq)+c\sin(aq)\right)\pi e^{-a\sqrt{p-q^2}}$$
$$\left[a>0, \quad p>q^2\right]$$ BI (202)(13)

3. $$\int_{-\infty}^{\infty} \frac{\cos[(b-1)t]-x\cos(bt)}{1-2x\cos t+x^2}\cos(ax)\,dx = \pi e^{-a\sin t}\sin(bt+a\cos t)$$
$$\left[a>0, \quad t^2<\pi^2\right]$$ BI (202)(14)

3.725

1. $$\int_0^{\infty} \frac{\sin(ax)\,dx}{x\left(\beta^2+x^2\right)} = \frac{\pi}{2\beta^2}\left(1-e^{-a\beta}\right) \qquad [\operatorname{Re}\beta>0, \quad a>0]$$ BI (172)(1)

2. $$\int_0^{\infty} \frac{\sin(ax)\,dx}{x\left(b^2-x^2\right)} = \frac{\pi}{2b^2}\left(1-\cos(ab)\right) \qquad [a>0]$$ BI (172)(4)

3. $$\int_0^{\infty} \frac{\sin(ax)\cos(bx)}{x\left(x^2+\beta^2\right)}\,dx = \frac{\pi}{2\beta^2}e^{-\beta b}\sinh(a\beta) \qquad [0<a<b]$$
$$= -\frac{\pi}{2\beta^2}e^{-a\beta}\cosh(b\beta)+\frac{\pi}{2\beta^2} \qquad [a>b>0]$$
ET I 19(4)

3.726

1.[11] $$\int_0^{\infty} \frac{x\sin(ax)\,dx}{b^3\pm b^2x+bx^2\pm x^3}$$
$$= \pm\frac{1}{4b}\left[e^{-ab}\overline{\operatorname{Ei}}(ab)-e^{ab}\operatorname{Ei}(-ab)-2\operatorname{ci}(ab)\sin(ab)+2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right]$$
$$+\frac{\pi e^{-ab}-\pi\cos(ab)}{4b}$$
$[a>0, \quad b>0;$ if the lower sign is taken, then the integral is a principal value integral$]$

ET I 65(21)a, BI(176)(10, 13)

2.[7] $$\int_0^{\infty} \frac{x^2\sin(ax)\,dx}{b^3\pm b^2x+bx^2\pm x^3}$$
$$= \frac{1}{4}\left[e^{ab}\operatorname{Ei}(-ab)-e^{-ab}\overline{\operatorname{Ei}}(ab)+2\operatorname{ci}(ab)\sin(ab)-2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right]$$
$$\pm\pi\left(e^{-ab}+\cos(ab)\right)$$
$[a>0, \quad b>0;$ if the lower sign is taken, then the integral is a principal value integral$]$

ET I 66(22), BI(176)(11, 14)

3.727

1. $$\int_0^\infty \frac{\cos(ax)}{b^4+x^4}\,dx = \frac{\pi\sqrt{2}}{4b^3}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}}+\sin\frac{ab}{\sqrt{2}}\right)$$
$[a>0, \quad b>0]$ BI(160)(25)a, ET I 9(19)

2.[8] $$\int_0^\infty \frac{\sin(ax)}{b^4-x^4}\,dx = \frac{1}{4b^3}\Big[2\sin(ab)\,\mathrm{ci}(ab) - 2\cos(ab)\left(\mathrm{si}(ab)+\frac{\pi}{2}\right) + e^{-ab}\,\mathrm{Ei}(ab) - e^{ab}\,\mathrm{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI (161)(12)

3. $$\int_0^\infty \frac{\cos(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b^3}\left[e^{-ab}+\sin(ab)\right]$$
$[a>0, \quad b>0]$ (cf. **3.723** 2 and **3.723** 9) BI (161)(16)

4. $$\int_0^\infty \frac{x\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2b^2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\sin\frac{ab}{\sqrt{2}}$$
$[a>0, \quad b>0]$ BI (160)(23)a

5. $$\int_0^\infty \frac{x\sin(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b^2}\left[e^{-ab}-\cos(ab)\right]$$
$[a>0, \quad b>0]$ BI (161)(13)

6.[11] $$\int_0^\infty \frac{x\cos(ax)}{b^4-x^4}\,dx = \frac{1}{4b^2}\left[2\cos(ab)\,\mathrm{ci}(ab) + 2\sin(ab)\left(\mathrm{si}(ab)+\frac{\pi}{2}\right) - e^{-ab}\overline{\mathrm{Ei}}(ab) - e^{ab}\,\mathrm{Ei}(-ab)\right]$$
$[a>0, \quad b>0]$ (cf. **3.723** 5 and **3.723** 11) BI (161)(17)

7. $$\int_0^\infty \frac{x^2\cos(ax)}{b^4+x^4}\,dx = \frac{\pi\sqrt{2}}{4b}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}}-\sin\frac{ab}{\sqrt{2}}\right)$$
$[a>0, \quad b>0]$ BI (160)(26)a

8.[11] $$\int_0^\infty \frac{x^2\sin(ax)\,dx}{b^4-x^4} = \frac{1}{4b}\Big[2\sin(ab)\,\mathrm{ci}(ab) - 2\cos(ab)\left(\mathrm{si}(ab)+\frac{\pi}{2}\right) - e^{-ab}\overline{\mathrm{Ei}}(ab) + e^{ab}\,\mathrm{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI (161)(14)

9. $$\int_0^\infty \frac{x^2\cos(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b}\left(\sin(ab)-e^{-ab}\right)$$
$[a>0, \quad b>0]$ BI (161)(18)

10. $$\int_0^\infty \frac{x^3\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}$$
$[a>0, \quad b>0]$ BI (160)(24)

11. $$\int_0^\infty \frac{x^3\sin(ax)}{b^4-x^4}\,dx = \frac{-\pi}{4}\left[e^{-ab}-\cos(ab)\right]$$
$[a>0, \quad b>0]$ BI (161)(15)

12.[7] $$\int_0^\infty \frac{x^3\cos(ax)\,dx}{b^4-x^4} = \frac{1}{4}\Big[2\cos(ab)\,\mathrm{ci}(ab) + 2\sin(ab)\left(\mathrm{si}(ab)+\frac{\pi}{2}\right) + e^{-ab}\overline{\mathrm{Ei}}(ab) + e^{ab}\,\mathrm{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI(161)(19)

13. $$\int_0^\infty \frac{x^3 \sin ax}{\left(x^2+b^2\right)^3}\,dx = \frac{\pi e^{-ab}}{16b}\left(3a - ba^2\right)$$

14. $$\int_0^\infty \frac{x^3 \sin ax}{\left(x^2+b^2\right)^4}\,dx = \frac{\pi e^{-ab} a}{96b^3}\left(3 + 3ab - a^2b^2\right)$$

3.728

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta e^{-a\gamma} - \gamma e^{-a\beta}\right)}{2\beta\gamma\left(\beta^2-\gamma^2\right)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$$
BI (175)(1)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(e^{-a\beta} - e^{-a\gamma}\right)}{2\left(\gamma^2-\beta^2\right)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$$
BI (174)(1)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta e^{-a\beta} - \gamma e^{-a\gamma}\right)}{2\left(\beta^2-\gamma^2\right)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$$
BI (175)(2)

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta^2 e^{-a\beta} - \gamma^2 e^{-a\gamma}\right)}{2\left(\beta^2-\gamma^2\right)} \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$$
BI (174)(2)

5. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(b\sin(ac) - c\sin(ab)\right)}{2bc\left(b^2-c^2\right)} \qquad [a>0, \quad b>0, \quad c>0]$$ BI (175)(3)

6. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(\cos(ab) - \cos(ac)\right)}{2\left(b^2-c^2\right)} \qquad [a>0]$$ BI (174)(3)

7. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(c\sin(ac) - b\sin(ab)\right)}{2\left(b^2-c^2\right)} \qquad [a>0, \quad b>0, \quad c>0]$$ BI (175)(4)

8. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(b^2\cos(ab) - c^2\cos(ac)\right)}{2\left(b^2-c^2\right)} \qquad [a>0, \quad b>0, \quad c>0]$$ BI (174)(4)

9. $$\int_0^\infty \frac{x\sin ax}{\left(b^2-x^2\right)\left(c^2+x^2\right)}\,dx = \frac{\pi}{2}\,\frac{e^{-ac} - \cos ba}{a^2+c^2} \qquad [a>0, \quad c>0, \quad b \text{ real}]$$

3.729

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab} \qquad [a>0, \quad b>0]$$ BI (170)(7)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4b}ae^{-ab} \qquad [a>0, \quad b>0]$$ BI (170)(3)

3. $$\int_0^\infty \cos(px)\frac{1-x^2}{\left(1+x^2\right)^2}\,dx = \frac{\pi p}{2}e^{-p}$$ BI (43)(10)a

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4}(2-ab)e^{-ab} \qquad [a>0, \quad b>0]$$ BI (170)(4)

3.731 **Notation**: $2A^2 = \sqrt{b^4+c^2}+b^2, \quad 2B^2 = \sqrt{b^4+c^2}-b^2,$

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2c}\frac{e^{-aA}\left(B\cos(aB)+A\sin(aB)\right)}{\sqrt{b^4+c^2}}$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(3)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2c}e^{-aA}\sin(aB)$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(1)

3. $$\int_0^\infty \frac{\left(x^2+b^2\right)\cos(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2}\frac{e^{-aA}\left(A\cos(aB)-B\sin(aB)\right)}{\sqrt{b^4+c^2}}$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(4)

4. $$\int_0^\infty \frac{x\left(x^2+b^2\right)\sin(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2}e^{-aA}\cos(aB)$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(2)

3.732

1. $$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2}-\frac{1}{\beta^2+(\gamma+x)^2}\right]\sin(ax)\,dx = \frac{\pi}{\beta}e^{-a\beta}\sin(a\gamma)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{ is not real}]$ ET I 65(16)

2. $$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2}+\frac{1}{\beta^2+(\gamma+x)^2}\right]\cos(ax)\,dx = \frac{\pi}{\beta}e^{-a\beta}\cos(a\gamma)$$
$[a>0, \quad |\operatorname{Im}\gamma|<\operatorname{Re}\beta]$ ET I 8(13)

3. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2}-\frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right]\sin(ax)\,dx = \pi e^{-a\beta}\cos(a\gamma)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{ is not real}]$ LI (175)(17)

4. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2}+\frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right]\cos(ax)\,dx = \pi e^{-a\beta}\sin(a\gamma)$$
$[a>0, \quad |\operatorname{Im}a|<\operatorname{Re}\beta]$ LI (176)(21)

3.733

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b^3}\exp\left(-ab\cos t\right)\frac{\sin\left(t+ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$ BI (176)(7)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b^2}\exp\left(-ab\cos t\right)\frac{\sin\left(ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$ BI(176)(5), ET I 66(23)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b}\exp\left(-ab\cos t\right)\frac{\sin\left(t-ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$ BI (176)(8)

4. $$\int_0^\infty \frac{x^3 \sin(ax)\,dx}{x^4 + 2b^2x^2\cos 2t + b^4} = \frac{\pi}{2}\exp(-ab\cos t)\,\frac{\sin(2t - ab\sin t)}{\sin 2t}$$
$$\left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}\right]$$ BI (176)(6)

5. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(x^4 + 2b^2x^2\cos 2t + b^4)} = \frac{\pi}{2b^4}\left[1 - \exp(-ab\cos t)\,\frac{\sin(2t + ab\sin t)}{\sin 2t}\right]$$
$$\left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}\right]$$ BI (176)(22)

3.734

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(b^4 + x^4)} = \frac{\pi}{2b^4}\left[1 - \exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}\right] \qquad [a > 0, \quad b > 0]$$ BI (172)(7)

2. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(b^4 - x^4)} = \frac{\pi}{4b^4}\left[2 - e^{-ab} - \cos(ab)\right] \qquad [a > 0, \quad b > 0]$$ BI (172)(10)

3.735 $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(b^2 + x^2)^2} = \frac{\pi}{2b^4}\left[1 - \frac{1}{2}e^{-ab}(2 + ab)\right] \qquad [a > 0, \quad b > 0]$$ WH, BI (172)(22)

3.736

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8b^5}\left[\sin(ab) + (2 + ab)e^{-ab}\right]$$
$$[a > 0, \quad b > 0]$$ BI (176)(5)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8b^4}\left[(1 + ab)e^{-ab} - \cos(ab)\right]$$
$$[a > 0, \quad b > 0]$$ BI (174)(5)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8b^3}\left[\sin(ab) - abe^{-ab}\right] \qquad [a > 0, \quad b > 0]$$ BI (175)(6)

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8b^2}\left[(1 - ab)e^{-ab} - \cos(ab)\right]$$
$$[a > 0, \quad b > 0]$$ BI (174)(6)

5. $$\int_0^\infty \frac{x^4\cos(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8b}\left[\sin(ab) + (ab - 2)e^{-ab}\right]$$
$$[a > 0, \quad b > 0]$$ BI (175)(7)

6. $$\int_0^\infty \frac{x^5\sin(ax)\,dx}{(b^2 + x^2)\,(b^4 - x^4)} = \frac{\pi}{8}\left[(ab - 3)e^{-ab} - \cos(ab)\right]$$
$$[a > 0, \quad b > 0]$$ BI (174)(7)

3.737

1.[8] $$\int_0^\infty \frac{\cos(ax)\,dx}{(b^2+x^2)^n} = \frac{\pi e^{-ab}}{(2b)^{2n-1}(n-1)!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2ab)^k}{k!(n-k-1)!}$$
$$= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{e^{-ab\sqrt{p}}}{\sqrt{p}}\right)\right]_{p=1}$$
$$= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{e^{-abp}}{(1+p)^n}\right)\right]_{p=1}$$
$[a>0, \quad b>0]$ GW(333)(67b), WA 209, WA 192

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(x^2+\beta^2)^{n+1}} = \frac{\pi a e^{-a\beta}}{2^{2n}n!\beta^{2n-1}}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!}$$
$$= \frac{\pi}{2}e^{-a\beta} \qquad [n=0, \quad \beta\geq 0]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ GW (333)(66c)

3. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(\beta^2+x^2)^{n+1}} = \frac{\pi}{2\beta^{2n+2}}\left[1-\frac{e^{-a\beta}}{2^n n!}F_n(a\beta)\right]$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad F_0(z)=1, \quad F_1(z)=z+2,\ldots,F_n(z)=(z+2n)F_{n-1}(z)-zF'_{n-1}(z)\right]$
GW (333)(66e)

4. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(b^2+x^2)^3} = \frac{\pi a}{16b^3}(1+ab)e^{-ab} \qquad [a>0, \quad b>0]$$ BI(170)(5), ET I 67(35)a

5. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(b^2+x^2)^4} = \frac{\pi a}{96b^5}\left(3+3ab+a^2b^2\right)e^{-ab} \qquad [a>0, \quad b>0]$$ BI(170)(6), ET I 67(35)a

6. $$\int_0^\infty \frac{x^3\sin ax}{(x^2+\beta^2)^{n+1}}\,dx = \frac{\pi e^{-a\beta}}{2^{2n}n!\beta^{2n-2}}\left[2^{n-1}(2n-3)!!(2-\beta a)\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{(2n-k-2)!2^k(\beta a)^{k-1}}{k!(n-k-1)!}\left[k(k+1)-2(k+1)\beta a+\beta^2a^2\right]\right]$$

3.738

1. $$\int_0^\infty \frac{x^{m-1}\sin(ax)}{x^{2n}+\beta^{2n}}\,dx = -\frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n}\exp\left[-a\beta\sin\frac{(2k-1)\pi}{2n}\right]$$
$$\times\cos\left\{\frac{(2k-1)m\pi}{2n}+a\beta\cos\frac{(2k-1)\pi}{2n}\right\}$$
$[m \text{ is even}], \qquad \left[a>0, \quad |\arg\beta|<\frac{\pi}{2n}, \quad 0<m\leq 2n\right]$ ET I 67(38)

2. $$\int_0^\infty \frac{x^{m-1}\cos(ax)}{x^{2n}+\beta^{2n}}\,dx = \frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n}\exp\left[-a\beta\sin\frac{(2k-1)\pi}{2n}\right]$$
$$\times\sin\left\{\frac{(2k-1)m\pi}{2n}+a\beta\cos\frac{(2k-1)\pi}{2n}\right\}$$
$[m \text{ is odd}], \qquad \left[a>0, \quad |\arg\beta|<\frac{\pi}{2n}, \quad 0<m<2n+1\right]$ BI(160)(29)a, ET I 10(29)

3.739

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\,(x^2+2^2)\,(x^2+4^2)\ldots(x^2+4n^2)} = \frac{\pi(-1)^n}{(2n)!2^{2n+1}}\left[2\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}e^{2(k-n)a}+(-1)^n\binom{2n}{n}\right]$$
$[a>0,\quad n\geq 0]$ LI(174)(8)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{(x^2+1^2)\,(x^2+3^2)\ldots[x^2+(2n+1)^2]} = \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}e^{(2k-2n-1)a} \qquad [a\geq 0,\quad n\geq 0]$$
$$= \frac{\pi 2^{-2n-1}}{(2n+1)\,(n!)^2} \qquad [a=0,\quad n\geq 0]$$
BI(175)(8)

3. $$\int_0^\infty \frac{x\sin(ax)\,dx}{(x^2+1^2)\,(x^2+3^2)\ldots[x^2+(2n+1)^2]} = \frac{\pi(-1)^n}{(2n+1)!2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}(2n-2k+1)e^{(2k-2n-1)a}$$
$[a>0,\quad n\geq 0]$ LI (174)(9)

4. $$\int_0^\infty \frac{\cos ax\,dx}{(x^2+2^2)\,(x^2+4^2)\ldots(x^2+4n^2)} = \frac{\pi 2^{1-2n}}{(2n)!}\sum_{k=1}^{n}(-1)^k k\binom{2n}{n-k}e^{-2ak}$$
$[n\geq 1,\quad a\geq 0]$

3.741

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x}\,dx = \frac{1}{4}\ln\left(\frac{a+b}{a-b}\right)^2 \qquad [a>0,\quad b>0,\quad a\neq b]$$
FI II 647

2. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{x}\,dx = \frac{\pi}{2} \qquad [a>b\geq 0]$$
$$= \frac{\pi}{4} \qquad [a=b>0]$$
$$= 0 \qquad [b>a\geq 0]$$
FI II 645

3. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2}\,dx = \frac{a\pi}{2} \qquad [0<a\leq b]$$
$$= \frac{b\pi}{2} \qquad [0<b\leq a]$$
BI (157)(1)

3.742

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}\left(e^{-|a-b|\beta} - e^{-(a+b)\beta}\right) \qquad [a>0,\quad b>0,\quad \operatorname{Re}\beta>0]$$
$$= \frac{\pi}{2\beta}e^{-a\beta}\sinh b\beta \qquad [\beta>0,\quad a\ge b\ge 0]$$
$$= \frac{\pi}{2\beta}e^{-b\beta}\sinh a\beta \qquad [\beta>0,\quad b\ge a\ge 0]$$

BI(162)(1)a, GW(333)(71a)

2. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{\beta^2+x^2}\,dx = \frac{1}{4\beta}e^{-a\beta}\left\{e^{b\beta}\operatorname{Ei}[\beta(a-b)] + e^{-b\beta}\operatorname{Ei}[\beta(a+b)]\right\}$$
$$-\frac{1}{4\beta}e^{a\beta}\left\{e^{b\beta}\operatorname{Ei}[-\beta(a+b)] + e^{-b\beta}\operatorname{Ei}[\beta(b-a)]\right\}$$

BI (162)(3)

3. $$\int_0^\infty \frac{\cos(ax)\cos(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}\left[e^{-|a-b|\beta} + e^{-(a+b)\beta}\right] \qquad [a>0,\quad b>0,\quad \operatorname{Re}\beta>0]$$
$$= \frac{\pi}{2\beta}e^{-a\beta}\cosh b\beta \qquad [\beta>0,\quad a\ge b\ge 0]$$
$$= \frac{\pi}{2\beta}e^{-b\beta}\cosh a\beta \qquad [\beta>0,\quad b\ge a\ge 0]$$

BI(163)(1)a, GW(333)(71c)

4. $$\int_0^\infty \frac{x\cos(ax)\cos(bx)}{\beta^2+x^2}\,dx = -\frac{1}{4}e^{a\beta}\left\{e^{b\beta}\operatorname{Ei}[-\beta(a+b)] + e^{-b\beta}\operatorname{Ei}[\beta(b-a)]\right\}$$
$$-\frac{1}{4}e^{-a\beta}\left\{e^{b\beta}\operatorname{Ei}[\beta(a-b)] + e^{-b\beta}\operatorname{Ei}[\beta(a+b)]\right\} \qquad [a\ne b]$$
$$= \infty \qquad [a=b]$$

BI (163)(2)

5. $$\int_0^\infty \frac{x\sin(ax)\cos(bx)}{x^2+\beta^2}\,dx = \frac{\pi}{2}e^{-a\beta}\cosh(b\beta) \qquad [0<b<a]$$
$$= \frac{\pi}{4}e^{-2a\beta} \qquad [0<b=a]$$
$$= -\frac{\pi}{2}e^{-b\beta}\sinh(a\beta) \qquad [0<a<b]$$

BI (162)(4)

6. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{p^2-x^2}\,dx = -\frac{\pi}{2p}\cos(ap)\sin(bp) \qquad [a>b>0]$$
$$= -\frac{\pi}{4p}\sin(2ap) \qquad [a=b>0]$$
$$= -\frac{\pi}{2p}\sin(ap)\cos(bp) \qquad [b>a>0]$$

BI (166)(1)

7. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{p^2-x^2}x\,dx = -\frac{\pi}{2}\cos(ap)\cos(bp) \qquad [a>b>0]$$
$$= -\frac{\pi}{4}\cos(2ap) \qquad [a=b>0]$$
$$= \frac{\pi}{2}\sin(ap)\sin(bp) \qquad [b>a>0]$$

BI (166)(2)

8. $$\int_0^\infty \frac{\cos(ax)\cos(bx)}{p^2-x^2}\,dx = \frac{\pi}{2p}\sin(ap)\cos(bp) \qquad [a>b>0]$$
$$= \frac{\pi}{4p}\sin(2ap) \qquad [a=b>0]$$
$$= \frac{\pi}{2p}\cos(ap)\sin(bp) \qquad [b>a>0]$$
BI (166)(3)

3.743

1. $$\int_0^\infty \frac{\sin(ax)}{\sin(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\sinh(a\beta)}{\sinh(b\beta)} \qquad [0<a<b,\quad \operatorname{Re}\beta>0]$$ ET I 80(21)

2. $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = -\frac{\pi}{2}\cdot\frac{\sinh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b,\quad \operatorname{Re}\beta>0]$$ ET I 81(30)

3. $$\int_0^\infty \frac{\cos(ax)}{\sin(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = \frac{\pi}{2}\cdot\frac{\cosh(a\beta)}{\sinh(b\beta)} \qquad [0<a<b,\quad \operatorname{Re}\beta>0]$$ ET I 23(37)

4. $$\int_0^\infty \frac{\cos(ax)}{\cos(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\cosh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b,\quad \operatorname{Re}\beta>0]$$ ET I 23(36)

5.[6] $$\mathrm{PV}\int_0^\infty \frac{\sin(ax)}{\sin x}\cdot\frac{dx}{b^2-x^2} = 0 \qquad \text{if } 0\le a\le 1$$
$$= \frac{\pi}{b}\sin(a-1)b \qquad \text{if } 1\le a\le 2$$
$[b \text{ real}, b/\pi\notin\mathbb{Z}]$

3.744[3] $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{dx}{x\,(x^2+\beta^2)} = \frac{\pi}{2\beta^2}\cdot\frac{\sinh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b,\quad \operatorname{Re}\beta>0]$$ ET I 82(32)

3.745[3] $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{dx}{x\,(c^2-x^2)} = 0 \qquad [0<a<b,\quad c>0]$$ ET I 82(31)

3.746

1. $$\int_0^\infty \frac{dx}{x^{n+1}}\prod_{k=0}^n \sin(a_k x) = \frac{\pi}{2}\prod_{k=1}^n a_k \qquad \left[a_0 > \sum_{k=1}^n a_k,\quad a_k>0\right]$$ FI II 646

2. $$\int_0^\infty \frac{\sin(ax)}{x^{n+1}}\,dx\prod_{k=1}^n \sin(a_k x)\prod_{j=1}^m \cos(b_j x) = \frac{\pi}{2}\prod_{k=1}^n a_k \qquad \left[a > \sum_{k=1}^n |a_k| + \sum_{j=1}^m |b_j|\right]$$ WH

3.747

1.[7] $$\int_0^{\pi/2} \frac{x^m}{\sin x}\,dx = \left(\frac{\pi}{2}\right)^m\left[\frac{1}{m}+\sum_{k=1}^\infty \frac{2^{2k-1}-1}{4^{2k-1}(m+2k)}\,\zeta(2k)\right] = 2\pi\boldsymbol{G} - \frac{7}{2}\,\zeta(3)$$
$[m=2]$ LI (206)(2)

2. $$\int_0^{\pi/2} \frac{x\,dx}{\sin x} = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\,dx}{\cos x} = 2\boldsymbol{G}$$ BI(204)(18), BI(206)(1), GW(333)(32)

3. $$\int_0^\infty \frac{x\,dx}{(x^2+b^2)\sin(ax)} = \frac{\pi}{2\sinh(ab)} \qquad [b>0]$$ GW (333)(79c)

4. $$\int_0^\pi x\tan x\,dx = -\pi\ln 2$$ BI (218)(4)

5. $\displaystyle\int_0^{\pi/2} x\tan x\,dx = \infty$ BI (205)(2)

6. $\displaystyle\int_0^{\pi/4} x\tan x\,dx = -\frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G} = 0.1857845358\ldots$ BI (204)(1)

7. $\displaystyle\int_0^{\pi/2} x\cot x\,dx = \frac{\pi}{2}\ln 2$ FI II 623

8. $\displaystyle\int_0^{\pi/4} x\cot x\,dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G} = 0.7301810584\ldots$ BI (204)(2)

9. $\displaystyle\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right)\tan x\,dx = \frac{1}{2}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\tan x\,dx = \frac{\pi}{2}\ln 2$ GW(333)(33b), BI(218)(12)

10. $\displaystyle\int_0^{\infty} \tan ax\frac{dx}{x} = \frac{\pi}{2}$ $[a > 0]$ LO V 279(5)

11. $\displaystyle\int_0^{\pi/2} \frac{x\cot x}{\cos 2x}\,dx = \frac{\pi}{4}\ln 2$ BI (206)(12)

3.748

1. $\displaystyle\int_0^{\pi/4} x^m\tan x\,dx = \frac{1}{2}\left(\frac{\pi}{4}\right)^m \sum_{k=1}^{\infty}\frac{\left(4^k - 1\right)\zeta(2k)}{4^{2k-1}(m+2k)}$ LI (204)(5)

2. $\displaystyle\int_0^{\pi/2} x^p\cot x\,dx = \left(\frac{\pi}{2}\right)^p\left(\frac{1}{p} - 2\sum_{k=1}^{\infty}\frac{1}{4^k(p+2k)}\,\zeta(2k)\right)$ LI (205)(7)

3. $\displaystyle\int_0^{\pi/4} x^m\cot x\,dx = \frac{1}{2}\left(\frac{\pi}{4}\right)^m\left(\frac{2}{m} - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^{2k-1}(m+2k)}\right)$ LI (204)(6)

3.749

1. $\displaystyle\int_0^{\infty} \frac{x\tan(ax)\,dx}{x^2+b^2} = \frac{\pi}{e^{2ab}+1}$ $[a > 0, \quad b > 0]$ GW (333)(79a)

2. $\displaystyle\int_0^{\infty} \frac{x\cot(ax)\,dx}{x^2+b^2} = \frac{\pi}{e^{2ab}-1}$ $[a > 0, \quad b > 0]$ GW (333)(79b)

3. $\displaystyle\int_0^{\infty} \frac{x\tan(ax)\,dx}{b^2-x^2} = \int_0^{\infty}\frac{x\cot(ax)\,dx}{b^2-x^2} = \int_0^{\infty}\frac{x\operatorname{cosec}(ax)\,dx}{b^2-x^2} = \infty$ BI (161)(7, 8, 9)

3.75 Combinations of trigonometric and algebraic functions

3.751

1. $\displaystyle\int_0^{\infty} \frac{\sin(ax)\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}}\left[\cos(a\beta) - \sin(a\beta) + 2\,C\left(\sqrt{a\beta}\right)\sin(a\beta) - 2\,S\left(\sqrt{a\beta}\right)\cos(a\beta)\right]$

$[a > 0, \quad |\arg\beta| < \pi]$ ET I 65(12)a

2.[9] $\displaystyle\int_0^{\infty} \frac{\cos(ax)\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}}\left[\cos(a\beta) + \sin(a\beta) - 2\,C\left(\sqrt{a\beta}\right)\cos(a\beta) - 2\,S\left(\sqrt{a\beta}\right)\sin(a\beta)\right]$

$[a > 0, \quad |\arg\beta| < \pi]$ ET I 8(9)a

3. $\displaystyle\int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}\,[\sin(au)+\cos(au)]$ $[a>0,\quad u>0]$ ET I 65(13)

4. $\displaystyle\int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}\,[\cos(au)-\sin(au)]$ $[a>0,\quad u>0]$ ET I 8(10)

3.752

1.[8] $\displaystyle\int_0^1 \sin(ax)\sqrt{1-x^2}\,dx = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a}\,\mathbf{H}_1(a)$

$[a>0]$ BI (149)(6)

2. $\displaystyle\int_0^1 \cos(ax)\sqrt{1-x^2}\,dx = \frac{\pi}{2a}\,J_1(a)$ KU 65(6)a

3.753

1.[8] $\displaystyle\int_0^1 \frac{\sin(ax)\,dx}{\sqrt{1-x^2}} = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} = \frac{\pi}{2}\,\mathbf{H}_0(a)$ $[a>0]$ BI (149)(9)

2. $\displaystyle\int_0^1 \frac{\cos(ax)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\,J_0(a)$ WA 30(7)a

3. $\displaystyle\int_1^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2-1}} = \frac{\pi}{2}\,J_0(a)$ $[a>0]$ WA 200(14)

4. $\displaystyle\int_1^\infty \frac{\cos(ax)}{\sqrt{x^2-1}}\,dx = -\frac{\pi}{2}\,Y_0(a)$ WA 200(15)

5. $\displaystyle\int_0^1 \frac{x\sin(ax)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\,J_1(a)$ $[a>0]$ WA 30(6)

3.754

1. $\displaystyle\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{\beta^2+x^2}} = \frac{\pi}{2}\,[I_0(a\beta)-\mathbf{L}_0(a\beta)]$ $[a>0,\quad \operatorname{Re}\beta>0]$ ET I 66(26)

2. $\displaystyle\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta)$ $[a>0,\quad \operatorname{Re}\beta>0]$

WA 191(1), GW(333)(78a)

3. $\displaystyle\int_0^\infty \frac{x\sin(ax)}{\sqrt{(\beta^2+x^2)^3}}\,dx = a\,K_0(a\beta)$ $[a>0,\quad \operatorname{Re}\beta>0]$ ET I 66(27)

3.755

1. $\displaystyle\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}-\beta}\,\sin(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta}$ $[a>0]$ ET I 66(31)

2. $\displaystyle\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}+\beta}\,\cos(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta}$ $[a>0,\quad \operatorname{Re}\beta>0]$ ET I 10(25)

3.756

1. $$\int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}} \prod_{k=2}^{n} \sin(a_k x)\, dx = 0 \qquad \left[a_k > 0, \quad a > \sum_{k=2}^{n} a_k\right]$$ ET I 80(22)

2. $$\int_0^\infty x^{\frac{n}{2}-1} \cos(ax) \prod_{k=1}^{n} \cos(a_k x)\, dx = 0 \qquad \left[a_k > 0, \quad a > \sum_{k=1}^{n} a_k\right]$$ ET I 22(26)

3.757

1.[11] $$\int_0^\infty \frac{\sin(ax)}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2a}} \qquad [a > 0]$$ BI (177)(1)

2.[11] $$\int_0^\infty \frac{\cos(ax)}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2a}} \qquad [a > 0]$$ BI (177)(2)

3.76–3.77 Combinations of trigonometric functions and powers

3.761

1. $$\int_0^1 x^{\mu-1} \sin(ax)\, dx = \frac{-i}{2\mu} \left[{}_1F_1(\mu;\mu+1;ia) - {}_1F_1(\mu;\mu+1;-ia)\right]$$
$$[a > 0, \quad \operatorname{Re}\mu > -1, \quad \mu \neq 0]$$
ET I 68(2)a

2.[8] $$\int_u^\infty x^{\mu-1} \sin x\, dx = \frac{i}{2}\left[e^{-\frac{\pi}{2}i\mu}\,\Gamma(\mu, iu) - e^{\frac{\pi}{2}i\mu}\,\Gamma(\mu, -iu)\right]$$
$$[\operatorname{Re}\mu < 1]$$
EH II 149(2)

3. $$\int_1^\infty \frac{\sin(ax)}{x^{2n}}\, dx = \frac{a^{2n-1}}{(2n-1)!}\left[\sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \sin\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^n \operatorname{ci}(a)\right]$$
$$[a > 0]$$
LI (203)(15)

4. $$\int_0^\infty x^{\mu-1} \sin(ax)\, dx = \frac{\Gamma(\mu)}{a^\mu} \sin\frac{\mu\pi}{2} = \frac{\pi \sec\frac{\mu\pi}{2}}{2a^\mu\,\Gamma(1-\mu)} \qquad [a > 0; \quad 0 < |\operatorname{Re}\mu| < 1]$$
FI II 809a, BI(150)(1)

5.[10] $$\int_0^\pi x^m \sin(nx)\, dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{\lfloor m/2\rfloor} (-1)^k \frac{m!}{(m-2k)!} (n\pi)^{m-2k} - (-1)^{\lfloor m/2\rfloor} \frac{m!\left\lfloor m - 2\left\lfloor\frac{m}{2}\right\rfloor - 1\right\rfloor}{n^{m+1}}$$
GW(333)(6)

6.[8] $$\int_0^1 x^{\mu-1} \cos(ax)\, dx = \frac{1}{2\mu}\left[{}_1F_1(\mu;\mu+1;ia) + {}_1F_1(\mu;\mu+1;-ia)\right]$$
$$[a > 0, \quad \operatorname{Re}\mu > 0]$$
ET I 11(2)

7. $$\int_u^\infty x^{\mu-1} \cos x\, dx = \frac{1}{2}\left[e^{-\frac{\pi}{2}i\mu}\,\Gamma(\mu, iu)s + e^{\frac{\pi}{2}i\mu}\,\Gamma(\mu, -iu)\right]$$
$$[\operatorname{Re}\mu < 1]$$
EH II 149(1)

8. $$\int_1^\infty \frac{\cos(ax)}{x^{2n+1}}\,dx = \frac{a^{2n}}{(2n)!}\left[\sum_{k=1}^{2n}\frac{(2n-k)!}{a^{2n-k+1}}\cos\left(a+(k-1)\frac{\pi}{2}\right)+(-1)^{n+1}\operatorname{ci}(a)\right]$$
$[a>0]$ LI (203)(16)

9.[8] $$\int_0^\infty x^{\mu-1}\cos(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\frac{\mu\pi}{2} = \frac{\pi\operatorname{cosec}\frac{\mu\pi}{2}}{2a^\mu\,\Gamma(1-\mu)} \qquad [a>0,\quad 0<\operatorname{Re}\mu<1]$$
FI II 809a, BI(150)(2)

10. $$\int_0^\pi x^m\cos(nx)\,dx = \frac{(-1)^n}{n^{m+1}}\sum_{k=0}^{\lfloor(m-1)/2\rfloor}(-1)^k\frac{m!}{(m-2k-1)!}(n\pi)^{m-2k-1}$$
$$+(-1)^{\lfloor(m+1)/2\rfloor}\frac{2[(m+1)/2]-m}{n^{m+1}}\cdot m!$$
GW (333)(7)

11. $$\int_0^{\pi/2} x^m\cos x\,dx = \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k\frac{m!}{(m-2k)!}\left(\frac{\pi}{2}\right)^{m-2k}+(-1)^{\lfloor m/2\rfloor}\left(2\left\lfloor\frac{m}{2}\right\rfloor-m\right)m!$$
GW (333)(9c)

12. $$\int_0^{2n\pi} x^m\cos kx\,dx = -\sum_{j=0}^{m-1}\frac{j!}{k^{j+1}}\binom{m}{j}(2n\pi)^{m-j}\cos\frac{j+1}{2}\pi$$ BI (226)(2)

3.762

1. $$\int_0^\infty x^{\mu-1}\sin(ax)\sin(bx)\,dx = \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left[|b-a|^{-\mu}-(b+a)^{-\mu}\right]$$
$[a>0,\quad b>0,\quad a\neq b,\quad -2<\operatorname{Re}\mu<1]$
(for $\mu=0$, see **3.741** 1, for $\mu=-1$, see **3.741** 3)
BI(149)(7), ET I 321(40)

2. $$\int_0^\infty x^{\mu-1}\sin(ax)\cos(bx)\,dx = \frac{1}{2}\sin\frac{\mu\pi}{2}\,\Gamma(\mu)\left[(a+b)^{-\mu}+|a-b|^{-\mu}\operatorname{sign}(a-b)\right]$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$ (for $\mu=0$ see **3.741** 2) BI(159)(8)a, ET I 321(41)

3. $$\int_0^\infty x^{\mu-1}\cos(ax)\cos(bx)\,dx = \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left[(a+b)^{-\mu}+|a-b|^{-\mu}\right]$$
$[a>0,\quad b>0,\quad 0<\operatorname{Re}\mu<1]$
ET I 20(17)

3.763

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^\nu}\,dx = \frac{1}{4}\cos\frac{\nu\pi}{2}\,\Gamma(1-\nu)\Big\{(c+a-b)^{\nu-1}-(c+a+b)^{\nu-1}$$
$$-|c-a+b|^{\nu-1}\operatorname{sign}(a-b-c)+|c-a-b|^{\nu-1}\operatorname{sign}(a+b-c)\Big\}$$
$[c>0,\quad 0<\operatorname{Re}\nu<4,\quad \nu\neq 1,2,3,\quad a\geq b>0]$ GW(333)(26a)a, ET I 79(13)

2. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x}\,dx = 0 \qquad [c < a-b \text{ and } c > a+b]$$
$$= \frac{\pi}{8} \qquad [c = a-b \text{ and } c = a+b]$$
$$= \frac{\pi}{4} \qquad [a-b < c < a+b]$$
$$[a \geq b > 0, \quad c > 0]$$ FI II 645

3. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^2}\,dx = \frac{1}{4}(c+a+b)\ln(c+a+b)$$
$$-\frac{1}{4}(c+a-b)\ln(c+a-b) - \frac{1}{4}|c-a-b|\ln|c-a-b|$$
$$\times \operatorname{sign}(a+b-c) + \frac{1}{4}|c-a+b|\ln|c-a+b|\operatorname{sign}(a-b-c)$$
$$[a \geq b > 0, \quad c > 0]$$ BI(157)(8)a, ET I 79(11)

4. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^3}\,dx = \frac{\pi bc}{2} \qquad [0 < c < a-b \text{ and } c > a+b]$$
$$= \frac{\pi bc}{2} - \frac{\pi(a-b-c)^2}{8} \qquad [a-b < c < a+b]$$
$$[a \geq b > 0, \quad c > 0]$$ BI(157)(20), ET I 79(12)

3.764

1. $$\int_0^\infty x^p \sin(ax+b)\,dx = \frac{1}{a^{p+1}}\,\Gamma(1+p)\cos\left(b + \frac{p\pi}{2}\right) \qquad [a > 0, \quad -1 < p < 0]$$ GW (333)(30a)

2. $$\int_0^\infty x^p \cos(ax+b)\,dx = -\frac{1}{a^{p+1}}\,\Gamma(1+p)\sin\left(b + \frac{\pi p}{2}\right)$$
$$[a > 0, \quad -1 < p < 0]$$ GW (333)(30b)

3.765

1.[10] $$\int_0^\infty \frac{\sin ax}{x^\nu(x+b)}\,dx$$
$$= a^{1+\nu} b\cos\frac{\pi\nu}{2}\,\Gamma(-1-\nu)\;{}_1F_2\left(1;\; 1+\frac{\nu}{2}, \frac{3}{2}+\frac{\nu}{2};\; -\frac{1}{4}a^2b^2\right)\operatorname{sign}(a)$$
$$-\frac{\pi\operatorname{cosec}(\pi\nu)\sin(ab)}{b^\nu} - a^\nu\,\Gamma(-\nu)\;{}_1F_2\left(1;\; 1+\frac{\nu}{2}, 1+\frac{\nu}{2};\; -\frac{1}{4}a^2b^2\right)\operatorname{sign}(a)\sin\frac{\pi\nu}{2}$$
$$[\operatorname{Im} a = 0, \quad -1 < \operatorname{Re} b < 2, \quad \arg b \neq \pi]$$ MC

2. $$\int_0^\infty \frac{\cos(ax)}{x^\nu(x+\beta)}\,dx = \frac{\Gamma(1-\nu)}{2\beta^\nu}\left[e^{ia\beta}\,\Gamma(\nu, ia\beta) + e^{-ia\beta}\,\Gamma(\nu, -ia\beta)\right]$$
$$[a > 0, \quad |\operatorname{Re}\nu| < 1, \quad |\arg\beta| < \pi]$$ ET II 221(52)

3.766

1.[10] $$\int_0^\infty \frac{x^{\mu-1}\sin ax}{1+x^2}\,dx$$
$$= -a^{2-\mu}\,\Gamma(\mu-2)\;{}_1F_2\left(1;\; \frac{3-\mu}{2}, \frac{4-\mu}{2};\; \frac{a^2}{4}\right)\operatorname{sign}(a)\sin\frac{\pi\mu}{2} + \frac{\pi}{2}\sec\frac{\pi\mu}{2}\sinh(a)$$
$$[\operatorname{Im} a = 0, \quad -1 < \operatorname{Re}\mu < 3]$$ MC

2. $$\int_0^\infty \frac{x^{\mu-1}\cos(ax)}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}\frac{\mu\pi}{2}\cosh a + \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left\{\exp\left[-a+i\pi(1-\mu)\right]\gamma(1-\mu,-a) - e^a\,\gamma(1-\mu,a)\right\}$$
$[a>0, \quad 0<\operatorname{Re}\mu<3]$ ET I 319(24)

3.[9] $$\int_0^\infty \frac{x^{2\mu+1}\sin(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2\mu}\sec(\mu\pi)\sinh(ab) + \frac{\sin(\mu\pi)}{2a^{2\mu}}\,\Gamma(2\mu)\left[\,{}_1F_1(1;1-2\mu;ab) + {}_1F_1(1;1-2\mu;-ab)\right]$$
$\left[a>0, \quad -\frac{3}{2}<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 220(39)

4.[9] $$\int_0^\infty \frac{x^{2\mu+1}\cos(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2\left(\mu+\frac{1}{2}\right)}\operatorname{cosec}\left[\left(\mu+\frac{1}{2}\right)\pi\right]\cosh(ab) + \frac{\cos\left[\left(\mu+\frac{1}{2}\right)\pi\right]}{2a^{2\left(\mu+\frac{1}{2}\right)}}\,\Gamma\left[2\left(\mu+\frac{1}{2}\right)\right]\left\{{}_1F_1\left(1;1-2\left(\mu+\frac{1}{2}\right);ab\right) + {}_1F_1\left(1;1-2\left(\mu+\frac{1}{2}\right);-ab\right)\right\}$$
$\left[a>0, \quad -1<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 221(56)

3.767

1. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = -\frac{\pi}{2}\gamma^{\beta-2}e^{-a\gamma}$$
$[a>0, \quad \operatorname{Re}\gamma>0, \quad 0<\operatorname{Re}\beta<2]$ BI (160)(20)

2. $$\int_0^\infty \frac{x^{\beta}\cos\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = \frac{\pi}{2}\gamma^{\beta-1}e^{-a\gamma}$$
$[a>0, \quad \operatorname{Re}\gamma>0, \quad |\operatorname{Re}\beta|<1]$ BI (160)(21)

3. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{x^2-b^2}\,dx = \frac{\pi}{2}b^{\beta-2}\cos\left(ab-\frac{\pi\beta}{2}\right)$$
$[a>0, \quad b>0, \quad 0<\operatorname{Re}\beta<2]$ BI (161)(11)

4. $$\int_0^\infty \frac{x^{\beta}\cos\left(ax-\frac{\beta\pi}{2}\right)}{x^2-b^2}\,dx = -\frac{\pi}{2}b^{\beta-1}\sin\left(ab-\frac{\pi\beta}{2}\right)$$
$[a>0, \quad b>0, \quad |\beta|<1]$ GW (333)(82)

3.768

1. $$\int_u^\infty (x-u)^{\mu-1}\sin(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\left(au+\frac{\mu\pi}{2}\right)$$
$[a>0, \quad 0<\operatorname{Re}\mu<1]$ ET II 203(19)

2. $$\int_u^\infty (x-u)^{\mu-1}\cos(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\left(au+\frac{\mu\pi}{2}\right)$$
$[a>0, \quad 0<\operatorname{Re}\mu<1]$ ET II 204(24)

3.[11] $$\int_0^1 (1-x)^\nu\sin(ax)\,dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}}\,C_\nu(a) = a^{-\nu-1/2}\,s_{\nu+1/2,1/2}(a)$$
$[a>0, \quad \operatorname{Re}\nu>-1]$ ET I 11(3)a

Here $C_\nu(a)$ is the Young's function given by:

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)}\left[\,{}_1F_1(1;\nu+1;ia) + {}_1F_1(1;\nu+1;-ia)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n a^{\nu+2n}}{\Gamma(\nu+2n+1)}.$$

4.[3] $$\int_0^1 (1-x)^\nu \cos(ax)\,dx = \frac{i}{2}a^{-\nu-1}\left\{\exp\left[\frac{i}{2}(\nu\pi-2a)\right]\gamma(\nu+1,-ia) - \exp\left[-\frac{i}{2}(\nu\pi-2a)\right]\gamma(\nu+1,ia)\right\}$$
$$= \Gamma(\nu+1)\sum_{n=0}^{\infty}\frac{\left(-a^2\right)^n}{\Gamma(\nu+2+2n)}$$
$[a>0, \quad \operatorname{Re}\nu > -1]$ ET I 11(3)a

5. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\sin(ax)\,dx = \frac{u^{\mu+\nu-1}}{2i}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\mu+\nu;iau) - {}_1F_1(\nu;\mu+\nu;-iau)\right]$$
$[a>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1, \quad \nu\neq 0]$ ET II 189(26)

6. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\cos(ax)\,dx = \frac{u^{\mu+\nu-1}}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\mu+\nu;iau) + {}_1F_1(\nu;\mu+\nu;-iau)\right]$$
$[a>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET II 189(32)

7. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\sin(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-1/2}\sin\frac{au}{2}\,\Gamma(\mu)\,J_{\mu-1/2}\left(\frac{au}{2}\right)$$
$[\operatorname{Re}\mu>0]$ ET II 189(25)

8. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-1/2}\Gamma(\mu)\left[\cos\frac{au}{2}\,J_{1/2-\mu}\left(\frac{au}{2}\right) - \sin\frac{au}{2}\,Y_{1/2-\mu}\left(\frac{au}{2}\right)\right]$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 203(20)

9. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\cos(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\cos\frac{au}{2}\,\Gamma(\mu)\,J_{\mu-\frac{1}{2}}\left(\frac{au}{2}\right)$$
$[\operatorname{Re}\mu>0]$ ET II 189(31)

10. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\Gamma(\mu)\left[\sin\frac{au}{2}\,J_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right) - \cos\frac{au}{2}\,Y_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right)\right]$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 204(25)

11.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\sin(ax)\,dx = -\frac{i}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\nu+\mu;ia) - {}_1F_1(\nu;\nu+\mu;-ia)\right]$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1, \quad \nu\neq 0]$ ET I 68 (5)a, ET I 317(5)

12.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\nu+\mu;ia) + {}_1F_1(\nu;\nu+\mu;-ia)\right]$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET I 11(5)

13. $$\int_0^1 x^\mu (1-x)^\mu \sin(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\,\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(a)\sin a$$
$[a>0, \quad \operatorname{Re}\mu > -1]$ ET I 68(4)

14. $$\int_0^1 x^\mu (1-x)^\mu \cos(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\,\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(a)\cos a$$
$[a>0, \quad \operatorname{Re}\mu > -1]$ ET I 11(4)

3.769

1. $$\int_0^\infty \left[(\beta+ix)^{-\nu}-(\beta-ix)^{-\nu}\right]\sin(ax)\,dx = -\frac{\pi i a^{\nu-1}e^{-a\beta}}{\Gamma(\nu)}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 70(15)

2. $$\int_0^\infty \left[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{\pi a^{\nu-1}e^{-a\beta}}{\Gamma(\nu)}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 13(19)

3. $$\int_0^\infty x\left[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}\right]\sin(ax)\,dx = -\frac{\pi a^{\nu-2}(\nu-1-a\beta)}{\Gamma(\nu)}e^{-a\beta}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 70(16)

4. $$\int_0^\infty x^{2n}\left[(\beta-ix)^{-\nu}-(\beta+ix)^{-\nu}\right]\sin(ax)\,dx = \frac{(-1)^n i}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}\,L_{2n}^{\nu-2n-1}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu]$ ET I 70(17)

5. $$\int_0^\infty x^{2n}\left[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{(-1)^n}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}\,L_{2n}^{\nu-2n-1}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu]$ ET I 13(20)

6. $$\int_0^\infty x^{2n+1}\left[(\beta+ix)^{-\nu}+(\beta-ix)^{-\nu}\right]\sin(ax)\,dx = \frac{(-1)^{n+1}}{\Gamma(\nu)}(2n+1)!\pi a^{\nu-2n-2}e^{-a\beta}\,L_{2n+1}^{\nu-2n-2}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad -1\le 2n+1<\operatorname{Re}\nu]$ ET I 70(18)

7. $$\int_0^\infty x^{2n+1}\left[(\beta+ix)^{-\nu}-(\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{(-1)^{n+1}}{\Gamma(\nu)}(2n+1)!\pi a^{\nu-2n-2}e^{-a\beta}\,L_{2n+1}^{\nu-2n-2}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu-1]$ ET I 13(21)

3.771

1. $$\int_0^\infty \left(\beta^2+x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\left[I_{-\nu}(a\beta)-\mathbf{L}_\nu(a\beta)\right]$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}, \quad \nu\ne -\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots\right]$ EH II 38a, ET I 68(6)

2. $$\int_0^\infty \left(\beta^2+x^2\right)^{\nu-\frac{1}{2}} \cos(ax)\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\,\Gamma\left(\nu+\frac{1}{2}\right) K_{-\nu}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}\right]$$
WA 191(1)a, GW(333)(78)a

3. $$\int_0^u x^{2\nu-1}\left(u^2-x^2\right)^{\mu-1}\sin(ax)\,dx$$
$$= \frac{a}{2}u^{2\mu+2\nu-1}\,\mathrm{B}\left(\mu,\nu+\frac{1}{2}\right)\ {}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\mu+\nu+\frac{1}{2};-\frac{a^2u^2}{4}\right)$$
$$\left[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 189(29)

4. $$\int_0^u x^{2\nu-1}\left(u^2-x^2\right)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}u^{2\mu+2\nu-2}\,\mathrm{B}(\mu,\nu)\ {}_1F_2\left(\nu;\frac{1}{2},\mu+\nu;-\frac{a^2u^2}{4}\right)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$
ET II 190(35)

5.[7] $$\int_0^\infty x\left(x^2+\beta^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{1}{\sqrt{\pi}}\beta\left(\frac{2\beta}{a}\right)^\nu \cos\nu\pi\,\Gamma\left(\nu+\frac{1}{2}\right)K_{\nu+1}(a\beta)$$
$$= \sqrt{\pi}\beta\left(\frac{2\beta}{a}\right)^\nu \frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)}K_{\nu+1}(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<0]$$
ET I 69(11)

6. $$\int_0^u \left(u^2-x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 69(7), WA 358(1)a

7. $$\int_u^\infty \left(x^2-u^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_{-\nu}(au)$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
EH II 81(12)a, ET I 69(8), WA 187(3)a

8. $$\int_0^u \left(u^2-x^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_\nu(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 11(8)

9. $$\int_u^\infty \left(x^2-u^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)Y_{-\nu}(au)$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
WA 187(4)a, EH II 82(13)a, ET I 11(9)

10. $$\int_0^u x\left(u^2-x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_{\nu+1}(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 69(9)

11. $$\int_u^\infty x\left(x^2-u^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)Y_{-\nu-1}(au)$$
$$\left[a>0,\quad u>0,\quad -\tfrac{1}{2}<\operatorname{Re}\nu<0\right]$$
ET I 69(10)

12.[7] $$\int_0^u x\left(u^2-x^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx=-\frac{u^{\nu+1}}{a^\nu}\,s_{(\nu-1)\nu+1}(au)$$
$$=\frac{1}{2}\left(\nu+\frac{1}{2}\right)^{-1}u^{2\nu+1}-\frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_{\nu+1}(au)$$
$$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET I 12(10)

13. $$\int_u^\infty x\left(x^2-u^2\right)^{\nu-1/2}\cos(ax)\,dx\,\frac{\sqrt{\pi}u}{2}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_{-\nu-1}(au)$$
$$\left[a>0,\quad u>0,\quad 0<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$ ET I 12(11)

3.772

1. $$\int_0^\infty\left(x^2+2\beta x\right)^{\nu-1/2}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(a\beta)\cos(a\beta)+Y_{-\nu}(a\beta)\sin(a\beta)\right]$$
$$\left[a>0,\quad |\arg\beta|<\pi,\quad \tfrac{1}{2}>\operatorname{Re}\nu>-\tfrac{3}{2}\right]$$ ET I 69(12)

2. $$\int_0^\infty\left(x^2+2\beta x\right)^{\nu-1/2}\cos(ax)\,dx$$
$$=-\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[Y_{-\nu}(a\beta)\cos(a\beta)-J_{-\nu}(a\beta)\sin(a\beta)\right]$$
$$\left[a>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET I 12(13)

3. $$\int_0^{2u}\left(2ux-x^2\right)^{\nu-1/2}\sin(ax)\,dx=\sqrt{\pi}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\sin(au)\,J_\nu(au)$$
$$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 69(13)a

4. $$\int_{2u}^\infty\left(x^2-2ux\right)^{\nu-1/2}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(au)\cos(au)-Y_{-\nu}(au)\sin(au)\right]$$
$$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET I 70(14)

5. $$\int_0^{2u}\left(2ux-x^2\right)^{\nu-1/2}\cos(ax)\,dx=\sqrt{\pi}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(au)\cos(au)$$
$$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 12(4)

6. $$\int_{2u}^\infty\left(x^2-2ux\right)^{\nu-1/2}\cos(ax)\,dx$$
$$=-\frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(au)\sin(au)+Y_{-\nu}(au)\cos(au)\right]$$
$$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET I 12(12)

3.773

1.[8] $$\int_0^\infty \frac{x^{2\nu}}{(x^2+\beta^2)^{\mu+1}}\sin(ax)\,dx$$
$$= \frac{1}{2}\beta^{2\nu-2\mu}a\,\mathrm{B}\left(1+\nu,\mu-\nu\right)\ {}_1F_2\left(\nu+1;\ \nu+1-\mu,\frac{3}{2};\ \frac{\beta^2a^2}{4}\right)$$
$$+\frac{\sqrt{\pi}a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}}\frac{\Gamma(\nu-\mu)}{\Gamma\left(\mu-\nu+\frac{3}{2}\right)}\ {}_1F_2\left(\mu+1;\ \mu-\nu+\frac{3}{2},\mu-\nu+1;\ \frac{\beta^2a^2}{4}\right)$$
$$= \frac{\sqrt{\pi}}{2\,\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\,G^{\,21}_{\,13}\left(\frac{a^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2},\frac{1}{2},0\end{matrix}\right)$$
$$[a>0,\quad \operatorname{Re}\beta>0,\quad -1<\operatorname{Re}\nu<\operatorname{Re}\mu+1]$$ ET I 71(28)a, ET II 234(17)

2.[8] $$\int_0^\infty \frac{x^{2m+1}\sin(ax)}{(z+x^2)^{n+1}}\,dx = \frac{(-1)^{n+m}}{n!}\cdot\frac{\pi}{2}\frac{d^n}{dz^n}\left(z^m e^{-a\sqrt{z}}\right)$$
$$[a>0,\quad 0\le m\le n,\quad |\arg z|<\pi]$$
ET I 68(39)

3. $$\int_0^\infty \frac{x^{2m+1}\sin(ax)\,dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^{m+1}\sqrt{\pi}}{2^n\beta^n\,\Gamma\left(n+\frac{1}{2}\right)}\frac{d^{2m+1}}{da^{2m+1}}\left[a^n\,K_n(a\beta)\right]$$
$$[a>0,\quad \operatorname{Re}\beta>0,\quad -1\le m\le n]$$
ET I 67(37)

4. $$\int_0^\infty \frac{x^{2\nu}\cos(ax)\,dx}{(x^2+\beta^2)^{\mu+1}} = \frac{1}{2}\beta^{2\nu-2\mu-1}\,\mathrm{B}\left(\nu+\frac{1}{2},\mu-\nu+\frac{1}{2}\right)\ {}_1F_2\left(\nu+\frac{1}{2};\nu-\mu+\frac{1}{2},\frac{1}{2};\frac{\beta^2a^2}{4}\right)$$
$$+\frac{\sqrt{\pi}a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}}\frac{\Gamma\left(\nu-\mu-\frac{1}{2}\right)}{\Gamma(\mu-\nu+1)}\ {}_1F_2\left(\mu+1;\mu-\nu+1,\mu-\nu+\frac{3}{2};\frac{\beta^2a^2}{4}\right)$$
$$= \frac{\sqrt{\pi}}{2\,\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\,G^{\,21}_{\,13}\left(\frac{a^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)$$
$$\left[a>0,\quad \operatorname{Re}\beta>0,\quad -\tfrac{1}{2}<\operatorname{Re}\nu<\operatorname{Re}\mu+1\right]$$ ET I 14(29)a, ET II 235(19)

5. $$\int_0^\infty \frac{x^{2m}\cos(ax)\,dx}{(z+x^2)^{n+1}} = (-1)^{m+n}\frac{\pi}{2\cdot n!}\cdot\frac{d^n}{dz^n}\left(z^{m-\frac{1}{2}}e^{-a\sqrt{z}}\right)$$
$$[a>0,\quad n+1>m\ge 0,\quad |\arg z|<\pi]$$
ET I 10(28)

6.[7] $$\int_0^\infty \frac{x^{2m}\cos(ax)\,dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^m\sqrt{\pi}}{2^n\beta^n\,\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{d^{2m}}{da^{2m}}\left\{a^n\,K_n(a\beta)\right\}$$
$$\left[a>0,\quad \operatorname{Re}\beta>0,\quad 0\le m<n+\tfrac{1}{2}\right]$$
ET I 14(28)

3.774

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2+b^2}\left(x+\sqrt{x^2+b^2}\right)^\nu} = \frac{\pi}{b^\nu\sin(\nu\pi)}\left[\sin\frac{\nu\pi}{2}\,I_\nu(ab)+\frac{i}{2}\,\mathbf{J}_\nu(iab)-\frac{i}{2}\,\mathbf{J}_\nu(-iab)\right]$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\nu>-1]$$
ET I 70(19)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{x^2+b^2}\left(x+\sqrt{x^2+b^2}\right)^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)}\left[\frac{1}{2}\,\mathbf{J}_\nu(iab) + \frac{1}{2}\,\mathbf{J}_\nu(-iab) - \cos\frac{\nu\pi}{2}\,I_\nu(ab)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu > -1]$$
ET I 12(15)

3. $$\int_0^\infty \frac{\left(x+\sqrt{x^2+\beta^2}\right)^\nu}{\sqrt{x\,(x^2+\beta^2)}}\sin(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu\, I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right) K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\tfrac{3}{2}\right]$$
ET I 71(23)

4. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x\,(x^2+\beta^2)}}\cos(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu\, I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right) K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$
ET I 12(17)

5. $$\int_0^\infty \frac{\left(\beta+\sqrt{x^2+\beta^2}\right)^\nu}{x^{\nu+\frac{1}{2}}\sqrt{x^2+\beta^2}}\sin(ax)\,dx = \frac{1}{\beta}\sqrt{\frac{2}{a}}\,\Gamma\left(\frac{3}{4}-\frac{\nu}{2}\right) W_{\frac{\nu}{2},\frac{1}{4}}(a\beta)\, M_{-\frac{\nu}{2},\frac{1}{4}}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\tfrac{3}{2}\right]$$
ET I 71(27)

6. $$\int_0^\infty \frac{\left(\beta+\sqrt{x^2+\beta^2}\right)^\nu}{x^{\nu+\frac{1}{2}}\sqrt{\beta^2+x^2}}\cos(ax)\,dx = \frac{1}{\beta\sqrt{2a}}\,\Gamma\left(\frac{1}{4}-\frac{\nu}{2}\right) W_{\frac{\nu}{2},-\frac{1}{4}}(a\beta)\, M_{-\frac{\nu}{2},-\frac{1}{4}}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET I 12(18)

3.775

1. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}+x\right)^\nu - \left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x^2+\beta^2}}\sin(ax)\,dx = 2\beta^\nu \sin\frac{\nu\pi}{2}\,K_\nu(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad |\operatorname{Re}\nu|<1]$$
ET I 70(20)

2. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}+x\right)^\nu + \left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x^2+\beta^2}}\cos(ax)\,dx = 2\beta^\nu \cos\frac{\nu\pi}{2}\,K_\nu(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad |\operatorname{Re}\nu|<1]$$
ET I 13(22)

3. $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu + \left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x^2-u^2}}\sin(ax)\,dx = \pi u^\nu\left[J_\nu(au)\cos\frac{\nu\pi}{2} - Y_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$$[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<1]$$
ET I 70(22)

4. $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu + \left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x^2-u^2}}\cos(ax)\,dx = -\pi u^\nu\left[Y_\nu(au)\cos\frac{\nu\pi}{2} + J_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$$[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<1]$$
ET I 13(25)

5. $$\int_0^u \frac{\left(x+i\sqrt{u^2-x^2}\right)^\nu+\left(x-i\sqrt{u^2-x^2}\right)^\nu}{\sqrt{u^2-x^2}}\sin(ax)\,dx=\frac{\pi}{2}u^\nu\operatorname{cosec}\frac{\nu\pi}{2}\left[\mathbf{J}_\nu(au)-\mathbf{J}_{-\nu}(au)\right]$$
$[a>0,\quad u>0]$ ET I 70(21)

6. $$\int_0^u \frac{\left(x+i\sqrt{u^2-x^2}\right)^\nu+\left(x-i\sqrt{u^2-x^2}\right)^\nu}{\sqrt{u^2-x^2}}\cos(ax)\,dx=\frac{\pi}{2}u^\nu\sec\frac{\nu\pi}{2}\left[\mathbf{J}_\nu(au)+\mathbf{J}_{-\nu}(au)\right]$$
$[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<1]$ ET I 13(24)

7.[6] $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x\left(x^2-u^2\right)}}\sin(ax)\,dx$$
$$=-\sqrt{\left(\frac{\pi}{2}\right)^3}au^\nu\left[J_{1/4+\nu/2}\left(\frac{au}{2}\right)Y_{1/4-\nu/2}\left(\frac{au}{2}\right)+J_{1/4-\nu/2}\left(\frac{au}{2}\right)Y_{1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{3}{2}\right]$ ET I 71(25)

8.[6] $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x\left(x^2-u^2\right)}}\cos(ax)\,dx$$
$$=-\sqrt{\left(\frac{\pi}{2}\right)^3}au^\nu\left[J_{-1/4+\nu/2}\left(\frac{au}{2}\right)Y_{-1/4-\nu/2}\left(\frac{au}{2}\right)+J_{-1/4-\nu/2}\left(\frac{au}{2}\right)Y_{-1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{3}{2}\right]$ ET I 13(26)

9. $$\int_0^\infty \frac{\left(x+\beta+\sqrt{x^2+2\beta x}\right)^\nu+\left(x+\beta-\sqrt{x^2+2\beta x}\right)^\nu}{\sqrt{x^2+2\beta x}}\sin(ax)\,dx$$
$$=\pi\beta^\nu\left[Y_\nu(\beta a)\sin\left(\beta a-\frac{\nu\pi}{2}\right)+J_\nu(\beta a)\cos\left(\beta a-\frac{\nu\pi}{2}\right)\right]$$
$[a>0,\quad |\arg\beta|<\pi,\quad |\operatorname{Re}\nu|<1]$ ET I 71(26)

10. $$\int_0^\infty \frac{\left(x+\beta+\sqrt{x^2+2\beta x}\right)^\nu+\left(x+\beta-\sqrt{x^2+2\beta x}\right)^\nu}{\sqrt{x^2+2\beta x}}\cos(ax)\,dx$$
$$=\pi\beta^\nu\left[J_\nu(\beta a)\sin\left(\beta a-\frac{\nu\pi}{2}\right)-Y_\nu(\beta a)\cos\left(\beta a-\frac{\nu\pi}{2}\right)\right]$$
$[a>0,\quad |\arg\beta|<\pi,\quad |\operatorname{Re}\nu|<1]$ ET I 13(23)

11. $$\int_0^{2u} \frac{\left(\sqrt{2u+x}+i\sqrt{2u-x}\right)^{4\nu}+\left(\sqrt{2u+x}-i\sqrt{2u-x}\right)^{4\nu}}{\sqrt{4u^2x-x^3}}\cos(ax)\,dx$$
$$=(4u)^{2\nu}\pi^{3/2}\sqrt{\frac{a}{2}}\,J_{\nu-1/4}(au)\,J_{-\nu-1/4}(au)$$
$[a>0,\quad u>0]$ ET I 14(27)

3.776

1. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\sin(ax)\,dx=\frac{a}{b^p}$$
$[a>0,\quad b>0,\quad p>0]$ BI (170)(1)

2. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\cos(ax)\,dx=\frac{p}{b^{p+1}}$$
$[a>0,\quad b>0,\quad p>0]$ BI (170)(2)

3.78–3.81 Rational functions of x and of trigonometric functions

3.781

1. $$\int_0^\infty \left(\frac{\sin x}{x} - \frac{1}{1+x}\right)\frac{dx}{x} = 1 - \boldsymbol{C}$$ (cf. **3.784** 4 and **3.781** 2) BI (173)(7)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x}\right)\frac{dx}{x} = -\boldsymbol{C}$$ BI (173)(8)

3.782

1. $$\int_0^u \frac{1-\cos x}{x}\,dx - \int_u^\infty \frac{\cos x}{x}\,dx = \boldsymbol{C} + \ln u$$ $[u > 0]$ GW (333)(31)

2. $$\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{a\pi}{2}$$ $[a \geq 0]$ BI (158)(1)

3. $$\int_{-\infty}^\infty \frac{1-\cos ax}{x(x-b)}\,dx = \pi\frac{\sin ab}{b}$$ $[a > 0, \quad b \text{ real}, \quad b \neq 0]$ ET II 253(48)

3.783

1. $$\int_0^\infty \left[\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)}\right]\frac{dx}{x} = \frac{1}{2}\boldsymbol{C} - \frac{3}{4}$$ BI (173)(19)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\boldsymbol{C}$$ EH I 17, BI(273)(21)

3.784

1. $$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{b}{a}$$ $[a > 0, \quad b > 0]$ FI II 635, GW(333)(20)

2. $$\int_0^\infty \frac{a\sin bx - b\sin ax}{x^2}\,dx = ab\ln\frac{a}{b}$$ $[a > 0, \quad b > 0]$ FI II 647

3. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{(b-a)\pi}{2}$$ $[a \geq 0, \quad b \geq 0]$ BI(158)(12), FI II 645

4. $$\int_0^\infty \frac{\sin x - x\cos x}{x^2}\,dx = 1$$ BI (158)(3)

5. $$\int_0^\infty \frac{\cos ax - \cos bx}{x(x+\beta)}\,dx = \frac{1}{\beta}\left[\operatorname{ci}(a\beta)\cos a\beta + \operatorname{si}(a\beta)\sin a\beta - \operatorname{ci}(b\beta)\cos b\beta - \operatorname{si}(b\beta)\sin b\beta + \ln\frac{b}{a}\right]$$
$[a > 0, \quad b > 0, \quad |\arg\beta| < \pi]$ ET II 221(49)

6. $$\int_0^\infty \frac{\cos ax + x\sin ax}{1+x^2}\,dx = \pi e^{-a}$$ $[a > 0]$ GW (333)(73)

7. $$\int_0^\infty \frac{\sin ax - ax\cos ax}{x^3}\,dx = \frac{\pi}{4}a^2\operatorname{sign} a$$ LI (158)(5)

8. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2\left(x^2+\beta^2\right)}\,dx = \frac{\pi\left[(b-a)\beta + e^{-b\beta} - e^{-a\beta}\right]}{2\beta^3}$$
$[a > 0, \quad b > 0, \quad |\arg\beta| < \pi]$
BI(173)(20)a, ET II 222(59)

9.[10] $$\int_0^\infty \frac{\cos mx}{1+a^2\,T_n(x)} = \frac{\pi}{2n\sqrt{1+a^2}} \sum_{k=1}^{n} e^{-m\sin u\sinh\phi}\left(\cos\beta\sin u\cosh\phi + \sin\beta\cos u\sinh\phi\right)$$

$$[u=(2k-1)\,\pi/\,(2n)\,,\quad \phi=\operatorname{arcsinh}(1/a),\quad \beta=m\cos u\cosh\phi,\quad 0<|a|<1]$$

3.785 $$\int_0^\infty \frac{1}{x}\sum_{k=1}^{n} a_k\cos b_k x\,dx = -\sum_{k=1}^{n} a_k\ln b_k \qquad \left[b_k>0,\quad \sum_{k=1}^{n} a_k=0\right]$$ FI II 649

3.786

1. $$\int_0^\infty \frac{(1-\cos ax)\sin bx}{x^2}\,dx = \frac{b}{2}\ln\frac{b^2-a^2}{b^2} + \frac{a}{2}\ln\frac{a+b}{a-b}$$
$[a>0,\quad b>0]$ ET I 81(29)

2.[11] $$\int_0^\infty \frac{(1-\cos ax)\cos bx}{x}\,dx = \ln\frac{\sqrt{|a^2-b^2|}}{b}$$ $[a>0,\quad b>0,\quad a\neq b]$ FI II 647

3.[11] $$\int_0^\infty \frac{(1-\cos ax)\cos bx}{x^2}\,dx = \frac{\pi}{2}(a-b) \qquad [a<b\le 0]$$
$$= 0 \qquad [0<a\le b]$$
ET I 20(16)

3.787

1. $$\int_0^\infty \frac{(\cos a-\cos nax)\sin mx}{x}\,dx = \frac{\pi}{2}\,(\cos a-1) \qquad [m>na>0]$$
$$= \frac{\pi}{2}\cos a \qquad [na>m]$$
BI(155)(7)

2. $$\int_0^\infty \frac{\sin^2 ax-\sin^2 bx}{x}\,dx = \frac{1}{2}\ln\frac{a}{b}$$ $[a>0,\quad b>0]$ GW (333)(20b)

3. $$\int_0^\infty \frac{x^3-\sin^3 x}{x^5}\,dx = \frac{13}{32}\pi$$ BI (158)(6)

4. $$\int_0^\infty \frac{\left(3-4\sin^2 ax\right)\sin^2 ax}{x}\,dx = \frac{1}{2}\ln 2$$ $[a \text{ real}, a\neq 0]$ HBI (155)(6)

3.788 $$\int_0^{\pi/2}\left(\frac{1}{x}-\cot x\right)dx = \ln\frac{\pi}{2}$$ GW (333)(61)a

3.789 $$\int_0^{\pi/2}\frac{4x^2\cos x+(\pi-x)x}{\sin x}\,dx = \pi^2\ln 2$$ LI (206)(10)

3.791

1. $$\int_0^{\pi/2}\frac{x\,dx}{1+\sin x} = \ln 2$$ GW (333)(55a)

2. $$\int_0^{\pi}\frac{x\cos x}{1+\sin x}\,dx = \pi\ln 2-4\boldsymbol{G}$$ GW (333)(55c)

3. $$\int_0^{\pi/2}\frac{x\cos x}{1+\sin x}\,dx = \pi\ln 2-2\boldsymbol{G}$$ GW (333)(55b)

4. $$\int_0^{\pi} \frac{\left(\frac{\pi}{2}-x\right)\cos x}{1-\sin x}\,dx = 2\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x}{1-\sin x}\,dx = \pi\ln 2 + 4\boldsymbol{G} = 5.8414484669\ldots$$

BI(207)(3), GW(333)(56c)

5. $$\int_0^{\pi/2} \frac{x^2\,dx}{1-\cos x} = -\frac{\pi^2}{4} + \pi\ln 2 + 4\boldsymbol{G} = 3.3740473667\ldots$$ BI (207)(3)

6. $$\int_0^{\pi} \frac{x^2\,dx}{1-\cos x} = 4\pi\ln 2$$ BI (219)(1)

7. $$\int_0^{\pi/2} \frac{x^{p+1}\,dx}{1-\cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^{p}(p+1)\left\{\frac{2}{p} - \sum_{k=1}^{\infty}\frac{1}{4^{2k-1}(p+2k)}\,\zeta(2k)\right\}$$

$[p>0]$ LI (207)(4)

8. $$\int_0^{\pi/2} \frac{x\,dx}{1+\cos x} = \frac{\pi}{2} - \ln 2$$ GW (333)(55a)

9. $$\int_0^{\pi/2} \frac{x\sin x\,dx}{1-\cos x} = \frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$$ GW (333)(56a)

10. $$\int_0^{\pi} \frac{x\sin x\,dx}{1-\cos x} = 2\pi\ln 2$$ GW (333)(56b)

11. $$\int_0^{\pi} \frac{x-\sin x}{1-\cos x}\,dx = \frac{\pi}{2} + \int_0^{\pi/2}\frac{x-\sin x}{1-\cos x}\,dx = 2$$ GW (333)(57a)

12. $$\int_0^{\pi/2} \frac{x\sin x}{1+\cos x}\,dx = -\frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$$ GW (333)(55b)

3.792

1. $$\int_{-\pi}^{\pi} \frac{dx}{1-2a\cos x + a^2} = \frac{2\pi}{1-a^2} \qquad \left[a^2<1\right]$$ FI II 485

2. $$\int_0^{\pi/2} \frac{x\cos x\,dx}{1+2a\sin x + a^2} = \frac{\pi}{2a}\ln(1+a) - \sum_{k=0}^{\infty}(-1)^k\frac{a^{2k}}{(2k+1)^2}$$

$\left[a^2<1\right]$ LI (241)(2)

3. $$\int_0^{\pi} \frac{x\sin x\,dx}{1-2a\cos x + a^2} = \frac{\pi}{a}\ln(1+a) \qquad \left[a^2<1,\quad a\neq 0\right]$$
$$= \frac{\pi}{a}\ln\left(1+\frac{1}{a}\right) \qquad \left[a^2<1\right]$$

BI (221)(2)

4. $$\int_0^{2\pi} \frac{x\sin x\,dx}{1-2a\cos x + a^2} = \frac{2\pi}{a}\ln(1-a) \qquad \left[a^2<1,\quad a\neq 0\right]$$
$$= \frac{2\pi}{a}\ln\left(1-\frac{1}{a}\right) \qquad \left[a^2>1\right]$$

BI (223)(4)

5. $$\int_0^{2\pi} \frac{x\sin nx\,dx}{1-2a\cos x + a^2} = \frac{2\pi}{1-a^2}\left[\left(a^{-n}-a^{n}\right)\ln(1-a) + \sum_{k=1}^{n-1}\frac{a^{-k}-a^{k}}{n-k}\right]$$

$\left[a^2<1,\quad a\neq 0\right]$ BI (223)(5)

6. $$\int_0^\infty \frac{\sin x}{1-2a\cos x+a^2}\cdot\frac{dx}{x}=\frac{\pi}{4a}\left[\left|\frac{1+a}{1-a}\right|-1\right] \qquad [a \text{ real}, \quad a\neq 0, \quad a\neq 1]$$

GW (333)(62b)

7.[8] $$\int_0^\infty \frac{\sin bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x}=\frac{\pi}{2}\frac{1+a-2a^{[b]+1}}{(1-a^2)(1-a)} \qquad [b\neq 0,1,2,\ldots]$$
$$=\frac{\pi}{2}\frac{1+a-a^b-a^{b+1}}{(1-a^2)(1-a)} \qquad [b=1,2,\ldots]; \qquad [0<a<1]$$

ET I 81(26)

8. $$\int_0^\infty \frac{\sin x\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x}=\frac{\pi}{2(1-a)}a^{[b]} \qquad [b\neq 0,1,2,\ldots]$$
$$=\frac{\pi}{2(1-a)}a^b+\frac{\pi}{4}a^{b-1} \qquad [b=1,2,3,\ldots];$$
$[0<a<1, \quad b>0]$; (for $b=0$, see **3.792** 6) ET I 19(5)

9. $$\int_0^\infty \frac{(1-a\cos x)\sin bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x}=\frac{\pi}{2}\cdot\frac{1-a^{[b]+1}}{1-a} \qquad [b\neq 1,2,3,\ldots]$$
$$=\frac{\pi}{2}\cdot\frac{1-a^b}{1-a}+\frac{\pi a^b}{4} \qquad [b=1,2,3,\ldots]$$
$[0<a<1, \quad b>0]$ ET I 82(33)

10.[3] $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\frac{dx}{\beta^2+x^2}=\frac{\pi}{2\beta(1-a^2)}\frac{1+ae^{-b\beta}}{1-ae^{-b\beta}}$$
$[a^2<1, \quad b\geq 0]$ BI (192)(1)

11. $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\frac{dx}{\beta^2-x^2}=\frac{a\pi}{\beta(1-a^2)}\frac{\sin b\beta}{1-2a\cos b\beta+a^2}$$
$[a^2<1, \quad b>0]$ BI (193)(1)

12. $$\int_0^\infty \frac{\sin bcx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2+x^2}=\frac{\pi}{2}\frac{e^{-\beta bc}-a^c}{(1-ae^{-b\beta})(1-ae^{b\beta})}$$
$[a^2<1, \quad b>0, \quad c>0]$ BI (192)(8)

13. $$\int_0^\infty \frac{\sin bx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2+x^2}=\frac{\pi}{2}\frac{1}{e^{b\beta}-a} \qquad [a^2<1, \quad b>0]$$
$$=\frac{\pi}{2a}\frac{1}{ae^{b\beta}-1} \qquad [a^2>1, \quad b>0]$$

BI (192)(2)

14. $$\int_0^\infty \frac{\sin bcx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2-x^2}=\frac{\pi}{2}\frac{a^c-\cos\beta bc}{1-2a\cos\beta b+a^2}$$
$[a^2<1, \quad b>0, \quad c>0]$ BI (193)(5)

15. $$\int_0^\infty \frac{\cos bcx}{1-2a\cos bx+a^2}\frac{dx}{\beta^2-x^2}=\frac{\pi}{2\beta(1-a^2)}\frac{(1-a^2)\sin\beta bc+2a^{c+1}\sin\beta b}{1-2a\cos\beta b+a^2}$$
$[a^2<1, \quad b>0, \quad c>0]$ BI (193)(9)

16. $$\int_0^\infty \frac{1-a\cos bx}{1-2a\cos bx+a^2}\frac{dx}{1+x^2}=\frac{\pi}{2}\frac{e^b}{e^b-a} \qquad [a^2<1, \quad b>0]$$ FI II 719

17. $$\int_0^\infty \frac{\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2}=\frac{\pi\left(e^{\beta-\beta b}+ae^{\beta b}\right)}{2\beta\left(1-a^2\right)\left(e^\beta-a\right)}$$
$[0\le b<1,\quad |a|<1,\quad \operatorname{Re}\beta>0]$ ET I 21(21)

18. $$\int_0^\infty \frac{\sin bx\sin x}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2}$$
$$=\frac{\pi}{2\beta}\frac{\sinh b\beta}{e^\beta-a} \qquad [0\le b<1]$$
$$=\frac{\pi}{4\beta\left(ae^\beta-1\right)}\left[a^m e^{\beta(m+1-b)}-e^{(1-b)\beta}\right]$$
$$-\frac{\pi}{4\beta\left(ae^{-\beta}-1\right)}\left[a^m e^{-(m+1-b)\beta}-e^{-(1-b)\beta}\right] \qquad [m\le b\le m+1]$$
$[0<a<1,\quad \operatorname{Re}\beta>0]$ ET I 81(27)

19. $$\int_0^\infty \frac{(\cos x-a)\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2}=\frac{\pi\cosh\beta b}{2\beta\left(e^\beta-a\right)}$$
$[0\le b<1,\quad |a|<1,\quad \operatorname{Re}\beta>0]$ ET I 21(23)

20. $$\int_0^\infty \frac{\sin x}{\left(1-2a\cos 2x+a^2\right)^{n+1}}\frac{dx}{x}=\int_0^\infty \frac{\tan x}{\left(1-2a\cos 2x+a^2\right)^{n+1}}\frac{dx}{x}$$
$$=\int_0^\infty \frac{\tan x}{\left(1-2a\cos 4x+a^2\right)^{n+1}}\frac{dx}{x}=\frac{\pi}{2\left(1-a^2\right)^{2n+1}}\sum_{k=0}^n\binom{n}{k}^2a^{2k}$$
BI (187)(14)

3.793

1.[3] $$\int_0^{2\pi}\frac{\sin nx-a\sin[(n+1)x]}{1-2a\cos x+a^2}x\,dx=-2\pi a^n\left[\ln(1-a)+\sum_{k=1}^n\frac{1}{ka^k}\right]$$
$[|a|<1]$ BI (223)(9)

2. $$\int_0^{2\pi}\frac{\cos nx-a\cos[(n+1)x]}{1-2a\cos x+a^2}x\,dx=2\pi a^n \qquad \left[a^2<1\right]$$ BI (223)(13)

3.794

1.[3] $$\int_0^\pi\frac{x\,dx}{1+a^2+2a\cos x}=\frac{\pi^2}{2\left(1-a^2\right)}+\frac{4}{\left(1-a^2\right)}\sum_{k=0}^\infty\frac{a^{2k+1}}{(2k+1)^2}$$
$\left[a^2<1\right]$

2. $$\int_0^{2\pi}\frac{x\sin nx}{1\pm a\cos x}\,dx=\frac{2\pi}{\sqrt{1-a^2}}\left[(\mp1)^n\frac{\left(1+\sqrt{1-a^2}\right)^n-\left(1-\sqrt{1-a^2}\right)^n}{a^n}\right.$$
$$\left.\times\ \ln\frac{2\sqrt{1\pm a}}{\sqrt{1+a}+\sqrt{1-a}}+\sum_{k=0}^{n-1}\frac{(\mp1)^k}{n-k}\frac{\left(1+\sqrt{1-a^2}\right)^k-\left(1-\sqrt{1-a^2}\right)^k}{a^k}\right]$$
$\left[a^2<1\right]$ BI (223)(2)

3.[3] $$\int_0^{2\pi}\frac{x\cos nx}{1\pm a\cos x}\,dx=\frac{2\pi^2}{\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{\mp a}\right)^n \qquad \left[a^2<1\right]$$ BI (223)(3)

4. $$\int_0^{\pi} \frac{x \sin x \, dx}{a + b \cos x} = \frac{\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a - b)} \qquad [a > |b| > 0]$$ GW (333)(53a)

5. $$\int_0^{2\pi} \frac{x \sin x \, dx}{a + b \cos x} = \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a + b)} \qquad [a > |b| > 0]$$ GW (333)(53b)

6. $$\int_0^{\infty} \frac{\sin x}{a \pm b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \qquad \left[a^2 > b^2\right]$$
$$= 0 \qquad \left[a^2 < b^2\right]$$
BI (181)(1)

3.795 $$\int_{-\infty}^{\infty} \frac{\left(b^2 + c^2 + x^2\right) x \sin ax - \left(b^2 - c^2 - x^2\right) c \sinh ac}{\left[x^2 + (b - c)^2\right]\left[x^2 + (b + c)^2\right](\cos ax + \cosh ac)} \, dx = \pi \qquad [c > b > 0]$$
$$= \frac{2\pi}{e^{ab} + 1} \qquad [b > c > 0]$$
$$[a > 0]$$ BI (202)(18)

3.796

1. $$\int_0^{\pi/2} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x \, dx = \mp \frac{\pi}{4} \ln 2 - \boldsymbol{G}$$ BI (207)(8, 9)

2. $$\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x \, dx = \frac{\pi}{4} \ln 2 - \frac{1}{2} \boldsymbol{G}$$ BI (204)(23)

3.797

1. $$\int_0^{\pi/4} \left(\frac{\pi}{4} - x \tan x\right) \tan x \, dx = \frac{1}{2} \ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8} \ln 2$$ BI (204)(8)

2. $$\int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x\right) \tan x \, dx}{\cos 2x} = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}$$ BI (204)(19)

3. $$\int_0^{\pi/4} \frac{\frac{\pi}{4} - x \tan x}{\cos 2x} \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}$$ BI (204)(20)

3.798

1.[8] $$\int_0^{\infty} \frac{\tan x}{a + b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
BI (181)(2)

2.[8] $$\int_0^{\infty} \frac{\tan x}{a + b \cos 4x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
BI (181)(3)

3.799

1. $$\int_0^{\pi/2} \frac{x \, dx}{(\sin x + a \cos x)^2} = \frac{a}{1 + a^2} \frac{\pi}{2} - \frac{\ln a}{1 + a^2} \qquad [a > 0]$$ BI (208)(5)

2. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + a\sin x)^2} = \frac{1}{1+a^2}\ln\frac{1+a}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1-a}{(1+a)(1+a^2)}$$ $[a>0]$ BI (204)(24)

3. $$\int_0^{\pi} \frac{a\cos x + b}{(a+b\cos x)^2}x^2\,dx = \frac{2\pi}{b}\ln\frac{2(a-b)}{a+\sqrt{a^2-b^2}}$$ $[a>|b|>0]$ GW (333)(58a)

3.811

1. $$\int_0^{\pi} \frac{\sin x}{1-\cos t_1\cos x}\cdot\frac{x\,dx}{1-\cos t_2\cos x} = \pi\operatorname{cosec}\frac{t_1+t_2}{2}\operatorname{cosec}\frac{t_1-t_2}{2}\ln\frac{1+\tan\frac{t_1}{2}}{1+\tan\frac{t_2}{2}}$$ (cf. **3.794** 4) BI (222)(5)

2. $$\int_0^{\pi/2} \frac{x\,dx}{(\cos x \pm \sin x)\sin x} = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$$ BI (208))(16, 17)

3. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + \sin x)\sin x} = -\frac{\pi}{8}\ln 2 + \boldsymbol{G}$$ BI (204)(29)

4. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + \sin x)\cos x} = \frac{\pi}{8}\ln 2$$ BI (204)(28)

5. $$\int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\frac{x\,dx}{\cos^2 x} = -\frac{\pi}{8}\ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2$$ BI (204)(30)

3.812

1. $$\int_0^{\pi} \frac{x\sin x\,dx}{a+b\cos^2 x} = \frac{\pi}{\sqrt{ab}}\arctan\sqrt{\frac{b}{a}} \qquad [a>0,\quad b>0]$$
$$= \frac{\pi}{2\sqrt{-ab}}\ln\frac{\sqrt{a}+\sqrt{-b}}{\sqrt{a}-\sqrt{-b}} \qquad [a>-b>0]$$
GW (333)(60a)

2. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{1+a\cos^2 x} = \frac{\pi}{a}\ln\frac{1+\sqrt{1+a}}{2}$$ $[a>-1,\quad a\neq 0]$ BI (207)(10)

3. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{1+a\sin^2 x} = \frac{\pi}{a}\ln\frac{2\left(1+a-\sqrt{1+a}\right)}{2}$$ $[a>-1,\quad a\neq 0]$ BI (207)(2)

4.[11] $$\int_0^{\pi} \frac{x\,dx}{a^2-\cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}} \qquad \left[a^2>1\right]$$
$$= 0 \qquad \left[\text{principal value for } 0<a^2<1\right]$$
$$= \text{divergent} \qquad [a=0]$$
BI (219)(10)

5.[7] $$\int_0^{\pi} \frac{x\sin x\,dx}{a^2-\cos^2 x} = \frac{\pi}{2a}\ln\left|\frac{1+a}{1-a}\right|$$ $[0<a<1]$ divergent if $a=0$ BI (219)(13)

6.[11] $$\int_0^\pi \frac{x\sin 2x\,dx}{a^2-\cos^2 x} = \pi\ln\left\{4\left(1-a^2\right)\right\} \qquad \left[\text{principal value for } 0\le a^2<1\right]$$
$$= 2\pi\ln\left[2\left(1-a^2+a\sqrt{a^2-1}\right)\right] \qquad \left[a^2>1\right]$$
$$= \text{divergent} \qquad \left[|a|=1\right]$$
BI (219)(19)

7. $$\int_0^{\pi/2} \frac{x\sin x\,dx}{\cos^2 t-\sin^2 x} = -2\operatorname{cosec} t\sum_{k=0}^{\infty}\frac{\sin(2k+1)t}{(2k+1)^2}$$ BI (207)(1)

8. $$\int_0^\pi \frac{x\sin x\,dx}{1-\cos^2 t\sin^2 x} = \pi(\pi-2t)\operatorname{cosec} 2t$$ BI (219)(12)

9. $$\int_0^\pi \frac{x\cos x\,dx}{\cos^2 t-\cos^2 x} = 4\operatorname{cosec} t\sum_{k=0}^{\infty}\frac{\sin(2k+1)t}{(2k+1)^2}$$ BI (219)(17)

10. $$\int_0^\pi \frac{x\sin x\,dx}{\tan^2 t+\cos^2 x} = \frac{\pi}{2}(\pi-2t)\cot t$$ BI (219)(14)

11. $$\int_0^\infty \frac{x\,(a\cos x+b)\sin x\,dx}{\cot^2 t+\cos^2 x} = 2a\pi\ln\cos\frac{t}{2}+\pi bt\tan t$$ BI (219)(18)

12.* $$\int_0^\pi \frac{x\sin x\cos x}{a-\sin^2 x}\,dx = -\pi\ln 2+\ln\left[1+\sqrt{\frac{a-1}{a}}\right] \qquad [a>1]$$

13.* $$\int_0^{\pi/2} \ln\left(a-\sin^2 x\right)\,dx = -\pi\ln 2+i\pi\ln\arccos\sqrt{a} \qquad [0<a<1]$$

14.* $$\mathrm{PV}\int_0^{\pi/2} \ln\left(\left|a-\sin^2 x\right|\right)\,dx = -\pi\ln 2 \qquad [0<a<1]$$

15.* $$\mathrm{PV}\int_0^{\pi/2} \ln\left(\left|a-\cos^2 x\right|\right)\,dx = -\pi\ln 2 \qquad [0<a<1]$$

3.813

1. $$\int_0^\pi \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{1}{4}\int_0^{2\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi^2}{2ab}$$
$$[a>0,\quad b>0]$$ GW (333)(36)

2. $$\int_0^\infty \frac{1}{\beta^2\sin^2 ax+\gamma^2\cos^2 ax}\cdot\frac{dx}{x^2+\delta^2} = \frac{\pi\sinh(2a\delta)}{4\delta\left(\beta^2\sinh^2(a\delta)-\gamma^2\cosh^2(a\delta)\right)}\left[\frac{\beta}{\gamma}-\frac{\gamma}{\beta}-\frac{2}{\sinh(2a\delta)}\right]$$
$$\left[\left|\arg\frac{\beta}{\gamma}\right|<\pi,\quad \operatorname{Re}\delta>0,\quad a>0\right]$$
GW(333)(81), ET II 222(63)

3. $$\int_0^\infty \frac{\sin x\,dx}{x\left(a^2\sin^2 x+b^2\cos^2 x\right)} = \frac{\pi}{2ab} \qquad [ab>0]$$ BI (181)(8)

4. $$\int_0^\infty \frac{\sin^2 x\,dx}{x\left(a^2\cos^2 x+b^2\sin^2 x\right)} = \frac{\pi}{2b(a+b)} \qquad [a>0,\quad b>0]$$ BI (181)(11)

5. $$\int_0^{\pi/2} \frac{x \sin 2x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b} \qquad [a > 0, \quad b > 0, \quad a \neq b]$$ GW (333)(52a)

6. $$\int_0^{\pi} \frac{x \sin 2x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a} \qquad [a > 0, \quad b > 0, \quad a \neq b]$$ GW (333)(52b)

7. $$\int_0^{\infty} \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{a(a+b)} \qquad [a > 0, \quad b > 0]$$ BI (182)(3)

8. $$\int_0^{\infty} \frac{\sin 2ax}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{x\, dx}{x^2 + \delta^2} = \frac{\pi}{2\left(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta)\right)} \left[\frac{\beta - \gamma}{\beta + \gamma} - e^{-2a\delta}\right]$$
$$\left[a > 0, \quad \left|\arg \frac{\beta}{\gamma}\right| < \pi, \quad \operatorname{Re} \delta > 0\right]$$
ET II 222(64), GW(333)(80)

9. $$\int_0^{\infty} \frac{(1 - \cos x) \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)} \qquad [a > 0, \quad b > 0]$$ BI (182)(7)a

10. $$\int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a(a+b)} \qquad [a > 0, \quad b > 0]$$ BI (182)(4)

11. $$\int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{2}{a+b} \qquad [a > 0, \quad b > 0]$$ BI (182)(1)

3.814

1. $$\int_0^{\pi/2} \frac{(1 - x \cot x)\ dx}{\sin^2 x} = \frac{\pi}{4}$$ BI (206)(9)

2. $$\int_0^{\pi/4} \frac{x \tan x\, dx}{(\sin x + \cos x) \cos x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2$$ BI (204)(30)

3. $$\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \frac{dx}{x} = \frac{\pi}{2ab} \qquad [a > 0, \quad b > 0]$$ BI (181)(9)

4. $$\int_0^{\pi/2} \frac{x \cot x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2a^2} \ln \frac{a+b}{b} \qquad [a > 0, \quad b > 0]$$ LI (208)(20)

5. $$\int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \tan x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{2} \int_0^{\pi} \frac{\left(\frac{\pi}{2} - x\right) \tan x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$
$$= \frac{\pi}{2b^2} \ln \frac{a+b}{a}$$
$$[a > 0, \quad b > 0]$$ GW (333)(59)

6. $$\int_0^{\infty} \frac{\sin^2 x \tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)} \qquad [a > 0, \quad b > 0]$$ BI (182)(6)

7. $$\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab} \qquad [a > 0, \quad b > 0]$$ BI (181)(10)a

8. $$\int_0^{\infty} \frac{\sin^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b} \qquad [a > 0, \quad b > 0]$$ BI (182)(2)a

9. $$\int_0^{\infty} \frac{\cos^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b} \qquad [a > 0, \quad b > 0]$$ BI (182)(5)a

10. $$\int_0^\infty \frac{\sin^2 x\cos x}{a^2\cos^2 2x+b^2\sin^2 2x}\cdot\frac{dx}{x\cos 4x}=-\frac{\pi}{8b}\frac{a}{a^2+b^2}\qquad [a>0,\quad b>0]$$ BI (186)(12)a

11. $$\int_0^\infty \frac{\sin x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x\cos 2x}=\frac{\pi}{2ab}\cdot\frac{b^2-a^2}{b^2+a^2}\qquad [a>0,\quad b>0]$$ BI (186)(4)a

12. $$\int_0^\infty \frac{\sin x\cos x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x\cos 2x}=\frac{\pi}{2a}\cdot\frac{b}{a^2+b^2}\qquad [a>0,\quad b>0]$$ BI (186)(7)a

13. $$\int_0^\infty \frac{\sin x\cos^2 x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x\cos 2x}=\frac{\pi}{2ab}\cdot\frac{b^2}{a^2+b^2}\qquad [a>0,\quad b>0]$$ BI (186)(8)a

14. $$\int_0^\infty \frac{\sin^3 x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x\cos 2x}=-\frac{\pi}{2b}\cdot\frac{a}{a^2+b^2}\qquad [a>0,\quad b>0]$$ BI (186)(10)

15. $$\int_0^\infty \frac{1-\cos x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x\sin x}=\frac{\pi}{2ab}\qquad [a>0,\quad b>0]$$ BI (186)(3)a

3.815

1. $$\int_0^{\pi/2}\frac{x\sin 2x\,dx}{\left(1+a\sin^2 x\right)\left(1+b\sin^2 x\right)}=\frac{\pi}{a-b}\ln\left\{\frac{1+\sqrt{1+b}}{1+\sqrt{1+a}}\cdot\frac{\sqrt{1+a}}{\sqrt{1+b}}\right\}$$
$[a>0,\quad b>0]$ (cf. **3.812** 3)
BI (208)(22)

2. $$\int_0^{\pi/2}\frac{x\sin 2x\,dx}{\left(1+a\sin^2 x\right)\left(1+b\cos^2 x\right)}=\frac{\pi}{a+ab+b}\ln\frac{\left(1+\sqrt{1+n}\right)\sqrt{1+a}}{1+\sqrt{1+a}}$$
$[a>0,\quad b>0]$ (cf. **3.812** 2 and 3)
BI (208)(24)

3. $$\int_0^{\pi/2}\frac{x\sin 2x\,dx}{\left(1+a\cos^2 x\right)\left(1+b\cos^2 x\right)}=\frac{\pi}{a-b}\ln\frac{1+\sqrt{1+a}}{1+\sqrt{1+b}}$$
$[a>0,\quad b>0]$ (cf. **3.812** 2)
BI (208)(23)

4. $$\int_0^{\pi/2}\frac{x\sin 2x\,dx}{\left(1-\sin^2 t_1\cos^2 x\right)\left(1-\sin^2 t_2\cos^2 x\right)}=\frac{2\pi}{\cos^2 t_1-\cos^2 t_2}\ln\frac{\cos\frac{t_1}{2}}{\cos\frac{t_2}{2}}$$
$[-\pi<t_1<\pi,\quad -\pi<t_2<\pi]$
BI (208)(21)

3.816

1. $$\int_0^\pi \frac{x^2\sin 2x}{\left(a^2-\cos^2 x\right)^2}\,dx=\pi^2\frac{\sqrt{a^2-1}-a}{a\left(a^2-1\right)}\qquad [a>1]$$ LI (220)(9)

2.[7] $$\int_0^\pi \frac{\left(a^2-1-\sin^2 x\right)\cos x}{\left(a^2-\cos^2 x\right)^2}x^2\,dx=\frac{\pi}{2}\ln\left|\frac{1-a}{1+a}\right|\qquad \left[a^2>1\right]$$ (cf. **3.812** 5) BI (220)(12)

3.[11] $$\int_0^\pi \frac{a\cos 2x-\sin^2 x}{\left(a+\sin^2 x\right)^2}x^2\,dx=-2\pi\ln\left[2\left(-a+\sqrt{a}\sqrt{a+1}\right)\right]$$
$\Big[a<-1$ and $a>0$. When $a>0$, can write $\sqrt{a}\sqrt{a+1}$ as $\sqrt{a(a+1)}.\Big]$ LI (220)(10)

4.[11] $$\int_0^\pi \frac{a\cos 2x+\sin^2 x}{\left(a-\sin^2 x\right)^2}x^2\,dx = 2\pi\ln\left[2\left(a-\sqrt{a}\sqrt{a+1}\right)\right]$$

$\left[a<0 \text{ and } a>1. \text{ When } a>1, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}.\right]$ (cf. **3.812** 6)

LI (220)(11)

3.817

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4}\cdot\frac{a^2+b^2}{a^3b^3} \qquad [ab>0]$$ BI (181)(12)

2. $$\int_0^\infty \frac{\sin x\cos x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4a^3b} \qquad [ab>0]$$ BI (182)(8)

3. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4ab^3} \qquad [ab>0]$$ BI (181)(15)

4. $$\int_0^\infty \frac{\sin x\cos^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4a^3b} \qquad [ab>0]$$ BI (182)(9)

5. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4}\cdot\frac{a^2+b^2}{a^3b^3} \qquad [ab>0]$$ BI (181)(13)

6. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4}\frac{a^2+b^2}{a^3b^3} \qquad [ab>0]$$ BI (181)(14)

7. $$\int_0^\infty \frac{\sin^2 x\tan x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4ab^3} \qquad [ab>0]$$ BI (182)(11)

8. $$\int_0^\infty \frac{\tan x\cos^2 2x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^2}\cdot\frac{dx}{x} = \frac{\pi}{4a^3b} \qquad [ab>0]$$ BI (182)(10)

3.818

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{3a^4+2a^2b^2+3b^4}{a^5b^5}$$

$[ab>0]$ BI (181)(16)

2. $$\int_0^\infty \frac{\sin x\cos x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{a^2+3b^2}{a^5b^3} \qquad [ab>0]$$ BI (182)(13)

3. $$\int_0^\infty \frac{\sin x\cos^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{a^2+3b^2}{a^5b^3} \qquad [ab>0]$$ BI (182)(14)

4. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{3a^2+b^2}{a^3b^5} \qquad [ab>0]$$ LI (181)(19)

5. $$\int_0^\infty \frac{\sin^3 x\cos x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{64}\cdot\frac{3a^2+b^2}{a^3b^5} \qquad [ab>0]$$ BI (182)(17)

6. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\,\frac{3a^4+2a^2b^2+3b^4}{a^5b^5}$$ $[ab>0]$ BI (181)(17)

7. $$\int_0^\infty \frac{\sin^2 x\tan x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{3a^2+b^2}{a^3b^5}$$ $[ab>0]$ BI (182)(16)

8. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 2x + b^2\sin^2 2x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{3a^4+2a^2b^2+3b^4}{a^5b^5}$$ $[ab>0]$ BI (181)(18)

9. $$\int_0^\infty \frac{\tan x\cos^2 2x}{\left(a^2\cos^2 2x + b^2\sin^2 2x\right)^3}\cdot\frac{dx}{x} = \frac{\pi}{16}\cdot\frac{a^2+3b^2}{a^5b^3}$$ $[ab>0]$ BI (182)(15)

3.819

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{5a^6+3a^4b^2+3a^2b^4+5b^6}{a^7b^7}$$ $[ab>0]$ BI (181)(20)

2. $$\int_0^\infty \frac{\sin x\cos x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^4+2a^2b^2+5b^4}{a^7b^5}$$ $[ab>0]$ BI (182)(18)

3. $$\int_0^\infty \frac{\sin x\cos^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^4+2a^2b^2+5b^4}{a^7b^5}$$ $[ab>0]$ BI (182)(19)

4. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{5a^4+a^2b^2+b^4}{a^5b^7}$$ $[ab>0]$ BI (181)(23)

5. $$\int_0^\infty \frac{\sin^3 x\cos x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^2+b^2}{a^5b^5}$$ $[ab>0]$ BI (182)(26)

6. $$\int_0^\infty \frac{\sin x\cos^3 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^2+5b^2}{a^7b^3}$$ $[ab>0]$ BI (182)(23)

7. $$\int_0^\infty \frac{\sin^3 x\cos^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^2+b^2}{a^5b^5}$$ $[ab>0]$ BI (182)(27)

8. $$\int_0^\infty \frac{\sin x\cos^4 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{a^2+5b^2}{a^7b^3}$$ $[ab>0]$ BI (182)(24)

9. $$\int_0^\infty \frac{\sin^5 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4}\cdot\frac{dx}{x} = \frac{\pi}{32}\cdot\frac{5a^2+b^2}{a^3b^7}$$ $[ab>0]$ BI (181)(24)

10. $$\int_0^\infty \frac{\sin^3 x\cos x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{128}\cdot\frac{5a^4+2a^2b^2+b^4}{a^5b^7}$$
$[ab>0]$ BI (182)(22)

11. $$\int_0^\infty \frac{\sin^5 x\cos^3 x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{512}\cdot\frac{5a^2+b^2}{a^3b^7}$$
$[ab>0]$ BI (182)(30)

12. $$\int_0^\infty \frac{\sin^2 x\tan x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{32}\cdot\frac{5a^4+2a^2b^2+b^4}{a^5b^7}$$
$[ab>0]$ BI (182)(21)

13. $$\int_0^\infty \frac{\sin^4 x\tan x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{32}\cdot\frac{5a^2+b^2}{a^3b^7}$$
$[ab>0]$ BI (182)(29)

14. $$\int_0^\infty \frac{\cos^2 2x\tan x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{32}\cdot\frac{a^4+2a^2b^2+5b^4}{a^7b^5}$$
$[ab>0]$ BI (182)(29)

15. $$\int_0^\infty \frac{\sin^3 4x\tan x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{8}\cdot\frac{a^2+b^2}{a^5b^5}$$
$[ab>0]$ BI (182)(28)

16. $$\int_0^\infty \frac{\cos^4 2x\tan x}{\left(a^2\cos^2 2x+b^2\sin^2 2x\right)^4}\cdot\frac{dx}{x}=\frac{\pi}{32}\cdot\frac{a^2+5b^2}{a^7b^3}$$
$[ab>0]$ BI (182)(25)

3.82–3.83 Powers of trigonometric functions combined with other powers

3.821

1. $$\int_0^\pi x\sin^p x\,dx=\frac{\pi^2}{2^{p+1}}\frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2}$$
$[p>-1]$ BI(218)(7), LO V 121(71)

2. $$\int_0^{r\pi} x\sin^n x\,dx=\frac{\pi^2}{2}\cdot\frac{(2m-1)!!}{(2m)!!}r^2$$
$[n=2m]$
$$=(-1)^{r+1}\pi\frac{(2m)!!}{(2m+1)!!}r$$
$[n=2m+1]$
$[r$ is a natural number$]$ GW (333)(8c)

3.[11] $$\int_0^{\pi/2} x\cos^n x\,dx=\frac{\pi^2}{8}\frac{(n-1)!!}{(n)!!}-\frac{1}{2^{n-2}}\sum_{k=0,m-k\text{ odd}}^{m-1}\binom{n}{k}\frac{1}{(n-2k)^2}$$
$[n=2m]$
$$=\frac{\pi}{2}\frac{(n-1)!!}{(n)!!}-\frac{1}{2^{n-1}}\sum_{k=0}^{m-1}\binom{n}{k}\frac{1}{(n-2k)^2}$$
$[n=2m-1]$
GW (333)(9b)

4. $$\int_0^\pi x\cos^{2m} x\,dx=\frac{\pi^2}{2}\frac{(2m-1)!!}{(2m)!!}$$
BI (218)(10)

5. $$\int_{r\pi}^{s\pi} x\cos^{2m} x\,dx=\frac{\pi^2}{2}\left(s^2-r^2\right)\frac{(2m-1)!!}{(2m)!!}$$
BI (226)(3)

6. $$\int_0^\infty \frac{\sin^p x}{x}\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2}\,\mathrm{B}\left(\frac{p}{2},\frac{p}{2}\right)$$
[p is a fraction with odd numerator and denominator] LO V 278, FI II 808

7. $$\int_0^\infty \frac{\sin^{2n+1} x}{x}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}$$ BI (151)(4)

8. $$\int_0^\infty \frac{\sin^{2n} x}{x}\,dx = \infty$$ BI (151)(3)

9. $$\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{a\pi}{2} \qquad [a>0]$$ LO V 307, 312, FI II 632

10. $$\int_0^\infty \frac{\sin^{2m} ax}{x^2}\,dx = \frac{(2m-3)!!}{(2m-2)!!}\cdot\frac{a\pi}{2} \qquad [a>0]$$ GW (333)(14b)

11. $$\int_0^\infty \frac{\sin^{2m+1} ax}{x^3}\,dx = \frac{(2m-3)!!}{(2m)!!}(2m+1)\frac{a^2\pi}{4} \qquad [a>0]$$ GW (333)(14d)

12. $$\int_0^\infty \frac{\sin^p x}{x^m}\,dx$$
$$= \frac{p}{m-1}\int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}}\cos x\,dx \qquad [p>m-1>0]$$
$$= \frac{p(p-1)}{(m-1)(m-2)}\int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}}\,dx - \frac{p^2}{(m-1)(m-2)}\int_0^\infty \frac{\sin^p x}{x^{m-2}}\,dx \qquad [p>m-1>1]$$
GW (333)(17)

13. $$\int_0^\infty \frac{\sin^{2n} px}{\sqrt{x}}\,dx = \infty$$ BI (177)(5)

14. $$\int_0^\infty \sin^{2n+1} px\frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$$ BI (177)(7)

3.822

1. $$\int_0^{\pi/2} x^p\cos^m x\,dx = -\frac{p(p-1)}{m^2}\int_0^{\pi/2} x^{p-2}\cos^m x\,dx + \frac{m-1}{m}\int_0^{\pi/2} x^p\cos^{m-2} x\,dx$$
$[m>1,\quad p>1]$ GW (333)(9a)

2. $$\int_0^\infty x^{-1/2}\cos^{2n+1}(px)\,dx = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$$ BI (177)(8)

3.823 $$\int_0^\infty x^{\mu-1}\sin^2 ax\,dx = -\frac{\Gamma(\mu)\cos\frac{\mu\pi}{2}}{2^{\mu+1}a^\mu} \qquad [a>0,\quad -2<\operatorname{Re}\mu<0]$$
ET I 319(15), GW(333)(19c)a

3.824

1. $$\int_0^\infty \frac{\sin^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}\left(1-e^{-2a\beta}\right) \qquad [a>0,\quad \operatorname{Re}\beta>0]$$ BI (160)(10)

2. $$\int_0^\infty \frac{\cos^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}\left(1+e^{-2a\beta}\right) \qquad [a>0,\quad \operatorname{Re}\beta>0]$$ BI (160)(11)

3.[7] $$\int_0^\infty \sin^{2m} x \frac{dx}{a^2+x^2} = \frac{(-1)^m}{2^{2m+1}} \cdot \frac{\pi}{2} \left\{ 2^{2m} \sinh^{2m} a - 2 \sum_{k=0}^{m} (-1)^k \binom{2m}{k} \sinh[2(m-k)a] \right\}$$
$[a > 0]$ BI (160)(12)

4.[7] $$\int_0^\infty \sin^{2m+1} x \frac{dx}{a^2+x^2} = \frac{(-1)^{m-1}}{2^{2m+2}a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a] \right.$$
$$\left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m+1-2k)a] \right\}$$
$[a > 0]$ BI (160)(14)

5.[7] $$\int_0^\infty \sin^{2m+1} x \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}} e^{-(2m+1)a} \sum_{k=0}^{m} (-1)^{m+k} \binom{2m+1}{k} e^{2ka}$$
$\left[|\arg a| < \frac{\pi}{2}\right], \quad m = 0, 1, 2, \ldots$

6.[7] $$\int_0^\infty \cos^{2m} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \binom{2m}{m} + \frac{\pi}{2^{2m}} \sum_{k=1}^{m} \binom{2m}{m+k} e^{-2ka}$$
$[a > 0]$ BI (160)(16)

7. $$\int_0^\infty \cos^{2m+1} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \sum_{k=1}^{m} \binom{2m+1}{m+k+1} e^{-(2k+1)a}$$
$[a > 0]$ BI (160)(17)

8. $$\int_0^\infty \cos^{2m+1} x \frac{x\,dx}{a^2+x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m-2k+1)a]$$
$$-\frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a]$$
BI (160)(18)

9. $$\int_0^\infty \frac{\cos^2 ax}{b^2-x^2}\,dx = \frac{\pi}{4b} \sin 2ab \qquad [a > 0, \quad b > 0]$$ BI (161)(10)

10. $$\int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2+x^2}\,dx = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \qquad [a > b]$$
$$= \frac{\pi}{16\beta} \left[1 - e^{-4a\beta} \right] \qquad [a = b]$$
$$= \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(a-b)\beta} - e^{-2a\beta} \right] \qquad [a < b]$$
$[a > 0, \quad b > 0]$, (cf. **3.824** 1 and 3) BI (162)(6)

11. $$\int_0^\infty \frac{x\sin 2ax\cos^2 bx}{\beta^2+x^2}\,dx = \frac{\pi}{8}\left[2e^{-2a\beta}+e^{-2(a+b)\beta}+e^{2(b-a)\beta}\right] \qquad [a>0]$$
$$= \frac{\pi}{8}\left[e^{-4a\beta}+2e^{-2a\beta}\right] \qquad [a=b]$$
$$= \frac{\pi}{8}\left[2e^{-2a\beta}+e^{-2(a+b)\beta}-e^{2(a-b)\beta}\right] \qquad [a<b]$$

LI (162)(5)

3.825

1. $$\int_0^\infty \frac{\sin^2 ax\,dx}{(b^2+x^2)(c^2+x^2)} = \frac{\pi\left(b-c+ce^{-2ab}-be^{-2ac}\right)}{4bc\,(b^2-c^2)}$$

$[a>0, \quad b>0, \quad c>0]$ BI (174)(15)

2. $$\int_0^\infty \frac{\cos^2 ax\,dx}{(b^2+x^2)(c^2+x^2)} = \frac{\pi\left(b-c+be^{-2ac}-ce^{-2ab}\right)}{4bc\,(b^2-c^2)}$$

$[a>0, \quad b>0, \quad c>0]$ BI (175)(14)

3.[3] $$\int_0^\infty \frac{\sin^2 ax\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi\,(c\sin 2ab-b\sin 2ac)}{4bc\,(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0, \quad b\neq c]$$

LI (174)(16)

4.[3] $$\int_0^\infty \frac{\cos^2 ax\,dx}{(b^2-x^2)(c^2-x^2)} = \frac{\pi\,(b\sin 2ac-c\sin 2ab)}{4bc\,(b^2-c^2)} \qquad [a>0, \quad b>0, \quad c>0, \quad b\neq c]$$

LI (175)(15)

3.826

1. $$\int_0^\infty \frac{\sin^2 ax\,dx}{x^2(b^2+x^2)} = \frac{\pi}{4b^2}\left[2a-\frac{1}{b}\left(1-e^{-2ab}\right)\right] \qquad [a>0, \quad b>0]$$ BI (172)(13)

2. $$\int_0^\infty \frac{\sin^2 ax\,dx}{x^2(b^2-x^2)} = \frac{\pi}{4b^2}\left(2a-\frac{1}{b}\sin 2ab\right) \qquad [a>0, \quad b>0]$$ BII (172)(14)

3.827

1.[8] $$\int_0^\infty \frac{\sin^3 ax}{x^\nu}\,dx = \frac{3-3^{\nu-1}}{4}a^{\nu-1}\cos\frac{\nu\pi}{2}\,\Gamma(1-\nu) \qquad [a<\operatorname{Re}\nu<4, \nu\neq 1,2,3]$$

GW (333)(19f)

2.[8] $$\int_0^\infty \frac{\sin^3 ax}{x}\,dx = \frac{\pi}{4}$$ LO V 277

3. $$\int_0^\infty \frac{\sin^3 ax}{x^2}\,dx = \frac{3}{4}a\ln 3$$ BI (156)(2)

4.[8] $$\int_0^\infty \frac{\sin^3 ax}{x^3}\,dx = \frac{3}{8}a^2\pi$$ BI(156)(7)a,LO V 312

5. $$\int_0^\infty \frac{\sin^4 ax}{x^2}\,dx = \frac{a\pi}{4} \qquad [a>0]$$ BI (156)(3)

6. $$\int_0^\infty \frac{\sin^4 ax}{x^3}\,dx = a^2\ln 2$$ BI (156)(8)

7. $\displaystyle\int_0^\infty \frac{\sin^4 ax}{x^4}\,dx = \frac{a^3\pi}{3}$ $[a>0]$ BI(156)(11), LO V 312

8. $\displaystyle\int_0^\infty \frac{\sin^5 ax}{x^2}\,dx = \frac{5}{16}a\,(3\ln 3 - \ln 5)$ BI (156)(4)

9. $\displaystyle\int_0^\infty \frac{\sin^5 ax}{x^3}\,dx = \frac{5}{32}a^2\pi$ $[a>0]$ BI (156)(9)

10. $\displaystyle\int_0^\infty \frac{\sin^5 ax}{x^4}\,dx = \frac{5}{96}a^3\,(25\ln 5 - 27\ln 3)$ BI (156)(12)

11. $\displaystyle\int_0^\infty \frac{\sin^5 ax}{x^5}\,dx = \frac{115}{384}a^4\pi$ $[a>0]$ BI(156)(13), LO V 312

12. $\displaystyle\int_0^\infty \frac{\sin^6 ax}{x^2}\,dx = \frac{3}{16}a\pi$ $[a>0]$ BI (156)(5)

13. $\displaystyle\int_0^\infty \frac{\sin^6 ax}{x^3}\,dx = \frac{3}{16}a^2\,(8\ln 2 - 3\ln 3)$ BI (156)(10)

14. $\displaystyle\int_0^\infty \frac{\sin^6 ax}{x^5}\,dx = \frac{1}{16}a^4\,(27\ln 3 - 32\ln 2)$ BI (156)(14)

15. $\displaystyle\int_0^\infty \frac{\sin^6 ax}{x^6}\,dx = \frac{11}{40}a^5\pi$ $[a>0]$ LO V 312

3.828 In **3.828** 1–21 the restrictions $a>0$, $b>0$, $c>0$ apply.

1.[8] $\displaystyle\int_0^\infty \frac{\sin ax\sin bx}{x}\,dx = \frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|$ $[a\neq b]$ FI II 647

2.[8] $\displaystyle\int_0^\infty \sin ax\sin bx\frac{dx}{x^2} = \frac{1}{2}\pi\min(a,b)$ BI (157)(1)

3.[8] $\displaystyle\int_0^\infty \frac{\sin^2 ax\sin bx}{x}\,dx = \frac{\pi}{4}$ $[b<2a]$

$\displaystyle = \frac{\pi}{8}$ $[b=2a]$

$\displaystyle = 0$ $[b>2a]$

BI (151)(10)

4.[8] $\displaystyle\int_0^\infty \frac{\sin^2 ax\cos bx}{x}\,dx = \frac{1}{4}\ln\frac{4a^2-b^2}{b^2}$ $[2a\neq b]$ BI (151)(12)

5.[8] $\displaystyle\int_0^\infty \frac{\sin^2 ax\cos 2bx}{x^2}\,dx = \frac{1}{2}\pi\max(0,a-b)$

6. $\displaystyle\int_0^\infty \frac{\sin 2ax\cos^2 bx}{x}\,dx = \frac{\pi}{2}$ $[a>b]$

$\displaystyle = \frac{3}{8}\pi$ $[a=b]$

$\displaystyle = \frac{\pi}{4}$ $[a<b]$

BI (151)(9)

7.[8] $$\int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x^2}\,dx = \frac{\pi}{16}\left(|b-2a-c| - |2a-b-c| + 2c\right)$$
$[a>0, \quad 0<c\le b]$

BI(157)(9)a, ET I 79(15)

8.[8] $$\int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x}\,dx = \frac{1}{8}\ln\left|\frac{(b+c)^2(2a-b+c)(2a+b-c)}{(b-c)^2(2a+b+c)(2a-b-c)}\right|$$
$[b\ne c, \quad 2a+c\ne b, \quad 2a+b\ne c, \quad 2a\ne b+c]$ LI (152)(2)

9. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^2}\,dx = \frac{\pi}{4}a \qquad [0\le a\le b]$$
$$= \frac{\pi}{4}b \qquad [0\le b\le a]$$

BI (157)(3)

10.[8] $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4}\,dx = \frac{1}{6}\pi \min\left(a^2,b^2\right)\left[3\max(a,b)-\min(a,b)\right]$$ BI (157)(27)

11.[8] $$\int_0^\infty \frac{\sin^2 ax \cos^2 bx}{x^2}\,dx = \frac{1}{4}\pi\left[a+\max(0,a-b)\right]$$ BI (157)(6)

12. $$\int_0^\infty \frac{\sin^3 ax \sin 3bx}{x^4}\,dx = \frac{a^3\pi}{2} \qquad [b>a]$$
$$= \frac{\pi}{16}\left[8a^3-9(a-b)^3\right] \qquad [a\le 3b\le 3a]$$ BI (157)(28)
$$= \frac{9b\pi}{8}\left(a^2-b^2\right) \qquad [3b\le a]$$ LI (157)(28)

13. $$\int_0^\infty \frac{\sin^3 ax \cos bx}{x}\,dx = 0 \qquad [b>3a]$$
$$= -\frac{\pi}{16} \qquad [b=3a]$$
$$= -\frac{\pi}{8} \qquad [3a>b>a]$$
$$= \frac{\pi}{16} \qquad [b=a]$$
$$= \frac{\pi}{4} \qquad [a>b]$$
$[a>0, \quad b>0]$ BI (151)(15)

14.[10] $$\int_0^\infty \frac{\sin^3 ax \cos 3bx}{x^2}\,dx = \frac{3}{16}\Big(a\ln 81 - 2(a-3b)\ln(a-3b) + 2(a-b)\ln(a-b)$$
$$+ 2(a+b)\ln(a+b) - 2(a+3b)\ln(a+3)\Big)$$
$[\operatorname{Im} a = 0, \quad \operatorname{Im} b = 0]$ MC

15. $$\int_0^\infty \frac{\sin^3 ax \cos bx}{x^3}\,dx = \frac{\pi}{8}\left(3a^2 - b^2\right) \qquad [b < a]$$
$$= \frac{\pi b^2}{4} \qquad [a = b]$$
$$= \frac{\pi}{16}(3a - b)^2 \qquad [a < b < 3a]$$
$$= 0 \qquad [3a < b]$$
$$[a > 0, \quad b > 0]$$ BI(157)(19), ET I 19(10)

16. $$\int_0^\infty \frac{\sin^3 ax \sin bx}{x^4}\,dx = \frac{b\pi}{24}\left(9a^2 - b^2\right) \qquad [0 < b \le a]$$
$$= \frac{\pi}{48}\left[24a^3 - (3a - b)^3\right] \qquad [0 < a \le b \le 3a]$$
$$= \frac{\pi a^3}{2} \qquad [0 < 3a \le b]$$
ET I 79(16)

17. $$\int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x}\,dx = \frac{\pi}{8} \qquad [2b > 3a]$$
$$= \frac{5\pi}{32} \qquad [2b = 3a]$$
$$= \frac{3\pi}{16} \qquad [3a > 2b > a]$$
$$= \frac{3\pi}{32} \qquad [2b = a]$$
$$= 0 \qquad [a > 2b]$$
$$[a > 0, \quad b > 0]$$ BI (151)(14)

18.[8] $$\int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x}\,dx = \frac{1}{16}\ln\left|\frac{(2a+b)^3(b-2a)^3(2a+3b)(3b-2a)}{9b^8}\right|$$
$$[2a \ne b, \quad 2a \ne 3b]$$ BI (151)(13)

19.[11] $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin^2 cx}{x}\,dx$$
$$= \frac{\pi}{32}\Big(4\,\mathrm{sign}(c) - 2\,\mathrm{sign}(2b+c) + 2\,\mathrm{sign}(2b-c) + \mathrm{sign}(2a-2b+c) - \mathrm{sign}(2a-2b-c)$$
$$+ 2\,\mathrm{sign}(2a-c) + \mathrm{sign}(2a+2b+c) - \mathrm{sign}(2a+2b-c) - 2\,\mathrm{sign}(2a+c)\Big)$$
$$[\mathrm{Im}\,a = 0, \quad \mathrm{Im}\,b = 0, \quad \mathrm{Im}\,c = 0]$$ MC

20. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx\,dx}{x^2}$$
$$= \frac{a-b-c}{16}\ln 4(a-b-c)^2 - \frac{a+b+c}{16}\ln 4(a+b+c)^2 + \frac{a+b-c}{16}\ln 4(a+b-c)^2$$
$$-\frac{a-b+c}{16}\ln 4(a-b+c)^2 + \frac{a+c}{8}\ln 4(a+c)^2 - \frac{a-c}{8}\ln 4(a-c)^2$$
$$+\frac{b+c}{8}\ln 4(b+c)^2 - \frac{b-c}{8}\ln 4(b-c)^2 - \frac{1}{2}c\ln 2c$$
$$[a > 0, \quad b > 0, \quad c > 0]$$ BI (157)(10)

21.[8] $$\int_0^\infty \frac{\sin^2 ax \sin^3 bx}{x^3}\,dx = \frac{3b^2\pi}{16} \qquad [2a > 3b]$$
$$= \frac{a^2\pi}{12} \qquad [2a = 3b]$$
$$= \frac{6b^2 - (3b-2a)^2}{32}\pi \qquad [3b > 2a > b]$$
$$= \frac{a^2\pi}{4} \qquad [b \geq 2a]$$

BI (157)(18)

3.829

1. $$\int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}}\,dx = \frac{\pi}{2^n(n+1)!}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{k}(n-2k)^{n+1}$$ GW (333)(63)

2. $$\int_0^\infty \left(1-\cos^{2m-1}x\right)\frac{dx}{x^2} = \int_0^\infty \left(1-\cos^{2m}x\right)\frac{dx}{x^2} = \frac{m\pi}{2^{2m}}\binom{2m}{m}$$ BI (158)(7, 8)

3.831

1. $$\int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x}\,dx = \frac{(2n-1)!!}{(2n)!!}\ln\frac{b}{a} \qquad [ab > 0, \quad n = 1, 2, \ldots]$$ FI II 651

2. $$\int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x}\,dx = \left[1 - \frac{(2n-1)!!}{(2n)!!}\right]\ln\frac{b}{a} \qquad [ab > 0, \quad n = 0, 1, \ldots]$$ FI II 651

3. $$\int_0^\infty \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x}\,dx = \ln\frac{b}{a} \qquad [ab > 0, \quad m = 0, 1, \ldots]$$ FI II

4. $$\int_0^\infty \frac{\cos^m ax \cos max - \cos^m bx \cos mbx}{x}\,dx = \left(1 - \frac{1}{2^m}\right)\ln\frac{b}{a}$$
$$[ab > 0, \quad m = 0, 1, \ldots]$$ LI (155)(8)

3.832

1. $$\int_0^{\pi/2} x\cos^{p-1} x \sin ax\,dx = \frac{\pi}{2^{p+1}}\Gamma(p)\frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right)\Gamma\left(\frac{p-a+1}{2}\right)}$$
$$[p > 0, \quad -(p+1) < a < p+1]$$

BI (205)(6)

2.[3] $$\int_0^\infty \sin^{2m+1} x \sin 2mx \frac{dx}{a^2+x^2} = \frac{(-1)^m\pi}{2^{2m+1}a}\left[\left(1-e^{-2a}\right)^{2m} - 1\right]\sinh a$$
$$[a > 0, \quad m = 0, 1. \ldots]$$ BI (162)(17)

3. $$\int_0^\infty \sin^{2m-1} x \sin[(2m-1)x]\frac{dx}{a^2+x^2} = \frac{(-1)^{m+1}\pi}{2^{2m}a}\left(1-e^{-2a}\right)^{2m-1}$$
$$[a > 0, \quad m = 1, 2, \ldots]$$ BI (162)(11)

4. $$\int_0^\infty \sin^{2m-1} x \sin[(2m+1)x]\frac{dx}{a^2+x^2} = \frac{(-1)^{m-1}\pi}{2^{2m}a}e^{-2a}\left(1-e^{-2a}\right)^{2m-1}$$
$$[a > 0, \quad m = 1, 2, \ldots]$$ BI (162)(12)

5. $$\int_0^\infty \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \sinh^{2m+1} a$$
$[a > 0]$ BI (162)(18)

6.[3] $$\int_0^\infty \sin^{2m} x \sin[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a \left[\left(1-e^{-2a}\right)^{2m} - \left(1+e^{-2a}\right)\right]$$
$[a \geq 0, \quad m = 0, 1, \ldots]$ BI (162)(13)

7. $$\int_0^\infty \sin^{2m} x \sin(2mx) \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} \left[\left(1-e^{-2a}\right)^{2m} - 1\right]$$
$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(14)

8. $$\int_0^\infty \sin^{2m} x \sin[(2m+2)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} \left(1-e^{-2a}\right)^{2m}$$
$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(15)

9. $$\int_0^\infty \sin^{2m} x \sin 4mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \sinh^{2m} a$$
$[a > 0, \quad m = 1, 2, \ldots]$ BI (162)(16)

10. $$\int_0^\infty \sin^{2m} x \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$
$[m = 1, 2, \ldots]$ GW (333)(15a)

11. $$\int_0^\infty \sin^{2m} x \cos[(2m-1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m} a} \left[\left(1-e^{-2a}\right)^{2m-1} - 1\right] \sinh a$$
$[a > 0, \quad m = 1, 2, \ldots]$ BI (162)(25)

12. $$\int_0^\infty \sin^{2m} x \cos(2mx) \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} \left(1-e^{-2a}\right)^{2m}$$
$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(26)

13. $$\int_0^\infty \sin^{2m} x \cos[(2m+2)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} \left(1-e^{-2a}\right)^{2m}$$
$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(27)

14. $$\int_0^\infty \sin^{2m} x \cos 4mx \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \sinh^{2m} a$$
$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(28)

15. $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2}$$
$[m = 0, 1, \ldots]$ GW (333)(15)

16.[3] $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$
$[m = 1, 2, \ldots]$ GW (333)(15b)

17. $$\int_0^\infty \sin^{2m-1} x \cos[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} \left[\left(1-e^{-2a}\right)^{2m-1} - 1\right]$$
$[m = 1, 2, \ldots, \quad a > 0]$ BI (162)(23)

18.[3] $$\int_0^\infty \sin^{2m+1} x \cos 2mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \left\{e^a \left[\left(1-e^{-2a}\right)^{2m+1} - 1\right] - e^{-a}\right\}$$
$[m = 0, 1, \ldots, \quad a \geq 0]$ BI (162)(29)

19. $$\int_0^\infty \sin^{2m-1} x \cos[(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} \left(1-e^{-2a}\right)^{2m-1}$$
$[m=1,2,\ldots, \quad a>0]$ BI (162)(24)

20. $$\int_0^\infty \sin^{2m+1} x \cos[2(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1}\pi}{2} e^{-2(2m+1)a} \sinh^{2m+1} a$$
$[m=0,1,\ldots, \quad a>0]$ BI (162)(30)

21. $$\int_0^\infty \cos^m x \sin mx \frac{x\,dx}{a^2+x^2} = \frac{1}{2^{m+1}a} \sum_{k=1}^m \binom{m}{k} \left[e^{-2ka}\,\mathrm{Ei}(2ka) - e^{2ka}\,\mathrm{Ei}(-2ka)\right]$$
$[a>0]$ BI (162)(8)

22. $$\int_0^\infty \cos^n sx \sin nsx \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{n+1}} \left[\left(1+e^{-2as}\right)^n - 1\right]$$
$[s>0, \quad \operatorname{Re} a>0, \quad n\geq 0]$ BI (163)(9)

23. $$\int_0^\infty \cos^n sx \sin nsx \frac{x\,dx}{a^2-x^2} = \frac{\pi}{2}\left(2^{-n} - \cos^n as \cos nas\right)$$
$[n=0,1,\ldots]$ BI (166)(10)

24. $$\int_0^\infty \cos^{m-1} x \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} e^{-2a} \left(1+e^{-2a}\right)^{m-1}$$
$[a>0, \quad m=1,2,\ldots]$ BI (163)(6)

25. $$\int_0^\infty \cos^m x \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{m+1}} e^{-a} \left(1+e^{-2a}\right)^m$$
$[m=0,1,\ldots, \quad a>0]$ BI (163)(10)

26.[3] $$\int_0^\infty \cos^m x \sin[(m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} \cosh a \left[\left(1+e^{-2a}\right)^{m-1} - 1\right]$$
$[m=0,1,\ldots, \quad a\geq 0]$ BI (163)(7)

27.[11] $$\int_0^\infty \cos^m x \sin(3mx) \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2} e^{-3ma} \cosh^m a$$
$[a>0, \quad m=1,2,\ldots]$ BI (163)(11)

28. $$\int_0^\infty \cos^n sx \cos nxs \frac{dx}{a^2+x^2} = \frac{\pi}{2^{n+1}a} \left(1+e^{-2as}\right)^n$$
$[n=0,1,\ldots]$ BI (163)(16)

29. $$\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2-x^2} = \frac{\pi}{2a} \cos^n as \sin nas$$
$[n=0,1,\ldots]$

30. $$\int_0^\infty \cos^{m-1} x \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^m a} e^{-2a} \left(1+e^{-2a}\right)^{m-1}$$
$[m=1,2,\ldots, \quad a>0]$ BI (163)(14)

31. $$\int_0^\infty \cos^m x \cos[(m-1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1}a} e^a \left[\left(1+e^{-2a}\right)^m - \left(1-e^{-2a}\right)\right]$$
$[m=0,1,\ldots, \quad a>0]$ BI (163)(15)

32. $$\int_0^\infty \cos^m x \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1}a} e^{-a} \left(1+e^{-2a}\right)^m$$
$[m = 0, 1, \ldots, \quad a > 0]$ BI (163)(17)

33. $$\int_0^\infty \sin^p x \cos x \frac{dx}{x^q} = \frac{p}{q-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^\infty \frac{\sin^{p+1} x}{x^{q-1}} dx \qquad [p > q-1 > 0]$$
$$= \frac{p(p-1)}{(q-1)(q-2)} \int_0^\infty \sin^{p-2} x \cos x \frac{dx}{x^{q-2}}$$
$$- \frac{(p+1)^2}{(q-1)(q-2)} \int_0^\infty \sin^p x \cos x \frac{dx}{x^{q-2}} \qquad [p > q-1 > 1]$$
GW (333)(18)

34. $$\int_0^\infty \cos^{2m} x \cos 2nx \sin x \frac{dx}{x} x = \int_0^\infty \cos^{2m-1} x \cos 2nx \sin \frac{dx}{x} x = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}$$
BI (152)(5, 6)

35. $$\int_0^\infty \cos^p ax \sin bx \cos x \frac{dx}{x} = \frac{\pi}{2} \qquad [b > ap, \quad p > -1]$$ BI (153)(12)

36. $$\int_0^\infty \cos^p ax \sin pax \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} \left(2^p - 1\right) \qquad [p > -1]$$ BI (153)(2)

37. $$\int_0^\infty \frac{dx}{x^2} \left(\prod_{k=1}^n \cos^{p_k} a_k x\right) \sin bx \sin x = \frac{\pi}{2} \qquad \left[b > \sum_{k=1}^n a_k p_k, \quad a_k > 0, \quad p_k > 0\right]$$
BI (157)(15)

3.833

1.[10] $$\int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} = \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!} \pi$$
BI (151)(24, 25)
$$= \frac{1}{2} \mathrm{B}\left(m + \frac{1}{2}, n + \frac{1}{2}\right)$$
GW (333)(24)

2. $$\int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}$$ LI (152)(4)

3.834

1. $$\int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left|\frac{1-a}{1+a}\right|^{2m-1} - \sum_{k=0}^{2m} (-1)^k \binom{m - \frac{1}{2}}{k} \left(\frac{4a}{(1+a)^2}\right)^k \right\}$$
$[|a| \neq 1]$ GW (333)(62a)

2. $$\int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1-2a\cos x+a^2)^p}\cdot\frac{dx}{x}$$
$$= \frac{n!\pi}{2^{n+1}(2m+n+1)!(1+a)^{2p}}\sum_{k=0}^{n}\frac{(-1)^k(2m+2n-2k+1)!!(2m+2k-1)!!}{k!(n-k)!}$$
$$\times F\left(m+n-k+\frac{3}{2}, p; 2m+n+2; \frac{4a}{(1+a)^2}\right)$$
$[a \neq \pm 1]$ GW (333)(62)

3.835

1. $$\int_0^\infty \frac{\cos^{2m} x\cos 2mx\sin x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{2}\frac{b^{2m-1}}{a(a+b)^{2m}}$$ $[ab>0]$ BI (182)(31)a

2. $$\int_0^\infty \frac{\cos^{2m-1} x\cos 2mx\sin x}{a^2\cos^2 x+b^2\sin^2 x}\cdot\frac{dx}{x} = \frac{\pi}{2a}\frac{b^{2m-1}}{(a+b)^{2m}}$$ $[ab>0]$ LI (182)(32)a

3.836

1. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$$ $[m \geq n]$ LI (159)(12)

2.[11] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos mx\,dx = \frac{n\pi}{2^n}\sum_{k=0}^{\lfloor \frac{1}{2}(m+n)\rfloor}\frac{(-1)^k(n+m-2k)^{n-1}}{k!(n-k)!}$$ $[0 \leq m < n]$
$$= 0$$ $[m \geq n \geq 2]$
$$= \frac{\pi}{4}$$ $[m = n = 1]$
GI(159)(14), ET I 20(11)

3. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin nx\cos x\frac{dx}{x} = \frac{\pi}{2}$$ $[n \geq 1]$ BI (159)(20)

4.[8] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin(anx)}{x}\,dx = \frac{\pi}{2}\left[1-\frac{1}{2^{n-1}n!}\sum_{k=0}^{\lfloor \frac{1}{2}n(1+a)\rfloor}(-1)^k\binom{n}{k}(n+an-2k)^n\right]$$
[all real a, $n \geq 1$] ET I 20(11)

5.[10] $$I_n(b) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx\,dx = n\left(2^{n-1}n!\right)^{-1}\sum_{k=0}^{\lfloor r\rfloor}(-1)^k\binom{n}{k}(n-b-2k)^{n-1}$$
where $0 \leq b < n$, $n \geq 1$, $r = (n-b)/2$, and $\lfloor r\rfloor$ is the largest integer contained in r
LO V 340(14)

6.[11] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos anx\,dx = 0$$ $[a \leq -1$ or $a \geq 1, \quad n \geq 2;$ for $n = 1$ see **3.741** 2$]$

3.837

1. $$\int_0^{\pi/2} \frac{x^2\,dx}{\sin^2 x} = \pi\ln 2$$ BI (206)(9)

2. $$\int_0^{\pi/4} \frac{x^2\,dx}{\sin^2 x} = -\frac{\pi^2}{16}+\frac{\pi}{4}\ln 2+\boldsymbol{G} = 0.8435118417\ldots$$ BI (204)(10)

3. $$\int_0^{\pi/4} \frac{x^2\,dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 - \boldsymbol{G}$$ GW (333)(35a)

4. $$\int_0^{\pi/4} \frac{x^{p+1}}{\sin^2 x}\,dx = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1)\left(\frac{\pi}{4}\right)^p \left\{\frac{1}{p} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)}\,\zeta(2k)\right\}$$ $[p>0]$ LI (204)(14)

5. $$\int_0^{\pi/2} \frac{x^2\cos x}{\sin^2 x}\,dx = -\frac{\pi^2}{4} + 4\boldsymbol{G} = 1.1964612764\ldots$$ BI (206)(7)

6. $$\int_0^{\pi/2} \frac{x^3\cos x}{\sin^3 x}\,dx = -\frac{\pi^3}{16} + \frac{3}{2}\pi\ln 2$$ BI (206)(8)

7. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\sin^{2n} x\,\frac{dx}{x^m} = 0$$ $\left[n > \frac{m-1}{2},\quad m>0\right]$ BI (180)(16)

8. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\sin^{2n+1} x\,\frac{dx}{x^m} = 0$$ $\left[n > \frac{m-2}{2},\quad m>0\right]$ BI (180)(17)

9. $$\int_0^{1} \frac{x\,dx}{\cos ax\cos[a(1-x)]} = \frac{1}{a}\operatorname{cosec} a\cdot\ln\sec a$$ $\left[a<\frac{\pi}{2}\right]$ BI (149)(20)

10.[3] $$\int_0^{\pi} \frac{x\sin(2n+1)x}{\sin x}\,dx = \frac{1}{2}\pi^2$$ $[n=0,1,2,\ldots]$

11.[3] $$\int_0^{\pi} \frac{x\sin 2nx}{\sin x}\,dx = -4\sum_{k=1}^{n}(2k-1)^{-2}$$ $[n=1,2,3,\ldots]$

3.838

1. $$\int_0^{\pi/2} \frac{x\cos^{p-1} x}{\sin^{p+1} x}\,dx = \frac{\pi}{2p}\sec\frac{\pi p}{2}$$ $[p<1]$ BI (206)(13)a

2. $$\int_0^{\pi/4} \frac{x\sin^{p-1} x}{\cos^{p+1} x}\,dx = \frac{\pi}{4p} - \frac{1}{2p}\beta\left(\frac{p+1}{2}\right)$$ $[p>-1]$ LI (204)(15)

3. $$\int_0^{\pi/4} \frac{x\sin^{2m-1} x}{\cos^{2m+1} x}\,dx = \frac{\pi}{8m}(1-\cos m\pi) + \frac{1}{2m}\sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{2m-2k-1}$$ BI (204)(17)

4. $$\int_0^{\pi/4} \frac{x\sin^{2m} x}{\cos^{2m+2} x}\,dx = \frac{1}{2(2m+1)}\left[\frac{\pi}{2} + (-1)^{m-1}\ln 2 + \sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{m-k}\right]$$ BI (204)(16)

3.839

1.[11] $$\int_0^{\pi/4} x\tan^2 x\,dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$ BI (204)(3)

2. $$\int_0^{\pi/4} x\tan^3 x\,dx = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8}\ln 2 - \frac{1}{2}\boldsymbol{G}$$ BI (204)(7)

3. $$\int_0^{\pi/4} \frac{x^2\tan x}{\cos^2 x}\,dx = \frac{1}{2}\ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16}$$ (cf. **3.839** 1) BI (204)(13)

4. $$\int_0^{\pi/4} \frac{x^2 \tan^2 x}{\cos^2 x}\, dx = \frac{1}{3}\left(1 - \frac{\pi}{4}\ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + \boldsymbol{G}\right) \qquad \text{(cf. } \mathbf{3.839}\ 2)$$ BI (204)(12)

5. $$\int_0^{\pi/2} x\cos^p x \tan x\, dx = \frac{\pi}{2^{p+1}p} \cdot \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2} \qquad [p > -1]$$ BI (205)(3)

6. $$\int_0^{\pi/2} x\sin^p x \cot x\, dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p}\,\mathrm{B}\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$
$$[p > -1]$$ BI (206)(11)

7. $$\int_0^{\infty} \sin^{2n} x \tan x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}$$ GW (333)(16)

8. $$\int_0^{\infty} \cos^s rx \tan qx \frac{dx}{x} = \frac{\pi}{2} \qquad [s > -1]$$ BI (151)(26)

9. $$\int_0^{\infty} \frac{\cos[(2n-1)x]}{\cos x} \cdot \left(\frac{\sin x}{x}\right)^{2n} dx = (-1)^{n-1}\frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1}\pi|B_{2n}|$$ BI (180)(15)

10. $$\int_0^{\infty} \tan^r px \frac{dx}{q^2+x^2} = \frac{\pi}{2q}\sec\frac{r\pi}{2}\tanh^r pq \qquad [r^2 < 1]$$ BI (160)(19)

3.84 Integrals containing $\sqrt{1-k^2\sin^2 x}$, $\sqrt{1-k^2\cos^2 x}$, and similar expressions

Notation: $k' = \sqrt{1-k^2}$

3.841

1. $$\int_0^{\infty} \sin x\sqrt{1-k^2\sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(8)

2. $$\int_0^{\infty} \sin x\sqrt{1-k^2\cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(20)

3. $$\int_0^{\infty} \tan x\sqrt{1-k^2\sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(9)

4. $$\int_0^{\infty} \tan x\sqrt{1-k^2\cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(21)

3.842

1.[11] $$\int_0^{\infty} \frac{\sin x}{\sqrt{1+\sin^2 x}}\frac{dx}{x}$$
$$= \int_0^{\infty} \frac{\tan x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x}$$
$$= \int_0^{\infty} \frac{\sin x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771$$
BI (183)(4, 5, 9, 10)

2. $$\int_u^{\frac{\pi}{2}} \frac{x\cos x\, dx}{\sqrt{\sin^2 x - \sin^2 u}} = \frac{\pi}{2}\ln(1+\cos u)$$ BI (226)(4)

3. $$\int_0^\infty \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \boldsymbol{K}(k)$$
BI (183)(12, 13, 21, 22)

4. $$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}\,dx = \frac{1}{2k^2}\left[-\pi k' + 2\,\boldsymbol{E}(k)\right]$$ BI (211)(1)

5. $$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\,dx = \frac{1}{2k^2}\left[\pi - 2\,\boldsymbol{E}(k)\right]$$ BI (214)(1)

6. $$\int_0^{\alpha} \frac{x\sin x\,dx}{\cos^2 x\sqrt{\sin^2\alpha-\sin^2 x}} = \frac{\pi\sin^2\frac{\alpha}{2}}{\cos^2\alpha}$$ LO III 284

7. $$\int_0^{\beta} \frac{x\sin x\,dx}{\left(1-\sin^2\alpha\sin^2 x\right)\sqrt{\sin^2\beta-\sin^2 x}} = \frac{\pi\ln\dfrac{\cos\alpha+\sqrt{1-\sin^2\alpha\sin^2\beta}}{2\cos\beta\cos^2\frac{\alpha}{2}}}{2\cos\alpha\sqrt{1-\sin^2\alpha\sin^2\beta}}$$ LO III 284

3.843

1. $$\int_0^\infty \tan x\sqrt{1-k^2\sin^2 2x}\,\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(10)

2. $$\int_0^\infty \tan x\sqrt{1-k^2\cos^2 2x}\,\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(22)

3.[11] $$\int_0^\infty \frac{\tan x}{\sqrt{1+\sin^2 2x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1+\cos^2 2x}}\frac{dx}{x} = \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771$$
BI (183)(6, 11)

4. $$\int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2 2x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x} = \boldsymbol{K}(k)$$ BI (183)(14, 23)

3.844

1. $$\int_0^\infty \frac{\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k)-\boldsymbol{E}(k)\right]$$ BI (185)(20)

2. $$\int_0^\infty \frac{\sin x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k)-\boldsymbol{E}(k)\right]$$ BI (185)(21)

3. $$\int_0^\infty \frac{\sin x\cos^3 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k)-2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(22)

4. $$\int_0^\infty \frac{\sin x\cos^4 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k)-2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(23)

5. $$\int_0^\infty \frac{\sin^3 x\cos x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k)-2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(24)

6. $$\int_0^\infty \frac{\sin^3 x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k)-2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(25)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2\cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (184)(16)

8. $$\int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2\cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(18)

3.845

1.[11] $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.5990701174$$ BI (185)(6)

2.[11] $$\int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.5990701174$$ BI (185)(7)

3.[11] $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$$ BU (184)(8)

3.846

1. $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(9)

2. $$\int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(10)

3. $$\int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(11)

4. $$\int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(12)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(13)

6. $$\int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(14)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (184)(9)

8. $$\int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(11)

3.847[11] $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$$

BI (185)(3, 4)

3.848

1. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (185)(15)

2. $$\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (184)(12)

3. $$\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(13)

4. $$\int_0^\infty \frac{\sin^2 4x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (184)(17)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(26)

6. $$\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (184)(19)

7. $$\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(20)

3.849

1.[11] $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.1779896649$$ BI (185)(8)

2.[11] $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8}\left[2\,\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.1497675293$$ BI (185)(5)

3.[11] $$\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$$ BI (184)(7)

3.85–3.88 Trigonometric functions of more complicated arguments combined with powers

3.851

5. $$\int_0^\infty \sin\left(ax^2\right)\cos(bx)\frac{dx}{x^2} = \frac{b\pi}{2}\left\{S\left(\frac{b}{2\sqrt{a}}\right) - C\left(\frac{b}{2\sqrt{a}}\right) + \sqrt{a\pi}\sin\left(\frac{b^2}{4a}+\frac{\pi}{4}\right)\right\}$$
$[a>0, \quad b>0]$, (cf. **3.691** 7)
ET I 23(3)a

3.852

1. $$\int_0^\infty \frac{\sin\left(ax^2\right)}{x^2}\,dx = \sqrt{\frac{a\pi}{2}} \qquad [a \geq 0]$$ BI (177)(10)a

2. $$\int_0^\infty \sin\left(ax^2\right)\cos\left(bx^2\right)\frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\sqrt{a+b}+\sqrt{a-b}\right) \qquad [a>b>0]$$
$$= \frac{1}{2}\sqrt{\pi a} \qquad [b=a\geq 0]$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\sqrt{a+b}-\sqrt{b-a}\right)$$
$[b>a>0]$, (cf. **3.852** 1) BI (177)(23)

3. $$\int_0^\infty \frac{\sin^2\left(a^2x^2\right)}{x^4}\,dx = \frac{2\sqrt{\pi}}{3}a^3 \qquad [a \geq 0]$$ GW (333)(19e)

4.[10] $$\int_0^\infty \frac{\sin^3\left(a^2x^2\right)}{x^2}\,dx = \frac{a}{4}\sqrt{\frac{\pi}{2}}\left(3-\sqrt{3}\right) \qquad \left[\operatorname{Im} a^2 = 0\right]$$ MC

5. $$\int_0^\infty \left(\sin^2 x - x^2\cos x^2\right)\frac{dx}{x^4} = \frac{1}{3}\sqrt{\frac{\pi}{2}}$$ BI (178)(8)

6. $$\int_0^\infty \left(\cos^2 x - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}$$ BI (173)(22)

3.853

1. $$\int_0^\infty \frac{\sin\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right) - \sin\left(a\beta^2\right)\right]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 219(33)a

2. $$\int_0^\infty \frac{\cos\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\cos\left(a\beta^2\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 221(51)a

3. $$\int_0^\infty \frac{x^2\sin\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\beta\pi}{2}\left[\sin\left(a\beta^2\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) + \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right] - \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 219(32)a

4. $$\int_0^\infty \frac{x^2\cos\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2}\left\{\cos\left(a\beta^2\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right\}$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 221(50)a

3.854

1. $$\int_0^\infty \left(\cos\left(ax^2\right) - \sin\left(ax^2\right)\right)\frac{dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b^3\sqrt{2}} \qquad [a>0, \quad b>0]$$
LI (178)(11)a, BI (168)(25)

2. $$\int_0^\infty \left(\cos\left(ax^2\right) + \sin\left(ax^2\right)\right)\frac{x^2\,dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b\sqrt{2}} \qquad [a>0, b>0]$$ LI (178)(12)

3. $$\int_0^\infty \left(\cos\left(ax^2\right) + \sin\left(ax^2\right)\right)\frac{x^2\,dx}{\left(x^4+b^4\right)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3}\left(a+\frac{1}{2b^2}\right)$$
$[a>0, \quad b>0]$ LI (178)(14)

4. $$\int_0^\infty \left(\cos\left(ax^2\right) - \sin\left(ax^2\right)\right)\frac{x^4\,dx}{\left(x^4+b^4\right)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b}\left(\frac{1}{2b^2}-a\right)$$
$[a>0, \quad b>0]$ BI (178)(15)

3.855

1. $$\int_0^\infty \frac{\sin\left(ax^2\right)}{\sqrt{\beta^2+x^4}}\,dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}}\,I_{\frac{1}{4}}\left(\frac{a\beta}{2}\right)K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0,\quad \operatorname{Re}\beta>0]$$ ET I 66(28)

2. $$\int_0^\infty \frac{\cos\left(ax^2\right)}{\sqrt{\beta^2+x^4}}\,dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}}\,I_{-\frac{1}{4}}\left(\frac{a\beta}{2}\right)K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0,\quad \operatorname{Re}\beta>0]$$ ET I 9(22)

3. $$\int_0^u \frac{\sin\left(a^2x^2\right)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{\frac{1}{4}}\left(\frac{a^2}{u^2}2\right)\right]^2 \qquad [a>0]$$ ET I 66(29)

4. $$\int_u^\infty \frac{\sin\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}}\,J_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\,Y_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)$$
$$[a>0]$$ ET I 66(30)

5. $$\int_0^u \frac{\cos\left(a^2x^2\right)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\right]^2$$ ET I 9(23)

6. $$\int_u^\infty \frac{\cos\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}}\,J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\,Y_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)$$ ET I 10(24)

3.856

1. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}}\sin\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\,I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\nu<\frac{3}{2},\quad |\arg\beta|<\frac{\pi}{4}\right]$$ ET I 71(23)

2. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}}\cos\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\,I_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)K_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\nu<\frac{3}{2},\quad |\arg\beta|<\frac{\pi}{4}\right]$$ ET I 12(16)

3. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}-x^2\right)^\nu}{\sqrt{\beta^4+x^4}}\cos\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\,I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[\operatorname{Re}\nu>-\frac{3}{2},\quad |\arg\beta|<\frac{\pi}{4}\right]$$
ET I 12(17)

4. $$\int_0^\infty \frac{\sin\left(a^2x^2\right)\,dx}{\sqrt{\beta^4+x^4}\sqrt{x^2+\sqrt{\beta^4+x^4}}} = \frac{\sinh\frac{a^2\beta^2}{2}}{\sqrt{2}\beta^2}\,K_0\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[|\arg\beta|<\frac{\pi}{4}\right]$$ ET I 66(32)

5. $$\int_0^\infty \frac{\cos\left(a^2x^2\right)\,dx}{\sqrt{\beta^4+x^4}\sqrt{\left(x^2+\sqrt{\beta^4+x^4}\right)^3}} = \frac{\sinh\frac{a^2\beta^2}{2}}{2\sqrt{2}\beta^4}\,K_1\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[|\arg\beta|<\frac{\pi}{4}\right]$$ ET I 10(27)

6. $$\int_0^\infty \frac{\sqrt{\sqrt{\beta^4+x^4}+x^2}}{\sqrt{\beta^4+x^4}} \sin\left(a^2x^2\right)\,dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2\beta^2}{2}} I_0\left(\frac{a^2\beta^2}{2}\right)$$
$$\left[|\arg\beta| < \frac{\pi}{4}\right]$$ ET I 67(33)

3.857

1. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2-R_1}{R_2+R_1}} \sin\left(ax^2\right)\,dx = \frac{1}{2\sqrt{b}} K_0(ac)\sin ab$$
$$\left[R_1=\sqrt{c^2+(b-x^2)^2},\quad R_2=\sqrt{c^2+(b+x^2)^2},\quad a>0,\quad c>0\right]$$ ET I 67(34)

2. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2+R_1}{R_2-R_1}} \cos\left(ax^2\right)\,dx = \frac{1}{2\sqrt{b}} K_0(ac)\cos ab$$
$$\left[R_1=\sqrt{c^2+(b-x^2)^2},\quad R_2=\sqrt{c^2+(b+x^2)^2},\quad a>0,\quad c>0\right]$$ ET I 10(26)

3.858

1. $$\int_u^\infty \frac{\left(x^2+\sqrt{x^4-u^4}\right)^\nu+\left(x^2-\sqrt{x^4-u^4}\right)^\nu}{\sqrt{x^4-u^4}} \sin\left(a^2x^2\right)\,dx$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}}u^{2\nu}\left[J_{\frac14+\frac\nu2}\left(\frac{a^2u^2}{2}\right)Y_{\frac14-\frac\nu2}\left(\frac{a^2u^2}{2}\right)+J_{\frac14-\frac\nu2}\left(\frac{a^2u^2}{2}\right)Y_{\frac14+\frac\nu2}\left(\frac{a^2u^2}{2}\right)\right]$$
$$\left[\operatorname{Re}\nu<\tfrac32\right]$$ ET I 71(25)

2. $$\int_u^\infty \frac{\left(x^2+\sqrt{x^4-u^4}\right)^\nu+\left(x^2-\sqrt{x^4-u^4}\right)^\nu}{\sqrt{x^4-u^4}} \cos\left(a^2x^2\right)\,dx$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}}u^{2\nu}\left[J_{-\frac14+\frac\nu2}\left(\frac{a^2u^2}{2}\right)Y_{-\frac14-\frac\nu2}\left(\frac{a^2u^2}{2}\right)+J_{-\frac14-\frac\nu2}\left(\frac{a^2u^2}{2}\right)Y_{-\frac14+\frac\nu2}\left(\frac{a^2u^2}{2}\right)\right]$$
$$\left[\operatorname{Re}\nu<\tfrac32\right]$$ ET I 13(26)

3.859 $$\int_0^\infty \left[\cos\left(x^{2^n}\right)-\frac{1}{1+x^{2^{n+1}}}\right]\frac{dx}{x} = -\frac{1}{2^n}\boldsymbol{C}$$ BI (173)(24)

3.861

1. $$\int_0^\infty \sin^{2n+1}\left(ax^2\right)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac12}}{2^{2n-m+\frac12}(2m-1)!!}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{2n+1}{n+k}(2k-1)^{m-\frac12}$$
$$\left[\begin{array}{l}\text{the + sign is taken when } m\equiv 0 \pmod 4 \text{ or } m\equiv 1 \pmod 4,\\ \text{the − sign is taken when } m\equiv 2 \pmod 4 \text{ or } m\equiv 3 \pmod 4\end{array}\right]$$ BI (177)(19)a

2. $$\int_0^\infty \sin^{2n}\left(ax^2\right)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac12}}{2^{2n-2m+1}(2m-1)!!}\sum_{k=1}^{n}(-1)^{k}\binom{2n}{n+k}k^{m-\frac12}$$
$$\left[\begin{array}{l}\text{the + sign is taken when } m\equiv 0 \pmod 4 \text{ or } m\equiv 3 \pmod 4,\\ \text{the − sign is taken when } m\equiv 2 \pmod 4 \text{ or } m\equiv 1 \pmod 4\end{array}\right]$$ BI (177)(18)a, LI (177)(18)

3.862 $$\int_0^\infty \left[\cos\left(ax^2\sqrt{n}\right)+\sin\left(ax^2\sqrt{n}\right)\right]\left(\frac{\sin^2 x}{x^2}\right)^n dx$$
$$= \frac{\sqrt{\pi}}{(2n-1)!!\sqrt{2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(n-2k+a\sqrt{n}\right)^{n-\frac12}$$
$$\left[a>\sqrt{n}>0\right]$$ BI (178)(9)

3.863

1. $$\int_0^\infty x^2\cos\left(ax^4\right)\sin\left(2bx^2\right)\,dx=-\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a}-\frac{\pi}{8}\right)J_{-\frac14}\left(\frac{b^2}{2a}\right)+\cos\left(\frac{b^2}{2a}-\frac{\pi}{8}\right)J_{\frac34}\left(\frac{b^2}{2a}\right)\right]$$
$[a>0,\quad b>0]$ ET I 25(22)

2. $$\int_0^\infty x^2\cos\left(ax^4\right)\cos\left(2bx^2\right)\,dx=-\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a}+\frac{\pi}{8}\right)J_{-\frac34}\left(\frac{b^2}{2a}\right)+\cos\left(\frac{b^2}{2a}+\frac{\pi}{8}\right)J_{-\frac14}\left(\frac{b^2}{2a}\right)\right]$$
$[a>0,\quad b>0]$ ET I 25(23)

3.864

1. $$\int_0^\infty \sin\frac{b}{x}\sin ax\frac{dx}{x}=\frac{\pi}{2}\,Y_0\left(2\sqrt{ab}\right)+K_0\left(2\sqrt{ab}\right)\qquad[a>0,\quad b>0]$$
WA 204(3)a

2. $$\int_0^\infty \cos\frac{b}{x}\cos ax\frac{dx}{x}=-\frac{\pi}{2}\,Y_0\left(2\sqrt{ab}\right)+K_0\left(2\sqrt{ab}\right)$$
$[a>0,\quad b>0]$
WA 204(4)a, ET I 24 (12)

3.865

1. $$\int_0^u \frac{\left(u^2-x^2\right)^{\mu-1}}{x^{2\mu}}\sin\frac{a}{x}\,dx=\frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac12}u^{\mu-\frac32}\,\Gamma(\mu)\,J_{\frac12-\mu}\left(\frac{a}{u}\right)$$
$[a>0,\quad u>0,\quad 0<\operatorname{Re}\mu<1]$
ET II 189(30)

2. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}}\sin\frac{a}{x}\,dx=\sqrt{\frac{\pi}{u}}a^{\frac12-\mu}\,\Gamma(\mu)\sin\frac{a}{2u}\,J_{\mu-\frac12}\left(\frac{a}{2u}\right)$$
$[a>0,\quad u>0,\quad \operatorname{Re}\mu>0]$
ET II 203(21)

3. $$\int_0^u \frac{\left(u^2-x^2\right)^{\mu-1}}{x^{2\mu}}\cos\frac{a}{x}\,dx=-\frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac12}\Gamma(\mu)u^{\mu-\frac32}\,Y_{\frac12-\mu}\left(\frac{a}{u}\right)$$
$[a>0,\quad u>0,\quad 0<\operatorname{Re}\mu<1]$
ET II 190(36)

4. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}}\cos\frac{a}{x}\,dx=\sqrt{\frac{\pi}{u}}a^{\frac12-\mu}\,\Gamma(\mu)\cos\frac{a}{2u}\,J_{\mu-\frac12}\left(\frac{a}{2u}\right)$$
$[a>0,\quad u>0,\quad \operatorname{Re}\mu>0]$
ET II 204(26)

3.866

1. $$\int_0^\infty x^{\mu-1}\sin\frac{b^2}{x}\sin\left(a^2x\right)\,dx=\frac{\pi}{4}\left(\frac{b}{a}\right)^\mu\operatorname{cosec}\frac{\mu\pi}{2}\left[J_\mu(2ab)-J_{-\mu}(2ab)+I_{-\mu}(2ab)-I_\mu(2ab)\right]$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$
ET I 322(42)

2. $$\int_0^\infty x^{\mu-1}\sin\frac{b^2}{x}\cos\left(a^2x\right)\,dx=\frac{\pi}{4}\left(\frac{b}{a}\right)^\mu\sec\frac{\mu\pi}{2}\left[J_\mu(2ab)+J_{-\mu}(2ab)+I_\mu(2ab)-I_{-\mu}(2ab)\right]$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$
ET I 322(43)

3. $$\int_0^\infty x^{\mu-1}\cos\frac{b^2}{x}\cos\left(a^2x\right)\,dx=\frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \operatorname{cosec}\frac{\mu\pi}{2}\left[J_{-\mu}(2ab)-J_\mu(2ab)+I_{-\mu}(2ab)-I_\mu(2ab)\right]$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$
ET I 322(44)

3.867

1. $$\int_0^1\frac{\cos ax-\cos\frac{a}{x}}{1-x^2}\,dx=\frac{1}{2}\int_0^\infty\frac{\cos ax-\cos\frac{a}{x}}{1-x^2}\,dx=\frac{\pi}{2}\sin a$$
$[a>0]$ GW (334)(7a)

2. $$\int_0^1\frac{\cos ax+\cos\frac{a}{x}}{1+x^2}\,dx=\frac{1}{2}\int_0^\infty\frac{\cos ax+\cos\frac{a}{x}}{1+x^2}\,dx=\frac{\pi}{2}e^{-a}$$
$[a>0]$ GW (334)(7b)

3.868

1. $$\int_0^\infty\sin\left(a^2x+\frac{b^2}{x}\right)\frac{dx}{x}=\pi\,J_0(2ab)$$
$[a>0,\quad b>0]$
GW (334)(11a), WA 200(16)

2. $$\int_0^\infty\cos\left(a^2x+\frac{b^2}{x}\right)\frac{dx}{x}=-\pi\,Y_0(2ab)$$
$[a>0,\quad b>0]$ GW (334)(11a)

3. $$\int_0^\infty\sin\left(a^2x-\frac{b^2}{x}\right)\frac{dx}{x}=0$$
$[a>0,\quad b>0]$ GW (334)(11b)

4. $$\int_0^\infty\cos\left(a^2x-\frac{b^2}{x}\right)\frac{dx}{x}=2\,K_0(2ab)$$
$[a>0,\quad b>0]$ GW (334)(11b)

3.869

1. $$\int_0^\infty\sin\left(ax-\frac{b}{x}\right)\frac{x\,dx}{\beta^2+x^2}=\frac{\pi}{2}\exp\left(-\alpha\beta-\frac{b}{\beta}\right)$$
$[a>0,\quad b>0,\quad \operatorname{Re}\beta>0]$
ET II 220(42)

2. $$\int_0^\infty\cos\left(ax-\frac{b}{x}\right)\frac{dx}{\beta^2+x^2}=\frac{\pi}{2\beta}\exp\left(-a\beta-\frac{b}{\beta}\right)$$
$[a>0,\quad b>0,\quad \operatorname{Re}\beta>0]$
ET II 222(58)

3.871

1. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x+\frac{b^2}{x}\right)\right]dx=\pi b^\mu\left[J_\mu(2ab)\cos\frac{\mu\pi}{2}-Y_\mu(2ab)\sin\frac{\mu\pi}{2}\right]$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu<1]$
ET I 319(17)

2. $$\int_0^\infty x^{\mu-1}\cos\left[a\left(x+\frac{b^2}{x}\right)\right]dx=-\pi b^\mu\left[J_\mu(2ab)\sin\frac{\mu\pi}{2}+Y_\mu(2ab)\cos\frac{\mu\pi}{2}\right]$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$
ET I 321(35)

3. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x-\frac{b^2}{x}\right)\right]dx=2b^\mu\,K_\mu(2ab)\sin\frac{\mu\pi}{2}$$
$[a>0,\quad b>0,\quad |\operatorname{Re}\mu|<1]$
ET I 319(16)

4. $$\int_0^\infty x^{\mu-1}\cos\left[a\left(x-\frac{b^2}{x}\right)\right]dx = 2b^\mu K_\mu(2ab)\cos\frac{\mu\pi}{2}$$
$[a>0, \quad b>0, \quad |\operatorname{Re}\mu|<1]$ ET I 321(36)

3.872

1. $$\int_0^1 \sin\left[a\left(x+\frac{1}{x}\right)\right]\sin\left[a\left(x-\frac{1}{x}\right)\right]\frac{dx}{1-x^2} = \frac{1}{2}\int_0^\infty \sin\left[a\left(x+\frac{1}{x}\right)\right]\sin\left[a\left(x-\frac{1}{x}\right)\right]\frac{dx}{1-x^2} = -\frac{\pi}{4}\sin 2a$$
$[a\geq 0]$ BI (149)(15), GW (334)(8a)

2. $$\int_0^1 \cos\left[a\left(x+\frac{1}{x}\right)\right]\cos\left[a\left(x-\frac{1}{x}\right)\right]\frac{dx}{1+x^2} = \frac{1}{2}\int_0^\infty \cos\left[a\left(x+\frac{1}{x}\right)\right]\cos\left[a\left(x-\frac{1}{x}\right)\right]\frac{dx}{1+x^2} = \frac{\pi}{4}e^{-2a}$$
$[a\geq 0]$ GW (334)(8b)

3.873

1. $$\int_0^\infty \sin\frac{a^2}{x^2}\cos b^2x^2\frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a}\left[\sin(2ab)+\cos(2ab)+e^{-2ab}\right]$$
$[a>0, \quad b>0]$ ET I 24(15)

2. $$\int_0^\infty \cos\frac{a^2}{x^2}\cos b^2x^2\frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a}\left[\cos(2ab)-\sin(2ab)+e^{-2ab}\right]$$
$[a>0, \quad b>0]$ ET I 24(16)

3.874

1. $$\int_0^\infty \sin\left(a^2x^2+\frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b}\sin\left(2ab+\frac{\pi}{4}\right)$$
$[a>0, \quad b>0]$ BI (179)(6)a, GW(334)(10a)

2. $$\int_0^\infty \cos\left(a^2x^2+\frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b}\cos\left(2ab+\frac{\pi}{4}\right)$$
$[a>0, \quad b>0]$ GI (179)(8)a, GW(334)(10a)

3. $$\int_0^\infty \sin\left(a^2x^2-\frac{b^2}{x^2}\right)\frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab}$$
$[a\geq 0, \quad b>0]$ GW (335)(10b)

4. $$\int_0^\infty \cos\left(a^2x^2-\frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab}$$
$[a\geq 0, \quad b>0]$ GW (334)(10b)

5. $$\int_0^\infty \sin\left(ax-\frac{b}{x}\right)^2\frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$
$[a>0, \quad b>0]$ BI (179)(13)a

6. $$\int_0^\infty \cos\left(ax-\frac{b}{x}\right)^2\frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$
$[a>0, \quad b>0]$ BI (179)(14)a

3.875

1. $$\int_u^\infty \frac{x\sin\left(p\sqrt{x^2-u^2}\right)}{x^2+a^2}\cos bx\,dx = \frac{\pi}{2}\exp\left(-p\sqrt{a^2+u^2}\right)\cosh ab$$
$[0<b<p]$ ET I 27(39)

2. $$\int_u^\infty \frac{x\sin\left(p\sqrt{x^2-u^2}\right)}{a^2+x^2-u^2}\cos bx\,dx = \frac{\pi}{2}e^{-ap}\cos\left(b\sqrt{u^2-a^2}\right)$$
$[0<b<p, \quad a>0]$ ET I 27(38)

3.[6] $$\int_0^\infty \frac{\sin\left(p\sqrt{a^2+x^2}\right)}{\left(a^2+x^2\right)^{3/2}}\cos bx\,dx = \frac{\pi p}{2a}e^{-ab}$$
$[0<p<b, \quad a>0]$ ET I 26(29)

3.876

1. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}\cos bx\,dx = \frac{\pi}{2}\,J_0\left(a\sqrt{p^2-b^2}\right) \qquad [0<b<p]$$
$$= 0 \qquad [b>p>0]$$
$[a>0]$ ET I 26(30)

2. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}\cos bx\,dx = -\frac{\pi}{2}\,Y_0\left(a\sqrt{p^2-b^2}\right) \qquad [0<b<p]$$
$$= K_0\left(a\sqrt{b^2-p^2}\right) \qquad [b>p>0]$$
$[a>0]$ ET I 26(34)

3. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{x^2+c^2}\cos bx\,dx = \frac{\pi}{2c}e^{-bc}\cos\left(p\sqrt{a^2-c^2}\right)$$
$[c>0, \quad b>p]$ ET I 26(33)

4. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\left(x^2+c^2\right)\sqrt{x^2+a^2}}\cos bx\,dx = \frac{\pi}{2c}\frac{e^{-bc}\sin\left(p\sqrt{a^2-c^2}\right)}{\sqrt{a^2-c^2}} \qquad [c\neq a]$$
$$= \frac{\pi}{2}e^{-ba}\frac{p}{a} \qquad [c=a]$$
$[b>p, \quad c>0]$ ET I 26(31)a

5.[6] $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{x^2+a^2}\cos bx\,dx = \frac{\pi}{2a}e^{-ab}$$
$[b>p>0; \quad a>0]$ ET I 27(35)a

6.[6] $$\int_0^\infty \frac{x\cos\left(p\sqrt{x^2+a^2}\right)}{x^2+a^2}\sin bx\,dx = \frac{\pi}{2}e^{-ab}$$
$[a>0, \quad b>p>0]$ ET I 85(29)a

7. $$\int_0^u \frac{\cos\left(p\sqrt{u^2-x^2}\right)}{\sqrt{u^2-x^2}}\cos bx\,dx = \frac{\pi}{2}\,J_0\left(u\sqrt{b^2+p^2}\right)$$
ET I 28(42)

8. $$\int_u^\infty \frac{\cos\left(p\sqrt{x^2-u^2}\right)}{\sqrt{x^2-u^2}}\cos bx\,dx = K_0\left(u\sqrt{p^2-b^2}\right) \qquad [0<b<|p|]$$
$$= -\frac{\pi}{2}\,Y_0\left(u\sqrt{b^2-p^2}\right) \qquad [b>|p|]$$
ET I 28(43)

3.877

1. $$\int_0^u \frac{\sin\left(p\sqrt{u^2-x^2}\right)}{\sqrt[4]{(u^2-x^2)^3}}\cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}}\,J_{\frac14}\left[\frac{u}{2}\left(\sqrt{b^2+p^2}-b\right)\right]J_{\frac14}\left[\frac{u}{2}\left(\sqrt{b^2+p^2}+b\right)\right]$$
$[b>0,\quad p>0]$ ET I 27(40)

2. $$\int_u^\infty \frac{\sin\left(p\sqrt{x^2-u^2}\right)}{\sqrt[4]{(x^2-u^2)^3}}\cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}}\,J_{\frac14}\left[\frac{u}{2}\left(b-\sqrt{b^2-p^2}\right)\right]Z_{\frac14}\left[\frac{u}{2}\left(b+\sqrt{b^2-p^2}\right)\right]$$
$[b>p>0]$ ET I 27(41)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^2-x^2}\right)}{\sqrt[4]{(u^2-x^2)^3}}\cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}}\,J_{-\frac14}\left[\frac{u}{2}\left(\sqrt{p^2+b^2}-b\right)\right]J_{-\frac14}\left[\frac{u}{2}\left(\sqrt{p^2+b^2}+b\right)\right]$$
$[u>0,\quad p>0]$ ET I 28(44)

4. $$\int_u^\infty \frac{\cos\left(p\sqrt{x^2-u^2}\right)}{\sqrt[4]{(x^2-u^2)^3}}\cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}}\,J_{-\frac14}\left[\frac{u}{2}\left(b-\sqrt{b^2-p^2}\right)\right]Y_{\frac14}\left[\frac{u}{2}\left(b+\sqrt{b^2-p^2}\right)\right]$$
$[b>p>0]$ ET I 28(45)

3.878

1. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^4+a^4}\right)}{\sqrt{x^4+a^4}}\cos bx^2\,dx = \frac12\sqrt{\left(\frac{\pi}{2}\right)^3 b}\,J_{-\frac14}\left[\frac{a^2}{2}\left(p-\sqrt{p^2-b^2}\right)\right]J_{\frac14}\left[\frac{a^2}{2}\left(p+\sqrt{p^2-b^2}\right)\right]$$
$[p>b>0]$ ET I 26(32)

2. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^4+a^4}\right)}{\sqrt{x^4+a^4}}\cos bx^2\,dx$$
$$= -\frac12\sqrt{\left(\frac{\pi}{2}\right)^3 b}\,J_{-\frac14}\left[\frac{a^2}{2}\left(p-\sqrt{p^2-b^2}\right)\right]Y_{\frac14}\left[\frac{a^2}{2}\left(p+\sqrt{p^2-b^2}\right)\right]$$
$[a>0,\quad p>b>0]$ ET I 27(36)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^4-x^4}\right)}{\sqrt{u^4-x^4}}\cos bx^2\,dx = \frac12\sqrt{\left(\frac{\pi}{2}\right)^3 b}\,J_{-\frac14}\left[\frac{u^2}{2}\left(\sqrt{p^2+b^2}-p\right)\right]J_{-\frac14}\left[\frac{u^2}{2}\left(\sqrt{p^2+b^2}+p\right)\right]$$
$[p>0,\quad b>0]$ ET I 28(46)

3.879 $$\int_0^\infty \sin ax^p\frac{dx}{x} = \frac{\pi}{2p}$$ $[a>0,\quad p>0]$ GW (334)(6)

3.881

1. $$\int_0^{\pi/2} x\sin(a\tan x)\,dx = \frac{\pi}{4}e^{-a}\left[\boldsymbol{C}+\ln 2a - e^{2a}\,\mathrm{Ei}(-2a)\right]$$
$[a>0]$ BI (205)(9)

2. $$\int_0^\infty \sin(a\tan x)\frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$$ $[a>0]$ BI (151)(6)

3. $$\int_0^\infty \sin(a\tan x)\cos x\frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$$ $[a>0]$ BI (151)(19)

4. $\displaystyle\int_0^\infty \cos(a\tan x)\sin x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(20)

5. $\displaystyle\int_0^\infty \sin(a\tan x)\sin 2x\frac{dx}{x} = \frac{1+a}{2}\pi e^{-a}$ $[a>0]$ BI (152)(11)

6. $\displaystyle\int_0^\infty \cos(a\tan x)\sin^3 x\frac{dx}{x} = \frac{1-a}{4}\pi e^{-a}$ $[a>0]$ BI (151)(23)

7. $\displaystyle\int_0^\infty \sin(a\tan x)\tan\frac{x}{2}\cos^2 x\frac{dx}{x} = \frac{1+a}{4}\pi e^{-a}$ $[a>0]$ BI (152)(13)

8. $\displaystyle\int_0^{\pi/2} \cos(a\tan x)\frac{x\,dx}{\sin 2x} = -\frac{\pi}{4}\operatorname{Ei}(-a)$ $[a>0]$ BI (206)(15)

9. $\displaystyle\int_0^{\pi/2} \sin(a\cot x)\frac{x\,dx}{\sin^2 x} = \frac{1-e^{-a}}{2a}\pi$ $[a>0]$ LI (206)(14)

10. $\displaystyle\int_0^{\pi/2} x\cos(a\tan x)\tan x\,dx = -\frac{\pi}{4}e^{-a}\left[\boldsymbol{C}+\ln 2a+e^{2a}\operatorname{Ei}(-2a)\right]$

$[a>0]$ BI (205)(10)

11. $\displaystyle\int_0^\infty \cos(a\tan x)\tan x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(21)

12. $\displaystyle\int_0^\infty \cos(a\tan x)\sin^2 x\tan x\frac{dx}{x} = \frac{1-a}{16}\pi e^{-a}$ $[a>0]$ BI (152)(15)

13. $\displaystyle\int_0^\infty \sin(a\tan x)\tan^2 x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (152)(9)

14. $\displaystyle\int_0^\infty \cos(a\tan 2x)\tan x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(22)

15. $\displaystyle\int_0^\infty \sin(a\tan 2x)\cos^2 2x\tan x\frac{dx}{x} = \frac{1+a}{4}\pi e^{-a}$ $[a>0]$ BI (152)(13)

16. $\displaystyle\int_0^\infty \sin(a\tan 2x)\tan x\tan 2x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (152)(10)

17. $\displaystyle\int_0^\infty \sin(a\tan 2x)\tan x\cot 2x\frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$ $[a>0]$ BI (180)(6)

3.882

1. $\displaystyle\int_0^\infty \sin\left(a\tan^2 x\right)\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\exp(-a\tanh b)-e^{-a}\right]$

$[a>0,\quad b>0]$ BI (160)(22)

2. $\displaystyle\int_0^\infty \cos\left(a\tan^2 x\right)\cos x\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\left[\cosh b\exp(-a\tanh b)-e^{-a}\sinh b\right]$

$[a>0,\quad b>0]$ BI (163)(3)

3. $\displaystyle\int_0^\infty \cos\left(a\tan^2 x\right)\operatorname{cosec} 2x\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2\sinh 2b}\exp(-a\tanh b)$

$[a>0,\quad b>0]$ BI (191)(10)

4. $$\int_0^\infty \cos\left(a\tan^2 x\right)\tan x\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2\cosh b}\left[e^{-a}\cosh b - \exp\left(-a\tanh b\right)\sinh b\right]$$
$[a>0, \quad b>0]$ BI (163)(4)

5.[11] $$\int_0^\infty \cos\left(a\tan^2 x\right)\cot x\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\coth b\exp\left(-a\tanh b\right) - e^{-a}\right]$$
$[a>0, \quad b>0]$ BI (163)(5)

6. $$\int_0^\infty \cos\left(a\tan^2 x\right)\cot 2x\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\coth 2b\exp\left(-a\tanh b\right) - e^{-a}\right]$$
$[a>0, \quad b>0]$ BI (191)(11)

3.883

1. $$\int_0^1 \cos\left(a\ln x\right)\frac{dx}{(1+x)^2} = \frac{a\pi}{2\sinh a\pi}$$
BI (404)(4)

2. $$\int_0^1 x^{\mu-1}\sin\left(\beta\ln x\right)\,dx = -\frac{\beta}{\beta^2+\mu^2}$$
$[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$ ET I 319(19)

3. $$\int_0^1 x^{\mu-1}\cos\left(\beta\ln x\right)\,dx = \frac{\mu}{\beta^2+\mu^2}$$
$[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$ ET I 321(38)

3.884[11] $$\int_{-\infty}^\infty \frac{\sin a\sqrt{|x|}}{x-b}\operatorname{sign} x\,dx = \pi\left[\exp\left(-a\sqrt{|-b|}\right) + \exp\left(-a\sqrt{|b|}\right)\right]$$
$[a>0, \quad \operatorname{Im} b \neq 0]$ ET II 253(46)

3.89–3.91 Trigonometric functions and exponentials

3.891

1. $$\int_0^{2\pi} e^{imx}\sin nx\,dx = 0 \qquad [m\neq n;\ \text{or}\ m=n=0]$$
$$= \pi i \qquad [m=n\neq 0]$$

2. $$\int_0^{2\pi} e^{imx}\cos nx\,dx = 0 \qquad [m\neq n]$$
$$= \pi \qquad [m=n\neq 0]$$
$$= 2\pi \qquad [m=n=0]$$

3.892

1.[11] $$\int_0^\pi e^{i\beta x}\sin^{\nu-1} x\,dx = \frac{\pi e^{i\beta\frac{\pi}{2}}}{2^{\nu-1}\nu\,\mathrm{B}\left(\frac{\nu+\beta+1}{2},\frac{\nu-\beta+1}{2}\right)}$$
$[\operatorname{Re}\nu > -1]$ NH 158, EH I 12(29)

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x}\cos^{\nu-1} x\,dx = \frac{\pi}{2^{\nu-1}\nu\,\mathrm{B}\left(\frac{\nu+\beta+1}{2},\frac{\nu-\beta+1}{2}\right)}$$
$[\operatorname{Re}\nu > -1]$ GW (335)(19)

3.[6] $$\int_0^{\pi/2} e^{i2\beta x}\sin^{2\mu}x\cos^{2\nu}x\,dx = \frac{1}{2^{2\mu+2\nu+1}}\left\{\exp\left[i\pi\left(\beta-\nu-\tfrac{1}{2}\right)\right]\mathrm{B}\left(\beta-\mu-\nu,2\nu+1\right)\right.$$
$$\times F\left(-2\mu,\beta-\mu-\nu;1+\beta-\mu+\nu;-1\right)+\exp\left[i\pi\left(\mu+\tfrac{1}{2}\right)\right]$$
$$\left.\times\ \mathrm{B}(\beta-\mu-\nu,2\mu+1)\,F\left(-2\nu,\beta-\mu-\nu;1+\beta+\mu-\nu;-1\right)\right\}$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ EH I 80(6)

4. $$\int_0^{\pi} e^{i2\beta x}\sin^{2\mu}x\cos^{2\nu}x\,dx = \frac{\pi\exp\left[i\pi(\beta-\nu)\right]F(-2\nu,\beta-\mu-\nu;1+\beta+\mu-\nu;-1)}{4^{\mu+\nu}(2\mu+1)\,\mathrm{B}(1-\beta+\mu+\nu,1+\beta+\mu-\nu)}$$
EH I 80(8)

5. $$\int_0^{\pi/2} e^{i(\mu+\nu)x}\sin^{\mu-1}x\cos^{\nu-1}x\,dx = e^{i\mu\frac{\pi}{2}}\,\mathrm{B}(\mu,\nu)$$
$$= \frac{1}{2^{\mu+\nu-1}}e^{i\mu\frac{\pi}{2}}\left\{\frac{1}{\mu}\,F(1-\nu,1;\mu+1;-1)+\frac{1}{\nu}\,F\left(1-\mu,1;\nu+1;-1\right)\right\}$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ EH I 80(7)

3.893

1.[8] $$\int_0^{\infty} e^{-px}\sin(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left(q\cos\lambda+p\sin\lambda\right)\qquad [\operatorname{Re}p>0]$$ BI (261)(3)

2.[8] $$\int_0^{\infty} e^{-px}\cos(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left(p\cos\lambda-q\sin\lambda\right)\qquad [\operatorname{Re}p>0]$$ BI (261)(4)

3. $$\int_0^{\infty} e^{-x\cos t}\cos\left(t-x\sin t\right)\,dx = 1$$ BI (261)(7)

4.[8] $$\int_0^{\infty}\frac{e^{-\beta x}\sin ax}{\sin bx}\,dx = \operatorname{Re}\left\{\frac{1}{2bi}\left[\psi\left(\frac{a+b}{2b}-i\frac{\beta}{2b}\right)-\psi\left(\frac{b-a}{2b}-i\frac{\beta}{2b}\right)\right]\right\}$$
$$[\operatorname{Re}\beta>0,\quad b\neq 0]$$ GW (335)(15)

5.[8] $$\int_0^{\infty}\frac{e^{-2px}\sin[(2n+1)x]}{\sin x}\,dx = \frac{1}{2p}+\sum_{k=1}^{n}\frac{p}{p^2+k^2}\qquad [\operatorname{Re}p>0]$$ BI (267)(15)

6.[8] $$\int_0^{\infty}\frac{e^{-px}\sin 2nx}{\sin x}\,dx = 2p\sum_{k=0}^{n-1}\frac{1}{p^2+(2k+1)^2}\qquad [\operatorname{Re}p>0]$$ GW (335)(15c)

7. $$\int_0^{\infty} e^{-px}\cos[(2n+1)x]\tan x\,dx = \frac{2n+1}{p^2+(2n+1)^2}+(-1)^n 2\sum_{k=0}^{n-1}\frac{(-1)^k(2k+1)}{p^2+(2k+1)^2}$$
$$[p>0]$$ LI (267)(16)

3.894 $$\int_{-\pi}^{\pi}\left[\beta+\sqrt{\beta^2-1}\cos x\right]^{\nu}e^{inx}\,dx = \frac{2\pi\,\Gamma(\nu+1)\,P_\nu^m(\beta)}{\Gamma(\nu+m+1)}$$
$$[\operatorname{Re}\beta>0]$$ ET I 157(15)

3.895

1. $$\int_0^{\infty} e^{-\beta x}\sin^{2m}x\,dx = \frac{(2m)!}{\beta(\beta^2+2^2)\,(\beta^2+4^2)\cdots[\beta^2+(2m)^2]}$$
$$[\operatorname{Re}\beta>0]$$ FI II 615, WA 620a

2.[10] $$\int_0^{\pi} e^{-px}\sin^{2m}x\,dx=\frac{(2m)!\,(1-e^{-p\pi})}{p\,(p^2+2^2)\,(p^2+4^2)\cdots[p^2+(2m)^2]}$$ GW (335)(4a)

3.[10] $$\int_0^{\pi/2} e^{-px}\sin^{2m}x\,dx$$
$$=\frac{(2m)!}{p\,(p^2+2^2)\,(p^2+4^2)\cdots[p^2+(2m)^2]}\times\left\{1-e^{-\frac{p\pi}{2}}\left[1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right]\right\}$$
BI (270)(4)

4. $$\int_0^{\infty} e^{-\beta x}\sin^{2m+1}x\,dx=\frac{(2m+1)!}{(\beta^2+1^2)\,(\beta^2+3^2)\cdots[\beta^2+(2m+1)^2]}$$
$[\operatorname{Re}\beta>0]$ FI II 615, WA 620a

5.[10] $$\int_0^{\pi} e^{-px}\sin^{2m+1}x\,dx=\frac{(2m+1)!\,(1+e^{-p\pi})}{(p^2+1^2)\,(p^2+3^2)\cdots[p^2+(2m+1)^2]}$$ GW (335)(4b)

6.[8] $$\int_0^{\pi/2} e^{-px}\sin^{2m+1}x\,dx$$
$$=\frac{(2m+1)!}{(p^2+1^2)\,(p^2+3^2)\cdots[p^2+(2m+1)^2]}\times\left\{1-pe^{\frac{-p\pi}{2}}\left[1+\frac{p^2+1^2}{3!}+\cdots+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right]\right\}$$
BI (270)(5)

7. $$\int_0^{\infty} e^{-px}\cos^{2m}x\,dx=\frac{(2m)!}{p\,(p^2+2^2)\cdots[p^2+(2m)^2]}\times\left\{1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right\}$$
$[p>0]$ BI (262)(3)

8.[10] $$\int_0^{\pi/2} e^{-px}\cos^{2m}x\,dx$$
$$=\frac{(2m)!}{p\,(p^2+2^2)\cdots[p^2+(2m)^2]}\times\left\{-e^{-p\frac{\pi}{2}}+1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right\}$$
BI (270)(6)

9.[7] $$\int_0^{\infty} e^{-px}\cos^{2m+1}x\,dx$$
$$=\frac{(2m+1)!p}{(p^2+1^2)\,(p^2+3^2)\cdots[p^2+(2m+1)^2]}\times\left\{1+\frac{p^2+1^2}{3!}+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)}{5!}+\cdots+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right\}$$
$[p>0]$ BI (262)(4)

10.[11] $$\int_0^{\pi/2} e^{-px}\cos^{2m+1} x\,dx = \frac{(2m+1)!}{(p^2+1^2)\,(p^2+3^2)\cdots[p^2+(2m+1)^2]} \times\left\{e^{-p\frac{\pi}{2}} + p\left[1+\frac{p^2+1^2}{3!}+\cdots+\frac{\left(p^2+1\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right]\right\}$$ BI (270)(7)

11.[8] $$\int_0^\infty e^{-\beta x}\sin^n ax\left\{\begin{matrix}\sin bx\\ \cos bx\end{matrix}\right\}dx = \frac{2^{-n-2}}{a(n+1)}e^{\frac{1}{4}(1\mp1+2n)\pi i} \times\left\{\left(\begin{matrix}\frac{b+na+i\beta}{2a}\\ n+1\end{matrix}\right)^{-1} \pm(-1)^n\left(\begin{matrix}\frac{b+na-i\beta}{2a}\\ n+1\end{matrix}\right)^{-1}\right\}$$ $[a>0,\quad b>0,\quad \operatorname{Re}\beta>0]$

12. $$\int_0^\infty e^{-ax}\cos^2 mx\,dx = \frac{a^2+2m^2}{a\,(a^2+4m^2)}$$ DW61 (861.06)

13. $$\int_0^\infty e^{-ax}\cos mx\cos nx\,dx = \frac{a\left(a^2+m^2+n^2\right)}{(a^2+(m-n)^2)\,(a^2+(m+n)^2)}$$ DW61 (861.15)

14. $$\int_0^\infty e^{-ax}\sin mx\cos nx\,dx = \frac{m\left(a^2+m^2-n^2\right)}{(a^2+(m-n)^2)\,(a^2+(m+n)^2)}$$ DW61 (861.14)

15. $$\int_0^\infty e^{-ax}\sin^2 mx\,dx = \frac{2m}{a\,(a^2+4m^2)} \qquad [a>0]$$ DW61 (861.10)

16. $$\int_0^\infty e^{-ax}\sin mx\sin nx\,dx = \frac{2amn}{[a^2+(m-n)^2]\,[a^2+(m+n)^2]}$$ DW61 (861.13)

3.896

1. $$\int_{-\infty}^\infty e^{-q^2x^2}\sin[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q}e^{-\frac{p^2}{4q^2}}\sin p\lambda$$ BI (269)(2)

2. $$\int_{-\infty}^\infty e^{-q^2x^2}\cos[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q}e^{-\frac{p^2}{4q^2}}\cos p\lambda$$ BI (269)(3)

3. $$\int_0^\infty e^{-ax^2}\sin bx\,dx = \frac{b}{2a}\exp\left(-\frac{b^2}{4a}\right)\ {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{b^2}{4a}\right)$$
$$= \frac{b}{2a}\ {}_1F_1\left(1;\frac{3}{2};-\frac{b^2}{4a}\right)$$ ET I 73(18)
$$= \frac{b}{2a}\sum_{k=1}^\infty\frac{1}{(2k-1)!!}\left(-\frac{b^2}{2a}\right)^{k-1} \qquad [a>0]$$ FI II 720

4. $$\int_0^\infty e^{-\beta x^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(-\frac{b^2}{4\beta}\right) \qquad [\operatorname{Re}\beta>0]$$ BI (263)(2)

3.897

1.[8] $$\int_0^\infty e^{-\beta x^2-\gamma x}\sin bx\,dx = -\frac{i}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma-ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right] - \exp\frac{(\gamma+ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ib}{2\sqrt{\beta}}\right)\right]\right\}$$ $[\operatorname{Re}\beta>0]$ ET I 74(27)

2. $$\int_0^\infty e^{-\beta x^2-\gamma x}\cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma-ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right] + \exp\frac{(\gamma+ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ib}{2\sqrt{\beta}}\right)\right]\right\}$$ $[\operatorname{Re}\beta>0]$ ET I 15(16)

3.898

1. $$\int_0^\infty e^{-\beta x^2}\sin ax\sin bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}}\right\}$$ $[\operatorname{Re}\beta>0]$ BI (263)(4)

2. $$\int_0^\infty e^{-\beta x^2}\cos ax\cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}}\right\}$$ $[\operatorname{Re}\beta>0]$ BI (263)(5)

3.[8] $$\int_0^\infty e^{-px^2}\sin^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left(1-e^{-\frac{a^2}{p}}\right)$$ $[\operatorname{Re}p>0]$ BI (263)(6)

3.899

1.[7] $$\int_0^\infty \frac{e^{p^2x^2}\sin[(2n+1)x]}{\sin x}\,dx = \frac{\sqrt{\pi}}{p}\left[\frac{1}{2}+\sum_{k=1}^{n} e^{-\left(\frac{k}{p}\right)^2}\right]$$ $[p>0]$ BI (267)(17)

2. $$\int_0^\infty \frac{e^{-p^2x^2}\cos[(4n+1)x]}{\cos x}\,dx = \frac{\sqrt{\pi}}{p}\left[\frac{1}{2}+\sum_{k=0}^{2n}(-1)^k e^{-\left(\frac{k}{p}\right)^2}\right]$$ $[p>0]$ BI (267)(18)

3. $$\int_0^\infty \frac{e^{-px^2}\,dx}{1-2a\cos x+a^2} = \frac{\sqrt{\frac{\pi}{p}}}{1-a^2}\left\{\frac{1}{2}+\sum_{k=1}^{\infty} a^k\exp\left(-\frac{k^2}{4p}\right)\right\}$$ $[a^2<1,\quad p>0]$ EI (266)(1)

$$= \frac{\sqrt{\frac{\pi}{p}}}{a^2-1}\left\{\frac{1}{2}+\sum_{k=1}^{\infty} a^{-k}\exp\left(-\frac{k^2}{4p}\right)\right\}$$ $[a^2>1,\quad p>0]$ LI (266)(1)

3.911

1. $$\int_0^\infty \frac{\sin ax}{e^{\beta x}+1}\,dx = \frac{1}{2a}-\frac{\pi}{2\beta\sinh\frac{a\pi}{\beta}}$$ $[a>0,\quad \operatorname{Re}\beta>0]$ BI (264)(1)

2. $$\int_0^\infty \frac{\sin ax}{e^{\beta x}-1}\,dx = \frac{\pi}{2\beta}\coth\left(\frac{\pi a}{\beta}\right)-\frac{1}{2a}$$ $[a>0,\quad \operatorname{Re}\beta>0]$ BI (264)(2), WH

3.[11] $$\int_0^\infty \frac{\sin ax}{e^x-1}e^{x/2}\,dx = \frac{1}{2}\pi\tanh(a\pi)$$ $[a>0]$ ET I 73(13)

4. $$\int_0^\infty \frac{\sin ax}{1-e^{-x}} e^{-nx}\,dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a}-1} - \sum_{k=1}^{n-1} \frac{a}{a^2+k^2}$$
$[a>0]$ BI (264)(8)

5. $$\int_0^\infty \frac{\sin ax}{e^{\beta x}-e^{\gamma x}}\,dx = \frac{1}{2i(\beta-\gamma)}\left[\psi\left(\frac{\beta+ia}{\beta-\gamma}\right) - \psi\left(\frac{\beta-ia}{\beta-\gamma}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ GW (335)(8)

6. $$\int_0^\infty \frac{\sin ax\,dx}{e^{\beta x}\left(e^{-x}-1\right)} = \frac{i}{2}\left[\psi(\beta+ia)-\psi(\beta-ia)\right]$$
$[\operatorname{Re}\beta>-1]$ ET 73(15)

3.912

1. $$\int_0^\infty e^{-\beta x}\left(1-e^{-\gamma x}\right)^{\nu-1}\sin ax\,dx = -\frac{i}{2\gamma}\left[\mathrm{B}\left(\nu,\frac{\beta-ia}{\gamma}\right) - \mathrm{B}\left(\nu,\frac{\beta+ia}{\gamma}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad \operatorname{Re}\nu>0, \quad a>0]$ ET I 73(17)

2. $$\int_0^\infty e^{-\beta x}\left(1-e^{-\gamma x}\right)^{\nu-1}\cos ax\,dx = \frac{1}{2\gamma}\left[\mathrm{B}\left(\nu,\frac{\beta-ia}{\gamma}\right) + \mathrm{B}\left(\nu,\frac{\beta+ia}{\gamma}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad \operatorname{Re}\nu>0, \quad a>0]$ ET I 15(10)

3.913

1. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x}\cos^\nu x\left(\beta^2 e^{ix}+\nu^2 e^{-ix}\right)^\mu\,dx = \frac{\pi\ {}_2F_1\left(-\mu,\frac{\beta}{2}-\frac{\nu}{2}-\frac{\mu}{2};1+\frac{\beta}{2}+\frac{\nu}{2}-\frac{\mu}{2};\frac{\beta^2}{\nu^2}\right)}{2^\nu(\nu+1)\,\mathrm{B}\left(1+\frac{\beta}{2}+\frac{\nu}{2}-\frac{\mu}{2},1-\frac{\beta}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}$$
$[\operatorname{Re}\nu>-1, \quad |\nu|>|\beta|]$ EH I 81(11)a

2.[11] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-iux}\cos^\mu x\left(a^2e^{ix}+b^2e^{-ix}\right)^\nu\,dx$$
$$= \frac{\pi b^{2\nu}\ {}_2F_1\left(-\nu,-\frac{u+\mu+\nu}{2};1+\frac{\mu-\nu-u}{2};\frac{a^2}{b^2}\right)}{2^\mu(\mu+1)\,\mathrm{B}\left(1-\frac{u+\nu-\mu}{2},1+\frac{u+\mu+\nu}{2}\right)} \qquad \left[\text{for } a^2<b^2\right]$$
$$= \frac{\pi a^{2\nu}\ {}_2F_1\left(-\nu,\frac{u-\mu-\nu}{2};1+\frac{\mu-\nu+u}{2};\frac{b^2}{a^2}\right)}{2^\mu(\mu+1)\,\mathrm{B}\left(1+\frac{u+\mu-\nu}{2},1+\frac{\mu+\nu-u}{2}\right)} \qquad \left[\text{for } b^2<a^2\right]$$
$[\operatorname{Re}\mu>-1]$ ET I 122(31)a

3.914

1. $$\int_0^\infty e^{-\beta\sqrt{\gamma^2+x^2}}\cos bx\,dx = \frac{\beta\gamma}{\sqrt{\beta^2+b^2}}\,K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ ET I 16(26)

2. $$\int_0^\infty \sqrt{\gamma^2+x^2}\,e^{-\beta\sqrt{\gamma^2+x^2}}\cos bx\,dx = \frac{\beta^2\gamma^2}{A^2}\,K_0(\gamma A) + \left(\frac{2\beta^2\gamma}{A^3}-\frac{\gamma}{A}\right)K_1(\gamma A)$$
$\left[A=\sqrt{\beta^2+b^2}\right]$

3. $$\int_0^\infty \left(\gamma^2+x^2\right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\, dx = \left(-\frac{3\beta\gamma^2}{A^2}+\frac{4\beta^3\gamma^2}{A^4}\right) K_0(\gamma A) + \left(-\frac{6\beta\gamma}{A^3}+\frac{8\beta^3\gamma}{A^5}+\frac{\beta^3\gamma^3}{A^3}\right) K_1(\gamma A) \quad \left[A=\sqrt{\beta^2+b^2}\right]$$

4. $$\int_0^\infty \frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \cos bx\, dx = K_0\left(\gamma\sqrt{\beta^2+b^2}\right) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad b>0]$$ ET I 16(27)

5. $$\int_0^\infty \left(\frac{1}{\beta\left(\gamma^2+x^2\right)^{3/2}}+\frac{1}{\gamma^2+x^2}\right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\, dx = \frac{1}{\beta\gamma}\sqrt{\beta^2+b^2}\, K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$ (6.726(4))

6. $$\int_0^\infty x e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx\, dx = \frac{b\beta\gamma^2}{\beta^2+b^2} K_2\left(\gamma\sqrt{\beta^2+b^2}\right)$$ ET I 175(35)

7. $$\int_0^\infty x\sqrt{\gamma^2+x^2} e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx\, dx = \left(-\frac{b\gamma^2}{A^2}+\frac{4b\beta^2\gamma^2}{A^4}\right) K_0(\gamma A) + \left(-\frac{2b\gamma}{A^3}+\frac{8b\beta^2\gamma}{A^5}+\frac{b\beta^2\gamma^3}{A^3}\right) K_1(\gamma A) \quad \left[A=\sqrt{\beta^2+b^2}\right]$$

8. $$\int_0^\infty \left(\gamma^2+x^2\right) e^{-\beta\sqrt{\gamma^2+x^2}} x\sin bx\, dx = \left(-\frac{12b\beta\gamma^2}{A^4}+\frac{24b\beta^3\gamma^2}{A^6}+\frac{b\beta^3\gamma^4}{A^4}\right) K_0(\gamma A) + \left(-\frac{24b\beta\gamma}{A^5}+\frac{48b\beta^3\gamma}{A^7}-\frac{3b\beta\gamma^3}{A^3}+\frac{8b\beta^3\gamma^3}{A^5}\right) K_1(\gamma A) \quad \left[A=\sqrt{\beta^2+b^2}\right]$$

9. $$\int_0^\infty \frac{x e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \sin bx\, dx = \frac{\gamma b}{\sqrt{\beta^2+b^2}} K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$ ET I 75(36)

10. $$\int_0^\infty \left(\frac{1}{\beta\left(\gamma^2+x^2\right)^{3/2}}+\frac{1}{\gamma^2+x^2}\right) e^{-\beta\sqrt{\gamma^2+x^2}} x\sin bx\, dx = \frac{b}{\beta} K_0\left(\gamma\sqrt{\beta^2+b^2}\right)$$ (6.726(3))

3.915

1. $$\int_0^\pi e^{a\cos x} \sin x\, dx = \frac{2}{a}\sinh a$$ GW (337)(15c)

2. $$\int_0^\pi e^{i\beta\cos x} \cos nx\, dx = i^n \pi\, J_n(\beta)$$ EH II 81(2)

3.[3] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\sin x} \cos^{2\nu} x\, dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) J_\nu(\beta) \qquad \left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ EH II 81(6)

4. $$\int_0^\pi e^{\pm\beta\cos x} \sin^{2\nu} x\, dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) I_\nu(\beta) \qquad \left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ GW (337)(15b)

5. $$\int_0^\pi e^{i\beta\cos x}\sin^{2\nu} x\,dx=\sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta)\qquad\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ WA 34(2), WA 60(6)

3.916

1. $$\int_0^{\pi/2} e^{-p^2\tan x}\frac{\sin\frac{x}{2}\sqrt{\cos x}}{\sin 2x}\,dx=\left[C(p)-\frac{1}{2}\right]^2+\left[S(p)-\frac{1}{2}\right]^2$$ NT 33(18)a

2. $$\int_0^{\pi/2}\frac{\exp(-p\tan x)\,dx}{\sin 2x+a\cos 2x+a}=-\frac{1}{2}e^{ap}\operatorname{Ei}(-ap)\qquad[p>0]\,,\qquad(\text{cf. }\mathbf{3552}\ 4\text{ and }6)$$ BI (273)(11)

3. $$\int_0^{\pi/2}\frac{\exp(-p\cot x)\,dx}{\sin 2x+a\cos 2x-a}=-\frac{1}{2}e^{-ap}\operatorname{Ei}(ap)\qquad[p>0]\,,\qquad(\text{cf. }\mathbf{3.552}\ 4\text{ and }6)$$ BI (273)(12)

4. $$\int_0^{\pi/2}\frac{\exp(-p\tan x)\sin 2x\,dx}{(1-a^2)-2a^2\cos 2x-(1+a^2)\cos^2 2x}=-\frac{1}{4}\left[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)\right]$$ $[p>0]$ BI (273)(13)

5. $$\int_0^{\pi/2}\frac{\exp(-p\cot x)\sin 2x\,dx}{(1-a^2)+2a^2\cos 2x-(1+a^2)\cos^2 2x}=-\frac{1}{4}\left[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)\right]$$ $[p>0]$ BI (273)(14)

3.917

1. $$\int_0^{\pi/2} e^{-2\beta\cot x}\cos^{\nu-1/2}x\sin^{-(\nu+1)}x\sin\left[\beta-\left(\nu-\frac{1}{2}\right)x\right]dx=\frac{\sqrt{\pi}}{2(2\beta)^\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta)$$ $\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ WA 186(7)

2. $$\int_0^{\pi/2} e^{-2\beta\cot x}\cos^{\nu-1/2}x\sin^{-(\nu+1)x}\cos\left[\beta-\left(\nu-\frac{1}{2}\right)x\right]dx=\frac{\sqrt{\pi}}{2(2\beta)^\nu}\Gamma\left(\nu+\frac{1}{2}\right)Y_\nu(\beta)$$ $\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ WA 186(8)

3.918

1. $$\int_0^{\pi/2}\frac{\cos^\mu x}{\sin^{2\mu+2}x}e^{i\gamma(\beta-\mu x)-2\beta\cot x}\,dx=\frac{i\gamma}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1)\,H^{(\varepsilon)}_{\mu+\frac{1}{2}}(\beta)$$ $\left[\varepsilon=1,2,\quad\gamma=(-1)^{\varepsilon+1},\quad\operatorname{Re}\beta>0,\quad\operatorname{Re}\mu>-1\right]$ GW (337)(16)

2. $$\int_0^{\pi/2}\frac{\cos^\mu x\sin(\beta-\mu x)}{\sin^{2\mu+2}x}e^{-2\beta\cot x}\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(\beta)$$ $[\operatorname{Re}\beta>0,\quad\operatorname{Re}\mu>-1]$ WH

3. $$\int_0^{\pi/2}\frac{\cos^\mu x\cos(\beta-\mu x)}{\sin^{2\mu+2}x}e^{-2\beta\cot x}\,dx=-\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\Gamma(\mu+1)\,Y_{\mu+\frac{1}{2}}(\beta)$$ $[\operatorname{Re}\beta>0,\quad\operatorname{Re}\mu>-1]$ GW (337)(17b)

3.919

1. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp(2\pi \cot x) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)}$$ BI (275)(6), LI (275)(6)

2. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp(\pi \cot x) - 1} = (-1)^{n-1} \frac{n}{2n+1}$$ BI (275)(7), LI (275)(7)

3.92 Trigonometric functions of more complicated arguments combined with exponentials

3.921[6]

1. $$\int_0^\infty e^{-\gamma x} \cos ax^2 (\cos \gamma x - \sin \gamma x)\, dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right)$$
$[a > 0, \quad \mathrm{Re}_\gamma \geq |\mathrm{Im}\, \gamma|]$ ET I 26(28)

2.[10] $$\int_0^{\pi/4} \prod_{n=1}^\infty \exp\left[-\frac{1}{n} \tan^{2n} x\right] = \frac{\pi}{2} - 1$$

3.[10] $$\int_0^{\pi/2} \exp\left[-\sum_{n=1}^\infty \frac{1}{n} \sin^{2n} x\right] = \int_0^{\pi/2} \exp\left[-\sum_{n=1}^\infty \frac{1}{n} \cos^{2n} x\right] = \frac{\pi}{4}$$

3.922

1. $$\int_0^\infty e^{-\beta x^2} \sin ax^2\, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\beta x^2} \sin ax^2\, dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} - \beta}{\beta^2 + a^2}}$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \sin\left(\frac{1}{2} \arctan \frac{a}{\beta}\right)$$
$[\mathrm{Re}\, \beta > 0, \quad a > 0]$ FI II 750, BI (263)(8)

2. $$\int_0^\infty e^{-\beta x^2} \cos ax^2\, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\beta x^2} \cos ax^2\, dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} + \beta}{\beta^2 + a^2}}$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \cos\left(\frac{1}{2} \arctan \frac{a}{\beta}\right)$$
$[\mathrm{Re}\, \beta > 0, \quad a > 0]$ FI II 750, BI (263)(9)

[In formulas **3.922** 3 and 4, $a > 0$, $b > 0$, $\mathrm{Re}\, \beta > 0$, and

$$A = \frac{b^2}{4(a^2 + \beta^2)}, \qquad B = \sqrt{\frac{1}{2}\left(\sqrt{\beta^2 + a^2} + \beta\right)}, \qquad C = \sqrt{\frac{1}{2}\left(\sqrt{\beta^2 + a^2} - \beta\right)}.$$

If a is complex, then $\mathrm{Re}\, \beta > |\mathrm{Im}\, a|$.]

3. $$\int_0^\infty e^{-\beta x^2} \sin ax^2 \cos bx\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \sin Aa - C \cos Aa)$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \sin\left\{\frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\}$$
LI (263)(10), GW (337)(5)

4. $$\int_0^\infty e^{-\beta x^2}\cos ax^2\cos bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta^2+a^2}}e^{-A\beta}\left(B\cos Aa+C\sin Aa\right)$$
$$=\frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}}\exp\left(-\frac{\beta b^2}{4\left(\beta^2+a^2\right)}\right)\cos\left\{\frac{1}{2}\arctan\frac{a}{\beta}-\frac{ab^2}{4\left(\beta^2+a^2\right)}\right\}$$
LI (263)(11), GW (337)(5)

3.923

1. $$\int_{-\infty}^\infty \exp\left[-\left(ax^2+2bx+c\right)\right]\sin\left(px^2+2qx+r\right)\,dx$$
$$=\frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}}\exp\frac{a\left(b^2-ac\right)-\left(aq^2-2bpq+cp^2\right)}{a^2+p^2}$$
$$\times\sin\left\{\frac{1}{2}\arctan\frac{p}{a}-\frac{p\left(q^2-pr\right)-\left(b^2p-2abq+a^2r\right)}{a^2+p^2}\right\}$$
$[a>0]$ GW (337)(3), BI (296)(6)

2. $$\int_{-\infty}^\infty \exp\left[-\left(ax^2+2bx+c\right)\right]\cos\left(px^2+2qx+r\right)\,dx$$
$$=\frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}}\exp\frac{a\left(b^2-ac\right)-\left(aq^2-2bpq+cp^2\right)}{a^2+p^2}$$
$$\times\cos\left\{\frac{1}{2}\arctan\frac{p}{a}-\frac{p\left(q^2-pr\right)-\left(b^2p-2abq+a^2r\right)}{a^2+p^2}\right\}$$
$[a>0]$ GW (337)(3), BI (269)(7)

3.924

1. $$\int_0^\infty e^{-\beta x^4}\sin bx^2\,dx=\frac{\pi}{4}\sqrt{\frac{b}{2\beta}}\exp\left(-\frac{b^2}{8\beta}\right)I_{\frac{1}{4}}\left(\frac{b^2}{8\beta}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ ET 73(22)

2. $$\int_0^\infty e^{-\beta x^4}\cos bx^2\,dx=\frac{\pi}{4}\sqrt{\frac{b}{2\beta}}\exp\left(-\frac{b^2}{8\beta}\right)I_{-\frac{1}{4}}\left(\frac{b^2}{8\beta}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ ET I 15(12)

3.925

1. $$\int_0^\infty e^{-\frac{p^2}{x^2}}\sin 2a^2x^2\,dx=\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{p^2}{x^2}}\sin 2a^2x^2\,dx=\frac{\sqrt{\pi}}{4a}e^{-2ap}\left(\cos 2ap+\sin 2ap\right)$$
$[a>0,\quad b>0]$ BI (268)(12)

2. $$\int_0^\infty e^{-\frac{p^2}{x^2}}\cos 2a^2x^2\,dx=\frac{1}{2}\int_{-\infty}^\infty e^{-\frac{p^2}{x^2}}\cos 2a^2x^2\,dx=\frac{\sqrt{\pi}}{4a}e^{-2ap}\left(\cos 2ap-\sin 2ap\right)$$
$[a>0,\quad b>0]$ BI (268)(13)

3.926 **Notation**:

$$u=\sqrt{\frac{\sqrt{a^2+\beta^2}+\beta}{2}},\qquad v=\sqrt{\frac{\sqrt{a^2+\beta^2}-\beta}{2}}$$

1. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)}\sin ax^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}\left[v\cos\left(2v\sqrt{\gamma}\right)+u\sin\left(2v\sqrt{\gamma}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$ BI (268)(14)

2. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)}\cos ax^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}\left[u\cos\left(2v\sqrt{\gamma}\right)-v\sin\left(2v\sqrt{\gamma}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$ BI (268)(15)

3.927 $$\int_0^\infty e^{-\frac{p}{x}}\sin^2\frac{a}{x}\,dx=a\arctan\frac{2a}{p}+\frac{p}{4}\ln\frac{p^2}{p^2+4a^2}\qquad [a>0,\quad p>0]$$ LI (268)(4)

3.928

1. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\sin\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx=\frac{\sqrt{\pi}}{2r}e^{-2rs\cos(A+B)}\sin\left\{A+2rs\sin(A+B)\right\}$$
BI (268)(22)

2. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\cos\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx=\frac{\sqrt{\pi}}{2r}e^{-2rs\cos(A+B)}\cos\left\{A+2rs\sin(A+B)\right\}$$
BI (268)(23)

3.929 $$\int_0^\infty\left[e^{-x}\cos\left(p\sqrt{x}\right)+pe^{-x^2}\sin px\right]\,dx=1$$ LI (268)(3)

Notation: For the formulas in **3.928**: $a^2+p^2>0$, $r=\sqrt[4]{a^4+p^4}$, $s=\sqrt[4]{b^4+q^4}$, $A=\frac{1}{2}\arctan\frac{a^2}{p^2}$, and $B=\frac{1}{2}\arctan\frac{b^2}{q^2}$.

3.93 Trigonometric and exponential functions of trigonometric functions

3.931

1. $$\int_0^{\pi/2}e^{-p\cos x}\sin\left(p\sin x\right)\,dx=\operatorname{Ei}(-p)-\operatorname{ci}(p)$$ NT 13(27)

2. $$\int_0^{\pi}e^{-p\cos x}\sin\left(p\sin x\right)\,dx=-\int_{-\pi}^{0}e^{-p\cos x}\sin\left(p\sin x\right)\,dx=-2\operatorname{shi}(p)$$ GW (337)(11b)

3. $$\int_0^{\pi/2}e^{-p\cos x}\cos\left(p\sin x\right)\,dx=-\operatorname{si}(p)$$ NT 13(26)

4. $$\int_0^{\pi/2}e^{-p\cos x}\cos\left(p\sin x\right)\,dx=\frac{1}{2}\int_0^{2\pi}e^{-p\cos x}\cos\left(p\sin x\right)\,dx=\pi$$ GW (337)(11a)

3.932

1. $$\int_0^{\pi}e^{p\cos x}\sin\left(p\sin x\right)\sin mx\,dx=\frac{1}{2}\int_0^{2\pi}e^{p\cos x}\sin\left(p\sin x\right)\sin mx\,dx=\frac{\pi}{2}\cdot\frac{p^m}{m!}$$
BI (277)(7), GW (337)(13a)

2. $$\int_0^{\pi} e^{p\cos x}\cos(p\sin x)\cos mx\,dx = \frac{1}{2}\int_0^{2\pi} e^{p\cos x}\cos(p\sin x)\cos mx\,dx = \frac{\pi}{2}\cdot\frac{p^m}{m!}$$
BI (277)(8), GW (337)(13b)

3.933 $$\int_0^{\pi} e^{p\cos x}\sin(p\sin x)\operatorname{cosec} x\,dx = \pi\sinh p$$ BI (278)(1)

3.934

1. $$\int_0^{\pi} e^{p\cos x}\sin(p\sin x)\tan\frac{x}{2}\,dx = \pi\left(1-e^p\right)$$ BI (271)(8)

2. $$\int_0^{\pi} e^{p\cos x}\sin(p\sin x)\cot\frac{x}{2}\,dx = \pi\left(e^p-1\right)$$ BI (272)(5)

3.935 $$\int_0^{\pi} e^{p\cos x}\cos(p\sin x)\frac{\sin 2nx}{\sin x}\,dx = \pi\sum_{k=0}^{n-1}\frac{p^{2k+1}}{(2k+1)!} \qquad [p>0]$$ LI (278)(3)

3.936

1. $$\int_0^{2\pi} e^{p\cos x}\cos(p\sin x - mx)\,dx = 2\int_0^{\pi} e^{p\cos x}\cos(p\sin x - mx)\,dx = \frac{2\pi p^m}{m!}$$
BI (277)(9), GW (337)(14a)

2. $$\int_0^{2\pi} e^{p\sin x}\sin(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\sin\frac{m\pi}{2} \qquad [p>0]$$ GW (337)(14b)

3. $$\int_0^{2\pi} e^{p\sin x}\cos(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\cos\frac{m\pi}{2} \qquad [p>0]$$ GW (337)(14b)

4. $$\int_0^{2\pi} e^{\cos x}\sin(mx - \sin x)\,dx = 0$$ WH

5. $$\int_0^{\pi} e^{\beta\cos x}\cos(ax + \beta\sin x)\,dx = \beta^{-a}\sin(a\pi)\,\gamma(a,\beta)$$ EH II 137(2)

3.937 **Notation**: In formulas **3.937** 1 and 2, $(b-p)^2+(a+q)^2>0$, $m=0,1,2,\ldots$, $A=p^2-q^2+a^2-b^2$, $B=2(pq+ab)$, $C=p^2+q^2-a^2-b^2$, and $D=2(ap+bq)$.

1.[11] $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\sin(a\cos x + b\sin x - mx)\,dx$$
$$= i\pi\left[(b-p)^2+(a+q)^2\right]^{-\frac{m}{2}}\left\{(A+iB)^{m/2}\,I_m\left(\sqrt{C-iD}\right) - (A-iB)^{m/2}\,I_m\left(\sqrt{C+iD}\right)\right\}$$
GW (337)(9b)

2. $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\cos(a\cos x + b\sin x - mx)\,dx$$
$$= \pi\left[(b-p)^2+(a+q)^2\right]^{-\frac{m}{2}}\left\{(A+iB)^{\frac{m}{2}}\,I_m\left(\sqrt{C-iD}\right) + (A-iB)^{\frac{m}{2}}\,I_m\left(\sqrt{C+iD}\right)\right\}$$
GW (337)(9a)

3. $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\sin(q\cos x - p\sin x + mx)\,dx = \frac{2\pi}{m!}\left(p^2+q^2\right)\frac{m}{2}\sin\left(m\arctan\frac{q}{p}\right)$$
GW (337)(12)

4. $$\int_0^{2\pi} \exp(p\cos x + q\sin x)\cos(q\cos x - p\sin x + mx)\,dx = \frac{2\pi}{m!}\left(p^2+q^2\right)\frac{m}{2}\cos\left(m\arctan\frac{q}{p}\right)$$ GW (337)(12)

3.938

1. $$\int_0^{\pi} e^{r(\cos px+\cos qx)}\sin(r\sin px)\sin(r\sin qx)\,dx = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{\Gamma(pk+1)\,\Gamma(qk+1)}r^{(p+q)k}$$ BI (277)(14)

2. $$\int_0^{\pi} e^{r(\cos px+\cos qx)}\cos(r\sin px)\cos(r\sin qx)\,dx = \frac{\pi}{2}\left(2+\sum_{k=1}^{\infty}\frac{r^{(p+q)k}}{\Gamma(pk+1)\,\Gamma(qk+1)}\right)$$ BI (277)(15)

3.939

1. $$\int_0^{\pi} e^{q\cos x}\frac{\sin rx}{1-2p^r\cos rx+p^{2r}}\sin(q\sin x)\,dx = \frac{\pi}{2pr}\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)}$$ $[r>0,\quad 0<p<1]$ BI (278)(15)

2.[3] $$\int_0^{\pi} e^{q\cos x}\frac{1-p^r\cos rx}{1-2p^r\cos rx+p^{2r}}\cos(q\sin x)\,dx = \frac{\pi}{2}\left[2+\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)}\right]$$ $[r>0, 0<p<1]$ BI (278)(16)

3. $$\int_0^{\pi/2}\frac{e^{p\cos 2x}\cos(p\sin 2x)\,dx}{\cos^2 x+q^2\sin^2 x} = \frac{\pi}{2q}\exp\left(p\frac{q-1}{q+1}\right)$$ BI (273)(8)

3.94–3.97 Combinations involving trigonometric functions, exponentials, and powers

3.941

1. $$\int_0^{\infty} e^{-px}\sin qx\frac{dx}{x} = \arctan\frac{q}{p}$$ $[p>0]$ BI (365)(1)

2. $$\int_0^{\infty} e^{-px}\cos qx\frac{dx}{x} = \infty$$ BI (365)(2)

3.942

1. $$\int_0^{\infty} e^{-px}\cos px\frac{x\,dx}{b^4+x^4} = \frac{\pi}{4b^2}\exp\left(-bp\sqrt{2}\right)$$ $[p>0,\quad b>0]$ BI (386)(6)a

2. $$\int_0^{\infty} e^{-px}\cos px\frac{x\,dx}{b^4-x^4} = \frac{\pi}{4b^2}e^{-bp}\sin bp$$ $[p>0,\quad b>0]$ BI (386)(7)a

3.943 $$\int_0^{\infty} e^{-\beta x}(1-\cos ax)\frac{dx}{x} = \frac{1}{2}\ln\frac{a^2+\beta^2}{\beta^2}$$ $[\operatorname{Re}\beta>0]$ BI (367)(6)

3.944

1. $$\int_0^{u} x^{\mu-1}e^{-\beta x}\sin\delta x\,dx = \frac{i}{2}(\beta+i\delta)^{-\mu}\gamma[\mu,(\beta+i\delta)u] - \frac{i}{2}(\beta-i\delta)^{-\mu}\gamma[\mu,(\beta-i\delta)u]$$ $[\operatorname{Re}\mu>-1]$ ET I 318(8)

2. $$\int_u^\infty x^{\mu-1}e^{-\beta x}\sin\delta x\,dx=\frac{i}{2}\left(\beta+i\delta\right)^{-\mu}\Gamma\left[\mu,\left(\beta+i\delta\right)u\right]-\frac{i}{2}\left(\beta-i\delta\right)^{-\mu}\Gamma\left[\mu,\left(\beta-i\delta\right)u\right]$$
$[\operatorname{Re}\beta>|\operatorname{Im}\delta|]$ ET I 318(9)

3. $$\int_0^u x^{\mu-1}e^{-\beta x}\cos\delta x\,dx=\frac{1}{2}\left(\beta+i\delta\right)^{-\mu}\gamma\left[\mu,\left(\beta+i\delta\right)u\right]+\frac{1}{2}\left(\beta-i\delta\right)^{-\mu}\gamma\left[\mu,\left(\beta-i\delta\right)u\right]$$
$[\operatorname{Re}\mu>0]$ ET I 320(28)

4. $$\int_u^\infty x^{\mu-1}e^{-\beta x}\cos\delta x\,dx=\frac{1}{2}\left(\beta+i\delta\right)^{-\mu}\Gamma\left[\mu,\left(\beta+i\delta\right)u\right]+\frac{1}{2}\left(\beta-i\delta\right)^{-\mu}\Gamma\left[\mu,\left(\beta-i\delta\right)u\right]$$
$[\operatorname{Re}\beta>|\operatorname{Im}\delta|]$ ET I 320(29)

5.[11] $$\int_0^\infty x^{\mu-1}e^{-\beta x}\sin\delta x\,dx=\frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}}\sin\left(\mu\arctan\frac{\delta}{\beta}\right)$$
$[\operatorname{Re}\mu>-1,\quad \operatorname{Re}\beta>|\operatorname{Im}\delta|]$
FI II 812, BI (361)(9)

6. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\cos\delta x\,dx=\frac{\Gamma(\mu)}{(\delta^2+\beta^2)^{\frac{\mu}{2}}}\cos\left(\mu\arctan\frac{\delta}{\beta}\right)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>|\operatorname{Im}\delta|]$
FI II 812, BI (361)(10)

7. $$\int_0^\infty x^{\mu-1}\exp\left(-ax\cos t\right)\sin\left(ax\sin t\right)\,dx=\Gamma(\mu)a^{-\mu}\sin(\mu t)$$
$\left[\operatorname{Re}\mu>-1,\quad a>0,\quad |t|<\frac{\pi}{2}\right]$
EH I 13(36)

8. $$\int_0^\infty x^{\mu-1}\exp\left(-ax\cos t\right)\cos\left(ax\sin t\right)\,dx=\Gamma(\mu)a^{-\mu}\cos(\mu t)$$
$\left[\operatorname{Re}\mu>-1,\quad a>0,\quad |t|<\frac{\pi}{2}\right]$
EH I 13(35)

9. $$\int_0^\infty x^{p-1}e^{-qx}\sin\left(qx\tan t\right)\,dx=\frac{1}{q^p}\Gamma(p)\cos^p t\sin pt$$
$\left[|t|<\frac{\pi}{2},\quad q>0\right]$ LO V 288(16)

10. $$\int_0^\infty x^{p-1}e^{-qx}\cos\left(qx\tan t\right)\,dx=\frac{1}{q^p}\Gamma(p)\cos^p(t)\cos pt$$
$\left[|t|<\frac{\pi}{2},\quad q>0\right]$ LO V 288(15)

11. $$\int_0^\infty x^n e^{-\beta x}\sin bx\,dx=n!\left(\frac{\beta}{\beta^2+b^2}\right)^{n+1}\sum_{0\le 2k\le n}(-1)^k\binom{n+1}{2k+1}\left(\frac{b}{\beta}\right)^{2k+1}$$
$$=(-1)^n\frac{\partial^n}{\partial\beta^n}\left(\frac{b}{b^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ GW (336)(3), ET I 72(3)

12. $$\int_0^\infty x^n e^{-\beta x}\cos bx\,dx = n!\left(\frac{\beta}{\beta^2+b^2}\right)^{n+1}\sum_{0\le 2k\le n+1}(-1)^k\binom{n+1}{2k}\left(\frac{b}{\beta}\right)^{2k}$$
$$= (-1)^n\frac{\partial^n}{\partial\beta^n}\left(\frac{\beta}{b^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ GW (336)(4), ET I 14(5)

13. $$\int_0^\infty x^{n-1/2}e^{-\beta x}\sin bx\,dx = (-1)^n\sqrt{\frac{\pi}{2}}\frac{d^n}{d\beta^n}\left(\frac{\sqrt{\sqrt{\beta^2+b^2}-\beta}}{\sqrt{\beta^2+b^2}}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ ET I 72(6)

14. $$\int_0^\infty x^{n-1/2}e^{-\beta x}\cos bx\,dx = (-1)^n\sqrt{\frac{\pi}{2}}\frac{d^n}{d\beta^n}\left(\frac{\sqrt{\sqrt{\beta^2+b^2}+\beta}}{\sqrt{\beta^2+b^2}}\right)$$
$[\operatorname{Re}\beta>0,\quad b>0]$ ET I 15(6)

3.945

1. $$\int_0^\infty\left(e^{-\beta x}\sin ax - e^{-\gamma x}\sin bx\right)\frac{dx}{x^r}$$
$$=\Gamma(1-r)\left\{\left(b^2+\gamma^2\right)\frac{r-1}{2}\sin\left[(r-1)\arctan\frac{b}{\gamma}\right]-\left(a^2+\beta^2\right)\frac{r-1}{2}\sin\left[(r-1)\arctan\frac{a}{\beta}\right]\right\}$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad r<2,\quad r\ne 1]$ BI(371)(6)

2. $$\int_0^\infty\left(e^{-\beta x}\cos ax - e^{-\gamma x}\cos bx\right)\frac{dx}{x^r}$$
$$=\Gamma(1-r)\left\{\left(a^2+\beta^2\right)\frac{r-1}{2}\cos\left[(r-1)\arctan\frac{a}{\beta}\right]-\left(b^2+\gamma^2\right)\frac{r-1}{2}\cos\left[(r-1)\arctan\frac{b}{\gamma}\right]\right\}$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad r<2,\quad r\ne 1]$ BI (371)(7)

3. $$\int_0^\infty\left(ae^{-\beta x}\sin bx - be^{-\gamma x}\sin ax\right)\frac{dx}{x^2} = ab\left[\frac{1}{2}\ln\frac{a^2+\gamma^2}{b^2+\beta^2}+\frac{\gamma}{a}\operatorname{arccot}\frac{\gamma}{a}-\frac{\beta}{b}\operatorname{arccot}\frac{\beta}{b}\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$ BI (368)(22)

3.946

1. $$\int_0^\infty e^{-px}\sin^{2m+1}ax\frac{dx}{x} = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\arctan\frac{(2m-2k+1)a}{p}$$
$[m=0,1,\ldots,\quad p>0]$ GW (336)(9a)

2. $$\int_0^\infty e^{-px}\sin^{2m}ax\frac{dx}{x} = \frac{(-1)^{m+1}}{2^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\ln\left[p^2+(2m-2k)^2a^2\right]-\frac{1}{2^{2m}}\binom{2m}{m}\ln p$$
$[m=1,2,\ldots,\quad p>0]$ GW (336)(9b)

3.947

1. $$\int_0^\infty e^{-\beta x}\sin\gamma x\sin ax\frac{dx}{x} = \frac{1}{4}\ln\frac{\beta^2+(a+\gamma)^2}{\beta^2+(a-\gamma)^2}$$
$[\operatorname{Re}\beta>|\operatorname{Im}\gamma|,\quad a>0]$ BI (365)(5)

2.[11] $$\int_0^\infty e^{-px}\sin ax\sin bx\frac{dx}{x^2}=\frac{|a+b|}{2}\arctan\left(\frac{|a+b|}{p}\right)-\frac{|a-b|}{2}\arctan\left(\frac{|a-b|}{p}\right)+\frac{p}{4}\ln\left(\frac{p^2+(a-b)^2}{p^2+(a+b)^2}\right)$$
$[p>0,\quad$ for $p=0$ see **3.741** 3$]$ BI (368)(1), FI II 744

3.[11] $$\int_0^\infty e^{-px}\sin ax\cos bx\frac{dx}{x}=\arctan\frac{a+b}{p}+\arctan\frac{a-b}{p}$$
$[a\geq 0,\quad p>0]$ GW (336)(10b)

3.948

1.[11] $$\int_0^\infty e^{-\beta x}(\sin ax-\sin bx)\frac{dx}{x}=\arctan\frac{a}{\beta}-\arctan\frac{br}{\beta}$$
$[\operatorname{Re}\beta>0]$, (cf. **3.951** 2)
BI (367)(7)

2. $$\int_0^\infty e^{-\beta x}(\cos ax-\cos bx)\frac{dx}{x}=\frac{1}{2}\ln\frac{b^2+\beta^2}{a^2+\beta^2}$$
$[\operatorname{Re}\beta>0]$, (cf. **3.951** 3)
BI (367)(8), FI II 748a

3. $$\int_0^\infty e^{-\beta x}(\cos ax-\cos bx)\frac{dx}{x^2}=\frac{\beta}{2}\ln\frac{a^2+\beta^2}{b^2+\beta^2}+b\arctan\frac{b}{\beta}-a\arctan\frac{a}{\beta}$$
$[\operatorname{Re}p>0]$ BI (368)(20)

4. $$\int_0^\infty e^{-\beta x}\left(\sin^2 ax-\sin^2 bx\right)\frac{dx}{x^2}=a\arctan\frac{2a}{p}-b\arctan\frac{2b}{p}-\frac{p}{4}\ln\frac{p^2+4a^2}{p^2+4b^2}$$
$[p>0]$ BI (368)(25)

5. $$\int_0^\infty e^{-\beta x}\left(\cos^2 ax-\cos^2 bx\right)\frac{dx}{x^2}=-a\arctan\frac{2a}{p}+b\arctan\frac{2b}{p}+\frac{p}{4}\ln\frac{p^2+4a^2}{p^2+4b^2}$$
$[p>0]$ BI (368)(26)

3.949

1. $$\int_0^\infty e^{-px}\sin ax\sin bx\sin cx\frac{dx}{x}=-\frac{1}{4}\arctan\frac{a+b+c}{p}+\frac{1}{4}\arctan\frac{a+b-c}{p}+\frac{1}{4}\arctan\frac{a-b+c}{p}+\frac{1}{4}\arctan\frac{-a+b+c}{p}$$
$[p>0]$ BI (365)(11)

2.[8] $$\int_0^\infty e^{-px}\sin^2 ax\sin bx\frac{dx}{x}=\frac{1}{2}\arctan\frac{b}{p}-\frac{1}{2}\left[\frac{1}{2}\arctan\frac{2pb}{p^2+4a^2-b^2}+s\frac{\pi}{2}\right]$$
$$\left[s=\begin{cases}1 & \text{for } p^2+4a^2-b^2<0\\ 0 & \text{for } p^2+4a^2-b^2\geq 0\end{cases}\right]$$
BI (365)(8)

3. $$\int_0^\infty e^{-px}\sin^2 ax\cos bx\frac{dx}{x}=\frac{1}{8}\ln\frac{\left[p^2+(2a+b)^2\right]\left[p^2+(2a-b)^2\right]}{\left(p^2+b^2\right)^2}$$
$[p>0]$ BI (365)(9)

4.[8] $$\int_0^\infty e^{-px}\sin ax\cos^2 bx\frac{dx}{x}=\frac{1}{2}\arctan\frac{a}{p}+\frac{1}{2}\left[\frac{1}{2}\arctan\frac{2pa}{p^2+4b^2-a^2}+s\frac{\pi}{2}\right]$$
$$\left[s=\begin{cases}1 & \text{for } p^2+4b^2-a^2<0\\ 0 & \text{for } p^2+4b^2-a^2\geq 0\end{cases}\right]$$
BI (365)(10)

5. $$\int_0^\infty e^{-px}\sin^2 ax\sin bx\sin cx\frac{dx}{x}=\frac{1}{8}\ln\frac{p^2+(b+c)^2}{p^2+(b-c)^2}$$
$$+\frac{1}{16}\ln\frac{\left[p^2+(2a-b+c)^2\right]\left[p^2+(2a+b-c)^2\right]}{\left[p^2+(2a+b+c)^2\right]\left[p^2+(2a-b-c)^2\right]}$$
$[p>0]$ BI (365)(15)

3.951

1. $$\int_0^\infty\left(1-e^{-x}\right)\cos x\frac{dx}{x}=\ln\sqrt{2}$$ FI II 745

2. $$\int_0^\infty\frac{e^{-\gamma x}-e^{-\beta x}}{x}\sin bx\,dx=\arctan\frac{(\beta-\gamma)b}{b^2+\beta\gamma}$$ $[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma\geq 0]$ BI (367)(3)

3. $$\int_0^\infty\frac{e^{-\gamma x}-e^{-\beta x}}{x}\cos bx\,dx=\frac{1}{2}\ln\frac{b^2+\beta^2}{b^2+\gamma^2}$$ $[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma\geq 0]$ BI (367)(4)

4.[11] $$\int_0^\infty\frac{e^{-\gamma x}-e^{-\beta x}}{x^2}\sin bx\,dx=\frac{b}{2}\ln\frac{b^2+\beta^2}{b^2+\gamma^2}+\beta\arctan\frac{b}{\beta}-\gamma\arctan\frac{b}{\gamma}$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$ BI (368)(21)a

5. $$\int_0^\infty\frac{x}{e^{\beta x}-1}\cos bx\,dx=\frac{1}{2b^2}-\frac{\pi^2}{2\beta^2}\operatorname{cosech}^2\frac{b\pi}{\beta}$$ $[\operatorname{Re}\beta>0]$ ET I 15(18)

6. $$\int_0^\infty\left(\frac{1}{e^x-1}-\frac{1}{x}\right)\cos bx\,dx=\ln b-\frac{1}{2}\left[\psi(ib)+\psi(-ib)\right]$$
$[b>0]$ ET I 15(9)

7. $$\int_0^\infty\frac{1-\cos ax}{e^{2\pi x}-1}\cdot\frac{dx}{x}=\frac{a}{4}+\frac{1}{2}\ln\frac{1-e^{-a}}{a}$$ $[a>0]$ BI (387)(10)

8. $$\int_0^\infty\left(e^{-\beta x}-e^{-\gamma x}\cos ax\right)\frac{dx}{x}=\frac{1}{2}\ln\frac{a^2+\gamma^2}{\beta^2}$$ $[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0]$ BI (367)(10)

9. $$\int_0^\infty\frac{\cos px-e^{-px}}{b^4+x^4}\frac{dx}{x}=\frac{\pi}{2b^4}\exp\left(-\frac{1}{2}bp\sqrt{2}\right)\sin\left(\frac{1}{2}bp\sqrt{2}\right)$$
$[p>0]$ BI (390)(6)

10. $$\int_0^\infty\left(\frac{1}{e^x-1}-\frac{\cos x}{x}\right)dx=\boldsymbol{C}$$ NT 65(8)

11. $$\int_0^\infty\left(ae^{-px}-\frac{e^{-qx}}{x}\sin ax\right)\frac{dx}{x}=\frac{a}{2}\ln\frac{a^2+q^2}{p^2}+q\arctan\frac{a}{q}-a$$
$[p>0,\quad q>0]$ BI (368)(24)

12. $$\int_0^\infty \frac{x^{2m}\sin bx}{e^x-1}\,dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{2}\coth b\pi - \frac{1}{2b}\right] \qquad [b>0]$$ GW (336)(15a)

13. $$\int_0^\infty \frac{x^{2m+1}\cos bx}{e^x-1}\,dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{2}\coth b\pi - \frac{1}{2b}\right]$$
$$[b>0]$$ GW (336)(15b)

14. $$\int_0^\infty \frac{x^{2m}\sin bx\,dx}{e^{(2n+1)cx}-e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{4c}\tanh\frac{b\pi}{2c} - \sum_{k=1}^{n}\frac{b}{b^2+(2k-1)^2c^2}\right]$$
$$[b>0]$$ GW (336)(14a)

15. $$\int_0^\infty \frac{x^{2m+1}\cos bx\,dx}{e^{(2n+1)cx}-e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{4c}\tanh\frac{b\pi}{2c} - \sum_{k=1}^{n}\frac{b}{b^2+(2k-1)^2c^2}\right]$$
$$[b>0]$$ GW (336)(14b)

16. $$\int_0^\infty \frac{x^{2m}\sin bx\,dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{4c}\coth\frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1}\frac{b}{b^2+(2k)^2c^2}\right]$$
$$[b>0, \quad c>0]$$ GW (336)(14c)

17. $$\int_0^\infty \frac{x^{2m+1}\cos bx\,dx}{e^{2ncx}-e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{4c}\coth\frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1}\frac{b}{b^2+(2k)^2c^2}\right]$$
$$[b>0, \quad c>0]$$ GW (336)(14d)

18. $$\int_0^\infty \frac{\cos ax-\cos bx}{e^{(2m+1)px}-e^{(2m-1)px}}\frac{dx}{x} = \frac{1}{2}\ln\frac{\cosh\frac{b\pi}{2p}}{\cosh\frac{a\pi}{2p}} - \frac{1}{2}\sum_{k=1}^{m}\ln\frac{b^2+(2k-1)^2p^2}{a^2+(2k-1)^2p^2}$$
$$[p>0]$$ GW (336)(16a)

19. $$\int_0^\infty \frac{\cos ax-\cos bx}{e^{2mpx}-e^{(2m-2)px}}\frac{dx}{x} = \frac{1}{2}\ln\frac{a\sinh\frac{b\pi}{2p}}{b\sinh\frac{a\pi}{2p}} - \frac{1}{2}\sum_{k=1}^{m-1}\ln\frac{b^2+4k^2p^2}{a^2+4k^2p^2}$$
$$[p>0]$$ GW (336)(16b)

20. $$\int_0^\infty \frac{\sin x\sin bx}{1-e^x}\cdot\frac{dx}{x} = \frac{1}{4}\ln\frac{(b+1)\sinh[(b-1)\pi]}{(b-1)\sinh[(b+1)\pi]} \qquad [b^2\neq 1]$$ LO V 305

21. $$\int_0^\infty \frac{\sin^2 ax}{1-e^x}\cdot\frac{dx}{x} = \frac{1}{4}\ln\frac{2a\pi}{\sinh 2a\pi}$$ LO V 306, BI (387)(5)

3.952

1. $$\int_0^\infty xe^{-p^2x^2}\sin ax\,dx = \frac{a\sqrt{\pi}}{4p^3}\exp\left(-\frac{a^2}{4p^2}\right)$$ BI (362)(1)

2. $$\int_0^\infty xe^{-p^2x^2}\cos ax\,dx = \frac{1}{2p^2} - \frac{a}{4p^3}\sum_{k=0}^{\infty}\frac{(-1)^k k!}{(2k+1)!}\left(\frac{a}{p}\right)^{2k+1}$$
$$[a>0]$$ BI (362)(2)

3. $$\int_0^\infty x^2 e^{-p^2x^2}\sin ax\,dx = \frac{a}{4p^4} + \frac{2p^2-a^2}{8p^5}\sum_{k=0}^\infty \frac{(-1)^k k!}{(2k+1)!}\left(\frac{a}{p}\right)^{2k+1}$$
$[a>0]$ BI (362)(4)

4. $$\int_0^\infty x^2 e^{-p^2x^2}\cos ax\,dx = \sqrt{\pi}\frac{2p^2-a^2}{8p^5}\exp\left(-\frac{a^2}{4p^2}\right)$$
BI (362)(5)

5. $$\int_0^\infty x^3 e^{-p^2x^2}\sin ax\,dx = \sqrt{\pi}\frac{6ap^2-a^3}{16p^7}\exp\left(-\frac{a^2}{4p^2}\right)$$
BI (362)(6)

6.[3] $$\int_0^\infty e^{-p^2x^2}\sin ax\frac{dx}{x} = \frac{a\sqrt{\pi}}{2p}\sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)}\left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2}\,\Phi\left(\frac{a}{2p}\right)$$
BI (365)(21)

7. $$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\sin\gamma x\,dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}}\Gamma\left(\frac{1+\mu}{2}\right)\,{}_1F_1\left(1-\frac{\mu}{2};\frac{3}{2};\frac{\gamma^2}{4\beta}\right)$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>-1]$ ET I 318(10)

8.[10] $$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\cos ax\,dx = \frac{1}{2}\beta^{-\mu/2}\Gamma\left(\frac{\mu}{2}\right)e^{-a^2/4\beta}\,{}_1F_1\left(-\frac{\mu}{2}+\frac{1}{2};\frac{1}{2};\frac{a^2}{4\beta}\right)$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>0,\quad a>0]$
ET I 320(30)

9. $$\int_0^\infty x^{2n}e^{-\beta^2x^2}\cos ax\,dx = (-1)^n\frac{\sqrt{\pi}}{2^{n+1}\beta^{2n+1}}\exp\left(-\frac{a^2}{8\beta^2}\right)D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right)$$
$$= (-1)^n\frac{\sqrt{\pi}}{(2\beta)^{2n+1}}\exp\left(-\frac{a^2}{4\beta^2}\right)H_{2n}\left(\frac{a}{2\beta}\right)$$
$\left[|\arg\beta|<\frac{\pi}{4},\quad a>0\right]$ WH, ET I 15(13)

10. $$\int_0^\infty x^{2n+1}e^{-\beta^2x^2}\sin ax\,dx = (-1)^n\frac{\sqrt{\pi}}{2^{n+\frac{3}{2}}\beta^{2n+2}}\exp\left(-\frac{a^2}{8\beta^2}\right)D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right)$$
$$= (-1)^n\frac{\sqrt{\pi}}{(2\beta)^{2n+2}}\exp\left(-\frac{a^2}{4\beta^2}\right)H_{2n+1}\left(\frac{a}{2\beta}\right)$$
$\left[|\arg\beta|<\frac{\pi}{4},\quad a>0\right]$ WH, ET I 74(23)

3.953

1. $$\int_0^\infty x^{\mu-1}e^{-\gamma x-\beta x^2}\sin ax\,dx$$
$$= -\frac{i}{2(2\beta)^{\frac{\mu}{2}}}\exp\left(\frac{\gamma^2-a^2}{8\beta}\right)\Gamma(\mu)\left\{\exp\left(-\frac{ia\gamma}{4\beta}\right)D_{-\mu}\left(\frac{\gamma-ia}{\sqrt{2\beta}}\right)-\exp\left(\frac{ia\gamma}{4\beta}\right)D_{-\mu}\left(\frac{\gamma+ia}{\sqrt{2\beta}}\right)\right\}$$
$[\operatorname{Re}\mu>-1,\quad \operatorname{Re}\beta>0,\quad a>0]$ ET I 318(11)

2. $$\int_0^\infty x^{\mu-1}e^{-\gamma x-\beta x^2}\cos ax\,dx$$
$$= \frac{1}{2(2\beta)^{\frac{\mu}{2}}}\exp\left(\frac{\gamma^2-a^2}{8\beta}\right)\Gamma(\mu)\left\{\exp\left(-\frac{ia\gamma}{4\beta}\right)D_{-\mu}\left(\frac{\gamma-ia}{\sqrt{2\beta}}\right)+\exp\left(\frac{ia\gamma}{4\beta}\right)D_{-\mu}\left(\frac{\gamma+ia}{\sqrt{2\beta}}\right)\right\}$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0,\quad a>0]$ ET I 16(18)

3. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\sin ax\,dx=\frac{i\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\exp\left[-\frac{(\gamma-ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right]\right.$$
$$\left.-(\gamma+ia)\exp\left[-\frac{(\gamma+ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\}$$
$[\operatorname{Re}\beta>0,\quad a>0]$ ET I 74(28)

4. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\cos ax\,dx=-\frac{\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\exp\frac{(\gamma-ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right]\right.$$
$$\left.+(\gamma+ia)\exp\frac{(\gamma+ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\}+\frac{1}{2\beta}$$
$[\operatorname{Re}\beta>0,\quad a>0]$ ET I 16(17)

3.954

1.[11] $$\int_0^\infty e^{-\beta x^2}\sin ax\frac{x\,dx}{\gamma^2+x^2}=-\frac{\pi}{4}e^{\beta\gamma^2}\left[2\sinh a\gamma+e^{-\gamma a}\,\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-e^{\gamma a}\,\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad a>0]$
ET I 74(26)a

2.[11] $$\int_0^\infty e^{-\beta x^2}\cos ax\frac{dx}{\gamma^2+x^2}=\frac{\pi}{4\gamma}e^{\beta\gamma^2}\left[2\cosh a\gamma-e^{-\gamma a}\,\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right)-e^{\gamma a}\,\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\gamma>0,\quad a>0]$
ET I 15(15)

3.955 $$\int_0^\infty x^\nu e^{-\frac{x^2}{2}}\cos\left(\beta x-\nu\frac{\pi}{2}\right)dx=\sqrt{\frac{\pi}{2}}e^{-\frac{\beta^2}{4}}\,D_\nu(\beta)\qquad[\operatorname{Re}\nu>-1]$$ EH II 120(4)

3.956 $$\int_0^\infty e^{-x^2}(2x\cos x-\sin x)\sin x\frac{dx}{x^2}=\sqrt{\pi}\frac{e-1}{2e}$$ BI (369)(19)

3.957

1. $$\int_0^\infty x^{\mu-1}\exp\left(\frac{-\beta^2}{4x}\right)\sin ax\,dx$$
$$=\frac{i}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}}\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\frac{\pi i}{4}}\sqrt{a}\right)-\exp\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\pi i/4}\sqrt{a}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu<1,\quad a>0]$ ET I 318(12)

2. $$\int_0^\infty x^{\mu-1}\exp\left(\frac{-\beta^2}{4x}\right)\cos ax\,dx$$
$$=\frac{1}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}}\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\pi i/4}\sqrt{a}\right)+\exp\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\pi i/4}\sqrt{a}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\mu<1,\quad a>0]$ ET I 320(32)a

3.958

1. $$\int_{-\infty}^\infty x^n e^{-(ax^2+bx+c)}\sin(px+q)\,dx=-\left(\frac{-1}{2a}\right)^n\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-p^2}{4a}-c\right)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2k)!k!}a^k$$
$$\times\sum_{j=0}^{n-2k}\binom{n-2k}{j}b^{n-2k-j}p^j\sin\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$
$[a>0]$ GW (37)(1b)

2. $$\int_{-\infty}^{\infty} x^n e^{-\left(ax^2+bx+c\right)} \cos(px+q)\,dx = \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!} a^k \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} p^j \cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$
$[a>0]$ GW (337)(1a)

3.959 $$\int_0^{\infty} x e^{-p^2x^2} \tan ax\,dx = \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2k^2}{p^2}\right)$$
$[p>0]$ BI (362)(15)

3.961

1. $$\int_0^{\infty} \exp\left(-\beta\sqrt{\gamma^2+x^2}\right) \sin ax \frac{x\,dx}{\sqrt{\gamma^2+x^2}} = \frac{a\gamma}{\sqrt{a^2+\beta^2}} K_1\left(\gamma\sqrt{a^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$
ET I 75(36)

2. $$\int_0^{\infty} \exp\left[-\beta\sqrt{\gamma^2+x^2}\right] \cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} = K_0\left(\gamma\sqrt{a^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$
ET I 17(27)

3.962

1. $$\int_0^{\infty} \frac{\sqrt{\sqrt{\gamma^2+x^2}-\gamma}\exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}} \sin ax\,dx = \sqrt{\frac{\pi}{2}} \frac{a\exp\left(-\gamma\sqrt{a^2+\beta^2}\right)}{\sqrt{\beta^2+a^2}\sqrt{\beta+\sqrt{a^2+\beta^2}}}$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$
ET I 75(38)

2. $$\int_0^{\infty} \frac{x\exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}\sqrt{\sqrt{\gamma^2+x^2}-\gamma}} \cos ax\,dx = \sqrt{\frac{\pi}{2}} \frac{\sqrt{\beta+\sqrt{a^2+\beta^2}}}{\sqrt{a^2+\beta^2}} \exp\left[-\gamma\sqrt{a^2+\beta^2}\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$
ET I 17(29)

3.963

1. $$\int_0^{\infty} e^{-\tan^2 x} \frac{\sin x}{\cos^2 x} \frac{dx}{x} = \frac{\sqrt{\pi}}{2}$$
BI (391)(1)

2. $$\int_0^{\pi/2} e^{-p\tan x} \frac{x\,dx}{\cos^2 x} = \frac{1}{p}\left[\operatorname{ci}(p)\sin p - \cos p\,\operatorname{si}(p)\right] \qquad [p>0] \qquad (\text{cf. } \mathbf{3.339})$$
BI (396)(3)

3.[8] $$\int_0^{\pi/2} x e^{-\tan^2 x} \sin 4x \frac{dx}{\cos^8 x} = -\frac{3}{2}\sqrt{\pi}$$
BI (396)(5)

4.[8] $$\int_0^{\pi/2} x e^{-\tan^2 x} \sin^3 2x \frac{dx}{\cos^8 x} = 2\sqrt{\pi}$$
BI (396)(6)

3.964

1. $$\int_0^{\pi/2} xe^{-p\tan x}\frac{p\sin x-\cos x}{\cos^3 x}\,dx=-\sin p\,\mathrm{si}(p)-\mathrm{ci}(p)\cos p$$ $[p>0]$ LI (396)(4)

2. $$\int_0^{\pi/2} xe^{-p\tan^2 x}\frac{p-\cos^2 x}{\cos^4 x\cot x}\,dx=\frac{1}{4}\sqrt{\frac{\pi}{p}}$$ $[p>0]$ BI (396)(7)

3.[8] $$\int_0^{\pi/2} xe^{-p\tan^2 x}\frac{p-2\cos^2 x}{\cos^6 x\cot x}\,dx=\frac{1+2p}{8p}\sqrt{\frac{\pi}{p}}$$ $[p>0]$ BI (396)(8)

3.965

1. $$\int_0^\infty xe^{-\beta x}\sin ax^2\sin\beta x\,dx=\frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}}e^{-\frac{\beta^2}{2a}}$$ $\left[|\arg\beta|<\frac{\pi}{4},\quad a>0\right]$ ET I 84(17)

2. $$\int_0^\infty xe^{-\beta x}\cos ax^2\cos\beta x\,dx=\frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}}e^{-\frac{\beta^2}{2a}}$$ $[a>0,\quad \mathrm{Re}\,\beta>|\mathrm{Im}\,\beta|]$ ET 26(27)

3.966

1. $$\int_0^\infty xe^{-px}\cos\left(2x^2+px\right)\,dx=0$$ $[p>0]$ BI (361)(16)

2. $$\int_0^\infty xe^{-px}\cos\left(2x^2-px\right)\,dx=\frac{p\sqrt{\pi}}{8}\exp\left(-\frac{1}{4}p^2\right)$$ $[p>0]$ BI (361)(17)

3. $$\int_0^\infty x^2e^{-px}\left[\sin\left(2x^2+px\right)+\cos\left(2x^2+px\right)\right]\,dx=0$$ $[p>0]$ BI (361)(18)

4. $$\int_0^\infty x^2e^{-px}\left[\sin\left(2x^2-px\right)-\cos\left(2x^2-px\right)\right]\,dx=\frac{\sqrt{\pi}}{16}\left(2-p^2\right)\exp\left(-\frac{1}{4}p^2\right)$$ BI (361)(19)

5.[3] $$\int_0^\infty x^{\mu-1}e^{-x}\cos\left(x+ax^2\right)\,dx=\frac{e^{\frac{1}{4a}}\,\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}}\cos\frac{\mu\pi}{4}\,D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$$ $[\mathrm{Re}\,\mu>0,\quad a>0]$ ET I 321(37)

6.[6] $$\int_0^\infty x^{\mu-1}e^{-x}\sin\left(x+ax^2\right)\,dx=\frac{e^{\frac{1}{4a}}\,\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}}\sin\frac{\mu\pi}{4}\,D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$$ $[\mathrm{Re}\,\mu>-1,\quad a>0]$ ET I 319(18)

3.967

1. $$\int_0^\infty e^{-\frac{\beta^2}{x^2}}\sin a^2x^2\frac{dx}{x^2}=\frac{\sqrt{\pi}}{2\beta}e^{-\sqrt{2}a\beta}\sin\left(\sqrt{2}a\beta\right)$$ $[\mathrm{Re}\,\beta>0,\quad a>0]$ ET I 75(30)a, BI(369)(3)a

2. $$\int_0^\infty e^{-\frac{\beta^2}{x^2}}\cos a^2x^2\frac{dx}{x^2}=\frac{\sqrt{\pi}}{2\beta}e^{-\sqrt{2}a\beta}\cos\left(\sqrt{2}a\beta\right)$$ $[\mathrm{Re}\,\beta>0,\quad a>0]$ BI (369)(4), ET I 16(20)

3. $$\int_0^\infty x^2 e^{-\beta x^2} \cos ax^2\, dx = \frac{\sqrt{\pi}}{4\sqrt[4]{(a^2+\beta^2)^3}} \cos\left(\frac{3}{2}\arctan\frac{a}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ ET I 14(3)a

3.968

1. $$\int_0^\infty e^{-\beta x^2} \sin ax^4\, dx = -\frac{\pi}{8}\sqrt{\frac{\beta}{a}}\left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\cos\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8} + Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\sin\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8}\right]$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 75(34)

2. $$\int_0^\infty e^{-\beta x^2} \cos ax^4\, dx = \frac{\pi}{8}\sqrt{\frac{\beta}{a}}\left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\sin\left(\frac{\beta^2}{8a}+\frac{\pi}{8}\right) - Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\cos\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8}\right]$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 16(24)

3.969

1. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\cos\left(2pqx^3\right) + q\sin\left(2pqx^3\right)\right] dx = \frac{\sqrt{\pi}}{2}$$
BI (363)(7)

2. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\sin\left(2pqx^3\right) - q\cos\left(2pqx^3\right)\right] dx = 0$$
BI (363)(8)

3.971 **Notation**: In formulas **3.971** 1 and 2, $p \geq 0$, $q \geq 0$, $r = \sqrt[4]{a^2+p^2}$, $s = \sqrt[4]{b^2+q^2}$, $A = \arctan\frac{a}{p}$, and $B = \arctan\frac{b}{q}$.

1. $$\int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2} = \frac{1}{2}\int_{-\infty}^\infty \exp\left(-px^2-\frac{q}{x^2}\right)\sin\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2}$$
$$= \frac{\sqrt{\pi}}{2s}\exp\left[-2rs\cos(A+B)\right]\sin\left[A+2rs\sin(A+B)\right]$$
BI (369)(16, 17)

2. $$\int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\cos\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2} = \frac{1}{2}\int_{-\infty}^\infty \exp\left(-px^2-\frac{q}{x^2}\right)\cos\left(ax^2+\frac{b}{x^2}\right)\frac{dx}{x^2}$$
$$= \frac{\sqrt{\pi}}{2s}\exp\left[-2rs\cos(A+B)\right]\cos\left[A+2rs\sin(A+B)\right]$$
BI (369)(15, 18)

3.972

1. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^4+x^4}\right]\sin ax^2\frac{dx}{\sqrt{\gamma^4+x^4}}$$
$$= \sqrt{\frac{a\pi}{8}}\, I_{1/4}\left[\frac{\gamma^2}{2}\left(\sqrt{\beta^2+a^2}-\beta\right)\right] K_{1/4}\left[\frac{\gamma^2}{4}\left(\sqrt{\beta^2+a^2}+\beta\right)\right]$$
$\left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right]$ ET I 75(37)

2. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^4+x^4}\right]\cos ax^2\frac{dx}{\sqrt{\gamma^4+x^4}}$$
$$= \sqrt{\frac{a\pi}{8}}\, I_{-1/4}\left[\frac{\gamma^2}{2}\left(\sqrt{\beta^2+a^2}-\beta\right)\right] K_{1/4}\left[\frac{\gamma^2}{4}\left(\sqrt{\beta^2+a^2}+\beta\right)\right]$$
$\left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right]$ ET I 17(28)

3.973

1. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\,\frac{dx}{x} = \frac{\pi}{2}\left(e^p - 1\right) \qquad [p>0, \quad a>0]$$ WH, FI II 725

2. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax + bx)\,\frac{x\,dx}{c^2+x^2} = \frac{\pi}{2}\exp\left(-cb + pe^{-ac}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad p>0]$$ BI (372)(3)

3. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax + bx)\,\frac{dx}{c^2+x^2} = \frac{\pi}{2c}\exp\left(-cb + pe^{-ac}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad p>0]$$ BI (372)(4)

4. $$\int_0^\infty \exp(p\cos x)\sin(p\sin x + nx)\,\frac{dx}{x} = \frac{\pi}{2}e^p \qquad [p>0]$$ BI (366)(2)

5. $$\int_0^\infty \exp(p\cos x)\sin(p\sin x)\cos nx\frac{dx}{x} = \frac{p^n}{n!}\cdot\frac{\pi}{4} + \frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{p^k}{k!}$$
$$[p>0]$$ LI (366)(3)

6. $$\int_0^\infty \exp(p\cos x)\cos(p\sin x)\sin nx\frac{dx}{x} = \frac{\pi}{2}\sum_{k=0}^{n-1}\frac{p^k}{k!} + \frac{p^n}{n!}\frac{\pi}{4}$$
$$[p>0]$$ LI (366)(4)

3.974

1. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab}\right)\right]}{2b\sinh ab}$$
$$[a>0, \quad b>0, \quad p>0]$$ BI (391)(4)

2. $$\int_0^\infty \left[1-\exp(p\cos ax)\cos(p\sin ax)\right]\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab}\right)\right]}{2\sinh ab}$$
$$[a>0, \quad b>0, \quad p>0]$$ BI (391)(5)

3. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax + ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab} - ab\right)\right]}{2b\sinh ab}$$
$$[a>0, \quad b>0, \quad p>0]$$ BI (391)(6)

4. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax + ax)\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab} - ab\right)\right]}{2\sinh ab}$$
$$[a>0, \quad b>0, \quad p>0]$$ BI (391)(7)

5. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\,\frac{x\,dx}{b^2-x^2} = \frac{\pi}{2}\left[1-\exp(p\cos ab)\cos(p\sin ab)\right]$$
$$[p>0, \quad a>0]$$ BI (378)(1)

6. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax)\,\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\exp(p\cos ab)\sin(p\sin ab)$$
$$[a>0, \quad b>0, \quad p>0]$$ BI (378)(2)

7. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\tan ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\cdot\tanh ab\left[\exp\left(pe^{-ab}\right)-e^p\right]$$
$[a>0,\quad b>0,\quad p>0]$ BI (372)(14)

8. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\cot ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\coth ab\left[e^p-\exp\left(pe^{-ab}\right)\right]$$
$[a>0,\quad b>0,\quad p>0]$ BI (372)(15)

9. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\operatorname{cosec} ab\left[e^p-\exp(p\cos ab)\cos(p\sin ab)\right]$$
$[a>0,\quad b>0,\quad p>0]$ BI (391)(12)

10. $$\int_0^\infty \left[1-\exp(p\cos ax)\cos(p\sin ax)\right]\operatorname{cosec} ax\frac{x\,dx}{b^2-x^2} = -\frac{\pi}{2}\exp(p\cos ab)\sin(p\sin ab)\operatorname{cosec} ab$$
$[a>0,\quad b>0,\quad p>0]$ BI (391)(13)

3.975

1. $$\int_0^\infty \frac{\sin\left(\beta\arctan\frac{x}{\gamma}\right)}{\left(\gamma^2+x^2\right)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}-1} = \frac{1}{2}\zeta(\beta,\gamma)-\frac{1}{4\gamma^\beta}-\frac{\gamma^{1-\beta}}{2(\beta-1)}$$
$[\operatorname{Re}\beta>1,\quad \operatorname{Re}\gamma>0]$ WH, ET I 26(7)

2. $$\int_0^\infty \frac{\sin(\beta\arctan x)}{\left(1+x^2\right)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}+1} = \frac{1}{2(\beta-1)}-\frac{\zeta(\beta)}{2^\beta}$$
$[\operatorname{Re}\beta>1]$ EH I 33(13)

3.976 $$\int_0^\infty \left(1+x^2\right)^{\beta-\frac{1}{2}}e^{-px^2}\cos\left[2px+(2\beta-1)\arctan x\right]dx = \frac{e^{-p}}{2p^\beta}\sin\pi\beta\,\Gamma(\beta)$$
$[\operatorname{Re}\beta>0,\quad p>0]$ WH

3.98–3.99 Combinations of trigonometric and hyperbolic functions

3.981

1. $$\int_0^\infty \frac{\sin ax}{\sinh\beta x}dx = \frac{\pi}{2\beta}\tanh\frac{a\pi}{2\beta}$$
$[\operatorname{Re}\beta>0,\quad a>0]$ BI (264)(16)

2. $$\int_0^\infty \frac{\sin ax}{\cosh\beta x}dx = -\frac{\pi}{2\beta}\tanh\frac{a\pi}{2\beta}-\frac{i}{2\beta}\left[\psi\left(\frac{\beta+ai}{4\beta}\right)-\psi\left(\frac{\beta-ai}{4\beta}\right)\right]$$
$[\operatorname{Re}\beta>0,\quad a>0]$
GW (335)(12), ET I 88(1)

3. $$\int_0^\infty \frac{\cos ax}{\cosh\beta x}dx = \frac{\pi}{2\beta}\operatorname{sech}\frac{a\pi}{2\beta}$$
$[\operatorname{Re}\beta>0,\quad \text{all real } a]$ BI (264)(14)

4. $$\int_0^\infty \sin ax\frac{\sinh\beta x}{\sinh\gamma x}dx = \frac{\pi}{2\gamma}\frac{\sinh\frac{a\pi}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}+\frac{i}{2\gamma}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right)-\psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right]$$
$[|\operatorname{Re}\beta|<\operatorname{Re}\gamma,\quad a>0]$ ET I 88(5)

5. $$\int_0^\infty \cos ax\frac{\sinh\beta x}{\sinh\gamma x}dx = \frac{\pi}{2\gamma}\frac{\sin\frac{\pi\beta}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$
$[|\operatorname{Re}\beta|<\operatorname{Re}\gamma]$ BI (265)(7)

6. $$\int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh \gamma x}\,dx = \frac{\pi}{\gamma}\frac{\sin\frac{\beta\pi}{2\gamma}\sinh\frac{a\pi}{2\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ BI (265)(2)

7. $$\int_0^\infty \cos ax \frac{\sinh \beta x}{\cosh \gamma x}\,dx = \frac{1}{4\gamma}\left[\left\{\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta-ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ia}{4\gamma}\right)\right\}\right.$$
$$\left.-\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)+\frac{2\pi\sin\frac{\pi\beta}{\gamma}}{\cos\frac{\pi\beta}{\gamma}+\cosh\frac{\pi a}{\gamma}}\right]$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ ET I 31(13)

8. $$\int_0^\infty \sin ax \frac{\cosh \beta x}{\sinh \gamma x}\,dx = \frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{\pi a}{\gamma}}{\cosh\frac{\pi a}{\gamma}+\cos\frac{\pi\beta}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ BI (265)(4)

9. $$\int_0^\infty \sin ax \frac{\cosh \beta x}{\cosh \gamma x}\,dx = \frac{i}{4\gamma}\left[\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ai}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)\right.$$
$$\left.-\psi\left(\frac{3\gamma-\beta-ai}{4\gamma}\right)-\frac{2\pi i\sinh\frac{\pi a}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}\right]$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ ET I 88(6)

10. $$\int_0^\infty \cos ax \frac{\cosh \beta x}{\cosh \gamma x}\,dx = \frac{\pi}{\gamma}\frac{\cos\frac{\beta\pi}{2\gamma}\cosh\frac{a\pi}{2\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad \text{all real } a]$ BI (265)(6)

11.[11] $$\int_0^{\pi/2} \cos^{2m} x \cosh \beta x\,dx = \frac{(2m)!\sinh\frac{\pi\beta}{2}}{\beta\left(\beta^2+2^2\right)\ldots\left[\beta^2+(2m)^2\right]}$$
$[\beta \neq 0]$ WA 620a

12.[11] $$\int_0^{\pi/2} \cos^{2m+1} x \cosh \beta x\,dx = \frac{(2m+1)!\cosh\frac{\pi\beta}{2}}{\left(\beta^2+1^2\right)\left(\beta^2+3^2\right)\ldots\left[\beta^2+(2m+1)^2\right]}$$ WA 620a

3.982

1. $$\int_0^\infty \frac{\cos ax}{\cosh^2\beta x}\,dx = \frac{a\pi}{2\beta^2\sinh\frac{a\pi}{2\beta}}$$ $[\operatorname{Re}\beta>0, \quad a>0]$ BI (264)(16)

2. $$\int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh^2 \gamma x}\,dx = \frac{\pi\left(a\sin\frac{\beta\pi}{2\gamma}\cosh\frac{a\pi}{2\gamma}-\beta\cos\frac{\beta\pi}{2\gamma}\sinh\frac{a\pi}{2\gamma}\right)}{\gamma^2\left(\cosh\frac{a\pi}{\gamma}-\cos\frac{\beta\pi}{\gamma}\right)}$$
$[|\operatorname{Re}\beta| < 2\operatorname{Re}\gamma, \quad a>0]$ ET I 88(9)

3.[11] $$\int_0^\infty \frac{\sin^2 x\cos ax}{\sinh^2 hx}\,dx = \frac{\pi}{4}\left\{\frac{a+2}{1-e^{-\pi(a+2)}}-\frac{2a}{1-e^{-\pi a}}+\frac{a-2}{1-e^{-\pi(a-2)}}\right\} = I(a)$$
$$\left[I(0)=\frac{1}{2}\left(\pi\coth\pi-1\right), \quad I(\pm 2)=\frac{1}{4}+\frac{\pi}{2}\left(\coth 2\pi-\coth\pi\right)\right]$$

3.983

1.[6] $$\int_0^\infty \frac{\cos ax\,dx}{b\cosh\beta x+c}=\frac{\pi\sin\left(\frac{a}{\beta}\operatorname{arccosh}\frac{c}{b}\right)}{\beta\sqrt{c^2-b^2}\sinh\frac{a\pi}{\beta}}\qquad [c>b>0]$$
$$=\frac{\pi\sinh\left(\frac{a}{\beta}\arccos\frac{c}{b}\right)}{\beta\sqrt{b^2-c^2}\sinh\frac{a\pi}{\beta}}\qquad [b>|c|>0]$$
$[\operatorname{Re}\beta>0,\quad a>0]$ GW (335)(13a)

2. $$\int_0^\infty \frac{\cos ax\,dx}{\cosh\beta x+\cos\gamma}=\frac{\pi}{\beta}\frac{\sinh\frac{a\gamma}{\beta}}{\sin\gamma\sinh\frac{a\pi}{\beta}}\qquad \left[\pi\operatorname{Re}\beta<\operatorname{Im}\overline{\beta}\gamma,\quad a>0\right]$$ BI (267)(3)

3.[3] $$\int_0^\infty \frac{\cos ax\,dx}{\cosh x-\cosh b}=-\pi\coth a\pi\frac{\sin ab}{\sinh b}\qquad [a>0,\quad b>0]$$ ET I 30(8)

4. $$\int_0^\infty \frac{\cos ax\,dx}{1+2\cosh\left(\sqrt{\frac{2}{3}\pi x}\right)}=\frac{\sqrt{\frac{\pi}{2}}}{1+2\cosh\left(\sqrt{\frac{2}{3}\pi a}\right)}\qquad [a>0]$$ ET I 30(9)

5. $$\int_0^\infty \frac{\sin ax\sinh\beta x}{\cosh\gamma x+\cos\delta}\,dx=\frac{\pi\left\{\sin\left[\frac{\beta}{\gamma}(\pi-\delta)\right]\sinh\left[\frac{a}{\gamma}(\pi+\delta)\right]-\sin\left[\frac{\beta}{\gamma}(\pi+\delta)\right]\sinh\left[\frac{a}{\gamma}(\pi-\delta)\right]\right\}}{\gamma\sin\delta\left(\cosh\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}\right)}$$
$[\pi\operatorname{Re}\gamma>|\operatorname{Re}\overline{\gamma}\delta|,\quad |\operatorname{Re}\beta|<\operatorname{Re}\gamma,\quad a>0]$ BI (267)(2)

6. $$\int_0^\infty \frac{\cos ax\cosh\beta x}{\cosh\gamma x+\cos b}\,dx=\frac{\pi\left\{\cos\left[\frac{\beta}{\gamma}(\pi-b)\right]\cosh\left[\frac{a}{\gamma}(\pi+b)\right]-\cos\left[\frac{\beta}{\gamma}(\pi+b)\right]\cosh\left[\frac{a}{\gamma}(\pi-b)\right]\right\}}{\gamma\sin b\left(\cosh\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}\right)}$$
$[|\operatorname{Re}\beta|<\operatorname{Re}\gamma,\quad 0<b<\pi,\quad a<0]$
BI (267)(6)

7. $$\int_0^\infty \frac{\cos ax\,dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{\nu+1}}=\Gamma(\nu+1-ai)e^{a\pi}\frac{Q_\nu^{ai}(\beta)}{\Gamma(\nu+1)}$$
$[\operatorname{Re}\nu>-1,\quad |\arg(\beta+1)|<\pi,\quad a>0]$
ET I 30(10)

3.984

1.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\sin ax\sinh cx}{\cosh x+\cos b}\,dx=\pi\frac{\cosh ab}{\sinh a\pi}\qquad [|b|\le\pi,\quad a\text{ real}]$$ BI (267)(1)

2.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\cos ax\cosh cx}{\cosh x+\cos b}\,dx=-\pi\cot b\frac{\sinh ab}{\sinh a\pi}\qquad [0<|b|<\pi,\quad a\text{ real}]$$ BI (267)(5)

3.[8] $$\int_0^\infty \frac{\sin ax\sinh\frac{x}{2}}{\cosh x+\cos\beta}\,dx=\frac{\pi\sinh a\beta}{2\sin\frac{\beta}{2}\cosh a\pi}\qquad [Re\beta<\pi,\quad a>0]$$ ET I 80(10)

4. $$\int_0^\infty \frac{\cos ax\cosh\frac{\beta}{2}x}{\cosh\beta x+\cosh\gamma}\,dx=\frac{\pi\cos\frac{a\gamma}{\beta}}{2\beta\cosh\frac{\gamma}{2}\cosh\frac{a\pi}{\beta}}\qquad \left[\pi\operatorname{Re}\beta>\left|\operatorname{Im}\left(\overline{\beta}\gamma\right)\right|\right]$$ ET I 31(16)

5. $$\int_0^\infty \frac{\sin ax\sinh\beta x}{\cosh 2\beta x+\cos 2ax}\,dx=\frac{a\pi}{4\left(a^2+\beta^2\right)}\qquad [a>0,\quad \operatorname{Re}\beta>0]$$ BI (267)(7)

6. $$\int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh 2\beta x + \cos 2ax}\, dx = \frac{\beta\pi}{4\left(a^2+\beta^2\right)} \qquad [\operatorname{Re}\beta > 0, \quad a > 0]$$ BI (267)(8)

7.[8] $$\int_0^\infty \frac{\sinh^{2\mu-1} x \cosh^{2\varrho-2\nu+1} x}{\left(\cosh^2 x - \beta \sinh^2 x\right)^\varrho}\, dx = \frac{1}{2}\,\mathrm{B}(\mu, \nu-\mu)\ {}_2F_1(\varrho, \mu; \nu; \beta)$$

$[\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ EH I 115(12)

3.985

1. $$\int_0^\infty \frac{\cos ax\, dx}{\cosh^\nu \beta x} = \frac{2^{\nu-2}}{\beta\,\Gamma(\nu)}\,\Gamma\left(\frac{\nu}{2}+\frac{ai}{2\beta}\right)\Gamma\left(\frac{\nu}{2}-\frac{ai}{2\beta}\right) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0, \quad a > 0]$$ ET I 30(5)

2. $$\int_0^\infty \frac{\cos ax\, dx}{\cosh^{2n} \beta x} = \frac{4^{n-1}\pi a}{2(2n-1)!\beta^2 \sinh \frac{a\pi}{2\beta}} \prod_{k=1}^{n-1}\left(\frac{a^2}{4\beta^2}+k^2\right)$$
$$= \frac{\pi a\left(a^2+2^2\beta^2\right)\left(a^2+4^2\beta^2\right)\cdots\left[a^2+(2n-2)^2\beta^2\right]}{2(2n-1)!\beta^{2n}\sinh\frac{a\pi}{2\beta}}$$
$[n \geq 2, \quad a > 0]$ ET I 30(3)

3. $$\int_0^\infty \frac{\cos ax\, dx}{\cosh^{2n+1} \beta x} = \frac{\pi 2^{2n-1}}{(2n)!\beta\cosh\frac{a\pi}{2\beta}} \prod_{k=1}^{n}\left[\frac{a^2}{4\beta^2}+\left(\frac{2k-1}{2}\right)^2\right]$$
$$= \frac{\pi\left(a^2+\beta^2\right)\left(a^2+3^2\beta^2\right)\cdots\left[a^2+(2n-1)^2\beta^2\right]}{2(2n)!\beta^{2n+1}\cosh\frac{a\pi}{2\beta}}$$
$[\operatorname{Re}\beta > 0, \quad n = 0, 1, \ldots, \text{ all real } a]$ ET I 30(4)

3.986

1. $$\int_0^\infty \frac{\sin\beta x \sin\gamma x}{\cosh\delta x}\, dx = \frac{\pi}{\delta}\cdot\frac{\sinh\frac{\beta\pi}{2\delta}\sinh\frac{\gamma\pi}{2\delta}}{\cosh\frac{\beta}{\delta}\pi+\cosh\frac{\gamma}{\delta}\pi} \qquad [|\operatorname{Im}(\beta+\gamma)| < \operatorname{Re}\delta]$$ BI (264)(19)

2. $$\int_0^\infty \frac{\sin\alpha x \cos\beta x}{\sinh\gamma x}\, dx = \frac{\pi\sinh\frac{\pi\alpha}{\gamma}}{2\gamma\left(\cosh\frac{\alpha\pi}{\gamma}+\cosh\frac{\beta\pi}{\gamma}\right)} \qquad [|\operatorname{Im}(\alpha+\beta)| < \operatorname{Re}\gamma]$$ LI (264)(20)

3. $$\int_0^\infty \frac{\cos\beta x \cos\gamma x}{\cosh\delta x}\, dx = \frac{\pi}{\delta}\cdot\frac{\cosh\frac{\beta\pi}{2\delta}\cosh\frac{\gamma\pi}{2\delta}}{\cosh\frac{\beta\pi}{\delta}+\cosh\frac{\gamma\pi}{\delta}} \qquad [|\operatorname{Im}(\beta+\gamma)| < \operatorname{Re}\delta]$$ BI (264)(21)

4.[3] $$\int_0^\infty \frac{\sin^2\beta x}{\sinh^2\pi x}\, dx = \frac{\beta}{\pi\left(e^{2\beta}-1\right)}+\frac{\beta-1}{2\pi} = \frac{\beta\coth\beta-1}{2\pi}$$

$[|\operatorname{Im}\beta| < \pi]$ EH I 44(3)

3.987

1. $$\int_0^\infty \sin ax\,(1-\tanh\beta x)\, dx = \frac{1}{a}-\frac{\pi}{2\beta\sinh\frac{\alpha\pi}{2\beta}} \qquad [\operatorname{Re}\beta > 0]$$ ET I 88(4)a

2. $$\int_0^\infty \sin ax\,(\coth\beta x-1)\, dx = \frac{\pi}{2\beta}\coth\frac{a\pi}{2\beta}-\frac{1}{a} \qquad [\operatorname{Re}\beta > 0]$$ ET I 88(3)

3.988

1. $$\int_0^{\pi/2} \frac{\cos ax \sinh(2b\cos x)}{\sqrt{\cos x}}\,dx = \frac{\pi}{2}\sqrt{\pi b}\, I_{\frac{a}{2}+\frac{1}{4}}(b)\, I_{-\frac{a}{2}+\frac{1}{4}}(b)$$ $[a>0]$ ET I 37(66)

2. $$\int_0^{\pi/2} \frac{\cos ax \cosh(2b\cos x)}{\sqrt{\cos x}}\,dx = \frac{\pi}{2}\sqrt{\pi b}\, I_{\frac{a}{2}-\frac{1}{4}}(b)\, I_{-\frac{a}{2}-\frac{1}{4}}(b)$$ $[a>0]$ ET I 37(67)

3. $$\int_0^{\infty} \frac{\cos ax\,dx}{\sqrt{\cosh x \cos b}} = \frac{\pi\, P_{-\frac{1}{2}+ia}(\cos b)}{\sqrt{2}\cosh a\pi}$$ $[a>0, \quad b>0]$ ET I 30(7)

3.989

1. $$\int_0^{\infty} \frac{\sin\frac{a^2x^2}{\pi}\sin bx}{\sinh ax}\,dx = \frac{\pi}{2a}\sin\frac{\pi b^2}{4a^2}\operatorname{cosech}\frac{\pi b}{2a}$$ $[a>0, \quad b>0]$ ET I 93(44)

2. $$\int_0^{\infty} \frac{\cos\frac{a^2x^2}{\pi}\sin bx}{\sinh ax}\,dx = \frac{\pi}{2a}\frac{\cosh\frac{\pi b}{a} - \cos\frac{\pi b^2}{4a^2}}{\sinh\frac{\pi b}{2a}}$$ $[a>0, \quad b>0]$ ET I 93(45)

3. $$\int_0^{\infty} \frac{\sin\frac{x^2}{\pi}\cos ax}{\cosh x}\,dx = \frac{\pi}{2}\frac{\cos\frac{a^2\pi}{4} - \frac{1}{\sqrt{2}}}{\cosh\frac{a\pi}{2}}$$ ET I 36(54)

4. $$\int_0^{\infty} \frac{\cos\frac{x^2}{\pi}\cos ax}{\cosh x}\,dx = \frac{\pi}{2}\cdot\frac{\sin\frac{a^2\pi}{4} + \frac{1}{\sqrt{2}}}{\cosh\frac{a\pi}{2}}$$ ET I 36(55)

5. $$\int_0^{\infty} \frac{\sin\left(\pi a x^2\right)\cos bx}{\cosh \pi x}\,dx = -\sum_{k=0}^{\infty}\exp\left[-\left(k+\tfrac{1}{2}\right)b\right]\sin\left[\left(k+\tfrac{1}{2}\right)^2\pi a\right] + \frac{1}{\sqrt{a}}\sum_{k=0}^{\infty}\exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\sin\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right]$$ $[a>0, \quad b>0]$ ET I 36(56)

6. $$\int_0^{\infty} \frac{\cos\left(\pi a x^2\right)\cos bx}{\cosh \pi x}\,dx = \sum_{k=0}^{\infty}(-1)^k\exp\left[-\left(k+\frac{1}{2}\right)b\right]\cos\left[\left(k+\frac{1}{2}\right)^2\pi a\right] + \frac{1}{\sqrt{a}}\sum_{k=0}^{\infty}\exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\cos\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right]$$ $[a>0, \quad b>0]$ ET I 36(57)

3.991

1. $$\int_0^{\infty} \sin\pi x^2 \sin ax \coth \pi x\,dx = \frac{1}{2}\tanh\frac{a}{2}\sin\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)$$ ET I 93(42)

2.[11] $$\int_0^{\infty} \cos\pi x^2 \sin ax \coth \pi x\,dx = \frac{1}{2}\tanh\frac{a}{2}\left[1 - \cos\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)\right]$$ ET I 93(43)

3.992

1. $$\int_0^\infty \frac{\sin \pi x^2 \cos ax}{1+2\cosh\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = -\sqrt{3} + \frac{\cos\left(\frac{\pi}{12}-\frac{a^2}{4\pi}\right)}{4\cosh\frac{a}{\sqrt{3}}-2}$$ ET I 37(60)

2. $$\int_0^\infty \frac{\cos \pi x^2 \cos ax}{1+2\cosh\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = 1 - \frac{\sin\left(\frac{\pi}{12}-\frac{a^2}{4\pi}\right)}{4\cosh\frac{a}{\sqrt{3}}-2}$$ ET I 37(61)

3.993 $$\int_0^\infty \frac{\sin^2 x + \cos x^2}{\cosh\left(\sqrt{\pi}x\right)}\cos ax\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\sin^2 a + \cos a^2}{\cosh\left(\sqrt{\pi}a\right)}$$ ET I 37(58)

3.994

1. $$\int_0^\infty \frac{\sin(2a\cosh x)\cos bx}{\sqrt{\cosh x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{\frac{1}{4}+\frac{ib}{2}}(a)\,Y_{\frac{1}{4}-\frac{ib}{2}}(a) + J_{\frac{1}{4}-\frac{ib}{2}}(a)\,Y_{\frac{1}{4}+\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 37(62)

2. $$\int_0^\infty \frac{\cos(2a\cosh x)\cos bx}{\sqrt{\cosh x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{-\frac{1}{4}+\frac{ib}{2}}(a)\,Y_{-\frac{1}{4}-\frac{ib}{2}}(a) + J_{-\frac{1}{4}-\frac{ib}{2}}(a)\,Y_{-\frac{1}{4}+\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 37(63)

3. $$\int_0^\infty \frac{\sin(2a\sinh x)\sin bx}{\sqrt{\sinh x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{\frac{1}{4}+\frac{ib}{2}}(a)\,K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 93(47)

4. $$\int_0^\infty \frac{\cos(2a\sinh x)\sin bx}{\sqrt{\sinh x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{-\frac{1}{4}+\frac{ib}{2}}(a)\,K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 93(48)

5. $$\int_0^\infty \frac{\sin(2a\sinh x)\cos bx}{\sqrt{\sinh x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)\,K_{\frac{1}{4}+\frac{ib}{2}}(a) + I_{\frac{1}{4}+\frac{ib}{2}}(a)\,K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 37(64)

6. $$\int_0^\infty \frac{\cos(2a\sinh x)\cos bx}{\sqrt{\sinh x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) + I_{-\frac{1}{4}+\frac{ib}{2}}(a)\,K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$
$[a>0,\quad b>0]$ ET I 37(65)

7. $$\int_0^\infty \sin(a\cosh x)\sin(a\sinh x)\,\frac{dx}{\sinh x} = \frac{\pi}{2}\sin a$$ $[a>0]$ BI (264)(22)

3.995

1. $$\int_0^{\pi/2} \frac{\sin\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\sin\frac{2ac}{b+c}$$
$[b>0,\quad c>0]$ BI (273)(9)

2. $$\int_0^{\pi/2} \frac{\cos\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\cos\frac{2ac}{b+c}$$
$[b>0,\quad c>0]$ BI (273)(10)

3.996

1. $$\int_0^\infty \sin(a\sinh x)\sinh\beta x\,dx = \sin\frac{\beta\pi}{2}\,K_\beta(a) \qquad [|\operatorname{Re}\beta|<1,\quad a>0]$$ EH II 82(26)

2. $$\int_0^\infty \cos(a\sinh x)\cosh\beta x\,dx = \cos\frac{\beta\pi}{2}\,K_\beta(a) \qquad [|\operatorname{Re}\beta|<1,\quad a>0]$$ WA 202(13)

3. $$\int_0^{\pi/2} \cos(a\sin x)\cosh(\beta\cos x)\,dx = \frac{\pi}{2}\,J_0\left(\sqrt{a^2-\beta^2}\right)$$ MO 40

4. $$\int_0^\infty \sin\left(a\cosh x-\tfrac{1}{2}\beta\pi\right)\cosh\beta x\,dx = \frac{\pi}{2}\,J_\beta(a) \qquad [|\operatorname{Re}\beta|<1,\quad a>0]$$ WA 199(12)

5. $$\int_0^\infty \cos\left(a\cosh x-\tfrac{1}{2}\beta\pi\right)\cosh\beta x\,dx = -\frac{\pi}{2}\,Y_\beta(a) \qquad [|\operatorname{Re}\beta|<1,\quad a>0]$$ WA 199(13)

3.997

1. $$\int_0^{\pi/2} \sin^\nu x\sinh(\beta\cos x)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{\beta}\right)^{\frac{\nu}{2}}\Gamma\left(\frac{\nu+1}{2}\right)\mathbf{L}_{\frac{\nu}{2}}(\beta)$$
$$[\operatorname{Re}\nu>-1]$$ EH II 38(53)

2. $$\int_0^{\pi} \sin^\nu x\cosh(\beta\cos x)\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\frac{\nu}{2}}\Gamma\left(\frac{\nu+1}{2}\right)I_{\frac{\nu}{2}}(\beta)$$
$$[\operatorname{Re}\nu>-1]$$ WH

3. $$\int_0^{\pi/2} \frac{dx}{\cosh(\tan x)\cos x\sqrt{\sin 2x}} = \sqrt{2\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}$$ BI (276)(13)

4. $$\int_0^{\pi/2} \frac{\tan^q x}{\cosh(\tan x)+\cos\lambda}\frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin\lambda}\sum_{k=1}^\infty (-1)^{k-1}\frac{\sin k\lambda}{k^q}$$
$$[q>0]$$ BI (275)(20)

4.11–4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.111

1. $$\int_0^\infty \frac{\sin ax}{\sinh \beta x} \cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\tanh \frac{a\pi}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ (cf. **3.981** 1) GW (336)(17a)

2. $$\int_0^\infty \frac{\cos ax}{\sinh \beta x} \cdot x^{2m+1}\, dx = (-1)^m \frac{\pi}{2\beta} \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\tanh \frac{a\pi}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ (cf. **3.981** 1) GW (336)(17b)

3. $$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \cdot x^{2m+1}\, dx = (-1)^{m+1} \frac{\pi}{2\beta} \cdot \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}}\right)$$
$[\operatorname{Re}\beta > 0]$ (cf. **3.981** 3) GW (336)(18b)

4. $$\int_0^\infty \frac{\cos ax}{\cosh \beta x} \cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}}\right)$$
$[\operatorname{Re}\beta > 0]$ (cf. **3.981** 3) GW (336)(18a)

5. $$\int_0^\infty x \frac{\sin 2ax}{\cosh \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\sinh \frac{a\pi}{\beta}}{\cosh^2 \frac{a\pi}{\beta}}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ BI (364)(6)a

6. $$\int_0^\infty x \frac{\cos 2ax}{\sinh \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\cosh^2 \frac{a\pi}{\beta}}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ BI (364)(1)a

7. $$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \frac{dx}{x} = 2 \arctan \left(\exp \frac{\pi a}{2\beta}\right) - \frac{\pi}{2}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ BI (387)(1), ET I 89(13), LI (298)(17)

4.112

1. $$\int_0^\infty \left(x^2 + \beta^2\right) \frac{\cos ax}{\cosh \frac{\pi x}{2\beta}}\, dx = \frac{2\beta^3}{\cosh^3 a\beta}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 32(19)

2. $$\int_0^\infty x \left(x^2 + 4\beta^2\right) \frac{\cos ax}{\sinh \frac{\pi x}{2\beta}}\, dx = \frac{6\beta^4}{\cosh^4 a\beta}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 32(20)

4.113

1. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x}\cdot\frac{dx}{x^2+\beta^2} = -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta\sin\pi\beta} + \frac{1}{2\beta^2}\left[\,{}_2F_1\left(1,-\beta;1-\beta;-e^{-a}\right) + {}_2F_1\left(1,\beta;1+\beta:-e^{-a}\right)\right]$$
$$= \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta\sin\pi\beta} - \sum_{k=1}^\infty \frac{(-1)^k e^{-ak}}{k^2-\beta^2}$$
$[\operatorname{Re}\beta>0, \quad \beta\neq 0,1,2,\ldots, \quad a>0]$ ET I 90(18)

2. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x}\cdot\frac{dx}{x^2+m^2} = \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m}\sum_{k=1}^{m-1}\frac{(-1)^k e^{-ka}}{m-k} + \frac{(-1)^m e^{-ma}}{2m}\ln\left(1+e^{-a}\right) + \frac{1}{2m!}\frac{d^{m-1}}{dz^{m-1}}\left[\frac{(1+z)^{m-1}}{z}\ln(1+z)\right]_{z=e^{-a}}$$
$[a>0]$ ET I 89(17)

3. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin ax}{\sinh \pi x}\frac{dx}{1+x^2} = -\frac{a}{2}\cosh a + \sinh a\ln\left(2\cosh\frac{a}{2}\right)$$
GW (336)(21b)

4. $$\int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\sinh a - \cosh a\arctan(\sinh a)$$
GW (336)(21a)

5. $$\int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{4}x}\cdot\frac{dx}{1+x^2} = -\frac{\pi}{\sqrt{2}}e^{-a} + \frac{\sinh a}{\sqrt{2}}\ln\frac{2\cosh a+\sqrt{2}}{2\cosh a-\sqrt{2}} + \sqrt{2}\cosh a\arctan\frac{\sqrt{2}}{2\sinh a}$$
$[a>0]$ LI (389)(1)

6. $$\int_0^\infty \frac{\sin ax}{\cosh \frac{\pi}{4}x}\cdot\frac{x\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{\sinh a}{\sqrt{2}}\ln\frac{2\cosh a+\sqrt{2}}{2\cosh a-\sqrt{2}} - \sqrt{2}\cosh a\arctan\left(\frac{1}{\sqrt{2}\sinh a}\right)$$
$[a>0]$ BI (388)(1)

7. $$\int_0^\infty \frac{\cos ax}{\sinh \pi x}\cdot\frac{x\,dx}{1+x^2} = -\frac{1}{2} + \frac{a}{2}e^{-a} + \cosh a\ln\left(1+e^{-a}\right)$$
$[a>0]$ BI (389)(14), ET I 32(24)

8. $$\int_0^\infty \frac{\cos ax}{\sinh \frac{\pi}{2}x}\cdot\frac{x\,dx}{1+x^2} = 2\sinh a\arctan\left(e^{-a}\right) + \frac{\pi}{2}e^{-a} - 1$$
$[a>0]$ BI (389)(11)

9.[11] $$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi e^{-a\beta}}{2\beta\cos(\beta\pi)} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{\left(k+\frac{1}{2}\right)^2-\beta^2}$$
$[\operatorname{Re}\beta>0, \quad a>0]$ ET I 32(26)

10.[11] $$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\cdot\frac{dx}{x^2+\left(m+\frac{1}{2}\right)^2} = \frac{(-1)^m e^{-a\beta}\left(a\beta+\frac{1}{2}\right)}{2\beta^2} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{\left(k+\frac{1}{2}\right)^2-\beta^2}$$
$[\operatorname{Re}\beta>0, \quad a>0]$ ET I 32(25)

11. $$\int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2\cosh\frac{a}{2} - \left[e^a \arctan\left(e^{-\frac{a}{2}}\right) + e^{-a}\arctan\left(e^{\frac{a}{2}}\right)\right]$$
$[a > 0]$ ET I 32(21)

12. $$\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{2}x} \cdot \frac{dx}{1+x^2} = ae^{-a} + \cosh a \ln\left(1+e^{-2a}\right)$$
$[a > 0]$ BI (388)(6)

13. $$\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{4}x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{2\sinh a}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}\sinh a}\right) - \frac{\cosh a}{\sqrt{2}}\ln\frac{2\cosh a+\sqrt{2}}{2\cosh a-\sqrt{2}}$$
$[a > 0]$ BI (388)(5)

4.114

1. $$\int_0^\infty \frac{\sin ax}{x}\frac{\sinh \beta x}{\sinh \gamma x}\,dx = \arctan\left(\tan\frac{\beta\pi}{2\gamma}\tanh\frac{a\pi}{2\gamma}\right)$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a > 0]$ BI (387)(6)a

2. $$\int_0^\infty \frac{\cos ax}{x}\frac{\sinh \beta x}{\cosh \gamma x}\,dx = \frac{1}{2}\ln\frac{\cosh\frac{a\pi}{2\gamma}+\sin\frac{\beta\pi}{2\gamma}}{\cosh\frac{a\pi}{2\gamma}-\sin\frac{\beta\pi}{2\gamma}}$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma]$ ET I 33(34)

4.115

1. $$\int_0^\infty \frac{x\sin ax}{x^2+b^2}\cdot\frac{\sinh \beta x}{\sinh \pi x}\,dx = \frac{\pi}{2}\frac{e^{-ab}\sin b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{ke^{-ak}\sin k\beta}{k^2-b^2}$$
$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0]$
BI (389)(23)

2. $$\int_0^\infty \frac{x\sin ax}{x^2+1}\cdot\frac{\sinh \beta x}{\sinh \pi x}\,dx = \frac{1}{2}e^{-a}\left(a\sin\beta - \beta\cos\beta\right) - \frac{1}{2}\sinh a\sin\beta\ln\left[1+2e^{-a}\cos\beta+e^{-2a}\right] + \cosh a\cos\beta\arctan\frac{\sin\beta}{e^a+\cos\beta}$$
$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$ LI (389)(10)

3. $$\int_0^\infty \frac{x\sin ax}{x^2+1}\cdot\frac{\sinh \beta x}{\sinh\frac{\pi}{2}x}\,dx = \frac{\pi}{2}e^{-a}\sin\beta + \frac{1}{2}\cos\beta\sinh a\ln\frac{\cosh a+\sin\beta}{\cosh a-\sin\beta} - \sin\beta\cosh a\arctan\left(\frac{\cos\beta}{\sinh a}\right)$$
$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0\right]$ BI (389)(8)

4. $$\int_0^\infty \frac{\cos ax}{x^2+b^2}\cdot\frac{\sinh \beta x}{\sinh \pi x}\,dx = \frac{\pi}{2b}\cdot\frac{e^{-ab}\sin b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{e^{-ak}\sin k\beta}{k^2-b^2}$$
$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0]$
BI (389)(22)

5. $$\int_0^\infty \frac{\cos ax}{x^2+1}\cdot\frac{\sinh \beta x}{\sinh \pi x}\,dx = \frac{1}{2}e^{-a}\left(a\sin\beta - \beta\cos\beta\right) + \frac{1}{2}\cosh a\sin\beta\ln\left(1+2e^{-a}\cos\beta+e^{-2a}\right) - \sinh a\cos\beta\arctan\frac{\sin\beta}{e^a+\cos\beta}$$
$[|\operatorname{Re}\beta| < \pi, \quad a > 0, \quad b > 0]$ BI (389)(20)a

6. $$\int_0^\infty \frac{\cos ax}{x^2+1}\cdot\frac{\sinh\beta x}{\sinh\frac{\pi}{2}x}\,dx=\frac{\pi}{2}e^{-a}\sin\beta-\frac{1}{2}\cosh a\cos\beta\ln\frac{\cosh a+\sin\beta}{\cosh a-\sin\beta}+\sinh a\sin\beta\arctan\frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta|<\frac{\pi}{2},\quad a>0,\quad b>0\right]$$
BI (389)(18)

7. $$\int_0^\infty \frac{\sin ax}{x^2+\frac{1}{4}}\cdot\frac{\sinh\beta x}{\cosh\pi x}\,dx=e^{-\frac{a}{2}}\left(a\sin\frac{\beta}{2}-\beta\cos\frac{\beta}{2}\right)-\sinh\frac{a}{2}\sin\frac{\beta}{2}\ln\left(1+2e^{-a}\cos\beta+e^{-2a}\right)$$
$$+\cosh\frac{a}{2}\cos\frac{\beta}{2}\arctan\frac{\sin\beta}{1+e^{-a}\cos\beta}$$
$$[|\operatorname{Re}\beta|<\pi,\quad a>0]$$
ET I 91(26)

8. $$\int_0^\infty \frac{\sin ax}{x^2+\beta^2}\cdot\frac{\cosh\gamma x}{\sinh\pi x}\,dx=\frac{1}{2\beta^2}-\frac{\pi}{2\beta}\cdot\frac{e^{-a\beta}\cos\beta\gamma}{\sin\beta\pi}+\sum_{k=1}^\infty(-1)^{k-1}\frac{e^{-ak}\cos k\gamma}{k^2-\beta^2}$$
$$[0\le\operatorname{Re}\beta,\quad |\operatorname{Re}\gamma|<\pi,\quad a>0]$$
BI (389)(21)

9. $$\int_0^\infty \frac{\sin ax}{x^2+1}\cdot\frac{\cosh\beta x}{\sinh\pi x}\,dx=-\frac{1}{2}e^{-a}\left(a\cos\beta+\beta\sin\beta\right)+\frac{1}{2}\sinh a\cos\beta\ln\left(1+2e^{-a}\cos\beta+e^{-2a}\right)$$
$$+\cosh a\sin\beta\arctan\frac{\sin\beta}{e^a+\cos\beta}$$
$$[|\operatorname{Re}\beta|<\pi,\quad a>0]$$
ET I 91(25), LI (389)(9)

10. $$\int_0^\infty \frac{\sin ax}{x^2+1}\cdot\frac{\cosh\beta x}{\sinh\frac{\pi}{2}x}\,dx=-\frac{\pi}{2}e^{-a}\cos\beta+\frac{1}{2}\sinh a\sin\beta\ln\frac{\cosh a+\sin\beta}{\cosh a-\sin\beta}+\cosh a\cos\beta\arctan\frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta|<\frac{\pi}{2},\quad a>0\right]$$
BI (389)(7)

11. $$\int_0^\infty \frac{x\cos ax}{x^2+b^2}\cdot\frac{\cosh\beta x}{\sinh\pi x}\,dx=\frac{\pi}{2}\cdot\frac{e^{-ab}\cos b\beta}{\sin b\pi}+\sum_{k=1}^\infty(-1)^k\frac{ke^{-ak}\cos k\beta}{k^2-b^2}$$
$$[|\operatorname{Re}\beta|<\pi,\quad a>0]$$
BI (389)(24)

12. $$\int_0^\infty \frac{x\cos ax}{x^2+1}\cdot\frac{\cosh\beta x}{\sinh\pi x}\,dx=\frac{1}{2}e^{-a}\left(a\cos\beta+\beta\sin\beta\right)$$
$$-\frac{1}{2}+\frac{1}{2}\cosh a\cos\beta\ln\left[1+2e^{-a}\cos\beta+e^{-2a}\right]$$
$$+\sinh a\sin\beta\arctan\frac{\sin\beta}{e^a+\cos\beta}$$
$$[|\operatorname{Re}\beta|<\pi,\quad a>0]$$
BI (389)(19)

13. $$\int_0^\infty \frac{x\cos ax}{x^2+1}\cdot\frac{\cosh\beta x}{\sinh\frac{\pi}{2}x}\,dx=-1+\frac{\pi}{2}e^{-a}\cos\beta+\frac{1}{2}\cosh a\sin\beta\ln\frac{\cosh a+\sin\beta}{\cosh a-\sin\beta}$$
$$+\sinh a\cos\beta\arctan\frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta|<\frac{\pi}{2},\quad a>0\right]$$
BI (389)(17)

14. $$\int_0^\infty \frac{\cos ax}{x^2+1}\cdot\frac{\cosh\beta x}{\cosh\frac{\pi}{2}x}\,dx = ae^{-a}\cos\beta+\beta e^{-a}\sin\beta+\sinh a\sin\beta\arctan\frac{e^{-2a}\sin 2\beta}{1+e^{-2a}\cos 2\beta}$$
$$+\frac{1}{2}\cosh a\cos\beta\ln\left(1+2e^{-2a}\cos 2\beta+e^{-4a}\right)$$
$$\left[|\operatorname{Re}\beta|<\frac{\pi}{2}, a>0\right]$$ ET I 34(37)

4.116

1.[6] $$\int_0^\infty x\cos 2ax\tanh x\,dx$$ the integral is divergent BI (364)(2)

2. $$\int_0^\infty \cos ax\tanh\beta x\frac{dx}{x}=\ln\coth\frac{a\pi}{4\beta}$$ $[\operatorname{Re}\beta>0,\quad a>0]$ BI (387)(8)

4.117

1. $$\int_0^\infty \frac{\sin ax}{1+x^2}\tanh\frac{\pi x}{2}\,dx = a\cosh a-\sinh a\ln(2\sinh a)$$
$[a>0]$ BI (388)(3)

2. $$\int_0^\infty \frac{\sin ax}{1+x^2}\tanh\frac{\pi x}{4}\,dx = -\frac{\pi}{2}e^a+\sinh a\ln\coth\frac{a}{2}+2\cosh a\arctan(e^a)$$ BI (388)(4)

3. $$\int_0^\infty \frac{\sin ax}{1+x^2}\coth\pi x\,dx = \frac{a}{2}e^{-a}-\sinh a\ln\left(1-e^{-a}\right)$$ $[a>0]$ BI (389)(5)

4. $$\int_0^\infty \frac{\sin ax}{1+x^2}\coth\frac{\pi}{2}x\,dx = \sinh a\ln\coth\frac{a}{2}$$ $[a>0]$ BI (389)(6)

5. $$\int_0^\infty \frac{x\cos ax}{1+x^2}\tanh\frac{\pi}{2}x\,dx = -ae^{-a}-\cosh a\ln\left(1-e^{-2a}\right)$$
$[a>0]$ BI (388)(7)

6. $$\int_0^\infty \frac{x\cos ax}{1+x^2}\tanh\frac{\pi}{4}x\,dx - \frac{\pi}{2}e^a+\cosh a\ln\coth\frac{a}{2}+2\sinh a\arctan(e^a)$$
$[a>0]$ BI (388)(8)

7. $$\int_0^\infty \frac{x\cos ax}{1+x^2}\coth\pi x\,dx = -\frac{a}{2}e^{-a}-\frac{1}{2}-\cosh a\ln\left(1-e^{-a}\right)$$ BI (389)(15)a, ET I 33(31)a

8. $$\int_0^\infty \frac{x\cos ax}{1+x^2}\coth\frac{\pi}{2}x\,dx = -1+\cosh a\ln\coth\frac{a}{2}$$ $[a>0]$ BI (389)(12)

9. $$\int_0^\infty \frac{x\cos ax}{1+x^2}\coth\frac{\pi}{4}x\,dx = -2+\frac{\pi}{2}e^{-a}+\cosh a\ln\coth\frac{a}{2}+2\sinh a\arctan\left(e^{-a}\right)$$
$[a>0]$ BI (389)(13)

4.118[8] $$\int_0^\infty \frac{x\sin ax}{\cosh^2 x}\,dx = \frac{\pi}{2}\frac{1}{\sinh\frac{1}{2}\pi a}\left(\frac{1}{2}\pi a\coth\frac{1}{2}\pi a-1\right)$$ ET I 89(14)

4.119 $$\int_0^\infty \frac{1-\cos px}{\sinh qx}\cdot\frac{dx}{x}=\ln\left(\cosh\frac{p\pi}{2q}\right)$$ BI (387)(2)a

4.121

1. $$\int_0^\infty \frac{\sin ax - \sin bx}{\cosh \beta x} \cdot \frac{dx}{x} = 2 \arctan \frac{\exp \frac{a\pi}{2\beta} - \exp \frac{b\pi}{2\beta}}{1 + \exp \frac{(a+b)\pi}{2\beta}}$$
$[\operatorname{Re} \beta > 0]$ GW (336)(19b)

2. $$\int_0^\infty \frac{\cos ax - \cos bx}{\sinh \beta x} \cdot \frac{dx}{x} = \ln \frac{\cosh \frac{b\pi}{2\beta}}{\cosh \frac{a\pi}{2\beta}}$$
$[\operatorname{Re} \beta > 0]$ GW (336)(19a)

4.122

1.[6] $$\int_0^\infty \frac{\cos \beta x \sin \gamma x}{\cosh \delta x} \cdot \frac{dx}{x} = \arctan \frac{\sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta\pi}{2\delta}}$$
$[\operatorname{Re} \delta > |\operatorname{Im} \beta| + |\operatorname{Im} \gamma|]$ ET I 93(46)a

2. $$\int_0^\infty \sin^2 ax \frac{\cosh \beta x}{\sinh x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\cosh 2a\pi + \cos \beta\pi}{1 + \cos \beta\pi}$$
$[|\operatorname{Re} \beta| < 1]$ BI (387)(7)

4.123

1. $$\int_0^\infty \frac{\sin x}{\cosh ax + \cos x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \arctan \frac{1}{a} - \frac{1}{a}$$
BI (390)(1)

2. $$\int_0^\infty \frac{\sin x}{\cosh ax - \cos x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{a}{1 + a^2} - \arctan \frac{1}{a}$$
BI (390)(2)

3. $$\int_0^\infty \frac{\sin 2x}{\cosh 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1 + 2a^2}{1 + a^2} - \arctan \frac{1}{a}$$
BI (390)(4)

4. $$\int_0^\infty \frac{\cosh ax \sin x}{\cosh 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{-1}{2a\,(1 + a^2)}$$
LI (390)(3)

5. $$\int_0^\infty \frac{\cos ax}{\cosh \pi x + \cos \pi\beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma\,(\cos \gamma\pi + \cos \beta\pi)} + \frac{1}{\sinh \beta\pi} \sum_{k=0}^\infty \left\{ \frac{e^{-(2k+1-\beta)a}}{\gamma^2 - (2k+1-\beta)^2} - \frac{e^{-(2k+1+\beta)a}}{\gamma^2 - (2k+1+\beta)^2} \right\}$$
$[0 < \operatorname{Re} \beta < 1, \quad \operatorname{Re} \gamma > 0, \quad a > 0]$ ET I 33(27)

6. $$\int_0^\infty \frac{\sin ax \sinh bx}{\cos 2ax + \cosh 2bx} x^{p-1}\,dx = \frac{\Gamma(p)}{(a^2 + b^2)^{\frac{p}{2}}} \sin \left(p \arctan \frac{a}{b} \right) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^p}$$
$[p > 0]$ BI (364)(8)

7. $$\int_0^\infty \sin ax^2 \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi x}{2}}{\cos \pi x + \cosh \pi x} \cdot x\,dx = \frac{1}{4} \left[\frac{\partial\, \vartheta_1(z \mid q)}{\partial z} \right]_{z=0, q=e^{-2a}}$$
$[a > 0]$ ET I 93(49)

4.124

1. $$\int_0^1 \frac{\cos px \cosh\left(q\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\,J_0\left(\sqrt{p^2-q^2}\right)$$ MO (40)

2. $$\int_u^\infty \cos ax \cosh\sqrt{\beta\left(u^2-x^2\right)}\cdot\frac{dx}{\sqrt{u^2-x^2}} = \frac{\pi}{2}\,J_0\left(\frac{u}{\sqrt{a^2-\beta^2}}\right)$$ ET I 34(38)

4.125

1. $$\int_0^\infty \sinh(a\sin x)\cos(a\cos x)\sin x\sin 2nx\frac{dx}{x} = \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\frac{\pi}{8}\left[1+\frac{a^2}{2n(2n+1)}\right]$$ LI (367)(14)

2. $$\int_0^\infty \cosh(a\sin x)\cos(a\cos x)\sin x\cos(2n-1)x\frac{dx}{x} = \frac{(-1)^{n-1}a^{2(n-1)}}{[2(n-1)]!}\frac{\pi}{8}\left[1-\frac{a^2}{2n(2n-1)}\right]$$ LI (367)(15)

3. $$\int_0^\infty \sinh(a\sin x)\cos(a\cos x)\cos x\cos 2nx\frac{dx}{x} = \frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1}}{(2n+1)!}\frac{3\pi}{8} + \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\frac{\pi}{8}$$ LI (367)(21)

4.126

1. $$\int_0^\infty \sin(a\cos bx)\sinh(a\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}\left[\cos(a\cos bc)\cosh(a\sin bc)-1\right]$$ $[b>0]$ BI (381)(2)

2. $$\int_0^\infty \sin(a\cos bx)\cosh(a\sin bx)\frac{dx}{c^2-x^2} = \frac{\pi}{2c}\cos(a\cos bc)\sinh(a\sin bc)$$ $[b>0,\quad c>0]$ BI (381)(1)

3. $$\int_0^\infty \cos(a\cos bx)\sinh(a\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}\left[a\cos bc-\sin(a\cos bc)\cosh(a\sin bc)\right]$$ $[b>0]$ BI (381)(4)

4. $$\int_0^\infty \cos(a\cos bx)\cosh(a\sin bx)\frac{dx}{c^2-x^2} = -\frac{\pi}{2c}\sin(a\cos bc)\sinh(a\sin bc)$$ $[b>0]$ BI (381)(3)

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

1. $$\int_0^\infty \sin ax\sinh^\nu\gamma x e^{-\beta x}\,dx = -\frac{i\,\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)}-\frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu+ai}{2\gamma}+1\right)}\right\}$$ $[\operatorname{Re}\nu>-2,\quad \operatorname{Re}\gamma>0,\quad |\operatorname{Re}(\gamma\nu)|<\operatorname{Re}\beta]$ ET I 91(30)a

2. $$\int_0^\infty \cos ax \sinh^\nu \gamma x e^{-\beta x}\,dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu-ai}{2\gamma}+1\right)}-\frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)}\right\}$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re}\beta]$ ET I 34(40)a

3. $$\int_0^\infty e^{-\beta x}\frac{\sin ax}{\sinh \gamma x}\,dx = \sum_{k=1}^\infty \frac{2a}{a^2+[\beta+(2k-1)\gamma]^2}$$ BI (264)(9)a
$$= \frac{1}{2\gamma i}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right)-\psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right]$$ $[\operatorname{Re}\beta > |\operatorname{Re}\gamma|]$ ET I 91(28)

4. $$\int_0^\infty e^{-x}\frac{\sin ax}{\sinh x}\,dx = \frac{\pi}{2}\coth\frac{a\pi}{2}-\frac{1}{a}$$ ET I 91(29)

4.132

1. $$\int_0^\infty \frac{\sin ax \sinh \beta x}{e^{\gamma x}-1}\,dx = -\frac{a}{2\left(a^2+\beta^2\right)}+\frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{2\pi a}{\gamma}}{\cosh\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}}$$
$$+\frac{i}{2\gamma}\left[\psi\left(\frac{\beta}{\gamma}+i\frac{a}{\gamma}+1\right)-\psi\left(\frac{\beta}{\gamma}-i\frac{a}{\gamma}+1\right)\right]$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, a > 0]$ ET I 92(33)

2. $$\int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x}-1}\,dx = -\frac{a}{2\left(a^2+\beta^2\right)}+\frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{2\pi a}{\gamma}}{\cosh\frac{2\pi a}{\gamma}-\cos\frac{2\pi\beta}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ BI (265)(5)a, ET I 92(34)

3. $$\int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x}+1}\,dx = \frac{a}{2\left(a^2+\beta^2\right)}-\frac{\pi}{\gamma}\cdot\frac{\sinh\frac{a\pi}{\gamma}\cos\frac{\beta\pi}{\gamma}}{\cosh\frac{2a\pi}{\gamma}-\cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ ET I 92(35)

4. $$\int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x}-1}\,dx = \frac{\beta}{2\left(a^2+\beta^2\right)}-\frac{\pi}{2\gamma}\cdot\frac{\sin\frac{2\pi\beta}{\gamma}}{\cosh\frac{2a\pi}{\gamma}-\cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ LI (265)(8)

5. $$\int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x}+1}\,dx = -\frac{\beta}{2\left(a^2+\beta^2\right)}+\frac{\pi}{\gamma}\frac{\sin\frac{\pi\beta}{\gamma}\cosh\frac{\pi a}{\gamma}}{\cosh\frac{2a\pi}{\gamma}-\cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ ET I 34(39)

4.133

1.[11] $$\int_0^\infty \sin ax \sinh \beta x \exp\left(-\frac{x^2}{4\gamma}\right)\,dx = \sqrt{\pi\gamma}\exp\left[\gamma\left(\beta^2-a^2\right)\right]\sin(2a\beta\gamma)$$
$[\operatorname{Re}\gamma > 0]$ ET I 92(37)

2.[11] $$\int_0^\infty \cos ax \cosh \beta x \exp\left(-\frac{x^2}{4\gamma}\right)\,dx = \sqrt{\pi\gamma}\exp\left[\gamma\left(\beta^2-a^2\right)\right]\cos(2a\beta\gamma)$$
$[\operatorname{Re}\gamma > 0]$ ET I 35(41)

4.134

1. $$\int_0^\infty e^{-\beta x^2}\left(\cosh x-\cos x\right)\,dx=\sqrt{\frac{\pi}{\beta}}\cosh\frac{1}{4\beta}\qquad[\operatorname{Re}\beta>0]$$ ME 24

2. $$\int_0^\infty e^{-\beta x^2}\left(\cosh x-\cos x\right)\,dx=\sqrt{\frac{\pi}{\beta}}\sinh\frac{1}{4\beta}\qquad[\operatorname{Re}\beta>0]$$ ME 24

4.135

1. $$\int_0^\infty \sin ax^2\cosh 2\gamma x e^{-\beta x^2}\,dx=\frac{1}{2}\sqrt[4]{\frac{\pi^2}{a^2+\beta^2}}\exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right)\sin\left(\frac{a\gamma^2}{a^2+\beta^2}+\frac{1}{2}\arctan\frac{a}{\beta}\right)$$
$$[\operatorname{Re}\beta>0]$$ LI (268)(7)

2. $$\int_0^\infty \cos ax^2\cosh 2\gamma x e^{-\beta x^2}\,dx=\frac{1}{2}\sqrt[4]{\frac{\pi^2}{a^2+\beta^2}}\exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right)\cos\left(\frac{a\gamma^2}{a^2+\beta^2}+\frac{1}{2}\arctan\frac{a}{\beta}\right)$$
$$[\operatorname{Re}\beta>0]$$ LI (268)(8)

4.136

1. $$\int_0^\infty \left(\sinh^2 x+\sin x^2\right)e^{-\beta x^4}\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\beta}}\,I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right)\cosh\frac{1}{8\beta}$$
$$[\operatorname{Re}\beta>0]$$ ME 24

2. $$\int_0^\infty \left(\sinh^2 x-\sin x^2\right)e^{-\beta x^4}\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\beta}}\,I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right)\sinh\frac{1}{8\beta}$$
$$[\operatorname{Re}\beta>0]$$ ME 24

3. $$\int_0^\infty \left(\cosh^2 x+\cos x^2\right)e^{-\beta x^4}\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\beta}}\,I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right)\cosh\frac{1}{8\beta}$$
$$[\operatorname{Re}\beta>0]$$ ME 24

4. $$\int_0^\infty \left(\cosh^2 x-\cos x^2\right)e^{-\beta x^4}\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\beta}}\,I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right)\sinh\frac{1}{8\beta}$$
$$[\operatorname{Re}\beta>0]$$ ME 24

4.137

1. $$\int_0^\infty \sin 2x^2\sinh 2x^2 e^{-\beta x^4}\,dx=\frac{\pi}{\sqrt[4]{128\beta^2}}\,J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}+\frac{\pi}{4}\right)$$
$$[\operatorname{Re}\beta>0]$$ MI 32

2. $$\int_0^\infty \sin 2x^2\cosh 2x^2 e^{-\beta x^4}\,dx=\frac{\pi}{\sqrt[4]{128\beta^2}}\,J_{\frac{1}{4}}\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}-\frac{\pi}{4}\right)$$
$$[\operatorname{Re}\beta>0]$$ MI 32

3. $$\int_0^\infty \cos 2x^2\sinh 2x^2 e^{-\beta x^4}\,dx=\frac{-\pi}{\sqrt[4]{128\beta^2}}\,J_{\frac{1}{4}}\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}-\frac{\pi}{4}\right)$$
$$[\operatorname{Re}\beta>0]$$ MI 32

4. $$\int_0^\infty \cos 2x^2 \cosh 2x^2 e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{128\beta^2}}\,J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}+\frac{\pi}{4}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4.138

1. $$\int_0^\infty \left(\sin^2 2x\cosh 2x^2 + \cos 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}}\,J_{\frac{1}{4}}\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty \left(\sin^2 2x\cosh 2x^2 - \cos 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}}\,J_{\frac{1}{4}}\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

3. $$\int_0^\infty \left(\cos^2 2x\cosh 2x^2 + \sin 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}}\,J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4. $$\int_0^\infty \left(\cos^2 2x\cosh 2x^2 - \sin 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}}\,J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.141

1. $$\int_0^\infty x e^{-\beta x^2}\cosh x \sin x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta}+\sin\frac{1}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty x e^{-\beta x^2}\sinh x \cos x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta}-\sin\frac{1}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

3. $$\int_0^\infty x^2 e^{-\beta x^2}\cosh x \cos x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{1}{2\beta}-\frac{1}{\beta}\sin\frac{1}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4. $$\int_0^\infty x^2 e^{-\beta x^2}\sinh x \sin x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\sin\frac{1}{2\beta}+\frac{1}{\beta}\cos\frac{1}{2\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4.142

1. $$\int_0^\infty xe^{-\beta x^2}\left(\sinh x+\sin x\right)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\cosh\frac{1}{4\beta}\qquad[\operatorname{Re}\beta>0]$$ ME 24

2. $$\int_0^\infty xe^{-\beta x^2}\left(\sinh x-\sin x\right)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\sinh\frac{1}{4\beta}\qquad[\operatorname{Re}\beta>0]$$ ME 24

3. $$\int_0^\infty x^2e^{-\beta x^2}\left(\cosh x+\cos x\right)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\left(\cosh\frac{1}{4\beta}+\frac{1}{2\beta}\sinh\frac{1}{4\beta}\right)$$
$[\operatorname{Re}\beta>0]$ ME 24

4. $$\int_0^\infty x^2e^{-\beta x^2}\left(\cosh x-\cos x\right)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}\left(\sinh\frac{1}{4\beta}+\frac{1}{2\beta}\cosh\frac{1}{4\beta}\right)$$
$[\operatorname{Re}\beta>0]$ ME 24

4.143

1. $$\int_0^\infty xe^{-\beta x^2}\left(\cosh x\sin x+\sinh x\cos x\right)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\cos\frac{1}{2\beta}$$
$[\operatorname{Re}\beta>0]$ MI 32

2. $$\int_0^\infty xe^{-\beta x^2}\left(\cosh x\sin x-\sinh x\cos x\right)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\sin\frac{1}{2\beta}$$
$[\operatorname{Re}\beta>0]$ MI 32

4.144 $$\int_0^\infty e^{-x^2}\sinh x^2\cos ax\frac{dx}{x^2}=\sqrt{\frac{\pi}{2}}e^{-\frac{a^2}{8}}-\frac{\pi a}{4}\left[1-\Phi\left(\frac{a}{\sqrt{8}}\right)\right]$$
$[a>0]$ ET I 35(44)

4.145

1. $$\int_0^\infty xe^{-\beta x^2}\cosh\left(2ax\sin t\right)\sin\left(2ax\cos t\right)\,dx=\frac{a}{2}\sqrt{\frac{\pi}{\beta^3}}\exp\left(-\frac{a^2}{\beta}\cos 2t\right)\cos\left(t-\frac{a^2}{\beta}\sin 2t\right)$$
$[\operatorname{Re}\beta>0]$ BI (363)(5)

2. $$\int_0^\infty xe^{-\beta x^2}\sinh\left(2ax\sin t\right)\cos\left(2ax\cos t\right)\,dx=\frac{a}{2}\sqrt{\frac{\pi}{\beta^3}}\exp\left(-\frac{a^2}{\beta}\cos 2t\right)\sin\left(t-\frac{a^2}{\beta}\sin 2t\right)$$
$[\operatorname{Re}\beta>0]$ BI (363)(6)

4.146[10]

1.[8] $$\int_0^\infty e^{-\beta x^2}\sinh ax\sin bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{a^2-b^2}{4\beta}\right)\sin\frac{ab}{2\beta}$$
$[\operatorname{Re}\beta>0]$

2.[8] $$\int_0^\infty e^{-\beta x^2}\cosh ax\cos bx\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{a^2-b^2}{4\beta}\right)\cos\frac{ab}{2\beta}$$
$[\operatorname{Re}\beta>0]$

3. $$\int_0^\infty xe^{-\beta x^2}\cosh ax\sin ax\,dx=\frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta}+\sin\frac{a^2}{2\beta}\right)$$
$[\operatorname{Re}\beta>0]$

4. $$\int_0^\infty xe^{-\beta x^2}\sinh ax\cos ax\,dx=\frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta}-\sin\frac{a^2}{2\beta}\right)$$
$[\operatorname{Re}\beta>0]$

5.[8] $$\int_0^\infty x^2e^{-\beta x^2}\cosh ax\sin ax\,dx=\frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\sin\frac{a^2}{2\beta}+\frac{a^2}{\beta}\cos\frac{a^2}{2\beta}\right)$$
$[\operatorname{Re}\beta>0]$

6.[8] $$\int_0^\infty x^2e^{-\beta x^2}\cosh ax\cos ax\,dx=\frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{a^2}{2\beta}-\frac{a^2}{\beta}\sin\frac{a^2}{2\beta}\right)$$
$[\operatorname{Re}\beta>0]$

4.2–4.4 Logarithmic Functions

4.21 Logarithmic functions

4.211

1. $$\int_e^\infty\frac{dx}{\ln\frac{1}{x}}=-\infty$$ BI (33)(9)

2. $$\int_0^u\frac{dx}{\ln x}=\operatorname{li}u$$ FI III 653, FI II 606

4.212

1.[7] $$\int_0^1\frac{dx}{a+\ln x}=e^{-a}\operatorname{Ei}(a)$$ $[a>0]$ BI (31)(4)

2. $$\int_0^1\frac{dx}{a-\ln x}=-e^{a}\operatorname{Ei}(-a)$$ $[a>0]$ BI (31)(5)

3.[7] $$\int_0^1\frac{dx}{(a+\ln x)^2}=-\frac{1}{a}+e^{-a}\operatorname{Ei}(a)$$ $[a\geq 0]$ BI (31)(14)

4. $$\int_0^1\frac{dx}{(a-\ln x)^2}=\frac{1}{a}+e^{a}\operatorname{Ei}(-a)$$ $[a>0]$ BI (31)(16)

5.[8] $$\int_0^1\frac{\ln x\,dx}{(a+\ln x)^2}=1+(1-a)e^{-a}\operatorname{Ei}(a)$$ $[a\geq 0]$ BI (31)(15)

6. $$\int_0^1\frac{\ln x\,dx}{(a-\ln x)^2}=1+(1+a)e^{a}\operatorname{Ei}(-a)$$ $[a>0]$ BI (31)(17)

7. $$\int_1^e\frac{\ln x\,dx}{(1+\ln x)^2}=\frac{e}{2}-1$$ BI (33)(10)

8.[7] $$\int_0^1 \frac{dx}{(a+\ln x)^n} = \frac{1}{(n-1)!} e^{-a}\,\mathrm{Ei}(a) - \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(n-k-1)!a^{k-n}$$

$[a \geq 0]$ BI (31))(22)

9. $$\int_0^1 \frac{dx}{(a-\ln x)^n} = \frac{(-1)^n}{(n-1)!} e^{a}\,\mathrm{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=1}^{n-1}(n-k-1)!(-a)^{k-n}$$

$[a > 0, \quad n \text{ odd}]$ BI (31)(23)

In integrals of the form $\int \frac{(\ln x)^m}{[a^n + (\ln x)^n]^l}\,dx$, it is convenient to make the substitution $x = e^{-t}$.

Results **4.212** 3, **4.212** 5, and **4.212** 8 [for $n > 1$] and **4.213** 6, **4.213** 8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z)\,dz}{(z-z_0)^n} = \frac{1}{(n-1)!}\left(\frac{d}{dz_0}\right)^{n-1}\left[\mathrm{PV}\int_0^a \frac{f(z)}{z-z_0}\,dz\right]$$

where $a > z_0 > 0, n = 1, 2, 3, \ldots$ and PV indicates the Cauchy principal value.

4.213

1. $$\int_0^1 \frac{dx}{a^2+(\ln x)^2} = \frac{1}{a}\left[\mathrm{ci}(a)\sin a - \mathrm{si}(a)\cos a\right]$$

$[a > 0]$ BI (31)(6)

2.[7] $$\int_0^1 \frac{dx}{a^2-(\ln x)^2} = \frac{1}{2a}\left[e^{-a}\overline{\mathrm{Ei}}(a) - e^{a}\,\mathrm{Ei}(-a)\right]$$

$[a > 0]$, (cf. **4.212** 1 and 2) BI (31)(8)

3. $$\int_0^1 \frac{\ln x\,dx}{a^2+(\ln x)^2} = \mathrm{ci}(a)\cos a + \mathrm{si}(a)\sin a$$

$[a > 0]$ BI (31)(7)

4.[7] $$\int_0^1 \frac{\ln x\,dx}{a^2-(\ln x)^2} = -\frac{1}{2}\left[e^{-a}\overline{\mathrm{Ei}}(a) + e^{a}\,\mathrm{Ei}(-a)\right]$$

$[a > 0]$, (cf. **4.212** 1 and 2) BI (31)(9)

5. $$\int_0^1 \frac{dx}{\left[a^2+(\ln x)^2\right]^2} = \frac{1}{2a^3}\left[\mathrm{ci}(a)\sin a - \mathrm{si}(a)\cos a\right] - \frac{1}{2a^2}\left[\mathrm{ci}(a)\cos a + \mathrm{si}(a)\sin a\right]$$

$[a > 0]$ LI (31)(18)

6.[8] $$\int_0^1 \frac{dx}{\left[a^2-(\ln x)^2\right]^2}$$ is divergent

7. $$\int_0^1 \frac{\ln x\,dx}{\left[a^2+(\ln x)^2\right]^2} = \frac{1}{2a}\left[\mathrm{ci}(a)\sin a - \mathrm{si}(a)\cos a\right] - \frac{1}{2a^2}$$

$[a > 0]$ BI (31)(19)

8.[8] $$\int_0^1 \frac{\ln x\,dx}{\left[a^2-(\ln x)^2\right]^2}$$ is divergent

4.214

1. $$\int_0^1 \frac{dx}{a^4-(\ln x)^4} = -\frac{1}{4a^3}\left[e^a\,\mathrm{Ei}(-a) - e^{-a}\overline{\mathrm{Ei}}(a) - 2\,\mathrm{ci}(a)\sin a + 2\,\mathrm{si}(a)\cos a\right]$$
$[a>0]$ BI (31)(10)

2. $$\int_0^1 \frac{\ln x\,dx}{a^4-(\ln x)^4} = -\frac{1}{4a^2}\left[e^a\,\mathrm{Ei}(-a) + e^{-a}\overline{\mathrm{Ei}}(a) - 2\,\mathrm{ci}(a)\cos a - 2\,\mathrm{si}(a)\sin a\right]$$
$[a>0]$ BI (31)(11)

3. $$\int_0^1 \frac{(\ln x)^2\,dx}{a^4-(\ln x)^4} = -\frac{1}{4a}\left[e^a\,\mathrm{Ei}(-a) - e^{-a}\overline{\mathrm{Ei}}(a) + 2\,\mathrm{ci}(a)\sin a - 2\,\mathrm{si}(a)\cos a\right]$$
$[a>0]$ BI (31)(12)

4.[7] $$\int_0^1 \frac{(\ln x)^3\,dx}{a^4-(\ln x)^4} = -\frac{1}{4}\left[e^a\,\mathrm{Ei}(-a) + e^{-a}\overline{\mathrm{Ei}}(a) + 2\,\mathrm{ci}(a)\cos a + 2\,\mathrm{si}(a)\sin a\right]$$
$[a>0]$ BI (31)(13)

4.215

1. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} dx = \Gamma(\mu)$$
$[\mathrm{Re}\,\mu>0]$ FI II 778

2. $$\int_0^1 \frac{dx}{\left(\ln\frac{1}{x}\right)^{\mu}} = \frac{\pi}{\Gamma(\mu)}\,\mathrm{cosec}\,\mu\pi$$
$[\mathrm{Re}\,\mu<1]$ BI (31)(1)

3. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\,dx = \frac{\sqrt{\pi}}{2}$$
BI (32)(1)

4. $$\int_0^1 \frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}$$
BI (32)(3)

4.216

1. $$\int_0^{1/e} \frac{dx}{\sqrt{(\ln x)^2-1}} = K_0(1)$$
GW (32)(2)

2.* $$\int_0^{1/e} \frac{dx}{\sqrt{-\ln x-1}} = \frac{\sqrt{\pi}}{e}$$

4.22 Logarithms of more complicated arguments

4.221

1. $$\int_0^1 \ln x\ln(1-x)\,dx = 2-\frac{\pi^2}{6}$$
BI (30)(7)

2. $$\int_0^1 \ln x\ln(1+x)\,dx = 2-\frac{\pi^2}{12}-2\ln 2$$
BI (30)(8)

3. $$\int_0^1 \ln\frac{1-ax}{1-a}\frac{dx}{\ln x} = -\sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \qquad [a<1]$$ BI (31)(3)

4.222

1. $$\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\,dx = (a-b)\pi \qquad [a>0,\quad b>0]$$ GW (322)(20)

2. $$\int_0^\infty \ln x \ln\frac{a^2+x^2}{b^2+x^2}\,dx = \pi(b-a)+\pi\ln\frac{a^a}{b^b} \qquad [a>0,\quad b>0]$$ BI (33)(1)

3. $$\int_0^\infty \ln x \ln\left(1+\frac{b^2}{x^2}\right)\,dx = \pi b\,(\ln b-1) \qquad [b>0]$$ BI (33)(2)

4. $$\int_0^\infty \ln\left(1+a^2x^2\right)\ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab)-b\right]$$
$$[a>0,\quad b>0]$$ BI (33)(3)

5. $$\int_0^\infty \ln\left(a^2+x^2\right)\ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[(a+b)\ln(a+b)-a\ln a-b\right]$$
$$[a>0,\quad b>0]$$ BI (33)(4)

6. $$\int_0^\infty \ln\left(1+\frac{a^2}{x^2}\right)\ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[(a+b)\ln(a+b)-a\ln a-b\ln b\right]$$
$$[a>0,\quad b>0]$$ BI (33)(5)

7. $$\int_0^\infty \ln\left(a^2+\frac{1}{x^2}\right)\ln\left(1+\frac{b^2}{x^2}\right)\,dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab)-b\ln b\right]$$
$$[a>0,\quad b>0]$$ BI (33)(7)

8.* $$\int_0^\infty \ln(1+ax)\,x^b e^{-x}\,dx = \sum_{m=0}^{b}\frac{b!}{(b-m)!}\left[\frac{(-1)^{b-m-1}}{a^{b-m}}e^{1/a}\,\mathrm{Ei}\left(-\frac{1}{a}\right)+\sum_{k=1}^{b-m}\frac{(k-1)!}{(-a)^{b-m-k}}\right]$$
$$[b>0,\quad \text{an integer}]$$

4.223

1. $$\int_0^\infty \ln\left(1+e^{-x}\right)\,dx = \frac{\pi^2}{12}$$ BI (256)(10)

2. $$\int_0^\infty \ln\left(1-e^{-x}\right)\,dx = -\frac{\pi^2}{6}$$ BI (256)(11)

3. $$\int_0^\infty \ln\left(1+2e^{-x}\cos t+e^{-2x}\right)\,dx = \frac{\pi^2}{6}-\frac{t^2}{2} \qquad [|t|<\pi]$$ BI (256)(18)

4.224

1. $$\int_0^u \ln\sin x\,dx = L\left(\frac{\pi}{2}-u\right)-L\left(\frac{\pi}{2}\right)$$ LO III 186(15)

2. $$\int_0^{\pi/4} \ln\sin x\,dx = -\frac{\pi}{4}\ln 2-\frac{1}{2}\boldsymbol{G}$$ BI (285)(1)

3. $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx = \frac{1}{2}\int_0^{\pi} \ln\sin x\,dx = -\frac{\pi}{2}\ln 2$ FI II 629,643

4. $\displaystyle\int_0^{u} \ln\cos x\,dx = -\,L(u)$ LO III 184(10)

5. $\displaystyle\int_0^{\pi/4} \ln\cos x\,dx = -\frac{\pi}{4}\ln 2 + \frac{1}{2}\boldsymbol{G}$ BI (286)(1)

6. $\displaystyle\int_0^{\pi/2} \ln\cos x\,dx = -\frac{\pi}{2}\ln 2$ BI 306(1)

7. $\displaystyle\int_0^{\pi/2} (\ln\sin x)^2\,dx = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right]$ BI (305)(19)

8. $\displaystyle\int_0^{\pi/2} (\ln\cos x)^2\,dx = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right]$ BI (306)(14)

9.[8] $\displaystyle\int_0^{\pi} \ln(a + b\cos x)\,dx = \pi\ln\frac{a+\sqrt{a^2-b^2}}{2}$ $[a \geq |b| > 0]$ GW (322)(15)

10. $\displaystyle\int_0^{\pi} \ln(1 \pm \sin x)\,dx = -\pi\ln 2 \pm 4\boldsymbol{G}$ GW (322)(16a)

11.[7] $\displaystyle\int_0^{\pi/2} \ln(1 + a\sin x)\,dx = \frac{\pi}{2}\ln\frac{a}{2} + 2\boldsymbol{G} + 2\sum_{k=1}^{\infty}\frac{b^k}{k}\sum_{n=1}^{k}\frac{(-1)^{n+1}}{2n-1}$ $[a > 0]$ $b = \dfrac{1-a}{1+a}$

$\displaystyle = -\frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$ $[a = 1]$

12. $\displaystyle\int_0^{\pi} \ln(1 + a\cos x)\,dx = \pi\ln\left(\frac{1+\sqrt{1-a^2}}{2}\right)$ $\left[a^2 \leq 1\right]$ BI (330)(1)

12 (1) $\displaystyle\int_0^{\pi} \ln(1 + a\cos x)^2\,dx = \begin{cases} 2\pi\ln\left(\dfrac{1+\sqrt{1-a^2}}{2}\right) & \text{for } a^2 \leq 1 \\ \frac{\pi}{2}\ln\frac{a^2}{4} & \text{for } a^2 \geq 1 \end{cases}$

13. $\displaystyle\int_0^{\pi/2} \ln\left(1 + 2a\sin x + a^2\right)\,dx = \sum_{k=0}^{\infty}\frac{2^{2k}\,(k!)^2}{(2k+1)\cdot(2k+1)!!}\left(\frac{2a}{1+a^2}\right)^{2k+1}$

$\left[a^2 \leq 1\right]$ BI (308)(24)

14.[11] $\displaystyle\int_0^{n\pi} \ln\left(a^2 - 2ab\cos x + b^2\right)\,dx = 2n\pi\ln\left[\max\left(|a|, |b|\right)\right]$

$[ab > 0]$ FI II 142, 163, 688

15.[8] $\displaystyle\int_0^{n\pi} \ln\left(1 - 2a\cos x + a^2\right)\,dx = 0$ $\left[a^2 \leq 1\right]$

$\displaystyle = n\pi\ln a^2$ $\left[a^2 \geq 1\right]$

4.225

1. $\displaystyle\int_0^{\pi/4} \ln(\cos x - \sin x)\,dx = -\frac{\pi}{8}\ln 2 - \frac{1}{2}\boldsymbol{G}$ GW (322)(9b)

2. $$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ GW (322)(9a)

3. $$\int_0^{2\pi} \ln(1 + a\sin x + b\cos x)\, dx = 2\pi \ln \frac{1+\sqrt{1-a^2-b^2}}{2}$$
$$\left[a^2+b^2<1\right]$$ BI (332)(2)

4. $$\int_0^{2\pi} \ln\left(1 + a^2 + b^2 + 2a\sin x + 2b\cos x\right)\, dx = 0 \qquad \left[a^2+b^2 \le 1\right]$$
$$= 2\pi \ln\left(a^2+b^2\right) \qquad \left[a^2+b^2 \ge 1\right]$$
BI (322)(3)

4.226

1. $$\int_0^{\pi/2} \ln\left(a^2 - \sin^2 x\right)^2\, dx = -2\pi \ln 2 \qquad \left[a^2 \le 1\right]$$
$$= 2\pi \ln \frac{a+\sqrt{a^2-1}}{2} = 2\pi\left(\operatorname{arccosh} a - \ln 2\right) \qquad [a>1]$$
FI II 644, 687

2. $$\int_0^{\pi/2} \ln\left(1 + a\sin^2 x\right)\, dx = \frac{1}{2}\int_0^{\pi} \ln\left(1 + a\sin^2 x\right)\, dx = \int_0^{\pi/2} \ln\left(1 + a\cos^2 x\right)\, dx$$
$$= \frac{1}{2}\int_0^{\pi} \ln\left(1 + a\cos^2 x\right)\, dx = \pi \ln \frac{1+\sqrt{1+a}}{2}$$
$$[a \ge -1]$$ BI (308)(15), GW(322)(12)

3. $$\int_0^{u} \ln\left(1 - \sin^2\alpha \sin^2 x\right)\, dx = (\pi - 2\theta)\ln\cot\frac{\alpha}{2} + 2u\ln\left(\frac{1}{2}\sin\alpha\right) - \frac{\pi}{2}\ln 2$$
$$+ L(\theta+u) - L(\theta-u) + L\left(\frac{\pi}{2} - 2u\right)$$
$$\left[\cot\theta = \cos\alpha\tan u; \quad -\pi \le \alpha \le \pi, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}\right]$$ LO III 287

4. $$\int_0^{\pi/2} \ln\left[1 - \cos^2 x\left(\sin^2\alpha - \sin^2\beta\sin^2 x\right)\right]\, dx = \pi \ln\left[\frac{1}{2}\left(\cos^2\frac{\alpha}{2} + \sqrt{\cos^4\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}}\right)\right]$$
$$[\alpha > \beta > 0]$$ LO III 283

5. $$\int_0^{u} \ln\left(1 - \frac{\sin^2 x}{\sin^2\alpha}\right)\, dx = -u\ln\sin^2\alpha - L\left(\frac{\pi}{2} - \alpha + u\right) + L\left(\frac{\pi}{2} - \alpha - u\right)$$
$$\left[-\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad |\sin u| \le |\sin\alpha|\right]$$
LO III 287

6. $$\int_0^{\pi/2} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\, dx = \frac{1}{2}\int_0^{\pi} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\, dx = \pi\ln\frac{a+b}{2}$$
$$[a>0, \quad b>0]$$ GW (322)(13)

7. $$\int_0^{\pi/2} \ln \frac{1+\sin t\cos^2 x}{1-\sin t\cos^2 x}\,dx = \pi \ln \frac{1+\sin\frac{t}{2}}{\cos\frac{t}{2}} = \pi \ln\cot\frac{\pi-t}{4}$$
$$\left[|t| < \frac{\pi}{2}\right]$$ LO III 283

4.227

1. $$\int_0^u \ln\tan x\,dx = L(u) + L\left(\frac{\pi}{2}-u\right) - L\left(\frac{\pi}{2}\right)$$ LO III 186(16)

2. $$\int_0^{\pi/4} \ln\tan x\,dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\tan x\,dx = -\boldsymbol{G}$$ BI (286)(11)

3. $$\int_0^{\pi/2} \ln(a\tan x)\,dx = \frac{\pi}{2}\ln a \qquad [a>0]$$ BI (307)(2)

4.[7] $$\int_0^{\pi/4} (\ln\tan x)^n\,dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}$$
$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^{n+1} |E_n| \qquad [n \text{ even}]$$
BI (286)(21)

5.[7] $$\int_0^{\pi/2} (\ln\tan x)^{2n}\,dx = 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|$$ BI (307)(15)

6. $$\int_0^{\pi/2} (\ln\tan x)^{2n+1}\,dx = 0$$ BI (307)(14)

7. $$\int_0^{\pi/4} (\ln\tan x)^2\,dx = \frac{\pi^3}{16}$$ BI (286)(16)

8. $$\int_0^{\pi/4} (\ln\tan x)^4\,dx = \frac{5}{64}\pi^5$$ BI (286)(19)

9. $$\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$$ BI (287)(1)

10. $$\int_0^{\pi/2} \ln(1+\tan x)\,dx = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$$ BI (308)(9)

11. $$\int_0^{\pi/4} \ln(1-\tan x)\,dx = \frac{\pi}{8}\ln 2 - \boldsymbol{G}$$ BI (287)(2)

12.[11] $$\int_0^{\pi/2} (\ln(1-\tan x))^2\,dx = \frac{\pi}{2}\ln 2 - 2\boldsymbol{G}$$ BI (308)(10)

13. $$\int_0^{\pi/4} \ln(1+\cot x)\,dx = \frac{\pi}{8}\ln 2 + \boldsymbol{G}$$ BI (287)(3)

14. $$\int_0^{\pi/4} \ln(\cot x - 1)\,dx = \frac{\pi}{8}\ln 2$$ BI (287)(4)

15. $$\int_0^{\pi/4} \ln(\tan x + \cot x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln(\tan x + \cot x)\, dx = \frac{\pi}{2}\ln 2$$ BI (287)(5), BI (308)(11)

16.[11] $$\int_0^{\pi/4} (\ln(\cot x - \tan x))^2\, dx = \frac{1}{2}\int_0^{\pi/2} (\ln(\cot x - \tan x))^2\, dx = \frac{\pi}{2}\ln 2$$ BI (287)(6), BI (308)(12)

17. $$\int_0^{\pi/2} \ln\left(a^2 + b^2\tan^2 x\right)\, dx = \frac{1}{2}\int_0^{\pi} \ln\left(a^2 + b^2\tan^2 x\right)\, dx = \pi\ln(a+b)$$ $[a > 0, \quad b > 0]$ GW (322)(17)

4.228

1. $$\int_0^{\pi/2} \ln\left(\sin t\sin x + \sqrt{1-\cos^2 t\sin^2 x}\right)\, dx = \frac{\pi}{2}\ln 2 - 2\,L\left(\frac{t}{2}\right) - 2\,L\left(\frac{\pi - t}{2}\right)$$ LO III 290

2. $$\int_0^{u} \ln\left(\cos x + \sqrt{\cos^2 x - \cos^2 t}\right)\, dx = -\left(\frac{\pi}{2} - t - \varphi\right)\ln\cos t + \frac{1}{2}\,L(u+\varphi) - \frac{1}{2}\,L(u-\varphi) - L(\varphi)$$ $\left[\cos\varphi = \dfrac{\sin u}{\sin t} \quad 0 \le u \le t \le \dfrac{\pi}{2}\right]$ LO III 290

3. $$\int_0^{t} \ln\left(\cos x + \sqrt{\cos^2 x - \cos^2 t}\right)\, dx = -\left(\frac{\pi}{2} - t\right)\ln\cos t$$ LO III 285

4. $$\int_0^{u} \ln\frac{\sin u + \sin t\cos x\sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t\cos x\sqrt{\sin^2 u - \sin^2 x}}\, dx = \pi\ln\left[\tan\frac{t}{2}\sin u + \sqrt{\tan^2\frac{t}{2}\sin^2 u + 1}\right]$$ $[t > 0, \quad u > 0]$ LO III 283

5. $$\int_0^{\pi/4} \sqrt{\ln\cot x}\, dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}$$ BI (297)(9)

6. $$\int_0^{\pi/4} \frac{dx}{\sqrt{\ln\cot x}} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}$$ BI (304)(24)

7. $$\int_0^{\pi/4} \ln\left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln\left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (287)(7), BI (308)(22)

8. $$\int_0^{\pi/4} \ln^2\left(\sqrt{\cot x} - \sqrt{\tan x}\right)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln^2\left(\sqrt{\cot x} - \sqrt{\tan x}\right)\, dx = \frac{\pi}{4}\ln 2 - \boldsymbol{G}$$ BI (287)(8), BI (308)(23)

4.229

1. $$\int_0^{1} \ln\left(\ln\frac{1}{x}\right)\, dx = -\boldsymbol{C}$$ FI II 807

2.[11] $$\mathrm{PV}\int_0^{1} \frac{dx}{\ln\left(\ln\dfrac{1}{x}\right)} = \mathrm{PV}\int_0^{\infty} \frac{e^{-u}}{\ln u}\, du \approx -0.154479$$ BI (31)(2)

3. $$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{\sqrt{\ln\frac{1}{x}}} = -\left(\boldsymbol{C}+2\ln 2\right)\sqrt{\pi}$$ BI (32)(4)

4.[11] $$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} dx = \psi(\mu)\,\Gamma(\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (30)(10)

If the integrand contains $(\ln\ln\frac{1}{x})$, it is convenient to make the substitution $\ln\frac{1}{x} = u$ so that $x = e^{-u}$.

5.[7] $$\int_0^1 \ln\left(a+\ln x\right)\,dx = \ln a - e^{-a}\operatorname{Ei}(a) \qquad [a>0]$$ BI (30)(5)

6. $$\int_0^1 \ln\left(a-\ln x\right)\,dx = \ln a - e^{a}\operatorname{Ei}(-a) \qquad [a>0]$$ BI (30)(6)

7. $$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\,dx = \frac{\pi}{2}\ln\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right)$$ BI (308)(28)

4.23 Combinations of logarithms and rational functions

4.231

1. $$\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$$ FI II 483a

2. $$\int_0^1 \frac{\ln x}{1-x}\,dx = -\frac{\pi^2}{6}$$ FI II 714

3. $$\int_0^1 \frac{x\ln x}{1-x}\,dx = 1-\frac{\pi^2}{6}$$ BI (108)(7)

4. $$\int_0^1 \frac{1+x}{1-x}\ln x\,dx = 1-\frac{\pi^2}{3}$$ BI (108)(9)

5.[11] $$\int_0^\infty \frac{\ln x\,dx}{(x+a)^2} = \frac{\ln a}{a} \qquad [0<a]$$ BI (139)(1)

6. $$\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = -\ln 2$$ BI (111)(1)

7.[7] $$\int_0^\infty \ln x\frac{dx}{(a^2+b^2x^2)^n} = \frac{\Gamma\left(n-\frac{1}{2}\right)\sqrt{\pi}}{4(n-1)!a^{2n-1}b}\left[2\ln\frac{a}{2b}-\boldsymbol{C}-\psi\left(n-\frac{1}{2}\right)\right]$$
$$[a>0,\quad b>0]$$ LI (139)(3)

8. $$\int_0^\infty \frac{\ln x\,dx}{a^2+b^2x^2} = \frac{\pi}{2ab}\ln\frac{a}{b} \qquad [ab>0]$$ BI (135)(6)

9. $$\int_0^\infty \frac{\ln px}{q^2+x^2}\,dx = \frac{\pi}{2q}\ln pq \qquad [p>0,\quad q>0]$$ BI (135)(4)

10. $$\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} = -\frac{\pi^2}{4ab} \qquad [ab>0]$$

11. $\displaystyle\int_0^a \frac{\ln x\,dx}{x^2+a^2} = \frac{\pi\ln a}{4a} - \frac{\boldsymbol{G}}{a}$ $[a>0]$ GW (324)(7b)

12. $\displaystyle\int_0^1 \frac{\ln x}{1+x^2}\,dx = -\int_1^\infty \frac{\ln x}{1+x^2}\,dx = -\boldsymbol{G}$ FI II 482, 614

13. $\displaystyle\int_0^1 \frac{\ln x\,dx}{1-x^2} = -\frac{\pi^2}{8}$ BI (108)(11)

14. $\displaystyle\int_0^1 \frac{x\ln x}{1+x^2}\,dx = -\frac{\pi^2}{48}$ GW (324)(7b)

15. $\displaystyle\int_0^1 \frac{x\ln x}{1-x^2}\,dx = -\frac{\pi^2}{24}$

16. $\displaystyle\int_0^1 \ln x\frac{1-x^{2n+2}}{(1-x^2)^2}\,dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^n \frac{n-k+1}{(2k-1)^2}$ BI (111)(5)

17. $\displaystyle\int_0^1 \ln x\frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^n (-1)^k\frac{n-k+1}{k^2}$ BI (111)(2)

18. $\displaystyle\int_0^1 \ln x\frac{1-x^{n+1}}{(1-x)^2}\,dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^n \frac{n-k+1}{k^2}$ BI (111)(3)

19.* $\displaystyle\int_0^1 \frac{x\ln x}{1+x}\,dx = -1+\frac{\pi^2}{2}$

20.* $\displaystyle\int_0^1 \frac{(1-x)\ln x}{1+x}\,dx = 1-\frac{\pi^2}{6}$

4.232

1. $\displaystyle\int_u^v \frac{\ln x\,dx}{(x+u)(x+v)} = \frac{\ln uv}{2(v-u)}\ln\frac{(u+v)^2}{4uv}$ BI (145)(32)

2. $\displaystyle\int_0^\infty \frac{\ln x\,dx}{(x+\beta)(x+\gamma)} = \frac{(\ln\beta)^2-(\ln\gamma)^2}{2(\beta-\gamma)}$ $[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi]$ ET II 218(24)

3. $\displaystyle\int_0^\infty \frac{\ln x}{x+a}\frac{dx}{x-1} = \frac{\pi^2+(\ln a)^2}{2(a+1)}$ $[a>0]$ BI (140)(10)

4.233

1.[3] $\displaystyle\int_0^1 \frac{\ln x\,dx}{1+x+x^2} = \frac{2}{9}\left[\frac{2\pi^2}{3}-\psi'\left(\frac{1}{3}\right)\right] = -0.7813024129\ldots$ LI (113)(1)

2.[3] $\displaystyle\int_0^1 \frac{\ln x\,dx}{1-x+x^2} = \frac{1}{3}\left[\frac{2\pi^2}{3}-\psi'\left(\frac{1}{3}\right)\right] = -1.17195361934\ldots$ LI (113)(2)

3.[11] $\displaystyle\int_0^1 \frac{x\ln x\,dx}{1+x+x^2} = -\frac{1}{9}\left[\frac{7\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right] = -0.15766014917\ldots$ LI (113)(2)

4.[3] $\displaystyle\int_0^1 \frac{x\ln x\,dx}{1-x+x^2} = \frac{1}{6}\left[\frac{5\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right] = -0.3118211319\ldots$ LI (113)(4)

5. $$\int_0^\infty \frac{\ln x\,dx}{x^2+2xa\cos t+a^2} = \frac{t\ln a}{a\sin t} \qquad [a>0, \quad 0<t<\pi]$$ GW (324)(13c)

4.234

1.[11] $$\int_1^\infty \frac{\ln x\,dx}{(1+x^2)^2} = \frac{\boldsymbol{G}}{2} - \frac{\pi}{8}$$ BI (144)(18)a

2. $$\int_0^1 \frac{x\ln x\,dx}{(1+x^2)^2} = -\frac{1}{4}\ln 2$$ BI (111)(4)

3. $$\int_0^\infty \frac{1+x^2}{(1-x^2)^2}\ln x\,dx = 0$$ BI (142)(2)a

4. $$\int_0^\infty \frac{1-x^2}{(1+x^2)^2}\ln x\,dx = -\frac{\pi}{2}$$ BI (142)(1)a

5. $$\int_0^1 \frac{x^2\ln x\,dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16\left(2+\sqrt{2}\right)}$$ BI (112)(21)

6. $$\int_0^\infty \frac{\ln x\,dx}{(a^2+b^2x^2)(1+x^2)} = \frac{b\pi}{2a(b^2-a^2)}\ln\frac{a}{b} \qquad [ab>0]$$ BI (317)(16)a

7. $$\int_0^\infty \frac{\ln x}{x^2+a^2}\cdot\frac{dx}{1+b^2x^2} = \frac{\pi}{2(1-a^2b^2)}\left(\frac{1}{a}\ln a + b\ln b\right)$$
$[a>0, \quad b>0]$ LI (140)(12)

8. $$\int_0^\infty \frac{x^2\ln x\,dx}{(a^2+b^2x^2)(1+x^2)} = \frac{a\pi}{2b(b^2-a^2)}\ln\frac{b}{a} \qquad [ab>0]$$ LI (140)(12), BI (317)(15)a

4.235

1. $$\int_0^\infty \ln x\frac{(1-x)x^{n-2}}{1-x^{2n}}\,dx = -\frac{\pi^2}{4n^2}\tan^2\frac{\pi}{2n} \qquad [n>1]$$ BI (135)(10)

2. $$\int_0^\infty \ln x\frac{\left(1-x^2\right)x^{m-1}}{1-x^{2n}}\,dx = -\frac{\pi^2\sin\left(\frac{m+1}{n}\right)\pi\sin\left(\frac{\pi}{n}\right)}{4n^2\sin^2\left(\frac{m\pi}{2n}\right)\sin^2\left(\frac{m+2}{2n}\pi\right)}$$ LI (135)(12)

3.[11] $$\int_0^\infty \ln x\frac{\left(1-x^2\right)x^{n-3}}{1-x^{2n}}\,dx = -\frac{\pi^2}{4n^2}\tan^2\left(\frac{\pi}{n}\right) \qquad [n>2]$$ BI (135)(11)

4. $$\int_0^1 \ln x\frac{x^{m-1}+x^{n-m-1}}{1-x^n}\,dx = -\frac{\pi^2}{n^2\sin^2\left(\frac{m}{n}\pi\right)} \qquad [n>m]$$ BI (108)(15)

4.236

1. $$\int_0^1 \left\{\frac{1+(p-1)\ln x}{1-x} + \frac{x\ln x}{(1-x)^2}\right\}x^{p-1}\,dx = -1+\psi'(p)$$
$[p>0]$ BI (111)(6)a, GW (326)(13)

2. $$\int_0^1 \left[\frac{1}{1-x} + \frac{x\ln x}{(1-x)^2}\right]dx = \frac{\pi^2}{6} - 1$$ GW (326)(13a)

4.24 Combinations of logarithms and algebraic functions

4.241

1. $$\int_0^1 \frac{x^{2n}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}-\ln 2\right)$$ BI (118)(5)a

2. $$\int_0^1 \frac{x^{2n+1}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n)!!}{(2n+1)!!}\left(\ln 2+\sum_{k=1}^{2n+1}\frac{(-1)^{k}}{k}\right)$$ BI (118)(5)a

3. $$\int_0^1 x^{2n}\sqrt{1-x^2}\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}-\frac{1}{2n+2}-\ln 2\right)$$ LI (117)(4), GW (324)(53a)

4. $$\int_0^1 x^{2n+1}\sqrt{1-x^2}\ln x\,dx = \frac{(2n)!!}{(2n+3)!!}\left(\ln 2+\sum_{k=1}^{2n+1}\frac{(-1)^{k}}{k}-\frac{1}{2n+3}\right)$$ BI (117)(5), GW (324)(53b)

5. $$\int_0^1 \ln x\cdot\sqrt{(1-x^2)^{2n-1}}\,dx = -\frac{(2n-1)!!}{4\cdot(2n)!!}\pi\left[\psi(n+1)+\boldsymbol{C}+\ln 4\right]$$ BI (117)(3)

6. $$\int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{4}\ln 2-\frac{1}{2}\boldsymbol{G}$$ BI (145)(1)

7. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\ln 2$$ FI II 614, 643

8. $$\int_1^\infty \frac{\ln x\,dx}{x^2\sqrt{x^2-1}} = 1-\ln 2$$ BI (144)(17)

9. $$\int_0^1 \sqrt{1-x^2}\ln x\,dx = -\frac{\pi}{8}-\frac{\pi}{4}\ln 2$$ BI (117)(1), GW (324)(53c)

10. $$\int_0^1 x\sqrt{1-x^2}\ln x\,dx = \tfrac{1}{3}\ln 2-\tfrac{4}{9}$$ BI (117)(2)

11. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{x\,(1-x^2)}} = -\frac{\sqrt{2\pi}}{8}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$ GW (324)(54a)

4.242

1. $$\int_0^\infty \frac{\ln x\,dx}{\sqrt{(a^2+x^2)\,(x^2+b^2)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab$$
$[a>b>0]$ BY (800.04)

2. $$\int_0^b \frac{\ln x\,dx}{\sqrt{(a^2+x^2)\,(b^2-x^2)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\ln ab-\frac{\pi}{2}\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\right]$$
$[a>0,\quad b>0]$ BY (800.02)

3. $$\int_b^\infty \frac{\ln x\,dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right]$$
$[a>0, \quad b>0]$ BY (800.06)

4. $$\int_0^b \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab - \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$
$[a>b>0]$ BY (800.01)

5. $$\int_b^a \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab$$
BY (800.03)

6. $$\int_a^\infty \frac{\ln x\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$
$[a>b>0]$ BY (800.05)

4.243 $$\int_0^1 \frac{x\ln x}{\sqrt{1-x^4}}\,dx = -\frac{\pi}{8}\ln 2$$
GW (324)(56b)

4.244

1. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8}\left[\Gamma\left(\frac{1}{3}\right)\right]^3$$
GW (324)(54b)

2. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 + \frac{\pi}{3\sqrt{3}}\right)$$
BI (118)(7)

3. $$\int_0^1 \frac{x\ln x\,dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}}\left(\frac{\pi}{3\sqrt{3}} - \ln 3\right)$$
BI (118)(8)

4.245

1. $$\int_0^1 \frac{x^{4n+1}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{8}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right)$$
GW (324)(56a)

2. $$\int_0^1 \frac{x^{4n+3}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n)!!}{4\cdot(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right)$$
GW (324)(56c)

4.246 $$\int_0^1 \left(1-x^2\right)^{n-\frac{1}{2}}\ln x\,dx = -\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{4}\left[2\ln 2 + \sum_{k=1}^{n}\frac{1}{k}\right]$$
GW (324)(55)

4.247

1.[6] $$\int_0^1 \frac{\ln x}{\sqrt[n]{1-x^{2n}}}\,dx = -\frac{\pi\,\mathrm{B}\left(\frac{1}{2n},\frac{1}{2n}\right)}{8n^2\sin\frac{\pi}{2n}}$$
$[n>1]$ GW (324)(54c)a

2.[6] $$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi\,\mathrm{B}\left(\frac{1}{2n},\frac{1}{2n}\right)}{8\sin\frac{\pi}{2n}}$$
GW (324)(54)

4.25 Combinations of logarithms and powers

4.251

1. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{\beta+x}\,dx = \frac{\pi\beta^{\mu-1}}{\sin\mu\pi}\left(\ln\beta - \pi\cot\mu\pi\right) \qquad [|\arg\beta|<\pi,\quad 0<\operatorname{Re}\mu<1]$$
BI (135)(1)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{a-x}\,dx = \pi a^{\mu-1}\left(\cot\mu\pi\ln a - \frac{\pi}{\sin^2\mu\pi}\right) \qquad [a>0,\quad 0<\operatorname{Re}\mu<1]$$
ET I 314(5)

3.[10] $$\int_0^1 \frac{x^{\mu-1}\ln x}{x+1}\,dx = \beta'(\mu) \qquad [\operatorname{Re}\mu>0]$$
GW (324)(6), ET I 314(3)

4. $$\int_0^1 \frac{x^{\mu-1}\ln x}{1-x}\,dx = -\psi'(\mu) = -\zeta(2,\mu) \qquad [\operatorname{Re}\mu>0]$$
BI (108)(8)

5.[11] $$\int_0^1 \ln x\frac{x^{2n}}{1+x}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k^2}$$
BI (108)(4)

6.[11] $$\int_0^1 \ln x\frac{x^{2n-1}}{1+x}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1}\frac{(-1)^{k}}{k^2}$$
BI (108)(5)

4.252

1. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{(x+\beta)(x+\gamma)}\,dx = \frac{\pi}{(\gamma-\beta)\sin\mu\pi}\left[\beta^{\mu-1}\ln\beta - \gamma^{\mu-1}\ln\gamma - \pi\cot\mu\pi\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\right]$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\mu<2,\quad \mu\neq 1]$ BI (140)(9)a, ET 314(6)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x\,dx}{(x+\beta)(x-1)} = \frac{\pi}{(\beta+1)\sin^2\mu\pi}\left[\pi - \beta^{\mu-1}\left(\sin\mu\pi\ln\beta - \pi\cos\mu\pi\right)\right]$$
$[|\arg\beta|<\pi,\quad 0<\operatorname{Re}\mu<2,\quad \mu\neq 1]$
BI (140)(11)

3. $$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^2}\,dx = -\frac{\pi^2}{4}\operatorname{cosec}^2\frac{p\pi}{2} \qquad [0<p<2]$$
(see also **4.254** 2)

4.[6] $$\int_0^\infty \frac{x^{\mu-1}\ln x}{(x+a)^2}\,dx = \frac{(1-\mu)a^{\mu-2}\pi}{\sin\mu\pi}\left(\ln a - \pi\cot\mu\pi + \frac{1}{\mu-1}\right)$$
$[|\arg a|<\pi\quad 0<\operatorname{Re}\mu<2\quad (\mu\neq 1)]$
GW (324)(13b)

4.253

1.[8] $$\int_0^1 x^{\mu-1}\left(1-x^r\right)^{\nu-1}\ln x\,dx = \frac{1}{r^2}\operatorname{B}\left(\frac{\mu}{r},\nu\right)\left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r}+\nu\right)\right]$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad r>0]$
GW (324)(3b)a, BI (107)(5)a

2. $$\int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}}\ln x\,dx = -\frac{\pi}{p}\operatorname{cosec}p\pi \qquad [0<p<1]$$
bi (319)(10)a

3. $$\int_u^\infty \frac{(x-u)^{\mu-1}\ln x\,dx}{x^\lambda} = u^{\mu-\lambda}\,\mathrm{B}(\lambda-\mu,\mu)\,[\ln u+\psi(\lambda)-\psi(\lambda-\mu)]$$
$[0<\operatorname{Re}\mu<\operatorname{Re}\lambda]$ ET II 203(18)

4.[11] $$\int_0^\infty \ln x\left(\frac{x}{a^2+x^2}\right)^p\frac{dx}{x} = \frac{\ln a}{2a^p}\,\mathrm{B}\left(\frac{p}{2},\frac{p}{2}\right)$$
$[a>0,\quad p>0]$ BI (140)(6)

5. $$\int_1^\infty (x-1)^{p-1}\ln x\,dx = \frac{\pi}{p}\operatorname{cosec}\pi p$$
$[-1<p<0]$ BI (289)(12)a

6.[7] $$\int_0^\infty \ln x\frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu}\left(\ln a-\boldsymbol{C}-\psi(\mu)\right)$$
$[\operatorname{Re}\mu>0,\quad a\neq 0,\quad |\arg a|<\pi]$
NT 68(7)

7.[7] $$\int_0^\infty \ln x\frac{dx}{(a+x)^{n+\frac12}} = \frac{2}{(2n-1)a^{n-\frac12}}\left(\ln a+2\ln 2-2\sum_{k=1}^{n-1}\frac{1}{2k-1}\right)$$
$[|\arg a|<\pi,\quad n=1,2,\ldots]$ BI (142)(5)

4.254

1. $$\int_0^1 \frac{x^{p-1}\ln x}{1-x^q}\,dx = -\frac{1}{q^2}\,\psi'\left(\frac{p}{q}\right)$$
$[p>0,\quad q>0]$ GW (324)(5)

2. $$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^q}\,dx = -\frac{\pi^2}{q^2\sin^2\dfrac{p\pi}{q}}$$
$[0<p<q]$ BI (135)(8)

3. $$\int_0^\infty \frac{\ln x}{x^q-1}\frac{dx}{x^p} = \frac{\pi^2}{q^2\sin^2\dfrac{p-1}{q}\pi}$$
$[p<1,\quad p+q>1]$ BI (140)(2)

4.[3] $$\int_0^1 \frac{x^{p-1}\ln x}{1+x^q}\,dx = \frac{1}{q^2}\beta'\left(\frac{p}{q}\right)$$
$[p>0,\quad q>0]$ GW (324)(7)

5. $$\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\,dx = -\frac{\pi^2}{q^2}\frac{\cos\dfrac{p\pi}{q}}{\sin^2\dfrac{p\pi}{q}}$$
$[0<p<q]$ BI (135)(7)

6. $$\int_0^1 \frac{x^{q-1}\ln x}{1-x^{2q}}\,dx = -\frac{\pi^2}{8q^2}$$
$[q>0]$ BI (108)(12)

4.255

1. $$\int_0^1 \ln x\frac{\left(1-x^2\right)x^{p-2}}{1+x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2\frac{\sin\frac{\pi}{2p}}{\cos^2\left(\frac{\pi}{2p}\right)}$$
$[p>1]$ BI (108)(13)

2. $$\int_0^1 \ln x\frac{\left(1+x^2\right)x^{p-2}}{1-x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2\sec^2\left(\frac{\pi}{2p}\right)$$
$[p>1]$ BI (108)(14)

3. $$\int_0^\infty \ln x\frac{1-x^p}{1-x^2}\,dx = \frac{\pi^2}{4}\tan^2\left(\frac{p\pi}{2}\right)$$
$[p<1]$ BI (140)(3)

4.256 $$\int_0^1 \ln\frac{1}{x}\,\frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2}\,\mathrm{B}\left(\frac{\mu}{n},\frac{m}{n}\right)\left[\psi\left(\frac{\mu+m}{n}\right)-\psi\left(\frac{\mu}{n}\right)\right]$$
$[\operatorname{Re}\mu > 0]$ LI (118)(12)

4.257

1. $$\int_0^\infty \frac{x^\nu \ln\frac{x}{\beta}\,dx}{(x+\beta)(x+\gamma)} = \frac{\pi\left[\gamma^\nu \ln\frac{\gamma}{\beta} + \pi\left(\beta^\nu-\gamma^\nu\right)\cot\nu\pi\right]}{\sin\nu\pi(\gamma-\beta)}$$
$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad |\operatorname{Re}\nu| < 1]$ ET II 219(30)

2. $$\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)\frac{dx}{x} = 0$$
$[q > 0]$ BI (140)(4)a

3. $$\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)^r\frac{dx}{q^2+x^2} = 0$$
$[q > 0]$ BI (140)(4)a

4. $$\int_0^\infty \ln x \ln\frac{x}{a}\,\frac{dx}{(x-1)(x-a)} = \frac{\left[4\pi^2+(\ln a)^2\right]\ln a}{6(a-1)}$$
$[a > 0]$ (for $a = 1$ see **4.261** 5) BI (141)(5)

5. $$\int_0^\infty \ln x \ln\frac{x}{a}\,\frac{x^p\,dx}{(x-1)(x-a)} = \frac{\pi^2\left[(a^p+1)\ln a - 2\pi\left(a^p-1\right)\cot p\pi\right]}{(a-1)\sin^2 p\pi}$$
$\left[p^2 < 1, \quad a > 0\right]$ BI (141)(6)

4.26–4.27 Combinations involving powers of the logarithm and other powers

4.261

1.[7] $$\int_0^1 (\ln x)^2\,\frac{dx}{1+2x\cos t+x^2} = \frac{t\left(\pi^2-t^2\right)}{6\sin t}$$
$[0 \le t \le \pi]$ BI (113)(7)

2. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{10\pi^3}{81\sqrt{3}}$$
GW (324)(16c)

3. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{8\pi^3}{81\sqrt{3}}$$
GW (324)(16b)

4. $$\int_0^\infty (\ln x)^2\,\frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2+(\ln a)^2\right]\ln a}{3(1+a)}$$
$[a > 0]$ BI (141)(1)

5. $$\int_0^\infty (\ln x)^2\,\frac{dx}{(1-x)^2} = \frac{2}{3}\pi^2$$
BI (139)(4)

6. $$\int_0^1 (\ln x)^2\,\frac{dx}{1+x^2} = \frac{\pi^3}{16}$$
BI (109)(3)

7. $$\int_0^1 (\ln x)^2\,\frac{1+x^2}{1+x^4}\,dx = \frac{1}{2}\int_0^\infty (\ln x)^2\,\frac{1+x^2}{1+x^4}\,dx = \frac{3\sqrt{2}}{64}\pi^3$$
BI (109)(5), BI (135)(13)

8.[11] $$\int_0^1 (\ln x)^2\,\frac{1-x}{1-x^6}\,dx = \frac{8\sqrt{3}\pi^3+351\,\zeta(3)}{486}$$

9. $$\int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right]$$ BI (118)(13)

10. $$\int_0^\infty (\ln x)^2 \frac{x^{\mu-1}}{1+x}\,dx = \frac{\pi^3\left(2-\sin^2\mu\pi\right)}{\sin^3\mu\pi} \qquad [0<\operatorname{Re}\mu<1]$$ ET I 315(10)

11.[7] $$\int_0^1 (\ln x)^2 \frac{x^n\,dx}{1+x} = 2\sum_{k=n}^\infty \frac{(-1)^{n+k}}{(k+1)^3} = (-1)^n\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^n \frac{(-1)^k}{k^3}\right)$$
$[n=0,1,\ldots]$ BI (109)(1)

12.[7] $$\int_0^1 (\ln x)^2 \frac{x^n\,dx}{1-x} = 2\sum_{k=n}^\infty \frac{1}{(k+1)^3} = 2\left(\zeta(3) - \sum_{k=1}^n \frac{1}{k^3}\right)$$
$[n=0,1,\ldots]$ BI (109)(2)

13.[11] $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{1-x^2} = 2\sum_{k=n}^\infty \frac{1}{(2k+1)^3} = \frac{7}{4}\zeta(3) - 2\sum_{k=1}^n \frac{1}{(2k-1)^3}$$
$[n=0,1,\ldots]$ BI (109)(4)

14. $$\int_0^\infty (\ln x)^2 \frac{x^{p-1}\,dx}{x^2+2x\cos t+1} = \frac{\pi\sin(1-p)t}{\sin t\sin p\pi}\left\{\pi^2 - t^2 + 2\pi\cot p\pi\left[\pi\cot p\pi + t\cot(1-p)t\right]\right\}$$
$[0<t<\pi, \quad 0<p<2, \quad p\neq 1]$
GW (324)(17)

15. $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2\cdot(2n)!!}\pi\left\{\frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}$$ GW (324)(60a)

16. $$\int_0^1 (\ln x)^2 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}\left\{-\frac{\pi^2}{12} - \sum_{k=1}^{2n+1}\frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}$$
GW (324)(60b)

17.[7] $$\int_0^1 (\ln x)^2 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\nu)\left\{[\psi(\mu)-\psi(\nu+\mu)]^2 + \psi'(\mu) - \psi'(\mu+\nu)\right\}$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET I 315(11)

18. $$\int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2}\,dx = 2(n+1)\zeta(3) - 2\sum_{k=1}^n \frac{n-k+1}{k^3}$$ LI (111)(8)

19. $$\int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = \frac{3}{2}(n+1)\zeta(3) - 2\sum_{k=1}^n (-1)^{k-1}\frac{n-k+1}{k^3}$$ LI (111)(7)

20.[7] $$\int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{(1-x^2)^2}\,dx = \frac{7}{4}(n+1)\zeta(3) - 2\sum_{k=1}^n \frac{n-k+1}{(2k-1)^3}$$
$[n=0,1,\ldots]$ LI (111)(9)

21. $$\int_0^1 (\ln x)^2 x^{p-1}(1-x^r)^{q-1}\,dx = \frac{1}{r^3}\mathrm{B}\left(\frac{p}{r},q\right)\left\{\psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r}+q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r}+q\right)\right]^2\right\}$$
$[p>0, \quad q>0, \quad r>0]$ GW (324)(8a)

4.262

1. $$\int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4$$ BI (109)(9)

2. $$\int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15}$$ BI (109)(11)

3. $$\int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{\left[\pi^2 + (\ln a)^2\right]^2}{4(a+1)} \qquad [a>0]$$ BI (141)(2)

4. $$\int_0^1 (\ln x)^3 \frac{x^n\,dx}{1+x} = (-1)^{n+1}\left[\frac{7\pi^4}{120} - 6\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)^4}\right] \qquad [n=1,2,\ldots]$$ BI (109)(10)

5. $$\int_0^1 (\ln x)^3 \frac{x^n\,dx}{1-x} = -\frac{\pi^4}{15} + 6\sum_{k=0}^{n-1}\frac{1}{(k+1)^4} \qquad [n=1,2,\ldots]$$ BI (109)(12)

6. $$\int_0^1 (\ln x)^3 \frac{x^{2n}\,dx}{1-x^2} = -\frac{\pi^4}{16} + 6\sum_{k=0}^{n-1}\frac{1}{(2k+1)^4} \qquad [n=1,2,\ldots]$$ BI (109)(14)

7. $$\int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2}\,dx = -\frac{(n+1)\pi^4}{15} + 6\sum_{k=1}^{n}\frac{n-k+1}{k^4}$$ BI (111)(11)

8. $$\int_0^1 (\ln x)^3 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = -\frac{7(n+1)\pi^4}{120} + 6\sum_{k=1}^{n}(-1)^{k-1}\frac{n-k+1}{k^4}$$ BI (111)(10)

9. $$\int_0^1 (\ln x)^3 \frac{1-x^{2n+2}}{(1-x^2)^2}\,dx = -\frac{(n+1)\pi^4}{16} + 6\sum_{k=1}^{n}\frac{n-k+1}{(2k-1)^4}$$ BI (111)(12)

4.263

1.[8] $$\int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a\left[\pi^2 + (\ln a)^2\right]\left[7\pi^2 + 3(\ln a)^2\right]}{15(1+a)} \qquad [a>0]$$ BI (141)(3)

2. $$\int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64}$$ BI (109)(17)

3. $$\int_0^1 (\ln x)^4 \frac{dx}{1+2x\cos t + x^2} = \frac{t\left(\pi^2 - t^2\right)\left(7\pi^2 - 3t^2\right)}{30\sin t} \qquad [|t|<\pi]$$ BI (113)(8)

4.264

1. $$\int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252}$$ BI (109)(20)

2. $$\int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63}$$ BI (109)(21)

3. $$\int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2+(\ln a)^2\right]^2\left[3\pi^2+(\ln a)^2\right]}{6(1+a)}$$ $[a>0]$ BI (141)(4)

4.265 $$\int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256}$$ BI (109)(25)

4.266

1. $$\int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240}$$ BI (109)(28)

2. $$\int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15}$$ BI (109)(29)

4.267

1. $$\int_0^1 \frac{1-x}{1+x}\frac{dx}{\ln x} = \ln\frac{2}{\pi}$$ BI (127)(3)

2. $$\int_0^1 \frac{(1-x)^2}{1+x^2}\frac{dx}{\ln x} = \ln\frac{\pi}{4}$$ BI (128)(2)

3.[8] $$\int_0^1 \frac{(1-x)^2}{1+2x\cos\dfrac{mx}{n}+x^2}\cdot\frac{dx}{\ln x}$$
$$= \frac{1}{\sin\left(\frac{m\pi}{n}\right)}\sum_{k=1}^{n-1}(-1)^k\sin\left(\frac{km\pi}{n}\right)\ln\frac{\left\{\Gamma\left(\frac{n+k+1}{2n}\right)\right\}^2\Gamma\left(\frac{k+2}{2n}\right)\Gamma\left(\frac{k}{2n}\right)}{\left\{\Gamma\left(\frac{k+1}{2n}\right)\right\}^2\Gamma\left(\frac{n+k}{2n}\right)\Gamma\left(\frac{n+k+2}{2n}\right)} \qquad [m+n \text{ is odd}]$$
$$= \frac{1}{\sin\left(\frac{m\pi}{n}\right)}\sum_{k=1}^{\lfloor\frac{1}{2}(n-1)\rfloor}(-1)^k\sin\left(\frac{km\pi}{n}\right)\ln\frac{\left\{\Gamma\left(\frac{n-k+1}{n}\right)\right\}^2\Gamma\left(\frac{k+2}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\left\{\Gamma\left(\frac{k+1}{n}\right)\right\}^2\Gamma\left(\frac{n-k}{n}\right)\Gamma\left(\frac{n-k+2}{n}\right)} \qquad [m+n \text{ is even}]$$
$[m<n]$ BI (130)(3)

4. $$\int_0^1 \frac{1-x}{1+x}\cdot\frac{1}{1+x^2}\cdot\frac{dx}{\ln x} = -\frac{\ln 2}{2}$$ BI (130)(16)

5. $$\int_0^1 \frac{1-x}{1+x}\cdot\frac{x^2}{1+x^2}\cdot\frac{dx}{\ln x} = \ln\frac{2\sqrt{2}}{\pi}$$ BI (130)(17)

6.[11] $$\int_0^1 (1-x)^p\frac{dx}{\ln x} = \sum_{k=1}^\infty(-1)^k\binom{p}{k}\ln(1+k)$$ $[p\geq 1]$ BI (123)(2)

7. $$\int_0^1 \left(\frac{1-x^p}{1-x}-p\right)\frac{dx}{\ln x} = \ln\Gamma(p+1)$$ GW (326)(10)

8. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \ln\frac{p}{q}$$ $[p>0,\quad q>0]$ FI II 647

9. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{\ln x}\cdot\frac{dx}{1+x} = \ln\frac{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}$$ $[p>0,\quad q>0]$ FI II 186

10. $$\int_0^1 \frac{x^{p-1}-x^{-p}}{(1+x)\ln x}\,dx = \frac{1}{2}\int_0^\infty \frac{x^{p-1}-x^{-p}}{(1+x)\ln x}\,dx = \ln\left(\tan\frac{p\pi}{2}\right)$$
$[0<p<1]$ FI II 816

11. $$\int_0^1 (x^p - x^q)\, x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q} \qquad [r>0, \quad p>0, \quad q>0]$$ LI (123)(5)

12. $$\int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k} \qquad \left[p>0, \quad q>0, \quad a^2<1\right]$$ BI (130)(15)

13. $$\int_0^1 (x^p - 1)\,(x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)} \qquad [p>-1, \quad q>-1, \quad p+q>-1]$$ GW (324)(19b)

14. $$\int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x}\, dx = \ln \frac{\Gamma\left(\frac{p}{2}+n+1\right)\Gamma\left(\frac{q+1}{2}+n\right)\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}+n+1\right)\Gamma\left(\frac{p+1}{2}+n\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p}{2}\right)}$$
$[p>0, \quad q>0]$ BI (127)(7)

15. $$\int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x}\, dx = \ln \frac{\Gamma(q+1)\,\Gamma(p+r+1)}{\Gamma(p+1)\,\Gamma(q+r+1)}$$
$[p>-1, \quad q>-1, \quad p+r>-1, \quad q+r>-1]$ GW (324)(23)

16. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r)\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right)\Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right)\Gamma\left(\frac{p}{2r}\right)} \qquad [p>0, \quad q>0, \quad r>0]$$ GW (324)(21)

17. $$\int_0^1 \frac{1-x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1}\, dx}{\ln x} = \ln \tan \frac{q\pi}{4p} \qquad [0<q<p]$$ BI (128)(6)

18. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x^r)\ln x}\, dx = \ln\left(\tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r}\right) \qquad [0<p<r, \quad 0<q<r]$$
GW (324)(22), BI (143)(2)

19. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1-x^r)\ln x}\, dx = \ln\left(\frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}}\right) \qquad [0<p<r, \quad 0<q<r]$$ BI (143)(4)

20. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right)\Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right)\Gamma\left(\frac{p}{2n}\right)} \qquad [p>0, \quad q>0]$$ BI (128)(11)

21. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \frac{1+x^2}{\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right)\Gamma\left(\frac{q+2}{4(2n+1)}\right)\Gamma\left(\frac{p+4n+2}{4(2n+1)}\right)\Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right)\Gamma\left(\frac{p+2}{4(2n+1)}\right)\Gamma\left(\frac{q+4n+2}{4(2n+1)}\right)\Gamma\left(\frac{p}{4(2n+1)}\right)}$$
$[p>0, \quad q>0]$ BI (128)(7)

22. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \cdot \frac{1+x^2}{\ln x}\, dx = \ln\left\{\tan \frac{p\pi}{4(2n+1)} \cdot \tan \frac{(p+2)\pi}{4(2n+1)} \cdot \cot \frac{q\pi}{4(2n+1)} \cdot \cot \frac{(q+2)\pi}{4(2n+1)}\right\}$$
$[0<p<4n, \quad 0<q<4n]$ BI (143)(5)

23. $$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x^{2n}}\,\frac{1-x^2}{\ln x}\,dx = \ln\frac{\sin\dfrac{p\pi}{2n}\cdot\sin\frac{(q+2)\pi}{2n}}{\sin\dfrac{q\pi}{2n}\cdot\sin\dfrac{(p+2)\pi}{2n}}$$
$[0<p<2n,\quad 0<q<2n]$ BI (143)(6)

24. $$\int_0^1 (1-x^p)\,(1-x^q)\,\frac{x^{r-1}\,dx}{\ln x} = \ln\frac{(p+q+r)r}{(p+r)(q+r)}$$
$[p>0,\quad q>0,\quad r>0]$ BI (123)(8)

25. $$\int_0^1 (1-x^p)\,(1-x^q)\,\frac{x^{r-1}\,dx}{(1-x)\ln x} = \ln\frac{\Gamma(p+r)\,\Gamma(q+r)}{\Gamma(p+q+r)\,\Gamma(r)}$$
$[r>0,\quad r+p>0,\quad r+q>0,\quad r+p+q>0]$ FI II 815a

26. $$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{dx}{\ln x} = \ln\frac{(p+q+1)(q+r+1)(r+p+1)}{(p+q+r+1)(p+1)(q+1)(r+1)}$$
$[p>-1,\quad q>-1,\quad r>-1,\quad p+q>-1,\quad p+r>-1,\quad q+r>-1,\quad p+q+r>-1]$
GW (324)(19c)

27. $$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{dx}{(1-x)\ln x} = \ln\frac{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(r+1)\,\Gamma(p+q+r+1)}{\Gamma(p+q+1)\,\Gamma(p+r+1)\,\Gamma(q+r+1)}$$
$[p>-1,\quad q>-1,\quad r>-1,\quad p+q>-1,\quad p+r>-1,\quad q+r>-1,\quad p+q+r>-1]$
FI II 815

28. $$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^{s-1}\,dx}{\ln x} = \ln\frac{(p+q+s)(p+r+s)(q+r+s)s}{(p+s)(q+s)(r+s)(p+q+r+s)}$$
$[p>0,\quad q>0,\quad r>0,\quad s>0]$
BI (123)(10)

29. $$\int_0^1 (1-x^p)\,(1-x^q)\,\frac{x^{s-1}\,dx}{(1-x^r)\ln x} = \ln\frac{\Gamma\left(\frac{p+s}{r}\right)\Gamma\left(\frac{q+s}{r}\right)}{\Gamma\left(\frac{s}{r}\right)\Gamma\left(\frac{p+q+s}{r}\right)}$$
$[p>0,\quad q>0,\quad r>0,\quad s>0]$
GW (324)(23a)

30. $$\int_0^\infty (1-x^p)\,(1-x^q)\,\frac{x^{s-1}\,dx}{(1-x^{p+q+2s})\ln x} = 2\int_0^1 (1-x^p)\,(1-x^q)\,\frac{x^{s-1}\,dx}{(1-x^{p+q+2s})\ln x}$$
$$= 2\ln\left(\sin\frac{s\pi}{p+q+2s}\,\operatorname{cosec}\frac{(p+s)\pi}{p+q+2s}\right)$$
$[s>0,\quad s+p>0,\quad s+p+q>0]$ GW (324)(23b)a

31. $$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^{s-1}\,dx}{(1-x)\ln x} = \ln\frac{\Gamma(p+s)\,\Gamma(q+s)\,\Gamma(r+s)\,\Gamma(p+q+r+s)}{\Gamma(p+q+s)4\,\Gamma(p+r+s)\,\Gamma(q+r+s)\,\Gamma(s)}$$
$[p>0,\quad q>0,\quad r>0,\quad s>0]$* BI (127)(11)

32. $$\int_0^1 (1-x^p)\,(1-x^q)\,(1-x^r)\,\frac{x^{s-1}\,dx}{(1-x^t)\ln x} = \ln\frac{\Gamma\left(\frac{p+s}{t}\right)\Gamma\left(\frac{q+s}{t}\right)\Gamma\left(\frac{r+s}{t}\right)\Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right)\Gamma\left(\frac{q+r+s}{t}\right)\Gamma\left(\frac{p+r+s}{t}\right)\Gamma\left(\frac{s}{t}\right)}$$
$[p>0,\quad q>0,\quad r>0,\quad s>0,\quad t>0]$* GW (324)(23b)

*In 4.267.31 the restrictions can be somewhat weakened by writing, for example, $s>0$, $p+s>0$, $q+s>0$, $r+s>0$, $p+q+s>0$, $p+r+s>0$, $q+r+s>0$, $p+q+r+s>0$, in **4.267** 31 and 32.

33. $$\int_0^1 \left\{ \frac{x^p - x^{p+q}}{1-x} - q \right\} \frac{dx}{\ln x} = \ln \frac{\Gamma(p+q+1)}{\Gamma(p+1)} \qquad [p > -1, \quad p+q > -1]$$ BI (127)(19)

34. $$\int_0^1 \left\{ \frac{x^\mu - x}{x-1} - x(\mu - 1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu) \qquad [\operatorname{Re} \mu > 0]$$ WH, BI (127)(18)

35. $$\int_0^1 \left\{ 1 - x - \frac{(1-x^p)(1-x^q)}{1-x} \right\} \frac{dx}{x \ln x} = -\ln \{\mathrm{B}(p,q)\}$$
$$[p > 0, \quad q > 0]$$ BI (130)(18)

36. $$\int_0^1 \left\{ \frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)} \right\} \frac{dx}{\ln x} = q \ln p$$
$$[p > 0]$$ BI (130)(20)

37. $$\int_0^1 \left\{ \frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1} \right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left(pq - \frac{1}{2} \right) \ln p$$
$$[p > 0, \quad q > 0]$$ BI (130)(22)

38. $$\int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x)\ln x} \, dx = \ln \mathrm{B}(p,q) \qquad [p > 0, \quad q > 0]$$ GW (324)(24)

39.[6] $$\int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^{n} \binom{n}{n-k} (-1)^{n-k} \ln(pk+1)$$
$$[n > 0, \quad pn > -1]$$
GW (324)(19d), BI (123)(12)a

40.[6] $$\int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln \Gamma[(n-k)p+1] \qquad [n > 1, \quad pn > -1]$$ BI (127)(12)

41. $$\int_0^1 (x^p - 1)^n x^{q-1} \frac{dx}{\ln x} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln[q + (n-k)p]$$
$$[n > 0, \quad q > 0, \quad pn > -q]$$
BI (123)(12)

42.[6] $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x)\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln \Gamma[(n-k)p+q]$$
$$[n > 1, \quad q > 0, \quad pn > -q]$$
BI (127)(13)

43.[10] $$\int_0^1 (x^p - 1)^n (x^q - 1)^m \frac{x^{r-1}\,dx}{\ln x} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \ln[r + (m-k)q + (n-j)p]$$
$$[n \geq 0, \quad m \geq 0, \quad n+m > 0, \quad r > 0, \quad pn + qm + r > 0]$$ BI (123)(16)

4.268

1. $$\int_0^1 \frac{(x^p - x^q)(1-x^r)}{(\ln x)^2} \, dx = (p+1)\ln(p+1) - (q+1)\ln(q+1)$$
$$-(p+r+1)\ln(p+r+1) + (q+r+1)\ln(q+r+1)$$
$$[p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1]$$ GW (324)(26)

2. $$\int_0^1 (x^p - x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1)\ln(2p+1) + (2q+1)\ln(2q+1) - 2(p+q+1)\ln(p+q+1)$$
$$\left[p > -\tfrac{1}{2}, \quad q > -\tfrac{1}{2}\right]$$ GW (324)(26a)

3. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r)\frac{dx}{(\ln x)^2}$$
$$= (p+q+1)\ln(p+q+1) + (q+r+1)\ln(q+r+1) + (p+r+1)\ln(p+r+1)$$
$$-(p+1)\ln(p+1) - (q+1)\ln(q+1) - (r+1)\ln(r+1) - (p+q+r)\ln(p+q+r)$$
$$[p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > 0]$$
BI (124)(4)

4. $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2}\sum_{k=0}^{n} (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q)$$
$$\left[q > 0, \quad p > -\frac{q}{n}\right]$$ BI (124)(14)

5. $$\int_0^1 (1-x^p)^n (1-x^q)^m x^{r-1} \frac{dx}{(\ln x)^2} = \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j}\right)\left(\sum_{k=0}^{m} (-1)^k \binom{m}{k}\right)$$
$$\times[(m-k)q + (n-j)p + r]\ln[(m-k)q + (n-j)p + r]$$
$$[r > 0, \quad mq + r > 0, \quad np + r > 0, \quad mq + np + r > 0]$$ BI (124)(8)

6. $$\int_0^1 \left[(q-r)x^{p-1} + (r-p)x^{q-1} + (p-q)x^{r-1}\right] \frac{dx}{(\ln x)^2}$$
$$= (q-r)p\ln p + (r-p)q\ln q + (p-q)r\ln r$$
$$[p > 0, \quad q > 0, \quad r > 0]$$ BI (124)(9)

7. $$\int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right.$$
$$\left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)}\right] \frac{dx}{(\ln x)^2} = \frac{1}{2}\left[\frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)}\right.$$
$$\left. + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)}\right]$$
$$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$$ BI (124)(16)

4.269

1. $$\int_0^1 \sqrt{\ln\frac{1}{x}} \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}$$ BI (115)(33)

2.[11] $$\int_0^1 \frac{dx}{\sqrt{\ln\frac{1}{x}\left(1+x^2\right)}} = \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$$ BI (133)(2)

3. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\,x^{p-1}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{p^3}} \qquad [p>0]$$ GW (324)(1c)

4. $$\int_0^1 \frac{x^{p-1}}{\sqrt{\ln\frac{1}{x}}}\,dx = \sqrt{\frac{\pi}{p}} \qquad [p>0]$$ BI (133)(1)

5. $$\int_0^1 \frac{\sin t - x^n\sin[(n+1)t] + x^{n+1}\sin nt}{1-2x\cos t + x^2}\cdot\frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}\sum_{k=1}^{n}\frac{\sin kt}{\sqrt{k}}$$
$$[|t|<\pi]$$ BI (133)(5)

6. $$\int_0^1 \frac{\cos t - x - x^{n-1}\cos nt + x^n\cos[(n-1)t]}{1-2x\cos t + x^2}\cdot\frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}\sum_{k=1}^{n-1}\frac{\cos kt}{\sqrt{k}}$$
$$[|t|<\pi]$$ BI (133)(6)

7. $$\int_u^v \frac{dx}{x\cdot\sqrt{\ln\frac{x}{u}\ln\frac{v}{x}}} = \pi \qquad [uv>0]$$ BI (145)(37)

4.271

1. $$\int_0^1 (\ln x)^{2n}\,\frac{dx}{1+x} = \frac{2^{2n}-1}{2^{2n}}\cdot(2n)!\,\zeta(2n+1)$$ BI (110)(1)

2. $$\int_0^1 (\ln x)^{2n-1}\,\frac{dx}{1+x} = \frac{1-2^{2n-1}}{2n}\pi^{2n}|B_{2n}| \qquad [n=1,2,\ldots]$$ BI (110)(2)

3. $$\int_0^1 (\ln x)^{2n-1}\,\frac{dx}{1-x} = -\frac{1}{n}2^{2n-2}\pi^{2n}|B_{2n}| \qquad [n=1,2,\ldots]$$ BI (110)(5), GW(324)(9a)

4. $$\int_0^1 (\ln x)^{p-1}\,\frac{dx}{1-x} = e^{i(p-1)\pi}\,\Gamma(p)\,\zeta(p) \qquad [p>1]$$ GW (324)(9b)

5. $$\int_0^1 (\ln x)^{n}\,\frac{dx}{1+x^2} = (-1)^n n!\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{n+1}}$$ BI (110)(11)

6. $$\int_0^1 (\ln x)^{2n}\,\frac{dx}{1+x^2} = \frac{1}{2}\int_0^\infty (\ln x)^{2n}\,\frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}}|E_{2n}|$$ GW (324)(10)a

7. $$\int_0^\infty \frac{(\ln x)^{2n+1}}{1+bx+x^2}\,dx = 0 \qquad [|b|<2]$$ BI (135)(2)

8. $$\int_0^1 (\ln x)^{2n}\,\frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}}\cdot(2n)!\,\zeta(2n+1) \qquad [n=1,2,\ldots]$$ BI (110)(12)

9. $$\int_0^\infty (\ln x)^{2n}\,\frac{dx}{1-x^2} = 0$$ BI (312)(7)a

10. $$\int_0^1 (\ln x)^{2n-1}\,\frac{dx}{1-x^2} = \frac{1}{2}\int_0^\infty (\ln x)^{2n-1}\,\frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n}\pi^{2n}|B_{2n}|$$
$$[n=1,2,\ldots]$$ BI (290)(17)a, BI(312)(6)a

11. $\displaystyle\int_0^1 (\ln x)^{2n-1}\,\frac{x\,dx}{1-x^2} = -\frac{1}{4n}\pi^{2n}|B_{2n}|$ $[n=1,2,\ldots]$ BI (290)(19)a

12. $\displaystyle\int_0^1 (\ln x)^{2n}\,\frac{1+x^2}{(1-x^2)^2}\,dx = \frac{2^{2n}-1}{2}\pi^{2n}|B_{2n}|$ $[n=1,2,\ldots]$ BI (296)(17)a

13. $\displaystyle\int_0^1 (\ln x)^{2n+1}\,\frac{(\cos 2a\pi - x)\,dx}{1-2x\cos 2a\pi + x^2} = -(2n+1)!\sum_{k=1}^{\infty}\frac{\cos 2ak\pi}{k^{2n+2}}$

$[a$ is not an integer$]$ LI (113)(10)

14.[6] $\displaystyle\int_0^\infty (\ln x)^n\,\frac{x^{\nu-1}\,dx}{a^2+2ax\cos t+x^2} = -\pi\operatorname{cosec} t\,\frac{d^n}{d\nu^n}\left[a^{\nu-2}\frac{\sin(\nu-1)t}{\sin\nu\pi}\right]$

$[a>0,\quad 0<\operatorname{Re}\nu<2,\quad 0<|t|<\pi]$ ET I 315(12)

15. $\displaystyle\int_0^1 (\ln x)^n\,\frac{x^{p-1}}{1-x^q}\,dx = -\frac{1}{q^{n+1}}\psi^{(n)}\left(\frac{p}{q}\right)$ $[p>0,\quad q>0]$ GW (324)(9)

16.[3] $\displaystyle\int_0^1 (\ln x)^n\,\frac{x^{p-1}}{1+x^q}\,dx = \frac{1}{q^{n+1}}\beta^{(n)}\left(\frac{p}{q}\right)$ $[p>0,\quad q>0]$ GW (324)(10)

4.272

1. $\displaystyle\int_0^1 \frac{\left[\ln\left(\frac{1}{x}\right)\right]^{q-1}dx}{1+2x\cos t+x^2} = \operatorname{cosec} t\,\Gamma(q)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k^q}$ $[|t|<\pi, q<1]$ LI (130)(1)

2. $\displaystyle\int_0^1 \left(\ln\frac{1}{x}\right)^{q-1}\frac{(1+x)\,dx}{1+2x\cos t+x^2} = \sec\frac{t}{2}\cdot\Gamma(q)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\left[\left(k-\frac{1}{2}\right)t\right]}{k^q}$

$\left[|t|<\pi,\quad q<\frac{1}{2}\right]$ LI (130)(5)

3.[9] $\displaystyle\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\mu}\frac{x^{\nu-1}\,dx}{1-2ax\cos t+x^2a^2} = \frac{\Gamma(\mu+1)}{a\sin t}\sum_{k=1}^{\infty}\frac{a^k\sin kt}{(\nu+k-1)^{\mu+1}}$

$[a>0,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad -\pi<t<\pi]$ BI (140)(14)a

4. $\displaystyle\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1}\frac{\cos\lambda - px}{1+p^2x^2-2px\cos\lambda}x^{q-1}\,dx = \Gamma(r)\sum_{k=1}^{\infty}\frac{p^{k-1}\cos k\lambda}{(q+k-1)^r}$

$[r>0,\quad q>0]$ BI (113)(11)

5. $\displaystyle\int_1^\infty (\ln x)^p\,\frac{dx}{x^2} = \Gamma(1+p)$ $[p>-1]$ BI (149)(1)

6. $\displaystyle\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1}x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\,\Gamma(\mu)$ $[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ BI (107)(3)

7. $\displaystyle\int_0^1 \left(\ln\frac{1}{x}\right)^{n-\frac{1}{2}}x^{\nu-1}\,dx = \frac{(2n-1)!!}{(2\nu)^n}\sqrt{\frac{\pi}{\nu}}$ $[\operatorname{Re}\nu>0]$ BI (107)(2)

8.[11] $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1+x}\,dx = \Gamma\left(3-\frac{1}{n}\right)\left(p^{\frac{1}{n}-3} - q^{\frac{1}{n}-3}\right)$$
$[\operatorname{Re}\nu > 0]$ BI (110)(4)

9. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1-x}\,dx = (n-1)!\,\zeta(n,\nu)$$
$[\operatorname{Re}\nu > 0]$ BI (110)(7)

10. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} (x-1)^n \left(a+\frac{nx}{x-1}\right) x^{a-1}\,dx = \Gamma(\mu)\sum_{k=0}^{n} \frac{(-1)^k n(n-1)\ldots(n-k+1)}{(a+n-k)^{\mu-1}k!}$$
$[\operatorname{Re}\mu > 0]$ LI (110)(10)

11. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{1-x^m}{1-x}\,dx = (n-1)!\sum_{k=1}^{m}\frac{1}{k^n}$$
LI (110)(9)

12. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} = \Gamma(\mu)\sum_{k=0}^{\infty}\frac{1}{(\nu+2k)^\mu} = \frac{1}{2^\mu}\,\Gamma(\mu)\,\zeta\left(\mu,\frac{\nu}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (110)(13)

13. $$\int_0^1 \frac{x^q - x^{-q}}{1-x^2}\left(\ln\frac{1}{x}\right)^p dx = \Gamma(p+1)\sum_{k=1}^{\infty}\left\{\frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}}\right\}$$
$[p > -1, \quad q^2 < 1]$ LI (326)(12)a

14. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1} \frac{x^{p-1}\,dx}{(1+x^q)^s} = \Gamma(r)\sum_{k=0}^{\infty}\binom{-s}{k}\frac{1}{(p+kq)^r}$$
$[p > 0, \quad q > 0, \quad r > 0, \quad 0 < s < r+2]$
GW (324)(11)

15. $$\int_0^1 \left(\ln\frac{1}{x}\right)^n (1+x^q)^m\, x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{1}{(p+kq)^{n+1}}$$
$[p > 0, \quad q > 0]$ BI (107)(6)

16. $$\int_0^1 \left(\ln\frac{1}{x}\right)^n (1-x^q)^m\, x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(p+kq)^{n+1}}$$
$[p > 0, \quad q > 0]$ BI (107)(7)

17. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \frac{1}{aq^p}\,\Gamma(p)\sum_{k=1}^{\infty}\frac{a^k}{k^p}$$
$[p > 0, \quad q > 0, \quad a < 1]$ LI (110)(8)

18. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{2-\frac{1}{n}} \left(x^{p-1} - x^{q-1}\right)\,dx = \frac{n}{n-1}\,\Gamma\left(\frac{1}{n}\right)\left(q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right)$$
$[q > p > 0]$ BI (133)(4)

19. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{2n-1} \frac{x^p - x^{-p}}{1-x^q}x^{q-1}\,dx = \frac{1}{p^{2n}}\sum_{k=n}^{\infty}\left(\frac{2p\pi}{q}\right)^k \frac{|B_{2k}|}{2k\cdot(2k-2n)!}$$
$\left[p < \frac{q}{2}\right]$ LI (110)(16)

4.273 $$\int_u^v \left(\ln\frac{x}{u}\right)^{p-1}\left(\ln\frac{v}{x}\right)^{q-1}\frac{dx}{x} = \mathrm{B}(p,q)\left(\ln\frac{v}{u}\right)^{p+q-1} \qquad [p>0,\quad q>0,\quad uv>0]$$ BI (145)(36)

4.274 $$\int_0^{\frac{1}{e}} \frac{\sqrt[q]{x}\,dx}{x\sqrt{-(1+\ln x)}} = \frac{\sqrt{q\pi}}{\sqrt[q]{e}} \qquad [q>0]$$ BI (145)(4)

4.275

1. $$\int_0^1 \left[\left(\ln\frac{1}{x}\right)^{q-1} - x^{p-1}(1-x)^{q-1}\right] dx = \frac{\Gamma(q)}{\Gamma(p+q)}\left[\Gamma(p+q)-\Gamma(p)\right]$$
$[p>0,\quad q>0]$ BI (107)(8)

2. $$\int_0^1 \left[x - \left(\frac{1}{1-\ln x}\right)^q\right]\frac{dx}{x\ln x} = -\psi(q) \qquad [q>0]$$ BI (126)(5)

4.28 Combinations of rational functions of $\ln x$ and powers

4.281

1. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1}{1-x}\right] dx = \boldsymbol{C}$$ BI (127)(15)

2. $$\int_1^\infty \frac{dx}{x^2(\ln p - \ln x)} = \frac{1}{p}\,\mathrm{li}(p)$$ LA 281(30)

3. $$\int_0^1 \frac{x^{p-1}\,dx}{q\pm\ln x} = \pm e^{\mp pq}\,\mathrm{Ei}\,(\pm pq) \qquad [p>0,\quad q>0]$$ LI (144)(11,12)

4. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x}\right] dx = -\psi(\mu) \qquad [\mathrm{Re}\,\mu>0]$$ WH

5. $$\int_0^1 \left[\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x}\right] dx = \ln p - \psi(q) \qquad [p>0,\quad q>0]$$ BI (127)(17)

6. $$\int_0^1 \left[\frac{1}{1-x^2} + \frac{1}{2x\ln x}\right]\frac{dx}{\ln x} = \frac{\ln 2}{2}$$ LI (130)(19)

7. $$\int_0^1 \left[q - \frac{1}{2} + \frac{(1-x)(1+q\ln x)+x\ln x}{(1-x)^2}x^{q-1}\right]\frac{dx}{\ln x} = \frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2}$$
$[q>0]$ BI (128)(15)

4.282

1. $$\int_0^1 \frac{\ln x}{4\pi^2+(\ln x)^2}\cdot\frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2}\boldsymbol{C}$$ BI (129)(1)

2. $$\int_0^1 \frac{1}{a^2+(\ln x)^2}\cdot\frac{dx}{1+x^2} = \frac{1}{2a}\,\beta\left(\frac{2a+\pi}{4\pi}\right) \qquad \left[a>-\frac{\pi}{2}\right]$$ BI (129)(9)

3. $$\int_0^1 \frac{1}{\pi^2+(\ln x)^2}\frac{dx}{1+x^2} = \frac{4-\pi}{4\pi}$$ BI (129)(6)

4. $$\int_0^1 \frac{\ln x}{\pi^2+(\ln x)^2}\cdot\frac{dx}{1-x^2} = \frac{1}{2}\left(\frac{1}{2}-\ln 2\right)$$ BI (129)(10)

5. $$\int_0^1 \frac{\ln x}{a^2+(\ln x)^2}\cdot\frac{x\,dx}{1-x^2}=\frac{1}{2}\left[\frac{\pi}{2a}+\ln\frac{\pi}{a}+\psi\left(\frac{a}{\pi}\right)\right]\qquad [a>0]$$ BI (129)(14)

6. $$\int_0^1 \frac{\ln x}{\pi^2+(\ln x)^2}\cdot\frac{x\,dx}{1-x^2}=\frac{1}{2}\left(\frac{1}{2}-\boldsymbol{C}\right)$$ BI (129)(13)

7. $$\int_0^1 \frac{1}{\pi^2+4(\ln x)^2}\cdot\frac{dx}{1+x^2}=\frac{\ln 2}{4\pi}$$ BI (129)(7)

8. $$\int_0^1 \frac{\ln x}{\pi^2+4(\ln x)^2}\cdot\frac{dx}{1-x^2}=\frac{2-\pi}{16}$$ BI (129)(11)

9.[10] $$\int_0^1 \frac{1}{\pi^2+16(\ln x)^2}\cdot\frac{dx}{1+x^2}=\frac{1}{8\pi\sqrt{2}}\left[\pi+2\ln\left(\sqrt{2}-1\right)\right]$$ BI (129)(8)

10. $$\int_0^1 \frac{\ln x}{\pi^2+16(\ln x)^2}\cdot\frac{dx}{1-x^2}=-\frac{\pi}{32\sqrt{2}}+\frac{1}{16}+\frac{1}{16\sqrt{2}}\ln\left(\sqrt{2}-1\right)$$ BI (129)(12)

11. $$\int_0^1 \frac{\ln x}{\left[a^2+(\ln x)^2\right]^2}\frac{dx}{1-x}=-\frac{\pi^2}{a^4}\sum_{k=1}^{\infty}|B_{2k}|\left(\frac{2\pi}{a}\right)^{2k-2}$$ BI (129)(4)

12. $$\int_0^1 \frac{\ln x}{\left[a^2+(\ln x)^2\right]^2}\frac{x\,dx}{1-x^2}=-\frac{\pi^2}{4a^4}\sum_{k=1}^{\infty}|B_{2k}|\left(\frac{\pi}{a}\right)^{2k-2}$$ BI (129)(16)

13. $$\int_0^1 \frac{x^p-x^{-p}}{x^2-1}\frac{dx}{q^2+(\ln x)^2}=\frac{2\pi}{q}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kp\pi}{2q+k\pi}\qquad [p^2<1]$$ BI (132)(13)a

4.283

1. $$\int_0^1 \left(\frac{x-1}{\ln x}-x\right)\frac{dx}{\ln x}=\ln 2-1$$ BI (132)(17)a

2. $$\int_0^1 \left(\frac{1}{\ln x}+\frac{1}{1-x}-\frac{1}{2}\right)\frac{dx}{\ln x}=\frac{\ln 2\pi}{2}-1$$ BI (127)(20)

3. $$\int_0^1 \left(\frac{1}{\ln x}+\frac{x}{1-x}+\frac{x}{2}\right)\frac{dx}{x\ln x}=\frac{\ln 2\pi}{2}$$ BI (127)(23)

4. $$\int_0^1 \left[\frac{1}{(\ln x)^2}-\frac{x}{(1-x)^2}\right]dx=\boldsymbol{C}-\frac{1}{2}$$ GW (326)(8a)

5. $$\int_0^1 \left(\frac{1}{1-x^2}+\frac{1}{2\ln x}-\frac{1}{2}\right)\frac{dx}{\ln x}=\frac{\ln 2-1}{2}$$ BI (128)(14)

6. $$\int_0^1 \left(\frac{1}{\ln x}+\frac{1}{2}\cdot\frac{1+x}{1-x}-\ln x\right)\frac{dx}{\ln x}=\frac{\ln 2\pi}{2}$$ BI (127)(22)

7. $$\int_0^1 \left[\frac{1}{1-\ln x}-x\right]\frac{dx}{x\ln x}=-\boldsymbol{C}$$ GW (326)(11a)

8. $$\int_0^1 \left[\frac{x^q-1}{x(\ln x)^2}-\frac{q}{\ln x}\right]dx=q\ln q-q\qquad [q>0]$$ BI (126)(2)

9. $$\int_0^1 \left[x + \frac{1}{a\ln x - 1}\right] \frac{dx}{x\ln x} = \ln\frac{a}{q} + \boldsymbol{C} \qquad [a > 0, \quad q > 0]$$ BI (126)(8)

10. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)}\right] \frac{x^{p-1}}{\ln x}\, dx = -\ln\Gamma(p) + \left(p - \frac{1}{2}\right)\ln p - p + \frac{\ln 2\pi}{2}$$
$$[p > 0]$$ GW (326)(9)

11. $$\int_0^1 \left[p - 1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x}\right) x^{p-1}\right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)\ln p + p - \frac{\ln 2\pi}{2}$$
$$[p > 0]$$ BI (127)(25)

12. $$\int_0^1 \left[-\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2}\right] dx = -\psi(p) + p - \frac{3}{2}$$
$$[p > 0]$$ GW (326)(8)

13. $$\int_0^1 \left[\left(p - \frac{1}{2}\right) x^3 + \frac{1}{2}\left(1 - \frac{1}{\ln x}\right)\left(x^{2p-1} - 1\right)\right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)(\ln p - 1)$$
$$[p > 0]$$ BI (132)(23)a

14. $$\int_0^1 \left[\left(q - \frac{1}{2}\right) \frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r}\right] \frac{dx}{\ln x} = (p-r)\left[\frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2}\right]$$
$$[q > 0]$$ BI (132)(13)

4.284

1. $$\int_0^1 \left[\frac{x^q - 1}{x(\ln x)^3} - \frac{q}{x(\ln x)^2} - \frac{q^2}{2\ln x}\right] dx = \frac{q^2}{2}\ln q - \frac{3}{4}q^2$$
$$[q > 0]$$ BI (126)(3)

2. $$\int_0^1 \left[\frac{x^q - 1}{x(\ln x)^4} - \frac{q}{x(\ln x)^3} - \frac{q^2}{2x(\ln x)^2} - \frac{q^3}{6\ln x}\right] dx = \frac{q^3}{6}\ln q - \frac{11}{36}q^3$$
$$[q > 0]$$ BI (126)(4)

4.285 $$\int_0^1 \frac{x^{p-1}\, dx}{(q + \ln x)^n} = \frac{p^{n-1}}{(n-1)!} e^{-pq}\,\mathrm{Ei}(pq) - \frac{1}{(n-1)!q^{n-1}} \sum_{k=1}^{n-1} (n-k-1)!(pq)^{k-1}$$
$$[p > 0, \quad q < 0]$$ BI (125)(21)

In integrals of the form $\int \frac{x^a (\ln x)^n\, dx}{[b \pm (\ln x)^m]^l}$, we should make the substitution $x = e^t$ or $x = e^{-t}$ and then seek the resulting integrals in **3.351**–**3.356**.

4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.291

1. $$\int_0^1 \frac{\ln(1+x)}{x}\, dx = \frac{\pi^2}{12}$$ FI II 483

2. $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$ FI II 714

3. $$\int_0^{1/2} \frac{\ln(1-x)}{x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (145)(2)

4. $$\int_0^1 \ln\left(1-\frac{x}{2}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (114)(18)

5. $$\int_0^1 \frac{\ln\dfrac{1+x}{2}}{1-x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (115)(1)

6. $$\int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \frac{1}{2}(\ln 2)^2$$ BI (114)(14)a

7.[7] $$\int_0^\infty \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{\pi}{4}\ln\left(1+a^2\right) - \int_0^a \frac{\ln u\,du}{1+u^2} \qquad [a>0]$$ GI II (2209)

8. $$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$ FI II 157

9. $$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$$ BI (136)(1)

10. $$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2 - \boldsymbol{G}$$ BI (114)(17)

11. $$\int_1^\infty \frac{\ln(x-1)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$ BI (144)(4)

12. $$\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2$$ BI (144)(4)

13. $$\int_0^\infty \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{6}.$$ BI (141)(9)a

14. $$\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{1}{a(a-b)}\ln\frac{a+b}{b} + \frac{2\ln 2}{b^2-a^2} \qquad [a\neq b,\quad ab>0]$$
$$= \frac{1}{2a^2}(1-\ln 2) \qquad [a=b]$$ LI (114)(5)a

15. $$\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{\ln\dfrac{a}{b}}{a(a-b)} \qquad [ab>0]$$ BI (139)(5)

16. $$\int_0^1 \ln(a+x)\frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}}\operatorname{arccot}\sqrt{a}\ln[(1+a)a] \qquad [a>0]$$ BI (114)(20)

17. $$\int_0^\infty \ln(a+x)\frac{dx}{(b+x)^2} = \frac{a\ln a - b\ln b}{b(a-b)} \qquad [a>0,\quad b>0,\quad a\neq b]$$ LI (139)(6)

18. $$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{1}{2}\arctan a\ln\left(1+a^2\right)$$ GI II (2195)

19. $$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\,dx = \frac{1}{2\sqrt{a}}\arctan\sqrt{a}\ln(1+a) \qquad [a>0]$$ BI (114)(21)

20. $$\int_0^1 \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[\frac{1}{2}(a+b)\ln(a+b) - b\ln b - a\ln 2\right]$$
$$[a>0, \quad b>0, \quad a\neq b]$$ BI (114)(22)

21. $$\int_0^\infty \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[a\ln a - b\ln b\right] \qquad [a>0, \quad b>0]$$ BI (139)(8)

22. $$\int_0^\infty \ln(a+x)\frac{x\,dx}{\left(b^2+x^2\right)^2} = \frac{1}{2\left(a^2+b^2\right)}\left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right)$$
$$[a>0, \quad b>0]$$ BI (139)(9)

23. $$\int_0^1 \ln(1+x)\frac{1+x^2}{(1+x)^4}\,dx = -\frac{1}{3}\ln 2 + \frac{23}{72}$$ LI (114)(12)

24. $$\int_0^1 \ln(1+x)\frac{1+x^2}{a^2+x^2}\cdot\frac{dx}{1+a^2x^2} = \frac{1}{2a\left(1+a^2\right)}\left[\frac{\pi}{2}\ln\left(1+a^2\right) - 2\arctan a\cdot\ln a\right]$$
$$[a>0]$$ LI (114)(11)

25. $$\int_0^1 \ln(1+x)\frac{1-x^2}{(ax+b)^2}\frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2}\left\{\frac{1}{a-b}\left[\frac{a+b}{ab}\ln(a+b) - \frac{1}{a}\ln b - \frac{1}{b}\ln a\right] + \frac{4\ln 2}{b^2-a^2}\right\}$$
$$\left[a>0, \quad b>0, \quad a^2\neq b^2\right]$$ LI (114)(13)

26. $$\int_0^\infty \ln(1+x)\frac{1-x^2}{(ax+b)^2}\cdot\frac{dx}{(bx+a)^2} = \frac{1}{ab\left(a^2-b^2\right)}\ln\frac{b}{a}$$
$$[a>0, \quad b>0]$$ LI (139)(14)

27. $$\int_0^1 \ln(1+ax)\frac{1-x^2}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\frac{(1+a)^2}{1+a^2}\ln(1+a) - \frac{1}{2}\cdot\frac{a}{1+a^2}\ln 2 - \frac{\pi}{4}\cdot\frac{a^2}{1+a^2}$$
$$[a>-1]$$ BI (114)(23)

28. $$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx = \frac{1}{a^2+b^2}\left(a\ln\frac{b}{a} - \frac{b\pi}{2}\right)$$
$$[a>0, \quad b>0]$$ BI (139)(11)

29. $$\int_0^\infty \ln^2(a-x)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{b\pi}{2}\right)$$
$$[a>0, \quad b>0]$$ BI (139)(12)

30. $$\int_0^\infty \ln^2(a-x)\frac{x\,dx}{\left(b^2+x^2\right)^2} = \frac{1}{a^2+b^2}\left(\ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right)$$
$$[a>0, \quad b>0]$$ BI (139)(10)

4.292

1. $$\int_0^1 \frac{\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln 2 \pm 2\boldsymbol{G}$$ GW (325)(20)

2. $$\int_0^1 \frac{x\ln(1\pm x)}{\sqrt{1-x^2}}\,dx = -1 \pm \frac{\pi}{2}$$ GW (325)(22c)

3. $$\int_{-a}^{a} \frac{\ln(1+bx)}{\sqrt{a^2-x^2}}\,dx = \pi\ln\frac{1+\sqrt{1-a^2b^2}}{2} \qquad \left[0 \le |b| \le \frac{1}{a}\right]$$ BI (145)(16, 17)a, GW (325)(21e)

4. $$\int_0^1 \frac{x\ln(1+ax)}{\sqrt{1-x^2}}\,dx = -1 + \frac{\pi}{2}\cdot\frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a}\arcsin a \qquad [|a| \le 1]$$
$$= -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a}\ln\left(a+\sqrt{a^2-1}\right) \qquad [a \ge 1]$$ GW (325)(22)

5. $$\int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin a\,(\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2}(\arccos a)^2$$
$[|a| \le 1]$ BI (120)(4), GW (325)(21a)

4.293

1. $$\int_0^1 x^{\mu-1}\ln(1+x)\,dx = \frac{1}{\mu}\left[\ln 2 - \beta(\mu+1)\right] \qquad [\operatorname{Re}\mu > -1]$$ BI (106)(4)a

2.[6] $$\int_1^\infty x^{\mu-1}\ln(1+x)\,dx = \frac{-1}{\mu}\left[\beta(-\mu) + \ln 2\right] \qquad [\operatorname{Re}\mu < 0]$$ ET I 315(17)

3. $$\int_0^\infty x^{\mu-1}\ln(1+x)\,dx = \frac{\pi}{\mu\sin\mu\pi} \qquad [-1 < \operatorname{Re}\mu < 0]$$ GW (325)(3)a

4. $$\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}$$ GW (325)(2b)

5. $$\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right]$$ GW (325)(2c)

6.[11] $$\int_0^1 x^{n-\frac{1}{2}}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1} + \frac{(-1)^n\cdot 4}{2n+1}\left[\frac{\pi}{4} - \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right]$$ GW (325)(2f)

7. $$\int_0^\infty x^{\mu-1}\ln|1-x|\,dx = \frac{\pi}{\mu}\cot(\mu\pi) \qquad [-1 < \operatorname{Re}\mu < 0]$$ BI (134)(4), ET I 315(18)

8. $$\int_0^1 x^{\mu-1}\ln(1-x)\,dx = -\frac{1}{\mu}\left[\psi(\mu+1) - \psi(1)\right] = -\frac{1}{\mu}\left[\psi(\mu+1) + \boldsymbol{C}\right]$$
$[\operatorname{Re}\mu > -1]$ ET I 316(19)

9.[7] $$\int_1^\infty x^{\mu-1}\ln(x-1)\,dx = \frac{1}{\mu}\left[\pi\cot(\mu\pi) + \psi(\mu+1) + \boldsymbol{C}\right]$$
$[\operatorname{Re}\mu < 0]$ ET I 316(20)

10. $$\int_0^\infty x^{\mu-1}\ln(1+\gamma x)\,dx = \frac{\pi}{\mu\gamma^\mu\sin\mu\pi} \qquad [-1<\operatorname{Re}\mu<0,\quad |\arg\gamma|<\pi]$$ BI (134)(3)

11.[11] $$\int_0^\infty \frac{x^{\mu-1}\ln(1+x)}{1+x}\,dx = -\frac{\pi}{\sin\mu\pi}\left[\boldsymbol{C}+\psi(1-\mu)\right] \qquad [-1<\operatorname{Re}\mu<1]$$ ET I 316(21)

12. $$\int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}}\,dx = -\frac{\ln 2}{2^\mu\mu}+\frac{2^\mu-1}{2^\mu\mu^2}$$ BI (114)(6)

13. $$\int_0^1 \frac{x^{\mu-1}\ln(1-x)}{(1-x)^{1-\nu}}\,dx = \mathrm{B}(\mu,\nu)\left[\psi(\nu)-\psi(\mu+\nu)\right] \qquad [\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ ET I 316(122)

14. $$\int_0^\infty \frac{x^{\mu-1}\ln(\gamma+x)}{(\gamma+x)^\nu}\,dx = \gamma^{\mu-\nu}\,\mathrm{B}(\mu,\nu-\mu)\left[\psi(\nu)-\psi(\nu-\mu)+\ln\gamma\right]$$
$$[0<\operatorname{Re}\mu<\operatorname{Re}\nu]$$ ET I 316(23)

4.294

1. $$\int_0^1 \ln(1+x)\frac{(p-1)x^{p-1}-px^{-p}}{x}\,dx = 2\ln 2-\frac{\pi}{\sin p\pi}$$
$$[0<p<1]$$ BI (114)(2)

2. $$\int_0^1 \ln(1+x)\frac{1+x^{2n+1}}{1+x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1}-\sum_{j=1}^{2n+1}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(7)

3. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1+x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1}-\sum_{j=1}^{2n}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(8)

4. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1-x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1}+\sum_{i=1}^{2n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(9)

5. $$\int_0^1 \ln(1+x)\frac{1-x^{2n+1}}{1-x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1}+\sum_{j=1}^{2n+1}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(10)

6. $$\int_0^1 \ln(1-x)\frac{1-(-1)^n x^n}{1-x}\,dx = \sum_{j=1}^{n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{1}{k}$$ BI (114)(15)

7. $$\int_0^1 \ln(1-x)\frac{1-x^n}{1-x}\,dx = -\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}\frac{1}{k}$$ BI (114)(16)

8. $$\int_0^\infty \ln^2(1-x)x^p\,dx = \frac{2\pi}{p+1}\cot p\pi \qquad [-2<p<-1]$$ BI (134)(13)a

9. $$\int_0^1 [\ln(1+x)]^n(1+x)^r\,dx = (-1)^{n-1}\frac{n!}{(r+1)^{n+1}}+2^{r+1}\sum_{k=0}^{n}\frac{(-1)^k n!\,(\ln 2)^{n-k}}{(n-k)!(r+1)^{k+1}}$$ LI (106)(34)a

10. $$\int_0^1 [\ln(1-x)]^n(1-x)^r\,dx = (-1)^n\frac{n!}{(r+1)^{n+1}} \qquad [r>-1]$$ BI (106)(35)a

11. $\int_0^1 \left(\ln \frac{1}{1-x^2}\right)^n x^{2q-1}\,dx = \frac{n!}{2}\,\zeta(n+1, q+1)$ $[-1 < q < 0]$ BI (311)(15)a

12. $\int_0^1 (\ln x)^{2n} \ln\left(1-x^2\right) \frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)}|B_{2n+2}|$ BI (309)(5)a

13.[6] $\int_0^1 \left[\ln \frac{1}{x}\right]^m \ln\left(1-x^2\right)\,dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}}$ $[m+1>0, \quad n+1>0]$

4.295

1. $\int_0^\infty \ln\left(\mu x^2+\beta\right) \frac{dx}{\gamma+x^2} = \frac{\pi}{\sqrt{\gamma}} \ln\left(\sqrt{\mu\gamma}+\sqrt{\beta}\right)$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg\gamma| < \pi]$ ET II 218(27)

2. $\int_0^1 \ln\left(1+x^2\right) \frac{dx}{x^2} = \frac{\pi}{2} - \ln 2$ GW (325)(2g)

3. $\int_0^\infty \ln\left(1+x^2\right) \frac{dx}{x^2} = \pi$ GW (325)(4c)

4. $\int_0^\infty \ln\left(1+x^2\right) \frac{dx}{(a+x)^2} = \frac{2a}{1+a^2}\left(\frac{\pi}{2a}+\ln a\right)$ $[a>0]$ BI (319)(6)a

5. $\int_0^1 \ln\left(1+x^2\right) \frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2 - \boldsymbol{G}$ BI (114)(24)

6. $\int_1^\infty \ln\left(1+x^2\right) \frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2 + \boldsymbol{G}$ BI (114)(5)

7. $\int_0^\infty \ln\left(a^2+b^2x^2\right) \frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\ln\frac{ag+bc}{g}$ $[a>0, \quad b>0, \quad c>0, \quad g>0]$ BI (136)(11-14)a

8. $\int_0^\infty \ln\left(a^2+b^2x^2\right) \frac{dx}{c^2-g^2x^2} = -\frac{\pi}{cg}\arctan\frac{bc}{ag}$ $[a>0, \quad b>0, \quad c>0, \quad g>0]$ BI (136)(15)a

9. $\int_0^\infty \frac{\ln\left(1+p^2x^2\right)-\ln\left(1+q^2x^2\right)}{x^2}\,dx = \pi(p-q)$ $[p>0, \quad q>0]$ FI II 645

10. $\int_0^1 \ln\frac{1+a^2x^2}{1+a^2}\frac{dx}{1-x^2} = -\left(\arctan a\right)^2$ BI (115)(2)

11. $\int_0^1 \ln\left(1-x^2\right)\frac{dx}{x} = -\frac{\pi^2}{12}$

12. $\int_0^\infty \ln^2\left(1-x^2\right)\frac{dx}{x^2} = 0$ BI (142)(9)a

13. $\int_0^1 \ln\left(1-x^2\right)\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2 - \boldsymbol{G}$ GW (325)(17)

14. $\int_1^\infty \ln\left(x^2-1\right)\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$ BI (144)(6)

15. $$\int_0^\infty \ln^2\left(a^2-x^2\right)\frac{dx}{b^2+x^2}=\frac{\pi}{b}\ln\left(a^2+b^2\right) \qquad [b>0]$$ BI (136)(16)

16. $$\int_0^\infty \ln^2\left(a^2-x^2\right)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx=-\frac{2b\pi}{a^2+b^2} \qquad [b>0]$$ BI (136)(20)

17. $$\int_0^1 \ln\left(1+x^2\right)\frac{dx}{x\left(1+x^2\right)}=\frac{1}{2}\left[\frac{\pi^2}{12}-\frac{1}{2}\left(\ln 2\right)^2\right]$$ BI (114)(25)

18. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{x\left(1+x^2\right)}=\frac{\pi^2}{12}$$ BI (141)(9)

19. $$\int_0^1 \ln\left(\cos^2 t+x^2\sin^2 t\right)\frac{dx}{1-x^2}=-t^2$$ BI (114)(27)a

20. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{dx}{(c+gx)^2}=\frac{2\ln b}{cg}+\frac{b^2}{a^2g^2+b^2c^2}\left(\frac{a}{b}\pi+2\frac{c}{g}\ln\frac{c}{g}+2\frac{a^2g}{b^2c}\ln\frac{a}{b}\right)$$
$$[a>0,\quad b>0,\quad c>0,\quad g>0]$$
BI (139)(16)a

21. $$\int_0^1 \ln\left(a^2+b^2x^2\right)\frac{dx}{(c+gx)^2}$$
$$=\frac{2}{c(c+g)}\ln a+\frac{b^2}{a^2g^2+b^2c^2}\left[\frac{2a}{b}\operatorname{arccot}\frac{a}{b}+\frac{cb^2-ga^2}{b^2(c+g)}\ln\frac{a^2+b^2}{a^2}-2\frac{c}{g}\ln\frac{c+g}{c}\right]$$
$$[a>0,\quad b>0,\quad c>0,\quad g>0]$$ BI (114)(28)a

22.[11] $$\int_0^\infty \frac{\ln\left(1+p^2x^2\right)}{r^2+q^2x^2}\,dx=\int_0^\infty \frac{\ln\left(p^2+x^2\right)}{q^2+r^2x^2}\,dx=\frac{\pi}{qr}\ln\frac{q+pr}{r}$$
$$[qr>0,\quad p>0]$$
FI II 745a, BI (318)(1)a, BI (318)(4)a

23. $$\int_0^\infty \frac{\ln\left(1+a^2x^2\right)}{b^2+c^2x^2}\frac{dx}{d^2+g^2x^2}=\frac{\pi}{b^2g^2-c^2d^2}\left[\frac{g}{d}\ln\left(1+\frac{ad}{g}\right)-\frac{c}{b}\ln\left(1+\frac{ab}{c}\right)\right]$$
$$\left[a>0,\quad b>0,\quad c>0,\quad d>0,\quad g>0,\quad b^2g^2\neq c^2d^2\right]$$ BI (141)(10)

24. $$\int_0^\infty \frac{\ln\left(1+a^2x^2\right)}{b^2+c^2x^2}\frac{x^2\,dx}{d^2+g^2x^2}=\frac{\pi}{b^2g^2-c^2d^2}\left[\frac{b}{c}\ln\left(1+\frac{ab}{c}\right)-\frac{d}{g}\ln\left(1+\frac{ad}{g}\right)\right]$$
$$\left[a>0,\quad b>0,\quad c>0,\quad d>0,\quad g>0,\quad b^2g^2\neq c^2d^2\right]$$ BI (141)(11)

25. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{dx}{\left(c^2+g^2x^2\right)^2}=\frac{\pi}{2c^3g}\left(\ln\frac{ag+bc}{g}-\frac{bc}{ag+bc}\right)$$
$$[a>0,\quad b>0,\quad c>0,\quad g>0]$$
GW (325)(18a)

26. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{x^2\,dx}{\left(c^2+g^2x^2\right)^2}=\frac{\pi}{2cg^3}\left(\ln\frac{ag+bc}{g}+\frac{bc}{ag+bc}\right)$$
$$[a>0,\quad b>0,\quad c>0,\quad g>0]$$
GW (325)(18b)

27. $$\int_0^1 \ln\left(1+ax^2\right)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}$$
$[a>0]$ BI (117)(6)

28. $$\int_0^1 \ln\left(1+a-ax^2\right)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}$$
$[a>0]$ BI (117)(7)

29. $$\int_0^1 \ln\left(1-a^2x^2\right)\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1-a^2}}{2}$$
$\left[a^2<1\right]$ BI (119)(1)

30.[6] $$\int_0^1 \ln\left(1-a^2x^2\right)\frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2$$
LI (120)(11)

31. $$\int_0^1 \ln\left(1-x^2\right)\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln\frac{k'}{k}\,\boldsymbol{K}(k) - \frac{\pi}{2}\,\boldsymbol{K}(k')$$
BI (120)(12)

32. $$\int_0^1 \ln\left(1\pm kx^2\right)\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2}\ln\frac{2\pm 2k}{\sqrt{k}}\,\boldsymbol{K}(k) - \frac{\pi}{8}\,\boldsymbol{K}(k')$$
BI (120)(8), BI (120)(14)

33. $$\int_0^1 \frac{\ln\left(1-k^2x^2\right)}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \ln k'\,\boldsymbol{K}(k)$$
BI (119)(27)

34. $$\int_0^1 \ln\left(1-k^2x^2\right)\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = \left(2-k^2\right)\boldsymbol{K}(k) - (2-\ln k')\,\boldsymbol{E}(k)$$
BI (119)(3)

35. $$\int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}}\ln\left(1-k^2x^2\right)\,dx = \frac{1}{k^2}\left(1+k'^2-k'^2\ln k'\right)\boldsymbol{K}(k) - (2-\ln k')\,\boldsymbol{E}(k)$$
BI (119)(7)

36. $$\int_{-1}^1 \ln\left(1-x^2\right)\frac{dx}{(a+bx)\sqrt{1-x^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$$
$[a>0, \quad b>0, \quad a\neq b]$ BI (145)(15)

37.[8] $$\int_0^1 \ln\left(1-x^2\right)\left(px^{p-1}-qx^{q-1}\right)\,dx = \psi\left(\frac{q}{2}+1\right) + \psi\left(\frac{p}{2}+1\right)$$
$[p>-2, \quad q>-2]$ BI (106)(15)

38. $$\int_0^1 \ln\left(1+ax^2\right)\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1+a}}{2}$$
$[a\geq -1]$ GW (325)(21b)

39. $$\int_0^1 \ln\left(1+x^2\right)x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln 2 - \beta\left(\frac{\mu}{2}+1\right)\right]$$
$[\operatorname{Re}\mu>-2]$ BI (106)(12)

40. $$\int_0^\infty \ln\left(1+x^2\right)x^{\mu-1}\,dx = \frac{\pi}{\mu\sin\frac{\mu\pi}{2}}$$
$[-2<\operatorname{Re}\mu<0]$

BI (311)(4)a, ET I 315(15)

41. $$\int_0^\infty \ln\left(1+x^2\right)\frac{x^{\mu-1}\,dx}{1+x} = \frac{\pi}{\sin\mu\pi}\left\{\ln 2-(1-\mu)\sin\frac{\mu\pi}{2}\,\beta\left(\frac{1-\mu}{2}\right)-(2-\mu)\cos\frac{\mu\pi}{2}\,\beta\left(\frac{2-\mu}{2}\right)\right\}$$
$[-2<\operatorname{Re}\mu<1]$ ET I 316(25)

4.296

1. $$\int_0^1 \ln\left(1+2x\cos t+x^2\right)\frac{dx}{x} = \frac{\pi^2}{6}-\frac{t^2}{2}$$ BI (114)(34)

2. $$\int_{-\infty}^\infty \ln\left(a^2-2ax\cos t+x^2\right)\frac{dx}{1+x^2} = \pi\ln\left(1+2a|\sin t|+a^2\right)$$ BI (145)(28)

3. $$\int_0^\infty \ln\left(1+2x\cos t+x^2\right)x^{\mu-1}\,dx = \frac{2\pi}{\mu}\frac{\cos\mu t}{\sin\mu\pi}$$ $[|t|<\pi,\quad -1<\operatorname{Re}\mu<0]$ ET I 316(27)

4. $$\int_0^\infty \ln\left(\frac{x^2+2ax\cos t+a^2}{x^2-2ax\cos t+a^2}\right)\frac{x\,dx}{x^2+b^2} = \frac{1}{2}\pi^2-\pi t+\pi\arctan\frac{\left(a^2-b^2\right)\cos t}{(a^2+b^2)\sin t+2ab}$$
$[a>0,\quad b>0,\quad 0<t<\pi]$

4.297

1. $$\int_0^1 \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = \frac{1}{a-b}\left[(a+b)\ln\frac{a+b}{2}-a\ln a-b\ln b\right]$$
$[a>0,\quad b>0]$ BI (115)(16)

2. $$\int_0^\infty \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = 0$$ $[ab>0]$ BI (139)(23)

3. $$\int_0^1 \ln\frac{1-x}{x}\frac{dx}{1+x^2} = \frac{\pi}{8}\ln 2$$ BI (115)(5)

4. $$\int_0^1 \ln\frac{1+x}{1-x}\frac{dx}{1+x^2} = \boldsymbol{G}$$ BI (115)(17)

5.[11] $$\int_0^\infty \ln\left(\frac{1+x}{1-x}\right)^2\frac{dx}{x\left(1+x^2\right)} = \frac{\pi^2}{2}$$ BI (141)(13)

6. $$\int_u^v \ln\frac{v+x}{u+x}\frac{dx}{x} = \frac{1}{2}\left(\ln\frac{v}{u}\right)^2$$ $[uv>0]$ BI (145)(33)

7. $$\int_0^\infty \frac{b\ln(1+ax)-a\ln(1+bx)}{x^2}\,dx = ab\ln\frac{b}{a}$$ $[a>0,\quad b>0]$ FI II 647

8. $$\int_0^1 \ln\frac{1+ax}{1-ax}\frac{dx}{x\sqrt{1-x^2}} = \pi\arcsin a$$ $[|a|\le 1]$ GW (325)(21c), BI (122)(2)

9. $$\int_u^v \ln\left(\frac{1+ax}{1-ax}\right)\frac{dx}{\sqrt{\left(x^2-u^2\right)\left(v^2-x^2\right)}} = \frac{\pi}{v}\,F\left(\arcsin av,\frac{u}{v}\right)$$
$[|av|<1]$ BI (145)(35)

10.[8] $$\mathrm{PV}\int_0^1 \ln\left|\frac{a+y}{a-y}\right|\frac{dy}{y\sqrt{1-y^2}} = \frac{\pi^2}{2}$$ $[0<a\le 1]$

4.298

1. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1+x}\,dx=\frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1)$$ BI (137)(1)

2. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n}}{1+x}\,dx=\frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1)$$ BI (137)(3)

3. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1-x}\,dx=\frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1)$$ BI (137)(2)

4. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n}}{1-x}\,dx=-\frac{\ln 2}{2n}-\frac{1}{4n^2}+\frac{1}{2n}\beta(2n+1)$$ BI (137)(4)

5. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1+x^2}\,dx=\frac{\ln 2}{2n}+\frac{1}{4n^2}-\frac{1}{2n}\beta(2n+1)$$ BI (137)(10)

6. $$\int_0^1 \ln\frac{1+x^2}{x}x^{2n}\,dx=\frac{1}{2n+1}\left\{(-1)^n\frac{\pi}{2}+\ln 2-\frac{1}{2n+1}+2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right\}$$ BI (294)(8)

7. $$\int_0^1 \ln\frac{1+x^2}{x}x^{2n-1}\,dx=\frac{1}{2n}\left\{(-1)^{n+1}\ln 2+\ln 2-\frac{1}{2n}+(-1)^{n+1}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right\}$$ BI (294)(9)a

8. $$\int_0^1 \ln\frac{1+x^2}{x}\frac{dx}{1+x^2}=\frac{\pi}{2}\ln 2$$ BI (115)(7)

9. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{dx}{1+x^2}=\pi\ln 2$$ BI (137)(8)

10. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{dx}{1-x^2}=0$$ BI (137)(9)

11. $$\int_0^1 \ln\frac{1-x^2}{x}\frac{dx}{1+x^2}=\frac{\pi}{4}\ln 2$$ BI (115)(9)

12. $$\int_1^\infty \ln\frac{1+x^2}{x+1}\frac{dx}{1+x^2}=\frac{3\pi}{8}\ln 2$$ BI (144)(8)

13. $$\int_0^1 \ln\frac{1+x^2}{x+1}\frac{dx}{1+x^2}=\frac{3\pi}{8}\ln 2-\boldsymbol{G}$$ BI (115)(18)

14. $$\int_1^\infty \ln\frac{1+x^2}{x-1}\frac{dx}{1+x^2}=\frac{3\pi}{8}\ln 2+\boldsymbol{G}$$ BI (144)(9)

15. $$\int_0^1 \ln\frac{1+x^2}{1-x}\frac{dx}{1+x^2}=\frac{3\pi}{8}\ln 2$$ BI (115)(19)

16. $$\int_0^\infty \ln\frac{1+x^2}{x^2}\frac{x\,dx}{1+x^2}=\frac{\pi^2}{12}$$ BI (138)(3)

17. $$\int_0^\infty \ln\frac{a^2+b^2x^2}{x^2}\frac{dx}{c^2+g^2x^2}=\frac{\pi}{cg}\ln\frac{ag+bc}{c}\qquad [a>0,\quad b>0,\quad c>0,\quad g>0]$$ BI (138)(6, 7, 9, 10)a

18. $$\int_0^\infty \ln\frac{a^2+b^2x^2}{x^2}\frac{dx}{c^2-g^2x^2}=\frac{1}{cg}\arctan\frac{ag}{bc}\qquad [a>0, b>0, c>0, g>0]$$ BI (138)(8, 11)a

19. $\displaystyle\int_0^\infty \ln\frac{1+x^2}{x^2}\frac{x^2\,dx}{(1+x^2)^2} = \frac{\pi}{4}(\ln 4 - 1)$ BI (139)(21)

20. $\displaystyle\int_0^1 \ln^2\left(\frac{1-x^2}{x^2}\right)\sqrt{1-x^2}\,dx = \pi$ FI II 643a

21. $\displaystyle\int_0^1 \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\frac{dx}{x} = \frac{1}{2}\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\frac{dx}{x} = -\frac{t^2}{2}$

$[|t|<\pi]$ BI (115)(23), BI (134)(15)

22. $\displaystyle\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}x^{p-1}\,dx = -\frac{2\pi(1-\cos pt)}{p\sin p\pi}$ $[0<|p|<1, \quad |t|<\pi]$ BI (134)(17)

23. $\displaystyle\int_0^1 \ln\frac{1+x^2\sin t}{1-x^2\sin t}\frac{dx}{\sqrt{1-x^2}} = \pi\ln\cot\left(\frac{\pi-t}{4}\right)$ $[|t|<\pi]$ GW (325)(21d)

4.299

1. $\displaystyle\int_0^\infty \ln\frac{(x+1)(x+a^2)}{(x+a)^2}\frac{dx}{x} = (\ln a)^2$ $[a>0]$ BI (134)(14)

2. $\displaystyle\int_0^1 \ln\frac{(1-ax)(1+ax^2)}{(1-ax^2)^2}\frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}}\arctan\sqrt{a}\ln(1+a)$

$[a>0]$ BI (115)(25)

3. $\displaystyle\int_0^1 \ln\frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2}\frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}}\arctan\sqrt{a}\ln(1+a)$

$[a>0]$ BI (115)(26)

4. $\displaystyle\int_0^1 \ln\frac{(x+1)(x+a^2)}{(x+a)^2}x^{\mu-1}\,dx = \frac{\pi(a^\mu-1)^2}{\mu\sin\mu\pi}$ $[a>0, \quad \operatorname{Re}\mu>0]$ BI (134)(16)

4.311

1.[11] $\displaystyle\int_0^\infty \frac{\ln(1+x^n)}{x^n}\,dx = \frac{\pi\operatorname{cosec}\left(\frac{\pi}{n}\right)}{n-1}$ $n=2,3,\ldots$

2. $\displaystyle\int_0^\infty \ln\left(1+x^3\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\ln 3$ LI (136)(8)

3. $\displaystyle\int_0^\infty \ln\left(1+x^3\right)\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9}$ LI (136)(6)

4. $\displaystyle\int_0^\infty \ln\left(1+x^3\right)\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 + \frac{\pi^2}{9}$ LI (136)(7)

5. $\displaystyle\int_0^\infty \ln\left(1+x^3\right)\frac{1-x}{1+x^3}\,dx = -\frac{2}{9}\pi^2$ BI (136)(9)

6.[8] $\displaystyle\int_0^\infty \left|1-\frac{x^3}{a^3}\right|\frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$

4.312

1. $$\int_0^\infty \ln\frac{1+x^3}{x^3}\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 + \frac{\pi^2}{9}$$ BI (138)(12)

2. $$\int_0^\infty \ln\frac{1+x^3}{x^3}\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9}$$ BI (138)(13)

4.313

1. $$\int_0^\infty \ln x \ln\left(1+a^2x^2\right)\frac{dx}{x^2} = \pi a\left(1-\ln a\right) \qquad [a>0]$$ BI (134)(18)

2. $$\int_0^\infty \ln\left(1+c^2x^2\right)\ln\left(a^2+b^2x^2\right)\frac{dx}{x^2} = 2\pi\left[\left(c+\frac{b}{a}\right)\ln(b+ac) - \frac{b}{a}\ln b - c\ln c\right]$$
$[a>0, \quad b>0, \quad c>0]$ BI (134)(20, 21)a

3. $$\int_0^\infty \ln\left(1+c^2x^2\right)\ln\left(a^2+\frac{b^2}{x^2}\right)\frac{dx}{x^2} = 2\pi\left[\frac{a+bc}{b}\ln(a+bc) - \frac{a}{b}\ln a - c\right]$$
$[a>0, \quad a+bc>0]$ BI (134)(22, 23)a

4. $$\int_0^\infty \ln x \ln\frac{1+a^2x^2}{1+b^2x^2}\frac{dx}{x^2} = \pi(a-b) + \pi\ln\frac{b^b}{a^a} \qquad [a>0, \quad b>0]$$ BI (134)(24)

5. $$\int_0^\infty \ln x \ln\frac{a^2+2bx+x^2}{a^2-2bx+x^2}\frac{dx}{x} = 2\pi\ln a\arcsin\frac{b}{a} \qquad [a\geq|b|]$$ BI (134)(25)

6. $$\int_0^\infty \ln(1+x)\frac{x\ln x - x - a}{(x+a)^2}\frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)} \qquad [a>0]$$ BI (141)(7)

7. $$\int_0^\infty \ln^2(1-x)\frac{x\ln x - x - a}{(x+a)^2}\frac{dx}{x} = \frac{\pi^2+(\ln a)^2}{1+a} \qquad [a>0]$$ LI (141)(8)

4.314

1.[11] $$\int_0^1 \ln(1+ax)\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \sum_{k=1}^\infty (-1)^{k+1}\frac{a^k}{k}\ln\frac{p+k}{q+k}$$
$[|a|<1, \quad p>0, \quad q>0]$ BI (123)(18)

2. $$\int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q}\right]\frac{dx}{x\ln(1+x)} = \ln\Gamma(q)$$
$[q>0]$ BI (143)(7)

3. $$\int_0^1 \frac{x\ln x + 1 - x}{x\left(\ln x\right)^2}\ln(1+x)dx = \ln\frac{4}{\pi}$$ BI (126)(12)

4. $$\int_0^1 \frac{\ln\left(1-x^2\right)dx}{x\left(q^2+(\ln x)^2\right)} = -\frac{\pi}{q}\ln\Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q}\ln 2q + \ln\frac{q}{\pi} - 1$$
$[q>0]$ LI (327)(12)a

4.315

1. $$\int_0^1 \ln(1+x)\,(\ln x)^{n-1}\,\frac{dx}{x} = (-1)^{n-1}(n-1)!\left(1-\frac{1}{2^n}\right)\zeta(n+1)$$ BI (116)(3)

2. $$\int_0^1 \ln(1+x)\,(\ln x)^{2n}\,\frac{dx}{x} = \frac{2^{2n+1}-1}{(2n+1)(2n+2)}\pi^{2n+2}|B_{2n+2}|$$ BI (116)(1)

3. $$\int_0^1 \ln(1-x)\,(\ln x)^{n-1}\,\frac{dx}{x} = (-1)^n(n-1)!\,\zeta(n+1)$$ BI (116)(4)

4. $$\int_0^1 \ln(1-x)\,(\ln x)^{2n}\,\frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)}\pi^{2n+2}\,|B_{2n+2}|$$ BI (116)(2)

4.316

1. $$\int_0^1 \ln\left(1-ax^r\right)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -\frac{1}{r^{p+1}}\,\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k}{k^{p+2}}$$
$[p>-1, \quad a<1, \quad r>0]$ BI (116)(7)

2. $$\int_0^1 \ln\left(1-2ax\cos t+a^2x^2\right)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -2\,\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k\cos kt}{k^{p+2}}$$ LI (116)(8)

4.317

1. $$\int_0^\infty \ln\frac{\sqrt{1+x^2}+a}{\sqrt{1+x^2}-a}\,\frac{dx}{\sqrt{1+x^2}} = \pi\arcsin a \qquad [|a|<1]$$ BI (142)(11)

2. $$\int_0^1 \ln\frac{\sqrt{1-a^2x^2}-x\sqrt{1-a^2}}{1-x}\,\frac{dx}{x} = \frac{1}{2}\,(\arcsin a)^2$$ BI (115)(32)

3. $$\int_0^1 \ln\frac{1+\cos t\sqrt{1-x^2}}{1-\cos t\sqrt{1-x^2}}\,\frac{dx}{x^2+\tan^2 v} = \pi\cot t\,\frac{\cos\dfrac{v-t}{2}}{\sin\dfrac{v+t}{2}}$$ BI (115)(30)

4. $$\int_0^1 \ln^2\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)\frac{x\,dx}{1-x^2} = \frac{\pi^2}{2}$$ BI (115)(31)

5. $$\int_0^1 \ln\left\{\sqrt{1+kx}+\sqrt{1-kx}\right\}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} = \frac{1}{4}\ln(4k)\,\boldsymbol{K}(k)+\frac{\pi}{8}\,\boldsymbol{K}\,(k')$$ BI (121)(8)

6. $$\int_0^1 \ln\left\{\sqrt{1+kx}-\sqrt{1-kx}\right\}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} = \frac{1}{4}\ln(4k)\,\boldsymbol{K}(k)+\frac{3}{8}\pi\,\boldsymbol{K}\,(k')$$ BI (121)(9)

7. $$\int_0^1 \ln\left\{1+\sqrt{1-k^2x^2}\right\}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} = \frac{1}{2}\ln k\,\boldsymbol{K}(k)+\frac{\pi}{4}\,\boldsymbol{K}\,(k')$$ BI (121)(6)

8. $$\int_0^1 \ln\left\{1-\sqrt{1-k^2x^2}\right\}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} = \frac{1}{2}\ln k\,\boldsymbol{K}(k)-\frac{3}{4}\pi\,\boldsymbol{K}\,(k')$$ BI (121)(7)

9. $$\int_0^1 \ln\frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}}\,\frac{dx}{1-x} = \pi\arcsin p \qquad \left[p^2<1\right]$$ BI (115)(29)

10. $$\int_0^1 \ln \frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \pi\, F(\arcsin q, k')$$
$[q^2 < 1]$ BI (122)(15)

11.[10] $$\int_{-\infty}^{\infty} \ln \left| \frac{1+2\sqrt{1+x^2}}{1-2\sqrt{1+x^2}} \right| \frac{dx}{\sqrt{1+x^2}} = \frac{\pi^2}{3}$$

4.318

1. $$\int_0^1 \frac{\ln(1-x^q)}{1+(\ln x)^2} \frac{dx}{x} = \pi \left[\ln \Gamma\left(\frac{q}{2\pi}+1\right) - \frac{\ln q}{2} + \frac{q}{2\pi}\left(\ln \frac{q}{2\pi} - 1\right)\right]$$
$[q > 0]$ BI (126)(11)

2. $$\int_0^{\infty} \ln(1+x^r) \left[\frac{(p-r)x^p - (q-r)x^q}{\ln x} + \frac{x^q - x^p}{(\ln x)^2} \right] \frac{dx}{x^{r+1}} = r \ln\left(\tan \frac{q\pi}{2r} \cot \frac{p\pi}{2r}\right)$$
$[p < r, \quad q < r]$ BI (143)(9)

In integrals containing $\ln(a+bx^r)$, it is useful to make the substitution $x^r = t$ and then to seek the resulting integral in the tables. For example,

$$\int_0^{\infty} x^{p-1} \ln(1+x^r)\, dx = \frac{1}{r}\int_0^{\infty} t^{\frac{p}{r}-1} \ln(1+t)\, dt = \frac{\pi}{p \sin \frac{p\pi}{r}}$$ (see **4.293** 3)

4.319

1. $$\int_0^{\infty} \ln\left(1-e^{-2a\pi x}\right) \frac{dx}{1+x^2} = -\pi \left[\frac{1}{2}\ln 2a\pi + a(\ln a - 1) - \ln \Gamma(a+1)\right]$$
$[a > 0]$ BI (354)(6)

2. $$\int_0^{\infty} \ln\left(1+e^{-2a\pi x}\right) \frac{dx}{1+x^2} = \pi \left[\ln \Gamma(2a) - \ln \Gamma(a) + a(1-\ln a) - \left(2a - \frac{1}{2}\right)\ln 2\right]$$
$[a > 0]$ BI (354)(7)

3. $$\int_0^{\infty} \ln \frac{a+be^{-px}}{a+be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q}$$
$\left[\frac{b}{a} > -1, \quad pq > 0\right]$ FI II 635, BI (354)(1)

4.321

1. $$\int_{-\infty}^{\infty} x \ln \cosh x\, dx = 0$$ BI (358)(2)a

2. $$\int_{-\infty}^{\infty} \ln \cosh x \frac{dx}{1-x^2} = 0$$ BI (138)(20)a

4.322

1.[11] $$\int_0^{\pi} x \ln \sin x\, dx = \frac{1}{2}\int_0^{\pi} x \ln \cos^2 x\, dx = -\frac{\pi^2}{2} \ln 2$$ BI (432)(1, 2) FI II 643

2. $$\int_0^{\infty} \frac{\ln \sin^2 ax}{b^2+x^2}\, dx = \frac{\pi}{b} \ln \frac{1-e^{-2ab}}{2}$$
$[a > 0, \quad b > 0]$ GW (338)(28b)

3. $$\int_0^\infty \frac{\ln\cos^2 ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\frac{1+e^{-2ab}}{2} \qquad [a>0,\quad b>0]$$ GW (338)(28a)

4. $$\int_0^\infty \frac{\ln\sin^2 ax}{b^2-x^2}\,dx = -\frac{\pi^2}{2b}+a\pi \qquad [a>0,\quad b>0]$$ BI (418)(1)

5.[11] $$\int_0^\infty \frac{\ln\cos^2 ax}{b^2-x^2}\,dx = \infty$$ BI (418)(2)

6. $$\int_0^\infty \frac{\ln\cos^2 x}{x^2}\,dx = -\pi$$ FI II 686

7.[7] $$\int_0^{\pi/4} \ln\sin x x^{\mu-1}\,dx = -\frac{1}{2\mu}\left(\frac{\pi}{4}\right)^\mu\left[\ln 2+\frac{2}{\mu}-\sum_{k=1}^\infty\frac{\zeta(2k)}{4^{2k-1}(\mu+2k)}\right]$$
$$[\operatorname{Re}\mu>0]$$ LI (425)(1)

8.[7] $$\int_0^{\pi/2} \ln\sin x x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\frac{\pi}{2}\right)^\mu\left[\frac{1}{\mu}-2\sum_{k=1}^\infty\frac{\zeta(2k)}{4^{k}(\mu+2k)}\right]$$
$$[\operatorname{Re}\mu>0]$$ LI (430)(1)

9. $$\int_0^{\pi/2} \ln(1-\cos x)\,x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\frac{\pi}{2}\right)^\mu\left[\frac{2}{\mu}-\sum_{k=1}^\infty\frac{\zeta(2k)}{4^{2k-1}(\mu+2k)}\right]$$
$$[\operatorname{Re}\mu>0]$$ LI (430)(2)

10. $$\int_0^\infty \ln\left(1\pm 2p\cos\beta x+p^2\right)\frac{dx}{q^2+x^2} = \frac{\pi}{q}\ln\left(1\pm pe^{-\beta q}\right) \qquad [p^2<1]$$
$$= \frac{\pi}{q}\ln\left(p\pm e^{-\beta q}\right) \qquad [p^2>1]$$
FI II 718a

4.323

1.[11] $$\int_0^\pi x\ln\tan^2 x\,dx = 0$$ BI (432)(3)

2. $$\int_0^\infty \frac{\ln\tan^2 ax}{b^2+x^2}\,dx = \frac{\pi}{b}\ln\tanh ab \qquad [a>0,\quad b>0]$$ GW (338)(28c)

3. $$\int_0^\infty \ln\left(\frac{1+\tan x}{1-\tan x}\right)^2\frac{dx}{x} = \frac{\pi^2}{2}$$ GW (338)(26)

4.324

1. $$\int_0^\infty \ln\left(\frac{1+\sin x}{1-\sin x}\right)^2\frac{dx}{x} = \pi^2$$ GW (338)(25)

2. $$\int_0^\infty \ln\frac{1+2a\cos px+a^2}{1+2a\cos qx+a^2}\frac{dx}{x} = \ln(1+a)\ln\frac{q^2}{p^2} \qquad [-1<a\le 1]$$
$$= \ln\left(1+\frac{1}{a}\right)\ln\frac{q^2}{p^2} \qquad [a<-1 \text{ or } a\ge 1]$$
GW (338)(27)

3. $$\int_0^\infty \ln\left(a^2\sin^2 px + b^2\cos^2 px\right)\frac{dx}{c^2+x^2} = \frac{\pi}{c}\left[\ln\left(a\sinh cp + b\cosh cp\right) - cp\right]$$
$$[a>0, \quad b>0, \quad c>0, \quad p>0]$$
GW (338)(29)

4.325

1.[3] $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x} = -\boldsymbol{C}\ln 2 + \sum_{k=2}^\infty (-1)^k\frac{\ln k}{k} = -\boldsymbol{C}\ln 2 + 0.159868905\cdots = -\frac{1}{2}(\ln 2)^2$$
GW (325)(25a)

2. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{x+e^{i\lambda}} = \sum_{k=1}^\infty \frac{(-1)^k}{k}e^{-ik\lambda}\left(\boldsymbol{C}+\ln k\right)$$ GW (325)(26)

3. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x)^2} = \int_1^\infty \ln\ln x\frac{dx}{(1+x)^2} = \frac{1}{2}\left[\psi\left(\frac{1}{2}\right)+\ln 2\pi\right] = \frac{1}{2}\left(\ln\frac{\pi}{2} - \boldsymbol{C}\right)$$
BI (147)(7)

4. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$ BI (148)(1)

5. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\frac{\sqrt[3]{2\pi}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}$$ BI (148)(2)

6. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right]$$ BI (148)(5)

7. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+2x\cos t+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+2x\cos t+x^2} = \frac{\pi}{2\sin t}\ln\frac{(2\pi)^{t/\pi}\,\Gamma\left(\frac{1}{2}+\frac{t}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{t}{2\pi}\right)}$$
BI (147)(9)

8. $$\int_0^1 \ln\ln\frac{1}{x}x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\boldsymbol{C}+\ln\mu\right) \qquad [\operatorname{Re}\mu>0]$$ BI (147)(1)

9. $$\int_1^\infty \ln\ln x\frac{x^{n-2}\,dx}{1+x^2+x^4+\cdots+x^{2n-2}}$$
$$= \frac{\pi}{2n}\tan\frac{\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \qquad [n \text{ is even}]$$
$$= \frac{\pi}{2n}\tan\frac{\pi}{2n}\ln\pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \qquad [n \text{ is odd}]$$
BI (148)(4)

10. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x^2)\sqrt{\ln\frac{1}{x}}} = \int_1^\infty \ln\ln x\frac{dx}{(1+x^2)\sqrt{\ln x}} = \sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}}\left[\ln(2k+1)+2\ln 2+\boldsymbol{C}\right]$$

BI (147)(4)

11. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{x^{\mu-1}\,dx}{\sqrt{\ln\frac{1}{x}}} = -\left(\boldsymbol{C}+\ln 4\mu\right)\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu>0]$$ BI (147)(3)

12. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\,\Gamma(\mu)\left[\psi(\mu)-\ln(\nu)\right]$$

$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ BI (147)(2)

4.326

1. $$\int_0^1 \ln(a-\ln x)\,x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln a - e^{a\mu}\operatorname{Ei}(-a\mu)\right] \qquad [\operatorname{Re}\mu>0, \quad a>0]$$ BI (107)(23)

2. $$\int_0^{\frac{1}{e}} \ln\left(2\ln\frac{1}{x}-1\right)\frac{x^{2\mu-1}}{\ln x}\,dx = -\frac{1}{2}\left[\operatorname{Ei}(-\mu)\right]^2 \qquad [\operatorname{Re}\mu>0]$$ BI (145)(5)

4.327

1. $$\int_0^1 \ln\left[a^2+(\ln x)^2\right]\frac{dx}{1+x^2} = \pi\ln\frac{2\,\Gamma\left(\frac{2a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2a+\pi}{4\pi}\right)} + \frac{\pi}{2}\ln\frac{\pi}{2}$$

$\left[a>-\frac{\pi}{2}\right]$ BI (147)(10)

2. $$\int_0^1 \ln\left[a^2+4(\ln x)^2\right]\frac{dx}{1+x^2} = \pi\ln\frac{2\,\Gamma\left(\frac{a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{a+\pi}{4\pi}\right)} + \frac{\pi}{2}\ln\pi$$

$[a>-\pi]$ BI (147)(16)a

3. $$\int_0^\infty \ln\left[a^2+(\ln x)^2\right]x^{\mu-1}\,dx = \frac{2}{\mu}\left[-\cos a\mu\,\operatorname{ci}(a\mu)-\sin a\mu\,\operatorname{si}(a\mu)+\ln a\right]$$

$[a>0, \quad \operatorname{Re}\mu>0]$ GW (325)(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains $\ln\ln\frac{1}{x}$, it is useful to make the substitution $\ln x = t$ and then seek the transformed integral in the tables.

4.33–4.34 Combinations of logarithms and exponentials

4.331

1. $$\int_0^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}\left(\boldsymbol{C}+\ln\mu\right) \qquad [\operatorname{Re}\mu>0]$$ BI (256)(2)

2. $$\int_1^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}\operatorname{Ei}(-\mu) \qquad [\operatorname{Re}\mu>0]$$ BI (260)(5)

3. $\int_0^1 e^{\mu x}\ln x\,dx = -\frac{1}{\mu}\int_0^1 \frac{e^{\mu x}-1}{x}\,dx$ $[\mu \neq 0]$ GW (324)(81a)

4.332

1. $\int_0^\infty \frac{\ln x\,dx}{e^x+e^{-x}-1} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right]$ (cf. **4.325** 6) BI (257)(6)

2. $\int_0^\infty \frac{\ln x\,dx}{e^x+e^{-x}+1} = \frac{\pi}{\sqrt{3}}\ln\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\sqrt{2\pi}\right]$ (cf. **4.325** 5) BI (257)(7)a, LI (260)(3)

4.333 $\int_0^\infty e^{-\mu x^2}\ln x\,dx = -\frac{1}{4}\left(\boldsymbol{C}+\ln 4\mu\right)\sqrt{\frac{\pi}{\mu}}$ $[\operatorname{Re}\mu > 0]$ BI (256)(8), FI II 807a

4.334 $\int_0^\infty \frac{\ln x\,dx}{e^{x^2}+1+e^{-x^2}} = \frac{1}{2}\sqrt{\frac{\pi}{3}}\sum_{k=1}^\infty (-1)^k \frac{\boldsymbol{C}+\ln 4k}{\sqrt{k}}\sin\frac{k\pi}{3}$ BI (357)(13)

4.335

1. $\int_0^\infty e^{-\mu x}(\ln x)^2\,dx = \frac{1}{\mu}\left[\frac{\pi^2}{6}+(\boldsymbol{C}+\ln\mu)^2\right]$ $[\operatorname{Re}\mu > 0]$ ET I 149(13)

2. $\int_0^\infty e^{-x^2}(\ln x)^2\,dx = \frac{\sqrt{\pi}}{8}\left[(\boldsymbol{C}+2\ln 2)^2+\frac{\pi^2}{2}\right]$ FI II 808

3.[7] $\int_0^\infty e^{-\mu x}(\ln x)^3\,dx = -\frac{1}{\mu}\left[(\boldsymbol{C}+\ln\mu)^3+\frac{\pi^2}{2}(\boldsymbol{C}+\ln\mu)-\psi''(1)\right]$ MI 26

4.336

1.[7] $\mathrm{PV}\int_0^\infty \frac{e^{-x}}{\ln x}\,dx = -0.154479567$ BI (260)(9)

2. $\int_0^\infty \frac{e^{-\mu x}\,dx}{\pi^2+(\ln x)^2} = \nu'(\mu)-e^{\mu}$ $[\operatorname{Re}\mu > 0]$ MI 26

4.337

1. $\int_0^\infty e^{-\mu x}\ln(\beta+x)dx = \frac{1}{\mu}\left[\ln\beta - e^{\mu\beta}\operatorname{Ei}(-\beta\mu)\right]$ $[|\arg\beta| < \pi,\quad \operatorname{Re}\mu > 0]$ BI (256)(3)

2. $\int_0^\infty e^{-\mu x}\ln(1+\beta x)dx = -\frac{1}{\mu}e^{\frac{\mu}{\beta}}\operatorname{Ei}\left(-\frac{\mu}{\beta}\right)$ $[|\arg\beta| < \pi,\quad \operatorname{Re}\mu > 0]$ ET I 148(4)

3. $\int_0^\infty e^{-\mu x}\ln|a-x|\,dx = \frac{1}{\mu}\left[\ln a - e^{-a\mu}\operatorname{Ei}(a\mu)\right]$ $[a > 0,\quad \operatorname{Re}\mu > 0]$ BI (256)(4)

4.[7] $\int_0^\infty e^{-\mu x}\ln\left|\frac{\beta}{\beta-x}\right|dx = \frac{1}{\mu}\left[e^{-\beta\mu}\operatorname{Ei}(\beta\mu)\right]$ $[\operatorname{Re}\mu > 0]$ MI 26

5.* $\int_0^\infty \ln(1+ax)x^\zeta e^{-x}\,dx = \sum_{\mu=0}^{\zeta}\frac{\zeta!}{(\zeta-\mu)!}\left[\frac{(-1)^{\zeta-\mu-1}}{a^{\zeta-\mu}}e^{1/a}\operatorname{Ei}\left(-\frac{1}{a}\right)+\sum_{k=1}^{\zeta-\mu}(k-1)!\left(-\frac{1}{a}\right)^{\zeta-\mu-k}\right]$

4.338

1. $\int_0^\infty e^{-\mu x}\ln\left(\beta^2+x^2\right)\,dx = \frac{2}{\mu}\left[\ln\beta - \operatorname{ci}(\beta\mu)\cos(\beta\mu)-\operatorname{si}(\beta\mu)\sin(\beta\mu)\right]$

$[\operatorname{Re}\beta > 0,\quad \operatorname{Re}\mu > 0]$ BI (256)(6)

2. $$\int_0^\infty e^{-\mu x}\ln^2\left(x^2-\beta^2\right)\,dx=\frac{2}{\mu}\left[\ln^2\beta-e^{\beta\mu}\,\mathrm{Ei}(-\beta\mu)-e^{\beta\mu}\,\mathrm{Ei}(\beta\mu)\right]$$
$[\operatorname{Im}\beta>0,\quad \operatorname{Re}\mu>0]$ BI (256)(5)

4.339 $$\int_0^\infty e^{-\mu x}\ln\left|\frac{x+1}{x-1}\right|dx=\frac{1}{\mu}\left[e^{-\mu}\left(\ln 2\mu+\gamma\right)-e^{\mu}\,\mathrm{Ei}(-2\mu)\right]$$
$[\operatorname{Re}\mu>0]$ MI 27

4.341 $$\int_0^\infty e^{-\mu x}\ln\frac{\sqrt{x+ai}+\sqrt{x-ai}}{\sqrt{2a}}\,dx=\frac{\pi}{4\mu}\left[\mathbf{H}_0(a\mu)-Y_0(a\mu)\right]$$
$[a>0,\quad \operatorname{Re}\mu>0]$ ET I 149(20)

4.342

1. $$\int_0^\infty e^{-2nx}\ln\left(\sinh x\right)\,dx=\frac{1}{2n}\left[\frac{1}{n}+\ln 2-2\,\beta(2n+1)\right]$$
BI (256)(17)

2. $$\int_0^\infty e^{-\mu x}\ln\left(\cosh x\right)\,dx=\frac{1}{\mu}\left[\beta\left(\frac{\mu}{2}\right)-\frac{1}{\mu}\right]\qquad[\operatorname{Re}\mu>0]$$
ET I 165(32)

3.[11] $$\int_0^\infty e^{-\mu x}\left[\ln\left(\sinh x\right)-\ln x\right]\,dx=\frac{1}{\mu}\left[\ln\frac{\mu}{2}-\frac{1}{\mu}-\psi\left(\frac{\mu}{2}\right)\right]$$
$[\operatorname{Re}\mu>0]$ ET I 165(33)

4.343 $$\int_0^\pi e^{\mu\cos x}\left[\ln\left(2\mu\sin^2 x\right)+\boldsymbol{C}\right]\,dx=-\pi\,K_0(\mu)$$
WA 95(16)

4.35–4.36 Combinations of logarithms, exponentials, and powers

4.351

1. $$\int_0^1(1-x)e^{-x}\ln x\,dx=\frac{1-e}{e}$$
BI (352)(1)

2. $$\int_0^1 e^{\mu x}\left(\mu x^2+2x\right)\ln x\,dx=\frac{1}{\mu^2}\left[(1-\mu)e^{\mu}-1\right]$$
BI (352)(2)

3. $$\int_1^\infty\frac{e^{-\mu x}\ln x}{1+x}\,dx=\frac{1}{2}e^{\mu}\left[\mathrm{Ei}(-\mu)\right]^2\qquad[\operatorname{Re}\mu>0]$$
NT 32(10)

4.352

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\ln x\,dx=\frac{1}{\mu^\nu}\,\Gamma(\nu)\left[\psi(\nu)-\ln\mu\right]\qquad[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$
BI (353)(3), ET I 315(10)a

2. $$\int_0^\infty x^n e^{-\mu x}\ln x\,dx=\frac{n!}{\mu^{n+1}}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\boldsymbol{C}-\ln\mu\right]$$
$[\operatorname{Re}\mu>0]$ ET I 148(7)

3. $$\int_0^\infty x^{n-\frac{1}{2}}e^{-\mu x}\ln x\,dx=\sqrt{\pi}\frac{(2n-1)!!}{2^n\mu^{n+\frac{1}{2}}}\left[2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)-\boldsymbol{C}-\ln 4\mu\right]$$
$[\operatorname{Re}\mu>0]$ ET I 148(10)

4. $\displaystyle\int_0^\infty x^{\mu-1}e^{-x}\ln x\,dx=\Gamma'(\mu)$ $[\operatorname{Re}\mu>0]$ GW (324)(83a)

4.353

1. $\displaystyle\int_0^\infty (x-\nu)x^{\nu-1}e^{-x}\ln x\,dx=\Gamma(\nu)$ $[\operatorname{Re}\nu>0]$ GW (324)(84)

2. $\displaystyle\int_0^\infty \left(\mu x-n-\frac{1}{2}\right)x^{n-\frac{1}{2}}e^{-\mu x}\ln x\,dx=\frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}}$

$[\operatorname{Re}\mu>0]$ BI (357)(2)

3. $\displaystyle\int_0^1 (\mu x+n+1)x^n e^{\mu x}\ln x\,dx=e^\mu\sum_{k=0}^{n}(-1)^{k-1}\frac{n!}{(n-k)!\mu^{k+1}}+(-1)^n\frac{n!}{\mu^{n+1}}$

$[\mu\neq 0]$ GW (324)(82)

4.354

1.[6] $\displaystyle\int_0^\infty \frac{x^{\nu-1}\ln x}{e^x+1}\,dx=\Gamma(\nu)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\nu}\left[\psi(\nu)-\ln k\right]$ $[\operatorname{Re}\nu>0]$

$\displaystyle=-\frac{1}{2}(\ln 2)^2$ $[\text{for } \nu=1]$

GW (324)(86a)

2.[7] $\displaystyle\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^2}\,dx=\Gamma(\nu)\sum_{k=2}^{\infty}\frac{(-1)^k(k-1)}{k^\nu}\left[\psi(\nu)-\ln k\right]$

$[\operatorname{Re}\nu>1]$ GW (324)(86b)

3. $\displaystyle\int_0^\infty \frac{(x-\nu)e^x-\nu}{(e^x+1)^2}x^{\nu-1}\ln x\,dx=\Gamma(\nu)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\nu}$ $[\operatorname{Re}\nu>0]$ GW (324)(87a)

4. $\displaystyle\int_0^\infty \frac{(x-2n)e^x-2n}{(e^x+1)^2}x^{2n-1}\ln x\,dx=\frac{2^{2n-1}-1}{2n}\pi^{2n}|B_{2n}|$

$[n=1,2,\ldots]$ GW (324)(87b)

5. $\displaystyle\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^n}\,dx=(-1)^n\frac{\Gamma(\nu)}{(n-1)!}\sum_{k=n}^{\infty}\frac{(-1)^k(k-1)!}{(k-n)!k^\nu}\left[\psi(\nu)-\ln k\right]$

$[\operatorname{Re}\nu>0]$ GW (324)(86c)

4.355

1. $\displaystyle\int_0^\infty x^2e^{-\mu x^2}\ln x\,dx=\frac{1}{8\mu}\left(2-\ln 4\mu-\boldsymbol{C}\right)\sqrt{\frac{\pi}{\mu}}$ $[\operatorname{Re}\mu>0]$ BI (357)(1)a

2. $\displaystyle\int_0^\infty x\left(\mu x^2-\nu x-1\right)e^{-\mu x^2+2\nu x}\ln x\,dx=\frac{1}{4\mu}+\frac{\nu}{4\mu}\sqrt{\frac{\pi}{\mu}}\exp\left(\frac{\nu^2}{\mu}\right)\left[1+\Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$

$[\operatorname{Re}\mu>0]$ BI (358)(1)

3. $\displaystyle\int_0^\infty \left(\mu x^2-n\right)x^{2n-1}e^{-\mu x^2}\ln x\,dx=\frac{(n-1)!}{4\mu^n}$ $[\operatorname{Re}\mu>0]$ BI (353)(4)

4. $$\int_0^\infty \left(2\mu x^2 - 2n - 1\right) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}}$$
$[\operatorname{Re}\mu > 0]$ BI (353)(5)

4.356

1. $$\int_0^\infty \exp\left[-\mu\left(\frac{x}{a} + \frac{a}{x}\right)\right] \ln x \frac{dx}{x} = 2 \ln a \, K_0(2\mu)$$
$[a > 0, \quad \operatorname{Re}\mu > 0]$ GW (324)(91)

2. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[2ax^2 - (2n+1)x - 2b\right] x^{n-\frac{1}{2}} \, dx$$
$$= 2\left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^\infty \frac{(n+k)!}{(n-k)!(2k)!!\left(2\sqrt{ab}\right)^k}$$
$[a > 0, \quad b > 0]$ BI (357)(4)

3. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[2ax^2 + (2n-1)x - 2b\right] \frac{dx}{x^{n+\frac{3}{2}}}$$
$$= 2\left(\frac{a}{b}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^\infty \frac{(n+k-1)!}{(n-k-1)!(2k)!!\left(2\sqrt{ab}\right)^k}$$
$[a > 0, \quad b > 0]$ BI (357)(11)

For $n = \frac{1}{2}$:

4. $$\int_0^\infty \exp\left(-ax - \frac{a}{x}\right) \ln x \frac{ax^2 - b}{x^2} \, dx = 2 \, K_0\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0]$ GW (324)(92c)

For $n = 0$:

5. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}} \, dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}$$
$[a > 0, \quad b > 0]$
BI (357)(7), GW(324)(92a)

For $n = -1$:

6. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}} \, dx = \frac{1 + 2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}$$
$[a > 0, \quad b > 0]$
LI (357)(6), GW (324)(92b)

7.[9] $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left(a - \frac{b}{x^2}\right) \, dx = K_0\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0]$

8.[9] $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\left[2ax^2-(2n+1)x-2b\right]x^{n-\frac{3}{2}}\,dx$$
$$=4\left(\frac{b}{a}\right)^{(2n+1)/4}K_{n+\frac{1}{2}}\left(2\sqrt{ab}\right)$$
$$=2\left(\frac{b}{a}\right)^{\frac{n}{2}}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!(2k)!!\left(2\sqrt{ab}\right)^k}$$
$$[n=0,1,\ldots,a>0,\quad b>0]$$

9.[9] $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln\left[\left(ax^2-b\right)\cos(\alpha\ln x)+\alpha x\sin(\alpha\ln x)\right]\frac{dx}{x^2}$$
$$=2\cos\left(\alpha\ln\sqrt{b/a}\right)K_{i\alpha}\left(2\sqrt{ab}\right)$$
$$[a>0,\quad b>0,\quad -\infty<\alpha<\infty]$$

10.[9] $$\int_0^\infty \exp\left(-ax-\frac{b}{x}\right)\ln x\left[\left(ax^2-b\right)\sin(\alpha\ln x)-\alpha x\cos(\alpha\ln x)\right]\frac{dx}{x^2}$$
$$=2\sin\left(\alpha\ln\sqrt{b/a}\right)K_{i\alpha}\left(2\sqrt{ab}\right)$$
$$[a>0,\quad b>0,\quad -\infty<\alpha<\infty]$$

11.[9] $$q\int_0^\infty x^\alpha\ln x\left[a-\frac{\alpha}{x}-\frac{b}{x^2}\right]\exp\left(-ax-\frac{b}{x}\right)dx=2\left(\frac{b}{a}\right)^{\alpha/2}K_\alpha\left(2\sqrt{ab}\right)$$
$$[a>0,\quad b>0,\quad -\infty<\alpha<\infty]$$

4.357

1. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\frac{1+ax^2-x^4}{x^2}\,dx=-\frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}$$
$$[a>0]$$ BI (357)(8)

2. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\frac{x^4+ax^2-1}{x^4}\,dx=\frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}\qquad [a>0]$$ BI (357)(9)

3. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right)\ln x\frac{x^4+3ax-1}{x^6}\,dx=\frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[a]{e}}$$
$$[a>0]$$ BI (357)(10)

4.358

1.[6] $$\int_1^\infty x^{\nu-1}e^{-\mu x}(\ln x)^m\,dx=\frac{\partial^m}{\partial\nu^m}\left\{\mu^{-\nu}\Gamma(\nu,\mu)\right\}\qquad [m=0,1,\ldots,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$
MI 26

2. $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln x)^2\,dx=\frac{\Gamma(\nu)}{\mu^\nu}\left\{[\psi(\nu)-\ln\mu]^2+\zeta(2,\nu)\right\}$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ MI 26

3.[9] $$\int_0^\infty x^{\nu-1}e^{-\mu x}(\ln x)^3\,dx=\frac{\Gamma(\nu)}{\mu^\nu}\left\{[\psi(\nu)-\ln\mu]^3+3\,\zeta(2,\nu)\,[\psi(\nu)-\ln\mu]-2\,\zeta(3,\nu)\right\}$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ MI 26

4.[7] $$\int_0^\infty x^{\nu-1}e^{-\mu x}\,(\ln x)^4\,dx = \frac{\Gamma(\nu)}{\nu}\Big\{[\psi(\nu)-\ln\mu]^4 + 6\,\zeta(2,\nu)\,[\psi(\nu)-\ln\mu]^2 - 8\;\zeta(3,\nu)\,[\psi(\nu)-\ln\mu] + 3\,[\zeta(2,\nu)]^2 + 6\,\zeta(4,\nu)\Big\}$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$

5.[3] $$\int_0^\infty x^{\nu-1}e^{-\mu x}\,(\ln x)^n\,dx = \frac{\partial^n}{\partial\nu^n}\left\{\mu^{-\nu}\,\Gamma(\nu)\right\}$$ $[n=0,1,2,\ldots]$

4.359

1. $$\int_0^\infty e^{-\mu x}\,\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \frac{1}{\mu}[\lambda(\mu,p-1)-\lambda(\mu,q-1)]$$
$[\operatorname{Re}\mu>0,\quad p>0,\quad q>0]$ MI 27

2.[11] $$\int_0^1 e^{\mu x}\,\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \sum_{k=0}^\infty \frac{\mu^k}{k!}\ln\frac{p+k}{q+k}$$ $[p>0,\quad q>0]$ BI (352)(9)

4.361

1. $$\int_0^\infty \frac{(x+1)e^{-\mu x}}{\pi^2+(\ln x)^2}\,dx = \nu'(\mu)-\nu''(\mu)$$ $[\operatorname{Re}\mu>0]$ MI 27

2. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{x\left[\pi^2+(\ln x)^2\right]} = e^\mu-\nu(\mu)$$ $[\operatorname{Re}\mu>0]$ MI 27

4.362

1. $$\int_0^1 xe^x\ln(1-x)\,dx = 1-e$$ BI (352)(5)a

2. $$\int_1^\infty e^{-\mu x}\ln(2x-1)\frac{dx}{x} = \frac{1}{2}\left[\operatorname{Ei}\left(-\frac{\mu}{2}\right)\right]^2$$ $[\operatorname{Re}\mu>0]$ ET I 148(8)

4.363

1. $$\int_0^\infty e^{-\mu x}\ln(a+x)\frac{\mu(x+a)\ln(x+a)-2}{x+a}\,dx = \frac{1}{4}\int_0^\infty e^{-\mu x}\ln^2(a-x)\frac{\mu(x-a)\ln^2(x-a)-4}{x-a}\,dx = (\ln a)^2$$
$[\operatorname{Re}\mu>0,\quad a>0]$ BI (354)(4, 5)

2. $$\int_0^1 x(1-x)(2-x)e^{-(1-x)^2}\ln(1-x)\,dx = \frac{1-e}{4e}$$ BI (352)(4)

4.364

1. $$\int_0^\infty e^{-\mu x}\ln[(x+a)(x+b)]\frac{dx}{x+a+b} = e^{(a+b)\mu}\left\{\operatorname{Ei}(-a\mu)\operatorname{Ei}(-b\mu)-\ln(ab)\operatorname{Ei}[-(a+b)\mu]\right\}$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$ BI (354)(11)

2. $$\int_0^\infty e^{-\mu x}\ln(x+a+b)\left(\frac{1}{x+a}+\frac{1}{x+b}\right)dx$$
$$=(1+\ln a\ln b)\ln(a+b)+e^{-(a+b)\mu}\{\mathrm{Ei}(-\alpha\mu)\,\mathrm{Ei}(-b\mu)\}$$
$$+(1-\ln(ab))\,\mathrm{Ei}[-(a+b)\mu]$$
$[a>0,\quad b>0,\quad \mathrm{Re}\,\mu>0]$ BI (354)(12)

4.365 $$\int_0^\infty\left[e^{-x}-\frac{x}{(1+x)^{p+1}\ln(1+x)}\right]\frac{dx}{x}=\ln p$$ $[p>0]$ BI (354)(15)

4.366

1. $$\int_0^\infty e^{-\mu x}\ln\left(1+\frac{x^2}{a^2}\right)\frac{dx}{x}=[\mathrm{ci}(a\mu)]^2+[\mathrm{si}(a\mu)]^2$$ $[\mathrm{Re}\,\mu>0]$ NT 32(11)a

2. $$\int_0^\infty e^{-\mu x}\ln\left|1-\frac{x^2}{a^2}\right|\frac{dx}{x}=\mathrm{Ei}(a\mu)\,\mathrm{Ei}(-a\mu)$$ $[\mathrm{Re}\,\mu>0]$ ME 18

3. $$\int_0^\infty xe^{-\mu x^2}\ln\left|\frac{1+x^2}{1-x^2}\right|dx=\frac{1}{\mu}\left[\cosh\mu\sinh(i\mu)-\sinh\mu\cosh(i\mu)\right]$$
$[\mathrm{Re}\,\mu>0]$; (cf. **4.339**) MI 27

4.367 $$\int_0^\infty xe^{-\mu x^2}\ln\frac{x+\sqrt{x^2+2\beta}}{\sqrt{2\beta}}\,dx=\frac{e^{\beta\mu}}{4\mu}K_0(\beta\mu)$$ $[|\arg\beta|<\pi,\quad \mathrm{Re}\,\mu>0]$ ET I 149(19)

4.368 $$\int_0^{2u}e^{-\mu x^2}\ln\frac{x^2\left(4u^2-x^2\right)}{u^4}\frac{dx}{\sqrt{4u^2-x^2}}=\frac{\pi}{2}e^{-2u^2\mu}\left[\frac{\pi}{2}\,Y_0\left(2iu^2\mu\right)-(\boldsymbol{C}-\ln 2)\,J_0\left(2iu^2\mu\right)\right]$$
$[\mathrm{Re}\,\mu>0]$ ET I 149(21)a

4.369

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\left[\psi(\nu)-\ln x\right]dx=\frac{\Gamma(\nu)\ln\mu}{\mu^\nu}$$ $[\mathrm{Re}\,\nu>0]$ ET I 149(12)

2. $$\int_0^\infty x^n e^{-\mu x}\left\{\left[\ln x-\tfrac{1}{2}\psi(n+1)\right]^2-\tfrac{1}{2}\psi'(n+1)\right\}dx$$
$$=\frac{n!}{\mu^{n+1}}\left\{\left[\ln\mu-\frac{1}{2}\psi(n+1)\right]^2+\frac{1}{2}\psi'(n+1)\right\}$$
$[\mathrm{Re}\,\mu>0]$ MI 26

4.37 Combinations of logarithms and hyperbolic functions

4.371

1. $$\int_0^\infty\frac{\ln x}{\cosh x}\,dx=\pi\ln\left[\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$ LI (260)(1)a

2. $$\int_0^\infty\frac{\ln x\,dx}{\cosh x+\cos t}=\frac{\pi}{\sin t}\ln\frac{(2\pi)^{t/\pi}\,\Gamma\left(\dfrac{\pi+t}{2\pi}\right)}{\Gamma\left(\dfrac{\pi-t}{2\pi}\right)}$$ $\left[t^2<\pi^2\right]$ BI (257)(7)a

3. $$\int_0^\infty \frac{\ln x\,dx}{\cosh^2 x} = \psi\left(\frac{1}{2}\right) + \ln\pi = \ln\pi - 2\ln 2 - \boldsymbol{C}$$ BI (257)(4)a

4.372

1. $$\int_1^\infty \ln x \frac{\sinh mx}{\sinh nx}\,dx = \frac{\pi}{2n}\tan\frac{m\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \qquad [m+n \text{ is odd}]$$
$$= \frac{\pi}{2n}\tan\frac{m\pi}{2n}\ln\pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \qquad [m+n \text{ is even}]$$
BI (148)(3)a

2. $$\int_1^\infty \ln x \frac{\cosh mx}{\cosh nx}\,dx$$
$$= \frac{\pi}{2n}\frac{\ln 2\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{n}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \qquad [m+n \text{ is odd}]$$
$$= \frac{\pi}{2n}\frac{\ln\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \qquad [m+n \text{ is even}]$$
BI (148)(6)a

4.373

1. $$\int_0^\infty \frac{\ln\left(a^2+x^2\right)}{\cosh bx}\,dx = \frac{\pi}{b}\left[2\ln\frac{2\,\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln\frac{2b}{\pi}\right] \qquad \left[b>0, \quad a>-\frac{\pi}{2b}\right].$$ BI (258)(11)a

2. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{\cosh\frac{\pi x}{2}} = 2\ln\frac{4}{\pi}$$ BI (258)(1)a

3. $$\int_0^\infty \ln\left(a^2+x^2\right)\frac{\sinh\left(\frac{2}{3}\pi x\right)}{\sinh\pi x}\,dx = 2\sin\frac{\pi}{3}\ln\frac{6\,\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)}$$
$$[a>-1].$$ BI (258)(12)

4. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{\sinh^2 ax} = \frac{2}{a}\left[\ln\frac{a}{\pi} + \frac{\pi}{2a} - \psi\left(\frac{\pi+a}{\pi}\right)\right]$$
$$[a>0]$$ BI (258)(5)

5. $$\int_0^\infty \ln\left(1+x^2\right)\frac{\cosh\left(\frac{\pi}{2}x\right)}{\sinh^2\left(\frac{\pi}{2}x\right)}\,dx = \frac{2\pi-4}{\pi}$$ BI (258)(3)

6. $$\int_0^\infty \ln\left(1+x^2\right)\frac{\cosh\left(\frac{\pi}{4}x\right)}{\sinh^2\left(\frac{\pi}{4}x\right)}\,dx = 4\sqrt{2} - \frac{16}{\pi} + \frac{8\sqrt{2}}{\pi}\ln\left(\sqrt{2}+1\right)$$ BI (258)(2)

4.374

1. $$\int_0^\infty \ln\left(\cos^2 t + e^{-2x}\sin^2 t\right)\frac{dx}{\sinh x} = -2t^2$$ BI (259)(10)a

2. $$\int_0^\infty \ln\left(a+be^{-2x}\right)\frac{dx}{\cosh^2 x} = \frac{2}{(b-a)}\left[\frac{a+b}{2}\ln(a+b) - a\ln a - b\ln 2\right]$$
$[a>0, \quad a+b>0]$ LI (259)(14)

4.375

1.[11] $$\int_0^\infty \ln\cosh\frac{x}{2}\frac{dx}{\cosh x} = \boldsymbol{G} - \frac{\pi}{4}\ln 2$$ BI (259)(11)

2. $$\int_0^\infty \ln\coth x\frac{dx}{\cosh x} = \frac{\pi}{2}\ln 2$$ BI (259)(16)

4.376

1. $$\int_0^\infty \frac{\ln x}{\sqrt{x}\cosh x}\,dx = 2\sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}}\{\ln(2k+1) + 2\ln 2 + \boldsymbol{C}\}$$ BI (147)(4)

2. $$\int_0^\infty \ln x\frac{(\mu+1)\cosh x - x\sinh x}{\cosh^2 x}x^\mu\,dx = 2\,\Gamma(\mu+1)\sum_{k=0}^\infty \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}}$$
$[\operatorname{Re}\mu > -1]$ BI (356)(10)

3. $$\int_0^\infty \ln x\frac{(n+1)\cosh x - x\sinh x}{\cosh^2 x}x^n\,dx = \frac{(-1)^n}{2^n}\beta^{(n)}\left(\frac{1}{2}\right)$$

4. $$\int_0^\infty \ln 2x\frac{n\sinh 2ax - ax}{\sinh^2 ax}x^{2n-1}\,dx = -\frac{1}{n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$
$[n=1,2,\ldots]$ BI (356)(9)a

5. $$\int_0^\infty \ln x\frac{ax\cosh ax - (2n+1)\sinh ax}{\sinh^2 ax}x^{2n}\,dx = 2\frac{2^{2n+1}-1}{(2a)^{2n+1}}(2n)!\,\zeta(2n+1)$$ BI (356)(14)

6. $$\int_0^\infty \ln x\frac{ax\cosh ax - 2n\sinh ax}{\sinh^2 ax}x^{2n-1}\,dx = \frac{2^{2n-1}-1}{2n}|B_{2n}|\left(\frac{\pi}{a}\right)^{2n}$$
$[n=1,2,\ldots, a>0]$ BI (356)(15)

7. $$\int_0^\infty \ln\frac{(2n+1)\cosh ax - ax\sinh ax}{\cosh^2 ax}x^{2n}\,dx = -\left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|$$
$[a>0]$ BI (356)(11)

8.[6] $$\int_0^\infty \ln x\frac{2ax\sinh ax - (2n+1)\cosh ax}{\cosh^3 ax}x^{2n}\,dx = \begin{cases}\dfrac{2}{a}\left(2^{2n-1}-1\right)\left(\dfrac{\pi}{2a}\right)^{2n}|B_{2n}| & n=1,2,\ldots\\[2ex] \dfrac{1}{a} & n=0\end{cases}$$
$[a>0]$ BI (356)(2)

9.[6] $$\int_0^\infty \ln x\frac{2ax\cosh ax - (2n+1)\sinh ax}{\sinh^3 ax}x^{2n}\,dx = \frac{1}{a}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$
$[a>0, \quad n=1,2,\ldots]$ BI (356)(6)a

10. $$\int_0^\infty \ln x \frac{x\sinh x - 6\sinh^2\left(\frac{x}{2}\right) - 6\cos^2\frac{t}{2}}{(\cosh x + \cos t)^2} x^2\,dx = \frac{\left(\pi - t^2\right)t}{3\sin t}$$
$[0 < t < \pi]$ BI (356)(16)a

11. $$\int_0^\infty \ln\left(1+x^2\right)\frac{\cosh\pi x + \pi x\sinh\pi x}{\cosh^2\pi x}\frac{dx}{x^2} = 4 - \pi$$ BI (356)(12)

12. $$\int_0^\infty \ln\left(1+4x^2\right)\frac{\cosh\pi x + \pi x\sinh\pi x}{\cosh^2\pi x}\frac{dx}{x^2} = 4\ln 2$$ BI (356)(13)

4.377 $$\int_0^\infty \ln 2x \frac{ax - n\left(1 - e^{-2ax}\right)}{\sinh^2 ax} x^{2n-1}\,dx = \frac{1}{2n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$
$[n = 1, 2, \ldots]$ LI (356)(8)a

4.38–4.41 Logarithms and trigonometric functions

4.381

1. $$\int_0^1 \ln x \sin ax\,dx = -\frac{1}{a}\left[\boldsymbol{C} + \ln a - \mathrm{ci}(a)\right]$$ $[a > 0]$ GW (338)(2a)

2. $$\int_0^1 \ln x \cos ax\,dx = -\frac{1}{a}\left[\mathrm{si}(a) + \frac{\pi}{2}\right]$$ $[a > 0]$ BI (284)(2)

3. $$\int_0^{2\pi} \ln x \sin nx\,dx = -\frac{1}{n}\left[\boldsymbol{C} + \ln(2n\pi) - \mathrm{ci}(2n\pi)\right]$$ GW (338)(1a)

4. $$\int_0^{2\pi} \ln x \cos nx\,dx = -\frac{1}{n}\left[\mathrm{si}(2n\pi) + \frac{\pi}{2}\right]$$ GW (338)(1b)

4.382

1. $$\int_0^\infty \ln\left|\frac{x+a}{x-a}\right|\sin bx\,dx = \frac{\pi}{b}\sin ab$$ $[a < 0, \quad b > 0]$ ET I 77(11)

2.[10] $$\int_0^\infty \ln\left|\frac{x+a}{x-a}\right|\cos bx\,dx = \frac{2}{b}\left[\cos(ab)\left\{\mathrm{si}(ab) + \frac{\pi}{2}\right\} - \sin(ab)\,\mathrm{ci}(ab)\right]$$
$[a > 0, \quad b > 0]$ ET I 18(9)

3. $$\int_0^\infty \ln\frac{a^2+x^2}{b^2+x^2}\cos cx\,dx = \frac{\pi}{c}\left(e^{-bc} - e^{-ac}\right)$$ $[a > 0, \quad b > 0, \quad c > 0]$
FI III 648a, BI (337)(5)

4. $$\int_0^\infty \ln\frac{x^2+x+a^2}{x^2-x+a^2}\sin bx\,dx = \frac{2\pi}{b}\exp\left(-b\sqrt{a^2 - \frac{1}{4}}\right)\sin\frac{b}{2}$$
$[b > 0]$ ET I 77(12)

5. $$\int_0^\infty \ln\frac{(x+\beta)^2+\gamma^2}{(x-\beta)^2+\gamma^2}\sin bx\,dx = \frac{2\pi}{b}e^{-\gamma b}\sin\beta b$$ $[\mathrm{Re}\,\gamma > 0, \quad |\mathrm{Im}\,\beta| \le \mathrm{Re}\,\gamma, \quad b > 0]$
ET I 77(13)

4.383

1. $$\int_0^\infty \ln\left(1+e^{-\beta x}\right)\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b\sinh\left(\dfrac{\pi b}{\beta}\right)} \qquad [\operatorname{Re}\beta>0,\quad b>0]$$ ET I 18(13)

2. $$\int_0^\infty \ln\left(1-e^{-\beta x}\right)\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b}\coth\left(\frac{\pi b}{\beta}\right) \qquad [\operatorname{Re}\beta>0,\quad b>0]$$ ET I 18(14)

4.384

1. $$\int_0^1 \ln(\sin\pi x)\sin 2n\pi x\,dx = 0$$ GW (338)(3a)

2.[7] $$\int_0^1 \ln(\sin\pi x)\sin(2n+1)\pi x\,dx = 2\int_0^{1/2} \ln(\sin\pi x)\sin(2n+1)\pi x\,dx$$
$$= \frac{2}{(2n+1)\pi}\left[\ln 2 - \frac{1}{2n+1} - 2\sum_{k=1}^{n}\frac{1}{2k-1}\right]$$
GW (338)(3b)

3.[6] $$\int_0^1 \ln(\sin\pi x)\cos 2n\pi x\,dx = 2\int_0^{1/2} \ln(\sin\pi x)\cos 2n\pi x\,dx$$
$$= -\ln 2 \qquad [n=0]$$
$$= -\frac{1}{2n} \qquad [n>0]$$
GW (338)(3c)

4. $$\int_0^1 \ln(\sin\pi x)\cos(2n+1)\pi x\,dx = 0$$ GW (338)(3d)

5. $$\int_0^{\pi/2} \ln\sin x\sin x\,dx = \ln 2 - 1$$ BI (305)(4)

6. $$\int_0^{\pi/2} \ln\sin x\cos x\,dx = -1$$ BI (305)(5)

7. $$\int_0^{\pi/2} \ln\sin x\cos 2nx\,dx = \begin{cases} -\dfrac{\pi}{4n}, & \text{for } n>0 \\ -\dfrac{\pi}{2}\ln 2, & \text{for } n=0 \end{cases}$$ LI (305)(6)

8. $$\int_0^{\pi} \ln\sin x\cos[2m(x-n)]\,dx = -\frac{\pi\cos 2mn}{2m}$$ LI (330)(8)

9. $$\int_0^{\pi/2} \ln\sin x\sin^2 x\,dx = \frac{\pi}{8}(1-\ln 4)$$ BI (305)(7)

10. $$\int_0^{\pi/2} \ln\sin x\cos^2 x\,dx = -\frac{\pi}{8}(1+\ln 4)$$ BI (305)(8)

11. $$\int_0^{\pi/2} \ln\sin x\sin x\cos^2 x\,dx = \frac{1}{9}(\ln 8 - 4)$$ BI (305)(9)

12. $$\int_0^{\pi/2} \ln\sin x\tan x\,dx = -\frac{\pi^2}{24}$$ BI (305)(11)

13. $$\int_0^{\pi/2} \ln\sin 2x \sin x\,dx = \int_0^{\pi/2} \ln\sin 2x \cos x\,dx = 2\,(\ln 2 - 1)$$ BI (305)(16, 17)

14. $$\int_0^{\pi} \frac{\ln(1+p\cos x)}{\cos x}\,dx = \pi \arcsin p \qquad [p^2 < 1]$$ FI II 484

15. $$\int_0^{\pi} \ln\sin x \frac{dx}{1-2a\cos x + a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^2}{2} \qquad [a^2<1]$$
$$= \frac{\pi}{a^2-1}\ln\frac{a^2-1}{2a^2} \qquad [a^2>1]$$
BI (331)(8)

16. $$\int_0^{\pi} \ln\sin bx \frac{dx}{1-2a\cos x + a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^{2b}}{2} \qquad [a^2<1]$$ BI (331)(10)

17. $$\int_0^{\pi} \ln\cos bx \frac{dx}{1-2a\cos x + a^2} = \frac{\pi}{1-a^2}\ln\frac{1+a^{2b}}{2} \qquad [a^2<1]$$ BI (331)(11)

18. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{1-2a\cos 2x + a^2} = \frac{1}{2}\int_0^{\pi} \ln\sin x \frac{dx}{1-2a\cos 2x + a^2}$$
$$= \frac{\pi}{2\,(1-a^2)}\ln\frac{1-a}{2} \qquad [a^2<1]$$
$$= \frac{\pi}{2\,(a^2-1)}\ln\frac{a-1}{2a} \qquad [a^2>1]$$
BI (321)(1), BI (331)(13)

19. $$\int_0^{\pi} \ln\sin bx \frac{dx}{1-2a\cos 2x + a^2} = \frac{\pi}{1-a^2}\ln\frac{1-a^{b}}{2} \qquad [a^2<1]$$ BI (331)(18)

20. $$\int_0^{\pi} \ln\cos bx \frac{dx}{1-2a\cos 2x + a^2} = \frac{\pi}{1-a^2}\ln\frac{1+a^{b}}{2} \qquad [a^2<1]$$ BI (331)(21)

21. $$\int_0^{\pi/2} \frac{\ln\cos x\,dx}{1-2p\cos 2x + p^2} = \frac{\pi}{2\,(1-p^2)}\ln\frac{1+p}{2} \qquad [p^2<1]$$
$$= \frac{\pi}{2\,(p^2-1)}\ln\frac{p+1}{2p} \qquad [p^2>1]$$
BI (321)(8)

22. $$\int_0^{\pi} \ln\sin x \frac{\cos x\,dx}{1-2a\cos x + a^2} = \frac{\pi}{2a}\frac{1+a^2}{1-a^2}\ln\left(1-a^2\right) - \frac{a\pi\ln 2}{1-a^2} \qquad [a^2<1]$$
$$= \frac{\pi}{2a}\frac{a^2+1}{a^2-1}\ln\frac{a^2-1}{a^2} - \frac{\pi\ln 2}{a\,(a^2-1)} \qquad [a^2>1]$$
LI (331)(9)

23. $$\int_0^{\pi} \ln\sin bx \frac{\cos x\,dx}{1-2a\cos 2x + a^2} = \int_0^{\pi} \ln\cos bx \frac{\cos x\,dx}{1-2a\cos 2x + a^2} = 0$$
$$[0<a<1]$$ BI (331)(19, 22)

24. $$\int_0^{\pi} \ln\sin x \frac{\cos^2 x\,dx}{1-2a\cos 2x + a^2} = \frac{\pi}{4a}\frac{1+a}{1-a}\ln(1-a) - \frac{\pi\ln 2}{2(1-a)} \qquad [0<a<1]$$
$$= \frac{\pi}{4a}\frac{a+1}{a-1}\ln\frac{a-1}{a} - \frac{\pi\ln 2}{2a(a-1)} \qquad [a>1]$$
BI (331)(16)

25. $$\int_0^{\pi/2} \ln\sin x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2} = \frac{1}{2}\int_0^{\pi} \ln\sin x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2}$$
$$= \frac{\pi}{2a\,(1-a^2)}\left\{\frac{1+a^2}{2}\ln(1-a) - a^2\ln 2\right\} \qquad [a^2<1]$$
$$= \frac{\pi}{2a\,(a^2-1)}\left\{\frac{1+a^2}{2}\ln\frac{a-1}{a} - \ln 2\right\} \qquad [a^2>1]$$
BI (321)(2), BI (331)(15), LI (321))(2)

26. $$\int_0^{\pi/2} \ln\cos x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2} = \frac{\pi}{2a\,(1-a^2)}\left\{\frac{1+a^2}{2}\ln(1+a) - a^2\ln 2\right\} \qquad [a^2<1]$$
$$= \frac{\pi}{2a\,(a^2-1)}\left\{\frac{1+a^2}{2}\ln\frac{1+a}{a} - \ln 2\right\} \qquad [a^2>1]$$
BI (321)(9)

4.385

1. $$\int_0^{\pi} \ln\sin x \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \qquad [a>0, \quad a>b]$$ BI (331)(6)

2. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{(a\sin x \pm b\cos x)^2} = \int_0^{\pi/2} \ln\cos x \frac{dx}{(a\cos x \pm b\sin x)^2}$$
$$= \frac{1}{b\,(a^2+b^2)}\left(\mp a\ln\frac{a}{b} - \frac{b\pi}{2}\right)$$
$$[a>0, \quad b>0]$$ BI (319)(1,6)a

3. $$\int_0^{\pi/2} \frac{\ln\sin x\,dx}{a^2\sin^2 x + b^2\cos^2 x} = \int_0^{\pi/2} \frac{\ln\cos x\,dx}{b^2\sin^2 x + a^2\cos^2 x} = \frac{\pi}{2ab}\ln\frac{b}{a+b}$$
$$[a>0, \quad b>0]$$ BI (317)(4, 10)

4. $$\int_0^{\pi/2} \ln\sin x \frac{\sin 2x\,dx}{\left(a\sin^2 x + b\cos^2 x\right)^2} = \int_0^{\pi/2} \ln\cos x \frac{\sin 2x\,dx}{\left(b\sin^2 x + a\cos^2 x\right)^2}$$
$$= \frac{1}{2b(b-a)}\ln\frac{a}{b}$$
$$[a>0, \quad b>0]$$ BI (319)(3, 7), LI (319)(3)

5. $$\int_0^{\pi/2} \ln\sin x \frac{a^2\sin^2 x - b^2\cos^2 x}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^2}\,dx = \int_0^{\pi/2} \ln\cos x \frac{a^2\cos^2 x - b^2\sin^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}\,dx$$
$$= \frac{\pi}{2b(a+b)}$$
$$[a>0, \quad b>0]$$ LI (319)(2, 8)

4.386

1. $$\int_0^{\pi/2} \ln\sin x \frac{\sin x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^{\pi/2} \frac{\cos x\ln\cos x}{\sqrt{1+\cos^2 x}}\,dx = -\frac{\pi}{8}\ln 2$$ BI (322)(1, 6)

2. $$\int_0^{\pi/2} \frac{\sin^3 x\ln\sin x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^{\pi/2} \frac{\cos^3 x\ln\cos x}{\sqrt{1+\cos^2 x}}\,dx = \frac{\ln 2 - 1}{4}$$ BI (322)(2, 7)

3. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{2}\boldsymbol{K}(k)\ln k - \frac{\pi}{4}\boldsymbol{K}(k')$$ BI (322)(3)

4. $$\int_0^{\pi/2} \frac{\ln\cos x\, dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2}\boldsymbol{K}(k)\ln\frac{k'}{k} - \frac{\pi}{4}\boldsymbol{K}(k')$$ BI (322)(9)

4.387

1. $$\int_0^{\pi/2} \ln\sin x \sin^\mu x \cos^\nu x\, dx = \int_0^{\pi/2} \ln\cos x \cos^\mu x \sin^\nu x\, dx$$
$$= \frac{1}{4}\mathrm{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right)\left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right)\right]$$
$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ GW (338)(6c)

2. $$\int_0^{\pi/2} \ln\sin x \sin^{\mu-1} x\, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\mu}{2}\right)}{4\,\Gamma\left(\frac{\mu+1}{2}\right)}\left[\psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right)\right]$$
$[\operatorname{Re}\mu > 0]$ GW (338)(6a)

3. $$\int_0^{\pi/2} \ln\sin x \cos^{\nu-1} x\, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}{4\,\Gamma\left(\frac{\nu+1}{2}\right)}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right]$$
$[\operatorname{Re}\nu > 0]$ GW (338)(6b)

4. $$\int_0^{\pi/2} \ln\sin x \sin^{2n} x\, dx = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}\left\{\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k} - \ln 2\right\}$$ FI II 811

5. $$\int_0^{\pi/2} \ln\sin x \sin^{2n+1} x\, dx = \frac{(2n)!!}{(2n+1)!!}\left\{\sum_{k=1}^{2n+1}\frac{(-1)^{k}}{k} + \ln 2\right\}$$ BI (305)(13)

6. $$\int_0^{\pi/2} \ln\sin x \cos^{2n} x\, dx = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}\left[\sum_{k=1}^{n}\frac{1}{k} + \ln 4\right]$$
$$= -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}\left[\boldsymbol{C} + \psi(n+1) + \ln 4\right]$$
BI (305)(14)

7. $$\int_0^{\pi/2} \ln\sin x \cos^{2n+1} x\, dx = -\frac{(2n)!!}{(2n+1)!!}\sum_{k=0}^{n}\frac{1}{2k+1}$$
$$= -\frac{(2n)!!}{2(2n+1)!!}\left[\psi\left(n+\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right)\right]$$
GW (338)(7b)

8. $$\int_0^{\pi/2} \ln\cos x \sin^{2n} x\, dx = -\frac{(2n-1)!!}{2^{n+1}\cdot n!}\frac{\pi}{2}\left\{\boldsymbol{C} + 2\ln 2 + \psi(n+1)\right\}$$
BI (306)(8)

9. $$\int_0^{\pi/2} \ln\cos x \cos^{2n} x\,dx = -\frac{(2n-1)!!}{2^n n!}\frac{\pi}{2}\left(\ln 2 + \sum_{k=1}^{2n}\frac{(-1)^k}{k}\right)$$ BI (306)(10)

10. $$\int_0^{\pi/2} \ln\cos x \cos^{2n} x\,dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!}\left[\ln 2 + \sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right]$$ BI (306)(9)

4.388

1. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n} x}{\cos^{2n+2} x}\,dx = \frac{1}{2n+1}\left[\frac{1}{2}\ln 2 + (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right]$$ BI (288)(1)

2. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\,dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n\ln 2 + \sum_{k=1}^{n-1}\frac{(-1)^k}{n-k}\right]$$ LI (288)(2)

3. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n} x}{\cos^{2n+2} x}\,dx = \frac{1}{2n+1}\left[-\frac{1}{2}\ln 2 + (-1)^{n+1}\frac{\pi}{4} + \sum_{k=0}^{n}\frac{(-1)^{k-1}}{2n-2k+1}\right]$$ BI (288)(10)

4. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\,dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n\ln 2 + \sum_{k=0}^{n-1}\frac{(-1)^k}{n-k}\right]$$ BI (288)(11)

5. $$\int_0^{\pi/2} \ln\sin x \frac{\sin^{p-1} x}{\cos^{p+1} x}\,dx = -\frac{\pi}{2p}\operatorname{cosec}\frac{p\pi}{2} \qquad [0 < p < 2]$$ BI (310)(4)

6. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{\tan^{p-1} x \sin 2x} = \frac{1}{4}\frac{\pi}{p-1}\sec\frac{p\pi}{2} \qquad [p^2 < 1]$$ BI (310)(3)

4.389

1. $$\int_0^{\pi} \ln\sin x \sin^{2n} 2x\cos 2x\,dx = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4n+2}$$ BI (330)(9)

2. $$\int_0^{\pi/4} \ln\sin x \cos^n 2x \sin 2x\,dx = -\frac{1}{4(n+1)}\{\boldsymbol{C} + \psi(n+2) + \ln 2\}$$ BI (285)(2)

3. $$\int_0^{\pi/4} \ln\cos x \cos^{\mu-1} 2x \tan 2x\,dx = \frac{1}{4(1-\mu)}\beta(\mu)$$ $[\operatorname{Re}\mu > 0]$ BI (286)(2)

4. $$\int_0^{\pi/2} \ln\sin x \sin^{\mu-1} x\cos x\,dx = \int_0^{\pi/2}\ln\cos x\cos^{\mu-1} x\sin x\,dx = -\frac{1}{\mu^2}$$ $[\operatorname{Re}\mu > 0]$ BI (306)(11)

5.[3] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\cos x\cos^p x\cos px\,dx = \frac{\pi}{2^{p+1}}[\boldsymbol{C} + \psi(p+1) - 2\ln 2]$$ $[p > -1]$

6. $$\int_0^{\pi/2} \ln\cos x\cos^{p-1} x\sin px\sin x\,dx = \frac{\pi}{2^{p+2}}\left[\boldsymbol{C} + \psi(p) - \frac{1}{p} - 2\ln 2\right]$$ $[p > 0]$ BI (306)(12)

4.391

1. $$\int_0^{\pi/4} (\ln\cos 2x)^n \cos^{p-1} 2x \tan x\, dx = \int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x \tan\left(\frac{\pi}{4} - x\right) dx = \frac{1}{2}\beta^{(n)}(p) \qquad [p > 0]$$ BI (286)(10), BI (285)(18)

2. $$\int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \frac{(-1)^n n!}{2}\,\zeta(n+1, p)$$ BI (285)(17)

3. $$\int_0^{\pi/4} (\ln\cos 2x)^{2n-1} \tan x\, dx = \frac{1 - 2^{2n-1}}{4n}\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots]$$ BI (286)(7)

4. $$\int_0^{\pi/4} (\ln\cos 2x)^{2n} \tan x\, dx = \frac{2^{2n}1}{2^{2n+1}}(2n)!\,\zeta(2n+1)$$ BI (286)(8)

4.392

1. $$\int_0^{\pi/4} \ln(\sin x\cos x)\,\frac{\sin^{2n} x}{\cos^{2n+2} x}\, dx = \frac{1}{2n+1}\left[(-1)^{n+1}\frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2\sum_{k=0}^{n-1}\frac{(-1)^{k-1}}{2n-2k-1}\right]$$ BI (294)(8)

2. $$\int_0^{\pi/4} \ln(\sin x\cos x)\,\frac{\sin^{2n-1} x}{\cos^{2n+1} x}\, dx = \frac{1}{2n}\left[(-1)^n\ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right]$$ BI (294)(9)

4.393

1. $$\int_0^{\pi/2} \ln\tan x \sin x\, dx = \ln 2$$ BI (307)(3)

2. $$\int_0^{\pi/2} \ln\tan x \cos x\, dx = -\ln 2$$ BI (307)(4)

3. $$\int_0^{\pi/2} \ln\tan x \sin^2 x\, dx = -\int_0^{\pi/2} \ln\tan x \cos^2 x\, dx = \frac{\pi}{4}$$ BI (307)(5, 6)

4. $$\int_0^{\pi/4} \frac{\ln\tan x}{\cos 2x}\, dx = -\frac{\pi^2}{8}$$ GW (338)(10b)a

5. $$\int_0^{\pi/2} \sin x \ln\cot\frac{x}{2}\, dx = \ln 2$$ LO III 290

4.394

1. $$\int_0^{\pi/2} \frac{\ln\tan x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2(1-a^2)}\ln\frac{1-a}{1+a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2(a^2-1)}\ln\frac{a-1}{a+1} \qquad [a^2 > 1]$$
BI (321)(15)

2. $$\int_0^{\pi/2} \frac{\ln\tan x \cos 2x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{4a}\frac{1+a^2}{1-a^2}\ln\frac{1-a}{1+a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{4a}\frac{a^2+1}{a^2-1}\ln\frac{a-1}{a+1} \qquad [a^2 > 1]$$
BI (321)(16)

3. $$\int_0^{\pi} \frac{\ln \tan bx\, dx}{1-2a\cos 2x+a^2} = \frac{\pi}{1-a^2} \ln \frac{1-a^b}{1+a^b} \qquad [0<a<1, \quad b>0]$$ BI (331)(24)

4. $$\int_0^{\pi} \frac{\ln \tan bx \cos x\, dx}{1-2a\cos 2x+a^2} = 0 \qquad [0<a<1]$$ BI (331)(25)

5. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x\, dx}{1-a\sin 2x} = -\frac{\arcsin a}{4a}(\pi+\arcsin a) \qquad [a^2 \le 1]$$ BI (291)(2,3)

6. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x\, dx}{1-a^2\sin^2 2x} = -\frac{\pi}{4a}\arcsin a \qquad [a^2<1]$$ BI (291)(9)

7. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x\, dx}{1+a^2\sin^2 2x} = -\frac{\pi}{4a}\operatorname{arcsinh} a = -\frac{\pi}{4a}\ln\left(a+\sqrt{1+a^2}\right)$$
$$[a^2<1]$$ BI (291)(10)

8. $$\int_0^{u} \frac{\sin x \ln \cot \frac{x}{2}}{1-\cos^2\alpha \sin^2 x}\, dx = \operatorname{cosec} 2\alpha \left\{\frac{\pi}{2}\ln 2 + L(\varphi-\alpha) - L(\varphi+\alpha) - L\left(\frac{\pi}{2}-2\alpha\right)\right\}$$
$$[\tan\varphi = \cot\alpha\cos u; \quad 0<u<\pi]$$
LO III 290

9. $$\int_0^{\pi/4} \frac{\ln \tan x \sin 2x\, dx}{1-\cos^2 t \sin^2 2x} = \operatorname{cosec} 2t \left[L\left(\frac{\pi}{2}-t\right) - \left(\frac{\pi}{2}-t\right)\ln 2\right]$$ LO III 290a

4.395

1. $$\int_0^{\pi/2} \frac{\ln \tan x\, dx}{\sqrt{1-k^2\sin^2 x}} = -\ln k'\, \boldsymbol{K}(k)$$ BI (322)(11)

2. $$\int_u^{\pi/4} \frac{\ln \tan x \sin 4x\, dx}{\left(\sin^2 u + \tan^2 \upsilon \sin^2 2x\right)\sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2}\frac{\cos^2\upsilon}{\sin u \sin \upsilon}\ln\frac{\sin\upsilon + \sqrt{1-\cos^2 u\cos^2\upsilon}}{\sin u\,(1+\sin\upsilon)}$$
$$\left[0<u<\frac{\pi}{2}, \quad 0<\upsilon<\frac{\pi}{2}\right]$$ LO III 285a

4.396

1. $$\int_0^{\pi/2} \ln(a\tan x)\sin^{\mu-1} 2x\, dx = 2^{\mu-2}\ln a \frac{\left\{\Gamma\left(\frac{a}{2}\right)\right\}^2}{\Gamma(a)} \qquad [a>0, \quad \operatorname{Re}\mu>0]$$ LI (307)(8)

2. $$\int_0^{\pi/2} \ln \tan x \cos^{2(\mu-1)x}\, dx = -\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(u-\frac{1}{2}\right)}{\Gamma(\mu)}\left[\boldsymbol{C} + \psi\left(\frac{2\mu-1}{2}\right) + \ln 4\right]$$
$$\left[\operatorname{Re}\mu > \tfrac{1}{2}\right]$$ BI (307)(9)

3. $$\int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cot x \sin[(q+1)x]\, dx = -\frac{\pi}{2}\left[\boldsymbol{C} + \psi(q+1)\right]$$
$$[q>-1]$$ BI (307)(11)

4. $$\int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cos[(q+1)x]\, dx = -\frac{\pi}{2q} \qquad [q>0]$$ BI (307)(10)

5. $$\int_0^{\pi/4} (\ln \tan x)^n \tan^p x\, dx = \frac{1}{2^{n+1}} B^{(n)}\left(\frac{p+1}{2}\right) \qquad [p>-1]$$ LI (286)(22)

6. $$\int_0^{\pi/2} (\ln\tan x)^{2n-1}\,\frac{dx}{\cos 2x} = \frac{1-2^{2n}}{2n}\pi^{2n}|B_{2n}| \qquad [n=1,2,\ldots]$$ BI (312)(6)

7. $$\int_0^{\pi/4} \ln\tan x\,\tan^{2n+1} x\,dx = \frac{(-1)^{n+1}}{4}\left[\frac{\pi^2}{12}+\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right]$$ GW (338)(8a)

4.397

1. $$\int_0^{\pi/2} \ln(1+p\sin x)\,\frac{dx}{\sin x} = \frac{\pi^2}{8}-\frac{1}{2}\left(\arccos p^2\right) \qquad \left[p^2<1\right]$$ BI (313)(1)

2. $$\int_0^{\pi/2} \ln(1+p\cos x)\,\frac{dx}{\cos x} = \frac{\pi^2}{8}-\frac{1}{2}\left(\arccos p\right)^2 \qquad \left[p^2<1\right]$$ BI (313)(8)

3. $$\int_0^{\pi} \ln(1+p\cos x)\,\frac{dx}{\cos x} = \pi\arcsin p \qquad \left[p^2<1\right]$$ BI (331)(1)

4. $$\int_0^{\pi/2} \frac{\cos x\ln(1+\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\,dx = \frac{L\left(\frac{\pi}{2}-\alpha\right)-\alpha\ln\sin\alpha}{\sin\alpha\cos\alpha} \qquad \left[0<\alpha<\frac{\pi}{2}\right]$$ LO III 291

5. $$\int_0^{\pi/2} \frac{\cos x\ln(1-\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\,dx = \frac{L\left(\frac{\pi}{2}-\alpha\right)+(\pi-\alpha)\ln\sin\alpha}{\sin\alpha\cos\alpha} \qquad \left[0<\alpha<\frac{\pi}{2}\right]$$ LO III 291

6. $$\int_0^{\pi} \ln\left(1-2a\cos x+a^2\right)\cos nx\,dx$$
$$= \frac{1}{2}\int_0^{2\pi} \ln\left(1-2a\cos x+a^2\right)\cos nx\,dx$$
$$= -\frac{\pi}{n}a^n \qquad \left[a^2<1\right]$$ BI (330)(11), BI (332)(5)
$$= -\frac{\pi}{na^n} \qquad \left[a^2>1\right]$$ GW (338)(13a)

7. $$\int_0^{\pi} \ln\left(1-2a\cos x+a^2\right)\sin nx\sin x\,dx = \frac{1}{2}\int_0^{2\pi} \ln\left(1-2a\cos x+a^2\right)\sin nx\sin x\,dx$$
$$= \frac{\pi}{2}\left(\frac{a^{n+1}}{n+1}-\frac{a^{n-1}}{n-1}\right) \qquad \left[a^2>1\right]$$ BI (330)(10), BI (332)(4)

8. $$\int_0^{\pi} \ln\left(1-2a\cos x+a^2\right)\sin nx\sin x\,dx = \frac{1}{2}\int_0^{2\pi} \ln\left(1-2a\cos x+a^2\right)\cos nx\cos x\,dx$$
$$= -\frac{\pi}{2}\left(\frac{a^{n+1}}{n+1}+\frac{a^{n-1}}{n-1}\right)$$

BI (330)(12), BI (332)(6)

9. $$\int_0^\pi \ln\left(1-2a\cos 2x+a^2\right)\cos(2n-1)x\,dx=0 \qquad [a^2<1]$$ BI (330)(15)

10. $$\int_0^\pi \ln\left(1-2a\cos 2x+a^2\right)\sin 2nx\sin x\,dx=0 \qquad [a^2<1]$$ BI (330)(13)

11. $$\int_0^\pi \ln\left(1-2a\cos 2x+a^2\right)\sin(2n-1)x\sin x\,dx=\frac{\pi}{2}\left(\frac{a^n}{n}-\frac{a^{n-1}}{n-1}\right) \qquad [a^2<1]$$ BI (330)(14)

12. $$\int_0^\pi \ln\left(1-2a\cos 2x+a^2\right)\cos 2nx\cos x\,dx=0 \qquad [a^2<1]$$ BI (330)(16)

13. $$\int_0^\pi \ln\left(1-2a\cos 2x+a^2\right)\cos(2n-1)x\cos x\,dx=-\frac{\pi}{2}\left(\frac{a^n}{n}+\frac{a^{n-1}}{n-1}\right) \qquad [a^2<1]$$ BI (330)(17)

14. $$\int_0^{\pi/2} \ln\left(1+2a\cos 2x+a^2\right)\sin^2 x\,dx=-\frac{a\pi}{4} \qquad [a^2<1]$$
$$=\frac{\pi\ln a^2}{4}-\frac{\pi}{4a} \qquad [a^2>1]$$
BI (309)(22), LI (309)(22)

15. $$\int_0^{\pi/2} \ln\left(1+2a\cos 2x+a^2\right)\cos^2 x\,dx=\frac{a\pi}{4} \qquad [a^2<1]$$
$$=\frac{\pi\ln a^2}{4}+\frac{\pi}{4a} \qquad [a^2>1]$$
BI (309)(23), LI (309)(23)

16. $$\int_0^\pi \frac{\ln\left(1-2a\cos x+a^2\right)}{1-2b\cos x+b^2}\,dx=\frac{2\pi\ln(1-ab)}{1-b^2} \qquad [a^2\le 1,\quad b^2<1]$$ BI (331)(26)

4.398

1. $$\int_0^\pi \ln\frac{1+2a\cos x+a^2}{1-2a\cos x+a^2}\sin(2n+1)x\,dx=(-1)^n\frac{2\pi a^{2n+1}}{2n+1} \qquad [a^2<1]$$ BI (330)(18)

2. $$\int_0^{2\pi} \ln\frac{1-2a\cos x+a^2}{1-2a\cos nx+a^2}\cos mx\,dx=2\pi\left(\frac{n}{m}a^{m/n}-\frac{a^m}{m}\right) \qquad [a^2\le 1]$$
$$=2\pi\left(\frac{n}{m}a^{-m/n}-\frac{a^{-m}}{m}\right) \qquad [a^2\ge 1]$$
BI (332)(9)

3. $$\int_0^\pi \ln\frac{1+2a\cos 2x+a^2}{1+2a\cos 2nx+a^2}\cot x\,dx=0$$ BI (331)(5), LI(331)(5)

4.399

1. $$\int_0^{\pi/2} \ln\left(1+a\sin^2 x\right)\sin^2 x\,dx=\frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2}-\frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad [a>-1]$$ BI (309)(14)

2. $$\int_0^{\pi/2} \ln\left(1+a\sin^2 x\right)\cos^2 x\,dx = \frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right)$$
$[a>-1]$ BI (309)(15)

3. $$\int_0^{\pi/2} \frac{\ln\left(1-\cos^2\beta\cos^2 x\right)}{1-\cos^2\alpha\cos^2 x}\,dx = -\frac{\pi}{\sin\alpha}\ln\frac{1+\sin\alpha}{\sin\alpha+\sin\beta}$$
$\left[0<\beta<\frac{\pi}{2},\quad 0<\alpha<\frac{\pi}{2}\right]$ LO III 285

4.411

1. $$\int_0^{\pi} \ln\frac{1+\sin x}{1+\cos\lambda\sin x}\frac{dx}{\sin x} = \lambda^2$$
$\left[\lambda^2<\pi^2\right]$ BI (331)(2)

2. $$\int_0^{\pi/2} \ln\frac{p+q\sin ax}{p-q\sin ax}\frac{dx}{\sin ax} = \int_0^{\pi/2} \ln\frac{p+q\cos ax}{p-q\cos ax}\frac{dx}{\cos ax} = \int_0^{\pi/2} \ln\frac{p+q\tan ax}{p-q\tan ax}\frac{dx}{\tan ax} = \pi\arcsin\frac{q}{p}$$
$[p>q>0]$
FI II 695a, BI (315)(5, 13,17)a

3. $$\int_0^{\pi/2} \frac{\cos x}{1-\cos^2\alpha\cos^2 x}\ln\frac{1+\cos\beta\cos x}{1-\cos\beta\cos x}\,dx = \frac{2\pi}{\sin 2\alpha}\ln\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$
$\left[0<\alpha\le\beta<\frac{\pi}{2}\right]$ LO III 284

4.412

1. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4}\pm x\right)\frac{dx}{\sin 2x} = \pm\frac{\pi^2}{8}$$
BI (293)(1)

2. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4}\pm x\right)\frac{dx}{\tan 2x} = \pm\frac{\pi^2}{16}$$
BI (293)(2)

3. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4}\pm x\right)(\ln\tan x)^{2n}\frac{dx}{\sin 2x} = \pm\frac{2^{2n+2}-1}{4(n+1)(2n+1)}\pi^{2n+2}|B_{2n+2}|$$
BI (294)(24)

4. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4}\pm x\right)(\ln\tan x)^{2n-1}\frac{dx}{\sin 2x} = \pm\frac{1-2^{2n+1}}{2^{2n+2}n}(2n)!\,\zeta(2n+1)$$
BI (294)(25)

5. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4}\pm x\right)(\ln\sin 2x)^{n-1}\frac{dx}{\tan 2x} = \frac{(-1)^{n-1}}{2}(n-1)!\,\zeta(n+1)$$
LI (294)(20)

4.413

1. $$\int_0^{\pi/2} \ln\left(p^2+q^2\tan^2 x\right)\frac{dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{\pi}{ab}\ln\frac{ap+bq}{a}$$
$[a>0,\quad b>0,\quad p>0,\quad q>0]$
BI (318)(1–4)a

2. $$\int_0^{\pi/2} \ln\left(1+q^2\tan^2 x\right)\frac{1}{p^2\sin^2 x+r^2\cos^2 x}\frac{dx}{s^2\sin^2 x+t^2\cos^2 x}$$
$$= \frac{\pi}{p^2t^2-s^2r^2}\left\{\frac{p^2-r^2}{pr}\ln\left(1+\frac{qr}{p}\right)+\frac{t^2-s^2}{st}\ln\left(1+\frac{qt}{s}\right)\right\}$$
$[q>0,\quad p>0,\quad r>0,\quad s>0,\quad t>0]$ BI (320)(18)

3. $$\int_0^{\pi/2} \ln\left(1+q^2\tan^2 x\right) \frac{\sin^2 x}{p^2\sin^2 x + r^2\cos^2 x}\,\frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{t}{s}\ln\left(1+\frac{qr}{p}\right) - \frac{r}{p}\ln\left(1+\frac{qt}{s}\right)\right\}$$
$[q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0]$ BI (320)(20)

4. $$\int_0^{\pi/2} \ln\left(1+q^2\tan^2 x\right) \frac{\cos^2 x}{p^2\sin^2 x + r^2\cos^2 x}\,\frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{p}{r}\ln\left(1+\frac{qr}{p}\right) - \frac{s}{t}\ln\left(1+\frac{qt}{s}\right)\right\}$$
$[q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0]$ BI (320)(21)

5. $$\int_0^{\pi} \frac{\ln\tan rx\,dx}{1-2p\cos x + p^2} = \frac{\pi}{1-p^2}\ln\frac{1-p^{2r}}{1+p^{2r}} \qquad \left[p^2<1\right]$$ BI (331)(12)

4.414

1. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \ln k'\,\boldsymbol{K}(k)$$ BI (323)(1)

2. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (323)(3)

3. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\cos^2 x}{dx}\sqrt{1-k^2\sin^2 x} = \frac{1}{k^2}\left[\left(1+k'^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(6)

4. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{dx}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} = \frac{1}{k'^2}\left[\left(k^2-2\right)\boldsymbol{K}(k) + \left(2+\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(9)

5. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x}{dx}\sqrt{\left(1-k^2\sin^2 x\right)^3} = \frac{1}{k^2k'^2}\left[\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(1+k'^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right]$$
BI (323)(10)

6. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\cos^2 x\,dx}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} = \frac{1}{k^2}\left[\left(1+k'^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(16)

7. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,dx = \left(1+k'^2\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$ BI (324)(18)

8. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sin^2 x\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{9k^2}\left\{\left(-2+11k^2-6k^4+3k'^2\ln k'\right)\boldsymbol{K}(k) + \left[2-10k^2-3\left(1-2k^2\right)\ln k'\right]\boldsymbol{E}(k)\right\}$$

BI (324)(20)

9. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\cos^2 x\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{9k^2}\left\{\left(2+7k^2-3k^4-3k'^2\ln k'\right)\boldsymbol{K}(k) - \left[2+8k^2-3\left(1+k^2\right)\ln k'\right]\boldsymbol{E}(k)\right\}$$

BI (324)(21), LI (324)(21)

10. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin x\cos x\,dx}{\sqrt{\left(1-k^2\sin^2 x\right)^{2n+1}}} = \frac{2}{(2n-1)^2k^2}\left\{\left[1+(2n-1)\ln k'\right]k'^{1-2n}-1\right\}$$

BI (324)(17)

4.415

1. $$\int_0^{\infty} \ln x\sin ax^2\,dx = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left(\ln 4a+\boldsymbol{C}-\frac{\pi}{2}\right) \qquad [a>0]$$ GW (338)(19)

2. $$\int_0^{\infty} \ln x\cos ax^2\,dx = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left(\ln 4a+\boldsymbol{C}-\frac{\pi}{2}\right) \qquad [a>0]$$ GW (338)(19)

4.416

1. $$\int_0^{\pi/2} \frac{\cos x\ln\left(1+\sqrt{\sin^2\beta-\cos^2\beta\tan^2\alpha\sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx = \operatorname{cosec}2\alpha\left\{(2\alpha+2\gamma-\pi)\ln\cos\beta+2\,L(\alpha)-2\,L(\gamma)+L(\alpha+\gamma)-L(\alpha-\gamma)\right\}$$

$$\left[\cos\gamma=\frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right]$$ LO III 291

2. $$\int_0^{\pi/2} \frac{\cos x\ln\left(1-\sqrt{\sin^2\beta-\cos^2\beta\tan^2\alpha\sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx = \operatorname{cosec}2\alpha\left\{(\pi+2\alpha-2\gamma)\ln\cos\beta+2\,L(\alpha)+2\,L(\gamma)-L(\alpha+\gamma)+L(\alpha-\gamma)\right\}$$

$$\left[\cos\gamma=\frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right]$$ LO III 291

3. $$\int_\beta^{\pi/2} \frac{\ln\left(\sin x+\sqrt{\sin^2 x-\sin^2\beta}\right)}{1-\cos^2\alpha\cos^2 x}\,dx = -\operatorname{cosec}\alpha\left\{\arctan\left(\frac{\tan\beta}{\sin\alpha}\right)\ln\sin\beta+\frac{\pi}{2}\ln\frac{1+\sin\alpha}{\sin\alpha+\sqrt{1-\cos^2\alpha\cos^2\beta}}\right\}$$

$$\left[0<\alpha<\pi,\quad 0<\beta<\frac{\pi}{2}\right]$$ LO III 285

4.[7] $$\int_0^{\pi/4} \ln\tan x\,(\ln\cos 2x)^{n-1}\tan 2x\,dx = \tfrac{1}{2}(-1)^n(n-1)!\left(1-2^{-(n+1)}\right)\zeta(n+1)$$

BI (287)(20)

4.42–4.43 Combinations of logarithms, trigonometric functions, and powers

4.421

1. $$\int_0^\infty \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2}\left(\boldsymbol{C} + \ln a\right) \qquad [a > 0]$$ FI II 810a

2. $$\int_0^\infty \ln ax \sin bx \frac{x\,dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln\left(a\beta'\right) - \frac{\pi}{4}\left[e^{b\beta'} \operatorname{Ei}\left(-b\beta'\right) + e^{-b\beta'} \operatorname{Ei}\left(b\beta'\right)\right]$$
$$[\beta' = \beta \operatorname{sign}\beta; \quad a > 0, \quad b > 0]$$
ET I 76(5), NT 27(10)a

3. $$\int_0^\infty \ln ax \cos bx \frac{\beta'\,dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln\left(a\beta'\right) + \frac{\pi}{4}\left[e^{b\beta'} \operatorname{Ei}\left(-b\beta'\right) - e^{-b\beta'} \operatorname{Ei}\left(b\beta'\right)\right]$$
$$[\beta' = \beta \operatorname{sign}\beta; \quad a > 0, \quad b > 0]$$
ET I 17(3), NT 27(11)a

4. $$\int_0^\infty \ln ax \sin bx \frac{x\,dx}{x^2 - c^2} = \frac{\pi}{2}\left\{-\operatorname{si}(bc)\sin bc + \cos bc\left[\ln ac - ci(bc)\right]\right\}$$
$$[a > 0, \quad b > 0, \quad c > 0]$$ BI (422)(5)

5. $$\int_0^\infty \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c}\left\{\sin bc\left[\operatorname{ci}(bc) - \ln ac\right] - \cos bc \operatorname{si}(bc)\right\}$$
$$[a > 0, \quad b > 0, \quad c > 0]$$ BI (422)(6)

4.422

1. $$\int_0^\infty \ln x \sin ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu} \sin\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a + \frac{\pi}{2}\cot\frac{\mu\pi}{2}\right]$$
$$[a > 0, \quad |\operatorname{Re}\mu| < 1]$$ BI (411)(5)

2. $$\int_0^\infty \ln x \cos ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu} \cos\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a - \frac{\pi}{2}\tan\frac{\mu\pi}{2}\right]$$
$$[a > 0, \quad 0 < \operatorname{Re}\mu < 1]$$ BI (411)(6)

4.423

1. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{a}{b}\left(\boldsymbol{C} + \frac{1}{2}\ln ab\right) \qquad [a > 0, \quad b > 0]$$ GW (338)(21a)

2. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}\left[(a-b)\left(\boldsymbol{C} - 1\right) + a\ln a - b\ln b\right]$$
$$[a > 0, \quad b > 0]$$ GW (338)(21b)

3. $$\int_0^\infty \ln x \frac{\sin^2 ax}{x^2}\,dx = -\frac{a\pi}{2}\left(\boldsymbol{C} + \ln 2a - 1\right) \qquad [a > 0]$$ GW (338)(20b)

4.424

1. $$\int_0^\infty (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2}\boldsymbol{C}^2 + \frac{\pi^3}{24} + \pi\boldsymbol{C}\ln a + \frac{\pi}{2}(\ln a)^2$$
$$[a > 0]$$ ET I 77(9), FI II 810a

2.[6] $$\int_0^\infty (\ln x)^2 \sin ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi'(\mu)+\psi^2(\mu)+\pi\,\psi(\mu)\cot\frac{\mu\pi}{2}-2\,\psi(\mu)\ln a - \pi\ln a\cot\frac{\mu\pi}{2}+(\ln a)^2-\frac{1}{4}\pi^2\right]$$
$[a>0,\quad 0<\operatorname{Re}\mu<1]$ ET I 77(10)

4.425

1. $$\int_0^\infty \ln(1+x)\cos ax\frac{dx}{x} = \frac{1}{2}\left\{[\operatorname{si}(a)]^2+[\operatorname{ci}(a)]^2\right\}$$ $[a>0]$ ET I 18(8)

2. $$\int_0^\infty \ln^2\left(\frac{b+x}{b-x}\right)\cos ax\frac{dx}{x} = -2\pi\operatorname{si}(ab)$$ $[a\geq 0,\quad b>0]$ ET I 18(11)

3. $$\int_0^\infty \ln\left(1+b^2x^2\right)\sin ax\frac{dx}{x} = -\pi\operatorname{Ei}\left(-\frac{a}{b}\right)$$ $[a>0,\quad b>0]$
GW (338)(24), ET I 77(14)

4. $$\int_0^1 \ln\left(1-x^2\right)\cos\left(p\ln x\right)\frac{dx}{x} = \frac{1}{2p^2}+\frac{\pi}{2p}\coth\frac{p\pi}{2}$$ LI (309)(1)a

4.426

1.[11] $$\int_0^\infty x\ln\frac{b^2+x^2}{c^2+x^2}\sin ax\,dx = \frac{\pi}{a^2}\left[(1+ac)e^{-ac}-(1+ab)e^{-ab}\right]$$
$[b\geq 0,\quad c\geq 0,\quad a>0]$ GW (338)(23)

2. $$\int_0^\infty \ln\frac{b^2x^2+p^2}{c^2x^2+p^2}\sin ax\frac{dx}{x} = \pi\left[\operatorname{Ei}\left(-\frac{ap}{c}\right)-\operatorname{Ei}\left(-\frac{ap}{b}\right)\right]$$
$[b>0,\quad c>0,\quad p>0,\quad a>0]$
ET I 77(15)

4.427 $$\int_0^\infty \ln\left(x+\sqrt{\beta^2+x^2}\right)\frac{\sin ax}{\sqrt{\beta^2+x^2}}\,dx = \frac{\pi}{2}K_0(a\beta)+\frac{\pi}{2}\ln(\beta)\left[I_0(a\beta)-\mathbf{L}(a\beta)\right]$$
$[\operatorname{Re}\beta>0,\quad a>0]$ ET I 77(16)

4.428

1. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2}\,dx = \pi b\ln 2-a\pi$$ $[a>0,\quad b>0]$ ET I 22(29)

2. $$\int_0^\infty \ln\left(4\cos^2 ax\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\cosh(bc)\ln\left(1+e^{-2ac}\right)$$
$\left[a<b<2a<\frac{\pi}{c}\right]$ ET I 22(30)

3. $$\int_0^\infty \ln\cos^2 ax\frac{\sin bx}{x\left(1+x^2\right)}\,dx = \pi\ln\left(1+e^{-2a}\right)\sinh b-\pi\ln 2\left(1-e^{-b}\right)$$
$[a>0,\quad b>0]$ ET I 82(36)

4. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2\left(1+x^2\right)}\,dx = -\pi\ln\left(1+e^{-2a}\right)\cosh b+\left(b+e^{-b}\right)\pi\ln 2-a\pi$$
$[a>0,\quad b>0]$ ET I 22(31)

4.429 $$\int_0^1 \frac{(1+x)x}{\ln x}\sin\left(\ln x\right)\,dx = \frac{\pi}{4}$$ BI (326)(2)a

4.431

1. $$\int_0^\infty \ln(2 \pm 2\cos x)\,\frac{\sin bx}{x^2+c^2}x\,dx = -\,\pi\sinh(bc)\ln(1 \pm e^{-c})$$
$[b>0, \quad c>0]$ ET I 22(32)

2. $$\int_0^\infty \ln(2 \pm 2\cos x)\,\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\cosh(bc)\ln\left(1 \pm e^{-c}\right)$$
$[b>0, \quad c>0]$ ET I 22(32)

3. $$\int_0^\infty \ln\left(1+2a\cos x+a^2\right)\frac{\sin bx}{x}\,dx = -\frac{\pi}{2}\sum_{k=1}^{[b]}\frac{(-a)^k}{k}\left[1+\operatorname{sign}(b-k)\right]$$
$[0<a<1, \quad b>0]$ ET I 82(25)

4. $$\int_0^\infty \ln\left(1-2a\cos x+a^2\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\ln\left(1-ae^{-c}\right)\cosh(bc) + \frac{\pi}{c}\sum_{k=1}^{\lfloor b\rfloor}\frac{a^k}{k}\sinh[c(b-k)]$$
$[|a|<1, \quad b>0, \quad c>0]$ ET I 22(33)

4.432

1. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \ln k'\,\boldsymbol{K}(k)$$
BI ((412, 414))(4)

2. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}x\,dx$$
$$= \frac{1}{k^2}\left\{\pi k'\left(1-\ln k'\right) + \left(2-k^2\right)\boldsymbol{K}(k) - \left(4-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (426)(3)

3. $$\int_0^{\pi/2} \ln\left(1-k^2\cos^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}x\,dx = \frac{1}{k^2}\left\{-\pi - \left(2-k^2\right)\boldsymbol{K}(k) + \left(4-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (426)(6)

4. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left\{\left(2-k^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (412)(5)

5. $$\int_0^\infty \ln\left(1-k^2\cos^2 x\right)\frac{\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (414)(5)

6. $$\int_0^\infty \ln\left(1 \pm k\sin^2 x\right) \frac{\sin x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln\left(1 \pm k\cos^2 x\right) \frac{\sin x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k\sin^2 x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k\cos^2 x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k\sin^2 2x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 2x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k^2\cos^2 2x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 2x}} \frac{dx}{x}$$
$$= \frac{1}{2}\ln\frac{2\left(1 \pm k\right)}{\sqrt{k}}\boldsymbol{K}(k) - \frac{\pi}{8}\boldsymbol{K}\left(k'\right)$$

BI (413)(1–6), BI (415)(1–6)

7. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right) \frac{\sin^3 x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(6)

8. $$\int_0^\infty \ln\left(1-k^2\cos^2 x\right) \frac{\sin^3 x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2-k^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (414)(6)a

9. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right) \frac{\sin x\cos^2 x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2-k^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(7)

10. $$\int_0^\infty \ln\left(1-k^2\cos^2 x\right) \frac{\sin x\cos^2 x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (414)(7)

11. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \ln k'\,\boldsymbol{K}(k)$$

BI ((412, 414))(9)

12. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right) \frac{\sin^2 x\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(8)

13. $$\int_0^\infty \ln\left(1-k^2\cos^2 x\right) \frac{\sin^2 x\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2-k^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (414)(8)

14. $$\int_0^\infty \ln\left(1-k^2\sin^2 x\right) \frac{\sin^2 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^\infty \ln\left(1-k^2\cos^2 x\right) \frac{\sin x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k'^2}\left\{\left(k^2-2\right)\boldsymbol{K}(k) + \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI ((412, 414))(13)

15. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x\cos x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} x\,dx = \frac{1}{k^2}\left\{\left(1+\ln k'\right)\frac{\pi}{k'} - \left(2+\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (426)(9)

16. $$\int_0^{\pi/2} \ln\left(1-k^2\cos^2 x\right) \frac{\sin x\cos x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} x\,dx = \frac{1}{k^2}\left\{-\pi + \left(2+\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (426)(15)

17. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x\cos x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin^3 x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2}\left\{\left(2-k^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(14), BI(414)(15)

18. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin^3 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(1-k^2 cos^2 x\right) \frac{\sin x\cos x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2k'^2}\left\{\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(2-k^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (412)(15), BI(414)(14)

19. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x\cos^2 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin^2 x\tan x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2}\left\{\left(2-k^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(16), BI(414)(17)

20. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin^2 x\tan x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin x\cos^2 x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2k'^2}\left\{\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(2-k^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (412)(17), BI(414)(16)

21. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\tan x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\tan x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k'^2}\left\{\left(k^2-2\right)\boldsymbol{K}(k) + \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI ((412, 414))(18)

22. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,\sin x\frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\,\sin x\frac{dx}{x}$$
$$= \left(2-k^2\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$

BI ((412, 414))(1)

23. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\sin x\cos x\cdot x\,dx$$
$$= \frac{1}{27k^2}\left\{3\pi k'^3\left(1-3\ln k'\right)+\left(22k'^2+6k^4-3k'^2\ln k'\right)\boldsymbol{K}(k)\right\} \quad -\left(2-k^2\right)\left(14-6\ln k'\right)\boldsymbol{E}(k)$$
BI (426)(1)

24. $$\int_0^{\pi/2} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\sin x\cos x\cdot x\,dx$$
$$= \frac{1}{27k^2}\left\{-3\pi-\left(22k'^2+6k^4-3k'^2\ln k'\right)\boldsymbol{K}(k)+\left(2-k^2\right)\left(14-6\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (426)(2)

25. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\tan x\frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\tan x\frac{dx}{x}$$
$$= \left(2-k^2\right)\boldsymbol{K}(k)-\left(2-\ln k'\right)\boldsymbol{E}(k)$$
((412,414))(2)

26. $$\int_0^{\infty} \ln\left(\sin^2 x+k'\cos^2 x\right)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \int_0^{\infty} \ln\left(\sin^2 x+k'\cos^2 x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(\sin^2 2x+k'\cos^2 2x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x}$$
$$= \frac{1}{2}\ln\left[\frac{2\left(\sqrt{k'}\right)^3}{1+k'}\right]\boldsymbol{K}(k)$$
BI (415)(19–21)

4.44 Combinations of logarithms, trigonometric functions, and exponentials

4.441

1.[7] $$\int_0^{\infty} e^{-qx}\sin px\ln x\,dx = \frac{1}{p^2+q^2}\left[q\arctan\frac{p}{q}-p\boldsymbol{C}-\frac{p}{c}\ln\left(p^2-q^2\right)\right]$$
$[q>0, \quad p>0]$ BI (467)(1)

2. $$\int_0^{\infty} e^{-qx}\cos px\ln x\,dx = -\frac{1}{p^2+q^2}\left[\frac{q}{2}\ln\left(p^2+q^2\right)+p\arctan\frac{p}{q}+q\boldsymbol{C}\right]$$
$[q>0]$ BI (467)(2)

4.442 $$\int_0^{\pi/2}\frac{e^{-p\tan x}\ln\cos x\,dx}{\sin x\cos x} = -\frac{1}{2}\left[\mathrm{ci}(p)\right]^2+\frac{1}{2}\left[\mathrm{si}(p)\right]^2 \qquad [\mathrm{Re}\,p>0]$$
NT 32(11)

4.5 Inverse Trigonometric Functions

4.51 Inverse trigonometric functions

4.511 $$\int_0^{\infty}\operatorname{arccot} px\operatorname{arccot} qx\,dx = \frac{\pi}{2}\left\{\frac{1}{p}\ln\left(1+\frac{p}{q}\right)+\frac{1}{q}\ln\left(1+\frac{q}{p}\right)\right\}$$
$[p>0, \quad q>0]$ BI (77)(8)

4.512 $\displaystyle\int_0^\pi \arctan(\cos x)\,dx = 0$ BI (345)(1)

4.52 Combinations of arcsines, arccosines, and powers

4.521

1. $\displaystyle\int_0^1 \frac{\arcsin x}{x}\,dx = \frac{\pi}{2}\ln 2$ FI II 614, 623

2. $\displaystyle\int_0^1 \frac{\arccos x}{1\pm x}\,dx = \mp\frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$ BI (231)(7, 8)

3. $\displaystyle\int_0^1 \arcsin x\frac{x}{1+qx^2}\,dx = \frac{\pi}{2q}\ln\frac{2\sqrt{1+q}}{1+\sqrt{1+q}}$ $[q>-1]$ BI (231)(1)

4. $\displaystyle\int_0^1 \arcsin x\frac{x}{1-p^2x^2}\,dx = \frac{\pi}{2p^2}\ln\frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}}$ $[p^2<1]$ LI (231)(3)

5. $\displaystyle\int_0^1 \arccos x\frac{dx}{\sin^2\lambda - x^2} = 2\operatorname{cosec}\lambda\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$ BI (231)(10)

6. $\displaystyle\int_0^1 \arcsin x\frac{dx}{x(1+qx^2)} = \frac{\pi}{2}\ln\frac{1+\sqrt{1+q}}{\sqrt{1+q}}$ $[q>-1]$ BI (235)(10)

7. $\displaystyle\int_0^1 \arcsin x\frac{x}{(1+qx^2)^2}\,dx = \frac{\pi}{4q}\frac{\sqrt{1+q}-1}{1+q}$ $[q>-1]$ BI (234)(2)

8. $\displaystyle\int_0^1 \arccos x\frac{x}{(1+qx^2)^2}\,dx = \frac{\pi}{4q}\frac{\sqrt{1+q}-1}{1+q}$ $[q>-1]$ BI (234)(4)

4.522

1. $\displaystyle\int_0^1 x\sqrt{1-k^2x^2}\arccos x\,dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\boldsymbol{K}(k) - 2\left(1+k'^2\right)\boldsymbol{E}(k)\right]$ BI (236)(9)

2. $\displaystyle\int_0^1 x\sqrt{1-k^2x^2}\arcsin x\,dx = \frac{1}{9k^2}\left[-\frac{3}{2}\pi k'^3 - k'^2\boldsymbol{K}(k) + 2\left(1+k'^2\right)\boldsymbol{E}(k)\right]$ BI (236)(1)

3. $\displaystyle\int_0^1 x\sqrt{k'^2+k^2x^2}\arcsin x\,dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\boldsymbol{K}(k) - 2\left(1+k'^2\right)\boldsymbol{E}(k)\right]$ BI(236)(5)

4. $\displaystyle\int_0^1 \frac{x\arcsin x}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right]$ BI (237)(1)

5. $\displaystyle\int_0^1 \frac{x\arccos x}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right]$ BI (240)(1)

6. $\displaystyle\int_0^1 \frac{x\arcsin x}{\sqrt{k'^2+k^2x^2}}\,dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right]$ BI (238)(1)

7. $\displaystyle\int_0^1 \frac{x\arccos x}{\sqrt{k'^2+k^2x^2}}\,dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right]$ BI (241)(1)

8. $$\int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda)\sqrt{1-x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$$ BI (243)(11)

9. $$\int_0^1 \frac{x \arcsin kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = -\frac{\pi}{2k} \ln k'$$ BI (239)(1)

10. $$\int_0^1 \frac{x \arccos kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = \frac{\pi}{2k} \ln(1+k)$$ BI (242)(1)

4.523

1. $$\int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1} \left[\frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!} \right]$$ BI (229)(1)

2. $$\int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n} \left[1 - \frac{(2n-1)!!}{2^n n!} \right]$$ BI (229)(2)

3. $$\int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!}$$ BI (229)(4)

4. $$\int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n} \frac{(2n-1)!!}{2^n n!}$$ BI (229)(5)

5. $$\int_{-1}^1 \left(1-x^2\right)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!}$$ BI (254)(2)

6. $$\int_{-1}^1 \left(1-x^2\right)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2} \frac{(2n-1)!!}{2^n n!}$$ BI (254)(3)

4.524

1. $$\int_0^1 (\arcsin x)^2 \frac{dx}{x^2\sqrt{1-x^2}} = \pi \ln 2$$ BI (243)(13)

2. $$\int_0^1 (\arccos x)^2 \frac{dx}{\left(\sqrt{1-x^2}\right)^3} = \pi \ln 2$$ BI (244)(9)

4.53–4.54 Combinations of arctangents, arccotangents, and powers

4.531

1. $$\int_0^1 \frac{\arctan x}{x} \, dx = \int_1^{\infty} \frac{\operatorname{arccot} x}{x} \, dx = \boldsymbol{G}$$ FI II 482, BI (253)(8)

2. $$\int_0^{\infty} \frac{\operatorname{arccot} x}{1 \pm x} \, dx = \pm \frac{\pi}{4} \ln 2 + \boldsymbol{G}$$ BI (248)(6, 7)

3. $$\int_0^1 \frac{\operatorname{arccot} x}{x(1+x)} \, dx = -\frac{\pi}{8} \ln 2 + \boldsymbol{G}$$ BI (235)(11)

4. $$\int_0^{\infty} \frac{\arctan x}{1-x^2} \, dx = -\boldsymbol{G}.$$ BI (248)(2)

5. $$\int_0^1 \arctan qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)}\arctan q \qquad [p>-1]$$ BI (243)(7)

6. $$\int_0^1 \operatorname{arccot} qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2}\arctan q + \frac{1}{1+p}\operatorname{arccot} q \qquad [p>-1]$$ BI (234)(10)

7. $$\int_0^1 \frac{\arctan x}{x(1+x^2)}\,dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (235)(12)

8. $$\int_0^\infty \frac{x\arctan x}{1+x^4}\,dx = \frac{\pi^2}{16}$$ BI (248)(3)

9. $$\int_0^\infty \frac{x\arctan x}{1-x^4}\,dx = -\frac{\pi}{8}\ln 2$$ BI (248)(4)

10.[11] $$\int_0^\infty \frac{x\operatorname{arccot} x}{1-x^4}\,dx = \frac{\pi}{8}\ln 2$$ BI (248)(12)

11. $$\int_0^\infty \frac{\operatorname{arccot} x}{x\sqrt{1+x^2}}\,dx = \int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{1+x^2}}\,dx = 2\boldsymbol{G}$$ BI (251)(3, 10)

12. $$\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$$ FI II 694

13. $$\int_0^1 \frac{x\arctan x\,dx}{\sqrt{(1+x^2)\left(1+k'^2x^2\right)}} = \frac{1}{k^2}\left[F\left(\frac{\pi}{4},k\right) - \frac{\pi}{2\sqrt{2\left(1+k'^2\right)}}\right]$$ BI (294)(14)

4.532

1. $$\int_0^1 x^p \arctan x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} - \beta\left(\frac{p}{2}+1\right)\right] \qquad [p>-2]$$ BI (229)(7)

2. $$\int_0^\infty x^p \arctan x\,dx = \frac{\pi}{2(p+1)}\operatorname{cosec}\frac{p\pi}{2} \qquad [-1>p>-2]$$ BI (246)(1)

3. $$\int_0^1 x^p \operatorname{arccot} x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} + \beta\left(\frac{p}{2}+1\right)\right] \qquad [p>-1]$$ BI (229)(8)

4. $$\int_0^\infty x^p \operatorname{arccot} x\,dx = -\frac{\pi}{2(p+1)}\operatorname{cosec}\frac{p\pi}{2} \qquad [-1<p<0]$$ BI (246)(2)

5. $$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^{2q}\arctan x\frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2}p}\frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)} \qquad [q>0]$$ BI (250)(10)

4.533

1. $$\int_0^\infty (1 - x\operatorname{arccot} x)\,dx = \frac{\pi}{4}$$ BI (246)(3)

2. $$\int_0^1 \left(\frac{\pi}{4} - \arctan x\right)\frac{dx}{1-x} = -\frac{\pi}{8}\ln 2 + \boldsymbol{G}$$ BI (232)(2)

3. $$\int_0^1 \left(\frac{\pi}{4} - \arctan x\right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}$$ BI (235)(25)

4. $$\int_0^1 \left(x \operatorname{arccot} x - \frac{1}{x} \arctan x\right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2$$ BI (232)(1)

4.534 $$\int_0^\infty (\arctan x)^2 \frac{dx}{x^2\sqrt{1+x^2}} = \int_0^\infty (\operatorname{arccot} x)^2 \frac{x\,dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4\boldsymbol{G}$$ BI (251)(9, 17)

4.535

1. $$\int_0^1 \frac{\arctan px}{1+p^2x}\,dx = \frac{1}{2p^2} \arctan p \ln\left(1+p^2\right)$$ BI (231)(19)

2. $$\int_0^1 \frac{\operatorname{arccot} px}{1+p^2x}\,dx = \frac{1}{p^2}\left\{\frac{\pi}{4} + \frac{1}{2}\operatorname{arccot} p\right\} \ln\left(1+p^2\right) \qquad [p>0]$$ BI (231)(24)

3. $$\int_0^\infty \frac{\arctan qx}{(p+x)^2}\,dx = -\frac{q}{1+p^2q^2}\left(\ln pq - \frac{\pi}{2}pq\right) \qquad [p>0, \quad q>0]$$ BI (249)(1)

4. $$\int_0^\infty \frac{\operatorname{arccot} qx}{(p+x)^2}\,dx = \frac{q}{1+p^2q^2}\left(\ln pq + \frac{\pi}{2pq}\right) \qquad [p>0, \quad q>0]$$ BI (249)(8)

5. $$\int_0^\infty \frac{x \operatorname{arccot} px}{q^2+x^2}\,dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \qquad [p>0, \quad q>0]$$ BI (248)(9)

6. $$\int_0^\infty \frac{x \operatorname{arccot} px\,dx}{x^2-q^2} = \frac{\pi}{4} \ln \frac{1+p^2q^2}{p^2q^2} \qquad [p>0, \quad q>0]$$ BI (248)(10)

7. $$\int_0^\infty \frac{\arctan px}{x\left(1+x^2\right)}\,dx = \frac{\pi}{2} \ln(1+p) \qquad [p \geq 0]$$ FI II 745

8. $$\int_0^\infty \frac{\arctan px}{x\left(1-x^2\right)}\,dx = \frac{\pi}{4} \ln\left(1+p^2\right) \qquad [p \geq 0]$$ BI (250)(6)

9. $$\int_0^\infty \arctan qx \frac{dx}{x\left(p^2+x^2\right)} = \frac{\pi}{2p^2} \ln(1+pq) \qquad [p>0, \quad q \geq 0]$$ BI (250)(3)

10. $$\int_0^\infty \arctan qx \frac{dx}{x\left(1-p^2x^2\right)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \qquad [p \geq 0]$$ BI (250)(6)

11. $$\int_0^\infty \frac{x \arctan qx}{\left(p^2+x^2\right)^2}\,dx = \frac{\pi q}{4p(1+pq)} \qquad [p>0, \quad q \geq 0]$$ BI (252)(12)a

12. $$\int_0^\infty \frac{x \operatorname{arccot} qx}{\left(p^2+x^2\right)^2}\,dx = \frac{\pi}{4p^2(1+pq)} \qquad [p>0, \quad q \geq 0]$$ BI (252)(20)a

13. $$\int_0^1 \frac{\arctan qx}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2} \ln\left(q + \sqrt{1+q^2}\right)$$ BI (244)(11)

14.[9] $$\int_{-\infty}^\infty \frac{x \arctan(\alpha x)\,dx}{\left(x^2+\beta^2\right)\left(x^2+\gamma^2\right)} = \begin{cases} \dfrac{\pi}{\beta^2-\gamma^2} \ln\left(\dfrac{1+|\alpha\beta|}{1+|\alpha\gamma|}\right) \operatorname{sign}(\alpha) & \text{for } \beta \neq \gamma \\ \dfrac{\pi\alpha}{2|\beta|\left(1+|\alpha\beta|\right)} & \text{for } \beta = \gamma \end{cases}$$

for α, β, γ real

15.[9] $$\int_{-\infty}^{\infty} \frac{x \arctan(\alpha/x)\, dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \dfrac{\pi}{\beta^2-\gamma^2} \ln\left(\dfrac{1+|\alpha/\gamma|}{1+|\alpha/\beta|}\right) \operatorname{sign}(\alpha) & (\alpha,\beta,\gamma \text{ real}; \quad \beta \neq \gamma) \\ \dfrac{\pi\alpha}{2\beta^2(|\beta|+|\alpha|)} & (\beta=\gamma) \end{cases}$$

4.536

1. $$\int_0^\infty \arctan qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q\pi \ln \frac{1+\sqrt{1+q^2}}{\sqrt{1+q^2}} + \frac{\pi}{2} \ln\left(q+\sqrt{1+q^2}\right) - \frac{\pi}{2} - \arctan q$$ BI (230)(7)

2. $$\int_0^\infty \frac{\arctan px - \arctan qx}{x}\, dx = \frac{\pi}{2} \ln \frac{p}{q} \qquad [p>0, \quad q>0]$$ FI II 635

3. $$\int_0^\infty \frac{\arctan px \arctan qx}{x^2}\, dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p q^q} \qquad [p>0, \quad q>0]$$ FI II 745

4.537

1.[8] $$\int_0^1 \arctan\left(\sqrt{1-x^2}\right) \frac{dx}{1-x^2\cos^2\lambda} = \frac{\pi}{2\cos\lambda} \ln\left[\cos\left(\frac{\pi-4\lambda}{8}\right) \operatorname{cosec}\left(\frac{\pi+4\lambda}{8}\right)\right]$$ BI (245)(9)

2. $$\int_0^1 \arctan\left(p\sqrt{1-x^2}\right) \frac{dx}{1-x^2} = \frac{1}{2}\pi \ln\left(p+\sqrt{1+p^2}\right)$$ $[p>0]$ BI (245)(10)

3. $$\int_0^1 \arctan\left(\tan\lambda\sqrt{1-k^2}x^2\right) \sqrt{\frac{1-x^2}{1-k^2x^2}}\, dx = \frac{\pi}{2k^2}\left[E(\lambda,k) - k'^2 F(\gamma,k)\right] - \frac{\pi}{2k^2}\cot\gamma\left(1-\sqrt{1-k^2\sin^2\gamma}\right)$$ BI (245)(12)

4. $$\int_0^1 \arctan\left(\tan\lambda\sqrt{1-k^2x^2}\right) \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx = \frac{\pi}{2} E(\lambda,k) - \frac{\pi}{2}\cot\lambda\left(1-\sqrt{1-k^2\sin^2\lambda}\right)$$ BI (245)(11)

5. $$\int_0^1 \frac{\arctan\left(\tan\lambda\sqrt{1-k^2x^2}\right)}{\sqrt{(1-x^2)(1-k^2x^2)}}\, dx = \frac{\pi}{2} F(\lambda,k)$$ BI (245)(13)

4.538

1. $$\int_0^\infty \arctan x^2 \frac{dx}{1+x^2} = \int_0^\infty \arctan x^3 \frac{dx}{1+x^2}$$ BI (252)(10, 11)
$$= \int_0^\infty \operatorname{arccot} x^2 \frac{dx}{1+x^2} = \int_0^\infty \operatorname{arccot} x^3 \frac{dx}{1+x^2} = \frac{\pi^2}{8}$$ BI (252)(18, 19)

2. $$\int_0^\infty \frac{1-x^2}{x^2} \arctan x^2\, dx = \frac{\pi}{2}\left(\sqrt{2}-1\right)$$ BI (244)(10)a

4.539 $$\int_0^\infty x^{s-1} \arctan\left(ae^{-x}\right) dx = 2^{-s-1}\Gamma(s) a\, \Phi\left(-a^2, s+1, \tfrac{1}{2}\right)$$ ET I 222(47)

4.541 $$\int_0^\infty \arctan\left(\frac{p\sin qx}{1+p\cos qx}\right) \frac{x\, dx}{1+x^2} = \frac{\pi}{2} \ln\left(1+pe^{-q}\right) \quad [p>-e^q]$$ BI (341)(14)a

4.55 Combinations of inverse trigonometric functions and exponentials

4.551

1.[9] $$\int_0^1 (\arcsin x)\, e^{-bx}\, dx = \frac{\pi}{2b}\left[I_0(b) - \mathbf{L}_0(b)\right] - \frac{\pi e^{-b}}{2b}$$ ET I 160(1)

2. $$\int_0^1 x\,(\arcsin x)\, e^{-bx}\, dx = \frac{\pi}{2b^2}\left[\mathbf{L}_0(b) - I_0(b) + b\,\mathbf{L}_1(b) - b\,I_1(b)\right] + \frac{1}{b}$$ ET I 161(2)

3.[9] $$\int_0^\infty \left(\arctan\frac{x}{a}\right) e^{-bx}\, dx = \frac{1}{b}\left[\mathrm{ci}(ab)\sin(ab) - \mathrm{si}(ab)\cos(ab)\right]$$
$[\operatorname{Re} b > 0]$ ET I 161(3)

4.[9] $$\int_0^\infty \left(\operatorname{arccot}\frac{x}{a}\right) e^{-bx}\, dx = \frac{1}{b}\left[\frac{\pi}{2} - \mathrm{ci}(ab)\sin(ab) + \mathrm{si}(ab)\cos(ab)\right]$$
$[\operatorname{Re} b > 0]$ ET I 161(4)

4.552 $$\int_0^\infty \frac{\arctan\dfrac{x}{q}}{e^{2\pi x} - 1}\, dx = \frac{1}{2}\left[\ln\Gamma(q) - \left(q - \frac{1}{2}\right)\ln q + q - \frac{1}{2}\ln 2\pi\right]$$
$[q > 0]$ WH

4.553 $$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arccot} x - e^{-px}\right)\frac{dx}{x} = \boldsymbol{C} + \ln p \qquad [p > 0]$$ NT 66(12)

4.56 A combination of the arctangent and a hyperbolic function

4.561 $$\int_{-\infty}^\infty \frac{\arctan e^{-x}}{\cosh^{2q} px}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\Pi(x)}{\cosh^{2q} px}\, dx = \frac{\sqrt{\pi^3}}{4p}\,\frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)}$$
$[q > 0]$ LI (282)(10)

4.57 Combinations of inverse and direct trigonometric functions

4.571 $$\int_0^{\pi/2} \arcsin(k\sin x)\,\frac{\sin x\, dx}{\sqrt{1 - k^2\sin^2 x}} = -\frac{\pi}{2k}\ln k'$$ BI (344)(2)

4.572 $$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arccot} x - \cos px\right) dx = \boldsymbol{C} + \ln p \qquad [p > 0]$$ NT 66(12)

4.573

1. $$\int_0^\infty \operatorname{arccot} qx \sin px\, dx = \frac{\pi}{2p}\left(1 - e^{-\frac{p}{q}}\right) \qquad [p > 0, \quad q > 0]$$ BI (347)(1)a

2. $$\int_0^\infty \operatorname{arccot} qx \cos px\, dx = \frac{1}{2p}\left[e^{-\frac{p}{q}}\,\mathrm{Ei}\left(\frac{p}{q}\right) - e^{\frac{p}{q}}\,\mathrm{Ei}\left(-\frac{p}{q}\right)\right]$$
$[p > 0, \quad q > 0]$ BI (347)(2)a

3. $$\int_0^\infty \operatorname{arccot} rx \frac{\sin px\, dx}{1 \pm 2q\cos px + q^2} = \pm\frac{\pi}{2pq}\ln\frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}} \qquad \left[p^2 < 1, \quad r > 0, \quad p > 0\right]$$
$$= \pm\frac{\pi}{2pq}\ln\frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \qquad \left[q^2 > 1, \quad r > 0, \quad p > 0\right]$$
BI (347)(10)

4. $$\int_0^\infty \operatorname{arccot} px \frac{\tan x\,dx}{q^2\cos^2 x + r^2\sin^2 x} = \frac{\pi}{2r^2}\ln\left(1+\frac{r}{q}\tanh\frac{1}{p}\right)$$
$[p>0, \quad q>0, \quad r>0]$ BI (347)(9)

4.574

1. $$\int_0^\infty \arctan\left(\frac{2a}{x}\right)\sin(bx)\,dx = \frac{\pi}{b}e^{-ab}\sinh(ab)$$
$[\operatorname{Re} a>0, \quad b>0]$ ET I 87(8)

2.[7] $$\int_0^\infty \arctan\frac{a}{x}\cos(bx)\,dx = \frac{1}{2b}\left[e^{-ab}\operatorname{Ei}(ab) - e^{ab}\operatorname{Ei}(-ab)\right]$$
$[a>0, \quad b>0]$ ET I 29(7)

3. $$\int_0^\infty \arctan\left[\frac{2ax}{x^2+c^2}\right]\sin(bx)\,dx = \frac{\pi}{b}e^{-b\sqrt{a^2+c^2}}\sinh(ab)$$
$[b>0]$ ET I 87(9)

4. $$\int_0^\infty \arctan\left(\frac{2}{x^2}\right)\cos(bx)\,dx = \frac{\pi}{b}e^{-b}\sin b$$
$[b>0]$ ET I 29(8)

4.575

1. $$\int_0^\pi \arctan\frac{p\sin x}{1-p\cos x}\sin nx\,dx = \frac{\pi}{2n}p^n$$
$[p^2<1]$ BI (345)(4)

2. $$\int_0^\pi \arctan\frac{p\sin x}{1-p\cos x}\sin nx\cos x\,dx = \frac{\pi}{4}\left(\frac{p^{n+1}}{n+1}+\frac{p^{n-1}}{n-1}\right)$$
$[p^2<1]$ BI (345)(5)

3. $$\int_0^\pi \arctan\frac{p\sin x}{1-p\cos x}\cos nx\sin x\,dx = \frac{\pi}{4}\left(\frac{p^{n+1}}{n+1}-\frac{p^{n-1}}{n-1}\right)$$
$[p^2<1]$ BI (345)(6)

4.576

1. $$\int_0^\pi \arctan\frac{p\sin x}{1-p\cos x}\frac{dx}{\sin x} = \frac{\pi}{2}\ln\frac{1+p}{1-p}$$
$[p^2<1]$ BI(346)(1)

2. $$\int_0^\pi \arctan\frac{p\sin x}{1-p\cos x}\frac{dx}{\tan x} = -\frac{\pi}{2}\ln\left(1-p^2\right)$$
$[p^2<1]$ BI(346)(3)

4.577

1. $$\int_0^{\pi/2} \arctan\left(\tan\lambda\sqrt{1-k^2\sin^2 x}\right)\frac{\sin^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2k^2}\left[F(\lambda,k)-E(\lambda,k)+\cot\lambda\left(1-\sqrt{1-k^2\sin^2\lambda}\right)\right]$$
BI (344)(4)

2. $$\int_0^{\pi/2} \arctan\left(\tan\lambda\sqrt{1-k^2\sin^2 x}\right)\frac{\cos^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2k^2}\left[E(\lambda,k)-k'^2F(\lambda,k)+\cot\lambda\left(\sqrt{1-k^2\sin^2\lambda}-1\right)\right]$$
BI (344)(5)

4.58 A combination involving an inverse and a direct trigonometric function and a power

4.581[10] $$\int_0^\infty \arctan x \cos px \frac{dx}{x} = \int_0^\infty \arctan \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2}\,\mathrm{Ei}(-p)$$
$[\mathrm{Re}(p) > 0]$ ET I 29(3), NT 25(13)

4.59 Combinations of inverse trigonometric functions and logarithms

4.591

1. $$\int_0^1 \arcsin x \ln x\, dx = 2 - \ln 2 - \frac{1}{2}\pi$$ BI (339)(1)

2. $$\int_0^1 \arccos x \ln x\, dx = \ln 2 - 2$$ BI (339)(2)

4.592 $$\int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1}$$ BI (339)(8)

4.593

1. $$\int_0^1 \arctan x \ln x\, dx = \frac{1}{2}\ln 2 - \frac{\pi}{4} + \frac{1}{48}\pi^2$$ BI (339)(3)

2. $$\int_0^1 \operatorname{arccot} x \ln x\, dx = -\frac{1}{48}\pi^2 - \frac{\pi}{4} - \frac{1}{2}\ln 2$$ BI (339)(4)

4.594 $$\int_0^1 \arctan x\, (\ln x)^{n-1} (\ln x + n)\, dx = \frac{n!}{(-2)^{n+1}} \left(2^{-n} - 1\right) \zeta(n+1)$$ BI (339)(7)

4.6 Multiple Integrals

4.60 Change of variables in multiple integrals

4.601

1. $$\iint\limits_{(\sigma)} f(x,y)\, dx\, dy = \iint\limits_{(\sigma')} f\left[\varphi(u,v), \psi(u,v)\right] |\Delta|\, du\, dv$$

   where $x = \varphi(u,v), y = \psi(u,v)$, and $\Delta = \dfrac{\partial\varphi}{\partial u}\dfrac{\partial\psi}{\partial v} - \dfrac{\partial\psi}{\partial u}\dfrac{\partial\varphi}{\partial v} \equiv \dfrac{D(\varphi,\psi)}{D(u,v)}$ is the Jacobian determinant of the functions φ and ψ.

2. $$\iiint\limits_{(V)} f(x,y,z)\, dx\, dy\, dz = \iiint\limits_{(V')} f\left[\varphi(u,v,w), \psi(u,v,w), \chi(u,v,w)|\Delta|\, du\, dv\, dw\right]$$

   where $x = \varphi(u,v,w)$, $y = \psi(u,v,w)$, and $z = \chi(u,v,w)$ and where

   $$\Delta = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial w} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\chi}{\partial u} & \frac{\partial\chi}{\partial v} & \frac{\partial\chi}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi,\psi,\chi)}{D(u,v,w)}$$

   is the Jacobian determinant of the functions φ, ψ, and χ.

   Here, we assume, both in (**4.601** 1) and in (**4.601** 2) that

(a) the functions φ, ψ, and χ and also their first partial derivatives are continuous in the region of integration;

(b) the Jacobian does not change sign in this region;

(c) there exists a one-to-one correspondence between the old variables x, y,z and the new ones u, v,w in the region of integration;

(d) when we change from the variables x, y,z to the variables u, v,w, the region V (resp. σ) is mapped into the region V' (resp. σ').

4.602 Transformation to polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi; \quad \frac{D(x,y)}{D(r,\varphi)} = r$$

4.603 Transformation to spherical coordinates:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \quad \frac{D(x,y,z)}{D(r,\theta,\varphi)} = r^2\sin\theta$$

4.61 Change of the order of integration and change of variables

4.611

1. $$\int_0^\alpha dx \int_0^x f(x,y)\,dy = \int_0^\alpha dy \int_y^\alpha f(x,y)\,dx$$

2. $$\int_0^\alpha dx \int_0^{\frac{\beta}{\alpha}x} f(x,y)\,dy = \int_0^\beta dy \int_{\frac{\alpha}{\beta}y}^\alpha f(x,y)\,dx$$

4.612

1. $$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f(x,y)\,dx$$

2. $$\int_0^{2p} dx \int_0^{q/p\sqrt{2px-x^2}} f(x,y)\,dy = \int_0^q dy \int_{p\left[1-\sqrt{1-(y/q)^2}\right]}^{p\left[1+\sqrt{1-(y/q)^2}\right]} f(x,y)\,dx$$

4.613

1. $$\int_0^{\alpha} dx \int_0^{\beta/(\beta+x)} f(x,y)\,dy = \int_0^{\beta/(\beta+\alpha)} dy \int_0^{\alpha} f(x,y)\,dx + \int_{\beta/(\beta+\alpha)}^{1} dy \int_0^{\beta(1-y)/y} f(x,y)\,dx$$

2. $$\int_0^{\alpha} dx \int_{\beta x}^{\delta-\nu x} f(x,y)\,dy = \int_0^{\alpha\beta} dy \int_0^{y/\beta} f(x,y)\,dx + \int_{\alpha\beta}^{\delta} dy \int_0^{(\delta-y)/\gamma} f(x,y)\,dx$$

$$\left[\alpha = \frac{\delta}{\beta+\gamma}, \quad a > 0, \quad \beta > 0, \quad \gamma > 0\right]$$

3. $$\int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha-x} f(x,y)\,dy = \int_0^{\alpha} dy \int_0^{2\sqrt{\alpha y}} f(x,y)\,dx + \int_{\alpha}^{3\alpha} dy \int_0^{3\alpha-y} f(x,y)\,dx$$

4. $$\int_0^{R} dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x,y)\,dy = \int_0^{R} dy \int_{\sqrt{R^2-y^2}}^{R} f(x,y)\,dx + \int_{R}^{2R} dy \int_0^{R} f(x,y)\,dx + \int_{2R}^{3R} dy \int_{y-2R}^{R} f(x,y)\,dx$$

4.614 $$\int_0^{\pi/2} d\varphi \int_0^{2R\cos\varphi} f(r,\varphi)\,dr = \int_0^{2R} dr \int_0^{\arccos\frac{r}{2R}} f(r,\varphi)\,d\varphi$$

4.615 $$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^{\pi/2} d\varphi \int_0^R f\left(r\cos\varphi, r\sin\varphi\right) r\,dr$$

4.616 $$\int_0^{2R} dx \int_0^{\sqrt{2R-x^2}} f(x,y)\,dy = \int_0^{\pi/2} d\varphi \int_0^{2R\cos\varphi} f\left(r\cos\varphi, r\sin\varphi\right) r\,dr$$

4.617 $$\int_\alpha^\beta dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy = \int_0^\beta dx \int_0^{\varphi_2(x)} f(x,y)dy - \int_0^\beta dx \int_0^{\varphi_1(x)} f(x,y)dy - \int_0^\alpha dx \int_0^{\varphi_2(x)} f(x,y)dy + \int_0^\alpha dx \int_0^{\varphi_1(x)} f(x,y)\,dy$$

$[\varphi_1(x) \le \varphi_2(x)$ for $\alpha \le x \le \beta]$

4.618 $$\int_0^\gamma dx \int_0^{\varphi(x)} f(x,y)\,dy = \int_0^\gamma dx \int_0^1 f\left[x, z\varphi(x)\right]\varphi(x)dz \qquad [y = z\varphi(x)]$$

$$= \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y)\,dy \qquad [x = \gamma z]$$

4.619 $$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x,y)\,dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0)\, f\left[x, y_0 + (y_1 - y_0)\,t\right]\,dt$$

$[y = y_0 + (y_1 - y_0)\,t]$

4.62 Double and triple integrals with constant limits

4.620 General formulas

1. $$\int_0^\pi d\omega \int_0^\infty f'\left(p\cosh x + q\cos\omega\sinh x\right)\sinh x\,dx = -\frac{\pi\,\mathrm{sign}\,p}{\sqrt{p^2-q^2}} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2}\right)$$

$$\left[p^2 > q^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 389

2. $$\int_0^{2\pi} d\omega \int_0^{\infty} f'\left[p\cosh x + (q\cos\omega + r\sin\omega)\sinh x\right]\sinh x\,dx = -\frac{2\pi\,\mathrm{sign}\,p}{\sqrt{p^2-q^2-r^2}} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

3. $$\int_0^{\pi}\int_0^{\pi} \frac{dx\,dy}{\sin x\sin^2 y} f'\left[\frac{p-q\cos x}{\sin x\sin y} + r\cot y\right] = -\frac{2\pi\,\mathrm{sign}\,p}{\sqrt{p^2-q^2-r^2}} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 280

4. $$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f'\left(p\cosh x\cosh y + q\sinh x\cosh y + r\sinh y\right)\cosh y\,dy = -\frac{2\pi\,\mathrm{sign}\,p}{\sqrt{p^2-q^2-r^2}} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

5. $$\int_0^{\infty} dx \int_0^{\pi} f\left(p\cosh x + q\cos\omega\sinh x\right)\sinh^2 x\sin\omega\,d\omega = 2\int_0^{\infty} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[\lim_{x\to+\infty} f(x) = 0\right]$$ LO III 391

6. $$\int_0^{\infty} dx \int_0^{2\pi} d\omega \int_0^{\pi} f\left[p\cosh x + (q\cos\omega + r\sin\omega)\sin\theta\sinh x\right]\sinh^2 x\sin\theta\,d\theta = 4\int_0^{\infty} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2-r^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

7. $$\int_0^{\infty} dx \int_0^{2\pi} d\omega \int_0^{\pi} f\left\{p\cosh x + \left[(q\cos\omega + r\sin\omega)\sin\theta + s\cosh\theta\right]\sinh x\right\}\sinh^2 x\sin\theta\,d\theta = 4\pi\int_0^{\infty} f\left(\mathrm{sign}\,p\sqrt{p^2-q^2-r^2-s^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[p^2 > q^2+r^2+s^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 391

4.621

1. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\,dx\,dy = \frac{\pi}{2\sqrt{1-k^2}}$$ LO I 252(90)

2. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\cos y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\,dx\,dy = \boldsymbol{K}(k)$$ LO I 252(91)

3. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin\alpha\sin y\,dx\,dy}{\sqrt{1-\sin^2\alpha\sin^2 x\sin^2 y}} = \frac{\pi\alpha}{2}$$ LO I 253

4.622

1. $$\int_0^\pi\int_0^\pi\int_0^\pi \frac{dx\,dy\,dz}{1-\cos x\cos y\cos z} = 4\pi\,\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right)$$ MO 137

2. $$\int_0^\pi\int_0^\pi\int_0^\pi \frac{dx\,dy\,dz}{3-\cos y\cos z-\cos x\cos z-\cos x\cos y} = \sqrt{3}\pi\,\boldsymbol{K}^2\left(\sin\frac{\pi}{12}\right)$$ MO 137

3. $$\int_0^\pi\int_0^\pi\int_0^\pi \frac{dx\,dy\,dz}{3-\cos x-\cos y-\cos z} = 4\pi\left[18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right]\boldsymbol{K}^2\left[\left(2-\sqrt{3}\right)\left(\sqrt{3}-\sqrt{2}\right)\right]$$ MO 137

4.623[3] $$\int_0^\infty\int_0^\infty \varphi\left(a^2x^2+b^2y^2\right)\,dx\,dy = \frac{\pi}{2ab}\int_0^\infty \varphi\left(x^2\right)x\,dx$$

4.624 $$\int_0^\pi\int_0^{2\pi} f\left(\alpha\cos\theta+\beta\sin\theta\cos\psi+\gamma\sin\theta\sin\psi\right)\sin\theta\,d\theta\,d\psi$$
$$= 2\pi\int_0^\pi f\left(R\cos p\right)\sin p\,dp = 2\pi\int_{-1}^1 f(Rt)\,dt$$
$$\left[R=\sqrt{\alpha^2+\beta^2+\gamma^2}\right]$$

4.625[8] $$p_l(a,b) = \int_0^a dx\int_0^b dy\left(x^2+y^2+1\right)^{-3/2}P_l\left(1/\sqrt{x^2+y^2+1}\right)$$

Then, for even and odd subscripts:

- $$p_{2l}(a,b) = \frac{1}{l(2l+1)2^{2l}}\frac{ab}{\sqrt{a^2+b^2+1}}\sum_{k=0}^{l-1}\frac{(-1)^{l-k-1}2^{2k}\binom{2l+2k}{l+k}\binom{l+k}{l-k-1}}{\binom{2k}{k}(2k+1)}$$
$$\times(2l+2k+1)\sum_{j=0}^{k}\frac{\binom{2j}{j}}{2^{2j}}\frac{1}{(a^2+b^2+1)^j}\left(\frac{1}{(a^2+1)^{k-j+1}}+\frac{1}{(b^2+1)^{k-j+1}}\right)$$
- $$p_{2l+1}(a,b) = \frac{1}{2^{2l+1}(2l+1)}\sum_{k=0}^{l}\frac{(-1)^{l+k}}{2^{2k}}\binom{l}{k}\binom{l+k+1}{k}\binom{2l+2k+1}{l+k}$$
$$\times\left\{\frac{1}{(b^2+1)^k}\frac{b}{\sqrt{b^2+1}}\arctan^{-1}\frac{a}{\sqrt{b^2+1}}+\frac{1}{(a^2+1)^k}\frac{a}{\sqrt{a^2+1}}\arctan^{-1}\frac{b}{\sqrt{a^2+1}}\right.$$
$$\left.+ab\sum_{j=1}^{k}\frac{2^{2j-1}}{j\binom{2j}{j}}\cdot\frac{1}{(a^2+b^2+1)^j}\left(\frac{1}{(a^2+1)^{k-j+1}}+\frac{1}{(b^2+1)^{k-j+1}}\right)\right\}$$

4.63–4.64 Multiple integrals

4.631 $$\int_p^x dt_{n-1}\int_p^{t_{n-1}} dt_{n-2}\ldots\int_p^{t_1} f(t)\,dt = \frac{1}{(n-1)!}\int_p^x (x-t)^{n-1}f(t)\,dt,$$
where $f(t)$ is continuous on the interval $[p,q]$ and $p\le x\le q$. FI II 692

4.632

1. $$\int\!\!\int\cdots\int\limits_{\substack{x_1\ge 0, x_2\ge 0,\dots,x_n\ge 0\\ x_1+x_2+\dots+x_n\le h}} dx_1\,dx_2\cdots dx_n=\frac{h^n}{n!}$$

[the volume of an n-dimensional simplex] FI III 472

2. $$\int\!\!\int\cdots\int\limits_{x_1^2+x_2^2+\dots+x_n^2\le R^2} dx_1\,dx_2\cdots dx_n=\frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}+1\right)}R^n$$ [the volume of an n-dimensional sphere]

FI III 473

4.633 $$\int\!\!\int\cdots\int\limits_{x_1^2+x_2^2+\dots+x_n^2\le 1}\frac{dx_1\,dx_2\cdots dx_n}{\sqrt{1-x_1^2-x_2^2-\dots-x_n^2}}=\frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\qquad [n>1]$$

[half-area of the surface of an $(n+1)$-dimensional sphere $x_1^2+x_2^2+\dots+x_{n+1}^2=1$] FI III 474

4.634[8] $$\int\!\!\int\cdots\int\limits_{\substack{x_1\ge 0, x_2\ge 0,\dots,x_n\ge 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\dots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\le 1}} x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\dots dx_n$$

$$=\frac{q_1^{p_1}q_2^{p_2}\dots q_n^{p_n}}{\alpha_1\alpha_2\dots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}+1\right)}$$

$[\alpha_i>0,\quad p_i>0,\quad q_i>0,\quad i=1,2,\dots,n]$ FI III 477

4.635

1.[8] $$\int\!\!\int\cdots\int\limits_{\substack{x_1\ge 0, x_2\ge 0,\dots,x_n\ge 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\dots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\ge 1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\dots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$

$$\times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\cdots dx_n$$

$$=\frac{q_1^{p_1}q_2^{p_2}\dots q_n^{p_n}}{\alpha_1\alpha_2\cdots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}\right)}\int_1^\infty f(x)x^{\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\dots+\frac{p_n}{\alpha_n}-1}\,dx$$

under the assumption that the integral on the right converges absolutely. FI III 487

2.[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\cdots,x_n\ge0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\le1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$
$$\times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\cdots dx_n$$
$$=\frac{q_1^{p_1}q_2^{p_2}\dots q_n^{p_n}}{\alpha_1\alpha_2\dots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_0^1 f(x)x^{\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}-1}\,dx$$

under the assumptions that the one-dimensional integral on the right converges absolutely and that the numbers q_i,α_i, and p_i are positive. FI III 479

In particular,

3.
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\dots,x_n\ge0\\ x_1+x_2+\cdots+x_n\le1}} x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}e^{-q(x_1+x_2+\cdots+x_n)}\,dx_1\,dx_2\dots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 x^{p_1+p_2+\cdots+p_n-1}e^{-qx}\,dx$$
$$[n>0,\quad p_1>0,\quad p_2>0,\dots,p_n>0]$$

4.[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\cdots,x_n\ge0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\le1}} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(1-x_1^{\alpha_1}-x_2^{\alpha_2}-\cdots-x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n$$
$$=\frac{\Gamma(1-\mu)}{\alpha_1\alpha_2\dots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1-\mu+\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$[p_1>0,\quad p_2>0,\dots,p_n>0,\quad \mu<1]$$ FI III 480

4.636

1.[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\dots,x_n\ge0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\ge1}} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n$$
$$=\frac{1}{\alpha_1\alpha_2\dots\alpha_n\left(\mu-\frac{p_1}{\alpha_1}-\frac{p_2}{\alpha_2}-\cdots-\frac{p_n}{\alpha_n}\right)}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$\left[p_1>0,\quad p_2>0,\dots,p_n>0;\quad \mu>\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right]$$ FI III 488

2.[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\cdots,x_n\ge0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\le1}} \frac{x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}}{(x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{1}{\alpha_1\alpha_2\ldots\alpha_n\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}-\mu\right)}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$\left[\mu<\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right]$$ FI III 480

3.[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,x_2\ge0,\ldots,x_n\ge0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\le1}} x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}\sqrt{\frac{1-x_1^{\alpha_1}-x_2^{\alpha_2}-\cdots-x_n^{\alpha_n}}{1+x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}}}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_1}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\alpha_2\ldots\alpha_n}\frac{1}{\Gamma(m)}\left\{\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)}-\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right\},$$
where $m=\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}$. FI III 480

4.637[8]
$$\iint\cdots\int\limits_{\substack{x_1\ge0,\quad x_2\ge0,\ldots,x_n\ge0\\ x_1+x_2+\cdots+x_n\le1}} f(x_1+x_2+\cdots+x_n)\frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}\,dx_1\,dx_2\ldots dx_n}{(q_1x_1+q_2x_2+\cdots+q_nx_n+r)^{p_1+p_2+\cdots+p_n}}$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1p_2+\ldots p_n)}\int_0^1 f(x)\frac{x^{p_1p_2+\cdots p_n-1}}{(q_1x+r)^{p_1}(q_2x+r)^{p_2}\ldots(q_nx+r)^{p_n}}\,dx,$$
$$[q_1\ge0,\quad q_2\ge0,\ldots,q_n\ge0;\quad r>0]$$
where $f(x)$ is continuous on the interval $(0,1)$.

4.638

1.
$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}e^{-(q_1x_1+q_2x_2+\cdots+q_nx_n)}}{(r_0+r_1x_1+r_2x_2+\cdots+r_nx_n)^s}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(s)}\int_0^\infty\frac{e^{r_0x}x^{s-1}\,dx}{(q_1r_1x)^{p_1}(q_1r_2x)^{p_2}\ldots(q_nr_nx)^{p_n}}$$
where p_i, q_i, r_i, and s are positive. This result is also valid for $r_0=0$, provided $p_1+p_2+\cdots+p_n>s$.

2.
$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{(r_0+r_1x_1+r_2x_2+\cdots+r_nx_n)^s}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)\Gamma(sp_1p_2-\cdots-p_n)}{r_1^{p_1}r_2^{p_2}\cdots r_n^{p_n}r_0^{s-p_1-p_2-\cdots-p_n}\Gamma(s)}$$
$$[p_i>0,\quad r_i>0,\quad s>0]$$

3.[8]
$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{[1+(r_1x_1)^{q_1}+(r_2x_2)^{q_2}+\cdots+(r_nx_n)^{q_n}]^s}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\ldots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1q_2\ldots q_nr_1^{p_1q_1}r_2^{p_2q_2}\ldots r_n^{p_nq_n}}\frac{\Gamma\left(s-\frac{p_1}{q_1}-\frac{p_2}{q_2}-\cdots-\frac{p_n}{q_n}\right)}{\Gamma(s)}$$
$$[p_i>0,\quad q_i>0,\quad r_i>0,\quad s>0]$$

4.639

1. $$\iint\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1}\cdots\int (p_1x_1+p_2x_2+\cdots+p_nx_n)^{2m}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{(2m-1)!!}{2^m}\frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}+m+1\right)}\left(p_1^2+p_2^2+\cdots+p_n^2\right)^m$$
FI III 482

2. $$\iint\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1}\cdots\int (p_1x_1+p_2x_2+\cdots+p_nx_n)^{2m+1}\,dx_1\,dx_2\ldots dx_n=0$$ FI III 483

4.641

1.[11] $$\iint\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1}\cdots\int e^{p_1x_1+p_2x_2+\cdots+p_nx_n}\,dx_1\,dx_2\ldots dx_n$$
$$=\sqrt{\pi^n}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma\left(\frac{n}{2}+k+1\right)}\left(\frac{p_1^2+p_2^2+\cdots+p_n^2}{4}\right)^k$$
FI III 483

2. $$\iint\limits_{x_1^2+x_2^2+\cdots+x_{2n}^2\le 1}\cdots\int e^{p_1x_1p_2x_2+\cdots p_{2n}x_{2n}}\,dx_1\,dx_2\ldots dx_{2n}=\frac{(2\pi)^n\,I_n\left(\sqrt{p_1^2+p_2^2+\cdots+p_{2n}^2}\right)}{\left(p_1^2+p_2^2+\cdots+p_{2n}^2\right)^{n/2}}$$
FI III 483a

4.642 $$\iint\limits_{x_1^2+x_2^2+\cdots+x_n^2\le R^2}\cdots\int f\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right)dx_1\,dx_2\ldots dx_n=\frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)}\int_0^R x^{n-1}f(x)\,dx,$$
where $f(x)$ is a function that is continuous on the interval $(0,R)$. FI III 485

4.643 $$\int_0^1\int_0^1\ldots\int_0^1 f(x_1x_2\cdots x_n)(1-x_1)^{p_1-1}(1-x_2)^{p_2-1}\ldots(1-x_n)^{p_n-1}$$
$$\times x_2^{p_1}x_3^{p_1+p_2}\cdots x_n^{p_1+p_2+\cdots+p_{n-1}}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(x)(1-x)^{p_1+p_2+\cdots+p_n-1}\,dx$$
under the assumption that the integral on the right converges absolutely. FI III 488

4.644 $$\overbrace{\iint\limits_{x_1^2+x_2^2+\cdots+x_n^2=1}\cdots\int}^{n-1} f(p_1x_1+p_2x_2+\cdots+p_nx_n)\frac{dx_1\,dx_2\ldots dx_{n-1}}{|x_n|}$$
$$=2\iint\limits_{x_1^2+x_2^2+\cdots+x_{n-1}^2\le 1}\cdots\int f(p_1x_1+p_2x_2+\cdots+p_nx_n)\frac{dx_1\,dx_2\cdots dx_{n-1}}{\sqrt{1-x_1^2-x_2^2-\cdots-x_{n-1}^2}}$$
$$=\frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi f\left(\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\cos x\right)\sin^{n-2}x\,dx\qquad[n\ge 3]$$
where $f(x)$ is continuous on the interval $\left\{-\sqrt{p_1^2+p_2^2+\cdots+p_n^2},\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\right\}$. FI III 489

4.645 Suppose that two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are continuous in a closed, bounded region D and that the smallest and greatest values of the function g in D are m and M, respectively. Let $\varphi(u)$ denote a function that is continuous for $m \le u \le$M. We denote by $\psi(u)$ the integral

1. $$\psi(u) = \iint\limits_{m \le g(x_1, x_2, \ldots, x_n) \le u} \cdots \int f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n,$$

over that portion of the region D on which the inequality $m \le g(x_1, x_2, \ldots, x_n) \le u$ is satisfied. Then

2. $$\iint\limits_{m \le g(x_1, x_2, \ldots, x_n) \le M} \cdots \int f(x_1, x_2, \ldots, x_n)\, \varphi[g(x_1, x_2, \ldots, x_n)]\, dx_1\, dx_2 \ldots dx_n$$
$$= (S)\int_m^M \varphi(u) d\,\psi(u) = (R)\int_m^M \varphi(u) \frac{d\,\psi(u)}{du}\, du$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative $\frac{d\psi}{du}$ exists and is continuous, the Riemann integral on the right exists.

M may be $+\infty$ in formulas **4.645** 2, in which case $\int_m^{+\infty}$ should be understood to mean $\lim\limits_{M \to +\infty} \int_m^M$.

4.646[8] $$\iint\limits_{\substack{x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \\ x_1 + x_2 + \cdots + x_n \le 1}} \cdots \int \frac{x_1^{p_1 - 1} x_2^{p_2 - 1} \ldots x_n^{p_n - 1}}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)^r}\, dx_1\, dx_2 \ldots dx_n$$
$$= \frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1)\Gamma(r)} \int_0^\infty \frac{x^{r-1}\, dx}{(1 + q_1 x)^{p_1} (1 + q_2 x)^{p_2} \cdots (1 + q_n x)^{p_n}}$$
$$= [p_1 > 0, \quad p_2 > 0, \ldots, p_n > 0, \quad q_1 > 0, \quad q_2 > 0, \ldots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0]$$

FI III 493

4.647 $$\iint\limits_{0 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le 1} \cdots \int \exp\left\{\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}\right\} dx_1\, dx_2 \ldots dx_n$$
$$= \frac{2\sqrt{\pi^n}}{n\,(p_1^2 + p_2^2 + \cdots + p_n^2)^{\frac{n}{4} - \frac{1}{2}}}\, I_{\frac{n}{2} - 1}\left(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\right)$$

FI III 495

4.648[8] $$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp\left[-\left(x_1 + x_2 + \cdots + x_n + \frac{\lambda^{n+1}}{x_1 x_2 \ldots x_n}\right)\right]$$
$$\times x c_1^{\frac{1}{n+1} - 1} x_2^{\frac{2}{n+1} - 1} \ldots x_n^{\frac{n}{n+1} - 1}\, dx_1\, dx_2 \cdots dx_n$$
$$= \frac{1}{\sqrt{n+1}} (2\pi)^{\frac{n}{2}} e^{-(n+1)\lambda}$$

FI III 496

This page intentionally left blank

5 Indefinite Integrals of Special Functions

5.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

5.11 Complete elliptic integrals

5.111

1. $$\int \boldsymbol{K}(k)k^{2p+3}\,dk = \frac{1}{(2p+3)^2}\left\{4(p+1)^2\int \boldsymbol{K}(k)k^{2p+1}\,dk + k^{2p+2}\left[\boldsymbol{E}(k)-(2p+3)\,\boldsymbol{K}(k)k'^2\right]\right\}$$ BY (610.04)

2. $$\int \boldsymbol{E}(k)k^{2p+3}\,dk = \frac{1}{4p^2+16p+15}\left\{4(p+1)^2\int \boldsymbol{E}(k)k^{2p+1}\,dk - \boldsymbol{E}(k)k^{2p+2}\left[(2p+3)k'^2-2\right] - k^{2p+2}k'^2\,\boldsymbol{K}(k)\right\}$$ BY (611.04)

5.112

1. $$\int \boldsymbol{K}(k)\,dk = \frac{\pi k}{2}\left[1+\sum_{j=1}^{\infty}\frac{[(2j)!]^2\,k^{2j}}{(2j+1)2^{4j}\,(j!)^4}\right]$$ BY (610.00)

2.[6] $$\int \boldsymbol{E}(k)\,dk = \frac{\pi k}{2}\left[1-\sum_{j=1}^{\infty}\frac{[(2j)!]^2\,k^{2j}}{(4j^2-1)\,2^{4j}\,(j!)^4}\right]$$ BY (611.00)

3. $$\int \boldsymbol{K}(k)k\,dk = \boldsymbol{E}(k)-k'^2\,\boldsymbol{K}(k)$$ BY (610.01)

4. $$\int \boldsymbol{E}(k)k\,dk = \frac{1}{3}\left[\left(1+k^2\right)\boldsymbol{E}(k)-k'^2\,\boldsymbol{K}(k)\right]$$ BY (611.01)

5. $$\int \boldsymbol{K}(k)k^3\,dk = \frac{1}{9}\left[\left(4+k^2\right)\boldsymbol{E}(k)-k'^2\left(4+3k^2\right)\boldsymbol{K}(k)\right]$$ BY (610.02)

6. $$\int \boldsymbol{E}(k)k^3\,dk = \frac{1}{45}\left[\left(4+k^2+9k^4\right)\boldsymbol{E}(k) - k'^2\left(4+3k^2\right)\boldsymbol{K}(k)\right]$$ BY 611.02)

7. $$\int \boldsymbol{K}(k)k^5\,dk = \frac{1}{225}\left[\left(64+16k^2+9k^4\right)\boldsymbol{E}(k) - k'^2\left(64+48k^2+45k^4\right)\boldsymbol{K}(k)\right]$$ BY (610.03)

8. $$\int \boldsymbol{E}(k)k^5\,dk = \frac{1}{1575}\left[\left(64+16k^2+9k^4+225k^6\right)\boldsymbol{E}(k) - k'^2\left(64+48k^2+45k^4\right)\boldsymbol{K}(k)\right]$$ BY (611.03)

9. $$\int \frac{\boldsymbol{K}(k)}{k^2}\,dk = -\frac{\boldsymbol{E}(k)}{k}$$ BY (612.05)

10. $$\int \frac{\boldsymbol{E}(k)}{k^2}\,dk = \frac{1}{k}\left[k'^2\,\boldsymbol{K}(k) - 2\,\boldsymbol{E}(k)\right]$$ BY (612.02)

11. $$\int \frac{\boldsymbol{E}(k)}{k'^2}\,dk = k\,\boldsymbol{K}(k)$$ BY (612.01)

12. $$\int \frac{\boldsymbol{E}(k)}{k^4}\,dk = \frac{1}{9k^3}\left[2\left(k^2-2\right)\boldsymbol{E}(k) + k'^2\,\boldsymbol{K}(k)\right]$$ BY (612.03)

13. $$\int \frac{k\,\boldsymbol{E}(k)}{k'^2}\,dk = \boldsymbol{K}(k) - \boldsymbol{E}(k)$$ BY (612.04)

5.113

1. $$\int \left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]\frac{dk}{k} = -\,\boldsymbol{E}(k)$$ BY (612.06)

2. $$\int \left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{k} = 2\,\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)$$ BY (612.09)

3. $$\int \left[\left(1+k^2\right)\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]\frac{dk}{k} = -k'^2\,\boldsymbol{K}(k)$$ BY (612.12)

4. $$\int \left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]\frac{dk}{k^2} = \frac{1}{k}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BY (612.07)

5. $$\int \left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{k^2k'^2} = \frac{1}{k}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$

6. $$\int \left[\left(1+k^2\right)\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{kk'^4} = \frac{\boldsymbol{E}(k)}{k'^2}$$ BY (612.13)

5.114 $$\int \frac{k\,\boldsymbol{K}(k)\,dk}{\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]^2} = \frac{1}{k'^2\,\boldsymbol{K}(k) - \boldsymbol{E}(k)}$$ BY (612.11)

5.115

1. $$\int \Pi\left(\frac{\pi}{2}, r^2, k\right)k\,dk = \left(k^2-r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right) - \boldsymbol{K}(k) + \boldsymbol{E}(k)$$ BY (612.14)

2. $$\int \left[\boldsymbol{K}(k) - \Pi\left(\frac{\pi}{2}, r^2, k\right)\right]k\,dk = k^2\,\boldsymbol{K}(k) - \left(k^2-r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right)$$ BY (612.15)

3. $$\int \left[\frac{\boldsymbol{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2}, r^2, k\right)\right]k\,dk = \left(k^2-r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right)$$ BY (612.16)

5.12 Elliptic integrals

5.121 $$\int_0^x \frac{F(x,k)\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{[F(x,k)]^2}{2} \qquad \left[0 < x \le \frac{\pi}{2}\right]$$ BY (630.01)

5.122[11] $$\int_0^x E(x,k)\sqrt{1-k^2\sin^2 x}\,dx = \frac{[E(x,k)]^2}{2}$$ BY (630.32)

5.123

1. $$\int_0^x F(x,k)\sin x\,dx = -\cos x\,F(x,k) + \frac{1}{k}\arcsin(k\sin x)$$ BY (630.11)

2. $$\int_0^x F(x,k)\cos x\,dx = \sin x\,F(x,k) + \frac{1}{k}\operatorname{arccosh}\sqrt{\frac{1-k^2\sin^2 x}{k'^2}} - \frac{1}{k}\operatorname{arccosh}\left(\frac{1}{k'}\right)$$ BY (630.21)

5.124

1. $$\int_0^x E(x,k)\sin x\,dx = -\cos x\,E(x,k) + \frac{1}{2k}\left[k\sin x\sqrt{1-k^2\sin^2 x} + \arcsin(k\sin x)\right]$$ BY (630.12)

2. $$\int_0^x E(x,k)\cos x\,dx = \sin x\,E(x,k) + \frac{1}{2k}\left[k\cos x\sqrt{1-k^2\sin^2 x} - k'^2\operatorname{arccosh}\sqrt{\frac{1-k^2\sin^2 x}{k'^2}} - k + k'^2\operatorname{arccosh}\left(\frac{1}{k'}\right)\right]$$ BY (630.22)

3.* $$\int_0^a \frac{x\,\boldsymbol{E}(x)\,dx}{(k'^2+k^2x^2)^2\sqrt{a^2-x^2}} = \frac{\pi}{4}\left(\frac{a\sqrt{1-a^2}}{(k'^2+k^2a^2)^2} + \frac{a^2\,E(\lambda,k)}{k'^2\,(k'^2+k^2a^2)^{3/2}} + \frac{(1-a^2)\,F(\lambda,k)}{(k'^2+k^2a^2)^{3/2}}\right)$$
$$\lambda = \arcsin\left(\frac{a}{\sqrt{k'^2+k^2a^2}}\right) \qquad k' = \sqrt{1-k^2} \qquad [0 < a < 1, \quad 0 < k < 1]$$

4.* $$\int_0^a \frac{x\,\boldsymbol{E}(x)\,dx}{(k^2-x^2)^2\sqrt{a^2-x^2}} = \frac{\pi}{4}\left(\frac{a\sqrt{1-a^2}}{k^2\,(k^2-a^2)} + \frac{F(\phi,k)}{k^2\sqrt{k^2-a^2}} + \frac{a^2\,E(\phi,k)}{k^2\,(k^2-a^2)^{3/2}}\right)$$
$$\phi = \arcsin\left(\frac{a}{k}\right) \qquad [0 < a < k < 1]$$

5.* $$\int_0^{\pi/2} \frac{E(x,k')\sin x\cos x\,dx}{\left(1-k'^2\cosh^2 v\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{1}{k'^2\sinh v\cosh v}\left\{\boldsymbol{E}(k')\operatorname{arctanh}\left(\frac{\tanh v}{k}\right) - \frac{\pi\tanh v}{2} - \frac{\pi}{2}\left[F(\phi,k) - E(\phi,k)\right]\right\}$$
$$\phi = \arcsin\left(\frac{\tanh v}{k}\right) \qquad k' = \sqrt{1-k^2} \qquad [0 < \tanh v < k < 1]$$

6.* $$\int_0^{\pi/2} \frac{E(x,k)\sin x\cos x\,dx}{\left(1-k^2\cos^2\psi\sin^2 x\right)\sqrt{1-k^2\sin^2 x}}$$
$$= \frac{1}{k^2\sin\psi\cos\psi}\left\{\boldsymbol{E}(k)\arctan\left(\frac{\tan\psi}{k'}\right) - \frac{\pi}{2}E(\beta,k) + \frac{\pi}{2}\frac{\tan\psi}{\sqrt{1-k^2\cos^2\psi}}\left(1-\sqrt{1-k^2\cos^2\psi}\right)\right\}$$
$$\beta = \arctan\left(\frac{\tan\psi}{k}\right) \quad k' = \sqrt{1-k^2} \qquad \left[0<k<1, \quad 0<\psi<\frac{\pi}{2}\right]$$

7.* $$\int_0^{\pi/2} \frac{E(x,k')\sin x\cos x\,dx}{\left(1+k'^2\sinh^2\mu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{-1}{k'^2\sinh\mu\cosh\mu}\left\{\boldsymbol{E}(k')\operatorname{arctanh}(k\tanh\mu) - \frac{\pi}{2}\left[F(\phi,k)-E(\phi,k)+\tanh\mu\sqrt{1+k'^2\sinh^2\mu}\right]\right.$$
$$\left. -\frac{\pi}{2}\coth\mu\left(1-\sqrt{1+k'^2\sinh^2\mu}\right)\right\}$$
$$\phi = \arcsin(\tanh\mu) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\mu<1]$$

8.* $$\int_0^{\pi/2} \frac{F(x,k')\sin x\cos x\,dx}{\left(1+k'^2\sinh^2\mu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{-1}{k'^2\sinh\mu\cosh\mu}\left[\boldsymbol{K}(k')\operatorname{arctanh}(k\tanh\mu) - \frac{\pi}{2}F(\phi,k)\right]$$
$$\phi = \arcsin(\tanh\mu) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\mu<1]$$

9.* $$\int_0^{\pi/2} \frac{F(x,k')\sin x\cos x\,dx}{\left(1-k'^2\cosh^2\nu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{1}{k'^2\sinh\nu\cosh\nu}\left[\boldsymbol{K}(k')\operatorname{arctanh}\left(\frac{\tanh\nu}{k}\right) - \frac{\pi}{2}F(\phi,k)\right]$$
$$\phi = \arcsin\left(\frac{\tanh\nu}{k}\right) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\nu<1]$$

10.* $$\int_0^{\pi/2} \frac{F(x,k)\sin x\cos x\,dx}{\left(1-k^2\cos^2\psi\sin^2 x\right)\sqrt{1-k^2\sin^2 x}}$$
$$= \frac{1}{k^2\sin\psi\cos\psi}\left[\boldsymbol{K}(k')\operatorname{arctanh}\left(\frac{\tan\psi}{k'}\right) - \frac{\pi}{2}F(\beta,k)\right]$$
$$\beta = \arctan\left(\frac{\tan\psi}{k'}\right) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\psi<1]$$

11.* $$\int_a^b \ln\left(\frac{\epsilon+x}{\epsilon-x}\right)\frac{x^2\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \frac{\pi}{\epsilon}\left(\epsilon^2-\sqrt{(\epsilon^2-a^2)(\epsilon^2-b^2)}\right) + \pi\beta\left[F(\phi,k)-E(\phi,k)\right]$$
$$\phi = \arcsin\left(\frac{\beta}{\epsilon}\right) \quad k = \frac{a}{b} \qquad [0<a<b<\epsilon]$$

5.125

1. $$\int_0^x \Pi\left(x,\alpha^2,k\right)\sin x\,dx$$
$$= -\cos x\,\Pi\left(x,\alpha^2,k\right) + \frac{1}{\sqrt{k^2-\alpha^2}}\arctan\left[\sqrt{\frac{k^2-\alpha^2}{1-k^2\sin^2 x}}\sin x\right] \qquad \left[\alpha^2<k^2\right]$$
$$= -\cos x\,\Pi\left(x,\alpha^2,k\right) + \frac{1}{\sqrt{\alpha^2-k^2}}\operatorname{arctanh}\left[\sqrt{\frac{\alpha^2-k^2}{1-k^2\sin^2 x}}\sin x\right] \qquad \left[\alpha^2>k^2\right]$$
BY (630.13)

2. $$\int_0^x \Pi\left(x,\alpha^2,k\right)\cos x\,dx = \sin x\,\Pi\left(x,\alpha^2,k\right) - f - f_0$$
where
$$f = \frac{1}{2\sqrt{\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)}}\arctan\left[\frac{2\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)+\left(1-\alpha^2\sin^2 x\right)\left(2k^2-\alpha^2-\alpha^2k^2\right)}{2\alpha^2\sqrt{\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)}\cos x\sqrt{1-k^2\sin^2 x}}\right]$$
for $\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)>0$;
$$= \frac{1}{2\sqrt{\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)}}\ln\left[\frac{2\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)+\left(1-\alpha^2\sin^2 x\right)\left(\alpha^2+\alpha^2k^2-2k^2\right)}{1-\alpha^2\sin^2 x} + \frac{2\alpha^2\sqrt{\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)}\cos x\sqrt{1-k^2\sin^2 x}}{1-\alpha^2\sin^2 x}\right]$$
for $\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)<0$,
f_0 is the value of f at $x=0$ BY (630.23)

Integration with respect to the modulus

5.126 $$\int F(x,k)k\,dk = E(x,k) - k'^2\,F(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x$$ BY (613.01)

5.127 $$\int E(x,k)k\,dk = \frac{1}{3}\left[\left(1+k^2\right)E(x,k) - k'^2\,F(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x\right]$$ BY (613.02)

5.128 $$\int \Pi\left(x,r^2,k\right)k\,dk = \left(k^2-r^2\right)\Pi\left(x,r^2,k\right) - F(x,k) + E(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x$$
BY (613.03)

5.13 Jacobian elliptic functions

5.131

1. $$\int \operatorname{sn}^m u\,du = \frac{1}{m+1}\left[\operatorname{sn}^{m+1}u\operatorname{cn}u\operatorname{dn}u + (m+2)\left(1+k^2\right)\int \operatorname{sn}^{m+2}u\,du - (m+3)k^2\int \operatorname{sn}^{m+4}u\,du\right]$$
SI 259, PE(567)

2. $$\int \operatorname{cn}^m u\,du = \frac{1}{(m+1)k'^2}\left[-\operatorname{cn}^{m+1}u\operatorname{sn}u\operatorname{dn}u + (m+2)\left(1-2k^2\right)\int \operatorname{cn}^{m+2}u\,du + (m+3)k^2\int \operatorname{cn}^{m+4}u\,du\right]$$ PE (568)

3. $$\int \operatorname{dn}^m u\,du = \frac{1}{(m+1)k'^2}\left[k^2\operatorname{dn}^{m+1}u\operatorname{sn}u\operatorname{cn}u + (m+2)\left(2-k^2\right)\int \operatorname{dn}^{m+2}u\,du - (m+3)\int \operatorname{dn}^{m+4}u\,du\right]$$ PE (569)

By using formulas **5.131**, we can reduce the integrals (for $m \neq 1$) $\int \operatorname{sn}^m u\,du$, $\int \operatorname{cn}^m u\,du$, and $\int \operatorname{dn}^m u\,du$ to the integrals **5.132**, **5.133** and **5.134**.

5.132

1. $$\int \frac{du}{\operatorname{sn}u} = \ln\frac{\operatorname{sn}u}{\operatorname{cn}u+\operatorname{dn}u}$$ H 87(164)
$$= \ln\frac{\operatorname{dn}u-\operatorname{cn}u}{\operatorname{sn}u}$$ SI 266(4)

2. $$\int \frac{du}{\operatorname{cn}u} = \frac{1}{k'}\ln\frac{k'\operatorname{sn}u+\operatorname{dn}u}{\operatorname{cn}u}$$ SI 266(5)

3. $$\int \frac{du}{\operatorname{dn}u} = \frac{1}{k'}\arctan\frac{k'\operatorname{sn}u-\operatorname{cn}u}{k'\operatorname{sn}u+\operatorname{cn}u}$$ H 88(166)
$$= \frac{1}{k'}\arccos\frac{\operatorname{cn}u}{\operatorname{dn}u}$$ JA
$$= \frac{1}{ik'}\ln\frac{\operatorname{cn}u+ik'\operatorname{sn}u}{\operatorname{dn}u}$$ SI 266(6)
$$= \frac{1}{k'}\arcsin\frac{k'\operatorname{sn}u}{\operatorname{dn}u}$$ JA

5.133

1. $$\int \operatorname{sn}u\,du = \frac{1}{k}\ln\left(\operatorname{dn}u - k\operatorname{cn}u\right)$$ H 87(161)
$$= \frac{1}{k}\operatorname{arccosh}\frac{\operatorname{dn}u-k^2\operatorname{cn}u}{1-k^2}$$ JA
$$= \frac{1}{k}\operatorname{arcsinh}\left(k\frac{\operatorname{dn}u-\operatorname{cn}u}{1-k^2}\right);$$ JA
$$= -\frac{1}{k}\ln\left(\operatorname{dn}u + k\operatorname{cn}u\right)$$ SI 365(1)

2. $$\int \operatorname{cn}u\,du = \frac{1}{k}\arccos\left(\operatorname{dn}u\right);$$ H 87(162)
$$= \frac{i}{k}\ln\left(\operatorname{dn}u - ik\operatorname{sn}u\right);$$ SI 265(2)a, ZH 87(162)
$$= \frac{1}{k}\arcsin\left(k\operatorname{sn}u\right)$$ JA

3. $\int \operatorname{dn} u\, du = \arcsin(\operatorname{sn} u)\,;$ H 87(163)

$= \operatorname{am} u = i \ln(\operatorname{cn} u - i \operatorname{sn} u)$ SI 266(3), ZH 87(163)

5.134

1. $\int \operatorname{sn}^2 u\, du = \frac{1}{k^2}\left[u - E(\operatorname{am} u, k)\right]$ PE (564)

2. $\int \operatorname{cn}^2 u\, du = \frac{1}{k^2}\left[E(\operatorname{am} u, k) - k'^2 u\right]$ PE (565)

3. $\int \operatorname{dn}^2 u\, du = E(\operatorname{am} u, k)$ PE (566)

5.135

1. $\int \frac{\operatorname{sn} u}{\operatorname{cn} u}\, du = \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u}$ SI 266(7)

$= \frac{1}{2k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}$ H 88(167)

2. $\int \frac{\operatorname{sn} u}{\operatorname{dn} u}\, du = \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u}$ SI 266(8)

$= \frac{1}{kk'} \operatorname{arccot} \frac{k \operatorname{cn} u}{k'}$

3. $\int \frac{\operatorname{cn} u}{\operatorname{sn} u}\, du = \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u}$ SI 266(10)

$= \frac{1}{2} \ln \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}$ H 88(168)

4. $\int \frac{\operatorname{cn} u}{\operatorname{dn} u}\, du = -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u}$ SI 266(9)

$= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u}$

5. $\int \frac{\operatorname{dn} u}{\operatorname{cn} u}\, du = \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u}$ H 88(172)

$= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}$ JA

6. $\int \frac{\operatorname{dn} u}{\operatorname{sn} u}\, du = \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}$ H 87(170)

5.136

1. $\int \operatorname{sn} u \operatorname{cn} u\, du = -\frac{1}{k^2} \operatorname{dn} u$

2. $\int \operatorname{sn} u \operatorname{dn} u\, du = -\operatorname{cn} u$

3. $\int \operatorname{cn} u \operatorname{dn} u\, du = \operatorname{sn} u$

5.137

1. $\int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u}\, du = \frac{1}{k'^2} \frac{\operatorname{dn} u}{\operatorname{cn} u}$ H 88(173)

2. $$\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u}\,du = -\frac{1}{k'^2}\frac{\operatorname{cn} u}{\operatorname{dn} u}$$ H 88(175)

3. $$\int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u}\,du = -\frac{\operatorname{dn} u}{\operatorname{sn} u}$$ H 88(174)

4. $$\int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u}\,du = \frac{\operatorname{sn} u}{\operatorname{dn} u}$$ H 88(177)

5. $$\int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u}\,du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}$$ H 88(176)

6. $$\int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u}\,du = \frac{\operatorname{sn} u}{\operatorname{cn} u}$$ H 88(178)

5.138

1. $$\int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u}\,du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u}$$ H 88(183)

2. $$\int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u}\,du = \frac{1}{k'^2}\ln \frac{\operatorname{dn} u}{\operatorname{cn} u}$$ H 88(182)

3. $$\int \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u}\,du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u}$$ H 88(184)

5.139

1.[11] $$\int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}\,du = \ln \operatorname{sn} u$$ H 88(179)

2. $$\int \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\,du = \ln \frac{1}{\operatorname{cn} u}$$ H 88(180)

3. $$\int \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\,du = -\frac{1}{k^2}\ln \operatorname{dn} u$$ H 88(181)

5.14 Weierstrass elliptic functions

The invariants g_1 and g_2 used below are defined in 8.161.

5.141

1. $$\int \wp(u)\,du = -\zeta(u)$$

2. $$\int \wp^2(u)\,du = \frac{1}{6}\wp'(u) + \frac{1}{12}g_2 u$$ H 120(192)

3. $$\int \wp^3(u)\,du = \frac{1}{120}\wp'''(u) - \frac{3}{20}g_2\,\zeta(u) + \frac{1}{10}g_3 u$$ H 120(193)

4.[8] $$\int \frac{du}{\wp(u)-\wp(v)} = \frac{1}{\wp'(v)}\left[2u\,\zeta(v) + \ln\frac{\sigma(u-v)}{\sigma(u+v)}\right] \qquad [\wp(v) \neq e_1, e_2, e_3] \qquad \text{(see } \mathbf{8.162}\text{)}$$ H 120(194)

5. $$\int \frac{\alpha\wp(u)+\beta}{\gamma\wp(u)+\delta}\,du = \frac{au}{\gamma} + \frac{\alpha\delta-\beta\gamma}{\gamma^2\wp'(v)}\left[\ln\frac{\sigma(u+v)}{\sigma(u-v)} - 2u\,\zeta(v)\right]$$

where $v = \wp^{-1}\left(\frac{-\delta}{\gamma}\right)$ H 120(195)

5.2 The Exponential Integral Function

5.21 The exponential integral function

5.211 $$\int_x^\infty \operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x)\,dx = \left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\operatorname{Ei}[-(\beta+\gamma)x] - x\operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta}\operatorname{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma}\operatorname{Ei}(-\beta x)$$
$[\operatorname{Re}(\beta+\gamma)>0]$ NT 53(2)

5.22 Combinations of the exponential integral function and powers

5.221

1. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^{n+1}}\,dx = \left[\frac{1}{x^n}-\frac{(-1)^n}{b^n}\right]\frac{\operatorname{Ei}[-a(x+b)]}{n} + \frac{e^{-ab}}{n}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{b^{n-k}}\int_x^\infty \frac{e^{-ax}}{x^{k+1}}\,dx$$
$[a>0,\quad b>0]$ NT 52(3)

2. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^2}\,dx = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{Ei}[-a(x+b)] - \frac{e^{-ab}\operatorname{Ei}(-ax)}{b}$$
$[a>0, b>0]$ NT 52(4)

3.* $$\int x\operatorname{Ei}(-ax)\,dx = \frac{x^2}{2}\operatorname{Ei}(-ax) + \frac{1}{2a^2}e^{-ax} + \frac{xe^{-ax}}{2a} \qquad [a>0]$$

4.* $$\int x^n\operatorname{Ei}(-ax)\,dx = \frac{x^{n+1}}{n+1}\operatorname{Ei}(-ax) + \frac{n!e^{-ax}}{(n+1)a^{n+1}}\sum_{k=0}^\infty\frac{(ax)^k}{k!}$$
$[a>0]$

5.* $$\int x\operatorname{Ei}(-ax)e^{-bx}\,dx = \frac{1}{b^2}\operatorname{Ei}[-(a+b)x] - \frac{1}{b^2}\operatorname{Ei}(-ax)e^{-bx} - \frac{x}{b}\operatorname{Ei}(-ax)e^{-bx} - \frac{1}{b(a+b)}e^{-(a+b)x}$$
$[a>0,\quad b>0]$

6.* $$\int \operatorname{Ei}^2(-ax)\,dx = x\operatorname{Ei}^2(-ax) + \frac{2}{a}\left[\operatorname{Ei}(-ax)e^{-ax} - \operatorname{Ei}(-2ax)\right]$$
$[a>0]$

7.* $$\int x\operatorname{Ei}^2(-ax)\,dx = \frac{x^2}{2}\operatorname{Ei}^2(-ax) + \left(\frac{1}{a^2}+\frac{x}{a}\right)\operatorname{Ei}(-ax)e^{-ax} - \frac{1}{a^2}\operatorname{Ei}(-2ax) + \frac{1}{a^2}e^{-2ax}$$
$[a>0]$

8.* $$\int_0^u \operatorname{Ei}(-ax)\,dx = u\operatorname{Ei}(-au) + \frac{e^{-au}-1}{a} \qquad [a>0]$$

9.* $$\int_0^\infty x\operatorname{Ei}\left(-\frac{x}{a}\right)\operatorname{Ei}\left(-\frac{x}{b}\right)\,dx = \left(\frac{a^2+b^2}{2}\right)\ln(a+b) - \frac{a^2}{2}\ln a - \frac{b^2}{2}\ln b - \frac{ab}{2}$$
$[a>0,\quad b>0]$

10.* $$\int_0^\infty x^2 \operatorname{Ei}\left(-\frac{x}{a}\right) \operatorname{Ei}\left(-\frac{x}{b}\right) dx = \frac{2}{3}\left[\left(a^3+b^3\right)\ln(a+b) - a^3\ln a - b^3 \ln b - \frac{ab}{a+b}\left(a^2-ab+b^2\right)\right]$$

$[a>0, \quad b>0]$

5.23 Combinations of the exponential integral and the exponential

5.231

1. $$\int_0^x e^x \operatorname{Ei}(-x)\, dx = -\ln x - \boldsymbol{C} + e^x \operatorname{Ei}(-x)$$ ET II 308(11)

1. $$\int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x)\, dx = -\frac{1}{\beta}\left\{e^{-\beta x}\operatorname{Ei}(-\alpha x) + \ln\left(1+\frac{\beta}{\alpha}\right) - \operatorname{Ei}[-(\alpha+\beta)x]\right\}$$ ET II 308(12)

5.3 The Sine Integral and the Cosine Integral

5.31

1. $$\int \cos\alpha x \operatorname{ci}(\beta x)\, dx = \frac{\sin\alpha x \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)}{2\alpha}$$ NT 49(1)

2. $$\int \sin\alpha x \operatorname{ci}(\beta x)\, dx = -\frac{\cos\alpha x \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)+\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$$ NT 49(2)

5.32

1. $$\int \cos\alpha x \operatorname{si}(\beta x)\, dx = \frac{\sin\alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)-\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$$ NT 49(3)

2. $$\int \sin\alpha x \operatorname{si}(\beta x)\, dx = -\frac{\cos\alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x+\beta x)-\operatorname{si}(\alpha x-\beta x)}{2\alpha}$$ NT 49(4)

5.33

1. $$\int \operatorname{ci}(\alpha x)\operatorname{ci}(\beta x)\, dx = x\operatorname{ci}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{2\alpha}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)\right)$$
$$+\frac{1}{2\beta}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x-\alpha x)\right) - \frac{1}{\alpha}\sin\alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x \operatorname{ci}(\alpha x)$$
NT 53(5)

2. $$\int \operatorname{si}(\alpha x)\operatorname{si}(\beta x)\, dx = x\operatorname{si}(\alpha x)\operatorname{si}(\beta x) - \frac{1}{2\beta}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)\right)$$
$$-\frac{1}{2\alpha}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x+\alpha x)\right) + \frac{1}{\alpha}\cos\alpha x \operatorname{si}(\beta x) + \frac{1}{\beta}\cos\beta x \operatorname{si}(\alpha x)$$
NT 54(6)

3. $$\int \operatorname{si}(\alpha x)\operatorname{ci}(\beta x)\, dx = x\operatorname{si}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{\alpha}\cos\alpha x \operatorname{ci}(\beta x)$$
$$-\frac{1}{\beta}\sin\beta x \operatorname{si}(\alpha x) - \left(\frac{1}{2\alpha}+\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x+\beta x) - \left(\frac{1}{2\alpha}-\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x-\beta x)$$
NT 54(10)

5.34

1. $$\int_x^\infty \operatorname{si}[a(x+b)]\frac{dx}{x^2} = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{si}[a(x+b)] - \frac{\cos ab\,\operatorname{si}(ax)+\sin ab\,\operatorname{ci}(ax)}{b}$$
$$[a>0, \quad b>0]$$ NT 52(6)

2. $$\int_x^\infty \operatorname{ci}[a(x+b)]\frac{dx}{x^2} = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{ci}[a(x+b)] + \frac{\sin ab\,\operatorname{si}(ax)-\cos ab\,\operatorname{ci}(ax)}{b}$$
$$[a>0, \quad b>0]$$ NT 52(5)

5.4 The Probability Integral and Fresnel Integrals

5.41[11] $$\int \Phi(\alpha x)\,dx = x\,\Phi(\alpha x) + \frac{e^{-\alpha^2x^2}}{\alpha\sqrt{\pi}}$$ NT 12(20)a

5.42 $$\int S(\alpha x)\,dx = x\,S(\alpha x) + \frac{\cos^2\alpha x^2}{\alpha\sqrt{2\pi}}$$ NT 12(22)a

5.43 $$\int C(\alpha x)\,dx = x\,C(\alpha x) - \frac{\sin^2\alpha x^2}{\alpha\sqrt{2\pi}}$$ NT 12(21)a

5.5 Bessel Functions

Notation: Z and $\mathfrak{Z}$ denote any of J, N, $H^{(1)}$, $H^{(2)}$. In formulae 5.52–5.56, $Z_p(x)$ and $\mathfrak{Z}_p(x)$ are arbitrary Bessel functions of the first, second, or third kinds.

5.51 $$\int J_p(x)\,dx = 2\sum_{k=0}^\infty J_{p+2k+1}(x)$$ JA, MO 30

5.52

1. $$\int x^{p+1}\,Z_p(x)\,dx = x^{p+1}\,Z_{p+1}(x)$$ WA 132(1)

2.[11] $$\int x^{-p}\,Z_{p+1}(x)\,dx = -x^{-p}\,Z_p(x)$$ WA 132(2)

5.53[10] $$\int \left[\left(\alpha^2-\beta^2\right)x - \frac{p^2-q^2}{x}\right] Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)\,dx$$
$$= \alpha x\,Z_{p+1}(\alpha x)\,\mathfrak{Z}_q(\beta x) - \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{q+1}(\beta x) - (p-q)\,Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)$$
$$= \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{q-1}(\beta x) - \alpha x\,Z_{p-1}(\alpha x)\,\mathfrak{Z}_q(\beta x) + (p-q)\,Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)$$
JA, MO 30, WA 134(7)

5.54

1.[10] $$\int x\,Z_p(\alpha x)\,\mathfrak{Z}_p(\beta x)\,dx = \frac{\alpha x\,Z_{p+1}(\alpha x)\,\mathfrak{Z}_p(\beta x) - \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p+1}(\beta x)}{\alpha^2-\beta^2}$$
$$= \frac{\beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p-1}(\beta x) - \alpha x\,Z_{p-1}(\alpha x)\,\mathfrak{Z}_p(\beta x)}{\alpha^2-\beta^2}$$
WA 134(8)

2. $$\int x\,[Z_p(\alpha x)]^2\,dx = \frac{x^2}{2}\left\{[Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x)\,Z_{p+1}(\alpha x)\right\}$$ WA 135(11)

3.* $$\int x\, Z_p(ax)\, \mathfrak{Z}_p(ax)\, dx = \frac{x^4}{4}\left[2\, Z_p(ax)\, \mathfrak{Z}_p(ax) - Z_{p-1}(ax)\, \mathfrak{Z}_{p+1}(ax) - Z_{p+1}(ax)\, \mathfrak{Z}_{p-1}(ax)\right]$$

5.55[10] $$\int \frac{1}{x}\, Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)\, dx = \alpha x \frac{Z_p(\alpha x)\, \mathfrak{Z}_{q+1}(\alpha x) - Z_{p+1}(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p^2 - q^2} + \frac{Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p+q}$$
$$= \alpha x \frac{Z_{p-1}(\alpha x)\, \mathfrak{Z}_q(\alpha x) - Z_p(\alpha x)\, \mathfrak{Z}_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p+q}$$

WA 135(13)

5.56

1. $$\int Z_1(x)\, dx = -\, Z_0(x)$$ JA

2. $$\int x\, Z_0(x)\, dx = x\, Z_1(x)$$ JA

6–7 Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

6.11 Forms containing $F(x,k)$

6.111 $$\int_0^{\pi/2} F(x,k)\cot x\,dx = \frac{\pi}{4}\,\boldsymbol{K}(k') + \frac{1}{2}\ln k\,\boldsymbol{K}(k)$$ BI (350)(1)

6.112

1. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1+k\sin^2 x}\,dx = \frac{1}{4k}\,\boldsymbol{K}(k)\ln\frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k}\,\boldsymbol{K}(k')$$ BI (350)(6)

2. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k\sin^2 x}\,dx = \frac{1}{4k}\,\boldsymbol{K}(k)\ln\frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k}\,\boldsymbol{K}(k')$$ BI (350)(7)

3. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 x}\,dx = -\frac{1}{2k^2}\ln k'\,\boldsymbol{K}(k)$$ BI (350)(2)a, BY(802.12)a

6.113

1. $$\int_0^{\pi/2} F\left(x,k'\right)\frac{\sin x\cos x\,dx}{\cos^2 x + k\sin^2 x} = \frac{1}{4(1-k)}\ln\frac{2}{(1+k)\sqrt{k}}\,\boldsymbol{K}(k')$$ BI (350)(5)

2. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 t\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{k^2\sin t\cos t}\left[\boldsymbol{K}(k)\arctan\left(k'\tan t\right) - \frac{\pi}{2}\,F(t,k)\right]$$ BI (350)(12)

6.114 $$\int_u^v F(x,k)\frac{dx}{\sqrt{\left(\sin^2 x-\sin^2 u\right)\left(\sin^2 v-\sin^2 x\right)}} = \frac{1}{2\cos u\sin v}\,\boldsymbol{K}(k)\,\boldsymbol{K}\left(\sqrt{1-\tan^2 u\cot^2 v}\right)$$ $\left[k^2 = 1-\cot^2 u\cdot\cot^2 v\right]$ BI (351)(9)

6.115 $$\int_0^1 F\left(\arcsin x,k\right)\frac{x\,dx}{1+kx^2} = \frac{1}{4k}\,\boldsymbol{K}(k)\ln\frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k}\,\boldsymbol{K}(k')$$ (cf. **6.112** 2) BI (466)(1)

This and similar formulas can be obtained from formulas **6.111–6.113** by means of the substitution $x = \arcsin t$.

6.12 Forms containing $\boldsymbol{E}(x,k)$

6.121 $$\int_0^{\pi/2} E(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 x}\,dx = \frac{1}{2k^2}\left\{\left(1+k'^2\right)\boldsymbol{K}(k) - (2+\ln k')\,\boldsymbol{E}(k)\right\}$$ BI (350)(4)

6.122 $$\int_0^{\pi/2} E(x,k)\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2}\left\{\boldsymbol{E}(k)\,\boldsymbol{K}(k) - \ln k'\right\}$$ BI (350)(10), BY (630.02)

6.123 $$\int_0^{\pi/2} E(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 t\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$
$$= -\frac{1}{k^2\sin t\cos t}\left[\boldsymbol{E}(k)\arctan\left(k'\tan t\right) - \frac{\pi}{2}\,E(t,k) + \frac{\pi}{2}\cot t\left(1-\sqrt{1-k^2\sin^2 t}\right)\right]$$
BI (350)(13)

6.124 $$\int_u^v E(x,k)\frac{dx}{\sqrt{\left(\sin^2 x-\sin^2 u\right)\left(\sin^2 v-\sin^2 x\right)}} = \frac{1}{2\cos u\sin v}\,\boldsymbol{E}(k)\,\boldsymbol{K}\left(\sqrt{1-\frac{tg^2 u}{tg^2 v}}\right)$$
$$+\frac{k^2\sin v}{2\cos u}\,\boldsymbol{K}\left(\sqrt{1-\frac{\sin^2 2u}{\sin^2 2v}}\right)$$
$$\left[k^2 = 1-\cot^2 u\cot^2 v\right]$$ BI (351)(10)

6.13 Integration of elliptic integrals with respect to the modulus

6.131 $$\int_0^1 F(x,k)k\,dk = \frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$$ BY (616.03)

6.132 $$\int_0^1 E(x,k)k\,dk = \frac{\sin^2 x+1-\cos x}{3\sin x}$$ BY (616.04)

6.133 $$\int_0^1 \Pi\left(x,r^2,k\right)k\,dk = \tan\frac{x}{2} - r\ln\sqrt{\frac{1+r\sin x}{1-r\sin x}} - r^2\,\Pi\left(x,r^2,0\right)$$ BY (616.05)

6.14–6.15 Complete elliptic integrals

6.141

1. $$\int_0^1 \boldsymbol{K}(k)\,dk = 2\boldsymbol{G}$$ FI II 755

2. $$\int_0^1 \boldsymbol{K}(k')\,dk = \frac{\pi^2}{4}$$ BY (615.03)

6.142 $$\int_0^1 \left(\boldsymbol{K}(k)-\frac{\pi}{2}\right)\frac{dk}{k} = \pi\ln 2 - 2\boldsymbol{G}$$ BY (615.05)

6.143[7] $$\int_0^1 \boldsymbol{K}(k)\frac{dk}{k'} = \boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{16\pi}\,\Gamma^4\left(\frac{1}{4}\right)$$ BY (615.08)

6.144 $$\int_0^1 \boldsymbol{K}(k)\frac{dk}{1+k} = \frac{\pi^2}{8}$$ BY (615.09)

6.145 $\displaystyle\int_0^1 \left(\boldsymbol{K}(k') - \ln\frac{4}{k}\right)\frac{dk}{k} = \frac{1}{12}\left[24\,(\ln 2)^2 - \pi^2\right]$ BY (615.13)

6.146 $\displaystyle n^2\int_0^1 k^n\,\boldsymbol{K}(k)\,dk = (n-1)^2\int_0^1 k^{n-2}\,\boldsymbol{K}(k)\,dk + 1$ BY (615.12)

6.147 $\displaystyle n\int_0^1 k^n\,\boldsymbol{K}(k')\,dk = (n-1)\int_0^1 k^{n-2}\,\boldsymbol{E}(k)\,dk$ $[n>1]$ (see **6.152**) BY (615.11)

6.148

1. $\displaystyle\int_0^1 \boldsymbol{E}(k)\,dk = \frac{1}{2} + \boldsymbol{G}$ BY (615.02)

2. $\displaystyle\int_0^1 \boldsymbol{E}(k')\,dk = \frac{\pi^2}{8}$ BY (615.04)

3.* $\displaystyle\int_0^1 \frac{\boldsymbol{E}(k)}{1+k}\,dk = 1$

6.149

1. $\displaystyle\int_0^1 \left(\boldsymbol{E}(k) - \frac{\pi}{2}\right)\frac{dk}{k} = \pi\ln 2 - 2\boldsymbol{G} + 1 - \frac{\pi}{2}$ BY (615.06)

2. $\displaystyle\int_0^1 \left(\boldsymbol{E}(k') - 1\right)\frac{dk}{k} = 2\ln 2 - 1$ BY (615.07)

3.* $\displaystyle\int_0^1 \frac{\boldsymbol{E}(k)}{1+k}\,dk = 1$

4.* $\displaystyle\int_0^1 \frac{dx}{x^3}\left(\sqrt{a-x^2}\,\boldsymbol{K}(x) - \frac{\boldsymbol{E}(x)}{\sqrt{1-x^2}} + \frac{\pi}{4}x^2\right) = -\frac{\pi}{4}\ln\left(\frac{4}{\sqrt{e}}\right)$

6.151 $\displaystyle\int_0^1 \boldsymbol{E}(k)\frac{dk}{k'} = \frac{1}{8}\left[4\,\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi^2}{\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right)}\right]$ BY (615.10)

6.152 $\displaystyle(n+2)\int_0^1 k^n\,\boldsymbol{E}(k')\,dk = (n+1)\int_0^1 k^n\,\boldsymbol{K}(k')\,dk$ $[n>1]$ (see **6.147**) BY (615.14)

6.153[6] $\displaystyle\int_0^a \frac{\boldsymbol{K}(k)k\,dk}{k'^2\sqrt{a^2-k^2}} = \frac{\pi}{4}\frac{1}{\sqrt{1-a^2}}\ln\left(\frac{1+a}{1-a}\right)$ $[0<a<1]$ LO I 252

6.154 $\displaystyle\int_0^{\pi/2} \frac{\boldsymbol{E}\left(p\sin x\right)}{1-p^2\sin^2 x}\sin x\,dx = \frac{\pi}{2\sqrt{1-p^2}}$ $\left[p^2>1\right]$ FI II 489

6.16 The theta function

6.161

1. $\displaystyle\int_0^\infty x^{s-1}\,\vartheta_2\left(0\mid ix^2\right)\,dx = 2^s\left(1-2^{-s}\right)\pi^{-\frac{s}{2}}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$

$[\operatorname{Re} s>2]$ ET I 339(20)

2. $\displaystyle\int_0^\infty x^{s-1}\left[\vartheta_3\left(0\mid ix^2\right)-1\right]\,dx = \pi^{-\frac{s}{2}}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$ $[\operatorname{Re} s>2]$ ET I 339(21)

3. $$\int_0^\infty x^{s-1}\left[1-\vartheta_4\left(0\mid ix^2\right)\right]\,dx=\left(1-2^{1-s}\right)\pi^{-\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$$
$[\operatorname{Re} s>2]$ ET I 339(22)

4. $$\int_0^\infty x^{s-1}\left[\vartheta_4\left(0\mid ix^2\right)+\vartheta_2\left(0\mid ix^2\right)-\vartheta_3\left(0\mid ix^2\right)\right]\,dx=-\left(2^s-1\right)\left(2^{1-s}-1\right)\pi^{-\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$$
ET I 339(24)

6.162

1.[11] $$\int_0^\infty e^{-ax}\,\vartheta_4\left(\frac{b\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)\,dx=\frac{l}{\sqrt{a}}\cosh\left(b\sqrt{a}\right)\operatorname{cosech}\left(l\sqrt{a}\right)$$
$[\operatorname{Re} a>0,\quad |b|\le l]$ ET I 224(1)a

2. $$\int_0^\infty e^{-ax}\,\vartheta_1\left(\frac{b\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)\,dx=-\frac{l}{\sqrt{a}}\sinh\left(b\sqrt{a}\right)\operatorname{sech}\left(l\sqrt{a}\right)$$
$[\operatorname{Re} a>0,\quad |b|\le l]$ ET I 224(2)a

3.[11] $$\int_0^\infty e^{-ax}\,\vartheta_2\left(\frac{(l+b)\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)\,dx=-\frac{l}{\sqrt{a}}\sinh\left(b\sqrt{a}\right)\operatorname{sech}\left(l\sqrt{a}\right)$$
$[\operatorname{Re} a>0,\quad |b|\le l]$ ET I 224(3)a

4.[11] $$\int_0^\infty e^{-ax}\,\vartheta_3\left(\frac{(l+b)\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)\,dx=\frac{l}{\sqrt{a}}\cosh\left(b\sqrt{a}\right)\operatorname{cosech}\left(l\sqrt{a}\right)$$
$[\operatorname{Re} a>0,\quad |b|\le l]$ ET I 224(4)a

6.163[10]

1. $$\int_0^\infty e^{-(a-\mu)x}\,\vartheta_3\left(\pi\sqrt{\mu}x\,|i\pi x\right)\,dx=\frac{1}{2\sqrt{a}}\left[\coth\left(\sqrt{a}+\sqrt{\mu}\right)+\coth\left(\sqrt{a}-\sqrt{\mu}\right)\right]$$
$[\operatorname{Re} a>0]$ ET I 224(7)a

2.[10] $$\int_0^\infty \vartheta_3\left(i\pi kx\mid i\pi x\right)e^{-\left(k^2+l^2\right)x}\,dx=\frac{\sinh 2l}{l\left(\cosh 2l-\cos 2k\right)}$$

6.164[11] $$\int_0^\infty\left[\vartheta_4\left(0\mid ie^{2x}\right)+\vartheta_2\left(0\mid ie^{2x}\right)-\vartheta_3\left(0\mid ie^{2x}\right)\right]e^{\frac{1}{2}x}\cos(ax)\,dx$$
$$=\frac{1}{2}\left(2^{\frac{1}{2}+ia}-1\right)\left(1-2^{\frac{1}{2}-ia}\right)\pi^{-\frac{1}{4}-\frac{1}{2}ia}\,\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}ia\right)\zeta\left(\tfrac{1}{2}+ia\right)$$
$[a>0]$ ET I 61(11)

6.165 $$\int_0^\infty e^{\frac{1}{2}x}\left[\vartheta_3\left(0\mid ie^{2x}\right)-1\right]\cos(ax)\,dx$$
$$=\frac{2}{1+4a^2}\left\{1+\left[\left(a^2+\tfrac{1}{4}\right)\pi^{-\frac{1}{2}ia-\frac{1}{4}}\,\Gamma\left(\tfrac{1}{2}ia+\tfrac{1}{4}\right)\zeta\left(ia+\tfrac{1}{2}\right)\right]\right\}$$
$[a>0]$ ET I 61(12)

6.17[10] Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi \left[1 - k^2 \cos\phi\right]^{-\left(j+\frac{1}{2}\right)} d\phi,$$

$$\alpha_m(j) = \frac{\pi}{(64)^m} \frac{j!}{(2j)!} \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m!}\right)^2, \qquad \lambda = \frac{\pi}{2}\sqrt{\frac{(2j+1)k^2}{1-k^2}},$$

then

$$\Omega_j(k) = \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} = \sqrt{\frac{\pi}{(2j+1)k^2}} \left(1-k^2\right)^{-j} \left[\operatorname{erf}\lambda + \frac{1}{2}(2j+1)^{-1}\left(1+\frac{1}{2k^2}\right)\right.$$
$$\times \left\{\operatorname{erf}\lambda - \left(\frac{2}{\sqrt{\pi}}\right)\left(\lambda e^{-\lambda^2}\right)\left(1+\frac{2}{3}\lambda^2\right)\right\} - \frac{1}{12}(2j+1)^{-2}\left(16+\frac{13}{k^2}+\frac{1}{k^4}\right)$$
$$\left.\times \left\{\operatorname{erf}\lambda - \left(\frac{2}{\sqrt{\pi}}\right)\left(\lambda e^{-\lambda^2}\right)\left(1+\frac{2}{3}\lambda^2+\frac{4}{15}\lambda^4\right)\right\} + \ldots\right]$$

while for large λ

$$\lim_{j\to\infty} \Omega_j(k) = \sqrt{\frac{\pi}{(2j+1)}k^2}\left(1-k^2\right)^{-j}$$
$$\times \left[1 + \frac{1}{2}(2j+1)^{-1}\left\{1+\frac{1}{2k^2}\right\} - \frac{4}{3}(2j+1)^{-2}\left\{1+\frac{13}{16k^2}+\frac{1}{16k^4}\right\} + \ldots\right]$$

2. Set

$$R_\mu(k,\alpha,\delta) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\delta-2\alpha-1}(\theta/2)\, d\theta}{[1-k^2\cos\theta]^{\mu+\frac{1}{2}}},$$
$$0 < k < 1, \quad \operatorname{Re}\delta > \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -1/2,$$
$$M_\nu(\mu,\alpha,\delta) = \frac{(-1)^\nu 2^\nu \left(\mu+\frac{1}{2}\right)_\nu}{\nu!} \frac{\Gamma(\alpha)\,\Gamma(\delta-\alpha+\nu)}{\Gamma(\delta+\nu)},$$
$$\text{with } (\lambda)_\nu = \Gamma(\lambda+\nu)/\Gamma(\lambda), \text{ and}$$
$$W_\nu(\mu,\alpha,\delta) = \frac{2^\nu \left(\mu+\frac{1}{2}\right)_\nu}{\nu!} \frac{\Gamma(\alpha+\nu)\,\Gamma(\delta-\alpha)}{\Gamma(\delta+\nu)},$$

then:

- for small k:

$$R_\mu(k,\alpha,\delta) = \left(1-k^2\right)^{-\left(\mu+\frac{1}{2}\right)} \sum_{\nu=0}^{\infty} \left[k^2/\left(1-k^2\right)\right]^\nu M_\nu(\mu,\alpha,\delta)$$
$$= \left(1+k^2\right)^{-\left(\mu+\frac{1}{2}\right)} \sum_{\nu=0}^{\infty} \left[k^2/\left(1+k^2\right)\right]^\nu W_\nu(\mu,\alpha,\delta),$$

- for k^2 close to 1:
$R_\mu(k,\alpha,\delta)$

$$= \left[\Gamma(\delta-\alpha)\,\Gamma\left(\mu+\alpha-\delta+\tfrac{1}{2}\right)\Gamma\left(\mu+\tfrac{1}{2}\right)\right]\left(2k^2\right)^{\alpha-\delta}\left(1-k^2\right)^{\delta-\alpha-\mu-\frac{1}{2}}$$

$$\times\left\{\Gamma\left(\delta-\alpha-\mu-\tfrac{1}{2}\right)\Gamma(\alpha)\left[\Gamma\left(\delta-\mu-\tfrac{1}{2}\right)\left(2k^2\right)^{\mu+\frac{1}{2}}\right]\right\}$$

$$\left[\mathrm{Re}\left(\mu+\alpha-\delta+\tfrac{1}{2}\right)\ \text{not an integer}\right]$$

$$= \left[2^{\mu+\frac{1}{2}}k^{2\mu+1}\,\Gamma\left(\mu+\tfrac{1}{2}\right)\Gamma(1-\alpha)\right]$$

$$\times\sum_{n=0}^{\infty}\left[\Gamma\left(\delta-\alpha+n\right)\Gamma(1-\alpha+n)\,\Gamma\left(\alpha-\delta+\mu-n+\tfrac{1}{2}\right)n!\right]\left[2k^2/\left(1-k^2\right)\right]^{\alpha-\delta+\mu-n+\frac{1}{2}}$$

$$\left[\alpha-\delta+\mu+\tfrac{1}{2}=m,\ \text{with } m \text{ a non-negative integer}\right]$$

6.2–6.3 The Exponential Integral Function and Functions Generated by It

6.21 The logarithm integral

6.211 $\displaystyle\int_0^1 \mathrm{li}(x)\,dx = -\ln 2$ BI (79)(5)

6.212

1. $\displaystyle\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)x\,dx = 0$ BI (255)(1)

2. $\displaystyle\int_0^1 \mathrm{li}(x)x^{p-1}\,dx = -\frac{1}{p}\ln(p+1)$ $[p>-1]$ BI (255)(2)

3. $\displaystyle\int_0^1 \mathrm{li}(x)\frac{dx}{x^{q+1}} = \frac{1}{q}\ln(1-q)$ $[q<1]$ BI (255)(3)

4. $\displaystyle\int_1^\infty \mathrm{li}(x)\frac{dx}{x^{q+1}} = -\frac{1}{q}\ln(q-1)$ $[q>1]$ BI (255)(4)

6.213

1. $\displaystyle\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)\sin\left(a\ln x\right)\,dx = \frac{1}{1+a^2}\left(a\ln a-\frac{\pi}{2}\right)$ $[a>0]$ BI (475)(1)

2. $\displaystyle\int_1^\infty \mathrm{li}\left(\frac{1}{x}\right)\sin\left(a\ln x\right)\,dx = -\frac{1}{1+a^2}\left(\frac{\pi}{2}+a\ln a\right)$ $[a>0]$ BI (475)(9)

3. $\displaystyle\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)\cos\left(a\ln x\right)\,dx = -\frac{1}{1+a^2}\left(\ln a+\frac{\pi}{2}a\right)$ $[a>0]$ BI (475)(2)

4. $\displaystyle\int_1^\infty \mathrm{li}\left(\frac{1}{x}\right)\cos\left(a\ln x\right)\,dx = \frac{1}{1+a^2}\left(\ln a-\frac{\pi}{2}a\right)$ $[a>0]$ BI (475)(10)

5. $\displaystyle\int_0^1 \mathrm{li}(x)\sin\left(a\ln x\right)\frac{dx}{x} = \frac{\ln\left(1+a^2\right)}{2a}$ $[a>0]$ BI(479)(1), ET I 98(20)a

6. $$\int_0^1 \operatorname{li}(x)\cos(a\ln x)\,\frac{dx}{x} = -\frac{\arctan a}{a}$$ BI (479)(2)

7. $$\int_0^1 \operatorname{li}(x)\sin(a\ln x)\,\frac{dx}{x^2} = \frac{1}{1+a^2}\left(a\ln a + \frac{\pi}{2}\right) \qquad [a>0]$$ BI (479)(3)

8. $$\int_1^\infty \operatorname{li}(x)\sin(a\ln x)\,\frac{dx}{x^2} = \frac{1}{1+a^2}\left(\frac{\pi}{2} - a\ln a\right) \qquad [a>0]$$ BI (479)(13)

9. $$\int_0^1 \operatorname{li}(x)\cos(a\ln x)\,\frac{dx}{x^2} = \frac{1}{1+a^2}\left(\ln a - \frac{\pi}{2}a\right) \qquad [a>0]$$ BI (479)(4)

10. $$\int_1^\infty \operatorname{li}(x)\cos(a\ln x)\,\frac{dx}{x^2} = -\frac{1}{1+a^2}\left(\ln a + \frac{\pi}{2}a\right) \qquad [a>0]$$ BI (479)(14)

11. $$\int_0^1 \operatorname{li}(x)\sin(a\ln x)\,x^{p-1}\,dx = \frac{1}{a^2+p^2}\left\{\frac{a}{2}\ln\left[(1+p)^2+a^2\right] - p\arctan\frac{a}{1+p}\right\}$$
$$[p>0]$$ BI (477)(1)

12. $$\int_0^1 \operatorname{li}(x)\cos(a\ln x)\,x^{p-1}\,dx = -\frac{1}{a^2+p^2}\left\{a\arctan\frac{a}{1+p} + \frac{p}{2}\ln\left[(1+p)^2+a^2\right]\right\}$$
$$[p>0]$$ BI (477)(2)

6.214

1. $$\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{p-1} dx = -\pi\cot p\pi\cdot\Gamma(p) \qquad [0<p<1]$$ BI (340)(1)

2. $$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right)(\ln x)^{p-1}\,dx = -\frac{\pi}{\sin p\pi}\,\Gamma(p) \qquad [0<p<1]$$ BI (340)(9)

6.215

1. $$\int_0^1 \operatorname{li}(x)\frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}}\,dx = -2\sqrt{\frac{\pi}{p}}\operatorname{arcsinh}\sqrt{p} = -2\sqrt{\frac{\pi}{p}}\ln\left(\sqrt{p}+\sqrt{p+1}\right)$$
$$[p>0]$$ BI (444)(3)

2. $$\int_0^1 \operatorname{li}(x)\frac{dx}{x^{p+1}\sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}}\arcsin\sqrt{p} \qquad [1>p>0]$$ BI (444)(4)

6.216

1. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1}\frac{ax}{x} = -\frac{1}{p}\,\Gamma(p) \qquad [0<p\le 1]$$ BI (444)(1)

2. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1}\frac{dx}{x^2} = -\frac{\pi\,\Gamma(p)}{\sin p\pi} \qquad [0<p<1]$$ BI (444)(2)

6.22–6.23 The exponential integral function

6.221 $$\int_0^p \operatorname{Ei}(\alpha x)\,dx = p\operatorname{Ei}(\alpha p) + \frac{1-e^{\alpha p}}{\alpha}$$ NT 11(7)

6.222 $$\int_0^\infty \operatorname{Ei}(-px)\operatorname{Ei}(-qx)\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q}$$ $[p>0, \quad q>0]$ FI II 653, NT 53(3)

6.223 $$\int_0^\infty \operatorname{Ei}(-\beta x)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu\beta^\mu}$$ $[\operatorname{Re}\beta \ge 0, \quad \operatorname{Re}\mu > 0]$ NT 55(7), ET I 325(10)

6.224

1. $$\int_0^\infty \operatorname{Ei}(-\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(1+\frac{\mu}{\beta}\right)$$ $[\operatorname{Re}(\beta+\mu) \ge 0, \quad \mu > 0]$
$$= -1/\beta$$ $[\mu = 0]$
FI II 652, NT 48(8)

2. $$\int_0^\infty \operatorname{Ei}(ax)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(\frac{\mu}{a}-1\right)$$ $[a>0, \quad \operatorname{Re}\mu > 0, \quad \mu > a]$
ET I 178(23)a, BI (283)(3)

6.225

1. $$\int_0^\infty \operatorname{Ei}\left(-x^2\right)e^{-\mu x^2}\,dx = -\sqrt{\frac{\pi}{\mu}}\operatorname{arcsinh}\sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}}\ln\left(\sqrt{\mu}+\sqrt{1+\mu}\right)$$
$[\operatorname{Re}\mu > 0]$ BI (283)(5), ET I 178(25)a

2. $$\int_0^\infty \operatorname{Ei}\left(-x^2\right)e^{px^2}\,dx = -\sqrt{\frac{\pi}{p}}\arcsin\sqrt{p}$$ $[1 > p > 0]$ NT 59(9)a

6.226

1. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x}\right)e^{-\mu x}\,dx = -\frac{2}{\mu}K_0\left(\sqrt{\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 34

2. $$\int_0^\infty \operatorname{Ei}\left(\frac{a^2}{4x}\right)e^{-\mu x}\,dx = -\frac{2}{\mu}K_0\left(a\sqrt{\mu}\right)$$ $[a>0, \quad \operatorname{Re}\mu > 0]$ MI 34

3. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right)e^{-\mu x^2}\,dx = \sqrt{\frac{\pi}{\mu}}\operatorname{Ei}\left(-\sqrt{\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 34

4. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right)e^{-\mu x^2+\frac{1}{4x^2}}\,dx = \sqrt{\frac{\pi}{\mu}}\left[\cos\sqrt{\mu}\operatorname{ci}\sqrt{\mu} - \sin\sqrt{\mu}\operatorname{si}\sqrt{\mu}\right]$$
$[\operatorname{Re}\mu > 0]$ MI 34

6.227

1. $$\int_0^\infty \operatorname{Ei}(-x)e^{-\mu x}x\,dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2}\ln(1+\mu)$$ $[\operatorname{Re}\mu > 0]$ MI 34

2. $$\int_0^\infty \left[\frac{e^{-ax}\,\mathrm{Ei}(ax)}{x-b} - \frac{e^{ax}\,\mathrm{Ei}(-ax)}{x+b}\right] dx = 0 \qquad [a>0, \quad b<0]$$
$$= \pi^2 e^{-ab} \qquad [a>0, \quad b>0]$$

ET II 253(1)a

6.228

1. $$\int_0^\infty \mathrm{Ei}(-x)e^x x^{\nu-1}\,dx = -\frac{\pi\,\Gamma(\nu)}{\sin\nu\pi} \qquad [0<\mathrm{Re}\,\nu<1]$$ ET II 308(13)

2. $$\int_0^\infty \mathrm{Ei}(-\beta x)e^{-\mu x}x^{\nu-1}\,dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu}\;{}_2F_1\left(1,\nu;\nu+1;\frac{\mu}{\beta+\mu}\right)$$
$$[|\arg\beta|<\pi, \quad \mathrm{Re}(\beta+\mu)>0, \quad \mathrm{Re}\,\nu>0]$$ ET II 308(14)

6.229 $$\int_0^\infty \mathrm{Ei}\left(-\frac{1}{4x^2}\right)\exp\left(-\mu x^2+\frac{1}{4x^2}\right)\frac{dx}{x^2} = 2\sqrt{\pi}\,(\cos\sqrt{\mu}\,\mathrm{si}\,\sqrt{\mu} - \sin\sqrt{\mu}\,\mathrm{ci}\,\sqrt{\mu})$$
$$[\mathrm{Re}\,\mu>0]$$ MI 34

6.231 $$\int_{-\ln a}^\infty \left[\mathrm{Ei}(-a) - \mathrm{Ei}\left(-e^{-x}\right)\right]e^{-\mu x}\,dx = \frac{1}{\mu}\,\gamma(\mu,a) \qquad [a<1, \quad \mathrm{Re}\,\mu>0]$$ MI 34

6.232

1. $$\int_0^\infty \mathrm{Ei}(-ax)\sin bx\,dx = -\frac{\ln\left(1+\frac{b^2}{a^2}\right)}{2b} \qquad [a>0, \quad b>0]$$ BI (473)(1)a

2. $$\int_0^\infty \mathrm{Ei}(-ax)\cos bx\,dx = -\frac{1}{b}\arctan\frac{b}{a} \qquad [a>0, \quad b>0]$$ BI (473)(2)a

6.233

1. $$\int_0^\infty \mathrm{Ei}(-x)e^{-\mu x}\sin\beta x\,dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\beta}{2}\ln\left[(1+\mu)^2+\beta^2\right] - \mu\arctan\frac{\beta}{1+\mu}\right\}$$
$$[\mathrm{Re}\,\mu>|\mathrm{Im}\,\beta|]$$ BI (473)(7)a

2. $$\int_0^\infty \mathrm{Ei}(-x)e^{-\mu x}\cos\beta x\,dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\mu}{2}\ln\left[(1+\mu)^2+\beta^2\right] + \beta\arctan\frac{\beta}{1+\mu}\right\}$$
$$[\mathrm{Re}\,\mu>|\mathrm{Im}\,\beta|]$$ BI (473)(8)a

6.234 $$\int_0^\infty \mathrm{Ei}(-x)\ln x\,dx = \boldsymbol{C}+1$$ NT 56(10)

6.24–6.26 The sine integral and cosine integral functions

6.241

1. $$\int_0^\infty \mathrm{si}(px)\,\mathrm{si}(qx)\,dx = \frac{\pi}{2p} \qquad [p\ge q]$$ BI II 653, NT 54(8)

2. $$\int_0^\infty \mathrm{ci}(px)\,\mathrm{ci}(qx)\,dx = \frac{\pi}{2p} \qquad [p\ge q]$$ FI II 653, NT 54(7)

3. $$\int_0^\infty \operatorname{si}(px)\operatorname{ci}(qx)\,dx = \frac{1}{4q}\ln\left(\frac{p+q}{p-q}\right)^2 + \frac{1}{4p}\ln\frac{\left(p^2-q^2\right)^2}{q^4} \qquad [p\neq q]$$
$$= \frac{1}{q}\ln 2 \qquad [p=q]$$
FI II 653, NT 54(10, 12)

6.242 $$\int_0^\infty \frac{\operatorname{ci}(ax)}{\beta+x}\,dx = -\frac{1}{2}\left\{[\operatorname{si}(a\beta)]^2 + [\operatorname{ci}(a\beta)]^2\right\} \qquad [a>0,\quad |\arg\beta|<\pi]$$ ET II 224(1)

6.243

1. $$\int_{-\infty}^\infty \frac{\operatorname{si}(a|x|)}{x-b}\operatorname{sign} x\,dx = \pi\operatorname{ci}(a|b|) \qquad [a>0,\quad b>0]$$ ET II 253(3)

2. $$\int_{-\infty}^\infty \frac{\operatorname{ci}(a|x|)}{x-b}\,dx = -\pi\operatorname{sign} b\cdot\operatorname{si}(a|b|) \qquad [a>0]$$ ET II 253(2)

6.244

1.[8] $$\int_0^\infty \operatorname{si}(px)\frac{x\,dx}{q^2+x^2} = \frac{\pi}{2}\operatorname{Ei}(-pq) \qquad [p>0,\quad q>0]$$ BI (255)(6)

2.[8] $$\int_0^\infty \operatorname{si}(px)\frac{x\,dx}{q^2-x^2} = -\frac{\pi}{2}\operatorname{ci}(pq) \qquad [p>0,\quad q>0]$$ BI (255)(6)

6.245

1. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2+x^2} = \frac{\pi}{2q}\operatorname{Ei}(-pq) \qquad [p>0,\quad q>0]$$ BI (255)(7)

2. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2-x^2} = \frac{\pi}{2q}\operatorname{si}(pq) \qquad [p>0,\quad q>0]$$ BI (255)(8)

6.246

1. $$\int_0^\infty \operatorname{si}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu}\sin\frac{\mu\pi}{2} \qquad [a>0,\quad 0<\operatorname{Re}\mu<1]$$
NT 56(9), ET I 325(12)a

2. $$\int_0^\infty \operatorname{ci}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu}\cos\frac{\mu\pi}{2} \qquad [a>0,\quad 0<\operatorname{Re}\mu<1]$$
NT 56(8), ET I 325(13)a

6.247

1. $$\int_0^\infty \operatorname{si}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\arctan\frac{\mu}{\beta} \qquad [\operatorname{Re}\mu>0]$$ NT 49(12), ET I 177(18)

2. $$\int_0^\infty \operatorname{ci}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\sqrt{1+\frac{\mu^2}{\beta^2}} \qquad [\operatorname{Re}\mu>0]$$ NT 49(11), ET I 178(19)a

6.248

1.[8] $$\int_0^\infty \operatorname{si}(x)e^{-\mu x^2}x\,dx = \frac{\pi}{4\mu}\left[\Phi\left(\frac{1}{2\sqrt{\mu}}\right)-1\right] \qquad [\operatorname{Re}\mu>0]$$ MI 34

2. $$\int_0^\infty \operatorname{ci}(x) e^{-\mu x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}}\,\operatorname{Ei}\left(-\frac{1}{4\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

6.249 $$\int_0^\infty \left[\operatorname{si}\left(x^2\right) + \frac{\pi}{2}\right] e^{-\mu x}\,dx = \frac{\pi}{\mu}\left\{\left[S\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2 + \left[C\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2\right\}$$
$$[\operatorname{Re}\mu > 0]$$ ME 26

6.251

1. $$\int_0^\infty \operatorname{si}\left(\frac{1}{x}\right) e^{-\mu x}\,dx = \frac{2}{\mu}\operatorname{kei}\left(2\sqrt{\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

2. $$\int_0^\infty \operatorname{ci}\left(\frac{1}{x}\right) e^{-\mu x}\,dx = -\frac{2}{\mu}\ker\left(2\sqrt{\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

6.252

1. $$\int_0^\infty \sin px \operatorname{si}(qx)\,dx = -\frac{\pi}{2p} \qquad [p^2 > q^2]$$
$$= -\frac{\pi}{4p} \qquad [p^2 = q^2]$$
$$= 0 \qquad [p^2 < q^2]$$
FI II 652, NT 50(8)

2.[6] $$\int_0^\infty \cos px \operatorname{si}(qx)\,dx = -\frac{1}{4p}\ln\left(\frac{p+q}{p-q}\right)^2 \qquad [p \neq 0, \quad p^2 \neq q^2]$$
$$= \frac{1}{q} \qquad [p = 0]$$
FI II 652, NT 50(10)

3. $$\int_0^\infty \sin px \operatorname{ci}(qx)\,dx = -\frac{1}{4p}\ln\left(\frac{p^2}{q^2} - 1\right)^2 \qquad [p \neq 0, \quad p^2 \neq q^2]$$
$$= 0 \qquad [p = 0]$$
FI II 652, NT 50(9)

4. $$\int_0^\infty \cos px \operatorname{ci}(qx)\,dx = -\frac{\pi}{2p} \qquad [p^2 > q^2]$$
$$= -\frac{\pi}{4p} \qquad [p^2 = q^2]$$
$$= 0 \qquad [p^2 < q^2]$$
FI II 654, NT 50(7)

6.253 $$\int_0^\infty \frac{\operatorname{si}(ax)\sin bx}{1 - 2r\cos x + r^2}\,dx = -\frac{\pi\left(r^m + r^{m+1}\right)}{4b(1-r)\left(1-r^2\right)} \qquad [b = a - m]$$
$$= -\frac{\pi\left(2 + 2r - r^m - r^{m+1}\right)}{4b(1-r)\left(1-r^2\right)} \qquad [b = a + m]$$
$$= -\frac{\pi r^{m+1}}{2b(1-r)\left(1-r^2\right)} \qquad [a - m - 1 < b < a - m]$$
$$= -\frac{\pi\left(1 + r - r^{m+1}\right)}{2b(1-r)\left(1-r^2\right)} \qquad [a + m < b < a + m + 1]$$
ET I 97(10)

6.254

1.* $$\int_0^\infty \operatorname{ci}(x)\sin^2 x\frac{dx}{x} = \frac{1}{2}\left[L_2\left(\frac{1}{2}\right) - L_2\left(-\frac{1}{2}\right)\right]$$

where $L_2(x)$ is the Euler dilogarithm defined as $L_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$ and this in turn can be expressed as $L_2(z) = \Phi(z,2,1)$ in terms of the Lerch function defined in 9.550, with z real.

2.[11] $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\cos bx \cdot \frac{dx}{x} = \frac{\pi}{2}\ln\frac{a}{b}\,\mathrm{H}(a-b)$$

$[a > 0, \quad b > 0, \quad \mathrm{H}(x)$ is the Heaviside step function$]$ ET I 41(11)

6.255

1. $$\int_{-\infty}^\infty \left[\cos ax\,\operatorname{ci}(a|x|) + \sin(a|x|)\operatorname{si}(a|x|)\right]\frac{dx}{x-b} = -\pi\left[\operatorname{sign} b\cos ab\operatorname{si}(a|b|) - \sin ab\operatorname{ci}(a|b|)\right]$$

$[a > 0]$ ET II 253(4)

2. $$\int_{-\infty}^\infty \left[\sin ax\,\operatorname{ci}(a|x|) - \operatorname{sign} x\cos ax\operatorname{si}(a|x|)\right]\frac{dx}{x-b} = -\pi\left[\sin(a|b|)\operatorname{si}(a|b|) + \cos ab\operatorname{ci}(a|b|)\right]$$

$[a > 0]$ ET II 253(5)

6.256

1. $$\int_0^\infty \left[\operatorname{si}^2(x) + \operatorname{ci}^2(x)\right]\cos ax\,dx = \frac{\pi}{a}\ln(1+a) \qquad [a > 0]$$

2.* $$\int_0^\infty \left[\operatorname{si}(x)\cos x - \operatorname{ci}(x)\sin x\right]^2\,dx = \frac{\pi}{2}$$

3.* $$\int_0^\infty \operatorname{si}^2(x)\cos(ax)\,dx = \frac{\pi}{2a}\log(1+a) \qquad [0 \le a \le 2]$$

4.* $$\int_0^\infty \operatorname{ci}^2(x)\cos(ax)\,dx = \frac{\pi}{2a}\log(1+a) \qquad [0 \le a \le 2]$$

6.257 $$\int_0^\infty \operatorname{si}\left(\frac{a}{x}\right)\sin bx\,dx = -\frac{\pi}{2b}\,J_0\left(2\sqrt{ab}\right) \qquad [b > 0]$$ ET I 42(18)

6.258

1. $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\sin bx\frac{dx}{x^2+c^2}$$

$$= \frac{\pi}{4c}\left\{e^{-bc}\left[\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)\right] + e^{bc}\left[\operatorname{Ei}(-ac) - \operatorname{Ei}(-bc)\right]\right\} \qquad [0 < b \le a, \quad c > 0]$$

$$= \frac{\pi}{4c}e^{-bc}\left[\operatorname{Ei}(ac) - \operatorname{Ei}(-ac)\right] \qquad [0 < a \le b, \quad c > 0]$$

BI (460)(1)

2. $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\cos bx\frac{x\,dx}{x^2+c^2}$$

$$= -\frac{\pi}{4}\left\{e^{-bc}\left[\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)\right] + e^{bc}\left[\operatorname{Ei}(-bc) - \operatorname{Ei}(-ac)\right]\right\} \qquad [0 < b \le a, \quad c > 0]$$

$$= \frac{\pi}{4}e^{-bc}\left[\operatorname{Ei}(-ac) - \operatorname{Ei}(ac)\right] \qquad [0 < a \le b, \quad c > 0]$$

BI (460)(2, 5)

6.259

1. $$\int_0^\infty \operatorname{si}(ax)\sin bx\frac{dx}{x^2+c^2} = \frac{\pi}{2c}\operatorname{Ei}(-ac)\sinh(bc) \qquad [0<b\le a,\quad c>0]$$
$$= \frac{\pi}{4c}e^{-cb}\left[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc)-\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)\right]$$
$$+\frac{\pi}{2c}\operatorname{Ei}(-bc)\sinh(bc) \qquad [0<a\le b,\quad c>0]$$
ET I 96(8)

2. $$\int_0^\infty \operatorname{ci}(ax)\sin bx\frac{x\,dx}{x^2+c^2} = -\frac{\pi}{2}\sinh(bc)\operatorname{Ei}(-ac) \qquad [0<b\le a,\quad c>0]$$
$$= -\frac{\pi}{2}\sinh(bc)\operatorname{Ei}(-bc)+\frac{\pi}{4}e^{-bc}[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc)$$
$$-\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)] \qquad [0<a\le b,\quad c>0]$$
BI (460)(3)a, ET I 97(15)a

3. $$\int_0^\infty \operatorname{ci}(ax)\cos bx\frac{dx}{x^2+c^2}$$
$$= \frac{\pi}{2c}\cosh bc\operatorname{Ei}(-ac) \qquad [0<b\le a,\quad c>0]$$
$$= \frac{\pi}{4c}\left\{e^{-bc}\left[\operatorname{Ei}(ac)+\operatorname{Ei}(-ac)-\operatorname{Ei}(bc)\right]+e^{bc}\operatorname{Ei}(-bc)\right\} \qquad [0<a\le b,\quad c>0]$$
BI (460)(4), ET I 41(15)

4.* $$\int_0^\infty [\operatorname{ci}(x)\sin x-\operatorname{Si}(x)\cos x]\sin x\frac{x\,dx}{a^2+x^2} = \frac{1}{8}\left[\operatorname{Ei}(a)e^{-a}-\operatorname{Ei}(-a)e^{a}\right]^2$$
$[a \text{ real}]$

5.* $$\int_0^\infty [\operatorname{ci}(x)\sin x-\operatorname{Si}(x)\cos x]^2\frac{x\,dx}{a^2+x^2} = \frac{\pi^3e^{-|a|}}{8a}\sinh(a)-\frac{\pi}{8|a|}\left[\operatorname{Ei}(a)e^{-a}-\operatorname{Ei}(-a)e^{a}\right]^2$$
$[a \text{ real}]$

6.261

1. $$\int_0^\infty \operatorname{si}(bx)\cos axe^{-px}\,dx = -\frac{1}{2\left(a^2+p^2\right)}\left[\frac{a}{2}\ln\frac{p^2+(a+b)^2}{p^2+(a-b)^2}+p\arctan\frac{2bp}{b^2-a^2-p^2}\right]$$
$[a>0,\quad b>0,\quad p>0]$ ET I 40(8)

2. $$\int_0^\infty \operatorname{si}(\beta x)\cos axe^{-\mu x}\,dx = -\frac{\arctan\dfrac{\mu+ai}{\beta}}{2(\mu+ai)}-\frac{\arctan\dfrac{\mu-ai}{\beta}}{2(\mu-ai)}$$
$[a>0,\quad \operatorname{Re}\mu>|\operatorname{Im}\beta|]$ ET I 40(9)

6.262

1. $$\int_0^\infty \operatorname{ci}(bx)\sin axe^{-\mu x}\,dx = \frac{1}{2\left(a^2+\mu^2\right)}\left\{\mu\arctan\frac{2a\mu}{\mu^2+b^2-a^2}-\frac{a}{2}\ln\frac{\left(\mu^2+b^2-a^2\right)^2+4a^2\mu^2}{b^4}\right\}$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$
ET I 98(16)a

2. $$\int_0^\infty \operatorname{ci}(bx)\cos ax e^{-px}\,dx = \frac{-1}{2\left(a^2+p^2\right)}\left\{\frac{p}{2}\ln\frac{\left[\left(b^2+p^2-a^2\right)^2+4a^2p^2\right]}{b^4}+a\arctan\frac{2ap}{b^2+p^2-a^2}\right\}$$
$[a>0,\quad b>0,\quad \operatorname{Re}p>0]$ ET I 41(16)

3. $$\int_0^\infty \operatorname{ci}(\beta x)\cos ax e^{-\mu x}\,dx = \frac{-\ln\left[1+\frac{(\mu+ai)^2}{\beta^2}\right]}{4(\mu+ai)}-\frac{\ln\left[1+\frac{(\mu-ai)^2}{\beta^2}\right]}{4(\mu-ai)}$$
$[a>0,\quad \operatorname{Re}\mu>|\operatorname{Im}\beta|]$ ET I 41(17)

6.263

1. $$\int_0^\infty [\operatorname{ci}(x)\cos x+\operatorname{si}(x)\sin x]\,e^{-\mu x}\,dx = \frac{-\frac{\pi}{2}-\mu\ln\mu}{1+\mu^2}$$
$[\operatorname{Re}\mu>0]$ ME 26a, ET I 178(21)a

2. $$\int_0^\infty [\operatorname{si}(x)\cos x-\operatorname{ci}(x)\sin x]\,e^{-\mu x}\,dx = \frac{-\frac{\pi}{2}\mu+\ln\mu}{1+\mu^2}$$
$[\operatorname{Re}\mu>0]$ ME 26a, ET I 178(20)a

3. $$\int_0^\infty [\sin x-x\operatorname{ci}(x)]\,e^{-\mu x}\,dx = \frac{\ln\left(1+\mu^2\right)}{2\mu^2}$$
$[\operatorname{Re}\mu>0]$ ME 26

6.264

1. $$\int_0^\infty \operatorname{si}(x)\ln x\,dx = \boldsymbol{C}+1$$
NT 46(10)

2. $$\int_0^\infty \operatorname{ci}(x)\ln x\,dx = \frac{\pi}{2}$$
NT 56(11)

6.27 The hyperbolic sine integral and hyperbolic cosine integral functions

6.271

1. $$\int_0^\infty \operatorname{shi}(x)e^{-\mu x}\,dx = \frac{1}{2\mu}\ln\frac{\mu+1}{\mu-1}=\frac{1}{\mu}\operatorname{arccoth}\mu$$
$[\operatorname{Re}\mu>1]$ MI 34

2.[11] $$\int_0^\infty \operatorname{chi}(x)e^{-\mu x}\,dx = -\frac{1}{2\mu}\ln\left(\mu^2-1\right)$$
$[\operatorname{Re}\mu>1]$ MI 34

6.272[11] $$\int_0^\infty \operatorname{chi}(x)e^{-px^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{p}}\operatorname{Ei}\left(\frac{1}{4p}\right)$$
$[p>0]$ MI 35

6.273

1.[11] $$\int_0^\infty [\cosh x\operatorname{shi}(x)-\sinh x\operatorname{chi}(x)]\,e^{-\mu x}\,dx = \frac{\ln\mu}{\mu^2-1}$$
$[\operatorname{Re}\mu>0]$ MI 35

2.[11] $$\int_0^\infty [\cosh x\operatorname{chi}(x)+\sinh x\operatorname{shi}(x)]\,e^{-\mu x}\,dx = \frac{\mu\ln\mu}{1-\mu^2}$$
$[\operatorname{Re}\mu>2]$ MI 35

6.274[11] $$\int_0^\infty [\cosh x\,\mathrm{shi}(x) - \sinh x\,\mathrm{chi}(x)]\, e^{-\mu x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}}\, e^{\frac{1}{4\mu}}\,\mathrm{Ei}\left(-\frac{1}{4\mu}\right)$$
$[\mathrm{Re}\,\mu > 0]$ MI 35

6.275 $$\int_0^\infty [x\,\mathrm{chi}(x) - \sinh x]\, e^{-\mu x}\, dx = -\frac{\ln\left(\mu^2 - 1\right)}{2\mu^2}$$
$[\mathrm{Re}\,\mu > 1]$ MI 35

6.276 $$\int_0^\infty [\cosh x\,\mathrm{chi}(x) + \sinh x\,\mathrm{shi}(x)]\, e^{-\mu x^2}\, x\, dx = \frac{1}{8}\sqrt{\frac{\pi}{\mu^3}}\exp\left(\frac{1}{4\mu}\right)\mathrm{Ei}\left(-\frac{1}{4\mu}\right)$$
$[\mathrm{Re}\,\mu > 0]$ MI 35

6.277

1. $$\int_0^\infty [\mathrm{chi}(x) + \mathrm{ci}(x)]\, e^{-\mu x}\, dx = -\frac{\ln\left(\mu^4 - 1\right)}{2\mu}$$
$[\mathrm{Re}\,\mu > 1]$ MI 34

2. $$\int_0^\infty [\mathrm{chi}(x) - \mathrm{ci}(x)]\, e^{-\mu x}\, dx = \frac{1}{2\mu}\ln\frac{\mu^2 + 1}{\mu^2 - 1}$$
$[\mathrm{Re}\,\mu > 1]$ MI 35

6.28–6.31 The probability integral

6.281

1.[6] $$\int_0^\infty [1 - \Phi(px)]\, x^{2q-1}\, dx = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2\sqrt{\pi}qp^{2q}}$$
$[\mathrm{Re}\,q > 0, \quad \mathrm{Re}\,p > 0]$
NT 56(12), ET II 306(1)a

2.[6] $$\int_0^\infty \left[1 - \Phi\left(at^\alpha \pm \frac{b}{t^\alpha}\right)\right] dt = \frac{2b}{\sqrt{\pi}}\left(\frac{b}{a}\right)^{\frac{1-\alpha}{2\alpha}}\left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab)\right] e^{\pm 2ab}$$
$[a > 0, \quad b > 0, \quad \alpha \neq 0]$

6.282

1. $$\int_0^\infty \Phi(qt)e^{-pt}\, dt = \frac{1}{p}\left[1 - \Phi\left(\frac{p}{2q}\right)\right]\exp\left(\frac{p^2}{4q^2}\right)$$
$\left[\mathrm{Re}\,p > 0, \quad |\arg q| < \frac{\pi}{4}\right]$
MO 175, EH II 148(11)

2. $$\int_0^\infty \left[\Phi\left(x + \frac{1}{2}\right) - \Phi\left(\frac{1}{2}\right)\right] e^{-\mu x + \frac{1}{4}}\, dx = \frac{1}{(\mu+1)(\mu+2)}\exp\frac{(\mu+1)^2}{4}\left[1 - \Phi\left(\frac{\mu+1}{2}\right)\right]$$
ME 27

6.283

1. $$\int_0^\infty e^{\beta x}\left[1 - \Phi\left(\sqrt{\alpha x}\right)\right] dx = \frac{1}{\beta}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1\right]$$
$[\mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,\beta < \mathrm{Re}\,\alpha]$ ET II 307(5)

2. $$\int_0^\infty \Phi\left(\sqrt{qt}\right) e^{-pt}\, dt = \frac{\sqrt{q}}{p}\frac{1}{\sqrt{p+q}}$$
$[\mathrm{Re}\,p > 0, \quad \mathrm{Re}(q+p) > 0]$
EH II 148(12)

6.284 $$\int_0^\infty \left[1 - \Phi\left(\frac{q}{2\sqrt{x}}\right)\right] e^{-px}\, dx = \frac{1}{p}e^{-q\sqrt{p}}$$
$\left[\mathrm{Re}\,p > 0, \quad |\arg q| < \frac{\pi}{4}\right]$
EF 147(235), EH II 148(13)

6.285

1. $$\int_0^\infty [1-\Phi(x)]\, e^{-\mu^2 x^2}\, dx = \frac{\arctan\mu}{\sqrt{\pi}\mu} \qquad [\operatorname{Re}\mu > 0]$$ MI 37

2. $$\int_0^\infty \Phi(iat) e^{-a^2t^2-st}\, dt = \frac{-1}{2ai\sqrt{\pi}} \exp\left(\frac{s^2}{4a^2}\right) \operatorname{Ei}\left(-\frac{s^2}{4a^2}\right) \qquad \left[\operatorname{Re} s > 0, \quad |\arg a| < \frac{\pi}{4}\right]$$ EH II 148(14)a

6.286

1. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{\mu^2 x^2} x^{\nu-1}\, dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\nu\beta^\nu}\, {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; \frac{\mu^2}{\beta^2}\right) \qquad \left[\operatorname{Re}^2\beta > \operatorname{Re}\mu^2, \quad \operatorname{Re}\nu > 0\right]$$ ET II 306(2)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1}\, dx = 2^{\frac{\nu}{2}-1} \sec\frac{\nu\pi}{2}\, \Gamma\left(\frac{\nu}{2}\right) \qquad [0 < \operatorname{Re}\nu < 1]$$ ET I 325(9)

6.287

1. $$\int_0^\infty \Phi(\beta x) e^{-\mu x^2} x\, dx = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \qquad \left[\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0\right]$$ ME 27a, ET I 176(4)

2. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{-\mu x^2} x\, dx = \frac{1}{2\mu}\left(1 - \frac{\beta}{\sqrt{\mu+\beta^2}}\right) \qquad \left[\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0\right]$$ NT 49(14), ET I 177(9)

3.* $$I = \int_{-\infty}^\infty \frac{r}{\sigma^2} \exp\left(\frac{r}{\sigma^2}\right) Q(rA)\, Q(rB)\, dr = \frac{1}{4} - \frac{1}{2\pi}\left[\alpha \arctan\left(\frac{A}{\alpha B}\right) + \beta \arctan\left(\frac{B}{\beta A}\right)\right] \qquad B \neq A$$
$$= \frac{1}{4} - \frac{1}{\pi}\alpha \arctan\frac{1}{\alpha} \qquad B = A$$
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad \alpha = \sqrt{\frac{\sigma^2 A^2}{1+\sigma^2 A^2}}, \quad \beta = \sqrt{\frac{\sigma^2 B^2}{1+\sigma^2 B^2}},$$

6.288 $$\int_0^\infty \Phi(iax) e^{-\mu x^2} x\, dx = \frac{ai}{2\mu\sqrt{\mu - a^2}} \qquad \left[a > 0, \quad \operatorname{Re}\mu > \operatorname{Re} a^2\right]$$ MI 37a

6.289

1. $$\int_0^\infty \Phi(\beta x) e^{(\beta^2-\mu^2)x^2} x\, dx = \frac{\beta}{2\mu\,(\mu^2-\beta^2)} \qquad \left[\operatorname{Re}^2\mu > \operatorname{Re}\beta^2, \quad |\arg\mu| < \frac{\pi}{4}\right]$$ ET I 176(5)

2. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{(\beta^2-\mu^2)x^2} x\, dx = \frac{1}{2\mu(\mu+\beta)} \qquad \left[\operatorname{Re}^2\mu > \operatorname{Re}\beta^2, \quad \arg\mu < \frac{\pi}{4}\right]$$ ET I 177(10)

3. $$\int_0^\infty \Phi\left(\sqrt{b-a}x\right) e^{-(a+\mu)x^2} x\,dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \qquad [\operatorname{Re}\mu > -a > 0, \quad b > a]$$ ME 27

6.291 $$\int_0^\infty \Phi(ix) e^{-\left(\mu x + x^2\right)} x\,dx = \frac{i}{\sqrt{\pi}}\left[\frac{1}{\mu} + \frac{\mu}{4}\operatorname{Ei}\left(-\frac{\mu^2}{4}\right)\right] \qquad [\operatorname{Re}\mu > 0]$$ MI 37

6.292 $$\int_0^\infty [1-\Phi(x)]\, e^{-\mu^2 x^2} x^2\,dx = \frac{1}{2\sqrt{\pi}}\left\{\frac{\arctan\mu}{\mu^3} - \frac{1}{\mu^2\left(\mu^2+1\right)}\right\}$$
$$\left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.293 $$\int_0^\infty \Phi(x) e^{-\mu x^2}\frac{dx}{x} = \frac{1}{2}\ln\frac{\sqrt{\mu+1}+1}{\sqrt{\mu+1}-1} = \operatorname{arccoth}\sqrt{\mu+1}$$
$$[\operatorname{Re}\mu > 0]$$ MI 37a

6.294

1. $$\int_0^\infty \left[1-\Phi\left(\frac{\beta}{x}\right)\right] e^{-\mu^2 x^2} x\,dx = \frac{1}{2\mu^2}\exp(-2\beta\mu) \qquad \left[|\arg\beta| < \frac{\pi}{4}, \quad |\arg\mu| < \frac{\pi}{4}\right]$$ ET I 177(11)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] e^{-\mu^2 x^2}\frac{dx}{x} = -\operatorname{Ei}(-2\mu) \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.295

1. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) dx = \frac{1}{\sqrt{\pi}\mu}\left[\sin 2\mu\,\operatorname{ci}(2\mu) - \cos 2\mu\,\operatorname{si}(2\mu)\right]$$
$$\left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) x\,dx = \frac{\pi}{2\mu}\left[\mathbf{H}_1(2\mu) - Y_1(2\mu)\right] - \frac{1}{\mu^2}$$
$$\left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

3. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) \frac{dx}{x} = \frac{\pi}{2}\left[\mathbf{H}_0(2\mu) - Y_0(2\mu)\right]$$
$$\left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.296 $$\int_0^\infty \left\{\left(x^2+a^2\right)\left[1-\Phi\left(\frac{a}{\sqrt{2}x}\right)\right] - \sqrt{\frac{2}{\pi}}ax\cdot e^{-\frac{a^2}{2x^2}}\right\} e^{-\mu^2 x^2} x\,dx = \frac{1}{2\mu^4}e^{-a\mu\sqrt{2}}$$
$$\left[|\arg\mu| < \frac{\pi}{4}, \quad a > 0\right]$$ MI 38a

6.297

1. $$\int_0^\infty \left[1-\Phi\left(\gamma x + \frac{\beta}{x}\right)\right] e^{\left(\gamma^2-\mu\right)x^2} x\,dx = \frac{1}{2\sqrt{\mu}\left(\sqrt{\mu}+\gamma\right)}\exp\left[-2\left(\beta\gamma + \beta\sqrt{\mu}\right)\right]$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$$ ET I 177(12)a

2. $$\int_0^\infty \left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right] \exp\left[-\left(\mu^2-a^2\right)x^2 + ab\right] x\,dx = \frac{e^{-b\mu}}{2\mu(\mu+a)}$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}\mu > 0]$$ MI 38

3. $$\int_0^\infty \left\{\left[1-\Phi\left(\frac{b-2ax^2}{2x}\right)\right]e^{-ab}+\left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right]e^{ab}\right\}e^{-\mu x^2}x\,dx=\frac{1}{\mu}\exp\left(-b\sqrt{a^2+\mu}\right)$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$ MI 38

6.298 $$\int_0^\infty \left\{2\cosh ab-e^{-ab}\,\Phi\left(\frac{b-2ax^2}{2x}\right)-e^{ab}\,\Phi\left(\frac{b+2ax^2}{2x}\right)\right\}e^{-(\mu-a^2)x^2}x\,dx=\frac{1}{\mu-a^2}\exp\left(-b\sqrt{\mu}\right)$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$ MI 38

6.299 $$\int_0^\infty \cosh(2\nu t)\exp\left[(a\cosh t)^2\right]\left[1-\Phi\left(a\cosh t\right)\right]dt=\frac{1}{2\cos(\nu\pi)}\exp\left(\tfrac{1}{2}a^2\right)K_\nu\left(a^2\right)$$
$\left[\operatorname{Re}a>0,\quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}\right]$ ET II 308(10)

6.311 $$\int_0^\infty [1-\Phi(ax)]\sin bx\,dx=\frac{1}{b}\left(1-e^{-\frac{b^2}{4a^2}}\right)$$
$[a>0,\quad b>0]$ ET I 96(4)

6.312 $$\int_0^\infty \Phi(ax)\sin bx^2\,dx=\frac{1}{4\sqrt{2\pi b}}\left(\ln\frac{b+a^2+a\sqrt{2b}}{b+a^2-a\sqrt{2b}}+2\arctan\frac{a\sqrt{2b}}{b-a^2}\right)$$
$[a>0,\quad b>0]$ ET I 96(3)

6.313

1. $$\int_0^\infty \sin(\beta x)\left[1-\Phi\left(\sqrt{\alpha x}\right)\right]dx=\frac{1}{\beta}-\left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}-\alpha\right]^{-\frac{1}{2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET II 307(6)

2. $$\int_0^\infty \cos(\beta x)\left[1-\Phi\left(\sqrt{\alpha x}\right)\right]dx=\left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{-\frac{1}{2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET II 307(7)

6.314

1. $$\int_0^\infty \sin(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]dx=b^{-1}\exp\left[-(2ab)^{\frac{1}{2}}\right]\cos\left[(2ab)^{\frac{1}{2}}\right]$$
$[\operatorname{Re}a>0,\quad b>0]$ ET II 307(8)

2. $$\int_0^\infty \cos(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]dx=-b^{-1}\exp\left[-(2ab)^{\frac{1}{2}}\right]\sin\left[(2ab)^{\frac{1}{2}}\right]$$
$[\operatorname{Re}a>0,\quad b>0]$ ET II 307(9)

6.315

1. $$\int_0^\infty x^{\nu-1}\sin(\beta x)\left[1-\Phi(\alpha x)\right]dx=\frac{\Gamma\left(1+\frac{1}{2}\nu\right)\beta}{\sqrt{\pi}(\nu+1)\alpha^{\nu+1}}\;{}_2F_2\left(\frac{\nu+1}{2},\frac{\nu}{2}+1;\frac{3}{2},\frac{\nu+3}{2};-\frac{\beta^2}{4\alpha^2}\right)$$
$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-1]$ ET II 307(3)

2. $$\int_0^\infty x^{\nu-1}\cos(\beta x)\left[1-\Phi(\alpha x)\right]dx=\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)}{\sqrt{\pi}\nu\alpha^{\nu}}\;{}_2F_2\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{1}{2},\frac{\nu}{2}+1;-\frac{\beta^2}{4\alpha^2}\right)$$
$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>0]$ ET II 307(4)

3. $$\int_0^\infty [1-\Phi(ax)]\cos bx\cdot x\,dx=\frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right)-\frac{1}{b^2}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right]$$
$[a>0,\quad b>0]$ ET I 40(5)

4. $$\int_0^\infty [\Phi(ax)-\Phi(bx)]\cos px\frac{dx}{x}=\frac{1}{2}\left[\mathrm{Ei}\left(-\frac{p^2}{4b^2}\right)-\mathrm{Ei}\left(\frac{p^2}{4a^2}\right)\right]$$
$[a>0,\quad b>0,\quad p>0]$ ET I 40(6)

5. $$\int_0^\infty x^{-\frac{1}{2}}\,\Phi\left(a\sqrt{x}\right)\sin bx\,dx=\frac{1}{2\sqrt{2\pi b}}\left\{\ln\left[\frac{b+a\sqrt{2b}+a^2}{b-a\sqrt{2b}+a^2}\right]+2\arctan\left[\frac{a\sqrt{2b}}{b-a^2}\right]\right\}$$
$[a>0,\quad b>0]$ ET I 96(3)

6.316 $$\int_0^\infty e^{\frac{1}{2}x^2}\left[1-\Phi\left(\frac{x}{\sqrt{2}}\right)\right]\sin bx\,dx=\sqrt{\frac{\pi}{2}}e^{\frac{b^2}{2}}\left[1-\Phi\left(\frac{b}{\sqrt{2}}\right)\right]$$
$[b>0]$ ET I 96(5)

6.317[6] $$\int_0^\infty e^{-a^2x^2}\,\Phi(iax)\sin bx\,dx=\frac{i}{a}\frac{\sqrt{\pi}}{2}e^{-\frac{b^2}{4a^2}}$$
$[b>0]$ ET I 96(2)

6.318 $$\int_0^\infty [1-\Phi(x)]\,\mathrm{si}(2px)\,dx=\frac{2}{\pi p}\left(1-e^{-p^2}\right)-\frac{2}{\sqrt{\pi}}\left(1-\Phi(p)\right)$$
$[p>0]$ NT 61(13)a

6.32 Fresnel integrals

6.321

1. $$\int_0^\infty\left[\frac{1}{2}-S(px)\right]x^{2q-1}\,dx=\frac{\sqrt{2}\,\Gamma\left(q+\frac{1}{2}\right)\sin\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}}$$
$\left[0<\mathrm{Re}\,q<\frac{3}{2},\quad p>0\right]$ NT 56(14)a

2. $$\int_0^\infty\left[\frac{1}{2}-C(px)\right]x^{2q-1}\,dx=\frac{\sqrt{2}\,\Gamma\left(q+\frac{1}{2}\right)\cos\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}}$$
$\left[0<\mathrm{Re}\,q<\frac{3}{2},\quad p>0\right]$ NT 56(13)a

6.322

1. $$\int_0^\infty S(t)e^{-pt}\,dt=\frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right]+\sin\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right]\right\}$$
MO 173a

2. $$\int_0^\infty C(t)e^{-pt}\,dt=\frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right]-\sin\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right]\right\}$$
MO 172a

6.323

1. $$\int_0^\infty S\left(\sqrt{t}\right)e^{-pt}\,dx=\frac{\left(\sqrt{p^2+1}-p\right)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}$$
EF 122(58)a

2. $$\int_0^\infty C\left(\sqrt{t}\right) e^{-pt}\,dt = \frac{\left(\sqrt{p^2+1}+p\right)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}$$ EF 122(58)a

6.324

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \sin 2px\,dx = \frac{1+\sin p^2 - \cos p^2}{4p} \qquad [p>0]$$ NT 61(12)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \sin 2px\,dx = \frac{1-\sin p^2 - \cos p^2}{4p} \qquad [p>0]$$ NT 61(11)a

6.325

1. $$\int_0^\infty S(x)\sin b^2x^2\,dx = \frac{\sqrt{\pi}}{b}2^{-\frac{5}{2}} \qquad \left[0<b^2<1\right]$$
$$= 0 \qquad \left[b^2>1\right]$$
ET I 98(21)a

2. $$\int_0^\infty C(x)\cos b^2x^2\,dx = \frac{\sqrt{\pi}}{b}2^{-\frac{5}{2}} \qquad \left[0<b^2<1\right]$$
$$= 0 \qquad \left[b^2>1\right]$$
ET I 42(22)

6.326

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \mathrm{si}(2px)\,dx = \left(\frac{\pi}{8}\right)^{1/2}(S(p)+C(p)-1) - \frac{1+\sin p^2-\cos p^2}{4p}$$
$$[p>0]$$ NT 61(15)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \mathrm{si}(2px)\,dx = \left(\frac{\pi}{8}\right)^{1/2}(S(p)-C(p)) - \frac{1-\sin p^2-\cos p^2}{4p}$$
$$[p>0]$$ NT 61(14)a

6.4 The Gamma Function and Functions Generated by It

6.41 The gamma function

6.411[11] $$\int_{-\infty}^\infty \Gamma(\alpha+x)\,\Gamma(\beta-x)\,dx = -i\pi 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$$
$$[\mathrm{Re}(\alpha+\beta)<1 \text{ and either } \mathrm{Im}\,\alpha<0<\mathrm{Im}\,\beta \text{ or } \mathrm{Im}\,\beta<0<\mathrm{Im}\,\alpha\,]$$
ET II 297(1)

$$= i\pi 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$$
$$[\mathrm{Re}(\alpha+\beta)<1, \quad \mathrm{Im}\,\alpha<0, \quad \mathrm{Im}\,\beta<0]$$
ET II 297(2)

$$= 0$$
$$[\mathrm{Re}(\alpha+\beta)<1, \quad \mathrm{Im}\,\alpha>0, \quad \mathrm{Im}\,\beta>0]$$
ET II 297(3)

6.412
$$\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds = 2\pi i \frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$
$[\operatorname{Re}\alpha, \quad \operatorname{Re}\beta, \quad \operatorname{Re}\gamma, \quad \operatorname{Re}\delta > 0]$ ET II 302(32)

6.413

1. $$\int_0^\infty |\Gamma(a+ix)\,\Gamma(b+ix)|^2\,dx = \frac{\sqrt{\pi}\,\Gamma(a)\,\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\,\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{2\,\Gamma\left(a+b+\frac{1}{2}\right)}$$
$[a>0, \quad b>0]$ ET II 302(27)

2. $$\int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2 dx = \frac{\sqrt{\pi}\,\Gamma(a)\,\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-a-\frac{1}{2}\right)}{2\,\Gamma(b)\,\Gamma\left(b-\frac{1}{2}\right)\Gamma(b-a)}$$
$\left[0<a<b-\frac{1}{2}\right]$ ET II 302(28)

6.414

1. $$\int_{-\infty}^\infty \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\,dx = 0$$
$[\operatorname{Im}\alpha \neq 0, \quad \operatorname{Re}(\alpha-\beta) < -1]$ ET II 297(4)

2. $$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$
$[\operatorname{Re}(\alpha+\beta) > 1]$ ET II 297(5)

3. $$\int_{-\infty}^\infty \frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(\alpha+x)\,\Gamma(\beta+x)}\,dx = 0$$
$[\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im}\gamma, \quad \operatorname{Im}\delta > 0]$ ET II 299(18)

4. $$\int_{-\infty}^\infty \frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(\alpha+x)\,\Gamma(\beta+x)}\,dx = \frac{\pm 2\pi^2 i\,\Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)]\,\Gamma(\alpha-\gamma)\,\Gamma(\alpha-\delta)\,\Gamma(\beta-\gamma)\,\Gamma(\beta-\delta)}$$
$[\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im}\gamma < 0, \quad \operatorname{Im}\delta < 0$. In the numerator, we take the plus sign if $\operatorname{Im}\gamma > \operatorname{Im}\delta$ and the minus sign if $\operatorname{Im}\gamma < \operatorname{Im}\delta$.] ET II 300(19)

5. $$\int_{-\infty}^\infty \frac{\Gamma(\alpha-\beta-\gamma+x+1)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)} = \frac{\pi\exp\left(\pm\frac{1}{2}\pi(\delta-\gamma)i\right)}{\Gamma(\beta+\gamma-1)\,\Gamma\left(\frac{1}{2}(\alpha+\beta)\right)\Gamma\left(\frac{1}{2}(\gamma-\delta+1)\right)}$$
$[\operatorname{Re}(\beta+\gamma) > 1, \quad \delta = \alpha-\beta-\gamma+1, \quad \operatorname{Im}\delta \neq 0$. The sign is plus in the argument if the exponential for $\operatorname{Im}\delta > 0$ and minus for $\operatorname{Im}\delta < 0$.] ET II 300(20)

6. $$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)}$$
$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3]$ ET II 300(21)

6.415

1. $$\int_{-\infty}^{-\infty} \frac{R(x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)}\int_0^1 R(t)\,dt$$
$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3, \quad R(x+1) = R(x)]$ ET II 301(24)

2. $$\int_{-\infty}^{\infty}\frac{R(x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}=\frac{\displaystyle\int_0^1 R(t)\cos\left[\tfrac{1}{2}\pi(2t+\alpha-\beta)\right]\,dt}{\Gamma\left(\dfrac{\alpha+\beta}{2}\right)\Gamma\left(\dfrac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$$[\alpha+\delta=\beta+\gamma,\quad \operatorname{Re}(\alpha+\beta+\gamma+\delta)>2,\quad R(x+1)=-R(x)]$$ ET II 301(25)

6.42 Combinations of the gamma function, the exponential, and powers

6.421

1. $$\int_{-\infty}^{\infty}\Gamma(\alpha+x)\,\Gamma(\beta-x)\exp\left[2(\pi n+\theta)xi\right]\,dx=2\pi i\,\Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta}\exp[(\beta-\alpha)i\theta]$$
$$\times\left[\eta_n(\beta)\exp(2n\pi\beta i)-\eta_n(-\alpha)\exp(-2n\pi\alpha i)\right]$$
$$\left[\operatorname{Re}(\alpha+\beta)<1,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n\text{ an integer},\quad \eta_n(\xi)=\begin{cases}0 & \text{if }\left(\frac{1}{2}-n\right)\operatorname{Im}\xi>0\\ \operatorname{sign}\left(\frac{1}{2}-n\right) & \text{if }\left(\frac{1}{2}-n\right)\operatorname{Im}\xi<0\end{cases}\right]$$
ET II 298(7)

2. $$\int_{-\infty}^{\infty}\frac{e^{\pi icx}\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+kx)\,\Gamma(\delta-kx)}=0$$
$$[\operatorname{Re}(\alpha+\beta+\gamma+\delta)>2,\quad c\text{ and }k\text{ are real},\quad |c|>|k|+1]$$ ET II 301(26)

3. $$\int_{-\infty}^{\infty}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx$$
$$=2\pi i\operatorname{sign}\left(n+\tfrac{1}{2}\right)\frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\exp[-(2\pi n+\pi-\theta)\alpha i+\theta i(\beta-1)]$$
$$\left[\operatorname{Re}(\beta-\alpha)>0,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n\text{ is an integer},\quad \left(n+\tfrac{1}{2}\right)\operatorname{Im}\alpha<0\right]$$ ET II 298(8)

4. $$\int_{-\infty}^{\infty}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx=0$$
$$\left[\operatorname{Re}(\beta-\alpha)>0,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n\text{ is an integer},\quad \left(n+\tfrac{1}{2}\right)\operatorname{Im}\alpha>0\right]$$ ET II 297(6)

6.422

1. $$\int_{-i\infty}^{i\infty}\Gamma(s-k-\lambda)\,\Gamma\left(\lambda+\mu-s+\tfrac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\tfrac{1}{2}\right)z^s\,ds$$
$$=2\pi i\,\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-k+\mu\right)z^{\lambda}e^{\frac{z}{2}}\,W_{k,\mu}(z)$$
$$\left[\operatorname{Re}(k+\lambda)<0,\quad \operatorname{Re}\lambda>|\operatorname{Re}\mu|-\tfrac{1}{2},\quad |\arg z|<\tfrac{3}{2}\pi\right]$$ ET II 302(29)

2. $$\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\alpha+s)\,\Gamma(-s)\,\Gamma(1-c-s)x^s\,ds=2\pi i\,\Gamma(\alpha)\,\Gamma(\alpha-c+1)\Psi(\alpha,c;x)$$
$$\left[-\operatorname{Re}\alpha<\gamma<\min\left(0,1-\operatorname{Re}c\right),\quad -\tfrac{3}{2}\pi<\arg x<\tfrac{3}{2}\pi\right]$$ EH I 256(5)

3. $$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s)\,\Gamma(\beta+s)t^s\,ds = 2\pi i\,\Gamma(\beta)(1+t)^{-\beta} \qquad [0>\gamma>\operatorname{Re}(1-\beta),\quad |\arg t|<\pi]$$

EH I 256, BU 75

4. $$\int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right)\Gamma(-t)\left(\sqrt{2}\right)^{t-p-2} z^t\,dt = 2\pi i e^{\frac{1}{4}z^2}\,\Gamma(-p)\,D_p(z)$$

$$\left[|\arg z|<\tfrac{3}{4}\pi,\quad p \text{ is not a positive integer}\right]$$ WH

5. $$\int_{-i\infty}^{i\infty} \Gamma(s)\,\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{4}-s\right)\Gamma\left(\tfrac{1}{2}\nu-\tfrac{1}{4}-s\right)\left(\frac{z^2}{2}\right)^s ds$$

$$= 2\pi i\cdot 2^{\frac{1}{4}-\frac{1}{2}\nu} z^{-\frac{1}{2}} e^{\frac{3}{4}z^2}\,\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{4}\right)\Gamma\left(\tfrac{1}{2}\nu-\tfrac{1}{4}\right)D_\nu(z)$$

$$\left[|\arg z|<\tfrac{3}{4}\pi,\quad \nu\neq\tfrac{1}{2},\quad -\tfrac{1}{2},\quad -\tfrac{3}{2},\ldots\right]$$ EH II 120

6.[3] $$\int_{c-i\infty}^{c+i\infty} \left(\tfrac{1}{2}x\right)^{-s}\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{2}s\right)\left[\Gamma\left(1+\tfrac{1}{2}\nu-\tfrac{1}{2}s\right)\right]^{-1} ds = 4\pi i\,J_\nu(x)$$

$$[x>0, -\operatorname{Re}\nu<c<1]$$ EH II 21(34)

7. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\,\Gamma(-s)\left(-\tfrac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi}\,H_\nu^{(1)}(z)$$

$$\left[|\arg(-iz)|<\frac{\pi}{2},\quad 0<\operatorname{Re}\nu<c\right]$$

EH II 83(34)

8. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\,\Gamma(-s)\left(\tfrac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi}\,H_\nu^{(2)}(z)$$

$$\left[|\arg(iz)|<\frac{\pi}{2},\quad 0<\operatorname{Re}\nu<c\right]$$

EH II 83(35)

9. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)}\,ds = 2\pi i\,J_\nu(x) \qquad [x>0,\quad \operatorname{Re}\nu>0]$$ EH II 83(36)

10. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\,\Gamma(-2\nu-s)\,\Gamma\left(\nu+s+\tfrac{1}{2}\right)(-2iz)^s\,ds = -\pi^{\frac{5}{2}} e^{-i(z-\nu\pi)}\sec(\nu\pi)(2z)^{-\nu}\,H_\nu^{(1)}(z)$$

$$\left[|\arg(-iz)|<\tfrac{3}{2}\pi,\quad 2\nu\neq\pm1,\quad \pm3\ldots\right]$$

EH II 83(37)

11. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\,\Gamma(-2\nu-s)\,\Gamma\left(\nu+s+\tfrac{1}{2}\right)(2iz)^s\,ds = \pi^{\frac{5}{2}} e^{i(z-\nu\pi)}\sec(\nu\pi)(2z)^{-\nu}\,H_\nu^{(2)}(z)$$

$$\left[|\arg(iz)|<\tfrac{3}{2}\pi,\quad 2\nu\neq\pm1,\quad \pm3\ldots\right]$$

EH II 84(38)

12. $$\int_{-i\infty}^{i\infty} \Gamma(s)\,\Gamma\left(\tfrac{1}{2}-s-\nu\right)\Gamma\left(\tfrac{1}{2}-s+\nu\right)(2z)^s\,ds = 2^{\frac{3}{2}}\pi^{\frac{3}{2}} i z^{\frac{1}{2}} e^z\sec(\nu\pi)\,K_\nu(z)$$

$$\left[|\arg z|<\tfrac{3}{2}\pi,\quad 2\nu\neq\pm1,\quad \pm3,\ldots\right]$$

EH II 84(39)

13. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s\,\Gamma(1+s)}x^{2s}\,ds = 4\pi\int_{2x}^{\infty}\frac{J_0(t)}{t}\,dt \qquad [x>0]$$ MO 41

14. $$\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s\,ds = 2\pi i\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;z)$$

[For $\arg(-z)<\pi$, the path of integration must separate the poles of the integrand at the points $s=0,1,2,3,\dots$ from the poles $s=-\alpha-n$ and $s=-\beta-n$ (for $n=0,1,2,\dots$).]

15. $$\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\,\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s\,ds = \frac{2\pi i\,\Gamma(\alpha)}{\Gamma(\gamma)}\;{}_1F_1(\alpha;\gamma;z)$$
$$\left[-\frac{\pi}{2}<\arg(-z)<\frac{\pi}{2},\quad 0>\delta>-\operatorname{Re}\alpha,\quad \gamma\neq 0,1,2,\dots\right]$$ EH I 62(15), EH I 256(4)

16. $$\int_{-i\infty}^{i\infty}\left[\frac{\Gamma\left(\frac12-s\right)}{\Gamma(s)}\right]^2 z^s\,ds = 2\pi i z^{\frac12}\left[2\pi^{-1}\,K_0\left(4z^{\frac14}\right)-\,Y_0\left(4z^{\frac14}\right)\right]$$
$$[z>0]$$ ET II 303(33)

17. $$\int_{-i\infty}^{i\infty}\frac{\Gamma\left(\lambda+\mu-s+\frac12\right)\Gamma\left(\lambda-\mu-s+\frac12\right)}{\Gamma(\lambda-k-s+1)}z^s\,ds = 2\pi i z^{\lambda}e^{-\frac{z}{2}}\;W_{k,\mu}(z)$$
$$\left[\operatorname{Re}\lambda>|\operatorname{Re}\mu|-\tfrac12,\quad |\arg z|<\frac{\pi}{2}\right]$$ ET II 302(30)

18. $$\int_{-i\infty}^{i\infty}\frac{\Gamma(k-\lambda+s)\,\Gamma\left(\lambda+\mu-s+\frac12\right)}{\Gamma\left(\mu-\lambda+s+\frac12\right)}z^s\,ds = 2\pi i\frac{\Gamma\left(k+\mu+\frac12\right)}{\Gamma(2\mu+1)}z^{\lambda}e^{-\frac{z}{2}}\;M_{k,\mu}(z)$$
$$\left[\operatorname{Re}(k-\lambda)>0,\quad \operatorname{Re}(\lambda+\mu)>-\tfrac12,\quad |\arg z|<\frac{\pi}{2}\right]$$ ET II 302(31)

19. $$\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma\left(b_j-s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-s\right)}z^s\,ds = 2\pi i\,G^{\,pq}_{\,mn}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$
$$\Bigg[p+q<2(m+n);\quad |\arg z|<\left(m+n-\tfrac12p-\tfrac12q\right)\pi;$$
$$\operatorname{Re}a_k<1,\quad k=1,\dots,n;\quad \operatorname{Re}b_j>0,\quad j=1,\dots,m\Bigg]$$ ET II 303(34)

6.423

1. $$\int_0^\infty e^{-\alpha x}\frac{dx}{\Gamma(1+x)} = \nu\left(e^{-\alpha}\right)$$ MI 39, EH III 222(16)

2. $$\int_0^\infty e^{-\alpha x}\frac{dx}{\Gamma(x+\beta+1)} = e^{\beta\alpha}\nu\left(e^{-\alpha},\beta\right)$$ MI 39, EH III 222(16)

3. $$\int_0^\infty e^{-\alpha x}\frac{x^m}{\Gamma(x+1)}\,dx = \mu\left(e^{-\alpha},m\right)\Gamma(m+1)\qquad [\operatorname{Re}m>-1]$$ MI 39, EH III 222(17)

4. $$\int_0^\infty e^{-\alpha x}\frac{x^m}{\Gamma(x+n+1)}\,dx = e^{n\alpha}\mu\left(e^{-\alpha},m,n\right)\Gamma(m+1)$$ MI 39, EH III 222(17)

6.424 $$\int_{-\infty}^\infty \frac{R(x)\exp[(2\pi n+\theta)xi]\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{\left[2\cos\left(\frac{\theta}{2}\right)\right]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\exp\left[\frac{1}{2}\theta(\beta-\alpha)i\right]\int_0^1 R(t)\exp(2\pi nti)\,dt$$
$[\mathrm{Re}(\alpha+\beta)>1,\quad -\pi<\theta<\pi,\quad n \text{ is an integer},\quad R(x+1)=R(x)]$ ET II 299(16)

6.43 Combinations of the gamma function and trigonometric functions

6.431

1. $$\int_{-\infty}^{-\infty}\frac{\sin rx\,dx}{\Gamma(p+x)\,\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\sin\frac{r(q-p)}{2}}{\Gamma(p+q-1)} \qquad [|r|<\pi]$$
$$= 0 \qquad [|r|>\pi]$$
$[r \text{ is real};\quad \mathrm{Re}(p+q)>1]$ MO 10a, ET II 298(9, 10)

2. $$\int_{-\infty}^{\infty}\frac{\cos rx\,dx}{\Gamma(p+x)\,\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\cos\frac{r(q-p)}{2}}{\Gamma(p+q-1)} \qquad [|r|<\pi]$$
$$= 0 \qquad [|r|>\pi]$$
$[r \text{ is real};\quad \mathrm{Re}(p+q)>1]$ MO 10a, ET II 299(13, 14)

6.432 $$\int_{-\infty}^{\infty}\frac{\sin(m\pi x)}{\sin(\pi x)}\frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = 0 \qquad [m \text{ is an even integer}]$$
$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad [m \text{ is an odd integer}]$$
$[\mathrm{Re}(\alpha+\beta)>1]$ ET II 298(11, 12)

6.433

1. $$\int_{-\infty}^{\infty}\frac{\sin\pi x\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\sin\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\,\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$[\alpha+\delta=\beta+\gamma,\quad \mathrm{Re}(\alpha+\beta+\gamma+\delta)>2]$ ET II 300(22)

2. $$\int_{-\infty}^{\infty}\frac{\cos\pi x\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\cos\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\,\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$[\alpha+\delta=\beta+\gamma,\quad \mathrm{Re}(\alpha+\beta+\gamma+\delta)>2]$ ET II 301(23)

6.44 The logarithm of the gamma function*

6.441

1. $$\int_p^{p+1} \ln\Gamma(x)\,dx = \frac{1}{2}\ln 2\pi + p\ln p - p$$ FI II 784

2. $$\int_0^1 \ln\Gamma(x)\,dx = \int_0^1 \ln\Gamma(1-x)\,dx = \frac{1}{2}\ln 2\pi$$ FI II 783

3. $$\int_0^1 \ln\Gamma(x+q)\,dx = \frac{1}{2}\ln 2\pi + q\ln q - q \qquad [q \geq 0]$$ NH 89(17), ET II 304(40)

4. $$\int_0^z \ln\Gamma(x+1)\,dx = \frac{z}{2}\ln 2\pi - \frac{z(z+1)}{2} + z\ln\Gamma(z+1) - \ln G(z+1),$$
$$\text{where } G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right) \prod_{k=1}^{\infty} \left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right)\right\}$$ WH

5. $$\int_0^n \ln\Gamma(\alpha+x)\,dx = \sum_{k=0}^{n-1}(a+k)\ln(a+k) - na + \frac{1}{2}n\ln(2\pi) - \frac{1}{2}n(n-1)$$
$$[a \geq 0; \quad n = 1, 2, \ldots]$$ ET II 304(41)

6.442 $$\int_0^1 \exp(2\pi nxi)\ln\Gamma(a+x)\,dx = (2\pi ni)^{-1}\left[\ln a - \exp(-2\pi nai)\,\mathrm{Ei}(2\pi nai)\right]$$
$$[a > 0; \quad n = \pm 1, \pm 2, \ldots]$$ ET II 304(38)

6.443

1. $$\int_0^1 \ln\Gamma(x)\sin 2\pi nx\,dx = \frac{1}{2\pi n}\left[\ln(2\pi n) + \boldsymbol{C}\right]$$ NH 203(5), ET II 304(42)

2. $$\int_0^1 \ln\Gamma(x)\sin(2n+1)\pi x\,dx = \frac{1}{(2n+1)\pi}\left[\ln\left(\frac{\pi}{2}\right) + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) + \frac{1}{2n+1}\right]$$
ET II 305(43)

3. $$\int_0^1 \ln\Gamma(x)\cos 2\pi nx\,dx = \frac{1}{4n}$$ NH 203(6), ET II 305(44)

4.[8] $$\int_0^1 \ln\Gamma(x)\cos(2n+1)\pi x\,dx = \frac{2}{\pi^2}\left[\frac{1}{(2n+1)^2}\left(\boldsymbol{C} + \ln 2\pi\right) + 2\sum_{k=2}^{\infty}\frac{\ln k}{4k^2 - (2n+1)^2}\right]$$ NH 203(6)

5. $$\int_0^1 \sin(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}\left[\ln a + \cos(2\pi na)\,\mathrm{ci}(2\pi na) - \sin(2\pi na)\,\mathrm{si}(2\pi na)\right]$$
$$[a > 0; \quad n = 1, 2, \ldots]$$ ET II 304(36)

6. $$\int_0^1 \cos(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}\left[\sin(2\pi na)\,\mathrm{ci}(2\pi na) + \cos(2\pi na)\,\mathrm{si}(2\pi na)\right]$$
$$[a > 0; \quad n = 1, 2, \ldots]$$ ET II 304(37)

*Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.

6.45 The incomplete gamma function

6.451

1. $$\int_0^\infty e^{-\alpha x}\,\gamma(\beta,x)\,dx=\frac{1}{\alpha}\,\Gamma(\beta)(1+\alpha)^{-\beta}\qquad[\beta>0]$$ MI 39

2. $$\int_0^\infty e^{-\alpha x}\,\Gamma(\beta,x)\,dx=\frac{1}{\alpha}\,\Gamma(\beta)\left[1-\frac{1}{(\alpha+1)^\beta}\right]\qquad[\beta>0]$$ MI 39

6.452

1. $$\int_0^\infty e^{-\mu x}\gamma\left(\nu,\frac{x^2}{8a^2}\right)\,dx=\frac{1}{\mu}2^{-\nu-1}\,\Gamma(2\nu)e^{(a\mu)^2}\,D_{-2\nu}(2a\mu)$$
$$\left[|\arg a|<\frac{\pi}{4},\quad \operatorname{Re}\nu>-\frac{1}{2},\quad \operatorname{Re}\mu>0\right]$$
ET I 179(36)

2. $$\int_0^\infty e^{-\mu x}\gamma\left(\frac{1}{4},\frac{x^2}{8a^2}\right)\,dx=\frac{2^{\frac{3}{4}}\sqrt{a}}{\sqrt{\mu}}e^{(a\mu)^2}\,K_{\frac{1}{4}}\left(a^2\mu^2\right)\qquad\left[|\arg a|<\frac{\pi}{4},\quad \operatorname{Re}\mu>0\right]$$ ET I 179(35)

6.453 $$\int_0^\infty e^{-\mu x}\,\Gamma\left(\nu,\frac{a}{x}\right)\,dx=2a^{\frac{1}{2}\nu}\mu^{\frac{1}{2}\nu-1}\,K_\nu\left(2\sqrt{\mu a}\right)\qquad\left[|\arg a|<\frac{\pi}{2},\quad \operatorname{Re}\mu>0\right]$$ ET I 179(32)

6.454 $$\int_0^\infty e^{-\beta x}\gamma\left(\nu,\alpha\sqrt{x}\right)\,dx=2^{-\frac{1}{2}\nu}\alpha^\nu\beta^{-\frac{1}{2}\nu-1}\,\Gamma(\nu)\exp\left(\frac{\alpha^2}{8\beta}\right)D_{-\nu}\left(\frac{\alpha}{\sqrt{2\beta}}\right)$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>0]$$
ET II 309(19), MI 39a

6.455

1. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\,\Gamma(\nu,\alpha x)\,dx=\frac{\alpha^\nu\,\Gamma(\mu+\nu)}{\mu(\alpha+\beta)^{\mu+\nu}}\;{}_2F_1\left(1,\mu+\nu;\mu+1;\frac{\beta}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta)>0,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}(\mu+\nu)>0]$$ ET II 309(16)

2. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\,\gamma(\nu,\alpha x)\,dx=\frac{\alpha^\nu\,\Gamma(\mu+\nu)}{\nu(\alpha+\beta)^{\mu+\nu}}\;{}_2F_1\left(1,\mu+\nu;\nu+1;\frac{\alpha}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta)>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}(\mu+\nu)>0]$$ ET II 308(15)

6.456

1. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\gamma\left(\nu,\frac{1}{4x}\right)\,dx=\sqrt{\pi}\frac{\gamma\left(2\nu,\sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39a

2. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\,\Gamma\left(\nu,\frac{1}{4x}\right)\,dx=\frac{\sqrt{\pi}\,\Gamma\left(2\nu,\sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39a

6.457

1. $$\int_0^\infty e^{-\alpha x}\frac{(4x)^\nu}{\sqrt{x}}\gamma\left(\nu+1,\frac{1}{4x}\right)\,dx=\sqrt{\pi}\frac{\gamma\left(2\nu+1,\sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39

2. $$\int_0^\infty e^{-\alpha x}\frac{(4x)^\nu}{\sqrt{x}}\,\Gamma\left(\nu+1,\frac{1}{4x}\right)\,dx=\sqrt{\pi}\frac{\Gamma\left(2\nu+1,\sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39

6.458 $$\int_0^\infty x^{1-2\nu}\exp\left(\alpha x^2\right)\sin(bx)\,\Gamma\left(\nu,\alpha x^2\right)\,dx=\pi^{\frac12}2^{-\nu}\alpha^{\nu-1}\,\Gamma\left(\tfrac32-\nu\right)\exp\left(\frac{b^2}{8\alpha}\right)D_{2\nu-2}\left[\frac{b}{(2\alpha)^{\frac12}}\right]$$
$$\left[|\arg\alpha|<\frac{3\pi}{2},\quad 0<\operatorname{Re}\nu<1\right]$$
ET II 309(18)

6.46–6.47 The function $\psi(x)$

6.461 $$\int_1^x \psi(t)\,dt=\ln\Gamma(x)$$

6.462 $$\int_0^1 \psi(\alpha+x)\,dx=\ln\alpha \qquad [\alpha>0]$$ ET II 305(1)

6.463 $$\int_0^\infty x^{-\alpha}\left[\boldsymbol{C}+\psi(1+x)\right]=-\pi\operatorname{cosec}(\pi\alpha)\,\zeta(\alpha) \qquad [1<\operatorname{Re}\alpha<2]$$ ET II 305(6)

6.464 $$\int_0^1 e^{2\pi nxi}\,\psi(\alpha+x)\,dx=e^{-2\pi n\alpha i}\operatorname{Ei}(2\pi n\alpha i) \qquad [\alpha>0;\quad n=\pm i,\pm 2,\ldots]$$ ET II 305(2)

6.465

1.[8] $$\int_0^1 \psi(x)\sin\pi x\,dx=-\frac{2}{\pi}\left[\boldsymbol{C}+\ln 2\pi+2\sum_{k=2}^\infty\frac{\ln k}{4k^2-1}\right]$$
(see **6.443** 4) NH 204

2. $$\int_0^1 \psi(x)\sin(2\pi nx)\,dx=-\frac12\pi \qquad [n=1,2,\ldots]$$ ET II 305(3)

6.466 $$\int_0^\infty\left[\psi(\alpha+ix)-\psi(\alpha-ix)\right]\sin xy\,dx=i\pi e^{-\alpha y}\left(1-e^{-y}\right)^{-1}$$
$[\alpha>0,\quad y>0]$ ET I 96(1)

6.467

1. $$\int_0^1 \sin(2\pi nx)\,\psi(\alpha+x)\,dx=\sin(2\pi n\alpha)\operatorname{ci}(2\pi n\alpha)+\cos(2\pi n\alpha)\operatorname{si}(2\pi n\alpha)$$
$[\alpha\geq 0;\quad n=1,2,\ldots]$ ET II 305(4)

2. $$\int_0^1 \cos(2\pi nx)\,\psi(\alpha+x)\,dx=\sin(2\pi n\alpha)\operatorname{si}(2\pi n\alpha)-\cos(2\pi n\alpha)\operatorname{ci}(2\pi n\alpha)$$
$[\alpha>0;\quad n=1,2,\ldots]$ ET II 305(5)

6.468 $$\int_0^1 \psi(x)\sin^2\pi x\,dx=-\frac12\left[\boldsymbol{C}+\ln(2\pi)\right]$$ NH 204

6.469

1. $$\int_0^1 \psi(x)\sin\pi x\cos\pi x\,dx=-\frac{\pi}{4}$$ NH 204

2.[8] $$\int_0^1 \psi(x)\sin\pi x\sin(n\pi x)\,dx=\frac{n}{1-n^2} \qquad [n \text{ is even}]$$
$$=\frac12\ln\frac{n-1}{n+1} \qquad [n>1 \text{ is odd}]$$
NH 204(8)a

6.471

1. $$\int_0^\infty x^{-\alpha}\left[\ln x-\psi(1+x)\right]\,dx=\pi\operatorname{cosec}(\pi\alpha)\,\zeta(\alpha)\qquad[0<\operatorname{Re}\alpha<1]$$ ET II 306(7)

2. $$\int_0^\infty x^{-\alpha}\left[\ln(1+x)-\psi(1+x)\right]\,dx=\pi\operatorname{cosec}(\pi\alpha)\left[\zeta(\alpha)-(\alpha-1)^{-1}\right]$$
$$[0<\operatorname{Re}\alpha<1]$$ ET II 306(8)

3. $$\int_0^\infty\left[\psi(x+1)-\ln x\right]\cos(2\pi xy)\,dx=\frac{1}{2}\left[\psi(y+1)-\ln y\right]$$ ET II 306(12)

6.472

1. $$\int_0^\infty x^{-\alpha}\left[(1+x)^{-1}-\psi'(1+x)\right]\,dx=-\pi\alpha\operatorname{cosec}(\pi\alpha)\left[\zeta(1+\alpha)-\alpha^{-1}\right]$$
$$[|\operatorname{Re}\alpha|<1]$$ ET II 306(9)

2. $$\int_0^\infty x^{-\alpha}\left[x^{-1}-\psi'(1+x)\right]\,dx=-\pi\alpha\operatorname{cosec}(\pi\alpha)\,\zeta(1+\alpha)$$
$$[-2<\operatorname{Re}\alpha<0]$$ ET II 306(10)

6.473 $$\int_0^\infty x^{-\alpha}\psi^{(n)}(1+x)\,dx=(-1)^{n-1}\frac{\pi\,\Gamma(\alpha+n)}{\Gamma(\alpha)\sin\pi\alpha}\,\zeta(\alpha+n)$$
$$[n=1,2,\ldots;\quad 0<\operatorname{Re}\alpha<1]$$
ET II 306(11)

6.5–6.7 Bessel Functions

6.51 Bessel functions

6.511

1. $$\int_0^\infty J_\nu(bx)\,dx=\frac{1}{b}\qquad[\operatorname{Re}\nu>-1,\quad b>0]$$ ET II 22(3)

2. $$\int_0^\infty Y_\nu(bx)\,dx=-\frac{1}{b}\tan\left(\frac{\nu\pi}{2}\right)\qquad[|\operatorname{Re}\nu|<1,\quad b>0]$$
WA 432(7), ET II 96(1)

3. $$\int_0^a J_\nu(x)\,dx=2\sum_{k=0}^\infty J_{\nu+2k+1}(a)\qquad[\operatorname{Re}\nu>-1]$$ ET II 333(1)

4. $$\int_0^a J_{\frac{1}{2}}(t)\,dt=2\,S\left(\sqrt{a}\right)$$ WA 599(4)

5. $$\int_0^a J_{-\frac{1}{2}}(t)\,dt=2\,C\left(\sqrt{a}\right)$$ WA 599(3)

6. $$\int_0^a J_0(x)\,dx=a\,J_0(a)+\frac{\pi a}{2}\left[J_1(a)\,\mathbf{H}_0(a)-J_0(a)\,\mathbf{H}_1(a)\right]$$
$$[a>0]$$ ET II 7(2)

7. $\displaystyle\int_0^a J_1(x)\,dx = 1 - J_0(a)$ $[a > 0]$ ET II 18(1)

8. $\displaystyle\int_a^\infty J_0(x)\,dx = 1 - a\,J_0(a) + \frac{\pi a}{2}\,[J_0(a)\,\mathbf{H}_1(a) - J_1(a)\,\mathbf{H}_0(a)]$

$[a > 0]$ ET II 7(3)

9. $\displaystyle\int_a^\infty J_1(x)\,dx = J_0(a)$ $[a > 0]$ ET II 18(2)

10. $\displaystyle\int_a^b Y_\nu(x)\,dx = 2\sum_{n=0}^\infty [\,Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)]$ ET II 339(46)

11. $\displaystyle\int_0^a I_\nu(x)\,dx = 2\sum_{n=0}^\infty (-1)^n\,I_{\nu+2n+1}(a)$ $[\operatorname{Re}\nu > -1]$ ET II 364(1)

12.* $\displaystyle\int_0^\infty K_0(ax) = \frac{\pi}{2a}$ $[a > 0]$

13.* $\displaystyle\int_0^\infty K_0^2(ax) = \frac{\pi^2}{4a}$ $[a > 0]$

6.512

1.[11] $$\int_0^\infty J_\mu(ax)\,J_\nu(bx)\,dx = b^\nu a^{-\nu-1}\frac{\Gamma\left(\dfrac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\dfrac{\mu-\nu+1}{2}\right)}\,F\left(\frac{\mu+\nu+1}{2},\frac{\nu-\mu+1}{2};\nu+1;\frac{b^2}{a^2}\right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu) > -1, \quad b < a.$

For $a > b$, the positions of μ and ν should be reversed.]

ET II 48(6)

2.[7] $$\begin{aligned}\int_0^\infty J_{\nu+n}(\alpha t)\,J_{\nu-n-1}(\beta t)\,dt &= \frac{\beta^{\nu-n-1}\,\Gamma(\nu)}{\alpha^{\nu-n}n!\,\Gamma(\nu-n)}\,F\left(\nu,-n;\nu-n;\frac{\beta^2}{\alpha^2}\right) && [0 < \beta < \alpha]\\ &= (-1)^n\frac{1}{2\alpha} && [0 < \beta = \alpha]\\ &= 0 && [0 < \alpha < \beta]\end{aligned}$$

$[\operatorname{Re}(\nu) > 0]$ MO 50

3.[8] $$\begin{aligned}\int_0^\infty J_\nu(\alpha x)\,J_{\nu-1}(\beta x)\,dx &= \frac{\beta^{\nu-1}}{\alpha^\nu} && [\beta < \alpha]\\ &= \frac{1}{2\beta} && [\beta = \alpha]\\ &= 0 && [\beta > \alpha]\end{aligned}$$

$[\operatorname{Re}\nu > 0]$ WA 444(8), KU (40)a

4. $$\begin{aligned}\int_0^\infty J_{\nu+2n+1}(ax)\,J_\nu(bx)\,dx &= b^\nu a^{-\nu-1}\,P_n^{(\nu,0)}\left(1-\frac{2b^2}{a^2}\right) && [\operatorname{Re}\nu > -1-n, \quad 0 < b < a]\\ &= 0 && [\operatorname{Re}\nu > -1-n, \quad 0 < a < b]\end{aligned}$$

ET II 47(5)

5. $$\int_0^\infty J_{\nu+n}(ax)\, Y_{\nu-n}(ax)\, dx = (-1)^{n+1}\frac{1}{2a} \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad a>0, \quad n=0,1,2,\ldots\right]$$
ET II 347(57)

6. $$\int_0^\infty J_1(bx)\, Y_0(ax)\, dx = -\frac{b^{-1}}{\pi}\ln\left(1-\frac{b^2}{a^2}\right) \qquad [0<b<a]$$
ET II 21(31)

7. $$\int_0^a J_\nu(x)\, J_{\nu+1}(x)\, dx = \sum_{n=0}^\infty \left[J_{\nu+n+1}(a)\right]^2 \qquad [\operatorname{Re}\nu > -1]$$
ET II 338(37)

8.[9] $$\int_0^\infty k\, J_n(ka)\, J_n(kb)\, dk = \frac{1}{a}\,\delta(b-a) \qquad [n=0,1,\ldots]$$
JAC 110

9.* $$\int_0^\infty K_0(ax)\, J_1(bx) = \frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right) \qquad [a>0, \quad b>0]$$

10.* $$\int_0^\infty K_0(ax)\, I_1(bx) = -\frac{1}{2b}\ln\left(1-\frac{b^2}{a^2}\right) \qquad [a>0, \quad b>0]$$

6.513

1. $$\int_0^\infty \left[J_\mu(ax)\right]^2 J_\nu(bx)\, dx = a^{2\mu}b^{-2\mu-1}\frac{\Gamma\left(\dfrac{1+\nu+2\mu}{2}\right)}{\left[\Gamma(\mu+1)\right]^2\Gamma\left(\dfrac{1+\nu-2\mu}{2}\right)} \times\left[F\left(\frac{1-\nu+2\mu}{2},\frac{1+\nu+2\mu}{2};\mu+1;\frac{1-\sqrt{1-\dfrac{4a^2}{b^2}}}{2}\right)\right]^2$$
$$[\operatorname{Re}\nu+\operatorname{Re}2\mu>-1, \quad 0<2a<b]$$
ET II 52(33)

2. $$\int_0^\infty \left[J_\mu(ax)\right]^2 K_\nu(bx)\, dx = \frac{b^{-1}}{2}\,\Gamma\left(\frac{2\mu+\nu+1}{2}\right)\Gamma\left(\frac{2\mu-\nu+1}{2}\right)\left[P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2$$
$$[2\operatorname{Re}\mu > |\operatorname{Re}\nu|-1, \quad \operatorname{Re}b > 2|\operatorname{Im}a|]$$
ET II 138(18)

3. $$\int_0^\infty I_\mu(ax)\, K_\mu(ax)\, J_\nu(bx)\, dx = \frac{e^{\mu\pi i}\,\Gamma\left(\dfrac{\nu+2\mu+1}{2}\right)}{b\,\Gamma\left(\dfrac{\nu-2\mu+1}{2}\right)}\, P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)$$
$$[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu+2\mu)>-1]$$
ET II 65(20)

4. $$\int_0^\infty J_\mu(ax)\, J_{-\mu}(ax)\, K_\nu(bx)\, dx = \frac{\pi}{2b}\sec\left(\frac{\nu\pi}{2}\right) P^{\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)$$
$$[|\operatorname{Re}\nu|<1, \quad \operatorname{Re}b > 2|\operatorname{Im}a|]$$
ET II 138(21)

5. $$\int_0^\infty [K_\mu(ax)]^2\, J_\nu(bx)\, dx = \frac{e^{2\mu\pi i}\,\Gamma\left(\dfrac{1+\nu+2\mu}{2}\right)}{b\,\Gamma\left(\dfrac{1+\nu-2\mu}{2}\right)} \left[Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) > -\frac{1}{2}\right]$ ET II 66(28)

6. $$\int_0^z J_\mu(x)\, J_\nu(z-x)\, dx = 2\sum_{k=0}^\infty (-1)^k\, J_{\mu+\nu+2k+1}(z)$$ $[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ WA 414(2)

7. $$\int_0^z J_\mu(x)\, J_{-\mu}(z-x)\, dx = \sin z$$ $[-1 < \operatorname{Re}\mu < 1]$ WA 415(4)

8. $$\int_0^z J_\mu(x)\, J_{1-\mu}(z-x)\, dx = J_0(z) - \cos(z)$$ $[-1 < \operatorname{Re}\mu < 2]$ WA 415(4)

9.* $$\int_0^\infty J_0^2(ax)\, J_1(bx) = \frac{1}{b}$$ $[b > 2a > 0]$
$$= \frac{2}{\pi b}\arcsin\left(\frac{b}{2a}\right)$$ $[2a > b > 0]$

6.514

1. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu(bx)\, dx = b^{-1}\, J_{2\nu}\left(2\sqrt{ab}\right)$$ $\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 57(9)

2. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\, dx = b^{-1}\left[Y_{2\nu}\left(2\sqrt{ab}\right) + \frac{2}{\pi}\, K_{2\nu}\left(\sqrt{2ab}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{3}{2}\right]$
ET II 110(12)

3. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx)\, dx = b^{-1} e^{\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}\left[2e^{\frac{1}{4}i\pi}\sqrt{ab}\right] + b^{-1} e^{-\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}\left[2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right]$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$
ET II 141(31)

4. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu(bx)\, dx = -\frac{2b^{-1}}{\pi}\left[K_{2\nu}\left(2\sqrt{ab}\right) - \frac{\pi}{2}\, Y_{2\nu}\left(2\sqrt{ab}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET II 62(37)a

5. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\, dx = -b^{-1}\, J_{2\nu}\left(2\sqrt{ab}\right)$$ $\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET II 110(14)

6. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu(bx)\, dx = -b^{-1} e^{\frac{1}{2}\nu\pi i}\, K_{2\nu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) - b^{-1} e^{-\frac{1}{2}\nu\pi i}\, K_{2\nu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right)$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$
ET II 143(37)

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\,dx = -2b^{-1}\left[\sin\left(\frac{3\nu\pi}{2}\right)\operatorname{ker}_{2\nu}\left(2\sqrt{ab}\right) + \cos\left(\frac{3\nu\pi}{2}\right)\operatorname{kei}_{2\nu}\left(2\sqrt{ab}\right)\right]$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET II 113(28)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \pi b^{-1} K_{2\nu}\left(2\sqrt{ab}\right) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$
ET II 146(54)

6.515

1. $$\int_0^\infty J_\mu\left(\frac{a}{x}\right) Y_\mu\left(\frac{a}{x}\right) K_0(bx)\,dx = -2b^{-1} J_{2\mu}\left(2\sqrt{ab}\right) K_{2\mu}\left(2\sqrt{ab}\right)$$
$$[a > 0, \quad \operatorname{Re} b > 0]$$
ET II 143(42)

2. $$\int_0^\infty \left[K_\mu\left(\frac{a}{x}\right)\right]^2 K_0(bx)\,dx = 2\pi b^{-1} K_{2\mu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) K_{2\mu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$
ET II 147(59)

3. $$\int_0^\infty H_\mu^{(1)}\left(\frac{a^2}{x}\right) H_\mu^{(2)}\left(\frac{a^2}{x}\right) J_0(bx)\,dx = 16\pi^{-2} b^{-1}\cos\mu\pi\, K_{2\mu}\left(2e^{\pi i/4}a\sqrt{b}\right) K_{2\mu}\left(2e^{-\pi i/4}a\sqrt{b}\right)$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad |\operatorname{Re}\mu| < \tfrac{1}{4}\right]$$
ET II 17(36)

6.516

1. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 58(16)

2. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx = -b^{-1}\mathbf{H}_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 111(18)

3. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi}{2} b^{-1}\left[I_\nu\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 144(45)

4.[10] $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{1}{b} J_\nu\left(\frac{a^2}{4b}\right)\cot(2\pi\nu) - \frac{1}{2b} J_{-\nu}\left(\frac{a^2}{4b}\right)\operatorname{cosec}(2\pi\nu)$$
$$- \frac{2^{3\nu-3}a^{2-2\nu}b^{\nu-2}}{\pi^{3/2}}\Gamma\left(\nu - \tfrac{1}{2}\right) {}_1F_2\left(1;\ \frac{3}{2}, \frac{3}{2} - \nu;\ \frac{a^4}{64b^2}\right)$$
$$[a > 0, \quad b > 0]$$
MC

5. $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx$$
$$= \frac{b^{-1}}{2}\left[\sec(\nu\pi) J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}(\nu\pi)\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - 2\cot(2\nu\pi)\mathbf{H}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET II 111(19)

6. $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[\operatorname{cosec}(2\nu\pi)\,\mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \cot(2\nu\pi)\,\mathbf{L}_\nu\left(\frac{a^2}{4b}\right) - \tan(\nu\pi)\,I_\nu\left(\frac{a^2}{4b}\right) - \frac{\sec(\nu\pi)}{\pi}\,K_\nu\left(\frac{a^2}{4b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}]$ ET II 144(46)

7. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{1}{4}\pi b^{-1}\sec(\nu\pi)\left[\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - Y_{-\nu}\left(\frac{a^2}{4b}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$
ET II 70(22)

8. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx = -\frac{1}{4}\pi b^{-1}\left[\sec(\nu\pi)\,J_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{cosec}(\nu\pi)\,\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) + 2\operatorname{cosec}(2\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}]$ ET II 114(34)

9. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{4\cos(\nu\pi)}\left\{K_\nu\left(\frac{a^2}{4b}\right) + \frac{\pi}{2\sin(\nu\pi)}\left[\mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]\right\}$$
$[\operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}]$ ET II 147(63)

10. $$\int_0^\infty I_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[I_\nu\left(\frac{a^2}{4b}\right) + \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$ ET II 147(60)

6.517 $$\int_0^z J_0\left(\sqrt{z^2 - x^2}\right)\,dx = \sin z$$ MO 48

6.518 $$\int_0^\infty K_{2\nu}\left(2z\sinh x\right)\,dx = \frac{\pi^2}{8\cos\nu\pi}\left(J_\nu^2(z) + N_\nu^2(z)\right) \qquad [\operatorname{Re} z > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}]$$ MO 45

6.519

1. $$\int_0^{\pi/2} J_{2\nu}\left(2z\cos x\right)\,dx = \frac{\pi}{2}J_\nu^2(z) \qquad [\operatorname{Re}\nu > -\tfrac{1}{2}]$$ WH

2. $$\int_0^{\pi/2} J_{2\nu}\left(2z\sin x\right)\,dx = \frac{\pi}{2}J_\nu^2(z) \qquad [\operatorname{Re}\nu > -\tfrac{1}{2}]$$ WA 42(1)a

6.52 Bessel functions combined with x and x^2

6.521

1. $$\int_0^1 x\,J_\nu(\alpha x)\,J_\nu(\beta x)\,dx = \frac{\beta\,J_{\nu-1}(\beta)\,J_\nu(\alpha) - \alpha\,J_{\nu-1}(\alpha)\,J_\nu(\beta)}{\alpha^2 - \beta^2} \qquad [\alpha \neq \beta, \quad \nu > -1]$$
$$= \frac{\alpha\,J_\nu(\beta)\,J'_\nu(\alpha) - \beta\,J_\nu(\alpha)\,J'_\nu(\beta)}{\beta^2 - \alpha^2} \qquad [\alpha \neq \beta, \quad \nu > -1]$$
WH

2.[10] $$\int_0^\infty x\,K_\nu(ax)\,J_\nu(bx)\,dx = \frac{b^\nu}{a^\nu\,(b^2+a^2)} \qquad [\operatorname{Re} a>0,\quad b>0,\quad \operatorname{Re}\nu>-1]$$

ET II 63(2)

3. $$\int_0^\infty x\,K_\nu(ax)\,K_\nu(bx)\,dx = \frac{\pi(ab)^{-\nu}\left(a^{2\nu}-b^{2\nu}\right)}{2\sin(\nu\pi)\,(a^2-b^2)} \qquad [|\operatorname{Re}\nu|<1,\quad \operatorname{Re}(a+b)>0]$$

ET II 145(48)

4. $$\int_0^a x\,J_\nu(\lambda x)\,K_\nu(\mu x)\,dx = \left(\mu^2+\lambda^2\right)^{-1}\left[\left(\frac{\lambda}{\mu}\right)^\nu + \lambda a\,J_{\nu+1}(\lambda a)\,K_\nu(\mu a) - \mu a\,J_\nu(\lambda a)\,K_{\nu+1}(\mu a)\right]$$

$[\operatorname{Re}\nu>-1]$ ET II 367(26)

5.* $$\int_0^\infty x\,K_1(ax) = \frac{\pi}{2a^2} \qquad [a>0]$$

6.* $$\int_0^\infty x\,K_0^2(ax) = \frac{1}{2a^2} \qquad [a>0]$$

7.* $$\int_0^\infty x\,K_1(ax)\,J_1(bx) = \frac{b}{a\,(a^2+b^2)} \qquad [a>0,\quad b>0]$$

8.* $$\int_0^\infty x\,K_0(ax)\,I_0(bx) = \frac{1}{a^2-b^2} \qquad [a>b>0]$$

9.* $$\int_0^\infty x\,K_1(ax)\,I_1(bx) = \frac{b}{a\,(a^2-b^2)} \qquad [a>b>0]$$

10.* $$\int_0^\infty x^2\,K_0(ax) = \frac{\pi}{2a^3} \qquad [a>0]$$

11.* $$\int_0^\infty x^2\,K_1(ax) = \frac{2}{a^3} \qquad [a>0]$$

12.* $$\int_0^\infty x^2\,K_0(ax)\,J_1(bx) = \frac{2b}{(a^2+b^2)^2} \qquad [a>0,\quad b>0]$$

13.* $$\int_0^\infty x^2\,K_1(ax)\,J_0(bx) = \frac{2a}{(a^2+b^2)^2} \qquad [a>b>0]$$

14.* $$\int_0^\infty x^2\,K_0(ax)\,I_1(bx) = \frac{2b}{(a^2-b^2)^2} \qquad [a>b>0]$$

15.* $$\int_0^\infty x^2\,K_1(ax)\,I_0(bx) = \frac{2a}{(a^2-b^2)^2} \qquad [a>b>0]$$

6.522 **Notation**: $\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}-\sqrt{(b-c)^2+a^2}\right]$, $\ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}+\sqrt{(b-c)^2+a^2}\right]$

1.[8] $$\int_0^\infty x\,[J_\mu(ax)]^2\,K_\nu(bx)\,dx = \Gamma\left(\mu+\tfrac{1}{2}\nu+1\right)\Gamma\left(\mu-\tfrac{1}{2}\nu+1\right)b^{-2}$$
$$\times\left(1+4a^2b^{-2}\right)^{-\frac{1}{2}}P^{-\mu}_{\frac{1}{2}\nu}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]P^{-\mu}_{-\frac{1}{2}\nu}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]$$

$[\operatorname{Re} b>2|\operatorname{Im} a|,\quad 2\operatorname{Re}\mu>|\operatorname{Re}\nu|-2]$ ET II 138(19)

2. $$\int_0^\infty x\,[K_\mu(ax)]^2\,J_\nu(bx)\,dx = \frac{2e^{2\mu\pi i}\,\Gamma\left(1+\frac{1}{2}\nu+\mu\right)}{b\left(4a^2+b^2\right)^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\nu-\mu\right)} \times Q^{-\mu}_{\frac{1}{2}\nu}\left(\sqrt{(1+4a^2b^{-2})}\right) Q^{-\mu}_{\frac{1}{2}\nu-1}\left(\sqrt{(1+4a^2b^{-2})}\right)$$
$$\left[b>0,\quad \operatorname{Re} a>0,\quad \operatorname{Re}\left(\tfrac{1}{2}\nu\pm\mu\right)>-1\right]$$ ET II 66(27)a

3.[11] $$\int_0^\infty x\,K_0(ax)\,J_\nu(bx)\,J_\nu(cx)\,dx = r_1^{-1}r_2^{-1}\,(r_2-r_1)^{\nu}\,(r_2-r_1)^{-\nu} = \frac{\ell_1^\nu}{\ell_2^\nu\,(\ell_2^2-\ell_1^2)},$$
$$\left[r_1=\sqrt{a^2+(b-c)^2},\quad r_2=\sqrt{a^2+(b+c)^2},\quad c>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re} a>|\operatorname{Im} b|\right]$$
ET II 63(6)

4.[10] $$\int_0^\infty x\,I_0(ax)\,K_0(bx)\,J_0(cx)\,dx = \left(a^4+b^4+c^4-2a^2b^2+2a^2c^2+2b^2c^2\right)^{-\frac{1}{2}}$$
$$[\operatorname{Re} b>\operatorname{Re} a,\quad c>0]$$ ET II 16(27)

alternatively, with a and c interchanged
$$\int_0^\infty x\,I_0(cx)\,K_0(bx)\,J_0(ax)\,dx = \frac{1}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} b>\operatorname{Re} c,\quad a>0]$$

5.[10] $$\int_0^\infty x\,J_0(ax)\,K_0(bx)\,J_0(cx)\,dx = \left(a^4+b^4+c^4-2a^2c^2+2a^2b^2+2b^2c^2\right)^{-\frac{1}{2}}$$
$$[\operatorname{Re} b>|\operatorname{Im} a|,\quad c>0]$$ ET II 15(25)

alternatively, with a and b interchanged
$$\int_0^\infty x\,J_0(bx)\,K_0(ax)\,J_0(cx)\,dx = \frac{1}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} a>|\operatorname{Im} b|,\quad c>0]$$

6. $$\int_0^\infty x\,J_0(ax)\,Y_0(ax)\,J_0(bx)\,dx = 0 \qquad [0<b<2a]$$
$$= -2\pi^{-1}b^{-1}\left[b^2-4a^2\right]^{-\frac{1}{2}} \qquad [0<2a<b<\infty]$$
ET II 15(21)

7. $$\int_0^\infty x\,J_\mu(ax)\,J_{\mu+1}(ax)\,K_\nu(bx)\,dx = \Gamma\left(\mu+\frac{3+\nu}{2}\right)\Gamma\left(\mu+\frac{3-\nu}{2}\right)b^{-2}\left(1+4a^2b^{-2}\right)^{-\frac{1}{2}}$$
$$\times P^{\frac{1}{2}\nu-\frac{1}{2}}_{-\mu}\left[\sqrt{1+4a^2b^{-2}}\right]P^{\frac{1}{2}\nu-\frac{1}{2}}_{-\mu-1}\left[\sqrt{1+4a^2b^{-2}}\right]$$
$$[\operatorname{Re} b>2|\operatorname{Im} a|,\quad 2\operatorname{Re}\mu>|\operatorname{Re}\nu|-3]$$ ET II 138(20)

8. $$\int_0^\infty x\,K_{\mu-\frac{1}{2}}(ax)\,K_{\mu+\frac{1}{2}}(ax)\,J_\nu(bx)\,dx$$
$$= -\frac{2e^{2\mu\pi i}\,\Gamma\left(\frac{1}{2}\nu+\mu+1\right)}{b\,\Gamma\left(\frac{1}{2}\nu-\mu\right)\left(b^2+4a^2\right)^{\frac{1}{2}}}\,Q^{-\mu+\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]Q^{-\mu-\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]$$
$$\left[b>0,\quad \operatorname{Re} a>0,\quad \operatorname{Re}\nu>-1,\quad |\operatorname{Re}\mu|<1+\tfrac{1}{2}\operatorname{Re}\nu\right]$$ ET II 67(29)a

9.[8] $$\int_0^\infty x\,I_{\frac{1}{2}\nu}(ax)\,K_{\frac{1}{2}\nu}(ax)\,J_\nu(bx)\,dx = b^{-1}\left(b^2+4a^2\right)^{-\frac{1}{2}}$$
$$[b>0,\quad \operatorname{Re} a>0,\quad \operatorname{Re}\nu>-1]$$
ET II 65(16)

10. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}(ax)\, Y_{\frac{1}{2}\nu}(ax)\, J_\nu(bx)\, dx$$
$$= 0 \qquad [a>0, \quad \operatorname{Re}\nu > -1, \quad 0<b<2a]$$
$$= -2\pi^{-1}b^{-1}\left(b^2-4a^2\right)^{-\frac{1}{2}} \qquad [a>0, \quad \operatorname{Re}\nu > -1, \quad 2a<b<\infty]$$
ET II 55(48)

11.[8] $$\int_0^\infty x\, J_{\frac{1}{2}(\nu+n)}(ax)\, J_{\frac{1}{2}(\nu-n)}(ax)\, J_\nu(bx)\, dx$$
$$= 2\pi^{-1}b^{-1}\left(4a^2-b^2\right)^{-\frac{1}{2}} T_n\left(\frac{b}{2a}\right) \qquad [a>0, \quad \operatorname{Re}\nu > -1, \quad 0<b<2a]$$
$$= 0 \qquad [a>0, \quad \operatorname{Re}\nu > -1, \quad 2a<b]$$
ET II 52(32)

12. $$\int_0^\infty x\, I_{\frac{1}{2}(\nu-\mu)}(ax)\, K_{\frac{1}{2}(\nu+\mu)}(ax)\, J_\nu(bx)\, dx = 2^{-\mu}a^{-\mu}b^{-1}\left(b^2+4a^2\right)^{-\frac{1}{2}}\left[b+\left(b^2+4a^2\right)^{\frac{1}{2}}\right]^\mu$$
$$[b>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-\mu)>-2]$$ ET II 66(23)

13.[8] $$\int_0^\infty x\, J_\mu\left(xa\sin\varphi\right) K_{\nu-\mu}\left(ax\cos\varphi\cos\psi\right) J_\nu\left(xa\sin\psi\right)\, dx = \frac{(\sin\varphi)^\mu\,(\sin\psi)^\nu\,(\cos\varphi)^{\nu-\mu}\,(\cos\psi)^{\mu-\nu}}{a^2\left(1-\sin^2\varphi\sin^2\psi\right)}$$
$$\left[a>0, \quad 0<\varphi<\frac{\pi}{2}, \quad 0<\psi<\frac{\pi}{2}, \quad \operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1\right]$$ ET II 64(10)

14.[8] $$\int_0^\infty x\, J_\mu\left(xa\sin\varphi\cos\psi\right) J_{\nu-\mu}(ax)\, J_\nu\left(xa\cos\varphi\sin\psi\right)\, dx$$
$$= -2\pi^{-1}a^{-2}\sin(\mu\pi)\,(\sin\varphi)^\mu\,(\sin\psi)^\nu\,(\cos\varphi)^{-\nu}\,(\cos\psi)^{-\mu}\left[\cos(\varphi+\psi)\cos(\varphi-\psi)\right]^{-1}$$
$$\left[a>0, \quad 0<\varphi<\frac{\pi}{2}, \quad 0<\psi<\tfrac{1}{2}\pi, \quad \operatorname{Re}\nu>-1\right]$$ ET II 54(39)

15.[10] $$\int_0^\infty x^{\nu+1}\, J_\nu(bx)\, K_\nu(ax)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\ell_2^2-\ell_1^2\right)^{2\nu+1}}$$
$$[\operatorname{Re}a > |\operatorname{Im}b|, \quad c>0]$$

16.[10] $$\int_0^\infty x^{\nu+1}\, I_\nu(cx)\, K_\nu(bx)\, J_\nu(ax)\, dx = \frac{2^{3\nu}(abc)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\ell_2^2-\ell_1^2\right)^{2\nu+1}}$$
$$[\operatorname{Re}b > |\operatorname{Im}a| + |\operatorname{Im}c|]$$

17.[11] $$\int_0^\infty t^{\nu-\mu-\rho+1}\, J_\mu(ct)\, J_\nu(bt)\, K_\rho(at)\, dt$$
$$= \frac{2^{1+\nu-\mu-\rho}}{c^\mu b^\nu a^\rho\,\Gamma\left(\mu-\nu+\rho\right)}\int_0^{\ell_1}\frac{x^{1+2\nu-2\rho}\left[\left(\ell_1^2-x^2\right)\left(\ell_2^2-x^2\right)\right]^{\mu-\nu+\rho-1}}{\left(b^2-x^2\right)^{\mu-\nu}}\, dx$$
$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}-\sqrt{(b-c)^2+a^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2}+\sqrt{(b-c)^2+a^2}\right]$$
$$[\operatorname{Re}a > |\operatorname{Im}b|, \quad c>0]$$

18.[11] $$\int_0^\infty t^{\mu-\nu+\rho+1}\, J_\mu(ct)\, J_\nu(bt)\, K_\rho(at)\, dt$$
$$= \frac{2^{1+\mu-\nu+\rho} a^\rho}{c^\mu b^\nu\, \Gamma(\nu-\mu-\rho)} \int_0^{\ell_1} \frac{x^{1+2\mu+2\rho} \left[\left(\ell_1^2 - x^2\right)\left(\ell_2^2 - x^2\right)\right]^{\nu-\mu-\rho-1}}{\left(c^2 - x^2\right)^{\nu-\mu}}\, dx$$
$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$
$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$

6.523 $$\int_0^\infty x\left[2\pi^{-1} K_0(ax) - Y_0(ax)\right] K_0(bx)\, dx = 2\pi^{-1}\left[\left(a^2+b^2\right)^{-1} + \left(b^2-a^2\right)^{-1}\right] \ln\frac{b}{a}$$
$[\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a+b) > 0]$ ET II 145(50)

6.524

1. $$\int_0^\infty x\, J_\nu^2(ax)\, J_\nu(bx)\, Y_\nu(bx)\, dx = 0 \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
$$= -(2\pi ab)^{-1} \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 352(14)

2. $$\int_0^\infty x\left[J_0(ax)\, K_0(bx)\right]^2 dx = \frac{\pi}{8ab} - \frac{1}{4ab}\arcsin\left(\frac{b^2-a^2}{b^2+a^2}\right)$$
$[a > 0, \quad b > 0]$ ET II 373(9)

6.525 **Notation**: $\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$

1.[10] $$\int_0^\infty x^2\, J_1(ax)\, K_0(bx)\, J_0(cx)\, dx = 2a\left(a^2+b^2-c^2\right)\left[\left(a^2+b^2+c^2\right)^2 - 4a^2c^2\right]^{-\frac{3}{2}}$$
$[c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0]$ ET II 15(26)

alternatively, with a and b interchanged
$$\int_0^\infty x^2\, J_1(bx)\, K_0(ax)\, J_0(cx)\, dx = \frac{2b\left(a^2+b^2-c^2\right)}{\left(\ell_2^2-\ell_1^2\right)^3} \qquad [\operatorname{Re} a > |\operatorname{Im} b|, \quad \operatorname{Re} b > 0, \quad c > 0]$$

2.[10] $$\int_0^\infty x^2\, I_0(ax)\, K_1(bx)\, J_0(cx)\, dx = 2b\left(b^2+c^2-a^2\right)\left[\left(a^2+b^2+c^2\right)^2 - 4a^2b^2\right]^{-\frac{3}{2}}$$
$[\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0]$ ET II 16(28)

3.[10] $$\int_0^\infty x^2\, I_0(cx)\, K_0(bx)\, J_0(ax)\, dx = \frac{2b\left(a^2+b^2-c^2\right)}{\left(\ell_2^2-\ell_1^2\right)^3} \qquad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

6.526

1. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = (2a)^{-1}\, J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 56(1)

2. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right)\, Y_\nu(bx)\, dx = (4a)^{-1}\left[Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \sec\left(\frac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1]$ ET II 109(9)

3. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{8a\cos\left(\frac{\nu\pi}{2}\right)}\left[\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - Y_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\nu>-1]$
ET II 140(27)

4. $$\int_0^\infty x\, Y_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = -(2a)^{-1}\,\mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad [a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\nu>-1]$$
ET II 61(35)

5. $$\int_0^\infty x\, Y_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{4a\sin(\nu\pi)}\left[\cos\left(\frac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right) J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a>0, \quad \operatorname{Re} b>0, \quad |\operatorname{Re}\nu|<1]$ ET II 141(28)

6. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = \frac{\pi}{4a}\left[I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}\nu>-1]$
ET II 68(9)

7. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right)\, Y_\nu(bx)\, dx = \frac{\pi}{4a}\left[\operatorname{cosec}(\nu\pi)\,\mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \cot(\nu\pi)\,\mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \frac{1}{\pi}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[\operatorname{Re} a>0, \quad b>0, \quad |\operatorname{Re}\nu|<1]$ ET II 112(25)

8. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{8a}\left\{\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi\operatorname{cosec}(\nu\pi)\left[\mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]\right\}$$
$[\operatorname{Re} a>0, \quad |\operatorname{Re}\nu|<1]$ ET II 146(52)

6.527

1. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, J_{\nu-\frac{1}{2}}\left(x^2\right)\, dx = \frac{1}{2}a\, J_{\nu+\frac{1}{2}}\left(a^2\right) \qquad \left[a>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 355(33)

2. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, J_{\nu+\frac{1}{2}}\left(x^2\right)\, dx = \frac{1}{2}a\, J_{\nu-\frac{1}{2}}\left(a^2\right) \qquad [a>0, \quad \operatorname{Re}\nu>-2]$$
ET II 355(35)

3. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, Y_{\nu+\frac{1}{2}}\left(x^2\right)\, dx = -\frac{1}{2}a\,\mathbf{H}_{\nu-\frac{1}{2}}\left(a^2\right) \qquad [a>0, \quad \operatorname{Re}\nu>-2]$$
ET II 355(36)

6.528 $$\int_0^\infty x\, K_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) I_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) J_\nu(bx)\, dx = K_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) I_{\frac{1}{4}\nu}\left(\frac{b^2}{4}\right)$$
$[b > 0, \quad \nu > -1]$ MO 183a

6.529

1. $$\int_0^\infty x\, J_\nu\left(2\sqrt{ax}\right) K_\nu\left(2\sqrt{ax}\right) J_\nu(bx)\, dx = \frac{1}{2} b^{-2} e^{-\frac{2a}{b}} \qquad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$$
ET II 70(23)

2. $$\int_0^a x\, J_\lambda(2a)\, I_\lambda(2x)\, J_\mu\left(2\sqrt{a^2-x^2}\right) I_\mu\left(2\sqrt{a^2-x^2}\right) dx$$
$$= \frac{a^{2\lambda+2\mu+2}}{2\,\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma(\lambda+\mu+2)} \times {}_1F_4\left(\frac{\lambda+\mu+1}{2}; \lambda+1, \mu+1, \lambda+\mu+1, \frac{\lambda+\mu+3}{2}; -a^4\right)$$
$[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\mu > -1]$ ET II 376(31)

6.53–6.54 Combinations of Bessel functions and rational functions

6.531

1.[10] $$\int_0^\infty \frac{Y_\nu(bx)}{x+a}\, dx$$
$$= -\pi\, J_\nu(ab)\cot(\pi\nu)\operatorname{cosec}(\pi\nu) - \pi\, J_{-\nu}(ab)\operatorname{cosec}^2(\pi\nu) + \frac{1}{\nu}\cot\frac{\pi\nu}{2}\; {}_1F_2\left(1;\ \frac{2-\nu}{2}, \frac{2+\nu}{2};\ -\frac{a^2b^2}{4}\right)$$
$$+ \frac{ab}{\nu^2-1}\; {}_1F_2\left(1;\ \frac{3-\nu}{2}, \frac{3+\nu}{2};\ -\frac{a^2b^2}{4}\right)\tan\frac{\pi\nu}{2}$$
$[\operatorname{Re}\nu < 1, \quad \arg a \neq \pi, \quad b > 0]$ MC

2. $$\int_0^\infty \frac{Y_\nu(bx)}{x-a}\, dx = \pi\left\{\cot(\nu\pi)\left[Y_\nu(ab) + \mathbf{E}_\nu(ab)\right] + \mathbf{J}_\nu(ab) + 2\left[\cot(\nu\pi)\right]^2\left[\mathbf{J}_\nu(ab) - J_\nu(ab)\right]\right\}$$
$[b > 0, \quad a > 0, \quad |\operatorname{Re}\nu| < 1]$
ET II 98(9)

3. $$\int_0^\infty \frac{K_\nu(bx)}{x+a}\, dx = \frac{\pi^2}{2}\left[\operatorname{cosec}(\nu\pi)\right]^2\left[I_\nu(ab) + I_{-\nu}(ab) - e^{-\frac{1}{2}i\nu\pi}\,\mathbf{J}_\nu(iab) - e^{\frac{1}{2}i\nu\pi}\,\mathbf{J}_{-\nu}(iab)\right]$$
$[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad |\operatorname{Re}\nu| < 1]$
ET II 128(5)

6.532

1.[11] $$\int_0^\infty \frac{J_\nu(x)}{x^2+a^2}\, dx = \frac{i}{a}\left[S_{0,\nu}(ia) - e^{-i\nu\pi/2}\, K_\nu(a)\right] = \frac{1}{a}\left[i\, s_{0,\nu}(ia) + \frac{\pi}{2}\sec\left(\frac{\nu\pi}{2}\right) I_\nu(a)\right]$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1]$

2. $$\int_0^\infty \frac{Y_\nu(x)}{x^2+a^2}\,dx = \frac{1}{\cos\frac{\nu\pi}{2}}\left[-\frac{\pi}{2a}\tan\left(\frac{\nu\pi}{2}\right)I_\nu(ab) - \frac{1}{a}K_\nu(ab) + \frac{b\sin\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\,{}_1F_2\left(1;\frac{3-\nu}{2},\frac{3+\nu}{2};\frac{a^2b^2}{4}\right)\right]$$
$[b>0,\quad \operatorname{Re}a>0,\quad |\operatorname{Re}\nu|<1]$ ET II 99(13)

3. $$\int_0^\infty \frac{Y_\nu(bx)}{x^2-a^2}\,dx = \frac{\pi}{2a}\left\{J_\nu(ab)+\tan\left(\frac{\nu\pi}{2}\right)\left\{\tan\left(\frac{\nu\pi}{2}\right)[\mathbf{J}_\nu(ab)-J_\nu(ab)]-\mathbf{E}_\nu(ab)-Y_\nu(ab)\right\}\right\}$$
$[b>0,\quad a>0,\quad |\operatorname{Re}\nu|<1]$
ET II 101(21)

4. $$\int_0^\infty \frac{x\,J_0(ax)}{x^2+k^2}\,dx = K_0(ak)$$ $[a>0,\quad \operatorname{Re}k>0]$ WA 466(5)

5. $$\int_0^\infty \frac{Y_0(ax)}{x^2+k^2}\,dx = -\frac{K_0(ak)}{k}$$ $[a>0,\quad \operatorname{Re}k>0]$ WA 466(6)

6. $$\int_0^\infty \frac{J_0(ax)}{x^2+k^2}\,dx = \frac{\pi}{2k}\left[I_0(ak)-\mathbf{L}_0(ak)\right]$$ $[a>0,\quad \operatorname{Re}k>0]$ WA 467(7)

6.533

1. $$\int_0^z J_p(x)\,J_q(z-x)\frac{dx}{x} = \frac{J_{p+q}(z)}{p}$$ $[\operatorname{Re}p>0,\quad \operatorname{Re}q>-1]$ WA 415(3)

2. $$\int_0^z \frac{J_p(x)}{x}\frac{J_q(z-x)}{z-x}\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\frac{J_{p+q}(z)}{z}$$ $[\operatorname{Re}p>0,\quad \operatorname{Re}q>0]$ WA 415(5)

3.[11] $$\int_0^\infty [J_0(ax)-1]\,J_1(bx)\frac{dx}{x^2} = -\frac{b}{4}\left[1+2\ln\frac{a}{b}\right] \qquad [0<b<a]$$
$$= -\frac{a^2}{4b} \qquad [0<a<b]$$
ET II 21(28)a

3b $$\int_0^\infty [J_0(ax)-1]\,J_1(bx)\frac{dx}{x} = \begin{cases}\frac{b}{2a}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2,\frac{b^2}{a^2}\right)-1 & [0<b<a]\\ \frac{2}{\pi}\boldsymbol{E}\left(\frac{b^2}{a^2}\right)-1 & [0<a<b]\end{cases}$$

4. $$\int_0^\infty [1-J_0(ax)]\,J_0(bx)\frac{dx}{x} = 0 \qquad [0<a<b]$$
$$= \ln\frac{a}{b} \qquad [0<b<a]$$
ET II 14(16)

6.534 $$\int_0^\infty \frac{x^3\,J_0(x)}{x^4-a^4}\,dx = \frac{1}{2}K_0(a)-\frac{1}{4}\pi\,Y_0(a)$$ $[a>0]$ ET II 340(5)

6.535 $$\int_0^\infty \frac{x}{x^2+a^2}\,[J_\nu(x)]^2\,dx = I_\nu(a)\,K_\nu(a)$$ $[\operatorname{Re}a>0,\quad \operatorname{Re}\nu>-1]$ ET II 342(26)

6.536 $$\int_0^\infty \frac{x^3\,J_0(bx)}{x^4+a^4}\,dx = \ker(ab)$$ $\left[b>0,\quad |\arg a|<\frac{1}{4}\pi\right]$
ET II 8(9), MO 46a

6.537 $$\int_0^\infty \frac{x^2\, J_0(bx)}{x^4+a^4}\, dx = -\frac{1}{a^2}\,\mathrm{kei}(ab) \qquad \left[b>0, \quad |\arg a| < \frac{\pi}{4}\right]$$ MO 46a

6.538

1. $$\int_0^\infty J_1(ax)\, J_1(bx) \frac{dx}{x^2} = \frac{a+b}{\pi}\left[E\left(\frac{2i\sqrt{ab}}{|b-a|}\right) - K\left(\frac{2i\sqrt{ab}}{|b-a|}\right)\right]$$
$$[a>0, \quad b>0]$$ ET II 21(30)

2.[8] $$\int_0^\infty x^{-1}\, J_{\nu+2n+1}(x)\, J_{\nu+2m+1}(x)\, dx = 0 \qquad [m \neq n \text{ with } m,n \text{ integers}, \nu > -1]$$
$$= (4n+2\nu+2)^{-1} \qquad [m=n, \quad \nu > -1]$$
EH II 64

6.539

1. $$\int_a^b \frac{dx}{x\,[J_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)}\right] \qquad [J_\nu(x) \neq 0 \quad \text{for } x \in [a,b]]$$ ET II 338(41)

2. $$\int_a^b \frac{dx}{x\,[Y_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)}\right] \qquad [Y_\nu(x) \neq 0 \quad \text{for } x \in [a,b]]$$
ET II 339(49)

3. $$\int_a^b \frac{dx}{x\, J_\nu(x)\, Y_\nu(x)} = \frac{\pi}{2}\ln\left[\frac{J_\nu(a)\, Y_\nu(b)}{J_\nu(b)\, Y_\nu(a)}\right]$$ ET II 339(50)

6.541

1. $$\int_0^\infty x\, J_\nu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2} = I_\nu(bc)\, K_\nu(ac) \qquad [0<b<a, \quad \operatorname{Re} c > 0, \quad \operatorname{Re}\nu > -1]$$
$$= I_\nu(ac)\, K_\nu(bc) \qquad [0<a<b, \quad \operatorname{Re} c > 0, \quad \operatorname{Re}\nu > -1]$$
ET II 49(10)

2.[8] $$\int_0^\infty x^{1-2n}\, J_\nu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2}$$
$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc)\, K_\nu(ac) - \frac{1}{2}\left(\frac{b}{a}\right)^\nu \frac{\pi}{\sin(\pi\nu)} \sum_{p=0}^{n-1} \frac{\left(a^2c^2/4\right)^p}{p!\,\Gamma(1-\nu+p)} \sum_{k=0}^{n-1-p} \frac{\left(b^2c^2/4\right)^k}{k!\,\Gamma(1-\nu+k)}\right]$$
$$[0<b<a]$$
$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc)\, K_\nu(ac) - \frac{1}{2\nu}\left(\frac{b}{a}\right)^\nu \sum_{p=0}^{n-1} \frac{\left(a^2c^2/4\right)^p}{p!(1-\nu)_p} \sum_{k=0}^{n-1-p} \frac{\left(b^2c^2/4\right)^k}{k!(1+\nu)_k}\right]$$
$$[n=1,2,\ldots, \quad \operatorname{Re}\nu > n-1, \quad \operatorname{Re} c > 0, \quad 0<b<a]$$

3.[8]
$$\int_0^\infty \frac{x^{\alpha-1}}{(x^2+z^2)^\rho}\, J_\mu(cx)\, J_\nu(cx)\, dx = \frac{1}{2}\left(\frac{c}{2}\right)^{2\rho-\alpha}$$
$$\times\, \Gamma\left[\begin{matrix}(\mu+\nu+\alpha)/2-\rho, 1+2\rho-\alpha\\ (\mu-\nu-\alpha)/2+\rho+1, (\mu+\nu-\alpha)/2+\rho+1, (\nu-\mu-\alpha)/2+\rho+1\end{matrix}\right]$$
$$\times\, {}_3F_4\left(\frac{1-\alpha}{2}+\rho, 1-\frac{\alpha}{2}+\rho, \rho; \rho+1-\frac{\mu+\nu+\alpha}{2}, \rho+1+\frac{\mu-\nu-\alpha}{2},\right.$$
$$\left.\rho+1+\frac{\mu+\nu-\alpha}{2}, \rho+1+\frac{\nu-\mu-\alpha}{2}; c^2z^2\right) + \frac{z^{\alpha-2\rho}}{2}\left(\frac{cz}{2}\right)^{\mu+\nu},$$
$$\Gamma\left[\begin{matrix}\rho-(\alpha+\mu+\nu)/2, (\alpha+\mu+\nu)/2\\ \rho, \mu+1, \nu+1\end{matrix}\right]\, {}_3F_4\left(\frac{1+\mu+\nu}{2}, 1+\frac{\mu+\nu}{2}\right.$$
$$\left.\frac{\alpha+\mu+\nu}{2}; 1-\rho+\frac{\alpha+\mu+\nu}{2}, \mu+1, \nu+1, \mu+\nu+1; c^2z^2\right)$$
$$\left[\Gamma\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right] = \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}, \quad c>0, \quad \operatorname{Re} z>0, \quad \operatorname{Re}(\alpha+\mu+\nu)>0; \quad \operatorname{Re}(\alpha-2\rho)>1\right]$$

6.542
$$\int_0^\infty \frac{J_\nu(ax)\, Y_\nu(bx) - J_\nu(bx)\, Y_\nu(ax)}{x\left\{[J_\nu(bx)]^2+[Y_\nu(bx)]^2\right\}}\, dx = -\frac{\pi}{2}\left(\frac{b}{a}\right)^\nu \qquad [0<b<a]$$
ET II 352(16)

6.543
$$\int_0^\infty J_\mu(bx)\left\{\cos\left[\frac{1}{2}(\nu-\mu)\pi\right] J_\nu(ax) - \sin\left[\frac{1}{2}(\nu-\mu)\pi\right] Y_\nu(ax)\right\}\frac{x\,dx}{x^2+r^2} = I_\mu(br)\, K_\nu(ar)$$
$$[\operatorname{Re} r>0, \quad a\ge b>0, \quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-2]$$

6.544

1.
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = -\frac{1}{a}\left[\frac{2}{\pi}\, K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - Y_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
EI II 357(47)

2.
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{1}{a}\, J_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \qquad \left[a>0, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 57(10)

3.
$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{1}{a}e^{\frac{1}{2}i\nu\pi}\, K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right) + \frac{1}{a}e^{-\frac{1}{2}i\nu\pi}\, K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)$$
$$\left[\operatorname{Re} b>0, \quad a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 142(32)

4.
$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{2}{a\pi}\left[K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) + \frac{\pi}{2}\, Y_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 62(38)

5.
$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{4}{a}\left[e^{\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right) + e^{-\frac{1}{2}i(\nu+1)\pi}\, K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)\right]$$
$$\left[\operatorname{Re} b>0, \quad a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 143(38)

6. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{i}{a}\left[e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right) - e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{-\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$ ET II 70(19)

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{2}{a}\left[\sin\left(\frac{3}{2}\pi\nu\right)\operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right)\operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$ ET II 113(29)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$ ET II 146(55)

6.55 Combinations of Bessel functions and algebraic functions

6.551[10]

1. $$\int_0^1 x^{1/2} J_\nu(xy)\,dx = \sqrt{2}y^{-3/2}\frac{\Gamma\left(\frac{3}{4}+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\nu\right)} + y^{-1/2}\left[\left(\nu-\tfrac{1}{2}\right) J_\nu(y)\, S_{-1/2,\nu-1}(y) - J_{\nu-1}(y)\, S_{1/2,\nu}(y)\right]$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET II 21(1)

2. $$\int_1^\infty x^{1/2} J_\nu(xy)\,dx = y^{-1/2}\left[J_{\nu-1}(y)\, S_{1/2,\nu}(y) + \left(\tfrac{1}{2}-\nu\right) J_\nu(y)\, S_{-1/2,\nu-1}(y)\right]$$
$[y > 0]$ ET II 22(2)

6.552

1. $$\int_0^\infty J_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = I_{\nu/2}\left(\tfrac{1}{2}ay\right) K_{\nu/2}\left(\tfrac{1}{2}ay\right)$$
$[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 23(11), WA 477(3), MO 44

2. $$\int_0^\infty Y_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = -\frac{1}{\pi}\sec\left(\tfrac{1}{2}\nu\pi\right) K_{\nu/2}\left(\tfrac{1}{2}ay\right)\left[K_{\nu/2}\left(\tfrac{1}{2}ay\right) + \pi\sin\left(\tfrac{1}{2}\nu\pi\right) I_{\nu/2}\left(\tfrac{1}{2}ay\right)\right]$$
$[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 100(18)

3. $$\int_0^\infty K_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = \frac{\pi^2}{8}\sec\left(\tfrac{1}{2}\nu\pi\right)\left\{\left[J_{\nu/2}\left(\tfrac{1}{2}ay\right)\right]^2 + \left[Y_{\nu/2}\left(\tfrac{1}{2}ay\right)\right]^2\right\}$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 128(6)

4. $$\int_0^1 J_\nu(xy)\frac{dx}{\left(1-x^2\right)^{1/2}} = \frac{\pi}{2}\left[J_{\nu/2}\left(\tfrac{1}{2}y\right)\right]^2$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 24(22)a

5. $$\int_0^1 Y_0(xy)\frac{dx}{\left(1-x^2\right)^{1/2}} = \frac{\pi}{2} J_0\left(\tfrac{1}{2}y\right) Y_0\left(\tfrac{1}{2}y\right)$$
$[y > 0]$ ET II 102(26)a

6. $$\int_1^\infty J_\nu(xy)\frac{dx}{\left(x^2-1\right)^{1/2}} = -\frac{\pi}{2} J_{\nu/2}\left(\tfrac{1}{2}y\right) Y_{\nu/2}\left(\tfrac{1}{2}y\right)$$
$[y > 0]$ ET II 24(23)a

7. $$\int_1^\infty Y_\nu(xy)\frac{dx}{(x^2-1)^{1/2}} = \frac{\pi}{4}\left\{\left[J_{\nu/2}\left(\tfrac{1}{2}y\right)\right]^2 - \left[Y_{\nu/2}\left(\tfrac{1}{2}y\right)\right]^2\right\}$$
$[y>0]$ ET II 102(27)

6.553 $$\int_0^\infty x^{-1/2}\, I_\nu(x)\, K_\nu(x)\, K_\mu(2x)\, dx = \frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu-\frac{1}{2}\mu\right)}{4\,\Gamma\left(\frac{3}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}+\nu-\frac{1}{2}\mu\right)}$$
$\left[|\operatorname{Re}\mu|<\frac{1}{2},\quad 2\operatorname{Re}\nu>|\operatorname{Re}\mu|-\frac{1}{2}\right]$
ET II 372(2)

6.554

1. $$\int_0^\infty x\, J_0(xy)\frac{dx}{(a^2+x^2)^{1/2}} = y^{-1}e^{-ay}$$
$[y>0,\quad \operatorname{Re}a>0]$ ET II 7(4)

2. $$\int_0^1 x\, J_0(xy)\frac{dx}{(1-x^2)^{1/2}} = y^{-1}\sin y$$
$[y>0]$ ET II 7(5)a

3. $$\int_1^\infty x\, J_0(xy)\frac{dx}{(x^2-1)^{1/2}} = y^{-1}\cos y$$
$[y>0]$ ET II 7(6)a

4. $$\int_0^\infty x\, J_0(xy)\frac{dx}{(x^2+a^2)^{3/2}} = a^{-1}e^{-ay}$$
$[y>0,\quad \operatorname{Re}a>0]$ ET II 7(7)a

5.[11] $$\int_0^\infty \frac{x^{\nu+1}\, J_\nu(ax)}{(x^4+4k^4)^{\nu+1/2}}\, dx = \frac{\left(\frac{1}{2}a\right)^\nu\sqrt{\pi}}{(2k)^{2\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)}\, J_\nu(ak)\, K_\nu(ak)$$
$\left[a>0,\quad |\arg k|>\frac{\pi}{4},\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$
WA 473(1)

6.555 $$\int_0^\infty x^{1/2}\, J_{2\nu-1}\left(ax^{1/2}\right)\, Y_\nu(xy)\, dx = -\frac{a}{2y^2}\,\mathbf{H}_{\nu-1}\left(\frac{a^2}{4y}\right)$$
$\left[a>0,\quad y>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$
ET II 111(17)

6.556 $$\int_0^\infty J_\nu\left[a\left(x^2+1\right)^{1/2}\right]\frac{dx}{\sqrt{x^2+1}} = -\frac{\pi}{2}\, J_{\nu/2}\left(\frac{a}{2}\right)\, Y_{\nu/2}\left(\frac{a}{2}\right)$$
$[\operatorname{Re}\nu>-1,\quad a>0]$ MO 46

6.56–6.58 Combinations of Bessel functions and powers

6.561

1. $$\int_0^1 x^\nu\, J_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[J_\nu(a)\,\mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\, J_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 333(2)a

2. $$\int_0^1 x^\nu\, Y_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[Y_\nu(a)\,\mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\, Y_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 338(43)a

3. $$\int_0^1 x^\nu\, I_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[I_\nu(a)\,\mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a)\, I_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 364(2)a

4. $$\int_0^1 x^\nu\, K_\nu(ax)\, dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}}\, \Gamma\left(\nu+\tfrac{1}{2}\right) \left[K_\nu(a)\, \mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a)\, K_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 367(21)a

5. $$\int_0^1 x^{\nu+1}\, J_\nu(ax)\, dx = a^{-1}\, J_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 333(3)a

6. $$\int_0^1 x^{\nu+1}\, Y_\nu(ax)\, dx = a^{-1}\, Y_{\nu+1}(a) + 2^{\nu+1} a^{-\nu-2} \pi^{-1}\, \Gamma(\nu+1)$$
$[\operatorname{Re}\nu > -1]$ ET II 339(44)a

7. $$\int_0^1 x^{\nu+1}\, I_\nu(ax)\, dx = a^{-1}\, I_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 365(3)a

8. $$\int_0^1 x^{\nu+1}\, K_\nu(ax)\, dx = 2^\nu a^{-\nu-2}\, \Gamma(\nu+1) - a^{-1}\, K_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 367(22)a

9. $$\int_0^1 x^{1-\nu}\, J_\nu(ax)\, dx = \frac{a^{\nu-2}}{2^{\nu-1}\, \Gamma(\nu)} - a^{-1}\, J_{\nu-1}(a)$$
ET II 333(4)a

10. $$\int_0^1 x^{1-\nu}\, Y_\nu(ax)\, dx = \frac{a^{\nu-2} \cot(\nu\pi)}{2^{\nu-1}\, \Gamma(\nu)} - a^{-1}\, Y_{\nu-1}(a)$$
$[\operatorname{Re}\nu < 1]$ ET II 339(45)a

11. $$\int_0^1 x^{1-\nu}\, I_\nu(ax)\, dx = a^{-1}\, I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1}\, \Gamma(\nu)}$$
ET II 365(4)a

12. $$\int_0^1 x^{1-\nu}\, K_\nu(ax)\, dx = 2^{-\nu} a^{\nu-2}\, \Gamma(1-\nu) - a^{-1}\, K_{\nu-1}(a)$$
$[\operatorname{Re}\nu < 1]$ ET II 367(23)a

13.[7] $$\int_0^1 x^\mu\, J_\nu(ax)\, dx = \frac{2^\mu\, \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{a^{\mu+1}\, \Gamma\left(\frac{\nu-\mu+1}{2}\right)} + a^{-\mu} \left\{(\mu+\nu-1)\, J_\nu(a)\, S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a)\, S_{\mu,\nu}(a)\right\}$$
$[a > 0, \quad \operatorname{Re}(\mu+\nu) > -1]$ ET II 22(8)a

14. $$\int_0^\infty x^\mu\, J_\nu(ax)\, dx = 2^\mu a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$$
$\left[-\operatorname{Re}\nu - 1 < \operatorname{Re}\mu < \frac{1}{2}, \quad a > 0\right]$ EH II 49(19)

15. $$\int_0^\infty x^\mu\, Y_\nu(ax)\, dx = 2^\mu \cot\left[\tfrac{1}{2}(\nu+1-\mu)\pi\right] a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$$
$\left[|\operatorname{Re}\nu| - 1 < \mu < \frac{1}{2}, \quad a > 0\right]$ ET II 97(3)a

16. $$\int_0^\infty x^\mu\, K_\nu(ax)\, dx = 2^{\mu-1} a^{-\mu-1}\, \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)$$
$[\operatorname{Re}(\mu+1\pm\nu) > 0, \quad \operatorname{Re} a > 0]$ EH II 51(27)

17. $$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}}\,dx = \frac{\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{2^{\nu-q}a^{q-\nu+1}\,\Gamma\left(\nu-\frac{1}{2}q+\frac{1}{2}\right)} \qquad \left[-1<\operatorname{Re}q<\operatorname{Re}\nu-\tfrac{1}{2}\right]$$

WA 428(1), KU 144(5)

18. $$\int_0^\infty \frac{Y_\nu(x)}{x^{\nu-\mu}}\,dx = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\nu\right)\sin\left(\frac{1}{2}\mu-\nu\right)\pi}{2^{\nu-\mu}\pi}$$

$$\left[|\operatorname{Re}\nu|<\operatorname{Re}(1+\mu-\nu)<\tfrac{3}{2}\right]$$

WA 430(5)

19. $$\int_0^1 x^{2m+n+1/2}\,K_{n+1/2}(\alpha x)\,dx = \sqrt{\frac{\pi}{2}}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\,\frac{\gamma(2m+n-k+1,\alpha)}{\alpha^{2m+n+3/2}2^k}$$ STR

6.562

1. $$\int_0^\infty x^\mu\,Y_\nu(bx)\frac{dx}{x+a} = (2a)^\mu\pi^{-1}\left\{\sin\left[\tfrac{1}{2}\pi(\mu-\nu)\right]\Gamma\left[\tfrac{1}{2}(\mu+\nu+1)\right]\Gamma\left[\tfrac{1}{2}(1+\mu-\nu)\right]S_{-\mu,\nu}(ab)\right.$$
$$\left.-2\cos\left[\tfrac{1}{2}\pi\left(\mu-\nu\right)\right]\Gamma\left(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu\right)\Gamma\left(1+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right)S_{-\mu-1,\nu}(ab)\right\}$$
$$\left[b>0,\quad |\arg a|<\pi,\quad \operatorname{Re}(\mu\pm\nu)>-1,\quad \operatorname{Re}\mu<\tfrac{3}{2}\right]$$ ET II 98(8)

2. $$\int_0^\infty \frac{x^\nu\,J_\nu(ax)}{x+k}\,dx = \frac{\pi k^\nu}{2\cos\nu\pi}\left[\mathbf{H}_{-\nu}(ak)-Y_{-\nu}(ak)\right] \qquad \left[-\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{3}{2},\quad a>0,\quad |\arg k|<\pi\right]$$

WA 479(7)

3. $$\int_0^\infty x^\mu\,K_\nu(bx)\frac{dx}{x+a}$$
$$= 2^{\mu-2}\,\Gamma\left[\tfrac{1}{2}(\mu+\nu)\right]\Gamma\left[\tfrac{1}{2}(\mu-\nu)\right]b^{-\mu}\;{}_1F_2\left(1;1-\frac{\mu+\nu}{2},1-\frac{\mu-\nu}{2};\frac{a^2b^2}{4}\right)$$
$$-2^{\mu-3}\,\Gamma\left[\tfrac{1}{2}\left(\mu-\nu-1\right)\right]\Gamma\left[\tfrac{1}{2}(\mu+\nu-1)\right]ab^{1-\mu}\;{}_1F_2\left(1;\frac{3-\mu-\nu}{2},\frac{3-\mu+\nu}{2};\frac{a^2b^2}{4}\right)$$
$$-\pi a^\mu\operatorname{cosec}[\pi(\mu-\nu)]\left\{K_\nu(ab)+\pi\cos(\mu\pi)\operatorname{cosec}[\pi(\nu+\mu)]\,I_\nu(ab)\right\}$$
$$[\operatorname{Re}b>0,\quad |\arg a|<\pi,\quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-1]$$ ET II 127(4)

6.563 $$\int_0^\infty x^{\varrho-1}\,J_\nu(bx)\frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho+\nu-\mu)\pi]\,\Gamma(\mu+1)}$$
$$\times\left\{\sum_{m=0}^\infty\frac{(-1)^m\left(\frac{1}{2}ab\right)^{\nu+2m}\Gamma(\varrho+\nu+2m)}{m!\,\Gamma(\nu+m+1)\,\Gamma\left(\varrho+\nu-\mu+2m\right)}\right.$$
$$\left.-\sum_{m=0}^\infty\frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m}\Gamma(\mu+m+1)}{m!\,\Gamma\left[\frac{1}{2}(\mu+\nu-\varrho+m+3)\right]}\,\frac{\sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{\Gamma\left[\frac{1}{2}(\mu-\nu-\varrho+m+3)\right]}\right\}$$
$$\left[b>0,\quad |\arg a|<\pi,\quad \operatorname{Re}(\varrho+\nu)>0,\quad \operatorname{Re}(\varrho-\mu)<\tfrac{5}{2}\right]$$ ET II 23(10), WA 479

6.564

1. $$\int_0^\infty x^{\nu+1}\,J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{2}{\pi b}}\,a^{\nu+\frac{1}{2}}\,K_{\nu+\frac{1}{2}}(ab) \qquad \left[\operatorname{Re}a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$

ET II 23(15)

2. $$\int_0^\infty x^{1-\nu}\, J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{\pi}{2b}}\, a^{\frac{1}{2}-\nu} \left[I_{\nu-\frac{1}{2}}(ab) - \mathbf{L}_{\nu-\frac{1}{2}}(ab) \right]$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 23(16)

6.565

1. $$\int_0^\infty x^{-\nu} \left(x^2+a^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\, dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\, I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right)$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
WA 477(4), ET II 23(17)

2. $$\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\, dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab}\, \Gamma\left(\nu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 24(18)

3. $$\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^{-\nu-\frac{3}{2}} J_\nu(bx)\, dx = \frac{b^\nu \sqrt{\pi}}{2^{\nu+1} a e^{ab}\, \Gamma\left(\nu+\frac{3}{2}\right)}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1\right]$$
ET II 24(19)

4. $$\int_0^\infty \frac{J_\nu(bx) x^{\nu+1}}{(x^2+a^2)^{\mu+1}}\, dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu\, \Gamma(\mu+1)}\, K_{\nu-\mu}(ab)$$
$$\left[-1 < \operatorname{Re}\nu < \operatorname{Re}\left(2\mu+\tfrac{3}{2}\right), \quad a > 0, \quad b > 0\right]$$
MO 43

5. $$\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^\mu Y_\nu(bx)\, dx = 2^{\nu-1} \pi^{-1} a^{2\mu+2} (1+\mu)^{-1}\, \Gamma(\nu) b^{-\nu}$$
$$\times\, {}_1F_2\left(1; 1-\nu, 2+\mu; \frac{a^2b^2}{4}\right) - 2^\mu a^{\mu+\nu+1} \left[\sin(\nu\pi)\right]^{-1}$$
$$\times\, \Gamma(\mu+1) b^{-1-\mu} \left[I_{\mu+\nu+1}(ab) - 2\cos(\mu\pi)\, K_{\mu+\nu+1}(ab)\right]$$
$$\left[b > 0, \quad \operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < -2\operatorname{Re}\mu\right]$$
ET II 100(19)

6.[10] $$\int_0^\infty x^{1-\nu} \left(x^2+a^2\right)^\mu Y_\nu(bx)\, dx = \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)}\, I_{-1-\mu+\nu}(ab) \cot[\pi(\mu-\nu)] \operatorname{cosec}(\pi\mu)$$
$$- \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu} \pi}{\Gamma(-\mu)}\, I_{1+\mu-\nu}(ab) \operatorname{cosec}[\pi(\mu-\nu)] \operatorname{cosec}(\pi\nu)$$
$$+ \frac{2^{-1-\nu} a^{2+2\mu} b^\nu}{(1+\mu)\pi} \cos(\pi\nu)\, \Gamma(-\mu)\, {}_1F_2\left(1;\ 2+\mu, 1+\nu;\ \frac{a^2b^2}{4}\right)$$
$$\left[\operatorname{Re}\nu < 1, \quad \operatorname{Re}(\nu-2\mu) > -3, \quad \arg a^2 \neq \pi, \quad b > 0\right]$$
MC

7. $$\int_0^\infty x^{1+\nu} \left(x^2+a^2\right)^\mu K_\nu(bx)\, dx = 2^\nu\, \Gamma(\nu+1) a^{\nu+\mu+1} b^{-1-\mu}\, S_{\mu-\nu,\mu+\nu+1}(ab)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1\right]$$
ET II 128(8)

8.[11] $$\int_0^\infty \frac{x^{\varrho-1}\,J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\,dx = \frac{a^\nu k^{\varrho+\nu-2\mu-2}\,\Gamma\left(\frac12\varrho+\frac12\nu\right)\Gamma\left(\mu+1-\frac12\varrho-\frac12\nu\right)}{2^{\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\ {}_1F_2\left(\frac{\varrho+\nu}{2};\frac{\varrho+\nu}{2}-\mu,\nu+1;\frac{a^2k^2}{4}\right)$$
$$+\frac{a^{2\mu+2-\varrho}\,\Gamma\left(\frac12\nu+\frac12\varrho-\mu-1\right)}{2^{2\mu+3-\varrho}\,\Gamma\left(\mu+2+\frac12\nu-\frac12\varrho\right)}$$
$$\times\ {}_1F_2\left(\mu+1;\mu+2+\frac{\nu-\varrho}{2},\mu+2-\frac{\nu+\varrho}{2};\frac{a^2k^2}{4}\right)$$
$$\left[a>0,\quad -\operatorname{Re}\nu<\operatorname{Re}\varrho<2\operatorname{Re}\mu+\tfrac72,\quad \operatorname{Re}k>0\right]$$ WA 477(1)

6.566

1. $$\int_0^\infty x^\mu\,Y_\nu(bx)\frac{dx}{x^2+a^2} = 2^{\mu-2}\pi^{-1}b^{1-\mu}$$
$$\times\cos\left[\frac{\pi}{2}(\mu-\nu+1)\right]\Gamma\left(\tfrac12\mu+\tfrac12\nu-\tfrac12\right)\Gamma\left(\tfrac12\mu-\tfrac12\nu-\tfrac12\right)$$
$$\times\ {}_1F_2\left(1;2-\frac{\mu+1+\nu}{2},2-\frac{\mu+1-\nu}{2};\frac{a^2b^2}{4}\right)$$
$$-\frac12\pi a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu+\nu+1)\right]\cot\left[\frac{\pi}{2}(\mu-\nu+1)\right]I_\nu(ab)$$
$$-a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu-\nu+1)\right]K_\nu(ab)$$
$$\left[b>0,\quad \operatorname{Re}a>0,\quad |\operatorname{Re}\nu|-1<\operatorname{Re}\mu<\tfrac52\right]$$ ET II 100(17)

2. $$\int_0^\infty x^{\nu+1}\,J_\nu(ax)\frac{dx}{x^2+b^2} = b^\nu\,K_\nu(ab) \qquad \left[a>0,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\tfrac32\right]$$
EH II 96(58)

3. $$\int_0^\infty x^\nu\,K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2 b^{\nu-1}}{4\cos\nu\pi}\left[\mathbf{H}_{-\nu}(ab)-Y_{-\nu}(ab)\right]$$
$$\left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu>-\tfrac12\right]$$
WA 468(9)

4. $$\int_0^\infty x^{-\nu}\,K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2}{4b^{\nu+1}\cos\nu\pi}\left[\mathbf{H}_\nu(ab)-Y_\nu(ab)\right]$$
$$\left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu<\tfrac12\right]$$
WA 468(10)

5. $$\int_0^\infty x^{-\nu}\,J_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi}{2b^{\nu+1}}\left[I_\nu(ab)-\mathbf{L}_\nu(ab)\right] \qquad \left[a>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\nu>-\tfrac52\right]$$
WA 468(11)

6.567

1. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^\mu J_\nu(bx)\,dx = 2^\mu\,\Gamma(\mu+1)b^{-(\mu+1)}\,J_{\nu+\mu+1}(b)$$
$$[b>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>-1]$$
ET II 26(33)a

2. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^{\mu} Y_\nu(bx)\,dx = b^{-(\mu+1)}\left[2^{\mu}\,\Gamma(\mu+1)\,Y_{\mu+\nu+1}(b)+2^{\nu+1}\pi^{-1}\,\Gamma(\nu+1)\,S_{\mu-\nu,\mu+\nu+1}(b)\right]$$
$[b>0,\quad \operatorname{Re}\mu>-1,\quad \operatorname{Re}\nu>-1]$ ET II 103(35)a

3. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} J_\nu(bx)\,dx = \frac{2^{1-\nu}\,S_{\nu+\mu,\mu-\nu+1}(b)}{b^{\mu+1}\,\Gamma(\nu)}$$
$[b>0,\quad \operatorname{Re}\mu>-1]$ ET II 25(31)a

4. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} Y_\nu(bx)\,dx = b^{-(\mu+1)}\left[2^{1-\nu}\pi^{-1}\cos(\nu\pi)\,\Gamma(1-\nu)\times S_{\mu+\nu,\mu-\nu+1}(b)-2^{\mu}\operatorname{cosec}(\nu\pi)\,\Gamma(\mu+1)\,J_{\mu-\nu+1}(b)\right]$$
$[b>0,\quad \operatorname{Re}\mu>-1,\quad \operatorname{Re}\nu<1]$ ET II 104(37)a

5. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} K_\nu(bx)\,dx = 2^{-\nu-2}b^{\nu}(\mu+1)^{-1}\,\Gamma(-\nu)\;{}_1F_2\left(1;\nu+1,\mu+2;\frac{b^2}{4}\right) +\pi 2^{\mu-1}b^{-(\mu+1)}\operatorname{cosec}(\nu\pi)\,\Gamma(\mu+1)\,I_{\mu-\nu+1}(b)$$
$[\operatorname{Re}\mu>-1,\quad \operatorname{Re}\nu<1]$ ET II 129(12)a

6. $$\int_0^1 x^{1-\nu}\,J_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\,\mathbf{H}_{\nu-\frac{1}{2}}(b)$$
$[b>0]$ ET II 24(24)a

7. $$\int_0^1 x^{1+\nu}\,Y_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\operatorname{cosec}(\nu\pi)\left[\cos(\nu\pi)\,J_{\nu+\frac{1}{2}}(b)-\mathbf{H}_{-\nu-\frac{1}{2}}(b)\right]$$
$[b>0,\quad \operatorname{Re}\nu>-1]$ ET II 102(28)a

8. $$\int_0^1 x^{1-\nu}\,Y_\nu(bx)\frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}}\left\{\cot(\nu\pi)\left[\mathbf{H}_{\nu-\frac{1}{2}}(b)-Y_{\nu-\frac{1}{2}}(b)\right]-J_{\nu-\frac{1}{2}}(b)\right\}$$
$[b>0,\quad \operatorname{Re}\nu<1]$ ET II 102(30)a

9. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[J_\nu\left(\frac{b}{2}\right)\right]^2$$
$\left[b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 24(25)a

10. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} Y_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)J_\nu\left(\frac{b}{2}\right)Y_\nu\left(\frac{b}{2}\right)$$
$\left[b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 102(31)a

11. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx = 2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)I_\nu\left(\frac{b}{2}\right)K_\nu\left(\frac{b}{2}\right)$$
$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 129(10)a

12. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} I_\nu(bx)\,dx = 2^{-\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[I_\nu\left(\frac{b}{2}\right)\right]^2$$
ET II 365(5)a

13. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^{-\nu}\frac{b^{\nu-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\sin b$$
$\left[b>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$ ET II 25(27)a

14. $$\int_1^\infty x^\nu \left(x^2-1\right)^{\nu-\frac{1}{2}} Y_\nu(bx)\,dx = 2^{\nu-2}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[J_\nu\left(\frac{b}{2}\right)J_{-\nu}\left(\frac{b}{2}\right)-Y_\nu\left(\frac{b}{2}\right)Y_{-\nu}\left(\frac{b}{2}\right)\right]$$
$\left[|\operatorname{Re}\nu|<\frac{1}{2},\quad b>0\right]$ ET II 103(32)a

15. $$\int_1^\infty x^\nu \left(x^2-1\right)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx = \frac{2^{\nu-1}}{\sqrt{\pi}}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[K_\nu\left(\frac{b}{2}\right)\right]^2$$
$\left[\operatorname{Re}b>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 129(11)a

16. $$\int_1^\infty x^{-\nu} \left(x^2-1\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = -2^{-\nu-1}\sqrt{\pi}b^{\nu}\,\Gamma\left(\frac{1}{2}-\nu\right)J_\nu\left(\frac{b}{2}\right)Y_\nu\left(\frac{b}{2}\right)$$
$\left[b>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET II 25(26)a

17.[8] $$\int_1^\infty x^{-\nu+1} \left(x^2-1\right)^{\nu-\frac{1}{2}} J_\nu(bx)\,dx = \frac{2^\nu}{\sqrt{\pi}}b^{-\nu-1}\,\Gamma\left(\frac{1}{2}+\nu\right)\cos b$$
$\left[b>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET II 25(28)

6.568

1. $$\int_0^\infty x^\nu\, Y_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\nu-1}\,J_\nu(ab)$$
$\left[a>0,\quad b>0,\quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{5}{2}\right]$ ET II 101(22)

2. $$\int_0^\infty x^\mu\, Y_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\mu-1}\,J_\nu(ab)+2^\mu\pi^{-1}a^{\mu-1}\cos\left[\frac{\pi}{2}(\mu-\nu+1)\right]$$
$$\times\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)S_{-\mu,\nu}(ab)$$
$\left[a>0,\quad b>0,\quad |\operatorname{Re}\nu|-1<\operatorname{Re}\mu<\frac{5}{2}\right]$ ET II (101)(25)

6.569 $$\int_0^1 x^\lambda(1-x)^{\mu-1}\,J_\nu(ax)\,dx$$
$$=\frac{\Gamma(\mu)\,\Gamma(1+\lambda+\nu)2^{-\nu}a^\nu}{\Gamma(\nu+1)\,\Gamma(1+\lambda+\mu+\nu)}$$
$$\times\,{}_2F_3\left(\frac{\lambda+1+\nu}{2},\frac{\lambda+2+\nu}{2};\nu+1,\frac{\lambda+1+\mu+\nu}{2},\frac{\lambda+2+\mu+\nu}{2};-\frac{a^2}{4}\right)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}(\lambda+\nu)>-1]$ ET II 193(56)a

6.571

1. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}}\pm x\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = a^\mu\, I_{\frac{1}{2}(\nu\mp\mu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\nu\pm\mu)}\left(\frac{ab}{2}\right)$$
$\left[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu<\frac{3}{2}\right]$ ET II 26(38)

2. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}}-x\right]^\mu Y_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$=a^\mu\left[\cot(\nu\pi)\,I_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right)-\operatorname{cosec}(\nu\pi)\,I_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)\right]$$
$\left[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}\mu>-\frac{3}{2},\quad |\operatorname{Re}\nu|<1\right]$ ET II 104(40)

3. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}}+x\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= \frac{\pi^2}{4}a^\mu \operatorname{cosec}(\nu\pi)\left[J_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)Y_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)-Y_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)J_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)\right]$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0]$ ET II 130(15)

6.572

1. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}+a\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab\,\Gamma(\nu+1)}W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\,M_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)$$
$[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}(\nu-\mu)>-1]$
ET II 26(40)

2. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}+a\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab}W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(iab)\,W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(-iab)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\mu+|\operatorname{Re}\nu|<1]$ ET II 130(18), BU 87(6a)

3. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}-a\right]^\mu Y_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= -\frac{1}{ab}W_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\left\{\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)}\tan\left(\frac{\nu-\mu}{2}\pi\right)M_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\right.$$
$$\left.+\sec\left(\frac{\nu-\mu}{2}\pi\right)W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\right\}$$
$\left[\operatorname{Re} a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}+\frac{1}{2}\operatorname{Re}\mu\right]$ ET II 105(42)

6.573

1. $$\int_0^\infty x^{\nu-M+1}J_\nu(bx)\prod_{i=1}^k J_{\mu_i}(a_i x)\,dx=0 \qquad M=\sum_{i=1}^k \mu_i$$
$$\left[a_i>0, \quad \sum_{i=1}^k a_i<b<\infty, \quad -1<\operatorname{Re}\nu<\operatorname{Re}M+\tfrac{1}{2}k-\tfrac{1}{2}\right]$$ ET II 54(42)

2. $$\int_0^\infty x^{\nu-M-1}J_\nu(bx)\prod_{i=1}^k J_{\mu_i}(a_i x)\,dx=2^{\nu-M-1}b^{-\nu}\Gamma(\nu)\prod_{i=1}^k\frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \qquad M=\sum_{i=1}^k\mu_i$$
$$\left[a_i>0, \quad \sum_{i=1}^k a_i<b<\infty, \quad 0<\operatorname{Re}\nu<\operatorname{Re}M+\tfrac{1}{2}k+\tfrac{3}{2}\right]$$ WA 460(16)a, ET II 54(43)

6.574

1.[8] $$\int_0^\infty J_\nu(\alpha t)\, J_\mu(\beta t) t^{-\lambda}\, dt = \frac{\alpha^\nu\, \Gamma\left(\dfrac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1}\, \Gamma\left(\dfrac{-\nu+\mu+\lambda+1}{2}\right) \Gamma(\nu+1)} \times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{\alpha^2}{\beta^2}\right)$$
$[\operatorname{Re}(\nu+\mu-\lambda+1)>0, \quad \operatorname{Re}\lambda > -1, \quad 0<\alpha<\beta]$ WA 439(2)a, MO 49

If we reverse the positions of ν and μ and at the same time reverse the positions of α and β, the function on the right-hand side of this equation will change. Thus, the right-hand side represents a function of $\dfrac{\alpha}{\beta}$ that is not analytic at $\dfrac{\alpha}{\beta}=1$.

For $\alpha=\beta$, we have the following equation:

2. $$\int_0^\infty J_\nu(\alpha t)\, J_\mu(\alpha t) t^{-\lambda}\, dt = \frac{\alpha^{\lambda-1}\, \Gamma(\lambda)\, \Gamma\left(\dfrac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda\, \Gamma\left(\dfrac{-\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\dfrac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\dfrac{\nu-\mu+\lambda+1}{2}\right)}$$
$[\operatorname{Re}(\nu+\mu+1) > \operatorname{Re}\lambda > 0, \quad \alpha>0]$
MO 49, WA 441(2)a

If $\mu-\nu+\lambda+1$ (or $\nu-\mu+\lambda+1$) is a negative integer, the right-hand side of equation **6.574** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.574** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

3.* $$\int_0^\infty J_\nu(\alpha t)\, J_\mu(\beta t) t^{-\lambda}\, dt = \frac{\beta^\nu\, \Gamma\left(\dfrac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda \alpha^{\mu-\lambda+1}\, \Gamma\left(\dfrac{\nu-\mu+\lambda+1}{2}\right) \Gamma(\nu+1)} \times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{-\nu+\mu-\lambda+1}{2}; \mu+1; \frac{\beta^2}{\alpha^2}\right)$$
$[\operatorname{Re}(\nu+\mu-\lambda+1)>0, \quad \operatorname{Re}\lambda > -1, \quad 0<\beta<\alpha]$ MO 50, WA 440(3)a

If $\mu-\nu+\lambda+1$ (or $\nu-\mu+\lambda+1$) is a negative integer, the right-hand side of equation **6.754** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.754** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

6.575

1.[11] $$\int_0^\infty J_{\nu+1}(\alpha t)\, J_\mu(\beta t) t^{\mu-\nu}\, dt = 0 \qquad [\alpha<\beta]$$
$$= \frac{\left(\alpha^2-\beta^2\right)^{\nu-\mu} \beta^\mu}{2^{\nu-\mu} \alpha^{\nu+1}\, \Gamma(\nu-\mu+1)} \qquad [\alpha \geq \beta]$$
$[\operatorname{Re}(\nu+1) > \operatorname{Re}\mu > -1]$ MO 51

2. $$\int_0^\infty \frac{J_\nu(x)\, J_\mu(x)}{x^{\nu+\mu}}\, dx = \frac{\sqrt{\pi}\, \Gamma(\nu+\mu)}{2^{\nu+\mu}\, \Gamma\left(\nu+\mu+\frac{1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right) \Gamma\left(\mu+\frac{1}{2}\right)}$$
$[\operatorname{Re}(\nu+\mu)>0]$ KU 147(17), WA 434(1)

6.576

1. $$\int_0^\infty x^{\mu-\nu+1}\, J_\mu(x)\, K_\nu(x)\, dx = \tfrac{1}{2}\,\Gamma(\mu-\nu+1) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) > -1]$$
ET II 370(47)

2.[11] $$\int_0^\infty x^{-\lambda}\, J_\nu(ax)\, J_\nu(bx)\, dx = \frac{a^\nu b^\nu\, \Gamma\left(\nu + \dfrac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma\left(\dfrac{1+\lambda}{2}\right)} \times F\left(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}\right)$$
$[a > 0, \quad b > 0, \quad 2\operatorname{Re}\nu + 1 > \operatorname{Re}\lambda > -1]$ ET II 47(4)

3. $$\int_0^\infty x^{-\lambda}\, K_\mu(ax)\, J_\nu(bx)\, dx = \frac{b^\nu\, \Gamma\left(\dfrac{\nu-\lambda+\mu+1}{2}\right) \Gamma\left(\dfrac{\nu-\lambda-\mu+1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1}\, \Gamma(1+\nu)} \times F\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(a \pm ib) > 0, \quad \operatorname{Re}(\nu-\lambda+1) > |\operatorname{Re}\mu|]$ EH II 52(31), ET II 63(4), WA 449(1)

4. $$\int_0^\infty x^{-\lambda}\, K_\mu(ax)\, K_\nu(bx)\, dx = \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)}\, \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \times F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right)$$
$[\operatorname{Re} a + b > 0, \quad \operatorname{Re}\lambda < 1 - |\operatorname{Re}\mu| - |\operatorname{Re}\nu|]$ ET II 145(49), EH II 93(36)

5. $$\int_0^\infty x^{-\lambda}\, K_\mu(ax)\, I_\nu(bx)\, dx = \frac{b^\nu\, \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{2^{\lambda+1}\, \Gamma(\nu+1) a^{-\lambda+\nu+1}} \times F\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu; \nu+1; \frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(\nu + 1 - \lambda \pm \mu) > 0, \quad a > b]$ EH II 93(35)

6. $$\int_0^\infty x^{-\lambda}\, Y_\mu(ax)\, J_\nu(bx)\, dx = \frac{2}{\pi} \sin\frac{\pi(\nu-\mu-\lambda)}{2} \int_0^\infty x^{-\lambda}\, K_\mu(ax)\, I_\nu(bx)\, dx$$
$[a > b, \quad \operatorname{Re}\lambda > -1, \quad \operatorname{Re}(\nu - \lambda + 1 \pm \mu) > 0]$ (see **6.576** 5) EH II 93(37)

7.[8] $$\int_0^\infty x^{\mu+\nu+1}\, J_\mu(ax)\, K_\nu(bx)\, dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}$$
$[\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1, \quad \operatorname{Re} b > |\operatorname{Im} a|]$
ET 137(16), EH II 93(36)

6.577

1.[8] $$\int_0^\infty x^{\nu-\mu+1+2n}\, J_\mu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\nu-\mu+2n}\, I_\mu(ac)\, K_\nu(bc)$$
$[a>0, \quad b>a, \quad \operatorname{Re} c>0, \quad 2+\operatorname{Re}\mu-2n>\operatorname{Re}\nu>-1-n, \quad n\geq 0 \text{ an integer}]$ ET II 49(13)

2.[8] $$\int_0^\infty x^{\mu-\nu+1+2n}\, J_\mu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\nu+2n}\, I_\nu(bc)\, K_\mu(ac)$$
$[b>0, \quad a>b, \quad \operatorname{Re}\nu-2n+2>\operatorname{Re}\mu>-n-1, \quad n\geq 0 \text{ an integer}]$ ET II 49(15)

6.578

1. $$\int_0^\infty x^{\varrho-1}\, J_\lambda(ax)\, J_\mu(bx)\, J_\nu(cx)\, dx = \frac{2^{\varrho-1} a^\lambda b^\mu c^{-\lambda-\mu-\varrho}\, \Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma\left(1-\dfrac{\lambda+\mu-\nu+\varrho}{2}\right)} \times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right)$$
$\left[\operatorname{Re}(\lambda+\mu+\nu+\varrho)>0, \quad \operatorname{Re}\varrho<\frac{5}{2}, \quad a>0, \quad b>0, \quad c>0, \quad c>a+b\right]$ ET II 351(9)

2. $$\int_0^\infty x^{\varrho-1}\, J_\lambda(ax)\, J_\mu(bx)\, K_\nu(cx)\, dx = \frac{2^{\varrho-2} a^\lambda b^\mu c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1)\,\Gamma(\mu+1)}\, \Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right) \times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right)$$
$[\operatorname{Re}(\varrho+\lambda+\mu)>|\operatorname{Re}\nu|, \quad \operatorname{Re} c>|\operatorname{Im} a|+|\operatorname{Im} b|]$ ET II 373(8)

3. $$\int_0^\infty x^{\lambda-\mu-\nu+1}\, J_\nu(ax)\, J_\mu(bx)\, J_\lambda(cx)\, dx = 0$$
$\left[\operatorname{Re}\lambda>-1, \quad \operatorname{Re}(\lambda-\mu-\nu)<\frac{1}{2}, \quad c>b>0, \quad 0<a<c-b\right]$ ET II 53(36)

4. $$\int_0^\infty x^{\lambda-\mu-\nu-1}\, J_\nu(ax)\, J_\mu(bx)\, J_\lambda(cx)\, dx = \frac{2^{\lambda-\mu-\nu-1} a^\nu b^\mu\, \Gamma(\lambda)}{c^\lambda\, \Gamma(\mu+1)\, \Gamma(\nu+1)}$$
$\left[\operatorname{Re}\lambda>0, \quad \operatorname{Re}(\lambda-\mu-\nu)<\frac{5}{2}, \quad c>b>0, \quad 0<a<c-b\right]$ ET II 53(37)

5. $$\int_0^\infty x^{1+\mu}\, Y_\mu(ax)\, J_\nu(bx)\, J_\nu(cx)\, dx = 0 \qquad [0<b<c, \quad 0<a<c-b]$$
ET II 352(13)

6.[11] $$\int_0^\infty x^{\mu+1}\, K_\mu(ax)\, J_\nu(bx)\, J_\nu(cx)\, dx = \frac{1}{\sqrt{2\pi}} a^\mu b^{-\mu-1} c^{-\mu-1} e^{-\left(\mu+\frac{1}{2}\right)\pi i} \left(u^2-1\right)^{-\frac{1}{2}\mu-\frac{1}{4}}\, Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(u)$$
$\left[2bcu=a^2+b^2+c^2, \quad \operatorname{Re} a>|\operatorname{Im} b|+|\operatorname{Im} c|, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1\right]$
WA 452(2), ET II 64(12)

7.[11] $$\int_0^\infty x^{\mu+1}\, I_\nu(ax)\, K_\mu(bx)\, J_\nu(cx)\, dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-\left(\mu-\frac{1}{2}\nu+\frac{1}{4}\right)\pi i} \left(v^2+1\right)^{-\frac{1}{2}\mu-\frac{1}{4}}\, Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iv),$$
$2acv=b^2-a^2+c^2 \qquad [\operatorname{Re} b>|\operatorname{Re} a|+|\operatorname{Im} c|; \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1]$ ET II 66(22)

8.[11] $$\int_0^\infty x^{1-\mu}\, J_\mu(ax)\, J_\nu(bx)\, J_\nu(cx)\, dx$$
$$= \sqrt{\frac{2}{\pi^3}} a^{-\mu}(bc)^{\mu-1}\,(\sinh u)^{\mu-\frac12} \sin[(\mu-\nu)\pi] e^{\left(\mu-\frac12\right)\pi i}\, Q^{\frac12-\mu}_{\nu-\frac12}(\cosh u) \qquad [a>b+c]$$
$$= \frac{1}{\sqrt{2\pi}} a^{-\mu}(bc)^{\mu-1}\,(\sin v)^{\mu-\frac12}\, P^{\frac12-\mu}_{\nu-\frac12}(\cos v) \qquad [|b-c|<a<b+c]$$
$$= 0 \qquad [0<a<|b-c|]$$
$$\left[2bc\cosh u = a^2-b^2-c^2, \quad 2bc\cos v = b^2+c^2-a^2, \quad b>0, \quad c>0; \quad \operatorname{Re}\nu>-1, \operatorname{Re}\mu>-\tfrac12\right]$$

9. $$\int_0^\infty J_\nu(ax)\, J_\nu(bx)\, J_\nu(cx) x^{1-\nu}\, dx = 0 \qquad [0<c\le|a-b| \text{ or } c\ge a+b]$$
$$= \frac{2^{\nu-1}\Delta^{2\nu-1}}{(abc)^\nu\,\Gamma\left(\nu+\frac12\right)\Gamma\left(\frac12\right)} \qquad [|a-b|<c<a+b]$$
$$\Delta = \tfrac14\sqrt{[c^2-(a-b)^2]\,[(a+b)^2-c^2]}, \quad \left[a>0, \quad b>0, \quad c>0; \quad \operatorname{Re}\nu>-\tfrac12\right]$$

($\Delta>0$ is equal to the area of a triangle whose sides are a, b, and c.)

10.[11] $$\int_0^\infty x^{\nu+1}\, K_\mu(ax)\, K_\mu(bx)\, J_\nu(cx)\, dx = \frac{\sqrt{\pi}c^\nu\,\Gamma(\nu+\mu+1)\,\Gamma(\nu-\mu+1)}{2^{3/2}(ab)^{\nu+1}\,(u^2-1)^{\frac12\nu+\frac14}}\, P^{-\nu-\frac12}_{\mu-\frac12}(u)$$
$$\left[2abu = a^2+b^2+c^2, \quad \operatorname{Re}(a+b)>|\operatorname{Im}c|, \quad \operatorname{Re}(\nu\pm\mu)>-1, \quad \operatorname{Re}\nu>-1\right]$$ ET II 67(30)

11.[11] $$\int_0^\infty x^{\nu+1}\, K_\mu(ax)\, I_\mu(bx)\, J_\nu(cx)\, dx = \frac{(ab)^{-\nu-1}c^\nu e^{-\left(\nu+\frac12\right)\pi i}\, Q^{\nu+\frac12}_{\mu-\frac12}(u)}{\sqrt{2\pi}\,(u^2-1)^{\frac12\nu+\frac14}} \qquad 2abu = a^2+b^2+c^2$$
$$[\operatorname{Re}a>|\operatorname{Re}b|+|\operatorname{Im}c|; \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1]$$ ET II 66(24)

12.[8] $$\int_0^\infty x^{\nu+1}\,[J_\nu(ax)]^2\, Y_\nu(bx)\, dx = 0 \qquad \left[0<b<2a, \quad |\operatorname{Re}\nu|<\tfrac12\right]$$
$$= \frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac12-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac12} \qquad \left[0<2a<b, \quad |\operatorname{Re}\nu|<\tfrac12\right]$$
ET II 109(3)

13. $$\int_0^\infty x^{\nu+1}\, J_\nu(ax)\, Y_\nu(ax)\, J_\nu(bx)\, dx$$
$$= 0 \qquad \left[a>0, \quad |\operatorname{Re}\nu|<\tfrac12, \quad 0<b<2a\right]$$
$$= -\frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac12-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac12} \qquad \left[a>0, \quad 2a<b<\infty, \quad |\operatorname{Re}\nu|<\tfrac12\right]$$
ET II 55(49)

14. $$\int_0^\infty x^{\nu+1}\, J_\mu(xa\sin\psi)\, J_\nu(xa\sin\varphi)\, K_\mu(xa\cos\varphi\cos\psi)\, dx$$
$$= \frac{2^\nu\,\Gamma(\mu+\nu+1)\,(\sin\varphi)^\nu\left(\cos\frac\alpha2\right)^{2\nu+1}}{a^{\nu+2}\,(\cos\psi)^{2\nu+2}}\, P^{-\mu}_\nu(\cos\alpha)$$
$$\left[\tan\tfrac12\alpha = \tan\psi\cos\varphi, \quad a>0, \quad \frac{\pi}{2}>\varphi>0, \quad 0<\psi<\frac{\pi}{2}, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1\right]$$
ET II 64(11)

15. $$\int_0^\infty x^{\nu+1}\, J_\nu(ax)\, K_\nu(bx)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[(a^2+b^2+c^2)^2-4a^2c^2\right]^{\nu+\frac{1}{2}}}$$
$$\left[\operatorname{Re} b > |\operatorname{Im} a|, \quad c>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 63(8)

16.[8] $$\int_0^\infty x^{\nu+1}\, I_\nu(ax)\, K_\nu(bx)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[(b^2-a^2+c^2)^2+4a^2c^2\right]^{\nu+\frac{1}{2}}}$$
$$\left[\operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|; \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 65(18)

6.579

1. $$\int_0^\infty x^{2\nu+1}\, J_\nu(ax)\, Y_\nu(ax)\, J_\nu(bx)\, Y_\nu(bx)\, dx$$
$$= \frac{a^{2\nu}\,\Gamma(3\nu+1)}{2\pi b^{4\nu+2}\,\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(2\nu+\frac{3}{2}\right)}\, F\left(\nu+\tfrac{1}{2}, 3\nu+1; 2\nu+\frac{3}{2}; \frac{a^2}{b^2}\right)$$
$$\left[0<a<b, \quad -\tfrac{1}{3} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ EH II 94(45), ET II 352(15)

2. $$\int_0^\infty x^{2\nu+1}\, J_\nu(ax)\, K_\nu(ax)\, J_\nu(bx)\, K_\nu(bx)\, dx$$
$$= \frac{2^{\nu-3}a^{2\nu}\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{3\nu+1}{2}\right)}{\sqrt{\pi}\, b^{4\nu+2}\,\Gamma(\nu+1)}\, F\left(\nu+\tfrac{1}{2}, \frac{3\nu+1}{2}; 2\nu+1; 1-\frac{a^4}{b^4}\right)$$
$$\left[0<a<b, \quad \operatorname{Re}\nu > -\tfrac{1}{3}\right]$$ ET II 373(10)

3. $$\int_0^\infty x^{1-2\nu}\, [J_\nu(ax)]^4\, dx = \frac{\Gamma(\nu)\,\Gamma(2\nu)}{2\pi\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2\Gamma(3\nu)} \qquad [\operatorname{Re}\nu > 0]$$ ET II 342(25)

4. $$\int_0^\infty x^{1-2\nu}\, [J_\nu(ax)]^2\, [J_\nu(bx)]^2\, dx = \frac{a^{2\nu-1}\,\Gamma(\nu)}{2\pi b\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)}\, F\left(\nu, \frac{1}{2}-\nu; 2\nu+\frac{1}{2}; \frac{a^2}{b^2}\right)$$
ET II 351(10)

6.581

1. $$\int_0^a x^{\lambda-1}\, J_\mu(x)\, J_\nu(a-x)\, dx = 2^\lambda \sum_{m=0}^\infty \frac{(-1)^m\,\Gamma(\lambda+\mu+m)\,\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)\,\Gamma(\mu+m+1)}\, J_{\lambda+\mu+\nu+2m}(a)$$
$$[\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > -1]$$
ET II 354(25)

2.[8] $$\int_0^a x^{\lambda-1}(a-x)^{-1}\, J_\mu(x)\, J_\nu(a-x)\, dx$$
$$= \frac{2^\lambda}{a\nu} \sum_{m=0}^\infty \frac{(-1)^m\,\Gamma(\lambda+\mu+m)\,\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)\,\Gamma(\mu+m+1)}(\lambda+\mu+\nu+2m)\, J_{\lambda+\mu+\nu+2m}(a)$$
$$[\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 354(27)

3. $$\int_0^a x^\mu (a-x)^\nu\, J_\mu(x)\, J_\nu(a-x)\, dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+1)}\, a^{\mu+\nu+\frac{1}{2}}\, J_{\mu+\nu+\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 354(28), EH II 46(6)

4. $$\int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x)\,dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\tfrac{1}{2}\right]$$
ET II 354(29)

5. $$\int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x)\,dx = \frac{2^\mu\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu)}{\sqrt{\pi}\,\Gamma(\mu+\nu+1)} a^\mu J_\nu(a)$$
$$\left[\operatorname{Re}\nu > \operatorname{Re}\mu > -\tfrac{1}{2}\right]$$
ET II 355(30)

6.582 $$\int_0^\infty x^{\mu-1}|x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x)\,dx = \frac{1}{\sqrt{\pi}}(2b)^{-\mu}\,\Gamma\left(\tfrac{1}{2}-\mu\right)\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)\,K_\nu(b)$$
$$\left[b>0, \quad \operatorname{Re}\mu < \tfrac{1}{2}, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|\right]$$
ET II 374(14)

6.583 $$\int_0^\infty x^{\mu-1}(x+b)^{-\mu} K_\mu(x+b) K_\nu(x)\,dx = \frac{\sqrt{\pi}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)}{2^\mu b^\mu\,\Gamma\left(\mu+\frac{1}{2}\right)} K_\nu(b)$$
$$\left[|\arg b| < \pi, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|\right]$$
ET II 374(15)

6.584

1.[8] $$\int_0^\infty \frac{x^{\varrho-1}\left[H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}\left(axe^{\pi i}\right)\right]}{(x^2-r^2)^{m+1}}\,dx = \frac{\pi i}{2^m m!}\left(\frac{d}{r\,dr}\right)^m \left[r^{\varrho-2} H_\nu^{(1)}(ar)\right]$$
$$\left[m=0,1,2,\ldots, \quad \operatorname{Im} r > 0, \quad a>0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m+\tfrac{7}{2}\right]$$
WA 465

2.[8] $$\int_0^\infty \left[\cos\frac{1}{2}(\varrho-\nu)\pi\, J_\nu(ax) + \sin\frac{1}{2}(\varrho-\nu)\pi\, Y_\nu(ax)\right] \frac{x^{\varrho-1}}{(x^2+k^2)^{m+1}}\,dx$$
$$= \frac{(-1)^{m+1}}{2^m\cdot m!}\left(\frac{d}{k\,dk}\right)^m \left[k^{\varrho-2} K_\nu(ak)\right]$$
$$\left[m=0,1,2,\ldots, \quad \operatorname{Re} k > 0, \quad a>0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m+\tfrac{7}{2}\right]$$
WA 466(2)

3. $$\int_0^\infty \{\cos\nu\pi\, J_\nu(ax) - \sin\nu\pi\, Y_\nu(ax)\} \frac{x^{1-\nu}\,dx}{(x^2+k^2)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m\cdot m! k^{\nu+m}}$$
$$\left[m=0,1,2,\ldots, \quad \operatorname{Re} k > 0, \quad a>0, \quad -2m-\tfrac{3}{2} < \operatorname{Re}\nu < 1\right]$$
WA 466(3)

4. $$\int_0^\infty \left\{\cos\left[\left(\tfrac{1}{2}\varrho-\tfrac{1}{2}\nu-\mu\right)\pi\right] J_\nu(ax) + \sin\left[\left(\tfrac{1}{2}\varrho-\tfrac{1}{2}\nu-\mu\right)\pi\right] Y_\nu(ax)\right\} \frac{x^{\varrho-1}}{(x^2+k^2)^{\mu+1}}\,dx$$
$$= \frac{\pi k^{\varrho-2\mu-2}}{2\sin\nu\pi\cdot\Gamma(\mu+1)}\left[\frac{\left(\frac{1}{2}ak\right)^\nu \Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu-\mu\right)}\ {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2}-\mu, \nu+1; \frac{a^2k^2}{4}\right)\right.$$
$$\left. - \frac{\left(\frac{1}{2}ak\right)^{-\nu}\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\,\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu-\mu\right)}\ {}_1F_2\left(\frac{\varrho-\nu}{2}; \frac{\varrho-\nu}{2}-\mu, 1-\nu; \frac{a^2k^2}{4}\right)\right]$$
$$\left[a>0, \quad \operatorname{Re} k > 0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2\operatorname{Re}\mu + \tfrac{7}{2}\right]$$
WA 407(1)

5.[8]
$$\int_0^\infty \left[\prod_{j=1}^n J_{\mu_j}(b_n x)\right]\left\{\cos\left[\frac{1}{2}\left(\varrho+\sum_j \mu_j-\nu\right)\pi\right] J_\nu(ax) + \sin\left[\frac{1}{2}\left(\varrho+\sum_j \mu_j-\nu\right)\pi\right] Y_\nu(ax)\right\}\frac{x^{\varrho-1}}{x^2+k^2}\,dx$$
$$= -\left[\prod_{j=1}^n I_{\mu_j}(b_n k)\right] K_\nu(ak)k^{\varrho-2}$$
$$\left[\operatorname{Re} k>0,\quad a>\sum_j |\operatorname{Re} b_j|,\quad \operatorname{Re}\left(\varrho+\sum_j \mu_j\right)>|\operatorname{Re}\nu|\right]$$ WA 472(9)

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.591

1. $$\int_0^\infty x^{2\nu+\frac12} J_{\nu+\frac12}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi}b^{-\nu-1}a^{\nu+\frac12} J_{1+2\nu}\left(\sqrt{2ab}\right) K_{1+2\nu}\left(\sqrt{2ab}\right)$$
$$[a>0,\quad \operatorname{Re} b>0,\quad \operatorname{Re}\nu>-1]$$
ET II 142(35)

2. $$\int_0^\infty x^{2\nu+\frac12} Y_{\nu+\frac12}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi}b^{-\nu-1}a^{\nu+\frac12} Y_{2\nu+1}\left(\sqrt{2ab}\right) K_{2\nu+1}\left(\sqrt{2ab}\right)$$
$$[a>0,\quad \operatorname{Re} b>0,\quad \operatorname{Re}\nu>-1]$$
ET II 143(41)

3. $$\int_0^\infty x^{2\nu+\frac12} K_{\nu+\frac12}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi}b^{-\nu-1}a^{\nu+\frac12} K_{2\nu+1}\left(e^{\frac14 i\pi}\sqrt{2ab}\right) K_{2\nu+1}\left(e^{-\frac14 i\pi}\sqrt{2ab}\right)$$
$$[\operatorname{Re} a>0,\quad \operatorname{Re} b>0]$$ ET II 146(56)

4. $$\int_0^\infty x^{-2\nu+\frac12} J_{\nu-\frac12}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi}b^{\nu-1}a^{\frac12-\nu} K_{2\nu-1}\left(\sqrt{2ab}\right) \times\left[\sin(\nu\pi)\, J_{2\nu-1}\left(\sqrt{2ab}\right)+\cos(\nu\pi)\, Y_{2\nu-1}\left(\sqrt{2ab}\right)\right]$$
$$[a>0,\quad \operatorname{Re} b>0,\quad \operatorname{Re}\nu<1]$$ ET II 142(34)

5. $$\int_0^\infty x^{-2\nu+\frac12} Y_{\nu-\frac12}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = -\sqrt{\frac{\pi}{2}}b^{\nu-1}a^{\frac12-\nu}\sec(\nu\pi)\, K_{2\nu-1}\left(\sqrt{2ab}\right) \times\left[J_{2\nu-1}\left(\sqrt{2ab}\right)-J_{1-2\nu}\left(\sqrt{2ab}\right)\right]$$
$$[a>0,\quad \operatorname{Re}\nu<1]$$ ET II 143(40)

6. $$\int_0^\infty x^{-2\nu+\frac12} J_{\frac12-\nu}\left(\frac{a}{x}\right) J_\nu(bx)\,dx$$
$$= -\frac12 i\operatorname{cosec}(2\nu\pi)b^{\nu-1}a^{\frac12-\nu}\left[e^{2\nu\pi i} J_{1-2\nu}(u)\, J_{2\nu-1}(v)-e^{-2\nu\pi i} J_{2\nu-1}(u)\, J_{1-2\nu}(v)\right]$$
$$\left[u=\left(\tfrac12 ab\right)^{\frac12} e^{\frac14\pi i},\quad v=\left(\tfrac12 ab\right)^{\frac12} e^{-\frac14\pi i},\quad a>0,\quad b>0,\quad -\tfrac12<\operatorname{Re}\nu<3\right]$$ ET II 58(12)

7. $$\int_0^\infty x^{-2\nu+\frac{1}{2}}\, K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) Y_\nu(bx)\, dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu}\, Y_{2\nu-1}\left(\sqrt{2ab}\right) K_{2\nu-1}\left(\sqrt{2ab}\right)$$
$$\left[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu>\tfrac{1}{6}\right]$$
ET II 113(30)

8. $$\int_0^\infty x^{\varrho-1}\, J_\mu(ax)\, J_\nu\left(\frac{b}{x}\right) dx = \frac{a^{\nu-\varrho} b^\nu\, \Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1}\,\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\varrho+1\right)}$$
$$\times\, {}_0F_3\left(\nu+1, \frac{\nu-\mu-\varrho}{2}+1, \frac{\nu+\mu-\varrho}{2}+1; \frac{a^2b^2}{16}\right)$$
$$+\frac{a^\mu b^{\mu+\varrho}\,\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1}\,\Gamma(\mu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\varrho+1\right)}$$
$$\times\, {}_0F_3\left(\mu+1, \frac{\mu-\nu+\varrho}{2}+1, \frac{\nu+\mu+\varrho}{2}+1; \frac{a^2b^2}{16}\right)$$
$$\left[a>0, \quad b>0, \quad -\operatorname{Re}\left(\mu+\tfrac{3}{2}\right)<\operatorname{Re}\varrho<\operatorname{Re}\left(\nu+\tfrac{3}{2}\right)\right]$$
WA 480(1)

6.592

1. $$\int_0^\infty x^\lambda (1-x)^{\mu-1}\, Y_\nu\left(a\sqrt{x}\right) dx = 2^{-\nu} a^\nu \cot(\nu\pi) \frac{\Gamma(\mu)\,\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)\,\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)}$$
$$\times\, {}_1F_2\left(\lambda+1+\tfrac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\tfrac{1}{2}\nu; -\frac{a^2}{4}\right)$$
$$-2^\nu a^{-\nu} \operatorname{cosec}(\nu\pi) \frac{\Gamma(\mu)\,\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\,\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}$$
$$\times\, {}_1F_2\left(\lambda-\tfrac{1}{2}\nu+1; 1-\nu, \lambda+1+\mu-\tfrac{1}{2}\nu; -\frac{a^2}{4}\right)$$
$$\left[\operatorname{Re}\lambda>-1+\tfrac{1}{2}|\operatorname{Re}\nu|, \quad \operatorname{Re}\mu>0\right]$$
ET II 197(76)a

2.[10] $$\int_0^1 x^\lambda (1-x)^{\mu-1}\, K_\nu\left(a\sqrt{x}\right) dx$$
$$= 2^{-\nu-1} a^{-\nu} \frac{\Gamma(\nu)\,\Gamma(\mu)\,\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}\, {}_1F_2\left(\lambda+1-\tfrac{1}{2}\nu; 1-\nu, \lambda+1+\mu-\tfrac{1}{2}\nu; \frac{a^2}{4}\right)$$
$$+2^{-1-\nu} a^\nu \frac{\Gamma(-\nu)\,\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)\Gamma(\mu)}{\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)}\, {}_1F_2\left(\lambda+1+\tfrac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; \frac{a^2}{4}\right)$$
$$= \frac{2^{\nu-1}}{a^\nu}\,\Gamma(\mu)\, G^{21}_{13}\left(\frac{a^2}{4}\,\middle|\, \begin{matrix}\frac{\nu}{2}-\lambda \\ \nu, 0, \frac{\nu}{2}-\lambda-\mu\end{matrix}\right)$$
OB 159 (3.16)

$$\left[\operatorname{Re}\lambda>-1+\tfrac{1}{2}|\operatorname{Re}\nu|, \quad \operatorname{Re}\mu>0\right]$$
ET II 198(87)a

3.[11] $$\int_1^\infty x^\lambda (x-1)^{\mu-1}\, J_\nu\left(a\sqrt{x}\right)\, dx = 2^{2\lambda} a^{-2\lambda}\, G^{20}_{13}\left(\frac{a^2}{4}\,\middle|\,\begin{matrix}0\\ -\mu, \lambda+\frac{1}{2}\nu, \lambda-\frac{1}{2}\nu\end{matrix}\right)\Gamma(\mu)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{3}{4}-\operatorname{Re}\lambda\right]$ ET II 205(36)a

4. $$\int_1^\infty x^\lambda (x-1)^{\mu-1}\, K_\nu\left(a\sqrt{x}\right)\, dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda}\, G^{30}_{13}\left(\frac{a^2}{4}\,\middle|\,\begin{matrix}0\\ -\mu, \frac{1}{2}\nu+\lambda, -\frac{1}{2}\nu+\lambda\end{matrix}\right)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re}\mu>0]$ ET II 209(60)a

5. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, J_\nu\left(a\sqrt{x}\right)\, dx = \pi\left[J_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2$$
$[\operatorname{Re}\nu>-1]$ ET II 194(59)a

6. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, I_\nu\left(a\sqrt{x}\right)\, dx = \pi\left[I_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2$$
$[\operatorname{Re}\nu>-1]$ ET II 197(79)

7. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, K_\nu\left(a\sqrt{x}\right)\, dx = \frac{1}{2}\pi\sec\left(\frac{1}{2}\nu\pi\right)\left[I_{\frac{\nu}{2}}\left(\frac{a}{2}\right)+I_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right] K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)$$
$[|\operatorname{Re}\nu|<1]$ ET II 198(85)a

8. $$\int_1^\infty x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}}\, K_\nu\left(a\sqrt{x}\right)\, dx = \left[K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2$$
$[\operatorname{Re} a>0]$ ET II 208(56)a

9. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, Y_\nu\left(a\sqrt{x}\right)\, dx = \pi\left\{\cot(\nu\pi)\left[J_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2 - \operatorname{cosec}(\nu\pi)\left[J_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$[|\operatorname{Re}\nu|<1]$ ET II 195(68)a

10. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, J_\nu\left(a\sqrt{x}\right)\, dx = \Gamma(\mu) 2^\mu a^{-\mu}\, J_{\nu-\mu}(a)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 205(34)a

11. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, J_{-\nu}\left(a\sqrt{x}\right)\, dx = \Gamma(\mu) 2^\mu a^{-\mu}\left[\cos(\nu\pi)\, J_{\nu-\mu}(a) - \sin(\nu\pi)\, Y_{\nu-\mu}(a)\right]$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 205(35)a

12. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, K_\nu\left(a\sqrt{x}\right)\, dx = \Gamma(\mu) 2^\mu a^{-\mu}\, K_{\nu-\mu}(a)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re}\mu>0]$ ET II 209(59)a

13. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, Y_\nu\left(a\sqrt{x}\right)\, dx = 2^\mu a^{-\mu}\, Y_{\nu-\mu}(a)\,\Gamma(\mu)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 206(40)a

14. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, H^{(1)}_\nu\left(a\sqrt{x}\right)\, dx = 2^\mu a^{-\mu}\, H^{(1)}_{\nu-\mu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Im} a>0]$ ET II 206(45)a

15. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}\, H^{(2)}_\nu\left(a\sqrt{x}\right)\, dx = 2^\mu a^{-\mu}\, H^{(2)}_{\nu-\mu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Im} a<0]$ ET II 207(48)a

16. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1}\, J_\nu\left(a\sqrt{x}\right)\, dx = \frac{2^{2-\nu}a^{-\mu}}{\Gamma(\nu)}\, s_{\mu+\nu-1,\mu-\nu}(a)$$
$[\operatorname{Re}\mu > 0]$ ET II 194(64)a

17. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1}\, Y_\nu\left(a\sqrt{x}\right)\, dx = \frac{2^{2-\nu}a^{-\mu}\cot(\nu\pi)}{\Gamma(\nu)}\, s_{\mu+\nu-1,\mu-\nu}(a) - 2^\mu a^{-\mu}\operatorname{cosec}(\nu\pi)\, J_{\mu-\nu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu < 1]$ ET II 196(75)a

6.593

1. $$\int_0^\infty \sqrt{x}\, J_{2\nu-1}\left(a\sqrt{x}\right) J_\nu(bx)\, dx = \frac{1}{2}ab^{-2}\, J_{\nu-1}\left(\frac{a^2}{4b}\right)$$
$[b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$ ET II 58(15)

2. $$\int_0^\infty \sqrt{x}\, J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \frac{\pi a}{4b^2}\left[I_{\nu-1}\left(\frac{a^2}{4b}\right) - \mathbf{L}_{\nu-1}\left(\frac{a^2}{4b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$ ET II 144(44)

6.594

1. $$\int_0^\infty x^\nu\, I_{2\nu-1}\left(a\sqrt{x}\right) J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \sqrt{\pi}2^{-\nu}a^{2\nu-1}b^{-2\nu-\frac{1}{2}}\, J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(65)

2. $$\int_0^\infty x^\nu\, I_{2\nu-1}\left(a\sqrt{x}\right) Y_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \sqrt{\pi}2^{-\nu-1}a^{2\nu-1}b^{-2\nu-\frac{1}{2}}\operatorname{cosec}(\nu\pi) \times\left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \cos(\nu\pi)\, J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi)\, Y_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(66)

3. $$\int_0^\infty x^\nu\, J_{2\nu-1}\left(a\sqrt{x}\right) K_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \pi^2 2^{-\nu-2}a^{2\nu-1}b^{-2\nu-\frac{1}{2}}\operatorname{cosec}(\nu\pi)\left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - Y_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(67)

6.595

1. $$\int_0^\infty x^{\nu+1}\, J_\nu(cx)\prod_{i=1}^n z_i^{-\mu_i}\, J_{\mu_i}(a_i z_i)\, dx = 0 \qquad z_i = \sqrt{x^2+b_i^2}$$
$$\left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c; \quad \operatorname{Re}\left(\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2}\right) > \operatorname{Re}\nu > -1\right]$$
EH II 52(33), ET II 60(26)

2. $$\int_0^\infty x^{\nu-1}\, J_\nu(cx)\prod_{i=1}^n z_i^{-\mu_i}\, J_{\mu_i}(a_i z_i)\, dx = 2^{\nu-1}\,\Gamma(\nu)c^{-\nu}\prod_{i=1}^n\left[b_i^{-\mu_i}\, J_{\mu_i}(a_i b_i)\right] \qquad z_i = \sqrt{x^2+b_i^2}$$
$$\left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c, \quad \operatorname{Re}\left(\frac{1}{2}n + \sum_{i=1}^n \mu_i + \frac{3}{2}\right) > \operatorname{Re}\nu > 0\right]$$
EH II 52(34), ET II 60(27)

6.596

1. $$\int_0^\infty J_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}}\,dx=\frac{2^\mu\,\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}\,J_{\nu-\mu-1}(\alpha z)$$
$$\left[\alpha>0,\quad \operatorname{Re}\left(\frac{1}{2}\nu-\frac{1}{4}\right)>\operatorname{Re}\mu>-1\right]$$
WA 457(5)

2. $$\int_0^\infty \frac{J_\nu\left(\alpha\sqrt{t^2+1}\right)}{\sqrt{t^2+1}}\,dt=-\frac{\pi}{2}\,J_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right)Y_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right)\qquad[\operatorname{Re}\nu>-1,\quad \alpha>0]$$
MO 46

3. $$\int_0^\infty K_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}}\,dx=\frac{2^\mu\,\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}\,K_{\nu-\mu-1}(\alpha z)$$
$$[\alpha>0,\quad \operatorname{Re}\mu>-1]$$
WA 457(6)

4.[8] $$\int_0^\infty J_\nu(\beta x)\frac{J_{\mu-1}\left\{\alpha\sqrt{x^2+z^2}\right\}}{(x^2+z^2)^{\frac{1}{2}\mu+\frac{1}{2}}}x^{\nu+1}\,dx=\frac{\alpha^{\mu-1}z^\nu}{2^{\mu-1}\,\Gamma(\mu)}\,K_\nu(\beta z)$$
$$[\alpha<\beta,\quad \operatorname{Re}(\mu+2)>\operatorname{Re}\nu>-1]$$
ET II 59(19)

5.[8] $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left\{\alpha\sqrt{x^2+z^2}\right\}}{\sqrt{(x^2+z^2)^\mu}}x^{\nu-1}\,dx=\frac{2^{\nu-1}\,\Gamma(\nu)}{\beta^\nu}\,\frac{J_\mu(\alpha z)}{z^\mu}$$
$$[\operatorname{Re}(\mu+2)>\operatorname{Re}\nu>0,\quad \beta>\alpha>0]$$
WA 459(12)

6.[6] $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{(x^2+z^2)^\mu}}x^{\nu+1}\,dx$$
$$=0\qquad[0<\alpha<\beta]$$
$$=\frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2-\beta^2}}{z}\right)^{\mu-\nu-1}J_{\mu-\nu-1}\left\{z\sqrt{\alpha^2-\beta^2}\right\}\qquad[\alpha>\beta>0]$$
$$[\operatorname{Re}\mu>\operatorname{Re}\nu>-1]$$
WA 415(1)

7.[8] $$\int_0^\infty J_\nu(\beta x)\frac{K_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{(x^2+z^2)^\mu}}x^{\nu+1}\,dx=\frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2+\beta^2}}{z}\right)^{\mu-\nu-1}K_{\mu-\nu-1}\left(z\sqrt{\alpha^2+\beta^2}\right)$$
$$\left[\alpha>0,\quad \beta>0,\quad \operatorname{Re}\nu>-1,\quad |\arg z|<\frac{\pi}{2}\right]$$
KU 151(31), WA 416(2)

8.[8] $$\int_0^\infty J_\nu(ux)\,K_\mu\left(v\sqrt{x^2-y^2}\right)\left(x^2-y^2\right)^{-\frac{\mu}{2}}x^{\nu+1}\,dx=\frac{\pi}{2}\exp\left[-i\pi\left(\mu-\nu-\frac{1}{2}\right)\right]\cdot\frac{u^\nu}{v^\mu}$$
$$\cdot\left[\frac{\sqrt{u^2+v^2}}{y}\right]^{\mu-\nu-1}H^{(2)}_{\mu-\nu-1}\left(y\sqrt{u^2+v^2}\right)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}\nu>-1, u>0,\quad v>0, y>0;\quad \left(x^2-y^2\right)^{\frac{1}{2}\alpha}=e^{\frac{1}{2}\alpha\pi i}\left(y^2-n^2\right)^{\frac{1}{2}\alpha}\text{ if } x<y\right]$$

9.[8] $$\int_0^\infty J_\nu(ux)\, H_\mu^{(2)}\left(v\sqrt{x^2+y^2}\right)\left(x^2+y^2\right)^{-\frac{\mu}{2}} x^{\nu+1}\, dx = \frac{u^\nu}{v^\mu}\left[\frac{\sqrt{v^2-u^2}}{y}\right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)}\left(y\sqrt{v^2-u^2}\right)$$

$[u<v]$

$$\left[\operatorname{Re}\mu > \operatorname{Re}\nu > -1, \quad u>0, \quad v>0, \quad y>0;, \quad \arg\sqrt{v^2-u^2}=0, \text{ for } v>u\right.$$

$$\left.\arg\left(v^2-u^2\right)^\sigma = -\pi\sigma \text{ for } v<u, \text{ where } \sigma=\frac{1}{2} \text{ or } \sigma=\frac{\mu-\nu-1}{2}\right]$$

MO 43

10.[8] $$\int_0^\infty J_\nu(\beta x)\, J_\mu\left(\alpha\sqrt{x^2+z^2}\right) J_\mu\left(\gamma\sqrt{x^2+z^2}\right)\frac{x^{\nu-1}}{(x^2+z^2)^\mu}\, dx = \frac{2^{\nu-1}\,\Gamma(\nu)}{\beta^\nu}\,\frac{J_\mu(\alpha z)}{z^\mu}\,\frac{J_\mu(\gamma z)}{z^\mu}$$

$\left[\alpha>0; \quad \beta>\alpha+\gamma; \quad \gamma>0, \quad \operatorname{Re}\left(2\mu+\frac{5}{2}\right)>\operatorname{Re}\nu>0\right]$ WA 459(14)

11.[8] $$\int_0^\infty J_\nu(\beta t) t^{\nu-1}\prod_{k=1}^n J_\mu\left(\alpha_k\sqrt{t^2+x^2}\right)\sqrt{(t^2+x^2)^{-n\mu}}\, dt = 2^{\nu-1}\beta^{-\nu}\,\Gamma(\nu)\prod_{k=1}^n\left[x^{-\mu}\, J_\mu\left(\alpha_k x\right)\right]$$

$$\left[x>0, \quad \alpha_1>0, \quad \alpha_2>0,\ldots,\alpha_n>0, \quad \beta>\prod_{k=1}^n\alpha_k; \quad \operatorname{Re}\left(n\mu+\frac{1}{2}n+\frac{1}{2}\right)>\operatorname{Re}\nu>0\right]$$

MO 43

12.[8] $$\int_0^\infty \frac{J_\mu^2\left(\sqrt{a^2+x^2}\right)}{(a^2+x^2)^\nu} x^{2\nu-2}\, dx = \frac{\Gamma\left(\nu-\frac{1}{2}\right)}{2a^{\nu+1}\sqrt{\pi}}\,\mathbf{H}_\nu(2a) \qquad \left[\operatorname{Re}\nu>\tfrac{1}{2}\right]$$ WA 457(8)

6.597 $$\int_0^\infty t^{\nu+1}\, J_\mu\left[b\left(t^2+y^2\right)^{\frac{1}{2}}\right]\left(t^2+y^2\right)^{-\frac{1}{2}\mu}\left(t^2+\beta^2\right)^{-1} J_\nu(at)\, dt = \beta^\nu\, J_\mu\left[b\left(y^2-\beta^2\right)^{\frac{1}{2}}\right]\left(y^2-\beta^2\right)^{-\frac{1}{2}\mu} K_\nu(a\beta)$$

$[a\ge b, \quad \operatorname{Re}\beta>0, \quad -1<\operatorname{Re}\nu<2+\operatorname{Re}\mu]$ EH II 95(56)

6.598 $$\int_0^1 x^{\frac{\mu}{2}}(1-x)^{\frac{\nu}{2}}\, J_\mu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{1-x}\right)\, dx = 2a^\mu b^\nu\left(a^2+b^2\right)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}\left(\sqrt{a^2+b^2}\right)$$

$[\operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>-1]$ EH II 46a

6.61 Combinations of Bessel functions and exponentials

6.611

1. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left[\sqrt{\alpha^2+\beta^2}-\alpha\right]^\nu}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\nu>-1, \quad \operatorname{Re}(\alpha\pm i\beta)>0]$$

EH II 49(18), WA 422(8)

2. $$\int_0^\infty e^{-\alpha x}\, Y_\nu(\beta x)\, dx = \left(\alpha^2+\beta^2\right)^{-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \times \left\{\beta^\nu \left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{-\nu} \cos(\nu\pi) - \beta^{-\nu}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{\nu}\right\}$$
$[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad |\operatorname{Re}\nu| < 1]$ MO 179, ET II 105(1)

3. $$\int_0^\infty e^{-\alpha x}\, K_\nu(\beta x)\, dx = \frac{\pi}{\beta\sin(\nu\pi)} \frac{\sin(\nu\theta)}{\sin\theta}$$
$\left[\cos\theta = \frac{\alpha}{\beta}; \quad \theta \to \frac{\pi}{2} \quad \text{for } \beta \to \infty\right]$
ET II 131(22)
$$= \frac{\pi \operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2-\beta^2}} \left[\beta^{-\nu}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^{\nu} - \beta^{\nu}\left(\sqrt{\alpha^2-\beta^2}+\alpha\right)^{-\nu}\right]$$
$[|\operatorname{Re}\nu| < 1, \quad \operatorname{Re}(\alpha+\beta) > 0]$
ET I 197(24), MO 180

4.[8] $$\int_0^\infty e^{-\alpha x}\, I_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left[\alpha-\sqrt{\alpha^2-\beta^2}\right]^{\nu}}{\sqrt{\alpha^2-\beta^2}}$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\alpha > |\operatorname{Re}\beta|]$
MO 180, ET I 195(1)

5. $$\int_0^\infty e^{-\alpha x}\, H_\nu^{(1,2)}(\beta x)\, dx = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^{\nu}}{\beta^\nu\sqrt{\alpha^2+\beta^2}} \left\{1 \pm \frac{i}{\sin(\nu\pi)}\left[\cos(\nu\pi) - \frac{\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)^{2\nu}}{b^{2\nu}}\right]\right\}$$
[$-1 < \operatorname{Re}\nu < 1$; a plus sign corresponds to the function $H_\nu^{(1)}$, a minus sign to the function $H_\nu^{(2)}$.]
MO 180, ET I188(54, 55)

6. $$\int_0^\infty e^{-\alpha x}\, H_0^{(1)}(\beta x)\, dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \left\{1 - \frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\}$$
$[\operatorname{Re}\alpha > |\operatorname{Im}\beta|]$ MO 180, ET I 188(53)

7. $$\int_0^\infty e^{-\alpha x}\, H_0^{(2)}(\beta x)\, dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \left\{1 + \frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\}$$
$[\operatorname{Re}\alpha > |\operatorname{Im}\beta|]$ MO 180, ET I 188(53)

8. $$\int_0^\infty e^{-\alpha x}\, Y_0(\beta x)\, dx = \frac{-2}{\pi\sqrt{\alpha^2+\beta^2}} \ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}$$
$[\operatorname{Re}\alpha > |\operatorname{Im}\beta|]$ MO 47, ET I 187(44)

9.[11] $$\int_0^\infty e^{-\alpha x}\, K_0(\beta x)\, dx = \frac{\arccos\frac{\alpha}{\beta}}{\sqrt{\beta^2-\alpha^2}}$$
$[\operatorname{Re}(\alpha+\beta) > 0]$ WA 424, ET II 131(22)
$$= \frac{1}{\sqrt{\alpha^2-\beta^2}} \ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}-1}\right)$$
$[\operatorname{Re}(\alpha+\beta) > 0]$
MO 48

10.[10] $$\int_a^b \alpha\, d\alpha \int_0^\infty dk\, J_1(k\alpha) e^{-k|\beta|} = \int_a^b \left(1 - \frac{|\beta|}{\sqrt{\alpha^2+\beta^2}}\right) d\alpha$$

(see **3.241** 6)

6.612

1. $$\int_0^\infty e^{-2\alpha x}\, J_0(x)\, Y_0(x)\, dx = \frac{\boldsymbol{K}\left[\alpha\left(\alpha^2+1\right)^{-\frac{1}{2}}\right]}{\pi\left(\alpha^2+1\right)^{\frac{1}{2}}} \qquad [\operatorname{Re}\alpha > 0]$$ ET II 347(58)

2. $$\int_0^\infty e^{-2\alpha x}\, I_0(x)\, K_0(x)\, dx = \frac{1}{2}\boldsymbol{K}\left[\left(1-\alpha^2\right)^{\frac{1}{2}}\right] \qquad [0 < \alpha < 1]$$
$$= \frac{1}{2\alpha}\boldsymbol{K}\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right] \qquad [1 < \alpha < \infty]$$
ET II 370(48)

3. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{\pi\sqrt{\gamma\beta}}\, Q_{\nu-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}\right)$$
$$\left[\operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ WA 426(2), ET II 50(17)

4. $$\int_0^\infty e^{-\alpha x}\left[J_0(\beta x)\right]^2 dx = \frac{2}{\pi\sqrt{\alpha^2+4\beta^2}}\boldsymbol{K}\left(\frac{2\beta}{\sqrt{\alpha^2+4\beta^2}}\right)$$ MO 178

5. $$\int_0^\infty e^{-2\alpha x} J_1^2(\beta x)\, dx = \frac{\left(2\alpha^2+\beta^2\right)\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) - 2\left(\alpha^2+\beta^2\right)\boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{\pi\beta^2\sqrt{\alpha^2+\beta^2}}$$ WA 428(3)

6. $$\int_0^\infty e^{-3x}\, I_l(x)\, I_m(x)\, I_n(x)\, dx = r_1 g + \frac{r_2}{\pi^2 g} + r_3$$
where
$$g = \frac{\sqrt{3}-1}{96\pi^3}\,\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right)$$
and

(lmn)	r_1	r_2	r_3
000	1	0	0
100	1	0	$-1/3$
110	$5/12$	$-1/2$	0
111	$-1/8$	$3/4$	0
200	$10/3$	2	-2
210	$3/8$	$-9/4$	$1/3$
211	$-2/3$	2	0
220	$73/36$	$-29/6$	0
221	$-15/16$	$21/8$	0
222	$5/8$	$-27/20$	0
300	$35/2$	21	-13
310	$-79/36$	$-85/6$	4
311	$-11/4$	$21/2$	$-2/3$
320	$319/48$	$-119/8$	$-1/3$
321	$-125/36$	$269/30$	0
322	$35/16$	$-213/40$	0
330	$50/3$	$-1046/25$	0
331	$-35/3$	$148/5$	0
332	$35/9$	$-1012/105$	0
333	$-35/16$	$1587/280$	0
400	$994/9$	$542/3$	-92
410	$-515/16$	$-879/8$	$115/3$
411	$-9/2$	$357/5$	-12
420	$12907/120$	$-13903/10$	-6
421	$-229/16$	$1251/40$	1
422	$35/3$	$-1024/35$	0
430	$2641/48$	$-28049/200$	$1/3$
431	$-1505/36$	$118051/1050$	0

(lmn)	r_1	r_2	r_3
432	$525/32$	$-4617/112$	0
433	$-595/72$	$8809/420$	0
440	$6025/36$	$-620161/1470$	0
441	$-29175/224$	$131379/400$	0
442	$2975/48$	$-31231/200$	0
443	$-539/32$	$119271/2800$	0
444	$77/8$	$-186003/7700$	0
500	$9287/12$	$3005/2$	$-2077/3$
510	$-189029/180$	$-138331/50$	348
511	$275/4$	$5751/10$	-150
520	$2897/16$	$-15123/20$	$-229/3$
521	$-937/12$	$27059/30$	24
522	$509/8$	$-4209/28$	0
530	$3589/18$	$-1993883/3075$	0
531	$-1329/8$	$297981/700$	$-4/3$
532	$2555/36$	$-187777/1050$	0
533	$-2233/48$	$164399/1400$	0
540	$18471/32$	$-28493109/19600$	$-1/3$
541	$-1390/3$	$286274/245$	0
542	$7777/32$	$-1715589/2800$	0
543	$-5621/72$	$4550057/23100$	0
544	$1155/32$	$-560001/6160$	0
550	$197045/108$	$-101441689/22050$	0
551	$-12023/8$	$18569853/4900$	0
552	$1683/2$	$-5718309/2695$	0
553	$-5159/16$	$2504541/3080$	0
554	$24563/312$	$-1527851/77000$	0
555	$-9251/208$	$12099711/107800$	0

6.613[11] $$\int_0^\infty e^{-xz}\, J_{\nu+\frac{1}{2}}\left(\frac{x^2}{2}\right)\, dx = \frac{\Gamma(\nu+1)}{\sqrt{\pi}}\, D_{-\nu-1}\left(ze^{\frac{\pi}{4}i}\right) D_{-\nu-1}\left(ze^{-\frac{\pi i}{4}}\right) \qquad [\operatorname{Re}\nu > -1]$$ MO 122

6.614

1. $$\int_0^\infty e^{-\alpha x}\, J_\nu\left(\beta\sqrt{x}\right)\, dx = \frac{\beta}{4}\sqrt{\frac{\pi}{\alpha^3}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$$= \frac{1}{\alpha}e^{-\beta^2/4\alpha} \qquad [\nu = 0]$$
MO 178

2. $$\int_0^\infty e^{-\alpha x}\, Y_{2\nu}\left(2\sqrt{\beta x}\right)\, dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}\left\{\cot(\nu\pi)\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\, M_{\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right) - \operatorname{cosec}(\nu\pi)\, W_{\frac{1}{2},nu}\left(\frac{\beta}{\alpha}\right)\right\}$$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$ ET I 188(50)a

3. $$\int_0^\infty e^{-\alpha x}\, I_{2\nu}\left(2\sqrt{\beta x}\right)\, dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\, M_{-\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$ ET I 197(20)a

4. $$\int_0^\infty e^{-\alpha x}\, K_{2\nu}\left(2\sqrt{\beta x}\right)\, dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}}\,\Gamma(\nu+1)\,\Gamma(1-\nu)\; W_{-\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$ ET I 199(37)a

5. $$\int_0^\infty e^{-\alpha x}\, K_1\left(\beta\sqrt{x}\right)\, dx = \frac{\beta}{8}\sqrt{\frac{\pi}{\alpha^3}}\exp\left(\frac{\beta^2}{8\alpha}\right)\left[K_1\left(\frac{\beta^2}{8\alpha}\right) - K_0\left(\frac{\beta^2}{8\alpha}\right)\right]$$
MO 181

6.615 $$\int_0^\infty e^{-\alpha x}\, J_\nu\left(2\beta\sqrt{x}\right) J_\nu\left(2\gamma\sqrt{x}\right)\, dx = \frac{1}{\alpha}\, I_\nu\left(\frac{2\beta\gamma}{\alpha}\right)\exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right) \qquad [\operatorname{Re}\nu > -1]$$
MO 178

6.616

1. $$\int_0^\infty e^{-\alpha x}\, J_0\left(\beta\sqrt{x^2+2\gamma x}\right)\, dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\exp\left[\gamma\left(\alpha - \sqrt{\alpha^2+\beta^2}\right)\right]$$
MO 179

2. $$\int_1^\infty e^{-\alpha x}\, J_0\left(\beta\sqrt{x^2-1}\right)\, dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\exp\left(-\sqrt{\alpha^2+\beta^2}\right)$$
MO 179

3. $$\int_{-\infty}^\infty e^{itx}\, H_0^{(1)}\left(r\sqrt{\alpha^2-t^2}\right)\, dt = -2i\frac{e^{i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$
$\left[0 \le \arg\sqrt{\alpha^2-t^2} < \pi, \quad 0 \le \arg\alpha < \pi; \quad r \text{ and } x \text{ are real}\right]$ MO 49

4. $$\int_{-\infty}^\infty e^{-itx}\, H_0^{(2)}\left(r\sqrt{\alpha^2-t^2}\right)\, dt = 2i\frac{e^{-i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$
$\left[-\pi < \arg\sqrt{\alpha^2-t^2} \le 0, \quad -\pi < \arg\alpha \le 0, \quad r \text{ and } x \text{ are real}\right]$ MO 49

5.[3] $$\int_{-1}^1 e^{-ax}\, I_0\left(b\sqrt{1-x^2}\right)\, dx = 2\left(a^2+b^2\right)^{-1/2}\sinh\sqrt{a^2+b^2}$$
$[a > 0, \quad b > 0]$

6.[8] $$\int_0^\infty e^{-xy}\, J_0\left[y\sqrt{1-x^2}\right]/(\alpha+y)\, dy = \sum_{n=0}^\infty n!\frac{P_n(x)}{\alpha^{n+1}}$$

6.617

1. $$\int_0^\infty K_{q-p}\left(2z\sinh x\right) e^{(p+q)x}\, dx = \frac{\pi^2}{4\sin[(p-q)\pi]}\left[J_p(z)\; Y_q(z) - J_q(z)\; Y_p(z)\right]$$
$[\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(p-q) < 1]$
MO 44

2. $$\int_0^\infty K_0\left(2z\sinh x\right) e^{-2px}\, dx = -\frac{\pi}{4}\left\{J_p(z)\frac{\partial\, Y_p(z)}{\partial p} - Y_p(z)\frac{\partial\, J_p(z)}{\partial p}\right\}$$
$[\operatorname{Re} z > 0]$ MO 44

6.618

1. $$\int_0^\infty e^{-\alpha x^2}\, J_\nu(\beta x)\, dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\exp\left(-\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > -1]$$
WA 432(5), ET II 29(8)

2. $$\int_0^\infty e^{-\alpha x^2}\, Y_\nu(\beta x)\, dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[\tan\frac{\nu\pi}{2}\, I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad |\operatorname{Re}\nu| < 1]$
WA 432(6), ET II 106(3)

3. $$\int_0^\infty e^{-\alpha x^2}\, K_\nu(\beta x)\, dx = \frac{1}{4}\sec\left(\frac{\nu\pi}{2}\right) \frac{\sqrt{\pi}}{\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$
EH II 51(28), ET II 132(24)

4. $$\int_0^\infty e^{-\alpha x^2}\, I_\nu(\beta x)\, dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\nu > -1, \quad \operatorname{Re}\alpha > 0]$$
EH II 92(27)

5. $$\int_0^\infty e^{-\alpha x^2}\, J_\mu(\beta x)\, J_\nu(\beta x)\, dx$$
$$= 2^{-\nu-\mu-1}\alpha^{-\frac{\nu+\mu+1}{2}}\beta^{\nu+\mu}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\, {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}\alpha > 0]$ EH II 50(21)a

6.62–6.63 Combinations of Bessel functions, exponentials, and powers

6.621 Notation:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} + \sqrt{(a-\rho)^2+z^2}\right]$$

1. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x) x^{\mu-1}\, dx$$
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu\,\Gamma(\nu+1)}\, F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$
WA 421(2)
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu\,\Gamma(\nu+1)}\left(1+\frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2}+1; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$
WA 421(3)
$$= \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}}\,\Gamma(\nu+1)}\, F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$[\operatorname{Re}(\nu+\mu) > 0, \quad \operatorname{Re}(\alpha+i\beta) > 0, \quad \operatorname{Re}(\alpha-i\beta) > 0]$
WA 421(3)
$$= \left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\mu}\Gamma(\nu+\mu)\, P_{\mu-1}^{-\nu}\left[\alpha\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\right]$$
$[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}(\nu+\mu) > 0]$
ET II 29(6)

2. $$\int_0^\infty e^{-\alpha x}\, Y_\nu(\beta x) x^{\mu-1}\, dx$$
$$= \cot \nu\pi \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}}\,\Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$$- \operatorname{cosec} \nu\pi \frac{\left(\frac{\beta}{2}\right)^{-\nu} \Gamma(\mu-\nu)}{\sqrt{(\alpha^2+\beta^2)^{\mu-\nu}}\,\Gamma(1-\nu)} F\left(\frac{\mu-\nu}{2}, \frac{1-\nu-\mu}{2}; 1-\nu; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$$[\operatorname{Re}\mu \geq |\operatorname{Re}\nu|, \quad \operatorname{Re}(\alpha \pm i\beta) > 0]$$
WA 421(4)
$$= -\frac{2}{\pi}\Gamma(\nu+\mu)\left(\beta^2+\alpha^2\right)^{-\frac{1}{2}\mu} Q_{\mu-1}^{-\nu}\left[\alpha\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\right]$$
$$[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|]$$
ET II 105(2)

3. $$\int_0^\infty x^{\mu-1} e^{-\alpha x}\, K_\nu(\beta x)\, dx = \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+\nu}} \frac{\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)} F\left(\mu+\nu, \nu+\frac{1}{2}; \mu+\frac{1}{2}; \frac{\alpha-\beta}{\alpha+\beta}\right)$$
$$[\operatorname{Re}\mu > |\operatorname{Re}\nu|, \quad \operatorname{Re}(\alpha+\beta) > 0]$$
ET II 131(23)a, EH II 50(26)

4. $$\int_0^\infty x^{m+1} e^{-\alpha x}\, J_\nu(\beta x)\, dx = (-1)^{m+1}\beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}}\left[\frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\sqrt{\alpha^2+\beta^2}}\right]$$
$$[\beta > 0, \quad \operatorname{Re}\nu > -m-2]$$ ET II 28(3)

5.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{-3/2}\, dx$$
$$= \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\left\{\frac{\ell_1}{2}\sqrt{a^2-\ell_1^2} + \frac{a^2}{2}\arcsin\left(\frac{\ell_1}{2}\right) + z\left[\sqrt{\rho^2-\ell_1^2}-\rho\right]\right\}$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

6.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{-1/2}\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\left[\rho - \sqrt{\rho^2-\ell_1^2}\right]$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

7.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{1/2}\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\frac{\ell_1\sqrt{a^2-\ell_1^2}}{\ell_2^2-\ell_1^2}$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

8.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{3/2}(\rho x)\, x^{1/2}\, dx = \sqrt{\frac{2}{\pi}}\frac{\ell_1^2\sqrt{\rho^2-\ell_1^2}}{\rho^{3/2}a\left(\ell_2^2-\ell_1^2\right)}$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

9.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{3/2}(\rho x)\, x^{-3/2}\, dx = \frac{1}{\sqrt{2\pi}}\frac{1}{\rho^{3/2}a}\left[a^2\arcsin\left(\frac{\ell_1}{a}\right) - \ell_1\sqrt{a^2-\ell_1^2}\right]$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

10.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{5/2}(\rho x)\, x^{-1/2}\, dx = \frac{1}{\sqrt{2\pi}}\frac{z}{\rho^{5/2}a}\left[\ell_1\sqrt{a^2-\ell_1^2}+\frac{2a^2\ell_1}{\sqrt{a^2-\ell_1^2}}-3a^2\arcsin\left(\frac{\ell_1}{a}\right)\right]$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

11.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{5/2}(\rho x)\, x^{-3/2}\, dx$$
$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho^{5/2}a}\left[\frac{\ell_1}{\sqrt{a^2-\ell_1^2}}\left(\frac{7a^2}{8}-a^2z^2-\frac{\ell_1^4}{4}-\frac{5a^2\ell_1^2}{8}\right)\right.$$
$$\left.-\frac{1}{2}\left(\ell_1^2+\ell_2^2\right)\ell_1\sqrt{a^2-\ell_1^2}+\arcsin\left(\frac{\ell_1}{a}\right)\left(\frac{3}{2}a^2z^2+\frac{1}{2}a^2\rho^2-\frac{3a^4}{8}\right)\right]$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

12.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{5/2}(\rho x)\, x^{-5/2}\, dx$$
$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\rho^{5/2}a}\left\{\frac{2\left[\rho^{5/2}-\left(\rho^2-\ell_1^2\right)^{5/2}\right]}{15}+za^2\arcsin\left(\frac{\ell_1}{a}\right)\left[\frac{3a^2}{8}-\frac{\rho^2}{2}-\frac{z^2}{2}\right]\right.$$
$$\left.+z\ell_1\sqrt{a^2-\ell_1^2}\left[\frac{\rho^2}{2}-\frac{3a^2}{8}+\frac{z^2}{6}-\frac{\ell_1^2}{4}\right]+\frac{z^3a^2\ell_1}{3\sqrt{a^2-\ell_1^2}}\right\}$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

13.[10] $$\int_0^\infty e^{-zx}\, J_2(ax)\, J_{3/2}(\rho x)\, x^{1/2}\, dx=\sqrt{\frac{2}{\pi}}a^2\rho^{3/2}\frac{\sqrt{\ell_2^2-\rho^2}}{\left(\ell_2^2-\ell_1^2\right)\ell_2^4}$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

14.[10] $$\int_0^\infty e^{-zx}\, J_2(ax)\, J_{3/2}(\rho x)\, x^{-1/2}\, dx=\sqrt{\frac{2}{\pi}}\frac{\rho^{3/2}}{a^2}\left[\frac{2}{3}-\frac{\sqrt{\rho^2-\ell_1^2}}{\rho}+\frac{\left(\rho^2-\ell_1^2\right)^{3/2}}{3\rho^3}\right]$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

15.[10] $$\int_0^\infty e^{-zx}\, J_3(ax)\, J_{1/2}(\rho x)\, x^{-1/2}\, dx$$
$$=\sqrt{\frac{2}{\pi\rho}}\frac{1}{3a^3}\left\{\rho\left[3a^2-4\rho^2+12z^2\right]-\sqrt{\rho^2-\ell_1^2}\left\{12\ell_2^2-16\rho^2+4\ell_1^2-3a^2\right\}\right\}$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

16.[10] $$\int_0^\infty e^{-zx}\, J_3(ax)\, J_{3/2}(\rho x)\, x^{1/2}\, dx$$
$$=\sqrt{\frac{2}{\pi}}\rho^{3/2}\left\{\frac{4}{a^3}\left[\frac{2}{3}-\frac{\sqrt{\rho^2-\ell_1^2}}{\rho}+\frac{\left(\rho^2-\ell_1^2\right)^{3/2}}{3\rho^2}\right]-\frac{a\sqrt{\ell_2^2-a^2}}{\left(\ell_2^2-\ell_1^2\right)\ell_2^3}\right\}$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

17.[10] $$\int_0^\infty e^{-zx}\, J_3(ax)\, J_{3/2}(\rho x)\, x^{-1/2}\, dx=\sqrt{\frac{2}{\pi}}\frac{\rho^{3/2}}{3a^3}\left[\sqrt{\ell_2^2-\rho^2}\left(\frac{4\rho^2\left(2\rho^2-\ell_1^2\right)-\ell_1^4}{\rho^4}\right)-8z\right]$$
$$[\arg a>0,\quad \arg\rho>0,\quad \arg z>0]$$

18.[10] $$\int_0^\infty e^{-zx}\, J_3(ax)\, J_{3/2}(\rho x)\, x^{-3/2}\, dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{\rho^{3/2}}{3a^3}\left\{a^2 - \frac{4}{5}\rho^2 + 4z^2 - \sqrt{\rho^2-\ell_1^2}\left[\frac{4\ell_2^2}{\rho} - \frac{24\rho}{5} + \frac{8\ell_1^2}{5\rho} - \frac{a^2}{\rho} + \frac{\ell_1^4}{5\rho^3}\right]\right\}$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

19.[10] $$\int_0^\infty e^{-zx}\, J_3(ax)\, J_{3/2}(\rho x)\, x^{-5/2}\, dx$$
$$= -\sqrt{\frac{2}{\pi}}\frac{\rho^{3/2}}{3a^3}\left\{\left(a^2 - \frac{4}{5}\rho^2\right)z + \frac{4z^3}{3}\right.$$
$$+\sqrt{\ell_2^2-\rho^2}\left[a^2 + \frac{32}{15}\rho^2 - \frac{12}{5}\ell_1^2 - \frac{4}{3}\ell_2^2 + \frac{2\ell_1^4}{5\rho^2} + \frac{a^4\ell_1^2}{16\rho^4} + \frac{a^2\ell_1^2}{24\rho^4} + \frac{\ell_1^6}{30\rho^4}\right]$$
$$\left. - \frac{a^6}{16\rho^3}\arcsin\left(\frac{\rho}{\ell_2}\right)\right\}$$
$$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$$

6.622

1. $$\int_0^\infty \left(J_0(x) - e^{-\alpha x}\right)\frac{dx}{x} = \ln 2\alpha \qquad [\alpha > 0]$$ NT 66(13)

2. $$\int_0^\infty \frac{e^{i(u+x)}}{u+x}\, J_0(x)\, dx = \frac{\pi}{2} i\, H_0^{(1)}(u)$$ MO 44

3.[8] $$\int_0^\infty e^{-x\cosh\alpha}\, I_\nu(x) x^{\mu-1}\, dx = \sqrt{\frac{2}{\pi}} e^{-\left(\mu-\frac{1}{2}\right)\pi i}\frac{Q_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(\cosh\alpha)}{\sinh^{\mu-\frac{1}{2}}\alpha}$$
$$[\mathrm{Re}(\mu+\nu) > 0, \quad \mathrm{Re}\,(\cosh\alpha) > 1]$$
WA 388(6)a

6.623

1. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x) x^\nu\, dx = \frac{(2\beta)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}} \qquad \left[\mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|\right]$$
WA 422(5)

2. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x) x^{\nu+1}\, dx = \frac{2\alpha(2\beta)^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}\,(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}} \qquad [\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|]$$
WA 422(6)

3. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x)\frac{dx}{x} = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\nu\beta^\nu}$$
$$[\mathrm{Re}\,\nu > 0; \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|] \qquad \text{(cf. } \mathbf{6.611}\ 1)$$ WA 422(7)

6.624

1. $$\int_0^\infty x e^{-\alpha x}\, K_0(\beta x)\, dx = \frac{1}{\alpha^2-\beta^2}\left\{\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\ln\left[\frac{\alpha}{\beta} + \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - 1}\right] - 1\right\}$$ MO 181

2. $$\int_0^\infty \sqrt{x}e^{-\alpha x}\, K_{\pm\frac{1}{2}}(\beta x)\, dx = \sqrt{\frac{\pi}{2\beta}}\frac{1}{\alpha+\beta}$$ MO 181

3. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-1/2}}\, K_\mu(t)t^\nu\, dt = \frac{\Gamma(\nu-\mu+1)}{(z^2-1)^{-\frac{1}{2}(\nu+1)}}e^{i\mu\pi}\, Q_\nu^\mu(z)$$
$[\mathrm{Re}\,(\nu\pm\mu)>-1]$ EH II 57(7)

4. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-1/2}}\, I_{-\mu}(t)t^\nu\, dt = \frac{\Gamma(-\nu-\mu)}{(z^2-1)^{\frac{1}{2}\nu}}\, P_\nu^\mu(z) \qquad [\mathrm{Re}(\nu+\mu)<0]$$ EH II 57(8)

5. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-\frac{1}{2}}}\, I_\mu(t)t^\nu\, dt = \frac{\Gamma(\nu+\mu+1)}{(z^2-1)^{-\frac{1}{2}(\nu+1)}}\, P_\nu^{-\mu}(z)$$
$[\mathrm{Re}(\nu+\mu)>-1]$ EH II 57(9)

6. $$\int_0^\infty e^{-t\cos\theta}\, J_\mu\left(t\sin\theta\right)t^\nu\, dt = \Gamma(\nu+\mu+1)\, P_\nu^{-\mu}\left(\cos\theta\right)$$
$\left[\mathrm{Re}(\nu+\mu)>-1, \quad 0\le\theta<\frac{1}{2}\pi\right]$
EH II 57(10)

7. $$\int_0^\infty \frac{J_\nu(bx)x^\nu}{e^{\pi x}-1}\, dx = \frac{(2b)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}\sum_{n=1}^\infty \frac{1}{\left(n^2\pi^2+b^2\right)^{\nu+\frac{1}{2}}}$$
$[\mathrm{Re}\,\nu>0, \quad |\mathrm{Im}\,b|<\pi]$ WA 423(9)

6.625

1. $$\int_0^1 x^{\lambda-\nu-1}(1-x)^{\mu-1}e^{\pm i\alpha x}\, J_\nu(\alpha x)\, dx = \frac{2^{-\nu}\alpha^\nu\,\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)\,\Gamma(\nu+1)}\; {}_2F_2\left(\lambda,\nu+\frac{1}{2};\lambda+\mu,2\nu+1;\pm 2i\alpha\right)$$
$[\mathrm{Re}\,\lambda>0, \quad \mathrm{Re}\,\mu>0]$ ET II 194(58)a

2. $$\int_0^1 x^\nu(1-x)^{\mu-1}e^{\pm i\alpha x}\, J_\nu(\alpha x)\, dx = \frac{(2\alpha)^\nu\,\Gamma(\mu)\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\mu+2\nu+1)}\; {}_1F_1\left(\nu+\frac{1}{2};\mu+2\nu+1;\pm 2i\alpha\right)$$
$\left[\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}\right]$ ET II 194(57)a

3. $$\int_0^1 x^\nu(1-x)^{\mu-1}e^{\pm\alpha x}\, J_\nu(\alpha x)\, dx = \frac{(2\alpha)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(\mu+2\nu+1)}\; {}_1F_1\left(\nu+\frac{1}{2};\mu+2\nu+1;\pm 2\alpha\right)$$
$\left[\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>-\frac{1}{2}\right]$
BU 9(16a), ET II 197(77)a

4. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}e^{\pm\alpha x}\, I_\nu(\alpha x)\, dx = \frac{\left(\frac{1}{2}\alpha\right)^\nu\,\Gamma\left(\lambda+\nu\right)\Gamma(\mu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu+\nu)} \times {}_2F_2\left(\nu+\frac{1}{2},\lambda+\nu;2\nu+1,\mu+\lambda+\nu;\pm 2\alpha\right)$$
$[\mathrm{Re}\,\mu>0, \quad \mathrm{Re}(\lambda+\nu)>0]$ ET II 197(78)a

5. $$\int_0^1 x^{\mu-\kappa}(1-x)^{2\kappa-1}\, I_{\mu-\kappa}\left(\frac{1}{2}xz\right)e^{-\frac{1}{2}xz}\, dx = \frac{\Gamma(2\kappa)}{\sqrt{\pi}\,\Gamma(1+2\mu)}e^{\frac{x}{2}}\, z^{-\kappa-\frac{1}{2}}\, M_{\kappa,u}(z)$$
$\left[\mathrm{Re}\left(\kappa-\frac{1}{2}-\mu\right)<0, \quad \mathrm{Re}\,\kappa>0\right]$
BU 129(14a)

6. $$\int_1^\infty x^{-\lambda}(x-1)^{\mu-1}e^{-\alpha x}\,I_\nu(\alpha x)\,dx=\frac{(2\alpha)^\lambda\,\Gamma(\mu)}{\sqrt{\pi}}\,G^{21}_{23}\left(2\alpha\,\middle|\,\begin{matrix}\frac12-\lambda,0\\-\mu,\nu-\lambda,-\nu-\lambda\end{matrix}\right)$$
$$\left[0<\operatorname{Re}\mu<\tfrac12+\operatorname{Re}\lambda,\quad \operatorname{Re}\alpha>0\right]$$
ET II 207(50)a

7. $$\int_1^\infty x^{-\lambda}(x-1)^{\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx=\Gamma(\mu)\sqrt{\pi}(2\alpha)^\lambda\,G^{30}_{23}\left(2\alpha\,\middle|\,\begin{matrix}0,\frac12-\lambda\\-\mu,\nu-\lambda,-\nu-\lambda\end{matrix}\right)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 208(55)a

8. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}\,I_\nu(\alpha x)\,dx=\frac{(2\alpha)^{\nu-\mu}\,\Gamma\left(\frac12-\mu+\nu\right)\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(1-\mu+2\nu)}$$
$$\times\,{}_1F_1\left(\frac12-\mu+\nu;1-\mu+2\nu;-2\alpha\right)$$
$$\left[0<\operatorname{Re}\mu<\tfrac12+\operatorname{Re}\nu,\quad \operatorname{Re}\alpha>0\right]$$
ET II 207(49)a

9. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx=\sqrt{\pi}\,\Gamma(\mu)(2\alpha)^{-\frac12\mu-\frac12}e^{-\alpha}\,W_{-\frac12\mu,\nu-\frac12\mu}(2\alpha)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 208(53)a

10. $$\int_1^\infty x^{-\mu-\frac12(x-1)\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx=\sqrt{\pi}\,\Gamma(\mu)(2\alpha)^{-\frac12}e^{-\alpha}\,W_{-\mu,\nu}(2\alpha)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 207(51)a

11.[3] $$\int_{-1}^1\left(1-x^2\right)^{-1/2}xe^{-ax}\,I_1\left(b\sqrt{1-x^2}\right)\,dx=\frac{2}{b}\left\{\sinh a-a\left(a^2+b^2\right)^{-1/2}\sinh\sqrt{a^2+b^2}\right\}$$
$$[a>0,\quad b>0]$$

6.626

1.[11] $$\int_0^\infty x^{\lambda-1}e^{-\alpha x}\,J_\mu(\beta x)\,J_\nu(\gamma x)\,dx=\frac{\beta^\mu\gamma^\nu}{\Gamma(\nu+1)}2^{-\nu-\mu}\alpha^{-\lambda-\mu-\nu}\sum_{m=0}^\infty\frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)}$$
$$\times F\left(-m,-\mu-m;\nu+1;\frac{\gamma^2}{\beta^2}\right)\left(-\frac{\beta^2}{4\alpha^2}\right)^m$$
$$[\operatorname{Re}(\lambda+\mu+\nu)>0,\quad \operatorname{Re}(\alpha\pm i\beta\pm i\gamma)>1]$$
EH II 48(15)

2. $$\int_0^\infty e^{-2\alpha x}\,J_\nu(\beta x)\,J_\mu(\beta x)x^{\nu+\mu}\,dx=\frac{\Gamma\left(\nu+\mu+\frac12\right)\beta^{\nu+\mu}}{\sqrt{\pi^3}}$$
$$\times\int_0^{\frac{\pi}{2}}\frac{\cos^{\nu+\mu}\varphi\cos(\nu-\mu)\varphi}{\left(\alpha^2+\beta^2\cos^2\varphi\right)^{\nu+\mu}\sqrt{\alpha^2+\beta^2\cos^2\varphi}}\,d\varphi$$
$$\left[\operatorname{Re}\alpha>|\operatorname{Im}\beta|,\quad \operatorname{Re}(\nu+\mu)>-\tfrac12\right]$$
WA 427(1)

3. $$\int_0^\infty e^{-2\alpha x}\,J_0(\beta x)\,J_1(\beta x)x\,dx=\frac{\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)-\boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{2\pi\beta\sqrt{\alpha^2+\beta^2}}$$
WA 427(2)

4. $$\int_0^\infty e^{-2\alpha x}\,I_0(\beta x)\,I_1(\beta x)x\,dx=\frac{1}{2\pi\beta}\left\{\frac{\alpha}{\alpha^2-\beta^2}\,\boldsymbol{E}\left(\frac{\beta}{\alpha}\right)-\frac{1}{\alpha}\,\boldsymbol{K}\left(\frac{\beta}{\alpha}\right)\right\}$$
$$[\operatorname{Re}\alpha>\operatorname{Re}\beta]$$
WA 428(5)

5.[10] $$\int_0^\infty x^{\nu-\mu+2n} e^{-zx}\, J_\mu(\alpha x)\, J_\nu(\rho x)\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^\nu \times \frac{1}{\Gamma\left(\mu-\nu-n+\frac{1}{2}\right)} \sum_{q=0}^\infty \frac{\Gamma\left(\nu+n+q+\frac{1}{2}\right)\left(\nu-\mu+n+\frac{1}{2}\right)_q}{q!\,\Gamma\left(\nu+q+\frac{1}{2}\right)} \times a^{-2q} \int_0^{\ell_1/\rho} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q$$

where $\ell_1 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}\right]$ $\quad \left[\mu > \nu+2n, \quad n = 0, 1, \ldots, \quad \nu > -\frac{1}{2}\right]$

6.627 $$\int_0^\infty \frac{x^{-1/2}}{x+a} e^{-x}\, K_\nu(x)\, dx = \frac{\pi e^a\, K_\nu(a)}{\sqrt{a}\cos(\nu\pi)} \qquad \left[|\arg a| < \pi, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$$ ET II 368(29)

6.628

1. $$\int_0^\infty e^{-x\cos\beta}\, J_{-\nu}(x\sin\beta)\, x^\mu\, dx = \Gamma(\mu-\nu+1)\, P_\mu^\nu(\cos\beta)$$
$$\left[0 < \beta < \frac{\pi}{2}, \quad \mathrm{Re}(\mu-\nu) > -1\right]$$
WA 424(3), WH

2. $$\int_0^\infty e^{-x\cos\beta}\, Y_\nu(x\sin\beta)\, x^\mu\, dx = -\frac{\sin\mu\pi}{\sin(\mu+\nu)\pi} \frac{\Gamma(\mu-\nu+1)}{\pi} \times \left[Q_\mu^\nu(\cos\beta + 0\cdot i)\, e^{\frac{1}{2}\nu\pi i} + Q_\mu^\nu(\cos\beta - 0\cdot i)\, e^{-\frac{1}{2}\nu\pi i}\right]$$
$$\left[\mathrm{Re}(\mu+\nu) > -1, \quad 0 < \beta < \frac{\pi}{2}\right]$$ WA 424(4)

3. $$\int_0^1 e^{\frac{xu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu}\, J_{\mu-\nu}\left(\frac{ixu}{2}\right) dx = 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{\mathrm{B}(2\nu, 2\mu-2\nu+1)}{\Gamma(\mu-\nu+1)} \frac{e^{\frac{u}{2}}}{u^{\nu+\frac{1}{2}}}\, M_{\nu,\mu}(u)$$
MO 118a

4.[8] $$\int_0^\infty e^{-x\cosh\alpha}\, I_\nu(x\sinh\alpha)\, x^\mu\, dx = \Gamma(\nu+\mu+1)\, P_\mu^{-\nu}(\cosh\alpha)$$
$$\left[\mathrm{Re}(\mu+\nu) > -1, \quad |\mathrm{Im}\,\alpha| < \tfrac{1}{2}\pi\right]$$
WA 423(1)

5. $$\int_0^\infty e^{-x\cosh\alpha}\, K_\nu(x\sinh\alpha)\, x^\mu\, dx = \frac{\sin\mu\pi}{\sin(\nu+\mu)\pi}\, \Gamma(\mu-\nu+1)\, Q_\mu^\nu(\cosh\alpha)$$
$$[\mathrm{Re}(\mu+1) > |\mathrm{Re}\,\nu|]$$ WA 423(2)

6. $$\int_0^\infty e^{-x\cosh\alpha}\, I_\nu(x) x^{\mu-1}\, dx = \frac{\cos\nu\pi}{\sin(\mu+\nu)\pi} \frac{Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh\alpha)}{\sqrt{\frac{\pi}{2}}\,(\sinh\alpha)^{\mu-\frac{1}{2}}}$$
$$[\mathrm{Re}(\mu+\nu) > 0, \quad \mathrm{Re}\,(\cosh\alpha) > 1]$$
WA 424(6)

7. $$\int_0^\infty e^{-x\cosh\alpha}\, K_\nu(x) x^{\mu-1}\, dx = \sqrt{\frac{\pi}{2}}\, \Gamma(\mu-\nu)\, \Gamma(\mu+\nu) \frac{P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh\alpha)}{(\sinh\alpha)^{\mu-\frac{1}{2}}}$$
$$[\mathrm{Re}\,\mu > |\mathrm{Re}\,\nu|, \quad \mathrm{Re}\,(\cosh\alpha) > -1]$$
WA 424(7)

6.629[8] $\displaystyle\int_0^\infty x^{-1/2}e^{-x\alpha\cos\varphi\cos\psi}\,J_\mu\left(\alpha x\sin\varphi\right)J_\nu\left(\alpha x\sin\psi\right)\,dx$

$$= \Gamma\left(\mu+\nu+\tfrac{1}{2}\right)\alpha^{-\frac{1}{2}}\,P^{-\mu}_{\nu-\frac{1}{2}}\left(\cos\varphi\right)P^{-\nu}_{\mu-\frac{1}{2}}\left(\cos\psi\right)$$

$$\left[\alpha>0,\quad 0<\varphi<\frac{\pi}{2},\quad 0<\psi<\frac{\pi}{2},\quad \operatorname{Re}(\mu+\nu)>-\frac{1}{2}\right]$$ ET II 50(19)

6.631

1. $$\int_0^\infty x^\mu e^{-\alpha x^2}\,J_\nu(\beta x)\,dx=\frac{\beta^\nu\,\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{2^{\nu+1}\alpha^{\frac{1}{2}(\mu+\nu+1)}\,\Gamma(\nu+1)}\;{}_1F_1\left(\frac{\nu+\mu+1}{2};\nu+1;-\frac{\beta^2}{4\alpha}\right)$$

BU 8(15)

$$=\frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{\beta\alpha^{\frac{1}{2}\mu}\,\Gamma(\nu+1)}\exp\left(-\frac{\beta^2}{8\alpha}\right)M_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}(\mu+\nu)>-1]$$

EH II 50(22), ET II 30(14), BU 14(13b)

2. $$\int_0^\infty x^\mu e^{-\alpha x^2}\,Y_\nu(\beta x)\,dx$$

$$=-\alpha^{-\frac{1}{2}\mu}\beta^{-1}\sec\left(\frac{\nu-\mu}{2}\pi\right)\exp\left(-\frac{\beta^2}{8\alpha}\right)$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)}\sin\left(\frac{\nu-\mu}{2}\pi\right)M_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)+W_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)\right\}$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-1,\quad \beta>0]$$ ET II 106(4)

3. $$\int_0^\infty x^\mu e^{-\alpha x^2}\,K_\nu(\beta x)\,dx=\frac{1}{2}\alpha^{-\frac{1}{2}\mu}\beta^{-1}\,\Gamma\left(\frac{1+\nu+\mu}{2}\right)\Gamma\left(\frac{1-\nu+\mu}{2}\right)\exp\left(\frac{\beta^2}{8\alpha}\right)W_{-\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$

$$[\operatorname{Re}\mu>|\operatorname{Re}\nu|-1]$$ ET II 132(25)

4.[11] $$\int_0^\infty x^{\nu+1}e^{-\alpha x^2}\,J_\nu(\beta x)\,dx=\frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right)\qquad[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-1]$$

WA 431(4), ET II 29(10)

5. $$\int_0^\infty x^{\nu-1}e^{-\alpha x^2}\,J_\nu(\beta x)\,dx=2^{\nu-1}\beta^{-\nu}\left[1-\gamma\left(\nu,\frac{\beta^2}{4\alpha}\right)\right]$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>0]$$ ET II 30(11)

6. $$\int_0^\infty x^{\nu+1}e^{\pm i\alpha x^2}\,J_\nu(\beta x)\,dx=\frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left[\pm i\left(\frac{\nu+1}{2}\pi-\frac{\beta^2}{4\alpha}\right)\right]$$

$$\left[\alpha>0,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2},\quad \beta>0\right]$$

ET II 30(12)

7. $$\int_0^\infty xe^{-\alpha x^2}\,J_\nu(\beta x)\,dx=\frac{\sqrt{\pi}\beta}{8\alpha^{\frac{3}{2}}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right)-I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right)\right]$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-2]$$ ET II 29(9)

8. $$\int_0^1 x^{n+1}e^{-\alpha x^2}\, I_n(2\alpha x)\, dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=-n}^{n} I_r(2\alpha)\right]$$
$[n = 0, 1, \ldots]$ ET II 365(8)a

9. $$\int_1^\infty x^{1-n}e^{-\alpha x^2}\, I_n(2\alpha x)\, dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=1-n}^{n-1} I_r(2\alpha)\right]$$
$[n = 1, 2, \ldots]$ ET II 367(20)a

10. $$\int_0^\infty e^{-x^2}x^{2n+\mu+1}\, J_\mu\left(2x\sqrt{z}\right)\, dx = \frac{n!}{2}e^{-z}z^{\frac{1}{2}\mu}\, L_n^\mu(z) \qquad [n = 0, 1, \ldots; \quad n + \operatorname{Re}\mu > -1]$$
BU 135(5)

6.632 $$\int_0^\infty x^{-\frac{1}{2}} \exp\left[-\left(x^2 + a^2 - 2ax\cos\varphi\right)^{\frac{1}{2}}\right]\left[x^2 + a^2 - 2ax\cos\varphi\right]^{-\frac{1}{2}} K_\nu(x)\, dx$$
$$= \pi a^{-\frac{1}{2}}\sec(\nu\pi)\, P_{\nu-\frac{1}{2}}(-\cos\varphi)\, K_\nu(a)$$
$\left[|\arg a| + |\operatorname{Re}\varphi| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 368(32)

6.633

1. $$\int_0^\infty x^{\lambda+1}e^{-\alpha x^2}\, J_\mu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{\beta^\mu\gamma^\nu\alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1}\,\Gamma(\nu+1)}\sum_{m=0}^\infty \frac{\Gamma\left(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1\right)}{m!\,\Gamma(m+\mu+1)}\left(-\frac{\beta^2}{4\alpha}\right)^m$$
$$\times F\left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2}\right)$$
$[\operatorname{Re}\alpha > 0, \operatorname{Re}(\mu + \nu + \lambda) > -2, \beta > 0, \quad \gamma > 0]$ EH II 49(20)a, ET II 51(24)a

2. $$\int_0^\infty e^{-\varrho^2x^2}\, J_p(\alpha x)\, J_p(\beta x)x\, dx = \frac{1}{2\varrho^2}\exp\left(-\frac{\alpha^2 + \beta^2}{4\varrho^2}\right) I_p\left(\frac{\alpha\beta}{2\varrho^2}\right)$$
$\left[\operatorname{Re} p > -1, \quad |\arg\varrho| < \frac{\pi}{4}, \quad \alpha > 0, \quad \beta > 0\right]$ KU 146(16)a, WA 433(1)

3. $$\int_0^\infty x^{2\nu+1}e^{-\alpha x^2}\, J_\nu(x)\, Y_\nu(x)\, dx = -\frac{1}{2\sqrt{\pi}}\alpha^{-\frac{3}{2}\nu-\frac{1}{2}}\exp\left(-\frac{1}{2\alpha}\right) W_{\frac{1}{2}\nu,\frac{1}{2}\nu}\left(\frac{1}{\alpha}\right)$$
$\left[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 347(59)

4. $$\int_0^\infty xe^{-\alpha x^2}\, I_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{2\alpha}\exp\left(\frac{\beta^2 - \gamma^2}{4\alpha}\right) J_\nu\left(\frac{\beta\gamma}{2\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$ ET II 63(1)

5. $$\int_0^\infty x^{\lambda-1}e^{-\alpha x^2}\, J_\mu(\beta x)\, J_\nu(\beta x)\, dx$$
$$= 2^{-\nu-\mu-1}\alpha^{-\frac{1}{2}(\nu+\lambda+\mu)}\beta^{\nu+\mu}\frac{\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\ {}_3F_3\left[\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{\mu}{2} + 1, \frac{\nu+\mu+\lambda}{2}; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right]$$
$[\operatorname{Re}(\nu + \lambda + \mu) > 0, \quad \operatorname{Re}\alpha > 0]$ WA 434, EH II 50(21)

6.634 $$\int_0^\infty x e^{-\frac{x^2}{2a}} \left[I_\nu(x) + I_{-\nu}(x)\right] K_\nu(x)\, dx = a e^a K_\nu(a) \qquad [\operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < 1]$$

ET II 371(49)

6.635

1. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} J_\nu(\beta x)\, dx = 2\, J_\nu\left(\sqrt{2\alpha\beta}\right) K_\nu\left(\sqrt{2\alpha\beta}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \beta > 0]$$ ET II 30(15)

2. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x}} Y_\nu(\beta x)\, dx = 2\, Y_\nu\left(\sqrt{2\alpha\beta}\right) K_\nu\left(\sqrt{2\alpha\beta}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \beta > 0]$$ ET II 106(5)

3. $$\int_0^\infty x^{-1} e^{-\frac{\alpha}{x} - \beta x} J_\nu(\gamma x)\, dx = 2\, J_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2} - \beta\right]^{\frac12}\right\} K_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2} + \beta\right]^{\frac12}\right\}$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0]$$
ET II 30(16)

6.636 $$\int_0^\infty x^{-\frac12} e^{-\alpha\sqrt{x}} J_\nu(\beta x)\, dx = \frac{\sqrt{2}}{\sqrt{\pi\beta}} \Gamma\left(\nu + \tfrac12\right) D_{-\nu-\frac12}\left(2^{-\frac12}\alpha e^{\frac14\pi i}\beta^{-\frac12}\right) D_{-\nu-\frac12}\left(2^{-\frac12}\alpha e^{-\frac14\pi i}\beta^{-\frac12}\right)$$
$$\left[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > -\tfrac12\right]$$
ET II 30(17)

6.637

1. $$\int_0^\infty \left(\beta^2 + x^2\right)^{-\frac12} \exp\left[-\alpha\left(\beta^2+x^2\right)^{\frac12}\right] J_\nu(\gamma x)\, dx$$
$$= I_{\frac12\nu}\left\{\frac12\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac12} - \alpha\right]\right\} K_{\frac12\nu}\left\{\frac12\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac12} + \alpha\right]\right\}$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 31(20)

2. $$\int_0^\infty \left(\beta^2 + x^2\right)^{-\frac12} \exp\left[-\alpha\left(\beta^2+x^2\right)^{\frac12}\right] Y_\nu(\gamma x)\, dx$$
$$= -\sec\left(\frac{\nu\pi}{2}\right) K_{\frac12\nu}\left\{\frac12\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac12} + \alpha\right]\right\}$$
$$\times\left(\frac{1}{\pi} K_{\frac12\nu}\left\{\frac12\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac12} + \alpha\right]\right\} + \sin\left(\frac{\nu\pi}{2}\right) I_{\frac12\nu}\left\{\frac12\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac12} - \alpha\right]\right\}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0, \quad |\operatorname{Re}\nu| < 1]$$ ET II 106(6)

3. $$\int_0^\infty \left(x^2 + \beta^2\right)^{-\frac12} \exp\left[-\alpha\left(x^2+\beta^2\right)^{\frac12}\right] K_\nu(\gamma x)\, dx$$
$$= \frac12 \sec\left(\frac{\nu\pi}{2}\right) K_{\frac12\nu}\left(\frac12\beta\left[\alpha + \left(\alpha^2-\gamma^2\right)^{\frac12}\right]\right) K_{\frac12\nu}\left(\frac12\beta\left[\alpha - \left(\alpha^2-\gamma^2\right)^{\frac12}\right]\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\gamma+\beta) > 0, \quad |\operatorname{Re}\nu| < 1]$$ ET II 132(26)

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers

6.641 $$\int_0^\infty \sqrt{x} e^{-\alpha x} J_{\pm\frac14}\left(x^2\right)\, dx = \frac{\sqrt{\pi\alpha}}{4}\left[\mathbf{H}_{\mp\frac14}\left(\frac{\alpha^2}{4}\right) - Y_{\mp\frac14}\left(\frac{\alpha^2}{4}\right)\right]$$ MI 42

6.642

1.[10] $$\int_0^\infty x^{-1}e^{-\alpha x}\,Y_\nu\left(\frac{2}{x}\right)\,dx = 2\,K_\nu\left(2\sqrt{a}\right)\,Y_\nu\left(2\sqrt{a}\right)$$
$[\operatorname{Re} a > 0]$ MC

2. $$\int_0^\infty x^{-1}e^{-\alpha x}\,H_\nu^{(1,2)}\left(\frac{2}{x}\right)\,dx = H_\nu^{(1,2)}\left(\sqrt{\alpha}\right)K_\nu\left(\sqrt{\alpha}\right)$$
MI 44, EH II 91(26)

6.643

1. $$\int_0^\infty x^{\mu-\frac{1}{2}}e^{-\alpha x}\,J_{2\nu}\left(2\beta\sqrt{x}\right)\,dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\beta\,\Gamma(2\nu+1)}e^{-\frac{\beta^2}{2\alpha}}\alpha^{-\mu}\,M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$
$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right]$
BU 14(13a), MI 42a

2. $$\int_0^\infty x^{\mu-\frac{1}{2}}e^{-\alpha x}\,I_{2\nu}\left(2\beta\sqrt{x}\right)\,dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)}\beta^{-1}e^{\frac{\beta^2}{2\alpha}}\alpha^{-\mu}\,M_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$
$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right]$ MI 45

3. $$\int_0^\infty x^{\mu-\frac{1}{2}}e^{-\alpha x}\,K_{2\nu}\left(2\beta\sqrt{x}\right)\,dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}{2\beta}e^{\frac{\beta^2}{2\alpha}}\alpha^{-\mu}\,W_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$
$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right]$, (cf. **6.631** 3)
MI 47a

4. $$\int_0^\infty x^{n+\frac{1}{2}\nu}e^{-\alpha x}\,J_\nu\left(2\beta\sqrt{x}\right)\,dx = n!\beta^\nu e^{-\frac{\beta^2}{\alpha}}\alpha^{-n-\nu-1}\,L_n^\nu\left(\frac{\beta^2}{\alpha}\right)$$
$[n+\nu > -1]$ MO 178a

5. $$\int_0^\infty x^{-\frac{1}{2}}e^{-\alpha x}\,Y_{2\nu}\left(\beta\sqrt{x}\right)\,dx = -\sqrt{\frac{\pi}{\alpha}}\frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)}\left[\sin(\nu\pi)\,I_\nu\left(\frac{\beta^2}{8\alpha}\right)+\frac{1}{\pi}\,K_\nu\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$\left[|\operatorname{Re}\nu| < \frac{1}{2}\right]$ MI 44

6. $$\int_0^\infty x^{\frac{1}{2}m}e^{-\alpha x}\,K_m\left(2\sqrt{x}\right)\,dx = \frac{\Gamma(m+1)}{2\alpha}\left(\frac{1}{\alpha}\right)^{\frac{1}{2}m-\frac{1}{2}}e^{\frac{1}{2\alpha}}\,W_{-\frac{1}{2}(m+1),-\frac{1}{2}m}\left(\frac{1}{\alpha}\right)$$
MI 48a

6.644 $$\int_0^\infty e^{-\beta x}\,J_{2\nu}\left(2a\sqrt{x}\right)J_\nu(bx)\,dx = \exp\left(-\frac{a^2\beta}{\beta^2+b^2}\right)J_\nu\left(\frac{a^2b}{\beta^2+b^2}\right)\frac{1}{\sqrt{\beta^2+b^2}}$$
$\left[\operatorname{Re}\beta > 0,\quad b > 0,\quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 58(17)

6.645

1. $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}}e^{-\alpha x}\,J_\nu\left(\beta\sqrt{x^2-1}\right)\,dx = I_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right]K_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)\right]$$
MO 179a

2. $$\int_1^\infty \left(x^2-1\right)^{\frac{1}{2}\nu}e^{-\alpha x}\,J_\nu\left(\beta\sqrt{x^2-1}\right)\,dx = \sqrt{\frac{2}{\pi}}\beta^\nu\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{1}{4}}K_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2}\right)$$
MO 179a

3.[3] $$\int_{-1}^{1}\left(1-x^2\right)^{-1/2} e^{-ax}\, I_1\left(b\sqrt{1-x^2}\right)\, dx = \frac{2}{b}\left(\cosh\sqrt{a^2+b^2}-\cosh a\right)$$

$[a>0, \quad b>0]$

6.646

1. $$\int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x}\, J_\nu\left(\beta\sqrt{x^2-1}\right)\, dx = \frac{\exp\left(-\sqrt{\alpha^2+\beta^2}\right)}{\sqrt{\alpha^2+\beta^2}}\left(\frac{\beta}{\alpha+\sqrt{\alpha^2+\beta^2}}\right)^\nu$$

$[\operatorname{Re}\nu>-1]$ EF 89(52), MO 179

2. $$\int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x}\, I_\nu\left(\beta\sqrt{x^2-1}\right)\, dx = \frac{\exp\left(-\sqrt{\alpha^2-\beta^2}\right)}{\sqrt{\alpha^2-\beta^2}}\left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^\nu$$

$[\operatorname{Re}\nu>-1, \quad \alpha>\beta]$ MO 180

3.[7] $$\int_b^\infty e^{-pt}\left(\frac{t-b}{t+b}\right)^{\nu/2} K_\nu\left[a\left(t^2-b^2\right)^{1/2}\right]\, dt = \frac{\Gamma(\nu+1)}{2sa^\nu}\left[x^\nu e^{-bx}\,\Gamma(-\nu,bx)-y^\nu e^{bs}\,\Gamma(-\nu,by)\right]$$

where $x=p-s, \quad y=p+s, \quad s=\left(p^2-a^2\right)^{1/2}$ $\quad [\operatorname{Re}(p+a)>0, \quad |\operatorname{Re}(\nu)|<1]$.

ME 39a

6.647

1. $$\int_0^\infty x^{-\lambda-\frac{1}{2}}\,(\beta+x)^{\lambda-\frac{1}{2}}\, e^{-\alpha x}\, K_{2\mu}\left[\sqrt{x(\beta+x)}\right]\, dx$$
$$= \frac{1}{\beta}e^{\frac{1}{2}\alpha\beta}\,\Gamma\left(\tfrac{1}{2}-\lambda+\mu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\mu\right)\, W_{\lambda,\mu}(z_1)\, W_{\lambda,\mu}(z_2)$$

$$z_1=\tfrac{1}{2}\beta\left(\alpha+\sqrt{\alpha^2-1}\right), \quad z_2=\tfrac{1}{2}\beta\left(\alpha-\sqrt{\alpha^2-1}\right)$$

$\left[|\arg\beta|<\pi, \quad \operatorname{Re}\alpha>-1, \quad \operatorname{Re}\lambda+|\operatorname{Re}\mu|<\tfrac{1}{2}\right]$ ET II 377(37)

2. $$\int_0^\infty (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x\cosh t}\, K_\nu\left[\sqrt{x(\alpha+x)}\right]\, dx$$
$$= \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right) e^{\frac{1}{2}\alpha\cosh t}\, K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^t\right) K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^{-t}\right)$$

$[-1<\operatorname{Re}\nu<1]$ ET II 377(36)

3.[11] $$\int_0^\alpha x^{\lambda-\frac{1}{2}}(\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x\sinh t}\, I_{2\mu}\left[\sqrt{x(\alpha-x)}\right]\, dx$$
$$= e^{-(\alpha/2)\sinh t}\frac{2\,\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha\left[\Gamma(2\mu+1)\right]^2}\, M_{\lambda,\mu}\left(\frac{1}{2}\alpha e^t\right) M_{-\lambda,\mu}\left(\frac{1}{2}\alpha e^{-t}\right)$$

$\left[\operatorname{Re}\mu>|\operatorname{Re}\lambda|-\tfrac{1}{2}\right]$ ET II 377(32)

6.648 $$\int_{-\infty}^\infty e^{\varrho x}\left(\frac{\alpha+\beta e^x}{\alpha e^x+\beta}\right)^\nu K_{2\nu}\left[\left(\alpha^2+\beta^2+2\alpha\beta\cosh x\right)^{\frac{1}{2}}\right]\, dx = 2\,K_{\nu+\varrho}(\alpha)\,K_{\nu-\varrho}(\beta)$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0]$ ET II 379(45)

6.649

1. $$\int_0^\infty K_{\mu-\nu}(2z\sinh x)\,e^{(\nu+\mu)x}\,dx = \frac{\pi^2}{4\sin[(\nu-\mu)\pi]}\,[J_\nu(z)\,Y_\mu(z) - J_\mu(z)\,Y_\nu(z)]$$
$$[\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(\nu-\mu) < 1]$$
MO 44

2. $$\int_0^\infty J_{\nu+\mu}(2x\sinh t)\,e^{(\nu-\mu)t}\,dt = K_\nu(x)\,I_\mu(x)$$
$$\left[\operatorname{Re}(\nu-\mu) < \tfrac{3}{2}, \quad \operatorname{Re}(\nu+\mu) > -1, \quad x > 0\right]$$ EH II 97(68)

3. $$\int_0^\infty Y_{\nu-\mu}(2x\sinh t)\,e^{-(\nu+\mu)t}\,dt = \frac{1}{\sin[\pi(\mu-\nu)]}\,\{I_\mu(x)\,K_\nu(x) - \cos[(\nu-\mu)\pi]\,I_\nu(x)\,K_\mu(x)\}$$
$$\left[|\operatorname{Re}(\nu-\mu)| < 1, \quad \operatorname{Re}(\nu+\mu) > -\tfrac{1}{2}, \quad x > 0\right]$$ EH II 97(73)

4. $$\int_0^\infty K_0(2z\sinh x)\,e^{-2\nu x}\,dx = -\frac{\pi}{4}\left\{J_\nu(z)\frac{\partial\,Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial\,J_\nu(z)}{\partial\nu}\right\}$$

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.651

1. $$\int_0^\infty x^{\lambda+\frac{1}{2}}e^{-\frac{1}{4}\alpha^2x^2}\,I_\mu\left(\tfrac{1}{4}\alpha^2x^2\right)J_\nu(\beta x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}2^{\lambda+1}\beta^{-\lambda-\frac{3}{2}}\,G^{21}_{23}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\,\begin{matrix}1-\mu, 1+\mu\\ h, \frac{1}{2}, k\end{matrix}\right)$$
$$h = \tfrac{3}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu, \quad k = \tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu$$
$$\left[|\arg\alpha| < \frac{\pi}{4}, \quad \beta > 0, \quad -\tfrac{3}{2} - \operatorname{Re}(2\mu+\nu) < \operatorname{Re}\lambda < 0\right]$$ ET II 68(8)

2. $$\int_0^\infty x^{\lambda+\frac{1}{2}}e^{-\frac{1}{4}\alpha^2x^2}\,K_\mu\left(\tfrac{1}{4}\alpha^2x^2\right)J_\nu(\beta x)\,dx$$
$$= \sqrt{\frac{\pi}{2}}2^{\lambda+1}\beta^{-\lambda-\frac{3}{2}}\,G^{12}_{23}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\,\begin{matrix}1-\mu, 1+\mu\\ h, \frac{1}{2}, k\end{matrix}\right)$$
$$h = \tfrac{3}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu, \quad k = \tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu$$
$$\left[|\arg\alpha| < \frac{\pi}{4}, \quad \operatorname{Re}(\lambda+\nu\pm 2\mu) > -\tfrac{3}{2}\right]$$ ET II 69(15)

3. $$\int_0^\infty x^{2\mu-\nu+1}e^{-\frac{1}{4}\alpha x^2}\,I_\mu\left(\tfrac{1}{4}\alpha x^2\right)J_\nu(\beta x)\,dx$$
$$= 2^{\mu-\nu+\frac{1}{2}}(\pi\alpha)^{-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+\mu\right)\frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)}\;{}_1F_1\left(\frac{1}{2}+\mu;\frac{1}{2}-\mu+\nu;-\frac{\beta^2}{2\alpha}\right)$$
$$\left[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > 2\operatorname{Re}\mu + \tfrac{1}{2} > -\tfrac{1}{2}\right]$$ ET II 68(6)

4. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} K_\mu\left(\tfrac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\, dx$$
$$= \sqrt{\pi} 2^\mu \alpha^{-2\mu-2\nu-2} \beta^\nu \frac{\Gamma(1+2\mu+\nu)}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)} {}_1F_1\left(1+2\mu+\nu; \mu+\nu+\frac{3}{2}; -\frac{\beta^2}{2\alpha^2}\right)$$
$$\left[|\arg\alpha| < \tfrac{1}{4}\pi, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -1, \quad \beta > 0\right]$$ ET II 69(13)

5. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha x^2} I_\mu\left(\tfrac{1}{2}\alpha x^2\right) K_\nu(\beta x)\, dx$$
$$= \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \beta^{-\mu-\frac{3}{2}} \alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} \Gamma(2\mu+\nu+1)\, \Gamma\left(\mu+\tfrac{1}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{k,m}\left(\frac{\beta^2}{4\alpha}\right)$$
$$2k = -3\mu-\nu-\tfrac{1}{2}, \quad 2m = \mu+\nu+\tfrac{1}{2}$$
$$\left[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1\right]$$ ET II 146(53)

6. $$\int_0^\infty x e^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu}\left(\tfrac{1}{4}\beta x^2\right) J_\nu(\gamma x)\, dx = 2\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}} \exp\left(-\frac{\alpha\gamma^2}{\alpha^2+\beta^2}\right) J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{\alpha^2+\beta^2}\right)$$
$$[\gamma > 0, \quad \operatorname{Re}\alpha > |\operatorname{Im}\beta|, \quad \operatorname{Re}\nu > -1]$$
ET II 56(2)

7. $$\int_0^\infty x e^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu}\left(\tfrac{1}{4}\alpha x^2\right) J_\nu(\beta x)\, dx = \left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{2}} \beta^{-1} \exp\left(-\frac{\beta^2}{2\alpha}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > -1]$$
ET II 67(3)

8. $$\int_0^\infty x^{1-\nu} e^{-\frac{1}{4}\alpha^2 x^2} I_\nu\left(\tfrac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\, dx = \sqrt{\frac{2}{\pi}} \frac{\beta^{\nu-1}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu}\left(\frac{\beta}{\alpha}\right)$$
$$\left[|\arg\alpha| < \tfrac{1}{4}\pi, \quad \beta > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 67(1)

9. $$\int_0^\infty x^{-\nu-1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu+1}\left(\tfrac{1}{4}\alpha^2 x^2\right) J_\nu(\beta x)\, dx = \sqrt{\frac{2}{\pi}} \beta^\nu \exp\left(-\frac{\beta^2}{4\alpha^2}\right) D_{-2\nu-3}\left(\frac{\beta}{\alpha}\right)$$
$$\left[|\arg\alpha| < \tfrac{1}{4}\pi, \quad \operatorname{Re}\nu > -1, \quad \beta > 0\right]$$
ET II 67(2)

6.652 $$\int_0^\infty x^{2\nu} e^{-\left(\frac{x^2}{8}+\alpha x\right)} I_\nu\left(\frac{x^2}{8}\right) dx = \frac{\Gamma(4\nu+1)}{2^{4\nu}\,\Gamma(\nu+1)} \frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}\left(\alpha^2\right)$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{4}\right) > 0\right]$$ MI 45

6.653

1. $$\int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}\left(a^2+b^2\right)\right] I_\nu\left(\frac{ab}{x}\right) \frac{dx}{x} = 2\, I_\nu(a)\, K_\nu(b) \qquad [0 < a < b]$$
$$= 2\, K_\nu(a)\, I_\nu(b) \qquad [0 < b < a]$$
$$[\operatorname{Re}\nu > -1]$$ WA 482(2)a, EH II 53(37), WA 482(3)a

2. $$\int_0^\infty \exp\left[-\frac{1}{2}x - \frac{1}{2x}\left(z^2+w^2\right)\right] K_\nu\left(\frac{zw}{x}\right) \frac{dx}{x} = 2\, K_\nu(z)\, K_\nu(w)$$
$$[|\arg z| < \pi, \quad |\arg w| < \pi, \quad \arg(z+w)] < \tfrac{1}{4}\pi$$ WA 483(1), EH II 53(36)

6.654 $$\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x}-\alpha x} K_\nu\left(\frac{\beta^2}{8x}\right) dx = \sqrt{4\pi}\alpha^{-\frac{1}{2}} K_{2\nu}\left(\beta\sqrt{\alpha}\right)$$ ME 39

6.655 $$\int_0^\infty x\left(\beta^2+x^2\right)^{-\frac{1}{2}} \exp\left(-\frac{\alpha^2\beta}{\beta^2+x^2}\right) J_\nu\left(\frac{\alpha^2 x}{\beta^2+x^2}\right) J_\nu(\gamma x)\, dx = \gamma^{-1} e^{-\beta\gamma} J_{2\nu}\left(2\alpha\sqrt{\gamma}\right)$$
$$\left[\operatorname{Re}\beta > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 58(14)

6.656

1. $$\int_0^\infty e^{-(\xi-z)\cosh t} J_{2\nu}\left[2(z\xi)^{\frac{1}{2}}\sinh t\right] dt = I_\nu(z)\, K_\nu(\xi)$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0\right]$$
EH II 98(78)

2. $$\int_0^\infty e^{-(\xi+z)\cosh t} K_{2\nu}\left[2(z\xi)^{\frac{1}{2}}\sinh t\right] dt = \frac{1}{2} K_\nu(z)\, K_\nu(\xi)\sec(\nu\pi)$$
$$\left[|\operatorname{Re}\nu| < \tfrac{1}{2}, \quad \operatorname{Re}\left(z^{\frac{1}{2}} + \xi^{\frac{1}{2}}\right)^2 \geq 0\right]$$
EH II 98(79)

6.66 Combinations of Bessel, hyperbolic, and exponential functions

Bessel and hyperbolic functions

6.661

1. $$\int_0^\infty \sinh(ax)\, K_\nu(bx)\, dx = \frac{\pi}{2}\frac{\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\sin\left[\nu\arcsin\left(\frac{a}{b}\right)\right]}{\sqrt{b^2-a^2}}$$
$$[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re}\nu| < 2]$$
ET II 133(32)

2. $$\int_0^\infty \cosh(ax)\, K_\nu(bx)\, dx = \frac{\pi\cos\left[\nu\arcsin\left(\frac{a}{b}\right)\right]}{2\sqrt{b^2-a^2}\cos\left(\frac{\nu\pi}{2}\right)} \qquad [\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re}\nu| < 1]$$
ET II 134(33)

6.662 **Notation**:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \qquad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$

1.[10] $$\int_0^\infty \cosh(\beta x)\, K_0(\alpha x)\, J_0(\gamma x)\, dx = \frac{\boldsymbol{K}(k)}{\sqrt{u+v}}$$
$$u = \frac{1}{2}\left\{\sqrt{\left(\alpha^2+\beta^2+\gamma^2\right)^2 - 4\alpha^2\beta^2}\right\} + \alpha^2 - \beta^2 - \gamma^2$$
$$v = \frac{1}{2}\left\{\sqrt{\left(\alpha^2+\beta^2+\gamma^2\right)^2 - 4\alpha^2\beta^2}\right\} - \alpha^2 + \beta^2 + \gamma^2$$
$$k^2 = v(u+v)^{-1} \qquad [\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \gamma > 0]$$
ET II 15(23)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,

$$\int_0^\infty \cosh(bx)\, K_0(cx)\, J_0(ax)\, dx = \frac{\boldsymbol{K}(k)}{\sqrt{\ell_2^2-\ell_1^2}}$$

$$k^2 = \frac{\ell_2^2-c^2}{\ell_2^2-\ell_1^2}, \qquad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a>0]$$

2.[10] $$\int_0^\infty \sinh(\beta x)\, K_1(\alpha x)\, J_0(\gamma x)\, dx = a^{-1}\left[u\,\boldsymbol{E}(k) - \boldsymbol{K}(k)\,\boldsymbol{E}(u) + \frac{\boldsymbol{K}(k)\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right]$$

$$\operatorname{cn}^2 u = 2\gamma^2\left\{\left[\left(\alpha^2+\beta^2+\gamma^2\right)^2 - 4\alpha^2\beta^2\right]^{\frac12} - \alpha^2+\beta^2+\gamma^2\right\}^{-1}$$

$$k^2 = \frac12\left\{1-\left(\alpha^2-\beta^2-\gamma^2\right)\left[\left(\alpha^2+\beta^2+\gamma^2\right)^2-4\alpha^2\beta^2\right]^{-\frac12}\right\}$$

$$[\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \gamma>0]$$

ET II 15(24)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,

$$\int_0^\infty \sinh(bx)\, K_1(cx)\, J_0(ax)\, dx = c^{-1}\left[u\,\boldsymbol{E}(k) - \boldsymbol{K}(k)\,\boldsymbol{E}(u) + \frac{\boldsymbol{K}(k)\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right]$$

$$\operatorname{cn}^2 u = \frac{a^2}{\ell_2^2-c^2}, \quad k^2 = \frac{\ell_2^2-c^2}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a>0]$$

6.663

1. $$\int_0^\infty K_{\nu\pm\mu}(2z\cosh t)\cosh\left[(\mu\mp\nu)\,t\right]\, dt = \frac12 K_\mu(z)\, K_\nu(z)$$

$[\operatorname{Re} z > 0]$ WA 484(1), EH II 54(39)

2. $$\int_0^\infty Y_{\mu+\nu}(2z\cosh t)\cosh[(\mu-\nu)t]\, dt = \frac{\pi}{4}\left[J_\mu(z)\, J_\nu(z) - Y_\mu(z)\, Y_\nu(z)\right]$$

$[z>0]$ EH II 96(64)

3. $$\int_0^\infty J_{\mu+\nu}(2z\cosh t)\cosh[(\mu-\nu)t]\, dt = -\frac{\pi}{4}\left[J_\mu(z)\, Y_\nu(z) + J_\nu(z)\, Y_\mu(z)\right]$$

$[z>0]$ EH II 97(65)

4. $$\int_0^\infty J_{\mu+\nu}(2z\sinh t)\cosh[(\mu-\nu)t]\, dt = \frac12\left[I_\nu(z)\, K_\mu(z) + I_\mu(z)\, K_\nu(z)\right]$$

$\left[\operatorname{Re}(\nu+\mu) > -1, \quad |\operatorname{Re}(\mu-\nu)| < \frac32, \quad z>0\right]$ EH II 97(71)

5. $$\int_0^\infty J_{\mu+\nu}(2z\sinh t)\sinh[(\mu-\nu)t]\, dt = \frac12\left[I_\nu(z)\, K_\mu(z) - I_\mu(z)\, K_\nu(z)\right]$$

$\left[\operatorname{Re}(\nu+\mu) > -1, \quad |\operatorname{Re}(\mu-\nu)| < \frac32, \quad z>0\right]$ EH II 97(72)

6.664

1. $$\int_0^\infty J_0(2z\sinh t)\sinh(2\nu t)\, dt = \frac{\sin(\nu\pi)}{\pi}\left[K_\nu(z)\right]^2 \qquad \left[|\operatorname{Re}\nu| < \tfrac34, \quad z>0\right]$$

EH II 97(69)

2. $$\int_0^\infty Y_0\left(2z\sinh t\right)\cosh(2\nu t)\,dt = -\frac{\cos(\nu\pi)}{\pi}\left[K_\nu(z)\right]^2 \qquad \left[|\mathrm{Re}\,\nu| < \tfrac{3}{4},\quad z>0\right]$$ EH II 97(70)

3. $$\int_0^\infty Y_0\left(2z\sinh t\right)\sinh(2\nu t)\,dt = \frac{1}{\pi}\left[I_\nu(z)\frac{\partial K_\nu(z)}{\partial\nu} - K_\nu(z)\frac{\partial I_\nu(z)}{\partial\nu}\right] - \frac{1}{\pi}\cos(\nu\pi)\left[K_\nu(z)\right]^2$$
$$\left[|\mathrm{Re}\,\nu| < \tfrac{3}{4},\quad z>0\right]$$ EH II 97(75)

4. $$\int_0^\infty K_0\left(2z\sinh t\right)\cosh 2\nu t\,dt = \frac{\pi^2}{8}\left\{J_\nu^2(z)+N_\nu^2(z)\right\} \qquad [\mathrm{Re}\,z>0]$$ MO 44

5. $$\int_0^\infty K_{2\mu}\left(z\sinh 2t\right)\coth^{2\nu} t\,dt = \frac{1}{4z}\,\Gamma\left(\frac{1}{2}+\mu-\nu\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)W_{\nu,\mu}(iz)\,W_{\nu,\mu}(-iz)$$
$$\left[|\arg z| \le \frac{\pi}{2},\quad |\mathrm{Re}\,\mu| + \mathrm{Re}\,\nu < \tfrac{1}{2}\right]$$
MO 119

6. $$\int_0^\infty \cosh(2\mu x)\,K_{2\nu}\left(2a\cosh x\right)\,dx = \frac{1}{2}\,K_{\mu+\nu}(a)\,K_{\mu-\nu}(a)$$
$$[\mathrm{Re}\,a>0]$$ ET II 378(42)

6.665 $$\int_0^\infty \operatorname{sech} x\cosh(2\lambda x)\,I_{2\mu}\left(a\operatorname{sech} x\right)\,dx = \frac{\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{2a\left[\Gamma(2\mu+1)\right]^2}\,M_{\lambda,\mu}(a)\,M_{-\lambda,\mu}(a)$$
$$\left[|\mathrm{Re}\,\lambda| - \mathrm{Re}\,\mu < \tfrac{1}{2}\right]$$ ET II 378(43)

Bessel, hyperbolic, and algebraic functions

6.666 $$\int_0^\infty x^{\nu+1}\sinh(\alpha x)\operatorname{cosech}(\pi x)\,J_\nu(\beta x)\,dx = \frac{2}{\pi}\sum_{n=1}^\infty(-1)^{n-1}n^{\nu+1}\sin(n\alpha)\,K_\nu(n\beta)$$
$$[|\mathrm{Re}\,\alpha| < \pi,\quad \mathrm{Re}\,\nu > -1]$$
ET II 41(3), WA 469(12)

6.667

1.[3] $$\int_0^a \frac{\cosh\left(\sqrt{a^2-x^2}\right)\sinh t\,I_{2\nu}(x)}{\sqrt{a^2-x^2}}\,dx = \frac{\pi}{2}\,I_\nu\left(\frac{1}{2}ae^t\right)I_\nu\left(\frac{1}{2}ae^{-t}\right)$$
$$\left[\mathrm{Re}\,\nu > -\tfrac{1}{2}\right]$$ ET II 365(10)

2. $$\int_0^a \frac{\cosh\left(\sqrt{a^2-x^2}\sinh t\right)K_{2\nu}(x)}{\sqrt{a^2-x^2}}\,dx = \frac{\pi^2}{4}\operatorname{cosec}(\nu\pi)\left[I_{-\nu}\left(ae^t\right)I_{-\nu}\left(ae^{-t}\right) - I_\nu\left(ae^t\right)I_\nu\left(ae^{-t}\right)\right]$$
$$\left[|\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$$ ET II 367(25)

Exponential, hyperbolic, and Bessel functions

6.668 **Notation**:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \qquad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$

1.[10] $$\int_0^\infty e^{-\alpha x}\sinh(\beta x)\,J_0(\gamma x)\,dx=(\alpha\beta)^{\frac12}r_1^{-1}r_2^{-1}(r_2-r_1)^{\frac12}(r_2+r_1)^{-\frac12}$$
$$r_1=\sqrt{\gamma^2+(\beta-\alpha)^2},\qquad r_2=\sqrt{\gamma^2+(\beta+\alpha)^2},\qquad [\operatorname{Re}\alpha>|\operatorname{Re}\beta|,\quad \gamma>0]$$ ET II 12(52)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,
$$\int_0^\infty e^{-cx}\sinh(bx)\,J_0(ax)\,dx=\frac{\ell_1}{\ell_2^2-\ell_1^2}$$
$$[\operatorname{Re}c>|\operatorname{Re}b|,\quad a>0]$$

2.[10] $$\int_0^\infty e^{-\alpha x}\cosh(\beta x)\,J_0(\gamma x)\,dx=(\alpha\beta)^{\frac12}r_1^{-1}r_2^{-1}(r_2-r_1)^{\frac12}(r_2+r_1)^{-\frac12}$$
$$r_1=\sqrt{\gamma^2+(\beta-\alpha)^2},\qquad r_2=\sqrt{\gamma^2+(\beta+\alpha)^2},\qquad [\operatorname{Re}\alpha>|\operatorname{Re}\beta|,\quad \gamma>0]$$ ET II 12(54)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,
$$\int_0^\infty e^{-cx}\cosh(bx)\,J_0(ax)\,dx=\frac{\ell_2}{\ell_2^2-\ell_1^2}$$
$$[\operatorname{Re}c>|\operatorname{Re}b|,\quad a>0]$$

6.669

1. $$\int_0^\infty\left[\coth\left(\frac12 x\right)\right]^{2\lambda}e^{-\beta\cosh x}\,J_{2\mu}(\alpha\sinh x)\,dx=\frac{\Gamma\left(\frac12-\lambda+\mu\right)}{\alpha\,\Gamma(2\mu+1)}M_{-\lambda,\mu}\left[\left(\alpha^2+\beta^2\right)^{\frac12}-\beta\right]$$
$$\times W_{\lambda,\mu}\left[\left(\alpha^2+\beta^2\right)^{\frac12}+\beta\right]$$
$$\left[\operatorname{Re}\beta>|\operatorname{Re}\alpha|,\quad \operatorname{Re}(\mu-\lambda)>-\tfrac12\right]$$ BU 86(5b)a, ET II 363(34)

2. $$\int_0^\infty\left[\coth\left(\frac12 x\right)\right]^{2\lambda}e^{-\beta\cosh x}\,Y_{2\mu}(\alpha\sinh x)\,dx$$
$$=-\frac{\sec[(\mu+\lambda)\pi]}{\alpha}W_{\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}+\beta\right)W_{-\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}-\beta\right)$$
$$-\frac{\tan[(\mu+\lambda)\pi]\,\Gamma\left(\frac12-\lambda+\mu\right)}{\alpha\,\Gamma(2\mu+1)}W_{\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}+\beta\right)M_{-\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}-\beta\right)$$
$$\left[\operatorname{Re}\beta>|\operatorname{Re}\alpha|,\quad \operatorname{Re}\lambda<\tfrac12-|\operatorname{Re}\mu|\right]$$ ET II 363(35)

3. $$\int_0^\infty e^{-\frac12(a_1a_2)t\cosh x}\left[\coth\left(\frac12 x\right)\right]^{2\nu}K_{2\mu}(t\sqrt{a_1a_2}\sinh x)\,dx$$
$$=\frac{\Gamma\left(\frac12+\mu-\nu\right)\Gamma\left(\frac12-\mu-\nu\right)}{2t\sqrt{a_1a_2}}W_{\nu,\mu}(a_1t)\,W_{\nu,\mu}(a_2t)$$
$$\left[\operatorname{Re}\nu<\operatorname{Re}\frac{1\pm2\mu}{2},\quad \operatorname{Re}\left[t\left(\sqrt{a_1}+\sqrt{a_2}\right)^2\right]>0\right]$$ BU 85(4a)

4. $$\int_0^\infty e^{-\frac12(a_1a_2)t\cosh x}\left[\coth\left(\frac{x}{2}\right)\right]^{2\nu}I_{2\mu}(t\sqrt{a_1a_2}\sinh x)\,dx=\frac{\Gamma\left(\frac12+\mu-\nu\right)}{t\sqrt{a_1a_2}\,\Gamma(1+2\mu)}W_{\nu,\mu}(a_1t)\,M_{\nu,\mu}(a_2t)$$
$$\left[\operatorname{Re}\left(\tfrac12+\mu-\nu\right)>0,\quad \operatorname{Re}\mu>0,\quad a_1>a_2\right]$$ BU 86(5c)

5. $$\int_{-\infty}^\infty e^{2\nu s-\frac{x-y}{2}\tanh s}\,I_{2\mu}\left(\frac{\sqrt{xy}}{\cosh s}\right)\frac{ds}{\cosh s}=\frac{\Gamma\left(\frac12+\mu+\nu\right)\Gamma\left(\frac12+\mu-\nu\right)}{\sqrt{xy}\,[\Gamma(1+2\mu)]^2}M_{\nu,\mu}(x)\,M_{-\nu,\mu}(y)$$
$$\left[\operatorname{Re}\left(\pm\nu+\tfrac12+\mu\right)>0\right]$$ BU 83(3a)a

6. $$\int_{-\infty}^{\infty} e^{2\nu s-\frac{x+y}{2}\tanh s}\, J_{2\mu}\left(\frac{\sqrt{xy}}{\cosh s}\right)\frac{ds}{\cosh s} = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\sqrt{xy}\,[\Gamma(1+2\mu)]^2}\, M_{\nu,\mu}(x)\, M_{\nu,\mu}(y)$$
$$\left[\operatorname{Re}\left(\mp\nu+\tfrac{1}{2}+\mu\right)>0\right]$$ BU 84(3b)a

6.67–6.68 Combinations of Bessel and trigonometric functions

6.671

1. $$\int_0^\infty J_\nu(\alpha x)\sin\beta x\,dx = \frac{\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} \qquad [\beta<\alpha]$$
$$= \infty \text{ or } 0 \qquad [\beta=\alpha]$$
$$= \frac{\alpha^\nu\cos\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta>\alpha]$$
$$[\operatorname{Re}\nu>-2]$$ WA 444(4)

2. $$\int_0^\infty J_\nu(\alpha x)\cos\beta x\,dx = \frac{\cos\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} \qquad [\beta<\alpha]$$
$$= \infty \text{ or } 0 \qquad [\beta=\alpha]$$
$$= \frac{-\alpha^\nu\sin\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta>\alpha]$$
$$[\operatorname{Re}\nu>-1]$$ WA 444(5)

3. $$\int_0^\infty Y_\nu(ax)\sin(bx)\,dx$$
$$= \cot\left(\frac{\nu\pi}{2}\right)\left(a^2-b^2\right)^{-\frac{1}{2}}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a, |\operatorname{Re}\nu|<2]$$
$$= \frac{1}{2}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\left(b^2-a^2\right)^{-\frac{1}{2}}$$
$$\times\left\{a^{-\nu}\cos(\nu\pi)\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu - a^\nu\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^{-\nu}\right\} \qquad [0<a<b, \quad |\operatorname{Re}\nu|<2]$$
ET I 103(33)

4. $$\int_0^\infty Y_\nu(ax)\cos(bx)\,dx$$
$$= \frac{\tan\left(\frac{\nu\pi}{2}\right)}{(a^2-b^2)^{\frac{1}{2}}}\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a, \quad |\operatorname{Re}\nu|<1]$$
$$= -\sin\left(\frac{\nu\pi}{2}\right)\left(b^2-a^2\right)^{-\frac{1}{2}}\left\{a^{-\nu}\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \cot(\nu\pi)\right.$$
$$\left.+a^\nu\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^{-\nu}\operatorname{cosec}(\nu\pi)\right\} \qquad [0<a<b, \quad |\operatorname{Re}\nu|<1]$$
ET I 47(29)

5. $$\int_0^\infty K_\nu(ax)\sin(bx)\,dx$$
$$= \frac{1}{4}\pi a^{-\nu}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\left(a^2+b^2\right)^{-\frac{1}{2}}\left\{\left[\left(b^2+a^2\right)^{\frac{1}{2}}+b\right]^\nu-\left[\left(b^2+a^2\right)^{\frac{1}{2}}-b\right]^\nu\right\}$$
$[\operatorname{Re} a>0,\quad b>0,\quad |\operatorname{Re}\nu|<2,\quad \nu\neq 0]$ ET I 105(48)

6. $$\int_0^\infty K_\nu(ax)\cos(bx)\,dx$$
$$= \frac{\pi}{4}\left(b^2+a^2\right)^{-\frac{1}{2}}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\left[b+\left(b^2+a^2\right)^{\frac{1}{2}}\right]^\nu+a^\nu\left[b+\left(b^2+a^2\right)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$[\operatorname{Re} a>0, b>0, |\operatorname{Re}\nu|<1]$ ET I 49(40)

7. $$\int_0^\infty J_0(ax)\sin(bx)\,dx = 0 \qquad [0<b<a]$$
$$= \frac{1}{\sqrt{b^2-a^2}} \qquad [0<a<b]$$
ET I 99(1)

8. $$\int_0^\infty J_0(ax)\cos(bx)\,dx = \frac{1}{\sqrt{a^2-b^2}} \qquad [0<b<a]$$
$$= \infty \qquad [a=b]$$
$$= 0 \qquad [0<a<b]$$
ET I 43(1)

9. $$\int_0^\infty J_{2n+1}(ax)\sin(bx)\,dx = (-1)^n\frac{1}{\sqrt{a^2-b^2}}\,T_{2n+1}\left(\frac{b}{a}\right) \qquad [0<b<a]$$
$$= 0 \qquad [0<a<b]$$
ET I 99(2)

10. $$\int_0^\infty J_{2n}(ax)\cos(bx)\,dx = (-1)^n\frac{1}{\sqrt{a^2-b^2}}\,T_{2n}\left(\frac{b}{a}\right) \qquad [0<b<a]$$
$$= 0 \qquad [0<a<b]$$
ET I 43(2)

11. $$\int_0^\infty Y_0(ax)\sin(bx)\,dx = \frac{2\arcsin\left(\frac{b}{a}\right)}{\pi\sqrt{a^2-b^2}} \qquad [0<b<a]$$
$$= \frac{2}{\pi}\frac{1}{\sqrt{b^2-a^2}}\ln\left[\frac{b}{a}-\sqrt{\frac{b^2}{a^2}-1}\right] \qquad [0<a<b]$$
ET I 103(31)

12. $$\int_0^\infty Y_0(ax)\cos(bx)\,dx = 0 \qquad [0<b<a]$$
$$= -\frac{1}{\sqrt{b^2-a^2}} \qquad [0<a<b]$$
ET I 47(28)

13. $$\int_0^\infty K_0(\beta x)\sin\alpha x\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}+1}\right)$$
$[\alpha>0,\quad \beta>0]$ WA 425(11)a, MO 48

14.[8] $$\int_0^\infty K_0(\beta x)\cos\alpha x\,dx = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}}$$
$[\alpha>0]$ WA 425(10)a, MO 48

6.672

1. $$\int_0^\infty J_\nu(ax)\,J_\nu(bx)\sin(cx)\,dx$$
$$= 0 \qquad [\operatorname{Re}\nu>-1,\quad 0<c<b-a,\quad 0<a<b]$$
$$= \frac{1}{2\sqrt{ab}}\,P_{\nu-\frac12}\left(\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu>-1,\quad b-a<c<b+a,\quad 0<a<b]$$
$$= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}}\,Q_{\nu-\frac12}\left(-\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu>-1,\quad b+a<c,\quad 0<a<b]$$
ET I 102(27)

2. $$\int_0^\infty J_\nu(x)\,J_{-\nu}(x)\cos(bx)\,dx = \frac12\,P_{\nu-\frac12}\left(\frac12 b^2-1\right) \qquad [0<b<2]$$
$$= 0 \qquad [2<b]$$
ET I 46(21)

3. $$\int_0^\infty K_\nu(ax)\,K_\nu(bx)\cos(cx)\,dx = \frac{\pi^2}{4\sqrt{ab}}\sec(\nu\pi)\,P_{\nu-\frac12}\left[\left(a^2+b^2+c^2\right)(2ab)^{-1}\right]$$
$[\operatorname{Re}(a+b)>0,\quad c>0,\quad |\operatorname{Re}\nu|<\frac12]$
ET I 50(51)

4. $$\int_0^\infty K_\nu(ax)\,I_\nu(bx)\cos(cx)\,dx = \frac{1}{2\sqrt{ab}}\,Q_{\nu-\frac12}\left(\frac{a^2+b^2+c^2}{2ab}\right)$$
$[\operatorname{Re}a>|\operatorname{Re}b|,\quad c>0,\quad \operatorname{Re}\nu>-\frac12]$
ET I 49(47)

5. $$\int_0^\infty \sin(2ax)\,[J_\nu(x)]^2\,dx = \frac12\,P_{\nu-\frac12}\left(1-2a^2\right) \qquad [0<a<1,\quad \operatorname{Re}\nu>-1]$$
$$= \frac{1}{\pi}\cos(\nu\pi)\,Q_{\nu-\frac12}\left(2a^2-1\right) \qquad [a>1,\quad \operatorname{Re}\nu>-1]$$
ET II 343(30)

6. $$\int_0^\infty \cos(2ax)\,[J_\nu(x)]^2\,dx = \frac{1}{\pi}\,Q_{\nu-\frac12}\left(1-2a^2\right) \qquad [0<a<1,\quad \operatorname{Re}\nu>-\tfrac12]$$
$$= -\frac{1}{\pi}\sin(\nu\pi)\,Q_{\nu-\frac12}\left(2a^2-1\right) \qquad [a>1,\quad \operatorname{Re}\nu>-\tfrac12]$$
ET II 344(32)

7. $$\int_0^\infty \sin(2ax)\,J_0(x)\,Y_0(x)\,dx = 0 \qquad [0<a<1]$$
$$= -\frac{\boldsymbol{K}\left[\left(1-a^{-2}\right)^{\frac12}\right]}{\pi a} \qquad [a>1]$$
ET II 348(60)

8. $$\int_0^\infty K_0(ax)\, I_0(bx)\cos(cx)\, dx = \frac{1}{\sqrt{c^2+(a+b)^2}}\, \boldsymbol{K}\left\{\frac{2\sqrt{ab}}{\sqrt{c^2+(a+b)^2}}\right\}$$
$[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0]$ ET I 49(46)

9. $$\int_0^\infty \cos(2ax)\, J_0(x)\, Y_0(x)\, dx = -\frac{1}{\pi}\, \boldsymbol{K}(a) \qquad [0 < a < 1]$$
$$= -\frac{1}{\pi a}\, \boldsymbol{K}\left(\frac{1}{a}\right) \qquad [a > 1]$$
ET II 348(61)

10. $$\int_0^\infty \cos(2ax)\, [Y_0(x)]^2\, dx = \frac{1}{\pi}\, \boldsymbol{K}\left(\sqrt{1-a^2}\right) \qquad [0 < a < 1]$$
$$= \frac{2}{\pi a}\, \boldsymbol{K}\left(\sqrt{1-\frac{1}{a^2}}\right) \qquad [a > 1]$$
ET II 348(62)

6.673

1. $$\int_0^\infty \left[J_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) - Y_\nu(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\sin(bx)\, dx$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 2]$$
$$= \frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b+\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 2]$$
ET I 104(39)

2. $$\int_0^\infty \left[Y_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) + J_\nu(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\cos(bx)\, dx$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 1]$$
$$= -\frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b+\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 1]$$
ET I 48(32)

3.* $$\int_0^{\pi/2} [\cos x\, I_0(a\cos x) + I_1(a\cos x)]\, dx = \frac{e^a-1}{a}$$

6.674

1. $$\int_0^a \sin(a-x)\, J_\nu(x)\, dx = a\, J_{\nu+1}(a) - 2\nu\sum_{n=0}^\infty (-1)^n\, J_{\nu+2n+2}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 334(12)

2. $$\int_0^a \cos(a-x)\, J_\nu(x)\, dx = a\, J_\nu(a) - 2\nu\sum_{n=0}^\infty (-1)^n\, J_{\nu+2n+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 336(23)

3. $$\int_0^a \sin(a-x)\, J_{2n}(x)\, dx = a\, J_{2n+1}(a) + (-1)^n 2n\left[\cos a - J_0(a) - 2\sum_{m=1}^n (-1)^m\, J_{2m}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 334(10)

4. $$\int_0^a \cos(a-x)\, J_{2n}(x)\, dx = a\, J_{2n}(a) - (-1)^n 2n \left[\sin a - 2\sum_{m=0}^{n-1} (-1)^m\, J_{2m+1}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 335(21)

5. $$\int_0^a \sin(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+2}(a) + (-1)^n (2n+1) \left[\sin a - 2\sum_{m=0}^{n} (-1)^m\, J_{2m+1}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 334(11)

6. $$\int_0^a \cos(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+1}(a) + (-1)^n (2n+1) \left[\cos a - J_0(a) - 2\sum_{m=1}^{n} (-1)^m\, J_{2m}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 336(22)

7. $$\int_0^z \sin(z-x)\, J_0(x)\, dx = z\, J_1(z)$$ WA 415(2)

8. $$\int_0^z \cos(z-x)\, J_0(x)\, dx = z\, J_0(z)$$ WA 415(1)

6.675

1. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -4]$
ET I 110(23)

2. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) \cos(bx)\, dx$$
$$= -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) + \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -2]$ ET I 53(22)a

3. $$\int_0^\infty J_0\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{1}{b} \cos\left(\frac{a^2}{4b}\right)$$ $[a > 0, \quad b > 0]$ ET I 110(22)

4. $$\int_0^\infty J_0\left(a\sqrt{x}\right) \cos(bx)\, dx = \frac{1}{b} \sin\left(\frac{a^2}{4b}\right)$$ $[a > 0, \quad b > 0]$ ET I 53(21)

6.676

1. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{x}\right) \sin(cx)\, dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \cos\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -2]$
ET I 111(29)a

2. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{x}\right) \cos(cx)\, dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \sin\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -1]$
ET I 54(27)

3. $$\int_0^\infty J_0\left(a\sqrt{x}\right) K_0\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{1}{2b} K_0\left(\frac{a^2}{2b}\right)$$ $[\operatorname{Re} a > 0, \quad b > 0]$ ET I 111(31)

4. $$\int_0^\infty J_0\left(\sqrt{ax}\right) K_0\left(\sqrt{ax}\right)\cos(bx)\,dx = \frac{\pi}{4b}\left[I_0\left(\frac{a}{2b}\right) - \mathbf{L}_0\left(\frac{a}{2b}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0]$ ET I 54(29)

5. $$\int_0^\infty K_0\left(\sqrt{ax}\right) Y_0\left(\sqrt{ax}\right)\cos(bx)\,dx = -\frac{1}{2b} K_0\left(\frac{a}{2b}\right) \qquad \left[\operatorname{Re}\sqrt{a} > 0, \quad b > 0\right]$$
ET I 54(30)

6. $$\int_0^\infty K_0\left(\sqrt{axe^{\frac{1}{4}\pi i}}\right) K_0\left(\sqrt{axe^{-\frac{1}{4}\pi i}}\right)\cos(bx)\,dx = \frac{\pi^2}{8b}\left[\mathbf{H}_0\left(\frac{a}{2b}\right) - Y_0\left(\frac{a}{2b}\right)\right]$$
$[\operatorname{Re} a > 0, b > 0]$ ET I 54(31)

6.677

1. $$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\sin(cx)\,dx = 0 \qquad [0 < c < b]$$
$$= \frac{\cos\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c]$$
ET I 113(47)

2. $$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\exp\left(-a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b]$$
$$= \frac{-\sin\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c]$$
ET I 57(48)a

3.[6] $$\int_0^\infty J_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{\cos z\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-\beta^2}} \qquad [0 < \beta < \alpha, \quad z > 0]$$
$$= 0 \qquad [0 < \alpha < \beta, \quad z > 0]$$
MO 47a

4. $$\int_0^\infty Y_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right) \qquad [0 < \beta < \alpha, \quad z > 0]$$
$$= -\frac{1}{\sqrt{\beta^2-\alpha^2}}\exp\left(-z\sqrt{\beta^2-\alpha^2}\right) \qquad [0 < \alpha < \beta, \quad z > 0]$$
MO 47a

5. $$\int_0^\infty K_0\left[\alpha\sqrt{x^2+\beta^2}\right]\cos(\gamma x)\,dx = \frac{\pi}{2\sqrt{\alpha^2+\gamma^2}}\exp\left(-\beta\sqrt{\alpha^2+\gamma^2}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0]$
ET I 56(43)

6. $$\int_0^a J_0\left(b\sqrt{a^2-x^2}\right)\cos(cx)\,dx = \frac{\sin\left(a\sqrt{b^2+c^2}\right)}{\sqrt{b^2+c^2}} \qquad [b > 0]$$
MO 48a, ET I 57(47)

7. $$\int_0^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\cosh\left(a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b, \quad a > 0]$$
$$= 0 \qquad [0 < b < c, \quad a > 0]$$
ET I 57(49)

8. $$\int_0^\infty H_0^{(1)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx=-i\frac{\exp\left(i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[\pi>\arg\sqrt{\beta^2-x^2}\geq 0,\quad \alpha>0,\quad \gamma>0\right]$$ ET I 59(59)

9. $$\int_0^\infty H_0^{(2)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx=\frac{i\exp\left(-i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[-\pi<\arg\sqrt{\beta^2-x^2}\leq 0,\quad \alpha>0,\quad \gamma>0\right]$$ ET I 58(58)

6.678 $$\int_0^\infty\left[K_0\left(2\sqrt{x}\right)+\frac{\pi}{2}\,Y_0\left(2\sqrt{x}\right)\right]\sin(bx)\,dx=\frac{\pi}{2b}\sin\left(\frac{1}{b}\right)\qquad[b>0]$$ ET I 111(34)

6.679

1. $$\int_0^\infty J_{2\nu}\left[2b\sinh\left(\frac{x}{2}\right)\right]\sin(bx)\,dx=-i\left[I_{\nu-ib}(a)\,K_{\nu+ib}(a)-I_{\nu+ib}(a)\,K_{\nu-ib}(a)\right]$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\nu>-1]$$ ET I 115(59)

2. $$\int_0^\infty J_{2\nu}\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx=I_{\nu-ib}(a)\,K_{\nu+ib}(a)+I_{\nu+ib}(a)\,K_{\nu-ib}(a)$$
$$\left[a>0,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET I 59(64)

3. $$\int_0^\infty J_{2\nu}\left[2a\cosh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx=-\frac{\pi}{2}\left[J_{\nu+ib}(a)\;Y_{\nu-ib}(a)+J_{\nu-ib}(a)\;Y_{\nu+ib}(a)\right]$$ ET I 59(63)

4. $$\int_0^\infty J_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\sin(bx)\,dx=\frac{2}{\pi}\sinh(\pi b)\left[K_{ib}(a)\right]^2$$
$$[a>0,\quad b>0]$$ ET I 115(58)

5. $$\int_0^\infty J_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx=\left[I_{ib}(a)+I_{-ib}(a)\right]K_{ib}(a)$$
$$[a>0,\quad b>0]$$ ET I 59(62)

6. $$\int_0^\infty Y_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx=-\frac{2}{\pi}\cosh(\pi b)\left[K_{ib}(a)\right]^2$$
$$[a>0,\quad b>0]$$ ET I 59(65)

7. $$\int_0^\infty K_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx=\frac{\pi^2}{4}\left\{\left[J_{ib}(a)\right]^2+\left[Y_{ib}(a)\right]^2\right\}$$
$$[\operatorname{Re}a>0,\quad b>0]$$ ET I 59(66)

6.681

1. $$\int_0^{\frac{\pi}{2}}\cos(2\mu x)\,J_{2\nu}\left(2a\cos x\right)\,dx=\frac{\pi}{2}\,J_{\nu+\mu}(a)\,J_{\nu-\mu}(a)\qquad\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 361(23)

2. $$\int_0^{\frac{\pi}{2}}\cos(2\mu x)\,Y_{2\nu}\left(2a\cos x\right)\,dx=\frac{\pi}{2}\left[\cot(2\nu\pi)\,J_{\nu+\mu}(a)\,J_{\nu-\mu}(a)-\operatorname{cosec}(2\nu\pi)\,J_{\mu-\nu}(a)\,J_{-\mu-\nu}(a)\right]$$
$$\left[|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET II 361(24)

3. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(2\mu x)\, I_{2\nu}\,(2a\cos x)\; dx = \frac{\pi}{2}\, I_{\nu-\mu}(a)\, I_{\nu+\mu}(a)$ $\quad\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET I 59(61)

4. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(\nu x)\, K_{\nu}\,(2a\cos x)\; dx = \frac{\pi}{2}\, I_0(a)\, K_{\nu}(a)$ $\quad[\operatorname{Re}\nu < 1]$ WA 484(3)

5. $\displaystyle\int_0^{\pi} J_0\,(2z\cos x)\cos 2nx\, dx = (-1)^n \pi J_n^2(z).$ MO 45

6. $\displaystyle\int_0^{\pi} J_0\,(2z\sin x)\cos 2nx\, dx = \pi J_n^2(z).$ WA 43(3), MO 45

7. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(2n\pi)\; Y_0\,(2a\sin x)\; dx = \frac{\pi}{2}\, J_n(a)\; Y_n(a)$ $\quad[n = 0, 1, 2, \ldots]$ ET II 360(16)

8. $\displaystyle\int_0^{\pi} \sin(2\mu x)\, J_{2\nu}\,(2a\sin x)\; dx = \pi\sin(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$

$[\operatorname{Re}\nu > -1]$ ET II 360(13)

9. $\displaystyle\int_0^{\pi} \cos(2\mu x)\, J_{2\nu}\,(2a\sin x)\; dx = \pi\cos(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 360(14)

10. $\displaystyle\int_0^{\frac{\pi}{2}} J_{\nu+\mu}\,(2z\cos x)\cos[(\nu-\mu)x]\, dx = \frac{\pi}{2}\, J_{\nu}(z)\, J_{\mu}(z)$ $\quad[\operatorname{Re}(\nu+\mu) > -1]$ MO 42

11. $\displaystyle\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x]\, I_{\mu+\nu}\,(2a\cos x)\; dx = \frac{\pi}{2}\, I_{\mu}(a)\, I_{\nu}(a)$ $\quad[\operatorname{Re}(\mu+\nu) > -1]$

WA 484(2), ET II 378(39)

12. $\displaystyle\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x]\, K_{\mu+\nu}\,(2a\cos x)\; dx = \frac{\pi^2}{4}\operatorname{cosec}[(\mu+\nu)\pi]\,[I_{-\mu}(a)\, I_{-\nu}(a) - I_{\mu}(a)\, I_{\nu}(a)]$

$[|\operatorname{Re}(\mu+\nu)| < 1]$ ET II 378(40)

13.[8] $\displaystyle\int_0^{\frac{\pi}{2}} K_{\nu-m}\,(2a\cos x)\cos[(m+\nu)x]\, dx = (-1)^m \frac{\pi}{2}\, I_m(a)\, K_{\nu}(a)$

$[|\operatorname{Re}(\nu-m)| < 1]$ WA 485(4)

6.682

1.[7] $\displaystyle\int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}\,(x\sin t)\sin^{\nu+\frac{1}{2}} t\, dt = \sqrt{\frac{\pi}{2x}}\, J_{\nu}(x)$

[ν may be zero, a natural number, one half, or a natural number plus one half; $x > 0$] MO 42a

2. $\displaystyle\int_0^{\frac{\pi}{2}} J_{\nu}\,(z\sin x)\sin^{\nu} x\cos^{2\nu} x\, dx = 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) z^{-\nu} J_{\nu}^2\left(\frac{z}{2}\right)$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ MO 42a

6.683

1. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin x)\, I_\mu(z\cos x)\tan^{\nu+1} x\, dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} J_\mu(z)$$
$[\operatorname{Re}\nu > \operatorname{Re}\mu > -1]$ WA 407(4)

2. $$\int_0^{\frac{\pi}{2}} J_\nu(z_1\sin x)\, J_\mu(z_2\cos x)\sin^{\nu+1} x\cos^{\mu+1} x\, dx = \frac{z_1^\nu z_2^\mu\, J_{\nu+\mu+1}\left(\sqrt{z_1^2+z_2^2}\right)}{\sqrt{(z_1^2+z_2^2)^{\nu+\mu+1}}}$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ WA 410(1)

3. $$\int_0^{\frac{\pi}{2}} J_\nu\left(z\cos^2 x\right) J_\mu\left(z\sin^2 x\right)\sin x\cos x\, dx = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k J_{\nu+\mu+2k+1}(z)$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ (see also **6.513** 6) WA 414(1)

4. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}(\cos\theta)^{2\nu+1}\, d\theta = \frac{s_{\mu+\nu,\nu-\mu+1}(z)}{2^{\mu-1}z^{\nu+1}\Gamma(\mu)}$$
$[\operatorname{Re}\nu > -1]$ WA 407(2)

5. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}\, d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\frac{2z}{\pi}}}$$
WA 407(3)

6. $$\int_0^{\frac{\pi}{2}} J_\mu(a\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\varrho+1}\, d\theta = 2^\varrho\,\Gamma(\varrho+1)a^{-\varrho-1} J_{\varrho+\mu+1}(a)$$
$[\operatorname{Re}\varrho > -1, \quad \operatorname{Re}\mu > -1]$
WA 406(1), EH II 46(5)

7. $$\int_0^{\frac{\pi}{2}} J_\nu(2z\sin\theta)(\sin\theta)^\nu(\cos\theta)^{2\nu}\, d\theta$$
$$= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m z^{\nu+2m}\,\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{m!\,\Gamma(\nu+m+1)\,\Gamma(2\nu+m+1)}$$
$$= \frac{1}{2}z^{-\nu}\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{1}{2}\right)[J_\nu(z)]^2 \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
EH II 47(10)

8. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{-2\nu}\, d\theta = 2^{-\nu}\frac{z^{\nu-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\sin z$$
$\left[-1 < \operatorname{Re}\nu < \frac{1}{2}\right]$ EH II 68(39)

9. $$\int_0^{\frac{\pi}{2}} J_\nu\left(z\sin^2\theta\right) J_\nu\left(z\cos^2\theta\right)(\sin\theta)^{2\nu+1}(\cos\theta)^{2\nu+1}\, d\theta = \frac{\Gamma\left(\frac{1}{2}+\nu\right) J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}}\,\Gamma(\nu+1)\sqrt{z}}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 409(1)

10. $$\int_0^{\frac{\pi}{2}} J_\mu\left(z\sin^2\theta\right) J_\nu\left(z\cos^2\theta\right) \sin^{2\mu+1}\theta\cos^{2\nu+1}\theta\, d\theta = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right) J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi}\,\Gamma(\mu+\nu+1)\sqrt{2z}}$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 417(1)

6.684

1.[8] $$\int_0^{\pi} (\sin x)^{2\nu} \frac{J_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\, dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{J_\nu(\beta)}{\beta^\nu}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 362(27)

2. $$\int_0^{\pi} (\sin x)^{2\nu} \frac{Y_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\, dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{Y_\nu(\beta)}{\beta^\nu}$$
$\left[|\alpha| < |\beta|, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 362(28)

6.685 $$\int_0^{\frac{\pi}{2}} \sec x\cos(2\lambda x)\, K_{2\mu}(a\sec x)\, dx = \frac{\pi}{2a}\, W_{\lambda,\mu}(a)\, W_{-\lambda,\mu}(a) \qquad [\operatorname{Re} a > 0]$$ ET II 378(41)

6.686

1. $$\int_0^\infty \sin\left(ax^2\right) J_\nu(bx)\, dx = -\frac{\sqrt{\pi}}{2\sqrt{a}}\sin\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)$$
$[a>0, b>0, \operatorname{Re}\nu > -3]$ ET II 34(13)

2. $$\int_0^\infty \cos\left(ax^2\right) J_\nu(bx)\, dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\cos\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)$$
$[a>0, \quad b>0, \quad \operatorname{Re}\nu > -1]$
ET II 38(38)

3. $$\int_0^\infty \sin\left(ax^2\right) Y_\nu(bx)\, dx$$
$$= -\frac{\sqrt{\pi}}{4\sqrt{a}}\sec\left(\frac{\nu\pi}{2}\right) \times\left[\cos\left(\frac{b^2}{8a}-\frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a}+\frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0, \quad b>0, \quad -3 < \operatorname{Re}\nu < 3]$ ET II 107(7)

4. $$\int_0^\infty \cos\left(ax^2\right) Y_\nu(bx)\, dx$$
$$= \frac{\sqrt{\pi}}{4\sqrt{a}}\sec\left(\frac{\nu\pi}{2}\right) \times\left[\sin\left(\frac{b^2}{8a}-\frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) + \cos\left(\frac{b^2}{8a}+\frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0, \quad b>0, \quad -1 < \operatorname{Re}\nu < 1]$ ET II 107(8)

5. $$\int_0^\infty \sin\left(ax^2\right) J_1(bx)\, dx = \frac{1}{b}\sin\frac{b^2}{4a} \qquad [a>0, \quad b>0]$$ ET II 19(16)

6. $$\int_0^\infty \cos\left(ax^2\right) J_1(bx)\,dx = \frac{2}{b}\sin^2\left(\frac{b^2}{8a}\right) \qquad [a>0, \quad b>0]$$ ET II 20(20)

7. $$\int_0^\infty \sin^2\left(ax^2\right) J_1(bx)\,dx = \frac{1}{2b}\cos\left(\frac{b^2}{8a}\right) \qquad [a>0, \quad b>0]$$ ET II 19(17)

6.687 $$\int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}\left(xe^{i\frac{\pi}{4}}\right) K_{2\nu}\left(xe^{-i\frac{\pi}{4}}\right)\,dx$$
$$= \frac{\Gamma\left(\frac{1}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)\sqrt{\pi}}{8\sqrt{a}}\, W_{\frac{1}{4},\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\frac{1}{4},\nu}\left(ae^{-i\frac{\pi}{2}}\right)$$
$$\left[a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{4}\right]$$ ET II 372(1)

6.688

1. $$\int_0^{\frac{\pi}{2}} J_\nu\left(\mu z\sin t\right)\cos\left(\mu x\cos t\right)\,dt = \frac{\pi}{2}\, J_{\frac{\nu}{2}}\left(\mu\frac{\sqrt{x^2+z^2}+x}{2}\right) J_{\frac{\nu}{2}}\left(\mu\frac{\sqrt{x^2+z^2}-x}{2}\right)$$
$$[\operatorname{Re}\nu>-1, \quad \operatorname{Re}z>0]$$ MO 46

2. $$\int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1}\cos\left(\beta\cos x\right) J_\nu\left(\alpha\sin x\right)\,dx = 2^{-\frac{1}{2}}\sqrt{\pi}\alpha^\nu\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} J_{\nu+\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}\right]$$
$$[\operatorname{Re}\nu>-1]$$ ET II 361(19)

3. $$\int_0^{\frac{\pi}{2}} \cos\left[(z-\zeta)\cos\theta\right] J_{2\nu}\left[2\sqrt{z\zeta}\sin\theta\right]\,d\theta = \frac{\pi}{2}\, J_\nu(z)\, J_\nu(\zeta)$$
$$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ EH II 47(8)

6.69–6.74 Combinations of Bessel and trigonometric functions and powers

6.691 $$\int_0^\infty x\sin(bx)\, K_0(ax)\,dx = \frac{\pi b}{2}\left(a^2+b^2\right)^{-\frac{3}{2}} \qquad [\operatorname{Re}a>0, \quad b>0]$$ ET I 105(47)

6.692

1. $$\int_0^\infty x\, K_\nu(ax)\, I_\nu(bx)\sin(cx)\,dx = -\frac{1}{2}(ab)^{-\frac{3}{2}}c\left(u^2-1\right)^{-\frac{1}{2}} Q^1_{\nu-\frac{1}{2}}(u), \qquad u=(2ab)^{-1}\left(a^2+b^2+c^2\right)$$
$$\left[\operatorname{Re}a>|\operatorname{Re}b|, \quad c>0, \quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$ ET I 106(54)

2. $$\int_0^\infty x\, K_\nu(ax)\, K_\nu(bx)\sin(cx)\,dx = \frac{\pi}{4}(ab)^{-\frac{3}{2}}c\left(u^2-1\right)^{-\frac{1}{2}}\Gamma\left(\tfrac{3}{2}+\nu\right)\Gamma\left(\tfrac{3}{2}-\nu\right) P^{-1}_{\nu-\frac{1}{2}}(u)$$
$$u=(2ab)^{-1}\left(a^2+b^2+c^2\right) \qquad \left[\operatorname{Re}(a+b)>0, \quad c>0, \quad |\operatorname{Re}\nu|<\tfrac{3}{2}\right]$$ ET I 107(61)

6.693

1. $$\int_0^\infty J_\nu(\alpha x)\sin\beta x\frac{dx}{x} = \frac{1}{\nu}\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right) \qquad [\beta\le\alpha]$$
$$= \frac{\alpha^\nu\sin\frac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta\ge\alpha]$$
$$[\operatorname{Re}\nu>-1]$$ WA 443(2)

2.[8] $$\int_0^\infty J_\nu(\alpha x)\cos\beta x\frac{dx}{x} = \frac{1}{\nu}\cos\left(\nu\arcsin\frac{\beta}{\alpha}\right) \qquad [\beta\le\alpha]$$
$$= \frac{\alpha^\nu\cos\frac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta\ge\alpha] \qquad [\operatorname{Re}\nu>0]$$

WA 443(3)

3. $$\int_0^\infty Y_\nu(ax)\sin(bx)\frac{dx}{x}$$
$$= -\frac{1}{\nu}\tan\left(\frac{\nu\pi}{2}\right)\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a, \quad |\operatorname{Re}\nu|<1]$$
$$= \frac{1}{2\nu}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\cos(\nu\pi)\left[b-\left(b^2-a^2\right)^{\frac12}\right]^\nu - a^\nu\left[b-\left(b^2-a^2\right)^{\frac12}\right]^{-\nu}\right\} \qquad [0<a<b, \quad |\operatorname{Re}\nu|<1]$$

ET I 103(35)

4. $$\int_0^\infty J_\nu(ax)\sin(bx)\frac{dx}{x^2}$$
$$= \frac{\sqrt{a^2-b^2}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu^2-1} - \frac{b\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu\left(\nu^2-1\right)} \qquad [0<b<a, \quad \operatorname{Re}\nu>0]$$
$$= \frac{-a^\nu\cos\left(\frac{\nu\pi}{2}\right)\left[b+\nu\sqrt{b^2-a^2}\right]}{\nu\left(\nu^2-1\right)\left[b+\sqrt{b^2-a^2}\right]^\nu} \qquad [0<a<b, \quad \operatorname{Re}\nu>0]$$

ET I 99(6)

5. $$\int_0^\infty J_\nu(ax)\cos(bx)\frac{dx}{x^2}$$
$$= \frac{a\cos\left[(\nu-1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu-1)} + \frac{a\cos\left[(\nu+1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu+1)} \qquad [0<b<a, \quad \operatorname{Re}\nu>1]$$
$$= \frac{a^\nu\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu-1)\left[b+\sqrt{b^2-a^2}\right]^{\nu-1}} - \frac{a^{\nu+2}\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu+1)\left[b+\sqrt{b^2-a^2}\right]^{\nu+1}} \qquad [0<a<b, \quad \operatorname{Re}\nu>1]$$

ET I 44(6)

6. $$\int_0^\infty J_0(\alpha x)\sin x\frac{dx}{x} = \frac{\pi}{2} \qquad [0<\alpha<1]$$
$$= \operatorname{arccosec}_\alpha \qquad [\alpha>1]$$

WH

7. $$\int_0^\infty J_0(x)\sin\beta x\frac{dx}{x} = \frac{\pi}{2} \qquad [\beta>1]$$
$$= \arcsin\beta \qquad \left[\beta^2<1\right]$$
$$= -\frac{\pi}{2} \qquad [\beta<-1]$$

8. $$\int_0^\infty \left[J_0(x)-\cos\alpha x\right]\frac{dx}{x} = \ln 2\alpha$$ NT 66(13)

9. $$\int_0^z J_\nu(x)\sin(z-x)\frac{dx}{x} = \frac{2}{\nu}\sum_{k=0}^\infty(-1)^k J_{\nu+2k+1}(z) \qquad [\operatorname{Re}\nu>0]$$ WA 416(4)

10. $$\int_0^z J_\nu(x)\cos(z-x)\frac{dx}{x} = \frac{1}{\nu}\,J_\nu(z) + \frac{2}{\nu}\sum_{k=1}^{\infty}(-1)^k\,J_{\nu+2k}(z)$$
$[\operatorname{Re}\nu > 0]$ WA 416(5)

6.694[10] $$\int_0^\infty \left[\frac{J_1(ax)}{x}\right]^2 \sin(bx)\,dx$$
$$= \frac{1}{2}b - \left(\frac{4a}{3\pi}\right)\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{b}{2a}\right) + \left(1-\frac{b^2}{4a^2}\right)\boldsymbol{K}\left(\frac{b}{2a}\right)\right] \qquad [0 \le b \le 2a]$$ ET I 102(22)
$$= \frac{1}{2}b - \frac{2b}{3\pi}\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{2a}{b}\right) - \left(1-\left(\frac{4a^2}{b^2}\right)^{-1}\right)\boldsymbol{K}\left(\frac{2a}{b}\right)\right] \qquad [0 \le 2a \le b]$$

6.695

1. $$\int_0^\infty \frac{\sin\alpha x}{\beta^2+x^2}\,J_0(ux)\,dx = \frac{\sinh\alpha\beta}{\beta}\,K_0(\beta u)$$
$[\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad u > \alpha]$ MO 46

2. $$\int_0^\infty \frac{\cos\alpha x}{\beta^2+x^2}\,J_0(ux)\,dx = \frac{\pi}{2}\frac{e^{-\alpha\beta}}{\beta}\,I_0(\beta u)$$
$[\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad -\alpha < u < \alpha]$
MO 46

3. $$\int_0^\infty \frac{x}{x^2+\beta^2}\sin(\alpha x)\,J_0(\gamma x)\,dx = \frac{\pi}{2}e^{-\alpha\beta}\,I_0(\gamma\beta)$$
$[\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad 0 < \gamma < \alpha]$
ET II 10(36)

4. $$\int_0^\infty \frac{x}{x^2+\beta^2}\cos(\alpha x)\,J_0(\gamma x)\,dx = \cosh(\alpha\beta)\,K_0(\beta\gamma)$$
$[\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \alpha < \gamma]$
ET II 11(45)

6.696 $$\int_0^\infty [1-\cos(\alpha x)]\,J_0(\beta x)\frac{dx}{x} = \operatorname{arccosh}\left(\frac{\alpha}{\beta}\right) \qquad [0 < \beta < \alpha]$$
$$= 0 \qquad [0 < \alpha < \beta]$$
ET II 11(43)

6.697

1. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta}\,J_0(x)\,dx = 2\int_0^\alpha \frac{\cos\beta u}{\sqrt{1-u^2}}\,du \qquad [0 \le \alpha \le 1]$$ WA 463(2)
$$= \pi\,J_0(\beta) \qquad [1 \le \alpha < \infty]$$ WA 463(1), ET II 345(42)

2. $$\int_0^\infty \frac{\sin(x+t)}{x+t}\,J_0(t)\,dt = \frac{\pi}{2}\,J_0(x)$$
$[x > 0]$ WA 475(4)

3. $$\int_0^\infty \frac{\cos(x+t)}{x+t}\,J_0(t)\,dt = -\frac{\pi}{2}\,Y_0(x)$$
$[x > 0]$ WA 475(5)

4. $$\int_{-\infty}^\infty \frac{|x|}{x+\beta}\sin[\alpha(x+\beta)]\,J_0(bx)\,dx = 0$$
$[0 \le \alpha < b]$ WA 464(5), ET II 345(43)a

5. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta}\left[J_{n+\frac{1}{2}}(x)\right]^2 dx = \pi\left[J_{n+\frac{1}{2}}(\beta)\right]^2$$
$[2 \le \alpha < \infty, \quad n = 0, 1, \ldots]$
ET II 346(45)

6. $$\int_{-\infty}^{\infty} \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x)\, J_{-n-\frac{1}{2}}(x)\, dx = \pi\, J_{n+\frac{1}{2}}(\beta)\, J_{-n-\frac{1}{2}}(\beta)$$
$$[2 \le \alpha < \infty, \quad n = 0, 1, \ldots]$$
ET II 346(46)

7. $$\int_{-\infty}^{\infty} \frac{J_\mu[a(z+x)]}{(z+x)^\mu} \frac{J_\nu[a(\zeta+x)]}{(\zeta+x)^\nu}\, dx = \frac{\Gamma(\mu+\nu)\sqrt{\pi}\sqrt{\frac{2}{a}}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \cdot \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}}$$
$$[\operatorname{Re}(\mu+\nu) > 0]$$
WA 463(3)

6.698

1. $$\int_0^\infty \sqrt{x}\, J_{\nu+\frac{1}{4}}(ax)\, J_{-\nu+\frac{1}{4}}(ax) \sin(bx)\, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a]$$
$$= 0 \qquad [0 < 2a < b]$$
ET I 102(26)

2. $$\int_0^\infty \sqrt{x}\, J_{\nu-\frac{1}{4}}(ax)\, J_{-\nu-\frac{1}{4}}(ax) \cos(bx)\, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a]$$
$$= 0 \qquad [0 < 2a < b]$$
ET I 46(24)

3. $$\int_0^\infty \sqrt{x}\, I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx)\, dx = \sqrt{\frac{\pi}{2b}}\, a^{-2\nu} \frac{\left(b+\sqrt{a^2+b^2}\right)^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu < \tfrac{5}{4}\right]$$
ET I 106(56)

4. $$\int_0^\infty \sqrt{x}\, I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx)\, dx = \sqrt{\frac{\pi}{2b}}\, a^{-2\nu} \frac{\left(b+\sqrt{a^2+b^2}\right)^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu < \tfrac{3}{4}\right]$$
ET I 50(49)

6.699

1. $$\int_0^\infty x^\lambda\, J_\nu(ax) \sin(bx)\, dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma\left(\frac{2+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda}{2}\right)} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right)$$
$$\left[0 < b < a, \quad -\operatorname{Re}\nu - 1 < 1 + \operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
$$= \left(\frac{1}{2}a\right)^\nu b^{-(\nu+\lambda+1)} \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sin\left[\pi\left(\frac{1+\lambda+\nu}{2}\right)\right]$$
$$\times F\left(\frac{2+\lambda+\nu}{2}, \frac{1+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right)$$
$$\left[0 < a < b, \quad -\operatorname{Re}\nu - 1 < 1 + \operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
ET I 100(11)

2. $$\int_0^\infty x^\lambda J_\nu(ax)\cos(bx)\,dx$$
$$= \frac{2^\lambda a^{-(1+\lambda)}\,\Gamma\left(\frac{1+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda+1}{2}\right)} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right) \qquad \left[0<b<a, \quad -\operatorname{Re}\nu < 1+\operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
$$= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)}\,\Gamma(1+\lambda+\nu)\cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right) \qquad \left[0<a<b, \quad -\operatorname{Re}\nu < 1+\operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
ET I 45(13)

3. $$\int_0^\infty x^\lambda K_\mu(ax)\sin(bx)dx = \frac{2^\lambda b\,\Gamma\left(\frac{2+\mu+\lambda}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$
$$[\operatorname{Re}(-\lambda\pm\mu) < 2, \quad \operatorname{Re} a > 0, \quad b > 0]$$
ET I 106(50)

4. $$\int_0^\infty x^\lambda K_\mu(ax)\cos(bx)\,dx = 2^{\lambda-1}a^{-\lambda-1}\,\Gamma\left(\frac{\mu+\lambda+1}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right)$$
$$\times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right)$$
$$[\operatorname{Re}(-\lambda\pm\mu) < 1, \quad \operatorname{Re} a > 0, \quad b > 0]$$
ET I 49(42)

5. $$\int_0^\infty x^\nu \sin(ax)\,J_\nu(bx)\,dx = \frac{\sqrt{\pi}2^\nu b^\nu\left(a^2-b^2\right)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0<b<a, \quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<b, \quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET II 32(4)

6. $$\int_0^\infty x^\nu \cos(ax)\,J_\nu(bx)\,dx = -2^\nu\frac{\sin(\nu\pi)}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+\nu\right)b^\nu\left(a^2-b^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0<b<a, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
$$= 2^\nu\frac{b^\nu}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+\nu\right)\left(b^2-a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0<a<b, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 36(29)

7. $$\int_0^\infty x^{\nu+1}\sin(ax)\,J_\nu(bx)\,dx$$
$$= -2^{1+\nu}a\frac{\sin(\nu\pi)}{\sqrt{\pi}}b^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)\left(a^2-b^2\right)^{-\nu-\frac{3}{2}} \qquad \left[0<b<a, \quad -\tfrac{3}{2}<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
$$= -\frac{2^{1+\nu}}{\sqrt{\pi}}ab^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)\left(b^2-a^2\right)^{-\nu-\frac{3}{2}} \qquad \left[0<a<b, \quad -\tfrac{3}{2}<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
ET II 32(3)

8. $$\int_0^\infty x^{\nu+1}\cos(ax)\,J_\nu(bx)\,dx = 2^{1+\nu}\sqrt{\pi}ab^\nu\frac{\left(a^2-b^2\right)^{-\nu-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}-\nu\right)} \qquad \left[0<b<a, \quad -1<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<b, \quad -1<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
ET II 36(28)

9. $$\int_0^1 x^\nu \sin(ax)\, J_\nu(ax)\, dx = \frac{1}{2\nu+1}\left[\sin a J_\nu(a) - \cos a\, J_{\nu+1}(a)\right]$$
$[\operatorname{Re}\nu > -1]$ ET II 334(9)a

10. $$\int_0^1 x^\nu \cos(ax)\, J_\nu(ax)\, dx = \frac{1}{2\nu+1}\left[\cos a J_\nu(a) + \sin a\, J_{\nu+1}(a)\right]$$
$\left[\operatorname{Re}\gamma > -\frac{1}{2}\right]$ ET II 335(20)

11. $$\int_0^\infty x^{1+\nu}\, K_\nu(ax)\sin(bx)\, dx = \sqrt{\pi}(2a)^\nu\, \Gamma\left(\frac{3}{2}+\nu\right) b\left(b^2+a^2\right)^{-\frac{3}{2}-\nu}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET I 105(49)

12. $$\int_0^\infty x^\mu\, K_\mu(ax)\cos(bx)\, dx = \frac{1}{2}\sqrt{\pi}(2a)^\mu\, \Gamma\left(\mu+\frac{1}{2}\right)\left(b^2+a^2\right)^{-\mu-\frac{1}{2}}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET I 49(41)

13. $$\int_0^\infty x^\nu\, Y_{\nu-1}(ax)\sin(bx)\, dx = 0 \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= \frac{2^\nu\sqrt{\pi}a^{\nu-1}b}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET I 104(36)

14. $$\int_0^\infty x^\nu\, Y_\nu(ax)\cos(bx)\, dx = 0 \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= -2^\nu\sqrt{\pi}a^\nu\frac{\left(b^2-a^2\right)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET I 47(30)

6.711

1. $$\int_0^\infty x^{\nu-\mu}\, J_\mu(ax)\, J_\nu(bx)\sin(cx)\, dx = 0 \qquad [0 < c < b-a, \quad -1 < \operatorname{Re}\nu < 1+\operatorname{Re}\mu]$$
ET I 103(28)

2. $$\int_0^\infty x^{\nu-\mu+1}\, J_\mu(ax)\, J_\nu(bx)\cos(cx)\, dx = 0$$
$[0 < c < b-a, \quad a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < \operatorname{Re}\mu]$ ET I 47(25)

3. $$\int_0^\infty x^{\nu-\mu-2}\, J_\mu(ax)\, J_\nu(bx)\sin(cx)\, dx = 2^{\nu-\mu-1}a^\mu b^{-\nu}\frac{c\,\Gamma(\nu)}{\Gamma(\mu+1)}$$
$[0 < a, \quad 0 < b, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu+3]$ ET I 103(29)

4. $$\int_0^\infty x^{\varrho-\mu-1}\, J_\mu(ax)\, J_\varrho(bx)\cos(cx)\, dx = 2^{\varrho-\mu-1}b^{-\varrho}a^\mu\frac{\Gamma(\varrho)}{\Gamma(\mu+1)}$$
$[b > 0, \quad a > 0, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\varrho < \operatorname{Re}\mu+2]$ ET I 47(26)

5. $$\int_0^\infty x^{1-2\nu}\sin(2ax)\,J_\nu(x)\;Y_\nu(x)\,dx = -\frac{\Gamma\left(\frac{3}{2}-\nu\right)a}{2\,\Gamma\left(2\nu-\frac{1}{2}\right)\Gamma(2-\nu)}\,F\left(\frac{3}{2}-\nu,\frac{3}{2}-2\nu;2-\nu;a^2\right)$$
$$\left[0<\operatorname{Re}\nu<\tfrac{3}{2},\quad 0<a<1\right]$$
ET II 348(63)

6.[10] $$\int_0^\infty \arg\sin(zx)x^{\nu-\mu-4}\,J_\mu(ax)\,J_\nu(\rho x)\;dx = z\frac{\Gamma(\nu)\,a^\mu\rho^{-\nu}}{2^{\mu-\nu+3}\,\Gamma(\mu+1)}\left[\frac{\rho^2}{\nu-1}-\frac{a^2}{\mu+1}-\frac{2z^2}{3}\right]$$

7.[10] $$\int_0^\infty \cos(zx)x^{\nu-\mu-3}\,J_\mu(ax)\,J_\nu(\rho x)\;dx = \frac{\Gamma(\nu)\,a^\mu\rho^{-\nu}}{2^{\mu-\nu+3}\,\Gamma(\mu+1)}\left[\frac{\rho^2}{\nu-1}-\frac{a^2}{\mu+1}-2z^2\right]$$

6.712

1. $$\int_0^\infty x^\nu\left[J_\nu(ax)\cos(ax)+Y_\nu(ax)\sin(ax)\right]\sin(bx)\,dx = \frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2+2ab\right)^{-\nu-\frac{1}{2}}$$
$$\left[b>0,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET I 104(40)

2. $$\int_0^\infty x^\nu\left[Y_\nu(ax)\cos(ax)-J_\nu(ax)\sin(ax)\right]\cos(bx)\,dx = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2+2ab\right)^{-\nu-\frac{1}{2}}$$
ET I 48(35)

3. $$\int_0^\infty x^\nu\left[J_\nu(ax)\cos(ax)-Y_\nu(ax)\sin(ax)\right]\sin(bx)\,dx$$
$$=0 \qquad \left[0<b<2a,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
$$=\frac{2^\nu\sqrt{\pi}b^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-2ab\right)^{-\nu-\frac{1}{2}} \qquad \left[2a<b,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET I 104(41)

4. $$\int_0^\infty x^\nu\left[J_\nu(ax)\sin(ax)+Y_\nu(ax)\cos(ax)\right]\cos(bx)\,dx$$
$$=0 \qquad \left[0<b<2a,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
$$=-\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-2ab\right)^{-\nu-\frac{1}{2}} \qquad \left[0<2a<b,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET I 48(33)

6.713

1. $$\int_0^\infty x^{1-2\nu}\sin(2ax)\left\{[J_\nu(x)]^2-[Y_\nu(x)]^2\right\}\,dx$$
$$=\frac{\sin(2\nu\pi)\,\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{3}{2}-2\nu\right)a}{\pi\,\Gamma(2-\nu)}\,F\left(\frac{3}{2}-\nu,\frac{3}{2}-2\nu;2-\nu;a^2\right)$$
$$\left[0<\operatorname{Re}\nu<\tfrac{3}{4},\quad 0<a<1\right]$$
ET II 348(64)

2. $$\int_0^\infty x^{2-2\nu}\sin(2ax)\left[J_\nu(x)\,J_{\nu-1}(x)-Y_\nu(x)\;Y_{\nu-1}(x)\right]\,dx$$
$$=-\frac{\sin(2\nu\pi)\,\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{5}{2}-2\nu\right)a}{\pi\,\Gamma(2-\nu)}\,F\left(\frac{3}{2}-\nu,\frac{5}{2}-2\nu;2-\nu;a^2\right)$$
$$\left[\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{5}{4},\quad 0<a<1\right]$$
ET II 348(65)

3. $$\int_0^\infty x^{2-2\nu}\sin(2ax)\left[J_\nu(x)\,Y_{\nu-1}(x)+Y_\nu(x)\,J_{\nu-1}(x)\right]dx$$
$$=-\frac{\Gamma\left(\frac{3}{2}-\nu\right)a}{\Gamma\left(2\nu-\frac{3}{2}\right)\Gamma(2-\nu)}\,F\left(\frac{3}{2}-\nu,\frac{5}{2}-2\nu;2-\nu;a^2\right)$$
$$\left[\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{5}{2},\quad 0<a<1\right]$$ ET II 349(66)

6.714

1. $$\int_0^\infty \sin(2ax)\left[x^\nu\,J_\nu(x)\right]^2dx$$
$$=\frac{a^{-2\nu}\,\Gamma\left(\frac{1}{2}+\nu\right)}{2\sqrt{\pi}\,\Gamma(1-\nu)}\,F\left(\frac{1}{2}+\nu,\frac{1}{2};1-\nu;a^2\right)\qquad\left[0<a<1,\quad|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
$$=\frac{a^{-4\nu-1}\,\Gamma\left(\frac{1}{2}+\nu\right)}{2\,\Gamma(1+\nu)\,\Gamma\left(\frac{1}{2}-2\nu\right)}\,F\left(\frac{1}{2}+\nu,\frac{1}{2}+2\nu;1+\nu;\frac{1}{a^2}\right)\qquad\left[a>1,\quad|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 343(31)

2. $$\int_0^\infty \cos(2ax)\left[x^\nu\,J_\nu(x)\right]^2dx$$
$$=\frac{a^{-2\nu}\,\Gamma(\nu)}{2\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\,F\left(\nu+\frac{1}{2},\frac{1}{2};1-\nu;a^2\right)$$
$$+\frac{\Gamma(-\nu)\,\Gamma\left(\frac{1}{2}+2\nu\right)}{2\pi\,\Gamma\left(\frac{1}{2}-\nu\right)}\,F\left(\frac{1}{2}+\nu,\frac{1}{2}+2\nu;1+\nu;a^2\right)\qquad\left[0<a<1,\quad-\tfrac{1}{4}<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
$$=-\frac{\sin(\nu\pi)a^{-4\nu-1}\,\Gamma\left(\frac{1}{2}+2\nu\right)}{\Gamma(1+\nu)\,\Gamma\left(\frac{1}{2}-\nu\right)}\,F\left(\frac{1}{2}+\nu,\frac{1}{2}+2\nu;1+\nu;\frac{1}{a^2}\right)\qquad\left[a>1,\quad-\tfrac{1}{4}<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET II 344(33)

6.715

1. $$\int_0^\infty \frac{x^\nu}{x+\beta}\sin(x+\beta)\,J_\nu(x)\,dx=\frac{\pi}{2}\sec(\nu\pi)\beta^\nu\,J_{-\nu}(\beta)$$
$$\left[|\arg\beta|<\pi,\quad|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET II 340(8)

2. $$\int_0^\infty \frac{x^\nu}{x+\beta}\cos(x+\beta)\,J_\nu(x)\,dx=-\frac{\pi}{2}\sec(\nu\pi)\beta^\nu\,Y_{-\nu}(\beta)$$
$$\left[|\arg\beta|<\pi,\quad|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET II 340(9)

6.716

1. $$\int_0^a x^\lambda\sin(a-x)\,J_\nu(x)\,dx=2a^{\lambda+1}\sum_{n=0}^\infty\frac{(-1)^n\,\Gamma(\nu-\lambda+2n)\,\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\,\Gamma(\nu+\lambda+3+2n)}(\nu+2n+1)\,J_{\nu+2n+1}(a)$$
$$[\operatorname{Re}(\lambda+\nu)>-1]$$ ET II 335(16)

2. $$\int_0^a x^\lambda\cos(a-x)\,J_\nu(x)\,dx=\frac{a^{\lambda+1}\,J_\nu(a)}{\lambda+\nu+1}+2a^{\lambda+1}$$
$$\times\sum_{n=1}^\infty\frac{(-1)^n\,\Gamma(\nu-\lambda+2n-1)\,\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\,\Gamma(\nu+\lambda+2n+2)}(\nu+2n)\,J_{\nu+2n}(a)$$
$$[\operatorname{Re}(\lambda+\nu)>-1]$$ ET II 335(26)

6.717 $$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^{\nu}(x+\beta)}\, J_{\nu+2n}(x)\, dx = \pi\beta^{-\nu}\, J_{\nu+2n}(\beta)$$
$$\left[1 \le a < \infty, n = 0, 1, 2, \ldots; \quad \operatorname{Re}\nu > -\tfrac{3}{2}\right]$$ ET II 345(44)

6.718

1. $$\int_0^{\infty} \frac{x^{\nu}}{x^2+\beta^2} \sin(\alpha x)\, J_{\nu}(\gamma x)\, dx = \beta^{\nu-1} \sinh(\alpha\beta)\, K_{\nu}(\beta\gamma)$$
$$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \tfrac{3}{2}\right]$$ ET II 33(8)

2. $$\int_0^{\infty} \frac{x^{\nu+1}}{x^2+\beta^2} \cos(\alpha x)\, J_{\nu}(\gamma x)\, dx = \beta^{\nu} \cosh(\alpha\beta)\, K_{\nu}(\beta\gamma)$$
$$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ ET II 37(33)

3. $$\int_0^{\infty} \frac{x^{1-\nu}}{x^2+\beta^2} \sin(\alpha x)\, J_{\nu}(\gamma x)\, dx = \frac{\pi}{2}\beta^{-\nu} e^{-\alpha\beta}\, I_{\nu}(\beta\gamma) \qquad \left[0 < \gamma \le \alpha, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 33(9)

4. $$\int_0^{\infty} \frac{x^{-\nu}}{x^2+\beta^2} \cos(\alpha x)\, J_{\nu}(\gamma x)\, dx = \frac{\pi}{2}\beta^{-\nu-1} e^{-\alpha\beta}\, I_{\nu}(\beta\gamma)$$
$$\left[0 < \gamma \le \alpha, \operatorname{Re}\beta > 0, \operatorname{Re}\nu > -\tfrac{3}{2}\right]$$
ET II 37(34)

6.719

1.[6] $$\int_0^{\alpha} \frac{\sin(\beta x)}{\sqrt{\alpha^2-x^2}}\, J_{\nu}(x)\, dx = \pi \sum_{n=0}^{\infty} (-1)^n\, J_{2n+1}(\alpha\beta)\, J_{\frac{1}{2}\nu+n+\frac{1}{2}}\left(\tfrac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n-\frac{1}{2}}\left(\tfrac{1}{2}\alpha\right)$$
$$[\operatorname{Re}\nu > -2]$$ ET II 335(17)

2. $$\int_0^{\alpha} \frac{\cos(\beta x)}{\sqrt{\alpha^2-x^2}}\, J_{\nu}(x)\, dx = \frac{\pi}{2}\, J_0(\alpha\beta) \left[J_{\frac{1}{2}\nu}\left(\tfrac{1}{2}\alpha\right)\right]^2 + \pi \sum_{n=1}^{\infty} (-1)^n\, J_{2n}(\alpha\beta)\, J_{\frac{1}{2}\nu+n}\left(\tfrac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n}\left(\tfrac{1}{2}\alpha\right)$$
$$[\operatorname{Re}\nu > -1]$$ ET II 336(27)

6.721

1. $$\int_0^{\infty} \sqrt{x}\, J_{\frac{1}{4}}\left(a^2x^2\right) \sin(bx)\, dx = 2^{-3/2} a^{-2} \sqrt{\pi b}\, J_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$$[b > 0]$$ ET I 108(1)

2. $$\int_0^{\infty} \sqrt{x}\, J_{-\frac{1}{4}}\left(a^2x^2\right) \cos(bx)\, dx = 2^{-3/2} a^{-2} \sqrt{\pi b}\, J_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$$[b > 0]$$ ET I 51(1)

3. $$\int_0^{\infty} \sqrt{x}\, Y_{\frac{1}{4}}\left(a^2x^2\right) \sin(bx)\, dx = -2^{-3/2} \sqrt{\pi b}\, a^{-2}\, \mathbf{H}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$ ET I 108(7)

4. $$\int_0^{\infty} \sqrt{x}\, Y_{-\frac{1}{4}}\left(a^2x^2\right) \cos(bx)\, dx = -2^{-3/2} \sqrt{\pi b}\, a^{-2}\, \mathbf{H}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$ ET I 52(7)

5. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{4}}\left(a^2x^2\right)\sin(bx)\,dx = 2^{-5/2}\sqrt{\pi^3 b}\,a^{-2}\left[I_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right]$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0\right]$$ ET I 109(11)

6. $$\int_0^\infty \sqrt{x}\, K_{-\frac{1}{4}}\left(a^2x^2\right)\cos(bx)\,dx = 2^{-5/2}\sqrt{\pi^3 b}\,a^{-2}\left[I_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right]$$
$$[b > 0]$$ ET I 52(10)

6.722

1. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{8}+\nu}\left(a^2x^2\right) I_{\frac{1}{8}-\nu}\left(a^2x^2\right)\sin(bx)\,dx = \sqrt{2\pi}\,b^{-3/2}\frac{\Gamma\left(\frac{5}{8}-\nu\right)}{\Gamma\left(\frac{5}{4}\right)}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{5}{8}, \quad |\arg a| < \frac{\pi}{4}, \quad b > 0\right]$$
ET I 109(13)

2.[10] $$\int_0^\infty \sqrt{x}\, J_{-\frac{1}{8}-\nu}\left(a^2x^2\right) J_{-\frac{1}{8}+\nu}\left(a^2x^2\right)\cos(bx)\,dx$$
$$= \frac{\sqrt{\pi}}{2^{3/4}a^{3/2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{8}-\nu\right)\Gamma\left(\frac{5}{8}+\nu\right)}\, {}_2F_3\left(\frac{3}{8}-\nu, \frac{3}{8}+\nu;\ \frac{3}{8}, \frac{3}{4}, \frac{7}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$
$$-\frac{1}{a^2}\sqrt{\frac{2b}{\pi}}\cos(\pi\nu)\ {}_2F_3\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu;\ \frac{1}{2}, \frac{7}{8}, \frac{9}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$
$$-\frac{b^{5/2}\nu}{15a^4}\sqrt{\frac{2}{\pi}}\sin(\pi\nu)\ {}_2F_3\left(1-\nu, 1+\nu;\ \frac{11}{8}, \frac{3}{2}, \frac{13}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$
$$\left[a^2 > 0, \quad \operatorname{Im} b = 0\right]$$ MC

3. $$\int_0^\infty \sqrt{x}\, J_{\frac{1}{8}-\nu}\left(a^2x^2\right) J_{\frac{1}{8}+\nu}\left(a^2x^2\right)\sin(bx)\,dx$$
$$= \sqrt{\frac{2}{\pi}}\,b^{-3/2}\left[e^{\pi i/8}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2e^{\pi i/2}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2e^{\pi i/2}}{8a^2}\right)\right.$$
$$\left. + e^{-i\pi/8}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2e^{-\pi i/2}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2e^{-\frac{\pi i}{2}}}{8a^2}\right)\right]$$
$$[b > 0]$$ ET I 108(6)

4. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{8}-\nu}\left(a^2x^2\right) I_{-\frac{1}{8}-\nu}\left(a^2x^2\right)\cos(bx)\,dx$$
$$= \sqrt{2\pi}\,b^{-3/2}\frac{\Gamma\left(\frac{3}{8}-\nu\right)}{\Gamma\left(\frac{3}{4}\right)}\, W_{\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$
$$\left[\operatorname{Re}\nu < \frac{3}{8}, \quad b > 0\right]$$ ET I 52(12)

6.723 $$\int_0^\infty x\, J_\nu\left(x^2\right)\left[\sin(\nu\pi)\, J_\nu\left(x^2\right) - \cos(\nu\pi)\, Y_\nu\left(x^2\right)\right] J_{4\nu}(4ax)\,dx = \frac{1}{4}\, J_\nu\left(a^2\right) J_{-\nu}\left(a^2\right)$$
$$[a > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 375(20)

6.724

1. $$\int_0^\infty x^{2\lambda}\, J_{2\nu}\left(\frac{a}{x}\right)\sin(bx)\,dx$$
$$= \frac{\sqrt{\pi}a^{2\nu}\,\Gamma(\lambda-\nu+1)b^{2\nu-2\lambda-1}}{4^{2\nu-\lambda}\,\Gamma(2\nu+1)\,\Gamma\left(\nu-\lambda+\frac{1}{2}\right)}\;{}_0F_3\left(2\nu+1,\nu-\lambda,\nu-\lambda+\frac{1}{2};\frac{a^2b^2}{16}\right)$$
$$+\frac{a^{2\lambda+2}\,\Gamma(\nu-\lambda-1)b}{2^{2\lambda+3}\,\Gamma(\nu+\lambda+2)}\;{}_0F_3\left(\frac{3}{2},\lambda-\nu+2,\lambda+\nu+2;\frac{a^2b^2}{16}\right)$$
$$\left[-\tfrac{5}{4}<\operatorname{Re}\lambda<\operatorname{Re}\nu,\quad a>0,\quad b>0\right]$$ ET I 109(15)

2. $$\int_0^\infty x^{2\lambda}\, J_{2\nu}\left(\frac{a}{x}\right)\cos(bx)\,dx$$
$$= 4^{\lambda-2\nu}\sqrt{\pi}a^{2\nu}b^{2\nu-2\lambda-1}\frac{\Gamma\left(\lambda-\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)\,\Gamma(\nu-\lambda)}\;{}_0F_3\left(2\nu+1,\nu-\lambda+\frac{1}{2},\nu-\lambda;\frac{a^2b^2}{16}\right)$$
$$+4^{-\lambda-1}a^{2\lambda+1}\frac{\Gamma\left(\nu-\lambda-\frac{1}{2}\right)}{\Gamma\left(\nu+\lambda+\frac{3}{2}\right)}\;{}_0F_3\left(\frac{1}{2},\lambda-\nu+\frac{3}{2},\nu+\lambda+\frac{3}{2};\frac{a^2b^2}{16}\right)$$
$$\left[-\tfrac{3}{4}<\operatorname{Re}\lambda<\operatorname{Re}\nu-\tfrac{1}{2},\quad a>0,\quad b>0\right]$$ ET I 53(14)

6.725

1. $$\int_0^\infty \frac{\sin(bx)}{\sqrt{x}}\,J_\nu\left(a\sqrt{x}\right)\,dx = -\sqrt{\frac{\pi}{b}}\sin\left(\frac{a^2}{8b}-\frac{\nu\pi}{4}-\frac{\pi}{4}\right)J_{\frac{\nu}{2}}\left(\frac{a^2}{8b}\right)$$
$$[\operatorname{Re}\nu>-3,\quad a>0,\quad b>0]$$
ET I 110(27)

2. $$\int_0^\infty \frac{\cos(bx)}{\sqrt{x}}\,J_\nu\left(a\sqrt{x}\right)\,dx = \sqrt{\frac{\pi}{b}}\cos\left(\frac{a^2}{8b}-\frac{\nu\pi}{4}-\frac{\pi}{4}\right)J_{\frac{1}{2}\nu}\left(\frac{a^2}{8b}\right)$$
$$[\operatorname{Re}\nu>-1,\quad a>0,\quad b>0]$$
ET I 54(25)

3. $$\int_0^\infty x^{\frac{1}{2}\nu}\,J_\nu\left(a\sqrt{x}\right)\sin(bx)\,dx = 2^{-\nu}a^\nu b^{-\nu-1}\cos\left(\frac{a^2}{4b}-\frac{\nu\pi}{2}\right)$$
$$\left[-2<\operatorname{Re}\nu<\tfrac{1}{2},\quad a>0,\quad b>0\right]$$
ET I 110(28)

4. $$\int_0^\infty x^{\frac{1}{2}\nu}\,J_\nu\left(a\sqrt{x}\right)\cos(bx)\,dx = 2^{-\nu}b^{-\nu-1}a^\nu\sin\left(\frac{a^2}{4b}-\frac{\nu\pi}{2}\right)$$
$$\left[-1<\operatorname{Re}\nu<\tfrac{1}{2},\quad a>0,\quad b>0\right]$$
ET I 54(26)

6.726

1. $$\int_0^\infty x\left(x^2+b^2\right)^{-\frac{1}{2}\nu}J_\nu\left(a\sqrt{x^2+b^2}\right)\sin(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}}a^{-\nu}b^{-\nu+\frac{3}{2}}c\left(a^2-c^2\right)^{\frac{1}{2}\nu-\frac{3}{4}}J_{\nu-\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right)\qquad\left[0<c<a,\quad\operatorname{Re}\nu>\tfrac{1}{2}\right]$$
$$= 0\qquad\left[0<a<c,\quad\operatorname{Re}\nu>\tfrac{1}{2}\right]$$
ET I 111(37)

2. $$\int_0^\infty \left(x^2+b^2\right)^{-\frac{1}{2}\nu} J_\nu\left(a\sqrt{x^2+b^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}}a^{-\nu}b^{-\nu+\frac{1}{2}}\left(a^2-c^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0<c<a,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<c,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 55(37)

3. $$\int_0^\infty x\left(x^2+b^2\right)^{\frac{1}{2}\nu} K_{\pm\nu}\left(a\sqrt{x^2+b^2}\right)\sin(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}}a^{\nu}b^{\nu+\frac{3}{2}}c\left(a^2+c^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} K_{-\nu-\frac{3}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$[\operatorname{Re}a>0,\quad \operatorname{Re}b>0,\quad c>0]$ ET I 113(45)

4.[11] $$\int_0^\infty \left(x^2+b^2\right)^{\mp\frac{1}{2}\nu} K_\nu\left(a\sqrt{x^2+b^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}}a^{\mp\nu}b^{\frac{1}{2}\mp\nu}\left(a^2+c^2\right)^{\pm\frac{1}{2}\nu-\frac{1}{4}} K_{\pm\nu-\frac{1}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$[\operatorname{Re}a>0,\quad \operatorname{Re}b>0,\quad c \text{ is real}]$ ET I 56(45)

5. $$\int_0^\infty \left(x^2+a^2\right)^{-\frac{1}{2}\nu} Y_\nu\left(b\sqrt{x^2+a^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{a\pi}{2}}(ab)^{-\nu}\left(b^2-c^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} Y_{\nu-\frac{1}{2}}\left(a\sqrt{b^2-c^2}\right) \qquad \left[0<c<b,\quad a>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
$$= -\sqrt{\frac{2a}{\pi}}(ab)^{-\nu}\left(c^2-b^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} K_{\nu-\frac{1}{2}}\left(a\sqrt{c^2-b^2}\right) \qquad \left[0<b<c,\quad a>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 56(41)

6.727

1.[9] $$\int_0^a \frac{\cos(cx)}{\sqrt{a^2-x^2}} J_\nu\left(b\sqrt{a^2-x^2}\right)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}-c\right)\right] J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}+c\right)\right]$$
$[\operatorname{Re}\nu>-1,\quad c>0,\quad a>0]$
ET I 113(48)

2. $$\int_a^\infty \frac{\sin(cx)}{\sqrt{x^2-a^2}} J_\nu\left(b\sqrt{x^2-a^2}\right)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2+b^2}\right)\right] J_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2+b^2}\right)\right]$$
$[0<b<c,\quad a>0,\quad \operatorname{Re}\nu>-1]$
ET I 113(49)

3. $$\int_a^\infty \frac{\cos(cx)}{\sqrt{x^2-a^2}} J_\nu\left(b\sqrt{x^2-a^2}\right)\,dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2-b^2}\right)\right] Y_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2-b^2}\right)\right]$$
$[0<b<c,\quad a>0,\quad \operatorname{Re}\nu>-1]$
ET I 58(54)

4.[8] $$\int_0^a \left(a^2-x^2\right)^{\frac{1}{2}\nu}\cos x\, I_\nu\left(\sqrt{a^2-x^2}\right)\,dx = \frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}$$
$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ WA 409(2)

6.728

1. $$\int_0^\infty x\sin\left(ax^2\right)J_\nu(bx)\,dx = \frac{\sqrt{\pi}b}{8a^{3/2}}\left[\cos\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right)-\sin\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0,\quad b>0,\quad \operatorname{Re}\nu>-4]$ ET II 34(14)

2. $$\int_0^\infty x\cos\left(ax^2\right)J_\nu(bx)\,dx = \frac{\sqrt{\pi}b}{8a^{3/2}}\left[\cos\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right)+\sin\left(\frac{b^2}{8a}-\frac{\nu\pi}{4}\right)J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0,\quad b>0,\quad \operatorname{Re}\nu>-2]$ ET II 38(39)

3. $$\int_0^\infty J_0(\beta x)\sin\left(\alpha x^2\right)x\,dx = \frac{1}{2\alpha}\cos\frac{\beta^2}{4\alpha}$$ $[\alpha>0,\quad \beta>0]$ MO 47

4. $$\int_0^\infty J_0(\beta x)\cos\left(\alpha x^2\right)x\,dx = \frac{1}{2\alpha}\sin\frac{\beta^2}{4\alpha}$$ $[\alpha>0,\quad \beta>0]$ MO 47

5. $$\int_0^\infty x^{\nu+1}\sin\left(ax^2\right)J_\nu(bx)\,dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}}\cos\left(\frac{b^2}{4a}-\frac{\nu\pi}{2}\right)$$
$\left[a>0,\quad b>0,\quad -2<\operatorname{Re}\nu<\frac{1}{2}\right]$
ET II 34(15)

6. $$\int_0^\infty x^{\nu+1}\cos\left(ax^2\right)J_\nu(bx)\,dx = \frac{b^\nu}{2^{\nu+1}a^{\nu+1}}\sin\left(\frac{b^2}{4a}-\frac{\nu\pi}{2}\right)$$
$\left[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<\frac{1}{2}\right]$
ET II 38(40)

6.729

1. $$\int_0^\infty x\sin\left(ax^2\right)J_\nu(bx)\,J_\nu(cx)\,dx = \frac{1}{2a}\cos\left(\frac{b^2+c^2}{4a}-\frac{\nu\pi}{2}\right)J_\nu\left(\frac{bc}{2a}\right)$$
$[a>0,\quad b>0,\quad c>0,\quad \operatorname{Re}\nu>-2]$
ET II 51(26)

2. $$\int_0^\infty x\cos\left(ax^2\right)J_\nu(bx)\,J_\nu(cx)\,dx = \frac{1}{2a}\sin\left(\frac{b^2+c^2}{4a}-\frac{\nu\pi}{2}\right)J_\nu\left(\frac{bc}{2a}\right)$$
$[a>0,\quad b>0,\quad c>0,\quad \operatorname{Re}\nu>-1]$
ET II 51(27)

6.731

1.[11] $$\int_0^\infty x\sin\left(ax^2\right)J_\nu\left(bx^2\right)J_{2\nu}(2cx)\,dx$$
$$= \frac{1}{2\sqrt{b^2-a^2}}\sin\left(\frac{ac^2}{b^2-a^2}\right)J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad [0<a<b,\quad \operatorname{Re}\nu>-1]$$
$$= \frac{1}{2\sqrt{a^2-b^2}}\cos\left(\frac{ac^2}{a^2-b^2}\right)J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad [0<b<a,\quad \operatorname{Re}\nu>-1]$$
ET II 356(41)a

2.[10] $$\int_0^\infty x\cos\left(ax^2\right) J_\nu\left(bx^2\right) J_{2\nu}(2cx)\,dx$$
$$= \frac{1}{2\sqrt{b^2-a^2}}\cos\left(\frac{ac^2}{b^2-a^2}\right) J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad \left[0<a<b, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
$$= \frac{1}{2\sqrt{a^2-b^2}}\sin\left(\frac{ac^2}{a^2-b^2}\right) J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad \left[0<b<a, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 356(42)a

6.732 $$\int_0^\infty x^2\cos\left(\frac{x^2}{2a}\right) Y_1(x)\,K_1(x)\,dx = -a^3\,K_0(a) \qquad [a>0]$$ ET II 371(52)

6.733

1. $$\int_0^\infty \sin\left(\frac{a}{2x}\right)\left[\sin x\,J_0(x)+\cos x\,Y_0(x)\right]\frac{dx}{x} = \pi\,J_0\left(\sqrt{a}\right)\,Y_0\left(\sqrt{a}\right)$$
$[a>0]$ ET II 346(51)

2. $$\int_0^\infty \cos\left(\frac{a}{2x}\right)\left[\sin x\,Y_0(x)-\cos x\,J_0(x)\right]\frac{dx}{x} = \pi\,J_0\left(\sqrt{a}\right)\,Y_0\left(\sqrt{a}\right)$$
$[a>0]$ ET II 347(52)

3. $$\int_0^\infty x\sin\left(\frac{a}{2x}\right) K_0(x)\,dx = \frac{\pi a}{2}\,J_1\left(\sqrt{a}\right)\,K_1\left(\sqrt{a}\right) \qquad [a>0]$$ ET II 368(34)

4. $$\int_0^\infty x\cos\left(\frac{a}{2x}\right) K_0(x)\,dx = -\frac{\pi a}{2}\,Y_1\left(\sqrt{a}\right)\,K_1\left(\sqrt{a}\right) \qquad [a>0]$$ ET II 369(35)

6.734 $$\int_0^\infty \cos\left(a\sqrt{x}\right) K_\nu(bx)\frac{dx}{\sqrt{x}}$$
$$= \frac{\pi}{2\sqrt{b}}\sec(\nu\pi)\left[D_{\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) + D_{\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right)\right]$$
$\left[\operatorname{Re} b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$ ET II 132(27)

6.735

1. $$\int_0^\infty x^{1/4}\sin\left(2a\sqrt{x}\right) J_{-\frac{1}{4}}(x)\,dx = \sqrt{\pi}a^{3/2}\,J_{\frac{3}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(10)

2. $$\int_0^\infty x^{1/4}\cos\left(2a\sqrt{x}\right) J_{\frac{1}{4}}(x)\,dx = \sqrt{\pi}a^{3/2}\,J_{-\frac{3}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(12)

3. $$\int_0^\infty x^{1/4}\sin\left(2a\sqrt{x}\right) J_{\frac{3}{4}}(x)\,dx = \sqrt{\pi}a^{3/2}\,J_{-\frac{1}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(11)

4. $$\int_0^\infty x^{1/4}\cos\left(2a\sqrt{x}\right) J_{-\frac{3}{4}}(x)\,dx = \sqrt{\pi}a^{3/2}\,J_{\frac{1}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(13)

6.736

1.[11] $$\int_0^\infty x^{-1/2}\sin x\cos\left(4a\sqrt{x}\right) J_0(x)\,dx = -2^{-3/2}\sqrt{\pi}\left[\cos\left(a^2-\frac{\pi}{4}\right) J_0\left(a^2\right) - \sin\left(a^2-\frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$
$[a>0]$ ET II 341(18)

2. $$\int_0^\infty x^{-1/2}\cos x\cos\left(4a\sqrt{x}\right) J_0(x)\,dx = -2^{-3/2}\sqrt{\pi}\left[\sin\left(a^2-\frac{\pi}{4}\right) J_0\left(a^2\right) + \cos\left(a^2-\frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$
$[a>0]$ ET II 342(22)

3. $$\int_0^\infty x^{-1/2}\sin x\sin\left(4a\sqrt{x}\right)J_0(x)\,dx=\sqrt{\frac{\pi}{2}}\cos\left(a^2+\frac{\pi}{4}\right)J_0\left(a^2\right)$$
$[a>0]$ ET II 341(16)

4. $$\int_0^\infty x^{-1/2}\cos x\sin\left(4a\sqrt{x}\right)J_0(x)\,dx=\sqrt{\frac{\pi}{2}}\cos\left(a^2-\frac{\pi}{4}\right)J_0\left(a^2\right)$$
$[a>0]$ ET II 342(20)

5. $$\int_0^\infty x^{-1/2}\sin x\cos\left(4a\sqrt{x}\right)Y_0(x)\,dx=2^{-3/2}\sqrt{\pi}\left[3\sin\left(a^2-\frac{\pi}{4}\right)J_0\left(a^2\right)-\cos\left(a^2-\frac{\pi}{4}\right)Y_0\left(a^2\right)\right]$$
$[a>0]$ ET II 347(55)

6. $$\int_0^\infty x^{-1/2}\cos x\cos\left(4a\sqrt{x}\right)Y_0(x)\,dx$$
$$=-2^{-3/2}\sqrt{\pi}\left[3\cos\left(a^2-\frac{\pi}{4}\right)J_0\left(a^2\right)+\sin\left(a^2-\frac{\pi}{4}\right)Y_0\left(a^2\right)\right]$$
$[a>0]$ ET II 347(56)

6.737

1. $$\int_0^\infty\frac{\sin\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}J_\nu(cx)\,dx=\frac{\pi}{2}J_{\frac12\nu}\left[\frac{b}{2}\left(a-\sqrt{a^2-c^2}\right)\right]J_{-\frac12\nu}\left[\frac{b}{2}\left(a+\sqrt{a^2-c^2}\right)\right]$$
$[a>0,\quad \operatorname{Re}b>0,\quad c>0,\quad a>c,\quad \operatorname{Re}\nu>-1]$ ET II 35(19)

2. $$\int_0^\infty\frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}J_\nu(cx)\,dx=-\frac{\pi}{2}J_{\frac12\nu}\left[\frac{b}{2}\left(a-\sqrt{a^2-c^2}\right)\right]Y_{-\frac12\nu}\left[\frac{b}{2}\left(a+\sqrt{a^2-c^2}\right)\right]$$
$[a>0,\quad \operatorname{Re}b>0,\quad c>0,\quad a>c,\quad \operatorname{Re}\nu>-1]$ ET II 39(44)

3. $$\int_0^a\frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}}J_\nu(cx)\,dx=\frac{\pi}{2}J_{\frac12\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}-b\right)\right]J_{\frac12\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}+b\right)\right]$$
$[c>0,\quad \operatorname{Re}\nu>-1]$ ET II 39(47)

4. $$\int_0^a x^{\nu+1}\frac{\cos\left(\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}}I_\nu(x)\,dx=\frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}\qquad[\operatorname{Re}\nu>-1]$$
ET II 365(9)

5. $$\int_0^\infty x^{\nu+1}\frac{\sin\left(a\sqrt{b^2+x^2}\right)}{\sqrt{b^2+x^2}}J_\nu(cx)\,dx$$
$$=\sqrt{\frac{\pi}{2}}b^{\frac12+\nu}c^\nu\left(a^2-c^2\right)^{-\frac14-\frac12\nu}J_{-\nu-\frac12}\left(b\sqrt{a^2-c^2}\right)\qquad\left[0<c<a,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\tfrac12\right]$$
$$=0\qquad\left[0<a<c,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\tfrac12\right]$$
ET II 35(20)

6. $$\int_0^\infty x^{\nu+1}\frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}\,J_\nu(cx)\,dx=-\sqrt{\frac{\pi}{2}}b^{\frac12+\nu}c^\nu\left(a^2-c^2\right)^{-\frac14-\frac12\nu}Y_{-\nu-\frac12}\left(b\sqrt{a^2-c^2}\right)$$
$$\left[0<c<a,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\frac12\right]$$
$$=\sqrt{\frac{2}{\pi}}b^{\frac12+\nu}c^\nu\left(c^2-a^2\right)^{-\frac14-\frac12\nu}K_{\nu+\frac12}\left(b\sqrt{c^2-a^2}\right)$$
$$\left[0<a<c,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\frac12\right]$$
ET II 39(45)

6.738

1. $$\int_0^a x^{\nu+1}\sin\left(b\sqrt{a^2-x^2}\right)J_\nu(x)\,dx=\sqrt{\frac{\pi}{2}}a^{\nu+\frac32}b\left(1+b^2\right)^{-\frac12\nu-\frac34}J_{\nu+\frac32}\left(a\sqrt{1+b^2}\right)$$
$$[\operatorname{Re}\nu>-1]$$
ET II 335(19)

2. $$\int_0^\infty x^{\nu+1}\cos\left(a\sqrt{x^2+b^2}\right)J_\nu(cx)\,dx$$
$$=\sqrt{\frac{\pi}{2}}ab^{\nu+\frac32}c^\nu\left(a^2-c^2\right)^{-\frac12\nu-\frac34}\left[\cos(\pi\nu)\,J_{\nu+\frac32}\left(b\sqrt{a^2-c^2}\right)-\sin(\pi\nu)\,Y_{\nu+\frac32}\left(b\sqrt{a^2-c^2}\right)\right]$$
$$\left[0<c<a,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<-\tfrac12\right]$$
$$=0$$
$$\left[0<a<c,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<-\tfrac12\right]$$
ET II 39(43)

6.739 $$\int_0^t x^{-1/2}\frac{\cos\left(b\sqrt{t-x}\right)}{\sqrt{t-x}}\,J_{2\nu}\left(a\sqrt{x}\right)\,dx=\pi\,J_\nu\left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}+b\right)\right]J_\nu\left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}-b\right)\right]$$
$$\left[\operatorname{Re}\nu>-\tfrac12\right]$$
EH II 47(7)

6.741

1. $$\int_0^1\frac{\cos\left(\mu\arccos x\right)}{\sqrt{1-x^2}}\,J_\nu(ax)\,dx=\frac{\pi}{2}\,J_{\frac12(\mu+\nu)}\left(\frac{a}{2}\right)J_{\frac12(\nu-\mu)}\left(\frac{a}{2}\right)$$
$$[\operatorname{Re}(\mu+\nu)>-1,\quad a>0]$$
ET II 41(54)

2. $$\int_0^1\frac{\cos\left[(\nu+1)\arccos x\right]}{\sqrt{1-x^2}}\,J_\nu(ax)\,dx=\sqrt{\frac{\pi}{a}}\cos\left(\frac{a}{2}\right)J_{\nu+\frac12}\left(\frac{a}{2}\right)$$
$$[\operatorname{Re}\nu>-1,\quad a>0]$$
ET II 40(53)

3. $$\int_0^1\frac{\cos\left[(\nu-1)\arccos x\right]}{\sqrt{1-x^2}}\,J_\nu(ax)\,dx=\sqrt{\frac{\pi}{a}}\sin\left(\frac{a}{2}\right)J_{\nu-\frac12}\left(\frac{a}{2}\right)$$
$$[\operatorname{Re}\nu>0,\quad a>0]$$
ET II 40(52)a

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

6.751 **Notation**: $\ell_1=\frac12\left[\sqrt{(b+c)^2+a^2}-\sqrt{(b-c)^2+a^2}\right]$, $\ell_2=\frac12\left[\sqrt{(b+c)^2+a^2}+\sqrt{(b-c)^2+a^2}\right]$

1. $$\int_0^\infty e^{-\frac{1}{2}ax}\sin(bx)\,I_0\left(\frac{1}{2}ax\right)\,dx=\frac{1}{\sqrt{2b}}\frac{1}{\sqrt{b^2+a^2}}\sqrt{b+\sqrt{b^2+a^2}}$$
$[\operatorname{Re}a>0,\quad b>0]$ ET I 105(44)

2. $$\int_0^\infty e^{-\frac{1}{2}ax}\cos(bx)\,I_0\left(\frac{1}{2}ax\right)\,dx=\frac{a}{\sqrt{2b}}\frac{1}{\sqrt{a^2+b^2}\sqrt{b+\sqrt{a^2+b^2}}}$$
$[\operatorname{Re}a>0,\quad b>0]$ ET I 48(38)

3.[10] $$\int_0^\infty e^{-bx}\cos(ax)\,J_0(cx)\,dx=\frac{\left[\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}+b^2+c^2-a^2\right]^{1/2}}{\sqrt{2}\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}}$$
$[c>0]$ ET II 11(46)

alternatively, with a and b interchanged,
$$\int_0^\infty e^{-ax}\cos(bx)\,J_0(cx)\,dx=\frac{\sqrt{\ell_2^2-b^2}}{\ell_2^2-\ell_1^2}\qquad[c>0]$$

6.752

1.[10] $$\int_0^\infty e^{-ax}\,J_0(bx)\sin(cx)\frac{dx}{x}=\arcsin\left(\frac{2c}{\sqrt{a^2+(c+b)^2}+\sqrt{a^2+(c-b)^2}}\right)=\arcsin\left(\frac{c}{\ell_2}\right)$$
$[\operatorname{Re}a>|\operatorname{Im}b|,\quad c>0]$ ET I 101(17)

2.[10] $$\int_0^\infty e^{-ax}\,J_1(cx)\sin(bx)\frac{dx}{x}=\frac{b}{c}(1-r)=\frac{b-\sqrt{b^2-\ell_1^2}}{c},$$
$$\left[b^2=\frac{c^2}{1-r^2}-\frac{a^2}{r^2},\quad c>0\right]$$
ET II 19(15)

Notation: For integrals 6.752 3–6.752 5 we define the auxiliary functions

$$\ell_1(a)\equiv\ell_1(a,\rho,z)=\frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2}-\sqrt{(a-\rho)^2+z^2}\right]$$
$$\ell_2(a)\equiv\ell_1(a,\rho,z)=\frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2}+\sqrt{(a-\rho)^2+z^2}\right]$$

when $a\geq 0$, $\rho\geq 0$, and $z\geq 0$.

3.[10] $$\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-zx}\,J_{\nu+1/2}(ax)\,J_{\nu+1}(\rho x)\sqrt{x}\,dx$$
$$=a^{-\nu-3/2}\rho^{-\nu-1}\frac{\ell_1^{2\nu+2}}{\sqrt{\rho^2-\ell_1^2}}\frac{a\left(\rho^2-\ell_1^2\right)}{\ell_1\left(\ell_2^2-\ell_1^2\right)}$$
$$=a^{\nu+1/2}\frac{\rho^{\nu+1}}{\ell_2^{2\nu+2}}\frac{\sqrt{\ell_2^2-a^2}}{\ell_2^2-\ell_1^2}\qquad[\operatorname{Re}z>|\operatorname{Im}a|+|\operatorname{Im}\rho|]$$

4.[10] $$\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-zx}\,J_{\nu+1/2}(ax)\,J_\nu(\rho x)\frac{dx}{\sqrt{x}}$$
$$= a^{\nu+1/2}\rho^\nu\int_0^{1/\ell_2}\frac{1}{\ell_2^{2\nu}}\frac{1}{\sqrt{1-a^2/\ell_2^2}}\,d\left(\frac{1}{\ell_2}\right)$$
$$= a^{-\nu-1/2}\rho^\nu\int_0^{a/\ell_2}x^{2\nu}\frac{dx}{\sqrt{1-x^2}} \qquad \left[\nu>-\tfrac{1}{2},\quad \operatorname{Re}z>|\operatorname{Im}a|+|\operatorname{Im}\rho|\right]$$

5.[10] $$\int_0^\infty e^{-zx}\sin(ax)\,J_1(\rho x)\frac{dx}{x^2}=\frac{\sqrt{\ell_2^2-a^2}\left(a-\sqrt{a^2-\ell_1^2}\right)^2}{2a\rho}+\frac{\rho}{2}\arcsin\left(\frac{a}{\ell_2}\right)$$
$$[\operatorname{Re}z>|\operatorname{Im}a|+|\operatorname{Im}\rho|]$$

6.753

1.[8] $$\int_0^\infty\frac{\sin(xa\sin\psi)}{x}e^{-xa\cos\varphi\cos\psi}\,J_\nu(xa\sin\varphi)\,dx=\nu^{-1}\left(\tan\frac{\varphi}{2}\right)^\nu\sin(\nu\psi)$$
$$\left[\operatorname{Re}\nu>-1,\quad a>0,\quad 0<\varphi<\frac{\pi}{2},\quad 0<\psi<\frac{\pi}{2}\right]$$ ET II 33(10)

2. $$\int_0^\infty\frac{\cos(xa\sin\psi)}{x}e^{-xa\cos\varphi\cos\psi}\,J_\nu(xa\sin\varphi)\,dx=\nu^{-1}\left(\tan\frac{\varphi}{2}\right)^\nu\cos(\nu\psi)$$
$$\left[\operatorname{Re}\nu>0,\quad a>0,\quad 0<\varphi,\quad \psi<\frac{\pi}{2}\right]$$
ET II 38(35)

3.[8] $$\int_0^\infty x^{\nu+1}e^{-sx}\sin(bx)\,J_\nu(ax)\,dx=-\frac{2(2a)^\nu}{\sqrt{\pi}}\,\Gamma(\nu+\tfrac{3}{2})R^{-2\nu-3}\left[b\cos(\nu+\tfrac{3}{2})\varphi+s\sin(\nu+\tfrac{3}{2})\varphi\right]$$
$$\left[\operatorname{Re}\nu>-\tfrac{3}{2},\quad \operatorname{Re}s>|\operatorname{Im}a|+|\operatorname{Im}b|,\right.$$
$$\left.R^4=\left(s^2+a^2-b^2\right)^2+4b^2s^2,\quad \varphi=\arg\left(s^2+a^2-b^2-2ibs\right)\right]$$

4.[8] $$\int_0^\infty x^{\nu+1}e^{-sx}\cos(bx)\,J_\nu(ax)\,dx=\frac{2(2a)^\nu}{\sqrt{\pi}}\,\Gamma(\nu+\tfrac{3}{2})R^{-2\nu-3}\left[s\cos(\nu+\tfrac{3}{2})\varphi-b\sin(\nu+\tfrac{3}{2})\varphi\right],$$
$$\left[\operatorname{Re}\nu>-1,\quad \operatorname{Re}s>|\operatorname{Im}a|+|\operatorname{Im}b|,\right.$$
$$\left.R^4=\left(s^2+a^2-b^2\right)^2+4b^2s^2,\quad \varphi=\arg\left(s^2+a^2-b^2-2ibs\right)\right]$$

5.[10] $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi}\sin(ax\sin\psi)\,J_\nu(ax\sin\varphi)\,dx$$
$$=2^\nu\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}a^{-\nu-1}(\sin\varphi)^\nu\left(\cos^2\psi+\sin^2\psi\cos^2\varphi\right)^{-\nu-\frac{1}{2}}\sin\left[\left(\nu+\tfrac{1}{2}\right)\beta\right]$$
$$\tan\frac{\beta}{2}=\tan\psi\cos\varphi \qquad \left[a>0,\quad 0<\varphi<\frac{\pi}{2},\quad 0<\psi<\frac{\pi}{2},\quad \operatorname{Re}\nu>-1\right]$$ ET II 34(12)

6. $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi}\cos(ax\sin\psi)\,J_\nu(ax\sin\varphi)\,dx$$
$$= 2^\nu\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}a^{-\nu-1}(\sin\varphi)^\nu\left(\cos^2\psi+\sin^2\psi\cos^2\varphi\right)^{-\nu-\frac{1}{2}}\cos\left[\left(\nu+\tfrac{1}{2}\right)\beta\right]$$
$$\tan\frac{\beta}{2}=\tan\psi\cos\varphi \qquad \left[a>0,\quad 0<\varphi,\quad \psi<\frac{\pi}{2},\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$$ ET II 38(37)

6.754

1. $$\int_0^\infty e^{-x^2}\sin(bx)\,I_0\left(x^2\right)\,dx=\frac{\sqrt{\pi}}{2^{3/2}}e^{-\frac{b^2}{8}}\,I_0\left(\frac{b^2}{8}\right) \qquad [b>0]$$ ET I 108(9)

2. $$\int_0^\infty e^{-ax}\cos\left(x^2\right)J_0\left(x^2\right)\,dx=\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[J_0\left(\frac{a^2}{16}\right)\cos\left(\frac{a^2}{16}-\frac{\pi}{4}\right)-Y_0\left(\frac{a^2}{16}\right)\cos\left(\frac{a^2}{16}+\frac{\pi}{4}\right)\right]$$
$$[a>0]$$ MI 42

3. $$\int_0^\infty e^{-ax}\sin\left(x^2\right)J_0\left(x^2\right)\,dx=\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[J_0\left(\frac{a^2}{16}\right)\sin\left(\frac{a^2}{16}-\frac{\pi}{4}\right)-Y_0\left(\frac{a^2}{16}\right)\sin\left(\frac{a^2}{16}+\frac{\pi}{4}\right)\right]$$
$$[a>0]$$ MI 42

6.755

1. $$\int_0^\infty x^{-\nu}e^{-x}\sin\left(4a\sqrt{x}\right)I_\nu(x)\,dx=\left(2^{3/2}a\right)^{\nu-1}e^{-a^2}\,W_{\frac{1}{2}-\frac{3}{2}\nu,\frac{1}{2}-\frac{1}{2}\nu}\left(2a^2\right)$$
$$[a>0,\quad \operatorname{Re}\nu>0]$$ ET II 366(14)

2. $$\int_0^\infty x^{-\nu-\frac{1}{2}}e^{-x}\cos\left(4a\sqrt{x}\right)I_\nu(x)\,dx=2^{\frac{3}{2}\nu-1}a^{\nu-1}e^{-a^2}\,W_{-\frac{3}{2}\nu,\frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 366(16)

3. $$\int_0^\infty x^{-\nu}e^{x}\sin\left(4a\sqrt{x}\right)K_\nu(x)\,dx=\left(2^{3/2}a\right)^{\nu-1}\pi\frac{\Gamma\left(\frac{3}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}e^{a^2}\,W_{\frac{3}{2}\nu-\frac{1}{2},\frac{1}{2}-\frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0,\quad 0<\operatorname{Re}\nu<\tfrac{3}{4}\right]$$ ET II 369(38)

4. $$\int_0^\infty x^{-\nu-\frac{1}{2}}e^{x}\cos\left(4a\sqrt{x}\right)K_\nu(x)\,dx=2^{\frac{3}{2}\nu-1}\pi a^{\nu-1}\frac{\Gamma\left(\frac{1}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}e^{a^2}\,W_{\frac{3}{2}\nu,-\frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0,\quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{4}\right]$$
ET II 369(42)

5. $$\int_0^\infty x^{\varrho-\frac{3}{2}}e^{-x}\sin\left(4a\sqrt{x}\right)K_\nu(x)\,dx=\frac{\sqrt{\pi}a\,\Gamma(\varrho+\nu)\,\Gamma(\varrho-\nu)}{2^{\varrho-2}\,\Gamma\left(\varrho+\frac{1}{2}\right)}\,{}_2F_2\left(\varrho+\nu,\varrho-\nu;\frac{3}{2},\varrho+\frac{1}{2};-2a^2\right)$$
$$[\operatorname{Re}\varrho>|\operatorname{Re}\nu|]$$ ET II 369(39)

6. $$\int_0^\infty x^{\varrho-1}e^{-x}\cos\left(4a\sqrt{x}\right)K_\nu(x)\,dx=\frac{\sqrt{\pi}\,\Gamma(\varrho+\nu)\,\Gamma(\varrho-\nu)}{2^{\varrho}\,\Gamma\left(\varrho+\frac{1}{2}\right)}\,{}_2F_2\left(\varrho+\nu,\varrho-\nu;\frac{1}{2},\varrho+\frac{1}{2};-2a^2\right)$$
$$[\operatorname{Re}\varrho>|\operatorname{Re}\nu|]$$ ET II 370(43)

7. $$\int_0^\infty x^{-1/2}e^{-x}\cos\left(4a\sqrt{x}\right)I_0(x)\,dx=\frac{1}{\sqrt{2\pi}}e^{-a^2}\,K_0\left(a^2\right)$$
$$[a>0]$$ ET II 366(15)

8. $$\int_0^\infty x^{-1/2}e^x \cos\left(4a\sqrt{x}\right) K_0(x)\,dx = \sqrt{\frac{\pi}{2}}e^{a^2}\, K_0\left(a^2\right) \qquad [a>0]$$ ET II 369(40)

9. $$\int_0^\infty x^{-1/2}e^{-x} \cos\left(4a\sqrt{x}\right) K_0(x)\,dx = \frac{1}{\sqrt{2}}\pi^{3/2}e^{-a^2}\, I_0\left(a^2\right)$$ ET II 369(41)

6.756

1. $$\int_0^\infty x^{-\frac{1}{2}}e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_\nu(bx)\,dx$$
$$= \frac{i}{\sqrt{2\pi b}}\,\Gamma\left(\nu+\frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right)\left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$
$$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1]$$ ET II 34(17)

2. $$\int_0^\infty x^{-\frac{1}{2}}e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_\nu(bx)\,dx$$
$$= \frac{1}{\sqrt{2\pi b}}\,\Gamma\left(\nu+\frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right)\left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$
$$\left[a>0, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 39(42)

3. $$\int_0^\infty x^{-1/2}e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_0(bx)\,dx = \frac{1}{2b}a\, I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right)$$
$$\left[|\arg a|<\frac{\pi}{4}, \quad b>0\right]$$ ET II 11(40)

4. $$\int_0^\infty x^{-1/2}e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_0(bx)\,dx = \frac{a}{2b}\, I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right)$$
$$\left[|\arg a|<\frac{\pi}{4}, \quad b>0\right]$$ ET II 12(49)

6.757

1. $$\int_0^\infty e^{-bx} \sin\left[a\left(1-e^{-x}\right)\right] J_\nu\left(ae^{-x}\right)\,dx$$
$$= 2\sum_{n=0}^\infty \frac{(-1)^n\,\Gamma(\nu-b+2n+1)\,\Gamma(\nu+b)}{\Gamma(\nu-b+1)\,\Gamma(\nu+b+2n+2)}(\nu+2n-1)\,J_{\nu+2n+1}(a)$$
$$[\operatorname{Re} b>-\operatorname{Re}\nu]$$ ET I 193(26)

2. $$\int_0^\infty e^{-bx} \cos\left[a\left(1-e^{-x}\right)\right] J_\nu\left(ae^{-x}\right)\,dx$$
$$= \frac{J_\nu(a)}{\nu+b} + \sum_{n=0}^\infty 2(-1)^n \frac{\Gamma(\nu-b+2n)\,\Gamma(\nu+b)}{\Gamma(\nu-b+1)\,\Gamma(\nu+b+2n+1)}(\nu+2n)\,J_{\nu+2n}(a)$$
$$[\operatorname{Re} b>-\operatorname{Re}\nu]$$ ET I 193(27)

6.758 $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta}\,(\cos\theta)^{\nu+\mu}\,(\lambda z)^{-\nu-\mu}\,J_{\nu+\mu}(\lambda z)\,d\theta$$
$$= \pi(2az)^{-\mu}(2bz)^{-\nu}\,J_\mu(az)\,J_\nu(bz); \lambda = \sqrt{2\cos\theta\left(a^2e^{i\theta}+b^2e^{-i\theta}\right)}$$
$$\lambda = \sqrt{2\cos\theta\left(a^2e^{i\theta}+b^2e^{-i\theta}\right)} \qquad [\operatorname{Re}(\nu+\mu)>-1]$$ EH II 48(12)

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

6.761 $$\int_0^\infty \cosh x \cos(2a\sinh x)\, J_\nu(be^x)\, J_\nu\left(be^{-x}\right)\, dx = \frac{J_{2\nu}\left(2\sqrt{b^2-a^2}\right)}{2\sqrt{b^2-a^2}} \qquad [0<a<b, \quad \operatorname{Re}\nu>-1]$$
$$= 0 \qquad [0<b<a, \quad \operatorname{Re}\nu>-1]$$

ET II 359(10)

6.762 $$\int_0^\infty \cosh x \sin(2a\sinh x)\left[J_\nu(be^x)\, Y_\nu\left(be^{-x}\right) - Y_\nu(be^x)\, J_\nu\left(be^{-x}\right)\right] dx$$
$$= 0 \qquad \left[0<a<b, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
$$= -\frac{2}{\pi}\cos(\nu\pi)\left(a^2-b^2\right)^{-1/2} K_{2\nu}\left[2\left(a^2-b^2\right)^{1/2}\right] \qquad \left[0<b<a, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$

ET II 360(12)

6.763 $$\int_0^\infty \cosh x \cos(2a\sinh x)\, Y_\nu(be^x)\, Y_\nu\left(be^{-x}\right)\, dx$$
$$= -\frac{1}{2}\left(b^2-a^2\right)^{-1/2} J_{2\nu}\left[2\left(b^2-a^2\right)^{1/2}\right] \qquad [0<a<b, \quad |\operatorname{Re}\nu|<1]$$
$$= \frac{2}{\pi}\cos(\nu\pi)\left(a^2-b^2\right)^{-1/2} K_{2\nu}\left[2\left(a^2-b^2\right)^{1/2}\right] \qquad [0<b<a, \quad |\operatorname{Re}\nu|<1]$$

ET II 360(11)

6.77 Combinations of Bessel functions and the logarithm, or arctangent

6.771 $$\int_0^\infty x^{\mu+\frac{1}{2}} \ln x\, J_\nu(ax)\, dx = \frac{2^{\mu-\frac{1}{2}}\,\Gamma\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right) a^{\mu+\frac{3}{2}}}\left[\psi\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)+\psi\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right)-\ln\frac{a^2}{4}\right]$$
$$\left[a>0, \quad -\operatorname{Re}\nu-\tfrac{3}{2}<\operatorname{Re}\mu<0\right]$$

ET II 32(25)

6.772

1. $$\int_0^\infty \ln x\, J_0(ax)\, dx = -\frac{1}{a}\left[\ln(2a)+\boldsymbol{C}\right]$$ WA 430(4)a, ET II 10(27)

2. $$\int_0^\infty \ln x\, J_1(ax)\, dx = -\frac{1}{a}\left[\ln\left(\frac{a}{2}\right)+\boldsymbol{C}\right]$$ ET II 19(11)

3. $$\int_0^\infty \ln\left(a^2+x^2\right) J_1(bx)\, dx = \frac{2}{b}\left[K_0(ab)+\ln a\right]$$ ET II 19(12)

4. $$\int_0^\infty J_1(tx)\ln\sqrt{1+t^4}\, dt = \frac{2}{x}\ker x$$ MO 46

6.773 $$\int_0^\infty \frac{\ln\left(x+\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}\, J_0(bx)\, dx = \left[\frac{1}{2}K_0^2\left(\frac{ab}{2}\right)+\ln a\, I_0\left(\frac{ab}{2}\right) K_0\left(\frac{ab}{2}\right)\right]$$
$$[a>0, \quad b>0]$$

ET II 10(28)

6.774 $$\int_0^\infty \ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\, J_0(bx)\frac{dx}{\sqrt{x^2+a^2}} = K_0^2\left(\frac{ab}{2}\right) \qquad [\operatorname{Re}a>0, \quad b>0]$$ ET II 10(29)

6.775 $$\int_0^\infty x\left[\ln\left(1+\sqrt{a^2+x^2}\right)-\ln x\right] J_0(bx)\, dx = \frac{1}{b^2}\left(1-e^{-ab}\right)$$
$$[\operatorname{Re}a>0, \quad b>0]$$

ET II 12(55)

6.776 $$\int_0^\infty x \ln\left(1+\frac{a^2}{x^2}\right) J_0(bx)\,dx = \frac{2}{b}\left[\frac{1}{b} - a\,K_1(ab)\right] \qquad [\operatorname{Re} a > 0, \quad b > 0]$$ ET II 10(30)

6.777 $$\int_0^\infty J_1(tx) \arctan t^2\,dt = -\frac{2}{x}\operatorname{kei} x$$ MO 46

6.78 Combinations of Bessel and other special functions

6.781 $$\int_0^\infty \operatorname{si}(ax)\,J_0(bx)\,dx = -\frac{1}{b}\arcsin\left(\frac{b}{a}\right) \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
ET II 13(6)

6.782

1. $$\int_0^\infty \operatorname{Ei}(-x)\,J_0\left(2\sqrt{zx}\right)\,dx = \frac{e^{-z}-1}{z}$$ NT 60(4)

2. $$\int_0^\infty \operatorname{si}(x)\,J_0\left(2\sqrt{zx}\right)\,dx = -\frac{\sin z}{z}$$ NT 60(6)

3. $$\int_0^\infty \operatorname{ci}(x)\,J_0\left(2\sqrt{zx}\right)\,dx = \frac{\cos z - 1}{z}$$ NT 60(5)

4. $$\int_0^\infty \operatorname{Ei}(-x)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = \frac{\operatorname{Ei}(-z) - \boldsymbol{C} - \ln z}{\sqrt{z}}$$ NT 60(7)

5. $$\int_0^\infty \operatorname{si}(x)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \operatorname{si}(z)}{\sqrt{z}}$$ NT 60(9)

6. $$\int_0^\infty \operatorname{ci}(z)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = \frac{\operatorname{ci}(z) - \boldsymbol{C} - \ln z}{\sqrt{z}}$$ NT 60(8)

7. $$\int_0^\infty \operatorname{Ei}(-x)\,Y_0\left(2\sqrt{zx}\right)\,dx = \frac{\boldsymbol{C} + \ln z - e^2\operatorname{Ei}(-z)}{\pi z}$$ NT 63(5)

6.783

1. $$\int_0^\infty x \operatorname{si}\left(a^2x^2\right) J_0(bx)\,dx = -\frac{2}{b^2}\sin\left(\frac{b^2}{4a^2}\right) \qquad [a > 0]$$ ET II 13(7)a

2. $$\int_0^\infty x \operatorname{ci}\left(a^2x^2\right) J_0(bx)\,dx = \frac{2}{b^2}\left[1 - \cos\left(\frac{b^2}{4a^2}\right)\right] \qquad [a > 0]$$ ET II 13(8)a

3. $$\int_0^\infty \operatorname{ci}\left(a^2x^2\right) J_0(bx)\,dx = \frac{1}{b}\left[\operatorname{ci}\left(\frac{b^2}{4a^2}\right) + \ln\left(\frac{b^2}{4a^2}\right) + 2\boldsymbol{C}\right]$$
$[a > 0]$ ET II 13(8)a

4. $$\int_0^\infty \operatorname{si}\left(a^2x^2\right) J_1(bx)\,dx = \frac{1}{b}\left[-\operatorname{si}\left(\frac{b^2}{4a^2}\right) - \frac{\pi}{2}\right] \qquad [a > 0]$$ ET II 20(25)a

6.784

1. $$\int_0^\infty x^{\nu+1}\left[1 - \Phi(ax)\right] J_\nu(bx)\,dx = a^{-\nu}\frac{\Gamma\left(\nu + \frac{3}{2}\right)}{b^2\,\Gamma(\nu+2)}\exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu+\frac{1}{2},\,\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re}\nu > -1\right]$$
ET II 92(22)

2. $$\int_0^\infty x^\nu\,[1-\Phi(ax)]\,J_\nu(bx)\,dx=\sqrt{\frac{2}{\pi}}\,\frac{a^{\frac12-\nu}\,\Gamma\left(\nu+\frac12\right)}{b^{3/2}\,\Gamma\left(\nu+\frac32\right)}\exp\left(-\frac{b^2}{8a^2}\right)M_{\frac12\nu-\frac14,\frac12\nu+\frac14}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a|<\frac{\pi}{4},\quad \operatorname{Re}\nu>-\frac12,\quad b>0\right]$$
ET II 92(23)

6.785 $$\int_0^\infty \frac{\exp\left(\frac{a^2}{2x}-x\right)}{x}\left[1-\Phi\left(\frac{a}{\sqrt{2x}}\right)\right]K_\nu(x)\,dx=\frac{\pi^{5/2}}{4}\sec(\nu\pi)\left\{[J_\nu(a)]^2+[Y_\nu(a)]^2\right\}$$
$$\left[\operatorname{Re}a>0,\quad |\operatorname{Re}\nu|<\tfrac12\right]$$ ET II 370(46)

6.786 $$\int_0^\infty x^{\nu-2\mu+2n+2}e^{x2}\,\Gamma\left(\mu,x^2\right)\,Y_\nu(bx)\,dx$$
$$=(-1)^n\frac{\Gamma\left(\frac32-\mu+\nu+n\right)\Gamma\left(\frac32-\mu+n\right)}{b\,\Gamma(1-\mu)}\exp\left(\frac{b^2}{8}\right)W_{\mu-\frac12\nu-n-1,\frac12\nu}\left(\frac{b^2}{4}\right)$$
$$\left[n\text{ is an integer},\quad b>0,\quad \operatorname{Re}(\nu-\mu+n)>-\tfrac32,\quad \operatorname{Re}(-\mu+n)>-\tfrac32,\quad \operatorname{Re}\nu<\tfrac12-2n\right]$$
ET II 108(2)

6.787 $$\int_0^\infty \frac{x^{\nu+2n-\frac12}}{\mathrm{B}(a+x,a-x)}\,J_\nu(bx)\,dx=0$$
$$\left[\pi\le b<\infty,\quad -1<\operatorname{Re}\nu<2a-2n-\tfrac72\right]$$ ET II 92(21)

6.79 Integration of Bessel functions with respect to the order

6.791

1. $$\int_{-\infty}^\infty K_{ix+iy}(a)\,K_{ix+iz}(b)\,dx=\pi\,K_{iy-iz}(a+b)\qquad [|\arg a|+|\arg b|<\pi]$$ ET II 382(21)

2. $$\int_{-\infty}^\infty J_{\nu-x}(a)\,J_{\mu+x}(a)\,dx=J_{\mu+\nu}(2a)\qquad [\operatorname{Re}(\mu+\nu)>1]$$ ET II 379(1)

3. $$\int_{-\infty}^\infty J_{\kappa+x}(a)\,J_{\lambda-x}(a)\,J_{\mu+x}(a)\,J_{\nu-x}(a)\,dx$$
$$=\frac{\Gamma(\kappa+\lambda+\mu+\nu+1)}{\Gamma(\kappa+\lambda+1)\,\Gamma(\lambda+\mu+1)\,\Gamma(\mu+\nu+1)\,\Gamma(\nu+\kappa+1)}$$
$$\times\,{}_4F_5\left(\frac{\kappa+\lambda+\mu+\nu+1}{2},\frac{\kappa+\lambda+\mu+\nu+1}{2},\frac{\kappa+\lambda+\mu+\nu}{2}+1,\frac{\kappa+\lambda+\mu+\nu}{2}+1;\right.$$
$$\left.\kappa+\lambda+\mu+\nu+1,\kappa+\lambda+1,\lambda+\mu+1,\mu+\nu+1,\nu+\kappa+1;-4a^2\right)$$
$$[\operatorname{Re}(\kappa+\lambda+\mu+\nu)>-1]$$ ET II 379(3)

6.792

1. $$\int_{-\infty}^\infty e^{\pi x}\,K_{ix+iy}(a)\,K_{ix+iz}(b)\,dx=\pi e^{-\pi z}\,K_{i(y-z)}(a-b)$$
$$[a>b>0]$$ ET II 382(22)

2. $$\int_{-\infty}^{\infty} e^{i\varrho x}\, K_{\nu+ix}(\alpha)\, K_{\nu-ix}(\beta)\, dx = \pi \left(\frac{\alpha e^{\rho}+\beta}{\alpha+\beta e^{\rho}}\right)^{\nu} K_{2\nu}\left(\sqrt{\alpha^2+\beta^2+2\alpha\beta\cosh\varrho}\right)$$
$$[|\arg\alpha|+|\arg\beta|+|\operatorname{Im}\varrho|<\pi]$$
ET II 382(23)

3. $$\int_{-\infty}^{\infty} e^{(\pi-\gamma)x}\, K_{ix+iy}(a)\, K_{ix+iz}(b)\, dx = \pi e^{-\beta y-\alpha z}\, K_{iy-iz}(c)$$
$[0<\gamma<\pi,\quad a>0,\quad b>0,\quad c>0,\quad \alpha,\beta,\gamma$—the angles of the triangle with sides a, b, $c]$

ET II 382(24), EH II 55(44)a

4.[11] $$\int_{-\infty}^{\infty} e^{-cxi}\, H^{(2)}_{\nu-ix}(a)\, H^{(2)}_{\nu+ix}(b)\, dx = 2i\left(\frac{h}{k}\right)^{2\nu} H^{(2)}_{2\nu}(hk)$$
$$h=\sqrt{ae^{\frac12 c}+be^{-\frac12 c}},\quad k=\sqrt{ae^{-\frac12 c}+be^{\frac12 c}}\qquad [a,b>0,\quad c \text{ is real}]$$
ET II 380(11)

5. $$\int_{-\infty}^{\infty} a^{-\mu-x}b^{-\nu+x}e^{cxi}\, J_{\mu+x}(a)\, J_{\nu-x}(b)\, dx$$
$$= \left[\frac{2\cos\left(\frac{c}{2}\right)}{a^2e^{-\frac12 ci}+b^2e^{\frac12 ci}}\right]^{\frac12\mu+\frac12\nu} \exp\left[\frac{c}{2}(\nu-\mu)i\right] J_{\mu+\nu}\left\{\left[2\cos\left(\frac{c}{2}\right)\left(a^2e^{-\frac12 ci}+b^2e^{\frac12 ci}\right)\right]^{1/2}\right\}$$
$$[a>0,\quad b>0,\quad |c|<\pi,\quad \operatorname{Re}(\mu+\nu)>1]$$
$$= 0$$
$$[a>0,\quad b>0,\quad |c|\geq\pi,\quad \operatorname{Re}(\mu+\nu)>1]$$
EH II 54(41), ET II 379(2)

6.793

1. $$\int_{-\infty}^{\infty} e^{-cxi}\left[J_{\nu-ix}(a)\, Y_{\nu+ix}(b)+Y_{\nu-ix}(a)\, J_{\nu+ix}(b)\right] dx = -2\left(\frac{h}{k}\right)^{2\nu} J_{2\nu}(hk)$$
$$h=\sqrt{ae^{\frac12 c}+be^{-\frac12 c}},\qquad k=\sqrt{ae^{-\frac12 c}+be^{\frac12 c}}\qquad [a,b>0,\quad \operatorname{Im}c=0]$$
ET II 380(9)

2. $$\int_{-\infty}^{\infty} e^{-cxi}\left[J_{\nu-ix}(a)\, J_{\nu+ix}(b)-Y_{\nu-ix}(a)\, Y_{\nu+ix}(b)\right] dx = 2\left(\frac{h}{k}\right)^{2\nu} Y_{2\nu}(hk)$$
$$h=\sqrt{ae^{\frac12 c}+be^{-\frac12 c}},\qquad k=\sqrt{ae^{-\frac12 c}+be^{\frac12 c}}\qquad [a,b>0,\quad \operatorname{Im}c=0]$$
ET II 380(10)

3.[10] $$\int_{-\infty}^{\infty} e^{i\gamma x}\operatorname{sech}(\pi x)\left[J_{-ix}(\alpha)\, J_{ix}(\beta)-J_{ix}(\alpha)\, J_{-ix}(\beta)\right] dx = 2i\,\mathrm{H}(\sigma)\operatorname{sign}(\beta-\alpha)\, J_0\left(\sigma^{1/2}\right)$$
$$\left[\alpha,\beta,\gamma\in\mathbb{R},\quad \alpha,\beta>0,\quad \sigma=\alpha^2+\beta^2-2\alpha\beta\cosh\gamma,\quad \mathrm{H}(\sigma) \text{ the Heaviside step function}\right]$$

6.794

1. $$\int_0^{\infty} K_{ix}(a)\, K_{ix}(b)\cosh[(\pi-\varphi)x]\, dx = \frac{\pi}{2}\, K_0\left(\sqrt{a^2+b^2-2ab\cos\varphi}\right)$$
EH II 55(42)

2. $$\int_0^{\infty} \cosh\left(\frac{\pi}{2}x\right) K_{ix}(a)\, dx = \frac{\pi}{2}\qquad [a>0]$$
ET II 382(19)

3. $$\int_0^\infty \cosh(\varrho x)\, K_{ix+\nu}(a)\, K_{-ix+\nu}(a)\, dx = \frac{\pi}{2}\, K_{2\nu}\left[2a\cos\left(\frac{\varrho}{2}\right)\right]$$
$[2|\arg a| + |\mathrm{Re}\,\varrho| < \pi]$ ET II 383(28)

4. $$\int_{-\infty}^\infty \mathrm{sech}\left(\frac{\pi}{2}x\right) J_{ix}(a)\, dx = 2\sin a$$
$[a > 0]$ ET II 380(6)

5. $$\int_{-\infty}^\infty \mathrm{cosech}\left(\frac{\pi}{2}x\right) J_{ix}(a)\, dx = -2i\cos a$$
$[a > 0]$ ET II 380(7)

6. $$\int_0^\infty \mathrm{sech}(\pi x)\left\{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\right\} dx = -\,Y_0(2a) - \mathbf{E}_0(2a)$$
$[a > 0]$ ET II 380(12)

7. $$\int_0^\infty x\sinh\left(\frac{\pi}{2}x\right) K_{ix}(a)\, dx = \frac{\pi a}{2}$$
$[a > 0]$ ET II 382(20)

8. $$\int_0^\infty x\tanh(\pi x)\, K_{ix}(\beta)\, K_{ix}(\alpha)\, dx = \frac{\pi}{2}\sqrt{\alpha\beta}\frac{\exp(-\beta-\alpha)}{\alpha+\beta}$$
$[|\arg\beta| < \pi, \quad |\arg\alpha| < \pi]$ ET II 175(4)

9. $$\int_0^\infty x\sinh(\pi x)\, K_{2ix}(\alpha)\, K_{ix}(\beta)\, dx = \frac{\pi^{3/2}\alpha}{2^{5/2}\sqrt{\beta}}\exp\left(-\beta - \frac{\alpha^2}{8\beta}\right)$$
$\left[\beta > 0, \quad |\arg\alpha| < \frac{\pi}{4}\right]$ ET II 175(5)

10. $$\int_0^\infty \frac{x\sinh(\pi x)}{x^2+n^2}\, K_{ix}(\alpha)\, K_{ix}(\beta)\, dx = \frac{\pi^2}{2}\, I_n(\beta)\, K_n(\alpha) \qquad [0 < \beta < \alpha; \quad n = 0, 1, 2, \ldots]$$
$$= \frac{\pi^2}{2}\, I_n(\alpha)\, K_n(\beta) \qquad [0 < \alpha < \beta; \quad n = 0, 1, 2, \ldots]$$
ET II 176(8)

11. $$\int_0^\infty x\sinh(\pi x)\, K_{ix}(\alpha)\, K_{ix}(\beta)\, K_{ix}(\gamma)\, dx = \frac{\pi^2}{4}\exp\left[-\frac{\gamma}{2}\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\gamma^2}\right)\right]$$
$\left[|\arg\alpha| + |\arg\beta| < \frac{\pi}{2}, \quad \gamma > 0\right]$
ET II 176(9)

12. $$\int_0^\infty x\sinh\left(\frac{\pi}{2}x\right) K_{\frac{1}{2}ix}(\alpha)\, K_{\frac{1}{2}ix}(\beta)\, K_{ix}(\gamma)\, dx = \frac{\pi^2\gamma}{2\sqrt{\gamma^2+4\alpha\beta}}\exp\left[-\frac{(\alpha+\beta)\sqrt{\gamma^2+4\alpha\beta}}{2\sqrt{\alpha\beta}}\right]$$
$[|\arg\alpha| + |\arg\beta| < \pi, \quad \gamma > 0]$
ET II 176(10)

13. $$\int_0^\infty x\sinh(\pi x)\, K_{\frac{1}{2}ix+\lambda}(\alpha)\, K_{\frac{1}{2}ix-\lambda}(\alpha)\, K_{ix}(\gamma)\, dx = 0 \qquad [0 < \gamma < 2\alpha]$$
$$= \frac{\pi^2\gamma}{2^{2\lambda+1}\alpha^{2\lambda}z}\left[(\gamma+z)^{2\lambda} + (\gamma-z)^{2\lambda}\right]$$
$$z = \sqrt{\gamma^2 - 4\alpha^2} \qquad [0 < 2\alpha < \gamma]$$
ET II 176(11)

6.795

1. $$\int_0^\infty \cos(bx)\, K_{ix}(a)\, dx = \frac{\pi}{2} e^{-a\cosh b} \qquad \left[|\mathrm{Im}\, b| < \frac{\pi}{2}, \quad a > 0\right]$$
EH II 55(46), ET II 175(2)

2. $$\int_0^\infty J_x(ax)\, J_{-x}(ax) \cos(\pi x)\, dx = \frac{1}{4}\left(1 - a^2\right)^{-1/2} \qquad [|a| < 1]$$
ET II 380(4)

3. $$\int_0^\infty x \sin(ax)\, K_{ix}(b)\, dx = \frac{\pi b}{2} \sinh a \exp\left(-b \cosh a\right) \qquad \left[|\mathrm{Im}\, a| < \frac{\pi}{2}, \quad b > 0\right]$$
ET II 175(1)

4. $$\int_{-\infty}^{-\infty} \frac{\sin[(\nu + ix)\pi]}{n + \nu + ix}\, K_{\nu+ix}(a)\, K_{\nu-ix}(b)\, dx = \pi^2\, I_n(a)\, K_{n+2\nu}(b) \qquad [0 < a < b; \quad n = 0, 1, \ldots]$$
$$= \pi^2\, K_{n+2\nu}(a)\, I_n(b) \qquad [0 < b < a; \quad n = 0, 1, \ldots]$$
ET II 382(25)

5. $$\int_0^\infty x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a)\, K_{ix}(b)\, dx = \frac{\pi^{3/2} b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right)$$
$$\left[|\arg a| < \frac{\pi}{2}, \quad b > 0\right]$$
ET II 175(6)

6.796

1. $$\int_{-\infty}^\infty \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)}\, J_{ix}(a)\, dx = -i \exp\left(ia \cosh b\right) \qquad [a > 0, \quad b > 0]$$
ET II 380(8)

2. $$\int_0^\infty \cos(bx) \cosh\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2} \cos\left(a \sinh b\right)$$
EH II 55(47)

3. $$\int_0^\infty \sin(bx) \sinh\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2} \sin\left(a \sinh b\right)$$
EH II 55(48)

4. $$\int_0^\infty \cos(bx) \cosh(\pi x) \left[K_{ix}(a)\right]^2\, dx = -\frac{\pi^2}{4}\, Y_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$$
$$[a > 0, \quad b > 0]$$
ET II 383(27)

5. $$\int_0^\infty \sin(bx) \sinh(\pi x) \left[K_{ix}(a)\right]^2\, dx = \frac{\pi^2}{4}\, J_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$$
$$[a > 0, \quad b > 0]$$
ET II 382(26)

6.797

1. $$\int_0^\infty x e^{\pi x} \sinh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$$
$$= i 2^\nu \sqrt{\pi}\, \Gamma\left(\tfrac{1}{2} + \nu\right) (ab)^\nu (a + b)^{-\nu}\, K_\nu(a + b)$$
$$[a > 0, \quad b > 0, \quad \mathrm{Re}\, \nu > 0]$$
ET II 381(14)

2. $$\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx = \frac{i \pi^{3/2} 2^\nu}{\Gamma\left(\frac{1}{2} - \nu\right)} (b - a)^{-\nu}\, H_\nu^{(2)}(b - a)$$
$$\left[0 < a < b, \quad 0 < \mathrm{Re}\, \nu < \tfrac{1}{2}\right]$$
ET II 381(15)

3. $$\int_0^\infty xe^{\pi x}\sinh(\pi x)\,\Gamma\left(\frac{\nu+ix}{2}\right)\Gamma\left(\frac{\nu-ix}{2}\right)H_{ix}^{(2)}(a)\,H_{ix}^{(2)}(b)\,dx = i\pi 2^{2-\nu}(ab)^\nu\left(a^2+b^2\right)^{-\frac{1}{2}\nu}H_\nu^{(2)}\left(\sqrt{a^2+b^2}\right)$$
$[a>0,\quad b>0,\quad \operatorname{Re}\nu>0]$ ET II 381(16)

4.[11] $$\int_0^\infty x\sinh(\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,dx = 2^{\lambda-1}\pi^{3/2}(ab)^\lambda(a+b)^{-\lambda}\,\Gamma\left(\lambda+\tfrac{1}{2}\right)K_\lambda(a+b)$$
$[|\arg a|<\pi,\quad \operatorname{Re}\lambda>0,\quad b>0]$
ET II 176(12)

5. $$\int_0^\infty x\sinh(2\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,dx = \frac{2^\lambda\pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2}-\lambda\right)}\left(\frac{ab}{|b-a|}\right)^\lambda K_\lambda\left(|b-a|\right)$$
$\left[a>0,\quad 0<\operatorname{Re}\lambda<\tfrac{1}{2},\quad b>0\right]$
ET II 176(13)

6. $$\int_0^\infty x\sinh(\pi x)\,\Gamma\left(\lambda+\tfrac{1}{2}ix\right)\Gamma\left(\lambda-\tfrac{1}{2}ix\right)K_{ix}(a)\,K_{ix}(b)\,dx = 2\pi^2\left(\frac{ab}{2\sqrt{a^2+b^2}}\right)K_{2\lambda}\left(\sqrt{a^2+b^2}\right)$$
$\left[|\arg a|<\dfrac{\pi}{2},\quad \operatorname{Re}\lambda>0,\quad b>0\right]$
ET II 177(14)

7. $$\int_0^\infty \frac{x\tanh(\pi x)\,K_{ix}(a)\,K_{ix}(b)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}ix\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}ix\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi ab}{a^2+b^2}}\exp\left(-\sqrt{a^2+b^2}\right)$$
$\left[|\arg a|<\dfrac{\pi}{2},\quad b>0\right],$ ET II 177(15)

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.811

1. $$\int_0^\infty \mathbf{H}_\nu(bx)\,dx = -\frac{\cot\left(\frac{\nu\pi}{2}\right)}{b}$$ $[-2<\operatorname{Re}\nu<0,\quad b>0]$ ET II 158(1)

2. $$\int_0^\infty \mathbf{H}_\nu\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{J_{2\nu}\left(2a\sqrt{b}\right)}{b}$$ $\left[a>0,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$
ET II 170(37)

3. $$\int_0^\infty \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\frac{dx}{x} = -\frac{1}{a\sqrt{b}}\,J_{2\nu-1}\left(2a\sqrt{b}\right)$$ $\left[a>0,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$
ET II 170(38)

6.812

1. $$\int_0^\infty \frac{\mathbf{H}_1(bx)\,dx}{x^2+a^2} = \frac{\pi}{2a}\left[I_1(ab)-\mathbf{L}_1(ab)\right]$$ $[\operatorname{Re}a>0,\quad b>0]$ ET II 158(6)

2. $$\int_0^\infty \frac{\mathbf{H}_\nu(bx)}{x^2+a^2}\,dx = -\frac{\pi}{2a\sin\left(\frac{\nu\pi}{2}\right)}\,\mathbf{L}_\nu(ab) + \frac{b\cot\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\;{}_1F_2\left(1;\frac{3-\nu}{2};\frac{3+\nu}{2};\frac{a^2b^2}{2}\right)$$
$[\operatorname{Re} a>0,\quad b>0,\quad |\operatorname{Re}\nu|<2]$ ET II 159(7)

6.813

1. $$\int_0^\infty x^{s-1}\,\mathbf{H}_\nu(ax)\,dx = \frac{2^{s-1}\,\Gamma\left(\frac{s+\nu}{2}\right)}{a^s\,\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}\tan\left(\frac{s+\nu}{2}\pi\right)$$
$\left[a>0,\quad -1-\operatorname{Re}\nu<\operatorname{Re} s<\min\left(\frac{3}{2},1-\operatorname{Re}\nu\right)\right]$ WA 429(2), ET I 335(52)

2. $$\int_0^\infty x^{-\nu-1}\,\mathbf{H}_\nu(x)\,dx = \frac{2^{-\nu-1}\pi}{\Gamma(\nu+1)}$$
$\left[\operatorname{Re}\nu>-\frac{3}{2}\right]$ ET II 383(2)

3. $$\int_0^\infty x^{-\mu-\nu}\,\mathbf{H}_\mu(x)\,\mathbf{H}_\nu(x)\,dx = \frac{2^{-\mu-\nu}\sqrt{\pi}\,\Gamma(\mu+\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}$$
$[\operatorname{Re}(\mu+\nu)>0]$ WA 435(2), ET II 384(8)

4. $$\int_0^1 x^{\nu+1}\,\mathbf{H}_\nu(ax)\,dx = \frac{1}{a}\,\mathbf{H}_{\nu+1}(a)$$
$\left[a>0,\quad \operatorname{Re}\nu>-\frac{3}{2}\right]$ ET II 158(2)a

5. $$\int_0^1 x^{1-\nu}\,\mathbf{H}_\nu(ax)\,dx = \frac{a^{\nu-1}}{2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \frac{1}{a}\,\mathbf{H}_{\nu-1}(a)$$
$[a>0]$ ET II 158(3)a

6.814

1. $$\int_0^\infty \frac{x^{\nu+1}\,\mathbf{H}_\nu(bx)}{(x^2+a^2)^{1-\mu}}\,dx = \frac{2^{\mu-1}\pi a^{\mu+\nu}b^{-\mu}}{\Gamma(1-\mu)\cos[(\mu+\nu)\pi]}\,[I_{-\mu-\nu}(ab)-\mathbf{L}_{\mu+\nu}(ab)]$$
$\left[\operatorname{Re} a>0,\quad b>0,\quad \operatorname{Re}\nu>-\frac{3}{2},\quad \operatorname{Re}(\mu+\nu)<\frac{1}{2},\quad \operatorname{Re}(2\mu+\nu)<\frac{3}{2}\right]$ ET II 159(8)

6.815

1. $$\int_0^1 x^{\frac{1}{2}\nu}(1-x)^{\mu-1}\,\mathbf{H}_\nu\left(a\sqrt{x}\right)\,dx = 2^\mu a^{-\mu}\,\Gamma(\mu)\,\mathbf{H}_{\mu+\nu}(a)$$
$\left[\operatorname{Re}\nu>-\frac{3}{2},\quad \operatorname{Re}\mu>0\right]$ ET II 199(88)a

2. $$\int_0^1 x^{\lambda-\frac{1}{2}\nu-\frac{3}{2}}(1-x)^{\mu-1}\,\mathbf{H}_\nu\left(a\sqrt{x}\right)\,dx = \frac{\mathrm{B}(\lambda,\mu)a^{\nu+1}}{2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_2F_3\left(1,\lambda;\frac{3}{2},\nu+\frac{3}{2},\lambda+\mu;-\frac{a^2}{4}\right)$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\mu>0]$ ET II 199(89)a

6.82 Combinations of Struve functions, exponentials, and powers

6.821

1.[6] $$\int_0^\infty e^{-\alpha x}\,\mathbf{H}_{-n-\frac{1}{2}}(\beta x)\,dx = (-1)^n\beta^{n+\frac{1}{2}}\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2+\beta^2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET I 206(6)

2.[6] $$\int_0^\infty e^{-\alpha x}\,\mathbf{L}_{-n-\frac{1}{2}}(\beta x)\,dx=\beta^{n+\frac{1}{2}}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2-\beta^2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Re}\beta|]$ ET I 208(26)

3. $$\int_0^\infty e^{-\alpha x}\,\mathbf{H}_0(\beta x)\,dx=\frac{2}{\pi}\frac{\ln\left(\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET II 205(1)

4. $$\int_0^\infty e^{-\alpha x}\,\mathbf{L}_0(\beta x)\,dx=\frac{2}{\pi}\frac{\arcsin\left(\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}}$$
$[\operatorname{Re}\alpha>|\operatorname{Re}\beta|]$ ET II 207(18)

6.822 $$\int_0^\infty e^{(\nu+1)x}\,\mathbf{H}_\nu\left(a\sinh x\right)\,dx=\sqrt{\frac{\pi}{a}}\operatorname{cosec}(\nu\pi)\left[\sinh\left(\frac{a}{2}\right)I_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right)-\cosh\left(\frac{a}{2}\right)I_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$[\operatorname{Re}a>0,\quad -2<\operatorname{Re}\nu<0]$
ET II 385(11)

6.823

1. $$\int_0^\infty x^\lambda e^{-\alpha x}\,\mathbf{H}_\nu(bx)\,dx=\frac{b^{\nu+1}\,\Gamma(\lambda+\nu+2)}{2^\nu a^{\lambda+\nu+2}\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_3F_2\left(1,\frac{\lambda+\nu}{2}+1,\frac{\lambda+\nu+3}{2};\frac{3}{2},\nu+\frac{3}{2};-\frac{b^2}{a^2}\right)$$
$[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}(\lambda+\nu)>-2]$
ET II 161(19)

2. $$\int_0^\infty x^\nu e^{-\alpha x}\,\mathbf{L}_\nu(\beta x)\,dx=\frac{(2\beta)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\sqrt{\alpha^2-\beta^2}\right)^{2\nu+1}}-\frac{\Gamma(2\nu+1)\left(\frac{\beta}{\alpha}\right)^\nu}{\sqrt{\frac{\pi}{2}}\alpha\left(\beta^2-\alpha^2\right)^{\frac{1}{2}\nu+\frac{1}{4}}}P^{-\nu-\frac{1}{2}}_{-\nu-\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)$$
$\left[\operatorname{Re}\alpha>|\operatorname{Re}\beta|,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$
ET I 209(35)a

6.824

1. $$\int_0^\infty t^\nu e^{-at}\,\mathbf{L}_{2\nu}\left(2\sqrt{t}\right)\,dt=\frac{1}{a^{2\nu+1}}e^{\frac{1}{a}}\,\Phi\left(\frac{1}{\sqrt{a}}\right)$$
MI 51

2. $$\int_0^\infty t^\nu e^{-at}\,\mathbf{L}_{-2\nu}\left(\sqrt{t}\right)\,dt=\frac{1}{\Gamma\left(\frac{1}{2}-2\nu\right)a^{2\nu+1}}e^{\frac{1}{a}}\gamma\left(\frac{1}{2}-2\nu,\frac{1}{a}\right)$$
MI 51

6.825 $$\int_0^\infty x^{s-1}e^{-\alpha^2x^2}\,\mathbf{H}_\nu(\beta x)\,dx=\frac{\beta^{\nu+1}\,\Gamma\left(\frac{1}{2}+\frac{s}{2}+\frac{\nu}{2}\right)}{2^{\nu+1}\sqrt{\pi}\alpha^{\nu+s+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_2F_2\left(1,\frac{\nu+s+1}{2};\frac{3}{2},\nu+\frac{3}{2};-\frac{\beta^2}{4\alpha^2}\right)$$
$\left[\operatorname{Re}s>-\operatorname{Re}\nu-1,\quad |\arg\alpha|<\frac{\pi}{4}\right]$
ET I 335(51)a, ET II 162(20)

6.83 Combinations of Struve and trigonometric functions

6.831 $$\int_0^\infty x^{-\nu}\sin(ax)\,\mathbf{H}_\nu(bx)\,dx=0 \qquad \left[0<b<a,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
$$=\sqrt{\pi}2^{-\nu}b^{-\nu}\frac{\left(b^2-a^2\right)^{\nu-\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \qquad \left[0<a<b,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 162(21)

6.832 $$\int_0^\infty \sqrt{x}\sin(ax)\,\mathbf{H}_{\frac{1}{4}}\left(b^2x^2\right)\,dx = -2^{-3/2}\sqrt{\pi}\frac{\sqrt{a}}{b^2}\,Y_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right)$$
$[a>0]$ ET I 109(14)

6.84–6.85 Combinations of Struve and Bessel functions

6.841 $$\int_0^\infty \mathbf{H}_{\nu-1}(ax)\,Y_\nu(bx)\,dx = -a^{\nu-1}b^{-\nu} \qquad \left[0<b<a,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<b,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 114(36)

6.842 $$\int_0^\infty \left[\mathbf{H}_0(ax) - Y_0(ax)\right] J_0(bx)\,dx = \frac{4}{\pi(a+b)}\,\boldsymbol{K}\left(\frac{|a-b|}{a+b}\right)$$
$[a>0,\quad b>0]$ ET II 15(22)

6.843

1. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{1}{b}\,Y_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<\tfrac{5}{4}\right]$$
ET II 164(10)

2. $$\int_0^\infty K_{2\nu}\left(2a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{2^\nu}{\pi b}\,\Gamma(\nu+1)\,S_{-\nu-1,\nu}\left(\frac{a^2}{b}\right)$$
$[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}\nu>-1]$
ET II 168(27)

6.844 $$\int_0^\infty \left[\cos\left(\frac{\mu-\nu}{2}\pi\right)J_\mu\left(a\sqrt{x}\right) - \sin\left(\frac{\mu-\nu}{2}\pi\right)Y_\mu\left(a\sqrt{x}\right)\right]K_\mu\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{1}{a^2}\,W_{\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right)W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right)$$
$\left[|\arg a|<\frac{\pi}{4},\quad b>0,\quad \operatorname{Re}\nu>|\operatorname{Re}\mu|-2\right]$ ET II 169(35)

6.845

1. $$\int_0^\infty \left[\mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - Y_{-\nu}\left(\frac{a}{x}\right)\right]J_\nu(bx)\,dx = \frac{4}{\pi b}\cos(\nu\pi)\,K_{2\nu}\left(2\sqrt{ab}\right)$$
$\left[|\arg a|<\pi,\quad b>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$
ET II 73(7)

2. $$\int_0^\infty \left[J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{x}\right)\right]\mathbf{H}_\nu(bx)\,dx = \frac{1}{b}\left[\frac{2}{\pi}\,K_{2\nu}\left(2a\sqrt{b}\right) - Y_{2\nu}\left(2a\sqrt{b}\right)\right]$$
$\left[a>0,\quad b>0,\quad -\tfrac{3}{2}<\operatorname{Re}\nu<0\right]$
ET II 170(39)

6.846 $$\int_0^\infty \left[\frac{2}{\pi}\,K_{2\nu}\left(2a\sqrt{x}\right) + Y_{2\nu}\left(2a\sqrt{x}\right)\right]\mathbf{H}_\nu(bx)\,dx = \frac{1}{b}\,J_\nu\left(\frac{a^2}{b}\right)$$
$\left[a>0,\quad b>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$
ET II 169(30)

6.847 $$\int_0^\infty \left[\cos\frac{\nu\pi}{2}\,J_\nu(ax) + \sin\frac{\nu\pi}{2}\,\mathbf{H}_\nu(ax)\right]\frac{dx}{x^2+k^2} = \frac{\pi}{2k}\left[I_\nu(ak) - \mathbf{L}_\nu(ak)\right]$$
$\left[a>0,\quad \operatorname{Re}k>0,\quad -\tfrac{1}{2}<\operatorname{Re}\nu<2\right]$
ET II 384(5)a, WA 467(8)

6.848

1. $$\int_0^\infty x\left[I_\nu(ax)-\mathbf{L}_{-\nu}(ax)\right]J_\nu(bx)\,dx=\frac{2}{\pi}\left(\frac{a}{b}\right)^{\nu-1}\cos(\nu\pi)\frac{1}{a^2+b^2}$$
$$\left[\operatorname{Re}a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
ET II 74(12)

2. $$\int_0^\infty x\left[\mathbf{H}_{-\nu}(ax)-Y_{-\nu}(ax)\right]J_\nu(bx)\,dx=2\frac{\cos(\nu\pi)}{a^\nu\pi}b^{\nu-1}\frac{1}{a+b}$$
$$\left[|\arg a|<\pi,\quad -\tfrac{1}{2}<\operatorname{Re}\nu,\quad b>0\right]$$
ET II 73(5)

6.849

1. $$\int_0^\infty x\,K_\nu(ax)\,\mathbf{H}_\nu(bx)\,dx=a^{-\nu-1}b^{\nu+1}\frac{1}{a^2+b^2}\qquad\left[\operatorname{Re}a>0,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$
ET II 164(12)

2. $$\int_0^\infty x\left[K_\mu(ax)\right]^2\mathbf{H}_0(bx)\,dx=-2^{-\mu-1}\pi a^{-2\mu}\frac{\left[(z+b)^{2\mu}+(z-b)^{2\mu}\right]}{bz}\sec(\mu\pi),$$
$$z=\sqrt{4a^2+b^2}\qquad\left[\operatorname{Re}a>0,\quad b>0,\quad |\operatorname{Re}\mu|<\tfrac{3}{2}\right]$$
ET II 166(18)

6.851

1. $$\int_0^\infty x\left\{\left[J_{\frac{1}{2}\nu}(ax)\right]^2-\left[Y_{\frac{1}{2}\nu}(ax)\right]^2\right\}\mathbf{H}_\nu(bx)\,dx$$
$$=0\qquad\left[0<b<2a,\quad -\tfrac{3}{2}<\operatorname{Re}\nu<0\right]$$
$$=\frac{4}{\pi b}\frac{1}{\sqrt{b^2-4a^2}}\qquad\left[0<2a<b,\quad -\tfrac{3}{2}<\operatorname{Re}\nu<0\right]$$
ET II 164(7)

2. $$\int_0^\infty x^{\nu+1}\left\{\left[J_\nu(ax)\right]^2-\left[Y_\nu(ax)\right]^2\right\}\mathbf{H}_\nu(bx)\,dx$$
$$=0\qquad\left[0<b<2a,\quad -\tfrac{3}{4}<\operatorname{Re}\nu<0\right]$$
$$=\frac{2^{3\nu+2}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac{1}{2}}\qquad\left[0<2a<b,\quad -\tfrac{3}{4}<\operatorname{Re}\nu<0\right]$$
ET II 163(6)

6.852

1. $$\int_0^\infty x^{1-\mu-\nu}\,J_\nu(x)\,\mathbf{H}_\mu(x)\,dx=\frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1)\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\nu>\tfrac{1}{2},\quad \operatorname{Re}(\mu+\nu)>1\right]$$
ET II 383(4)

2. $$\int_0^\infty x^{\mu-\nu+1}\,Y_\mu(ax)\,\mathbf{H}_\nu(bx)\,dx$$
$$=0\qquad\left[0<b<a,\quad \operatorname{Re}(\nu-\mu)>0,\quad -\tfrac{3}{2}<\operatorname{Re}\mu<\tfrac{1}{2}\right]$$
$$=\frac{2^{1+\mu-\nu}a^\mu b^{-\nu}}{\Gamma(\nu-\mu)}\left(b^2-a^2\right)^{\nu-\mu-1}\qquad\left[0<a<b,\quad \operatorname{Re}(\nu-\mu)>0,\quad -\tfrac{3}{2}<\operatorname{Re}\mu<\tfrac{1}{2}\right]$$
ET II 163(3)

3. $$\int_0^\infty x^{\mu+\nu+1}\, K_\mu(ax)\, \mathbf{H}_\nu(bx)\, dx = \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}}\, \Gamma\left(\mu+\nu+\frac{3}{2}\right) F\left(1, \mu+\nu+\frac{3}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) > -\frac{3}{2}\right]$ ET II 165(13)

6.853

1. $$\int_0^\infty x^{1-\mu} \left[\sin(\mu\pi)\, J_{\mu+\nu}(ax) + \cos(\mu\pi)\, Y_{\mu+\nu}(ax)\right] \mathbf{H}_\nu(bx)\, dx$$
$$= 0 \qquad \left[0 < b < a, \quad 1 < \operatorname{Re}\mu < \tfrac{3}{2}, \quad \operatorname{Re}\nu > -\tfrac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \tfrac{1}{2}\right]$$
$$= \frac{b^\nu \left(b^2 - a^2\right)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu}\, \Gamma(\mu)} \qquad \left[0 < a < b, \quad 1 < \operatorname{Re}\mu < \tfrac{3}{2}, \quad \operatorname{Re}\nu > -\tfrac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \tfrac{1}{2}\right]$$
ET II 163(4)

2. $$\int_0^\infty x^{\lambda+\frac{1}{2}} \left[I_\mu(ax) - \mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\, dx$$
$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}}\, G^{22}_{33}\left(\frac{b^2}{a^2}\,\middle|\, \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu+\lambda) > -\frac{3}{2}, \quad -\operatorname{Re}\nu - \frac{5}{2} < \operatorname{Re}(\lambda-\mu) < 1\right]$ ET II 76(21)

3. $$\int_0^\infty x^{\lambda+\frac{1}{2}} \left[\mathbf{H}_\mu(ax) - Y_\mu(ax)\right] J_\nu(bx)\, dx$$
$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}}\, G^{23}_{33}\left(\frac{b^2}{a^2}\,\middle|\, \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$$
$\left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda+\mu) < 1, \quad \operatorname{Re}(\lambda+\nu) + \frac{3}{2} > |\operatorname{Re}\mu|\right]$ ET II 73(6)

4. $$\int_0^\infty \sqrt{x} \left[I_{\nu-\frac{1}{2}}(ax) - \mathbf{L}_{\nu-\frac{1}{2}}(ax)\right] J_\nu(bx)\, dx = \sqrt{\frac{2}{\pi}}\, a^{\nu-\frac{1}{2}} b^{-\nu} \frac{1}{\sqrt{a^2+b^2}}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET II 74(11)

5. $$\int_0^\infty x^{\mu-\nu+1} \left[I_\mu(ax) - \mathbf{L}_\mu(ax)\right] J_\nu(bx)\, dx = \frac{2^{\mu-\nu+1} a^{\mu-1} b^{\nu-2\mu-1}}{\sqrt{\pi}\, \Gamma\left(\nu-\mu+\frac{1}{2}\right)} F\left(1, \frac{1}{2}; \nu-\mu+\frac{1}{2}; -\frac{b^2}{a^2}\right)$$
$\left[-1 < 2\operatorname{Re}\mu + 1 < \operatorname{Re}\nu + \frac{1}{2}, \quad \operatorname{Re} a > 0, \quad b > 0\right]$ ET II 74(13)

6. $$\int_0^\infty x^{\mu-\nu+1} \left[I_\mu(ax) - \mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\, dx = \frac{2^{\mu-\nu+1} a^{-\mu-1} b^{\nu-1}}{\Gamma\left(\frac{1}{2}-\mu\right) \Gamma\left(\frac{1}{2}+\nu\right)} F\left(1, \frac{1}{2}+\mu; \frac{1}{2}+\nu; -\frac{b^2}{a^2}\right)$$
$\left[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\mu > -1, \quad b > 0\right]$ ET II 75(18)

6.854

1. $$\int_0^\infty x\, \mathbf{H}_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{2^{1-\frac{1}{2}\nu} a\pi}\, S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -2\right]$
ET II 150(75)

2. $$\int_0^\infty x\,\mathbf{H}_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\,dx = -\frac{1}{2a}\,Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad \left[a>0,\quad b>0,\quad -2<\operatorname{Re}\nu<\tfrac{3}{2}\right]$$

ET II 73(3)

6.855

1. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[I_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)-\mathbf{L}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx = 2^{\frac{3}{2}}\frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi}b^{\nu+1}}\,J_{2\nu+1}\left(\sqrt{2ab}\right)K_{2\nu+1}\left(\sqrt{2ab}\right)$$
$$\left[\operatorname{Re}a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$

ET II 76(22)

2. $$\int_0^\infty \left[\mathbf{H}_{-\nu-1}\left(\frac{a}{x}\right)-Y_{-\nu-1}\left(\frac{a}{x}\right)\right] J_\nu(bx)\frac{dx}{x} = -\frac{4}{\pi\sqrt{ab}}\cos(\nu\pi)\,K_{-2\nu-1}\left(2\sqrt{ab}\right)$$
$$\left[|\arg a|<\pi,\quad b>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$

ET II 74(8)

3. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[\mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)-Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx$$
$$= -2^{5/2}\pi^{-3/2}a^{\nu+\frac{1}{2}}b^{-\nu-1}\sin(\nu\pi)\,K_{2\nu+1}\left(\sqrt{2ab}e^{\frac{1}{4}\pi i}\right)K_{2\nu+1}\left(\sqrt{2ab}e^{-\frac{1}{4}\pi i}\right)$$
$$\left[|\arg a|<\pi,\quad b>0,\quad -1<\operatorname{Re}\nu<-\tfrac{1}{6}\right]$$

ET II 74(9)

6.856 $$\int_0^\infty x\,Y_\nu\left(a\sqrt{x}\right)K_\nu\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{1}{2b^2}\exp\left(-\frac{a^2}{2b}\right)$$
$$\left[b>0,\quad |\arg a|<\frac{\pi}{4},\quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$

ET II 169(32)

6.857

1. $$\int_0^\infty x\exp\left(\frac{a^2x^2}{8}\right)K_{\frac{1}{2}\nu}\left(\frac{a^2x^2}{8}\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{2}{\sqrt{\pi}}a^{-\frac{\nu}{2}-1}b^{\frac{\nu}{2}-1}\cos\left(\frac{\nu\pi}{2}\right)\Gamma\left(-\frac{1}{2}\nu\right)\exp\left(\frac{b^2}{2a^2}\right)W_{k,m}\left(\frac{b^2}{a^2}\right)$$
$$k=\tfrac{1}{4}\nu,\qquad m=\tfrac{1}{2}+\tfrac{1}{4}\nu\qquad \left[|\arg a|<\tfrac{3}{4}\pi,\quad b>0,\quad -\tfrac{3}{2}<\operatorname{Re}\nu<0\right]$$

ET II 167(24)

2. $$\int_0^\infty x^{\sigma-2}\exp\left(-\frac{1}{2}a^2x^2\right)K_\mu\left(\frac{1}{2}a^2x^2\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{\sqrt{\pi}}{2^{\nu+2}}a^{-\nu-\sigma}b^{\nu+1}\frac{\Gamma\left(\frac{\nu+\sigma}{2}+\mu\right)\Gamma\left(\frac{\nu+\sigma}{2}-\mu\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{\nu+\sigma}{2}\right)}$$
$$\times\,{}_3F_3\left(1,\frac{\nu+\sigma}{2}+\mu,\frac{\nu+\sigma}{2}-\mu;\frac{3}{2},\nu+\frac{3}{2},\frac{\nu+\sigma}{2};-\frac{b^2}{4a^2}\right)$$
$$\left[b>0,\quad |\arg a|<\frac{\pi}{4},\quad \operatorname{Re}(\sigma+\nu)>2|\operatorname{Re}\mu|\right]$$

ET II 167(23)

6.86 Lommel functions

6.861

1. $$\int_0^\infty x^{\lambda-1}\, S_{\mu,\nu}(x)\, dx = \frac{\Gamma\left[\frac{1}{2}(1+\lambda+\mu)\right]\Gamma\left[\frac{1}{2}(1-\lambda-\mu)\right]\Gamma\left[\frac{1}{2}(1+\mu+\nu)\right]\Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]}{2^{2-\lambda-\mu}\,\Gamma\left[\frac{1}{2}(\nu-\lambda)+1\right]\Gamma\left[1-\frac{1}{2}(\lambda+\nu)\right]}$$
$$\left[-\operatorname{Re}\mu < \operatorname{Re}\lambda + 1 < \tfrac{5}{2}\right] \qquad \text{ET II 385(17)}$$

6.862

1. $$\int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}}(u-x)^{\sigma-1}\, s_{\mu,\nu}\left(a\sqrt{x}\right)\, dx$$
$$= \Gamma(\sigma)\frac{a^{\mu+1}u^{\lambda+\sigma}\,\Gamma(\lambda+1)}{(\mu-\nu+1)\,(\mu+\nu+1)\,\Gamma(\lambda+\sigma+1)}$$
$$\times\ {}_2F_3\left(1, 1+\lambda; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}, \lambda+\sigma+1; -\frac{a^2u}{4}\right)$$
$$[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\sigma > 0] \qquad \text{ET II 199(92)}$$

2. $$\int_u^\infty x^{\frac{1}{2}\nu}(x-u)^{\mu-1}\, s_{\lambda,\nu}\left(a\sqrt{x}\right)\, dx = \frac{\mathrm{B}\left[\mu, \frac{1}{2}(1-\lambda-\nu)-\mu\right] u^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{a^\mu}\, S_{\lambda+\mu,\mu+\nu}\left(a\sqrt{u}\right)$$
$$\left[\left|\arg\left(a\sqrt{u}\right)\right| < \pi, \quad 0 < 2\operatorname{Re}\mu < 1 - \operatorname{Re}(\lambda+\nu)\right] \qquad \text{ET II 211(71)}$$

6.863 $$\int_0^\infty \sqrt{x}e^{-\alpha x}\, s_{\mu,\frac{1}{4}}\left(\frac{x^2}{2}\right)\, dx = 2^{-2\mu-1}\sqrt{\alpha}\,\Gamma\left(2\mu+\frac{3}{2}\right) S_{-\mu-1,\frac{1}{4}}\left(\frac{\alpha^2}{2}\right)$$
$$\left[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -\tfrac{3}{4}\right] \qquad \text{ET I 209(38)}$$

6.864 $$\int_0^\infty \exp[(\mu+1)x]\, s_{\mu,\nu}\left(a\sinh x\right)\, dx = 2^{\mu-2}\pi\operatorname{cosec}(\mu\pi)\,\Gamma(\varrho)\,\Gamma(\sigma)$$
$$\times\left[I_\varrho\left(\frac{a}{2}\right) I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right) I_{-\sigma}\left(\frac{a}{2}\right)\right]$$
$$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1 \qquad [a > 0, \quad -2 < \operatorname{Re}\mu < 0] \qquad \text{ET II 386(22)}$$

6.865 $$\int_0^\infty \sqrt{\sinh x}\cosh(\nu x)\, S_{\mu,\frac{1}{2}}\left(a\cosh x\right)\, dx = \frac{\mathrm{B}\left(\frac{1}{4}-\frac{\mu+\nu}{2}, \frac{1}{4}-\frac{\mu-\nu}{2}\right)}{\sqrt{a}\,2^{\mu+\frac{3}{2}}}\, S_{\mu+\frac{1}{2},\nu}(a)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu + |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$\text{ET II 388(31)}$$

6.866

1. $$\int_0^\infty x^{-\mu-1}\cos(ax)\, s_{\mu,\nu}(x)\, dx$$
$$= 0 \qquad [a > 1]$$
$$= 2^{\mu-\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)\left(1-a^2\right)^{\frac{1}{2}\mu+\frac{1}{4}} P^{\mu-\frac{1}{2}}_{\nu-\frac{1}{2}}(a) \qquad [0 < a < 1]$$
$$\text{ET II 386(18)}$$

2. $$\int_0^\infty x^{-\mu}\sin(ax)\, S_{\mu,\nu}(x)\, dx = 2^{-\mu-\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(1-\frac{\mu+\nu}{2}\right)\Gamma\left(1-\frac{\mu-\nu}{2}\right)\left(a^2-1\right)^{\frac{1}{2}\mu-\frac{1}{4}} P^{\mu-\frac{1}{2}}_{\nu-\frac{1}{2}}(a)$$
$$[a > 1, \quad \operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|]$$
$$\text{ET II 387(23)}$$

6.867

1. $$\int_0^{\pi/2} \cos(2\mu x)\, S_{2\mu-1,2\nu}\left(a\cos x\right)\, dx$$
$$= \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1-\mu-\nu)\,\Gamma\left(1-\mu+\nu\right)} \left[J_{\mu+\nu}\left(\frac{a}{2}\right) Y_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) Y_{\mu+\nu}\left(\frac{a}{2}\right)\right]$$
$$[\operatorname{Re}\mu > -2, \quad |\operatorname{Re}\nu| < 1]$$ ET II 388(29)

2. $$\int_0^{\pi/2} \cos\left[(\mu+1)\,x\right] s_{\mu,\nu}\left(a\cos x\right)\, dx = 2^{\mu-2}\pi\,\Gamma(\varrho)\,\Gamma(\sigma)\, J_\varrho\left(\frac{a}{2}\right) J_\sigma\left(\frac{a}{2}\right)$$
$$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1 \qquad [\operatorname{Re}\mu > -2]$$ ET II 386(21)

6.868 $$\int_0^{\pi/2} \frac{\cos(2\mu x)}{\cos x}\, S_{2\mu,2\nu}\left(a\sec x\right)\, dx = \frac{\pi 2^{2\mu-1}}{a}\, W_{\mu,\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\mu,\nu}\left(ae^{-i\frac{\pi}{2}}\right)$$
$$[|\arg a| < \pi, \quad \operatorname{Re}\mu < 1]$$ ET II 388(30)

6.869

1. $$\int_0^\infty x^{1-\mu-\nu}\, J_\nu(ax)\, S_{\mu,-\mu-2\nu}(x)\, dx = \frac{\sqrt{\pi}a^{\nu-1}\,\Gamma(1-\mu-\nu)}{2^{\mu+2\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(a^2-1\right)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$$
$$\left[a > 1, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re}(\mu+\nu) < 1\right]$$
ET II 388(28)

2. $$\int_0^\infty x^{-\mu}\, J_\nu(ax)\, s_{\nu+\mu,-\nu+\mu+1}(x)\, dx$$
$$= 2^{\nu-1}\,\Gamma(\nu) a^{-\nu}\left(1-a^2\right)^\mu \qquad \left[0 < a < 1, \quad \operatorname{Re}\mu > -1, \quad -1e < \operatorname{Re}\nu < \tfrac{3}{2}\right]$$
$$= 0 \qquad \left[1 < a, \quad \operatorname{Re}\mu > -1, \quad -1 < \operatorname{Re}\nu < \tfrac{3}{2}\right]$$
ET II 388(28)

3. $$\int_0^\infty x\, K_\nu(bx)\, s_{\mu,\frac{1}{2}\nu}\left(ax^2\right)\, dx = \frac{1}{4a}\,\Gamma\left(\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\mu-\frac{1}{2}\nu+1\right) S_{-\mu-1,\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$$\left[\operatorname{Re}\mu > \tfrac{1}{2}|\operatorname{Re}\nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0\right]$$ ET II 151(78)

6.87 Thomson functions

6.871

1. $$\int_0^\infty e^{-\beta x}\operatorname{ber} x\, dx = \frac{\left(\sqrt{\beta^4+1}+\beta^2\right)^{1/2}}{\sqrt{2\left(\beta^4+1\right)}}$$ ME 40

2. $$\int_0^\infty e^{-\beta x}\operatorname{bei} x\, dx = \frac{\left(\sqrt{\beta^4+1}-\beta^2\right)^{1/2}}{\sqrt{2\left(\beta^4+1\right)}}$$ ME 40

6.872

1. $$\int_0^\infty e^{-\beta x}\,\mathrm{ber}_\nu\left(2\sqrt{x}\right)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu\pi}{4}\right)\right.$$
$$\left.-\,J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right]$$
MI 49

2. $$\int_0^\infty e^{-\beta x}\,\mathrm{bei}_\nu\left(2\sqrt{x}\right)\,dx=\frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu}{4}\pi\right)\right.$$
$$\left.-\,J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right]$$
MI 49

3. $$\int_0^\infty e^{-\beta x}\,\mathrm{ber}\left(2\sqrt{x}\right)\,dx=\frac{1}{\beta}\cos\frac{1}{\beta}$$ ME 40

4. $$\int_0^\infty e^{-\beta x}\,\mathrm{bei}\left(2\sqrt{x}\right)\,dx=\frac{1}{\beta}\sin\frac{1}{\beta}$$ ME 40

5. $$\int_0^\infty e^{-\beta x}\,\mathrm{ker}\left(2\sqrt{x}\right)\,dx=-\frac{1}{2\beta}\left[\cos\frac{1}{\beta}\,\mathrm{ci}\,\frac{1}{\beta}+\sin\frac{1}{\beta}\,\mathrm{si}\,\frac{1}{\beta}\right]$$ MI 50

6. $$\int_0^\infty e^{-\beta x}\,\mathrm{kei}\left(2\sqrt{x}\right)\,dx=-\frac{1}{2\beta}\left[\sin\frac{1}{\beta}\,\mathrm{ci}\,\frac{1}{\beta}-\cos\frac{1}{\beta}\,\mathrm{si}\,\frac{1}{\beta}\right]$$ MI 50

7. $$\int_0^\infty e^{-\beta x}\,\mathrm{ber}_\nu\left(2\sqrt{x}\right)\mathrm{bei}_\nu\left(2\sqrt{x}\right)\,dx=\frac{1}{2\beta}\,J_\nu\left(\frac{2}{\beta}\right)\sin\left(\frac{2}{\beta}+\frac{3\nu\pi}{2}\right)$$
$[\mathrm{Re}\,\nu>-1]$ MI 49

6.873 $$\int_0^\infty\left[\mathrm{ber}_\nu^2\left(2\sqrt{x}\right)+\mathrm{bei}_\nu^2\left(2\sqrt{x}\right)\right]e^{-\beta x}\,dx=\frac{1}{\beta}\,I_\nu\left(\frac{2}{\beta}\right)$$
$[\mathrm{Re}\,\nu>-1]$ ME 40

6.874

1. $$\int_0^\infty\frac{e^{-\beta x}}{\sqrt{x}}\,\mathrm{ber}_{2\nu}\left(2\sqrt{2x}\right)\,dx=\sqrt{\frac{\pi}{\beta}}\,J_\nu\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}-\frac{3\pi}{4}+\frac{3\nu\pi}{2}\right)$$
$\left[\mathrm{Re}\,\nu>-\frac{1}{2}\right]$ MI 49

2. $$\int_0^\infty\frac{e^{-\beta x}}{\sqrt{x}}\,\mathrm{bei}_{2\nu}\left(2\sqrt{2x}\right)\,dx=\sqrt{\frac{\pi}{\beta}}\,J_\nu\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}-\frac{3\pi}{4}+\frac{3\nu\pi}{2}\right)$$
$\left[\mathrm{Re}\,\nu>-\frac{1}{2}\right]$ MI 49

3. $$\int_0^\infty x^{\frac{\nu}{2}}\,\mathrm{ber}_\nu\left(\sqrt{x}\right)e^{-\beta x}\,dx=\frac{2^{-\nu}}{\beta^{1+\nu}}\cos\left(\frac{1}{4\beta}+\frac{3\nu\pi}{4}\right)\qquad[\mathrm{Re}\,\nu>-1]$$ ME 40

4. $$\int_0^\infty x^{\frac{\nu}{2}}\,\mathrm{bei}_\nu\left(\sqrt{x}\right)e^{-\beta x}\,dx=\frac{2^{-\nu}}{\beta^{1+\nu}}\sin\left(\frac{1}{4\beta}+\frac{3\nu\pi}{4}\right)\qquad[\mathrm{Re}\,\nu>-1]$$ ME 40

6.875

1. $$\int_0^\infty e^{-\beta x}\left[\ker\left(2\sqrt{x}\right)-\frac{1}{2}\ln x\,\mathrm{ber}\left(2\sqrt{x}\right)\right]dx=\frac{1}{\beta}\left[\ln\beta\cos\frac{1}{\beta}+\frac{\pi}{4}\sin\frac{1}{\beta}\right]$$ MI 50

2. $$\int_0^\infty e^{-\beta x}\left[\mathrm{kei}\left(2\sqrt{x}\right)-\frac{1}{2}\ln x\,\mathrm{bei}\left(2\sqrt{x}\right)\right]dx=\frac{1}{\beta}\left[\ln\beta\sin\frac{1}{\beta}-\frac{\pi}{4}\cos\frac{1}{\beta}\right]$$ MI 50

6.876

1. $$\int_0^\infty x\,\mathrm{kei}\,x\,J_1(ax)\,dx=-\frac{1}{2a}\arctan a^2 \qquad [a>0]$$ ET II 21(32)

2. $$\int_0^\infty x\ker x\,J_1(ax)\,dx=\frac{1}{2a}\ln\sqrt{(1+a^4)} \qquad [a>0]$$ ET II 21(33)

6.9 Mathieu Functions

Notation: $k^2=q$. For definition of the coefficients $A_p^{(m)}$ and $B_p^{(m)}$, see section 8.6.

6.91 Mathieu functions

6.911

1. $$\int_0^{2\pi}\mathrm{ce}_m(z,q)\,\mathrm{ce}_p(z,q)\,dz=0 \qquad [m\neq p]$$ MA

2. $$\int_0^{2\pi}\left[\mathrm{ce}_{2n}(z,q)\right]^2\,dz=2\pi\left[A_0^{(2n)}\right]^2+\pi\sum_{r=1}^\infty\left[A_{2r}^{(2n)}\right]^2=\pi$$ MA

3. $$\int_0^{2\pi}\left[\mathrm{ce}_{2n+1}(z,q)\right]^2\,dz=\pi\sum_{r=0}^\infty\left[A_{2r+1}^{(2n+1)}\right]^2=\pi$$ MA

4. $$\int_0^{2\pi}\mathrm{se}_m(z,q)\,\mathrm{se}_p(z,q)\,dz=0 \qquad [m\neq p]$$ MA

5. $$\int_0^{2\pi}\left[\mathrm{se}_{2n+1}(z,q)\right]^2\,dz=\pi\sum_{r=0}^\infty\left[B_{2r+1}^{(2n+1)}\right]^2=\pi$$ MA

6. $$\int_0^{2\pi}\left[\mathrm{se}_{2n+2}(z,q)\right]^2\,dz=\pi\sum_{r=0}^\infty\left[B_{2r+2}^{(2n+2)}\right]^2=\pi$$ MA

7. $$\int_0^{2\pi}\mathrm{se}_m(z,q)\,\mathrm{ce}_p(z,q)\,dz=0 \qquad [m=1,2,\ldots;\quad p=1,2,\ldots]$$ MA

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.921

1. $$\int_0^\pi\cosh\left(2k\cos u\sinh z\right)\mathrm{ce}_{2n}(u,q)\,du=\frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}(-1)^n\,\mathrm{Ce}_{2n}(z,-q)$$
$$[q>0]$$ MA

2. $$\int_0^\pi \cosh(2k\sin u\cosh z)\,\mathrm{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}(0,q)}(-1)^n\,\mathrm{Ce}_{2n}(z,-q)$$
$[q>0]$ MA

3. $$\int_0^\pi \sinh(2k\sin u\cosh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0,q)}(-1)^n\,\mathrm{Ce}_{2n+1}(z,-q)$$
$[q>0]$ MA

4. $$\int_0^\pi \sinh(2k\cos u\sinh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi k A_1^{(2n+1)}}{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}(-1)^{n+1}\,\mathrm{Se}_{2n+1}(z,-q)$$
$[q>0]$ MA

5. $$\int_0^\pi \sinh(2k\sin u\sin z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0,q)}\,\mathrm{se}_{2n+1}(z,q)$$
$[q>0]$ MA

6.922

1. $$\int_0^\pi \cos u\cosh z\cos(2k\sin u\sinh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi A_1^{(2n+1)}}{2\,\mathrm{ce}_{2n+1}(0,q)}\,\mathrm{Ce}_{2n+1}(z,q)$$
$[q>0]$ MA

2. $$\int_0^\pi \sin u\sinh z\cos(2k\cos u\cosh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi {B_1}^{(2n+1)}}{2\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}\,\mathrm{Se}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^\pi \sin u\sinh z\sin(2k\cos u\cosh z)\,\mathrm{se}_{2n+2}(u,q)\,du = -\frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}\,\mathrm{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

4. $$\int_0^\pi \cos u\cosh z\sin(2k\sin u\sinh z)\,\mathrm{se}_{2n+2}(u,q)\,du = \frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}(0,q)}\,\mathrm{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

5. $$\int_0^\pi \sin u\cosh z\cosh(2k\cos u\sinh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi {B_1}^{(2n+1)}}{2\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}(-1)^n\,\mathrm{Ce}_{2n+1}(z,-q)$$
$[q>0]$ MA

6. $$\int_0^\pi \cos u\sinh z\cosh(2k\sin u\cosh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi A_1^{(2n+1)}}{2\,\mathrm{ce}_{2n+1}(0,q)}(-1)^n\,\mathrm{Se}_{2n+1}(z,-q)$$
$[q>0]$ MA

7. $$\int_0^\pi \sin u\cosh z\sinh(2k\cos u\sinh z)\,\mathrm{se}_{2n+2}(u,q)\,du = \frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}(-1)^{n+1}\,\mathrm{Se}_{2n+2}(z,-q)$$
$[q>0]$ MA

8. $$\int_0^{\pi} \cos u \sinh z \sinh(2k \sin u \cosh z)\, \mathrm{se}_{2n+2}(u,q)\, du = \frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}(0,q)} (-1)^n\, \mathrm{Se}_{2n+2}(z,-q)$$
$[q>0]$ MA

6.923

1. $$\int_0^{\infty} \sin(2k \cosh z \cosh u) \sinh z \sinh u\, \mathrm{Se}_{2n+1}(u,q)\, du = -\frac{\pi {B_1}^{(2n+1)}}{4\,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi, q\right)} \mathrm{Se}_{2n+1}(z,q)$$
$[q>0]$ MA

2. $$\int_0^{\infty} \cos(2k \cosh z \cosh u) \sinh z \sinh u\, \mathrm{Se}_{2n+1}(u,q)\, du = -\frac{\pi {B_1}^{(2n+1)}}{4\,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi, q\right)} \mathrm{Gey}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^{\infty} \sin(2k \cosh z \cosh u) \sinh z \sinh u\, \mathrm{Se}_{2n+2}(u,q)\, du = -\frac{k\pi {B_2}^{(2n+2)}}{4\,\mathrm{se}'_{2n+2}\left(\frac{1}{2}\pi, q\right)} \mathrm{Gey}_{2n+2}(z,q)$$
$[q>0]$ MA

4. $$\int_0^{\infty} \cos(2k \cosh z \cosh u) \sinh z \sinh u\, \mathrm{Se}_{2n+2}(u,q)\, du = -\frac{k\pi {B_2}^{(2n+2)}}{4\,\mathrm{se}_{2n+2}\left(\frac{1}{2}\pi, q\right)} \mathrm{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

5. $$\int_0^{\infty} \sin(2k \cosh z \cosh u)\, \mathrm{Ce}_{2n}(u,q)\, du = \frac{\pi A_0^{(2n)}}{2\,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \mathrm{Ce}_{2n}(z,q)$$
$[q>0]$ MA

6. $$\int_0^{\infty} \cos(2k \cosh z \cosh u)\, \mathrm{Ce}_{2n}(u,q)\, du = -\frac{\pi A_0^{(2n)}}{2\,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \mathrm{Fey}_{2n}(z,q)$$
$[q>0]$ MA

7. $$\int_0^{\infty} \sin(2k \cosh z \cosh u)\, \mathrm{Ce}_{2n+1}(u,q)\, du = \frac{k\pi A_1^{(2n+1)}}{2\,\mathrm{ce}'_{2n+1}\left(\frac{1}{2}\pi, q\right)} \mathrm{Fey}_{2n+1}(z,q)$$
$[q>0]$ MA

8. $$\int_0^{\infty} \cos(2k \cosh z \cosh u)\, \mathrm{Ce}_{2n+1}(u,q)\, du = \frac{k\pi A_1^{(2n+1)}}{2\,\mathrm{ce}'_{2n+1}\left(\frac{1}{2}\pi, q\right)} \mathrm{Ce}_{2n+1}(z,q)$$
$[q>0]$ MA

6.924

1. $$\int_0^{\pi} \cos(2k \cos u \cos z)\, \mathrm{ce}_{2n}(u,q)\, du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \mathrm{ce}_{2n}(z,q)$$
$[q>0]$ MA

2. $$\int_0^{\pi} \sin(2k \cos u \cos z)\, \mathrm{ce}_{2n+1}(u,q)\, du = -\frac{\pi k A_1^{(2n+1)}}{\mathrm{ce}'_{2n+1}\left(\frac{1}{2}\pi, q\right)} \mathrm{ce}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^\pi \cos(2k\cos u\cosh z)\,\mathrm{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Ce}_{2n}(z,q)$$
$[q>0]$ MA

4. $$\int_0^\pi \cos(2k\sin u\sinh z)\,\mathrm{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}(0,q)}\,\mathrm{Ce}_{2n}(z,q)$$
$[q>0]$ MA

5. $$\int_0^\pi \sin(2k\cos u\cosh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = -\frac{\pi k A_1^{(2n+1)}}{\mathrm{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Ce}_{2n+1}(z,q)$$
$[q>0]$ MA

6. $$\int_0^\pi \sin(2k\sin u\sinh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}''_{2n+1}(0,q)}\,\mathrm{Se}_{2n+1}(z,q)$$
$[q>0]$ MA

6.925 **Notation**: $z_1 = 2k\sqrt{\cosh^2\xi - \sin^2\eta}$, and $\tan\alpha = \tanh\xi\tan\eta$

1. $$\int_0^{2\pi} \sin[z_1\cos(\theta-\alpha)]\,\mathrm{ce}_{2n}(\theta,q)\,d\theta = 0.$$
MA

2. $$\int_0^{2\pi} \cos[z_1\cos(\theta-\alpha)]\,\mathrm{ce}_{2n}(\theta,q)\,d\theta = \frac{2\pi A_0^{(2n)}}{\mathrm{ce}_{2n}(0,q)\,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Ce}_{2n}(\xi,q)\,\mathrm{ce}_{2n}(\eta,q)$$
MA

3. $$\int_0^{2\pi} \sin[z_1\cos(\theta-\alpha)]\,\mathrm{ce}_{2n+1}(\theta,q)\,d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\mathrm{ce}_{2n+1}(0,q)\,\mathrm{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Ce}_{2n+1}(\xi,q)\,\mathrm{ce}_{2n+1}(\eta,q)$$
MA

4. $$\int_0^{2\pi} \cos[z_1\cos(\theta-\alpha)]\,\mathrm{ce}_{2n+1}(\theta,q)\,d\theta = 0$$
MA

5. $$\int_0^{2\pi} \sin[z_1\cos(\theta-\alpha)]\,\mathrm{se}_{2n+1}(\theta,q)\,d\theta = \frac{2\pi k B_1^{(2n+1)}}{\mathrm{se}_{2n+1}(0,q)\,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Se}_{2n+1}(\xi,q)\,\mathrm{se}_{2n+1}(\eta,q)$$
MA

6. $$\int_0^{2\pi} \cos[z_1\cos(\theta-\alpha)]\,\mathrm{se}_{2n+1}(\theta,q)\,d\theta = 0$$
MA

7. $$\int_0^{2\pi} \sin[z_1\cos(\theta-\alpha)]\,\mathrm{se}_{2n+2}(\theta,q)\,d\theta = 0$$
MA

8. $$\int_0^{2\pi} \cos[z_1\cos(\theta-\alpha)]\,\mathrm{se}_{2n+2}(\theta,q)\,d\theta = \frac{2\pi k^2 B_2{}^{(2n+2)}}{\mathrm{se}'_{2n+2}(0,q)\,\mathrm{se}'_{2n+2}\left(\frac{1}{2}\pi,q\right)}\,\mathrm{Se}_{2n+2}(\xi,q)\,\mathrm{se}_{2n+2}(\eta,q)$$
MA

6.926 $$\int_0^\pi \sin u\sin z\sin(2k\cos u\cos z)\,\mathrm{se}_{2n+2}(u,q)\,du = -\frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}\,\mathrm{se}_{2n+2}(z,q)$$
$[q>0]$ MA

6.93 Combinations of Mathieu and Bessel functions

6.931

1. $$\int_0^{\pi} J_0\left\{k\left[2\left(\cos 2u+\cos 2z\right)\right]^{1/2}\right\}\operatorname{ce}_{2n}(u,q)\,du=\frac{\pi\left[A_0^{(2n)}\right]^2}{\operatorname{ce}_{2n}(0,q)\operatorname{ce}_{2n}\left(\frac{\pi}{2},q\right)}\operatorname{ce}_{2n}(z,q)$$ MA

2. $$\int_0^{2\pi} Y_0\left\{k\left[2\left(\cos 2u+\cosh 2z\right)\right]^{1/2}\right\}\operatorname{ce}_{2n}(u,q)\,du=\frac{2\pi\left[A_0^{(2n)}\right]^2}{\operatorname{ce}_{2n}(0,q)\operatorname{ce}_{2n}\left(\frac{\pi}{2},q\right)}\operatorname{Fey}_{2n}(z,q)$$ MA

6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems

Notation: Particular solutions of the Helmholtz equation in three-dimensional infinite space

$$\nabla^2\Psi+k^2\Psi=0$$

in Cartesian (x,y,z), spherical (r,θ,ϕ), and cylindrical (ρ,z,ϕ) coordinates are

$$\Psi_{k_xk_yk_z}(x,y,z)\propto e^{i(k_xx+k_yy+k_zz)}\quad\text{with}\quad k^2=k_x^2+k_y^2+k_z^2$$

$$\Psi_{lm}(r,\theta,\phi)\propto e^{im\phi}\sqrt{\frac{k}{r}}\,Z_{l+1/2}(kr)\,P_l^m(\cos\theta)$$

$$\Psi_{mk_z}(\rho,z,\phi)\propto e^{i(m\phi+k_zz)}\,Z_{l+1/2}\left(\rho\sqrt{k^2-k_z^2}\right)$$

with $P_l^m(\cos\theta)$ the associated Legendre function, Z is any Bessel function, $m=0,1,\ldots,l$; $l\in\mathbb{N}$, $r^2=\rho^2+z^2$, $\rho=r\sin\theta$, $z=r\cos\theta$, $\phi=\operatorname{arccot}(x/y)$, and $k_t^2=k^2-k_z^2$.

6.941

1. $$\int_{-k}^{k} e^{i\rho z}\,J_m\left(\rho\sqrt{k^2-\rho^2}\right)P_l^m\left(\frac{p}{k}\right)dp=i^{l-m}\sqrt{\frac{2\pi k}{r}}\,J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$
$$[\rho>0,\quad l\geq m\geq 0]$$

2. $$\int_{-\infty}^{\infty} e^{-i\rho z}\,J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)dz=i^{m-l}\sqrt{\frac{2\pi r}{k}}\,J_m\left(\rho\sqrt{k^2-\rho^2}\right)P_l^m\left(\frac{\rho}{k}\right)$$
$$[\rho>0,\quad l\geq m\geq 0]$$

3. $$\int_0^{\infty} J_m\left(\rho k_t\right)\cos\left[k_xx+m\arcsin\left(\frac{x}{\rho}\right)\right]dx$$
$$=\frac{(-1)^m}{\sqrt{k_t^2-k_x^2}}\cos\left[y\sqrt{k_t^2-k_x^2}+m\arccos\left(\frac{k_x}{k_t}\right)\right]\qquad\left[k_x^2<k_t^2\right]$$
$$=0\qquad\left[k_x^2>k_t^2\right]$$

4. $$\int_0^\infty Y_m(\rho k_t)\cos\left[k_x x + m\arcsin\left(\frac{x}{\rho}\right)\right]dx$$
$$= \frac{(-1)^m}{\sqrt{k_t^2-k_x^2}}\sin\left[y\sqrt{k_t^2-k_x^2}+m\arccos\left(\frac{k_x}{k_t}\right)\right] \qquad [k_x^2<k_t^2]$$
$$= \frac{(-1)^m}{\sqrt{k_x^2-k_t^2}}\exp\left[-y\sqrt{k_x^2-k_t^2}-m\,\mathrm{sign}\,(k_x)\,\mathrm{arccosh}\left(\frac{|k_x|}{k_t}\right)\right] \qquad [k_x^2>k_t^2]$$

5. $$\int_{-\infty}^\infty H_{l+1/2}^{(j)}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dx = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,H_m^{(j)}\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)$$
$$[\rho>0]$$

The result is true for $j=1$ if $\pi > \arg\sqrt{k^2-k_z^2}\geq 0$, for $j=2$ if $-\pi<\arg\sqrt{k^2-k_z^2}\leq 0$.

6. $$\int_{-\infty}^\infty H_m^{(j)}\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z = i^{l-m}\sqrt{\frac{2\pi k}{r}}\,H_{l+1/2}^{(j)}(kr)\,P_l^m\left(\frac{z}{r}\right)$$

The result is true for $j=1$ if $\pi > \arg\sqrt{k^2-k_z^2}\geq 0$, for $j=2$ if $-\pi<\arg\sqrt{k^2-k_z^2}\leq 0$.

7. $$\int_{-\infty}^\infty J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dz = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,J_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2<k^2]$$
$$= 0 \qquad [k_z^2>k^2]$$

8. $$\int_{-k}^k J_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z = i^{l-m}\sqrt{\frac{2\pi k}{r}}\,J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$

9. $$\int_{-\infty}^\infty Y_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dz = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,Y_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2<k^2]$$
$$= -2i^{m-l}\sqrt{\frac{2r}{k\pi}}\,K_m\left(\rho\sqrt{k_z^2-k^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2>k^2]$$

10. $$i^{l-m}\int_{-k}^k Y_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z$$
$$-\frac{4}{\pi}\int_k^\infty \cos\left[k_z z+\tfrac{1}{2}\pi(m-l)\right]P_l^m\left(\frac{k_z}{k}\right)K_m\left(\rho\sqrt{k_z^2-k^2}\right)e^{ik_z z}\,dk_z$$
$$= \sqrt{\frac{2\pi k}{r}}\,Y_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$

7.1–7.2 Associated Legendre Functions

7.11 Associated Legendre functions

7.111 $$\int_{\cos\varphi}^{1} P_\nu(x)\,dx = \sin\varphi\, P_\nu^{-1}(\cos\varphi)$$ MO 90

7.112

1. $$\int_{-1}^{1} P_n^m(x)\,P_k^m(x)\,dx = 0 \qquad [n \neq k]$$
$$= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \qquad [n = k]$$
SM III 185, WH

2. $$\int_{-1}^{1} Q_n^m(x)\,P_k^m(x)\,dx = (-1)^m \frac{1-(-1)^{n+k}(n+m)!}{(k-n)(k+n+1)(n-m)!}$$ EH I 171(18)

3. $$\int_{-1}^{1} P_\nu(x)\,P_\sigma(x)\,dx$$
$$= \frac{2\pi\sin\pi(\sigma-\nu) + 4\sin(\pi\nu)\sin(\pi\sigma)\,[\psi(\nu+1)-\psi(\sigma+1)]}{\pi^2(\sigma-\nu)\,(\sigma+\nu+1)} \qquad [\sigma+\nu+1 \neq 0]$$ EH I 170(7)
$$= \frac{\pi^2 - 2\,(\sin\pi\nu)^2\,\psi'(\nu+1)}{\pi^2\left(\nu+\frac{1}{2}\right)} \qquad [\sigma = \nu]$$ EH I 170(9)a

4. $$\int_{-1}^{1} Q_\nu(x)\,Q_\sigma(x)\,dx = \frac{[\psi(\nu+1)-\psi(\sigma+1)]\,[1+\cos(\pi\sigma)\cos(\nu\pi)] - \frac{\pi}{2}\sin\pi(\nu-\sigma)}{(\sigma-\nu)\,(\sigma+\nu+1)}$$
$$[\sigma+\nu+1 \neq 0; \quad \nu, \quad \sigma \neq -1,-2,-3,\ldots]$$
EH I 170(11)
$$= \frac{\frac{1}{2}\pi^2 - \psi'(\nu+1)\left[1+(\cos\nu\pi)^2\right]}{2\nu+1}$$
$$[\nu = \sigma, \quad \nu \neq -1,-2,-3,\ldots]$$
EH I 170(12)

5. $$\int_{-1}^{1} P_\nu(x)\,Q_\sigma(x)\,dx = \frac{1-\cos\pi(\sigma-\nu) - 2\pi^{-1}\sin(\pi\nu)\cos(\pi\sigma)\,[\psi(\nu+1)-\psi(\sigma+1)]}{(\nu-\sigma)\,(\nu+\sigma+1)}$$
$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0, \quad \sigma \neq \nu]$$
EH I 170(13)
$$= -\frac{\sin(2\nu\pi)\,\psi'(\nu+1)}{\pi(2\nu+1)}$$
$$[\operatorname{Re}\nu > 0, \quad \sigma = \nu]$$
EH I 171(14)

7.113 **Notation**: $$A = \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}$$

1. $$\int_{0}^{1} P_\nu(x)\,P_\sigma(x)\,dx = \frac{A\sin\frac{\pi\sigma}{2}\cos\frac{\pi\nu}{2} - A^{-1}\sin\frac{\pi\nu}{2}\cos\frac{\pi\sigma}{2}}{\frac{1}{2}\pi(\sigma-\nu)(\sigma+\nu+1)}$$ EH I 171(15)

2. $$\int_0^1 Q_\nu(x)\,Q_\sigma(x)\,dx = \frac{\psi(\nu+1)-\psi(\sigma+1)-\frac{\pi}{2}\left[\left(A-A^{-1}\right)\sin\frac{\pi(\sigma+\nu)}{2}\left(A+A^{-1}\right)\sin\frac{\pi(\sigma-\nu)}{2}\right]}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}\nu>0,\quad \operatorname{Re}\sigma>0]$ EH I 171(16)

3. $$\int_0^1 P_\nu(x)\,Q_\sigma(x)\,dx = \frac{A^{-1}\cos\frac{\pi(\nu-\sigma)}{2}-1}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}\nu>0,\quad \operatorname{Re}\sigma>0]$ EH I 171(17)

7.114

1. $$\int_1^\infty P_\nu(x)\,Q_\sigma(x)\,dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}(\sigma-\nu)>0,\quad \operatorname{Re}(\sigma+\nu)>-1]$
ET II 324(19)

2. $$\int_1^\infty Q_\nu(x)\,Q_\sigma(x)\,dx = \frac{\psi(\sigma+1)-\psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}(\nu+\sigma)>-1;\quad \sigma,\nu\neq-1,-2,-3,\ldots]$ EH I 170(5)

3. $$\int_1^\infty \left[Q_\nu(x)\right]^2\,dx = \frac{\psi'(\nu+1)}{2\nu+1}$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}\right]$ EH I 170(6)

7.115 $$\int_1^\infty Q_\nu(x)\,dx = \frac{1}{\nu(\nu+1)}$$
$[\operatorname{Re}\nu>0]$ ET II 324(18)

7.12–7.13 Combinations of associated Legendre functions and powers

7.121 $$\int_{\cos\varphi}^1 x\,P_\nu(x)\,dx = \frac{-\sin\varphi}{(\nu-1)(\nu+2)}\left[\sin\varphi P_\nu(\cos\varphi)+\cos\varphi\,P_\nu^1(\cos\varphi)\right]$$
MO 90

7.122

1. $$\int_0^1 \frac{\left[P_n^m(x)\right]^2}{1-x^2}\,dx = \frac{1}{2m}\frac{(n+m)!}{(n-m)!}$$
$[0<m\leq n]$ MO 74

2. $$\int_0^1 \left[P_\nu^\mu(x)\right]^2\frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu\,\Gamma(1-\mu+\nu)}$$
$[\operatorname{Re}\mu<0,\quad \nu+\mu \text{ is a positive integer}]$
EH I 172(26)

3. $$\int_0^1 \left[P_\nu^{n-\nu}(x)\right]^2\frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu)\,\Gamma(1-n+2\nu)}$$
$[n=0,1,2,\ldots;\quad \operatorname{Re}\nu>n]$
ET II 315(9)

7.123 $$\int_{-1}^1 P_n^m(x)\,P_n^k(x)\frac{dx}{1-x^2} = 0$$
$[0\leq m\leq n,\quad 0\leq k\leq n;\quad m\neq k]$
MO 74

7.124 $$\int_{-1}^1 x^k(z-x)^{-1}\left(1-x^2\right)^{\frac{1}{2}m}P_n^m(x)\,dx = (-2)^m\left(z^2-1\right)^{\frac{1}{2}m}Q_n^m(z)\cdot z^k$$
$[m\leq n; k=0,1,\ldots,n-m;$
z is in the complex plane with a cut along the interval $(-1,1)$ on the real axis$]$
ET II 279(26)

7.125 $$\int_{-1}^{1}\left(1-x^2\right)^{\frac{1}{2}m} P_k^m(x)\,P_l^m(x)\,P_n^m(x)\,dx=(-1)^m\pi^{-3/2}\frac{(k+m)!(l+m)!\,(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!}$$
$$\times\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(t-k+\frac{1}{2}\right)\Gamma\left(t-l+\frac{1}{2}\right)\Gamma\left(t-n+\frac{1}{2}\right)}{(s-l)!(s-n)!\,\Gamma\left(s+\frac{3}{2}\right)}$$
$[2s=k+l+n+m$ and $2t=k+l-n-m$ are both even
$l\geq m,\quad m\leq k-l-m\leq n\leq k+l+m]$

ET II 280(32)

7.126

1. $$\int_0^1 P_\nu(x)x^\sigma\,dx=\frac{\sqrt{\pi}2^{-\sigma-1}\,\Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)}\qquad[\operatorname{Re}\sigma>-1]$$ EH I 171(23)

2. $$\int_0^1 x^\sigma\,P_\nu^m(x)\,dx=\frac{(-1)^m\pi^{1/2}2^{-2m-1}\,\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma(1+m+\nu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}m\right)\Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}\right)\Gamma(1-m+\nu)}$$
$$\times\,{}_3F_2\left(\frac{m+\nu+1}{2},\frac{m-\nu}{2},\frac{m}{2}+1;m+1,\frac{3+\sigma+m}{2};1\right)$$
$[\operatorname{Re}\sigma>-1;\quad m=0,1,2,\ldots]$ ET II 313(2)

3. $$\int_0^1 x^\sigma\,P_\nu^\mu(x)\,dx=\frac{\pi^{1/2}2^{2\mu-1}\,\Gamma\left(\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{3+\sigma-\mu}{2}\right)}\,{}_3F_2\left(\frac{\nu-\mu+1}{2},-\frac{\mu+\nu}{2},1-\frac{\mu}{2};1-\mu,\frac{3+\sigma-\mu}{2};1\right)$$
$[\operatorname{Re}\sigma>-1,\quad \operatorname{Re}\mu<2]$ ET II 313(3)

4. $$\int_1^\infty x^{\mu-1}\,Q_\nu(ax)\,dx=e^{\mu\pi i}\,\Gamma(\mu)a^{-\mu}\left(a^2-1\right)^{\frac{1}{2}\mu}\,Q_\nu^{-\mu}(a)$$
$[|\arg(a-1)|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}(\nu-\mu)>-1]$ ET II 325(26)

7.127 $$\int_{-1}^{1}(1+x)^\sigma\,P_\nu(x)\,dx=\frac{2^{\sigma+1}\left[\Gamma(\sigma+1)\right]^2}{\Gamma(\sigma+\nu+2)\,\Gamma(1+\sigma-\nu)}\qquad[\operatorname{Re}\sigma>-1]$$ ET II 316(15)

7.128

1. $$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}\,(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{3}{2}}\,P_\nu^\mu(x)\,dx$$
$$=-\frac{\Gamma\left(\mu-\frac{1}{2}\right)(z-1)^{\mu-\frac{1}{2}}\,(z+1)^{-1/2}}{\pi^{1/2}e^{2\mu\pi i}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu-1)}$$
$$\times\left\{Q_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]Q_{-\nu-1}^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]+Q_\nu^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]Q_{-\nu-1}^{\mu}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]\right\}$$
$[-\frac{1}{2}<\operatorname{Re}\mu<1,$
z is in the complex plane with a cut along the interval $(-1,1)$ of the real axis]

ET II 317(20)

2. $$\int_{-1}^{1}(1-x)^{-\frac{1}{2}\mu}\,(1+x)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{1}{2}}\,P_\nu^\mu(x)\,dx$$
$$=\frac{2e^{-2\mu\pi i}\,\Gamma\left(\frac{1}{2}+\mu\right)}{\pi^{1/2}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)}(z-1)^\mu\,Q_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]Q_{-\nu-1}^{\mu}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]$$
$[-\frac{1}{2}<\operatorname{Re}\mu<1,$
z is in the complex plane with a cut along the interval $(-1,1)$ of the real axis]

ET II 316(18)

7.129 $$\int_{-1}^{1} P_\nu(x)\, P_\lambda(x)(1+x)^{\lambda+\nu}\, dx = \frac{2^{\lambda+\nu+1}\left[\Gamma(\lambda+\nu+1)\right]^4}{\left[\Gamma(\lambda+1)\,\Gamma(\nu+1)\right]^2\,\Gamma(2\lambda+2\nu+2)}$$
$[\mathrm{Re}(\nu+\lambda+1)>0]$ EH I 172(30)

7.131

1. $$\int_1^\infty (x-1)^{-\frac{1}{2}\mu}\,(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{1}{2}}\,P_\nu^\mu(x)\,dx$$
$$= \pi^{1/2}\frac{\Gamma(-\mu-\nu)\,\Gamma\left(1-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\mu\right)}(z-1)^\mu\left\{P_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]\right\}^2$$
$[\mathrm{Re}(\mu+\nu)<0,\quad \mathrm{Re}(\mu-\nu)<1,\quad |\arg(z+1)|<\pi]$ ET II 321(6)

2. $$\int_1^\infty (x-1)^{-\frac{1}{2}\mu}\,(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{3}{2}}\,P_\nu^\mu(x)\,dx$$
$$= \frac{\pi^{1/2}\,\Gamma(1-\mu-\nu)\,\Gamma\left(2-\mu+\nu\right)(z-1)^{\mu-\frac{1}{2}}\,(z+1)^{-1/2}}{\Gamma\left(\frac{3}{2}-\mu\right)}\,P_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]P_\nu^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]$$
$[\mathrm{Re}\,\mu<1,\quad \mathrm{Re}(\mu+\nu)<1,\quad \mathrm{Re}(\mu-\nu)<2,\quad |\arg(1+z)|<\pi]$ ET II 321(7)

7.132

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-1}P_\nu^\mu(x)\,dx = \frac{\pi 2^\mu\,\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{\Gamma\left(\lambda+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(-\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$
$[2\,\mathrm{Re}\,\lambda>|\mathrm{Re}\,\mu|]$ ET II 316(16)

2. $$\int_1^\infty\left(x^2-1\right)^{\lambda-1}P_n^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\lambda-\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\lambda-\frac{1}{2}\mu\right)}$$
$[\mathrm{Re}\,\lambda>\mathrm{Re}\,\mu,\quad \mathrm{Re}(1-2\lambda-\nu)>0,\quad \mathrm{Re}(2-2\lambda+\nu)>0]$ ET II 320(2)

3.[9] $$\int_1^\infty\left(x^2-1\right)^{\lambda-1}Q_\nu^\mu(x)\,dx = e^{\mu\pi i}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{2^{2-\mu}\,\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\lambda+\frac{1}{2}\nu\right)}$$
$[|\mathrm{Re}\,\mu|<2\,\mathrm{Re}\,\lambda<\mathrm{Re}\,\nu+2]$
ET II 324(23)

4. $$\int_0^1 x^\sigma\left(1-x^2\right)^{-\frac{1}{2}\mu}P_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2}\right)}$$
$[\mathrm{Re}\,\mu<1,\quad \mathrm{Re}\,\sigma>-1]$ EH I 172(24)

5. $$\int_0^1 x^\sigma\left(1-x^2\right)^{\frac{1}{2}m}P_\nu^m(x)\,dx = \frac{(-1)^m 2^{-m-1}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)\Gamma\left(1+m+\nu\right)}{\Gamma(1-m+\nu)\,\Gamma\left(1+\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}m+\frac{1}{2}\nu\right)}$$
$[\mathrm{Re}\,\sigma>-1,\quad m$ is a positive integer$]$ EH I 172(25), ET II 313(4)

6. $$\int_0^1\left(1-x^2\right)^\eta P_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(1+\eta-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)}{\Gamma(1-\mu)\,\Gamma\left(\frac{3}{2}+\eta+\frac{1}{2}\sigma-\frac{1}{2}\mu\right)}$$
$$\times\ {}_3F_2\left(\frac{\nu-\mu+1}{2},-\frac{\mu+\nu}{2},1+\eta-\frac{\mu}{2};1-\mu,\frac{3+\sigma-\mu}{2}+\eta;1\right)$$
$\left[\mathrm{Re}\left(\eta-\frac{1}{2}\mu\right)>-1,\mathrm{Re}\,\sigma>-1\right]$ ET II 314(6)

7. $$\int_1^\infty x^{-\rho}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{2^{\rho+\mu-2}\,\Gamma\left(\frac{\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{\rho+\mu-\nu-1}{2}\right)}{\sqrt{\pi}\,\Gamma(\rho)}$$
$[\operatorname{Re}\mu<1,\quad \operatorname{Re}(\rho+\mu+\nu)>0,\quad \operatorname{Re}(\rho+\mu-\nu)>1]$ ET II 320(3)

7.133

1. $$\int_u^\infty Q_\nu(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}\left(u^2-1\right)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(u)$$
$[|\arg(u-1)|<\pi,\quad 0<\operatorname{Re}\mu<1+\operatorname{Re}\nu]$ MO 90a

2. $$\int_u^\infty \left(x^2-1\right)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}\left(u^2-1\right)^{\frac{1}{2}\lambda+\frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(u)$$
$[|\arg(u-1)|<\pi,\quad 0<\operatorname{Re}\mu<1+\operatorname{Re}(\nu-\lambda)]$ ET II 204(30)

7.134

1. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{2^{\lambda+\mu}\,\Gamma(\lambda)\,\Gamma(-\lambda-\mu-\nu)\,\Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)\,\Gamma(1-\lambda-\mu)}$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}(\lambda+\mu+\nu)<0,\quad \operatorname{Re}(\lambda+\mu-\nu)<1]$ ET II 321(4)

2. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = -\frac{2^{\lambda-\mu}\sin\pi\nu\,\Gamma(\lambda-\mu)\,\Gamma(-\lambda+\mu-\nu)\,\Gamma(1-\lambda+\mu+\nu)}{\pi\,\Gamma(1-\lambda)}$$
$[\operatorname{Re}(\lambda-\mu)>0,\quad \operatorname{Re}(\mu-\lambda-\nu)>0,\quad \operatorname{Re}(\mu-\lambda+\nu)>-1]$ ET II 321(5)

7.135

1. $$\int_{-1}^1 \left(1-x^2\right)^{-\frac{1}{2}\mu}(z-x)^{-1} P_{\mu+n}^\mu(x)\,dx = 2e^{-i\mu\pi}\left(z^2-1\right)^{-\frac{1}{2}\mu} Q_{\mu+n}^\mu(z)$$
$[n=0,1,2,\ldots,\quad \operatorname{Re}\mu+n>-1,$ z is in the complex plane with a cut along the interval $(-1,1)$ of the real axis.] ET II 316(17)

2. $$\begin{aligned}\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{\mu/2}(x+z)^{-\rho} P_\nu^\mu(x)\,dx &= \frac{2^{\lambda+\mu-\rho}\,\Gamma(\lambda-\rho)\,\Gamma(\rho-\lambda-\mu-\nu)\,\Gamma(\rho-\lambda-\mu+\nu+1)}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)\,\Gamma(1+\rho-\lambda-\mu)}\\ &\quad\times {}_3F_2\left(\rho,\rho-\lambda-\mu-\nu,\rho-\lambda-\mu+\nu+1;\rho-\lambda+1,\rho-\lambda-\mu+1;\frac{1+z}{2}\right)\\ &\quad+\frac{\Gamma(\rho-\lambda)\,\Gamma(\lambda)}{\Gamma(\rho)\,\Gamma(1-\mu)}2^\mu(z+1)^{\lambda-\rho}\ {}_3F_2\left(\lambda,-\mu-\nu,1-\mu+\nu;1-\mu,1-\rho+\lambda;\frac{1+z}{2}\right)\end{aligned}$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}(\rho-\lambda-\mu-\nu)>0,\quad \operatorname{Re}(\rho-\lambda-\mu+\nu+1)>0,\quad |\arg(z+1)|<\pi]$
ET II 322(9)

3. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{-\mu/2}(x+z)^{-\rho}\,P_\nu^\mu(x)\,dx$$
$$= -\frac{\sin(\nu\pi)\,\Gamma(\lambda-\mu-\rho)\,\Gamma\left(\rho-\lambda+\mu-\nu\right)\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\,\Gamma\left(1+\rho-\lambda\right)}$$
$$\times\ {}_3F_2\left(\rho,\rho-\lambda+\mu-\nu,\rho-\lambda+\mu+\nu+1;1+\rho-\lambda,1+\rho-\lambda+\mu;\frac{1+z}{2}\right)$$
$$+\frac{\Gamma(\lambda-\mu)\,\Gamma\left(\rho-\lambda+\mu\right)}{\Gamma(\rho)\,\Gamma(1-\mu)}(z+1)^{\lambda-\rho-\mu}$$
$$\times\ {}_3F_2\left(\lambda-\mu,-\nu,\nu+1;1+\lambda-\mu-\rho,1-\mu;\frac{1+z}{2}\right)$$
$$[\mathrm{Re}(\lambda-\mu)>0,\quad \mathrm{Re}\left(\rho-\lambda+\mu-\nu\right)>0,\quad \mathrm{Re}(\rho-\lambda+\mu+\nu+1)>0,\quad |\arg(z+1)|<\pi]$$
ET II 322(10)

7.136

1. $$\int_{-1}^1 \left(1-x^2\right)^{\lambda-1}\left(1-a^2x^2\right)^{\mu/2}P_\nu(ax)\,dx$$
$$=\frac{\pi 2^\mu\,\Gamma(\lambda)}{\Gamma\left(\frac12+\lambda\right)\Gamma\left(\frac12-\frac12\mu-\frac12\nu\right)\Gamma\left(1-\frac12\mu+\frac12\nu\right)}\ {}_2F_1\left(-\frac{\mu+\nu}{2},\frac{1-\mu+\nu}{2};\frac12+\lambda;a^2\right)$$
$$[\mathrm{Re}\,\lambda>0,\quad -1<a<1]$$
ET II 318(31)

2. $$\int_1^\infty \left(x^2-1\right)^{\lambda-1}\left(a^2x^2-1\right)^{\mu/2}P_\nu^\mu(ax)\,dx$$
$$=\frac{\Gamma(\lambda)\,\Gamma\left(1-\lambda-\frac12\mu+\frac12\nu\right)\Gamma\left(\frac12-\lambda-\frac12\mu-\frac12\nu\right)}{\Gamma\left(1-\frac12\mu+\frac12\nu\right)\Gamma\left(\frac12-\frac12\nu-\frac12\mu\right)\Gamma(1-\lambda-\mu)}$$
$$\times 2^{\mu-1}a^{\mu-\nu-1}\ {}_2F_1\left(\frac{1-\mu+\nu}{2},1-\lambda-\frac{\mu-\nu}{2};1-\lambda-\mu;1-\frac{1}{a^2}\right)$$
$$[\mathrm{Re}\,a>0,\quad \mathrm{Re}\,\lambda>0,\quad \mathrm{Re}(\nu-\mu-2\lambda)>-2,\quad \mathrm{Re}(2\lambda+\mu+\nu)<1]$$
ET II 325(25)

3. $$\int_1^\infty \left(x^2-1\right)^{\lambda-1}\left(a^2x^2-1\right)^{-\frac12\mu}Q_\nu^\mu(ax)\,dx=\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma(\lambda)\,\Gamma\left(1-\lambda+\frac{\mu+\nu}{2}\right)2^{\mu-2}e^{\mu\pi i}a^{-\mu-\nu-1}}{\Gamma\left(\nu+\frac32\right)}$$
$$\times\ {}_2F_1\left(\frac{\mu+\nu+1}{2},1-\lambda+\frac{\mu+\nu}{2};\nu+\frac32;a^{-2}\right)$$
$$[|\arg(a-1)|<\pi,\quad \mathrm{Re}\,\lambda>0,\quad \mathrm{Re}(2\lambda-\mu-\nu)<2]$$
ET II 325(27)

7.137

1. $$\int_1^\infty x^{-\frac12\mu-\frac12}(x-1)^{-\mu-\frac12}(1+ax)^{\frac12\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$=\pi^{-1/2}e^{-\mu\pi i}\,\Gamma\left(\tfrac12-\mu\right)a^{\frac12\mu}\left\{Q_\nu^\mu\left[(1+a)^{1/2}\right]\right\}^2$$
$$\left[|\arg a|<\pi,\quad \mathrm{Re}\,\mu<\tfrac12,\quad \mathrm{Re}(\mu+\nu)>-1\right]$$
ET II 325(28)

2. $$\int_1^\infty x^{-\frac12\mu-\frac12}(x-1)^{-\mu-\frac32}(1+ax)^{\frac12\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$=-\pi^{-1/2}e^{-\mu\pi i}\,\Gamma\left(-\mu-\tfrac12\right)a^{\frac12\mu+\frac12}\left(1+a^2\right)^{-1/2}Q_\nu^{\mu+1}\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]$$
$$\left[|\arg a|<\pi,\quad \mathrm{Re}\,\mu<-\tfrac12,\quad \mathrm{Re}(\mu+\nu+2)>0\right]$$
ET II 326(29)

3. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx = \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\left\{P_\nu^\mu\left[(1+a)^{1/2}\right]\right\}^2$$
$$\left[\operatorname{Re}\mu<\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 319(32)

4. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}\,(1+ax)^{\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(-\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu+\frac{1}{2}}\,P_\nu^{\mu+1}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^2\right]$$
$$\left[\operatorname{Re}\mu<-\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 319(33)

5. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}+\mu\right)a^{-\frac{1}{2}\mu}\,P_\nu^\mu\left[(1+a)^{1/2}\right]P_\nu^{-\mu}\left[(1+a)^{1/2}\right]$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 319(34)

6. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}\,(1+ax)^{-\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \frac{1}{2}\pi^{1/2}\,\Gamma\left(\mu-\tfrac{1}{2}\right)a^{\frac{1}{2}-\frac{1}{2}\mu}\,(1+a)^{-1/2}\left\{P_\nu^{1-\mu}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^{1/2}\right]\right\}$$
$$+(\mu+\nu)(1-\mu+\nu)\,P_\nu^{-\mu}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^{1/2}\right]$$
$$\left[\operatorname{Re}\mu>\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 319(35)

7. $$\int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\,P_\nu^\mu\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]$$
$$\left[\operatorname{Re}\mu<\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 320(38)

8. $$\int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}\,(1+ax)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= \frac{1}{2}\pi^{1/2}\,\Gamma\left(-\mu-\tfrac{1}{2}\right)(1+a)^{-1/2}\,a^{\frac{1}{2}\mu+\frac{1}{2}}$$
$$\times\left\{P_\nu^{\mu+1}\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]+P_\nu^\mu\left[(1+a)^{1/2}\right]Q_\nu^{\mu+1}\left[(1+a)^{1/2}\right]\right\}$$
$$\left[\operatorname{Re}\mu<-\tfrac{1}{2},\quad |\arg a|<\pi\right]$$ ET II 320(39)

9. $$\int_0^y (y-x)^{\mu-1}\left[x\left(1+\tfrac{1}{2}\gamma x\right)\right]^{-\frac{1}{2}\lambda}P_\nu^\lambda(1+\gamma x)\,dx$$
$$= \Gamma(\mu)\left(\frac{2}{\gamma}\right)^{\frac{1}{2}\mu}\left[y\left(1+\frac{1}{2}\gamma y\right)\right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}P_\nu^{\lambda-\mu}(1+\gamma y)$$
$$[\operatorname{Re}\lambda<1,\quad \operatorname{Re}\mu>0,\quad |\arg\gamma y|<\pi]$$ ET II 193(52)

10. $$\int_0^y (y-x)^{\mu-1}x^{\sigma+\frac{1}{2}\lambda-1}\left(1+\tfrac{1}{2}\gamma x\right)^{-\frac{1}{2}\lambda}P_\nu^\lambda(1+\gamma x)\,dx$$
$$= \frac{\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}\lambda}\Gamma(\sigma)\,\Gamma(\mu)y^{\sigma+\mu-1}}{\Gamma(1-\lambda)\,\Gamma(\sigma+\mu)}\;{}_3F_2\left(-\nu,1+\nu,\sigma;1-\lambda,\sigma+\mu;-\frac{1}{2}\gamma y\right)$$
$$[\operatorname{Re}\sigma>0,\quad \operatorname{Re}\mu>0,\quad |\gamma y|<1]$$ ET II 193(53)

11. $$\int_0^y (y-x)^{\mu-1}[x(1-x)]^{-\frac{1}{2}\lambda}\, P_\nu^\lambda(1-2x)\,dx = \Gamma(\mu)[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}\, P_\nu^{\lambda-\mu}(1-2y)$$
$[\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1]$ ET II 193(54)

12. $$\int_0^y (y-x)^{\mu-1}x^{\sigma+\frac{1}{2}\lambda-1}(1-x)^{-\frac{1}{2}\lambda}\, P_\nu^\lambda(1-2x)\,dx = \frac{\Gamma(\mu)\,\Gamma(\sigma)y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\,\Gamma(1-\lambda)}\; {}_3F_2\left(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; y\right)$$
$[\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1]$ ET II 193(155)

7.138 $$\int_0^\infty (a+x)^{-\mu-\nu-2}\, P_\mu\left(\frac{a-x}{a+x}\right) P_\nu\left(\frac{a-x}{a+x}\right) dx = \frac{a^{-\mu-\nu-1}\left[\Gamma(\mu+\nu+1)\right]^4}{\left[\Gamma(\mu+1)\,\Gamma(\nu+1)\right]^2\Gamma(2\mu+2\nu+2)}$$
$[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\nu) > -1]$ ET II 326(3)

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

1. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{a^{-\lambda-\mu}e^{-a}}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)}\, G_{23}^{31}\left(2a\,\middle|\,\begin{matrix}1+\mu, 1\\ \lambda+\mu, -\nu, 1+\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0]$ ET II 323(13)

2. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} Q_\nu^\mu(x)\,dx = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\,\Gamma(\nu-\mu+1)}a^{-\lambda-\mu}e^{-a}\, G_{23}^{22}\left(2a\,\middle|\,\begin{matrix}1+\mu, 1\\ \lambda+\mu, \nu+1, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0]$ ET II 325(24)

3. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = -\pi^{-1}\sin(\nu\pi)a^{\mu-\lambda}e^{-a}\, G_{23}^{31}\left(2a\,\middle|\,\begin{matrix}1, 1-\mu\\ \lambda-\mu, 1+\nu, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda-\mu) > 0]$ ET II 323(15)

4. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} Q_\nu^\mu(x)\,dx = \frac{1}{2}e^{\mu\pi i}a^{\mu-\lambda}e^{-a}\, G_{23}^{22}\left(2a\,\middle|\,\begin{matrix}1-\mu, 1\\ \lambda-\mu, \nu+1, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu) > 0]$ ET II 323(14)

5. $$\int_1^\infty e^{-ax}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = 2^{1/2}\pi^{-1/2}a^{\mu-\frac{1}{2}}\, K_{\nu+\frac{1}{2}}(a)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\mu < 1]$ ET II 323(11), MO 90

7.142 $$\int_1^\infty e^{-\frac{1}{2}ax}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\,dx = \frac{2}{a}\, W_{\mu,\nu}(a) \qquad \left[\operatorname{Re}\mu < 1, \quad \nu-\tfrac{1}{2} \neq 0, \pm1, \pm2, \ldots\right]$$
BU 79(34), MO 118

7.143

1. $$\int_0^\infty [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x}\, P_\nu^\mu(1+2x)\, dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta}\, K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right)$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 179(1)

2. $$\int_0^\infty \left(1+\frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x}\, P_\nu^\mu(1+2x)\, dx = \frac{e^{\frac{1}{2}\beta}}{\beta}\, W_{\mu,\nu+\frac{1}{2}}(\beta)$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 179(2)

7.144

1. $$\int_0^\infty e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu}\, Q_\nu^\mu(1+x)\, dx$$
$$= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left\{\frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu}\sin(\mu\pi)}\, E\left(-\nu,\nu+1,\lambda+\mu;\mu+1:2\beta\right) - \frac{\sin\left[(\mu+\nu)\pi\right]}{2^{1-\mu}\beta^\lambda\sin(\mu\pi)}\, E\left(\nu-\mu+1,-\nu-\mu,\lambda:1-\mu:2\beta\right)\right\}$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0]$ ET I 181(16)

2. $$\int_0^\infty e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu}\, Q_\nu^\mu(1+x)\, dx = -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu}\sin(\mu\pi)}\, E(-\nu,\nu+1,\lambda-\mu:1-\mu:2\beta) - \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu}\beta^\lambda\sin(\mu\pi)}\, E(\mu+\nu+1,\mu-\nu,\lambda:1+\mu:2\beta)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu)] > 0$ ET I 181(17)

7.145

1. $$\int_0^\infty \frac{e^{-\beta x}}{1+x}\, P_\nu\left[\frac{1}{(1+x)^2}-1\right] dx = \frac{e^\beta}{\beta}\, W_{\nu+\frac{1}{2},0}(\beta)\, W_{-\nu-\frac{1}{2},0}(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 180(6)

2. $$\int_0^\infty x^{-1} e^{-\beta x}\, Q_{-\frac{1}{2}}\left(1+2x^{-2}\right) dx = \frac{\pi^2}{8}\left\{\left[J_0\left(\frac{1}{2}\beta\right)\right]^2 + \left[Y_0\left(\frac{1}{2}\beta\right)\right]^2\right\}$$
$[\operatorname{Re}\beta > 0]$ ET II 327(5)

3. $$\int_0^\infty x^{-1} e^{-ax}\, Q_\nu\left(1+2x^{-2}\right) dx = \frac{1}{2}\left[\Gamma(\nu+1)\right]^2 a^{-1}\, W_{-\nu-\frac{1}{2},0}(ai)\, W_{-\nu-\frac{1}{2},0}(-ai)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -1]$ ET II 327(6)

7.146

1. $$\int_0^\infty x^{-\frac{1}{2}\mu} e^{-\beta x}\, P_\nu^\mu\left(\sqrt{1+x}\right) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{5}{4}} e^{\frac{\beta}{2}}\, W_{\frac{1}{2}\mu+\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(\beta)$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 180(7)

2. $$\int_0^\infty x^{-\frac{1}{2}\mu} \frac{e^{-\beta x}}{\sqrt{1+x}}\, P_\nu^\mu\left(\sqrt{1+x}\right) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{3}{4}} e^{\frac{\beta}{2}}\, W_{\frac{1}{2}\mu+\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(\beta)$$
$[\operatorname{Re}\mu < 1, \operatorname{Re}\beta > 0]$ ET I 180(8)a

3. $$\int_0^\infty \sqrt{x}e^{-\beta x}\, P_\nu^{1/4}\left(\sqrt{1+x^2}\right) P_\nu^{-1/4}\left(\sqrt{1+x^2}\right)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\, H^{(1)}_{\nu+\frac{1}{2}}\left(\frac{1}{2}\beta\right) H^{(2)}_{\nu+\frac{1}{2}}\left(\frac{1}{2}\beta\right)$$
$[\operatorname{Re}\beta > 0]$ ET I 180(9)

7.147 $$\int_0^\infty x^{\lambda-1}\left(x^2+a^2\right)^{\frac{1}{2}\nu} e^{-\beta x}\, P_\nu^\mu\left[\frac{x}{(x^2+a^2)^{1/2}}\right] dx$$
$$= \frac{2^{-\nu-2}a^{\lambda+\nu}}{\pi\,\Gamma(-\mu-\nu)}\, G^{32}_{24}\left(\frac{a^2\beta^2}{4}\,\middle|\,\begin{matrix}1-\frac{\lambda}{2}, \frac{1-\lambda}{2}\\ 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2}\end{matrix}\right)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\lambda>0]$ ET II 327(7)

7.148 $$\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu+\nu-1}\exp\left(-\frac{1-x}{1+x}y\right) P_\nu^\mu(x)\, dx = 2^\nu y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y}\, W_{\frac{1}{2}\mu-\nu-\frac{1}{2},\frac{1}{2}\mu}(y)$$
$[\operatorname{Re} y > 0]$ ET II 317(21)

7.149 $$\int_1^\infty \left(\alpha^2+\beta^2+2\alpha\beta x\right)^{-1/2}\exp\left[-\left(\alpha^2+\beta^2+2\alpha\beta x\right)^{1/2}\right] P_\nu(x)\, dx$$
$$= 2\pi^{-1}(\alpha\beta)^{-1/2}\, K_{\nu+\frac{1}{2}}(\alpha)\, K_{\nu+\frac{1}{2}}(\beta)$$
$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0]$ ET II 323(16)

7.15 Combinations of associated Legendre and hyperbolic functions

7.151

1. $$\int_0^\infty (\sinh x)^{\alpha-1}\, P_\nu^{-\mu}(\cosh x)\, dx = \frac{2^{-1-\mu}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\alpha+1\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1+\frac{1}{2}\mu-\frac{1}{2}\alpha\right)}$$
$[\operatorname{Re}(\alpha+\mu)>0, \quad \operatorname{Re}(\nu-\alpha+2)>0, \quad \operatorname{Re}(1-\alpha-\nu)>0]$ EH I 172(28)

2. $$\int_0^\infty (\sinh x)^{\alpha-1}\, Q_\nu^\mu(\cosh x)\, dx = \frac{e^{i\mu\pi}2^{\mu-\alpha}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\alpha\right)}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha\right)}$$
$$\times\,\Gamma\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\mu\right)\Gamma\left(\tfrac{1}{2}\alpha-\tfrac{1}{2}\mu\right)$$
$[\operatorname{Re}(\alpha\pm\mu)>0, \quad \operatorname{Re}(\nu-\alpha+2)>0]$ EH I 172(29)

7.152 $$\int_0^\infty e^{-\alpha x}\sinh^{2\mu}\left(\tfrac{1}{2}x\right) P_{2n}^{-2\mu}\left[\cosh\left(\tfrac{1}{2}x\right)\right] dx = \frac{\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\alpha-n-\mu)\,\Gamma\left(\alpha+n-\mu+\frac{1}{2}\right)}{4^\mu\sqrt{\pi}\,\Gamma(\alpha+n+\mu+1)\,\Gamma\left(\alpha-n+\mu+\frac{1}{2}\right)}$$
$\left[\operatorname{Re}\alpha > n+\operatorname{Re}\mu, \quad \operatorname{Re}\mu > -\frac{1}{4}\right]$ ET I 181(15)

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.161

1. $$\int_0^1 x^{\lambda-1}\left(1-x^2\right)^{-\frac{1}{2}\mu}\sin(ax)\,P_\nu^\mu(x)\,dx = \frac{\pi^{1/2}2^{\mu-\lambda-1}\,\Gamma(\lambda+1)\,a}{\Gamma\left(1+\frac{\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{3+\lambda-\mu+\nu}{2}\right)} \times {}_2F_3\left(\frac{1+\lambda}{2},1+\frac{\lambda}{2};\frac{3}{2},1+\frac{\lambda-\mu-\nu}{2},\frac{3+\lambda-\mu+\nu}{2};-\frac{a^2}{4}\right)$$
$[\operatorname{Re}\lambda>-1,\quad \operatorname{Re}\mu<1]$ ET II 314(7)

2. $$\int_0^1 x^{\lambda-1}\left(1-x^2\right)^{-\frac{1}{2}\mu}\cos(ax)\,P_\nu^\mu(x)\,dx = \frac{\pi^{1/2}2^{\mu-\lambda}\,\Gamma(\lambda)}{\Gamma\left(1+\frac{\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)} \times {}_2F_3\left(\frac{\lambda}{2},\frac{\lambda+1}{2};\frac{1}{2},\frac{1+\lambda-\mu-\nu}{2},1+\frac{\lambda-\mu+\nu}{2};-\frac{a^2}{4}\right)$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\mu<1]$ ET II 314(8)

3. $$\int_0^\infty \left(x^2-1\right)^{\frac{1}{2}\mu}\sin(ax)\,P_\nu^\mu(x)\,dx = \frac{2^\mu\pi^{1/2}a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\,S_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a)$$
$\left[a>0,\quad \operatorname{Re}\mu<\frac{3}{2},\quad \operatorname{Re}(\mu+\nu)<1\right]$ ET II 320(1)

7.162

1. $$\int_a^\infty P_\nu\left(2x^2a^{-2}-1\right)\sin(bx)\,dx = -\frac{\pi a}{4\cos(\nu\pi)}\left\{\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2-\left[J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2\right\}$$
$[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<0]$ ET II 326(1)

2. $$\int_a^\infty P_\nu\left(2x^2a^{-2}-1\right)\cos(bx)\,dx = -\frac{\pi}{4}a\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)-Y_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)Y_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]$$
$[a>0,\quad b>0,\quad -1<\operatorname{Re}\nu<0]$ ET II 326(2)

3. $$\int_0^\infty \left(x^2+2\right)^{-1/2}\sin(ax)\,P_\nu^{-1}\left(x^2+1\right)\,dx = 2^{-1/2}\pi^{-1}a\sin(\nu\pi)\left[K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)\right]^2$$
$[a>0,\quad -2<\operatorname{Re}\nu<1]$ ET I 98(22)

4. $$\int_0^\infty \left(x^2+2\right)^{-1/2}\sin(ax)\,Q_\nu^1\left(x^2+1\right)\,dx = -2^{-3/2}\pi a\,K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)I_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)$$
$\left[a>0,\quad \operatorname{Re}\nu>-\frac{3}{2}\right]$ ET 98(23)

5. $$\int_0^\infty \cos(ax)\, P_\nu\left(1+x^2\right)\, dx = -\frac{\sqrt{2}}{\pi}\sin(\nu\pi)\left[K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right)\right]^2$$
$[a>0, \quad -1<\operatorname{Re}\nu<0]$ ET I 42(23)

6. $$\int_0^\infty \cos(ax)\, Q_\nu\left(1+x^2\right)\, dx = \frac{\pi}{\sqrt{2}}\, K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right) I_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right)$$
$[a>0, \quad \operatorname{Re}\nu>-1]$ ET I 42(24)

7. $$\int_0^1 \cos(ax)\, P_\nu\left(2x^2-1\right)\, dx = \frac{\pi}{2}\, J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right)$$
$[a>0]$ ET I 42(25)

7.163

1. $$\int_a^\infty \left(x^2-a^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} \sin(bx)\, P_0^{\frac{1}{2}-\nu}\left(ax^{-1}\right)\, dx = b^{-\nu-\frac{1}{2}}\cos\left(ab-\frac{\nu\pi}{2}+\frac{\pi}{4}\right)$$
$\left[a>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET I 98(24)

2. $$\int_0^1 x^{-1}\cos(ax)\, P_\nu\left(2x^{-2}-1\right)\, dx = -\frac{1}{2}\pi\operatorname{cosec}(\nu\pi)\ {}_1F_1\left((\nu+1;1;ai)\right)\ {}_1F_1\left(\nu+1;1;-ai\right)$$
$[a>0, \quad -1<\operatorname{Re}\nu<0]$ ET II 327(4)

7.164

1. $$\int_0^\infty x^{1/2}\sin(bx)\left[P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right)\right]^2 dx = \frac{\sqrt{\frac{2}{\pi}}a^{-1}b^{-1/2}}{\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{5}{4}<\operatorname{Re}\nu<\frac{1}{4}\right]$ ET II 327(8)

2. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right) Q_{\nu-1}^{-1/4}\left(\sqrt{1+a^2x^2}\right) dx$$
$$= \frac{\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4}\pi i}\,\Gamma\left(\nu+\frac{5}{4}\right)}{ab^{\frac{1}{2}}\,\Gamma\left(\nu+\frac{3}{4}\right)}\, I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-\frac{5}{4}\right]$ ET II 328(9)

3. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right) P_{\nu-1}^{-1/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{5}{4}-\nu\right)}\, K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{5}{4}<\operatorname{Re}\nu<\frac{5}{4}\right]$ ET II 328(10)

4. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right) P_\nu^{-3/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{7}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{7}{4}<\operatorname{Re}\nu<\frac{3}{4}\right]$ ET II 328(11)

5. $$\int_0^\infty x^{1/2}\cos(bx)\left[P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)\right]^2 dx = \frac{a^{-1}\left(\frac{\pi b}{2}\right)^{-1/2}}{\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(-\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re} a>0,\quad b>0,\quad -\tfrac{3}{4}<\operatorname{Re}\nu<-\tfrac{1}{4}\right]$$
ET II 328(12)

6. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)Q_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)dx$$
$$=\frac{\sqrt{\frac{\pi}{2}}e^{\frac{1}{4}\pi i}\,\Gamma\left(\nu+\frac{3}{4}\right)}{ab^{1/2}\,\Gamma\left(\nu+\frac{5}{4}\right)}\,I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a>0,\quad b>0,\quad \operatorname{Re}\nu>-\tfrac{3}{4}\right]$$ ET II 328(13)

7. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right)P_\nu^{3/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$=\frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re} a>0,\quad b>0,\quad -\tfrac{5}{4}<\operatorname{Re}\nu<\tfrac{1}{4}\right]$$ ET II 328(14)

8. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)P_{\nu-1}^{1/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$=\frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)}K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right)K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a>0,\quad b>0,\quad |\operatorname{Re}\nu|<\tfrac{3}{4}\right]$$ ET II 329(15)

7.165 $$\int_0^\infty \cos(ax)\,P_\nu(\cosh x)\,dx$$
$$=-\frac{\sin(\nu\pi)}{4\pi^2}\,\Gamma\left(\frac{1+\nu+i\alpha}{2}\right)\Gamma\left(\frac{1+\nu-i\alpha}{2}\right)\Gamma\left(-\frac{\nu+i\alpha}{2}\right)\Gamma\left(-\frac{\nu-i\alpha}{2}\right)$$
$$[a>0,\quad -1<\operatorname{Re}\nu<0]$$ ET II 329(18)

7.166 $$\int_0^\pi P_\nu^{-\mu}(\cos\varphi)\sin^{\alpha-1}\varphi\,d\varphi=\frac{2^{-\mu}\pi\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$
$$[\operatorname{Re}(\alpha\pm\mu)>0]$$ MO 90, EH I 172(27)

7.167 $$\int_0^a P_\nu^{-\mu}(\cos x)\,P_\nu^{-\eta}[\cos(a-x)]\left[\frac{\sin(a-x)}{\sin x}\right]^\eta\frac{dx}{\sin x}=\frac{2^\eta\,\Gamma(\mu-\eta)\,\Gamma\left(\eta+\frac{1}{2}\right)(\sin a)^\eta}{\sqrt{\pi}\,\Gamma(\eta+\mu+1)}P_\nu^{-\mu}(\cos a)$$
$$\left[\operatorname{Re}\mu>\operatorname{Re}\eta>-\tfrac{1}{2}\right]$$ ET II 329(16)

7.17 A combination of an associated Legendre function and the probability integral

7.171 $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}\mu}\exp\left(a^2x^2\right)[1-\Phi(ax)]\,P_\nu^\mu(x)\,dx$$
$$=\pi^{-1}2^{\mu-1}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)a^{\mu-\frac{3}{2}}e^{\frac{a^2}{2}}\,W_{\frac{1}{4}-\frac{1}{2}\mu,\frac{1}{4}+\frac{1}{2}\nu}\left(a^2\right)$$
$$[\operatorname{Re} a>0,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+\nu)>-1,\quad \operatorname{Re}(\mu-\nu)>0]$$
ET II 324(17)

7.18 Combinations of associated Legendre and Bessel functions

7.181

1. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x)x^{1/2}\,Y_\nu(ax)\,dx = 2^{-1/2}a^{-1}\left[\cos\left(\tfrac{1}{2}a\right)J_\nu\left(\tfrac{1}{2}a\right)-\sin\left(\tfrac{1}{2}a\right)Y_\nu\left(\tfrac{1}{2}a\right)\right]$$
$$\left[a>0,\quad \operatorname{Re}\nu<\tfrac{1}{2}\right]$$ ET II 108(3)a

2. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x)x^{1/2}\,J_\nu(ax)\,dx = -\frac{1}{\sqrt{2}a}\left[\cos\left(\tfrac{1}{2}a\right)Y_\nu\left(\tfrac{1}{2}a\right)+\sin\left(\tfrac{1}{2}a\right)J_\nu\left(\tfrac{1}{2}a\right)\right]$$
$$\left[|\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$ ET II 344(36)a

7.182

1. $$\int_1^\infty x^\nu\left(x^2-1\right)^{\frac{1}{2}\lambda-\frac{1}{2}}P_\lambda^{\lambda-1}(x)\,J_\nu(ax)\,dx = \frac{2^{\lambda+\nu}a^{-\lambda}\,\Gamma\left(\frac{1}{2}+\nu\right)}{\pi^{1/2}\,\Gamma(1-\lambda)}\,S_{\lambda-\nu,\lambda+\nu}(a)$$
$$\left[a>0,\quad \operatorname{Re}\nu<\tfrac{5}{2},\quad \operatorname{Re}(2\lambda+\nu)<\tfrac{3}{2}\right]$$ ET II 345(38)a

2. $$\int_1^\infty x^{\frac{1}{2}-\mu}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)\,J_\nu(ax)\,dx$$
$$= -2^{-3/2}\pi^{1/2}a^{\mu-\frac{1}{2}}\left[J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)Y_\nu\left(\frac{a}{2}\right)+Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)J_\nu\left(\frac{a}{2}\right)\right]$$
$$\left[-\tfrac{1}{4}<\operatorname{Re}\mu<1,\quad a>0,\quad |\operatorname{Re}\nu|<\tfrac{1}{2}+2\operatorname{Re}\mu\right]$$ ET II 344(37)a

3. $$\int_1^\infty x^{\frac{1}{2}-\mu}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)\,Y_\nu(ax)\,dx$$
$$= 2^{-3/2}\pi^{1/2}a^{\mu-\frac{1}{2}}\left[J_\nu\left(\frac{a}{2}\right)J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)-Y_\nu\left(\frac{a}{2}\right)Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$$\left[-\tfrac{1}{4}<\operatorname{Re}\mu<1,\quad a>0,\quad \operatorname{Re}(2\mu-\nu)>-\tfrac{1}{2}\right]$$ ET II 349(67)a

4. $$\int_0^1 x^{\frac{1}{2}-\mu}\left(1-x^2\right)^{-\frac{1}{2}\mu}P^\mu_\nu(x)\,J_{\nu+\frac{1}{2}}(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{\mu-\frac{1}{2}}\,J_{\frac{1}{2}-\mu}\left(\tfrac{1}{2}a\right)J_{\nu+\frac{1}{2}}\left(\tfrac{1}{2}a\right)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu-\nu)<2\right]$$ ET II 337(33)a

5. $$\int_1^\infty x^{\frac{1}{2}-\mu}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)\,K_\nu(ax)\,dx = (2\pi)^{-1/2}a^{\mu-\frac{1}{2}}\,K_\nu\left(\tfrac{1}{2}a\right)K_{\mu-\frac{1}{2}}\left(\tfrac{1}{2}a\right)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}a>0\right]$$ ET II 135(5)a

6. $$\int_1^\infty x^{\mu+\frac{1}{2}}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)\,K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-3/2}e^{-\frac{1}{2}a}\,W_{\mu,\nu}(a)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}a>0\right]$$ ET II 135(3)a

7. $$\int_1^\infty x^{\mu-\frac{3}{2}}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{1}{2}}(x)\,K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-1/2}e^{-\frac{1}{2}a}\,W_{\mu-1,\nu}(a)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}a>0\right]$$ ET II 135(4)a

8. $$\int_1^\infty x^{\mu-\frac{1}{2}}\left(x^2-1\right)^{-\frac{1}{2}\mu}P^\mu_{\nu-\frac{3}{2}}(x)\,K_\nu(ax)\,dx = \sqrt{\frac{\pi}{2}}a^{-1}e^{-\frac{1}{2}a}\,W_{\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\mu<1\right]$$ ET II 135(6)a

9. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) K_\nu(ax)\,dx = \pi^{-1/2}a^{-\nu}2^{\nu-1}\left[K_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} a > 0\right]$$ ET II 136(11)a

10. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) Y_\nu(ax)\,dx$$
$$= \pi^{1/2}2^{\nu-2}a^{-\nu}\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - Y_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)Y_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu + |2\operatorname{Re}\mu + 1| < \tfrac{3}{2}\right]$$ ET II 108(5)a

11. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) J_\nu(ax)\,dx$$
$$= -2^{\nu-2}a^{-\nu}\pi^{1/2}\sec(\mu\pi)\left\{\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 - \left[J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu - \tfrac{3}{2} < 2\operatorname{Re}\mu < \tfrac{1}{2} - \operatorname{Re}\nu\right]$$ ET II 345(39)a

12. $$\int_1^\infty x\left(x^2-1\right)^{-\frac{1}{2}\nu} P_\mu^\nu\left(2x^2-1\right) K_\nu(ax)\,dx = 2^{-\nu}a^{\nu-1}\,K_{\mu+1}(a)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu < 1\right]$$ ET II 136(10)a

13. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu} P_\mu^\nu\left(1+2x^2a^{-2}\right) K_\nu(xy)\,dx = 2^{-\nu}ay^{-\nu-1}\,S_{2\nu,2\mu+1}(ay)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1\right]$$ ET II 135(7)

14. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu}\left[(\mu-\nu)\,P_\mu^\nu\left(1+2x^2a^{-2}\right) + (\mu+\nu)\,P_{-\mu}^\nu\left(1+2x^2a^{-2}\right)\right] K_\nu(xy)\,dx$$
$$= 2^{1-\nu}\mu y^{-\nu-2}\,S_{2\nu+1,2\mu}(ay)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1\right]$$ ET II 136(8)

15. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu-1}\left[P_\mu^\nu\left(1+2x^2a^{-2}\right) + P_{-\mu}^\nu\left(1+2x^2a^{-2}\right)\right] K_\nu(xy)\,dx = 2^{1-\nu}y^{-\nu}\,S_{2\nu-1,2\mu}(ay)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\nu < 1\right]$$ ET II 136(9)

16. $$\int_0^\infty x^{1/2}\left(x^2+2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}\left(x^2+1\right) J_\nu(xy)\,dx = \frac{y^{-1/2}2^{\frac{1}{2}-\nu}\pi^{-1/2}\left[K_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right)\right]^2}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}$$
$$\left[-\tfrac{3}{2} - \operatorname{Re}\nu < \operatorname{Re}\mu < \operatorname{Re}\nu + \tfrac{1}{2}, \quad y > 0\right]$$ ET II 44(1)

17. $$\int_0^\infty x^{1/2}\left(x^2+2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} Q_\mu^{\nu+\frac{1}{2}}\left(x^2+1\right) J_\nu(xy)\,dx$$
$$= 2^{-\nu-\frac{1}{2}}\pi^{1/2}e^{\left(\nu+\frac{1}{2}\right)\pi i}y^\nu\,K_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right) I_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right)$$
$$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -\tfrac{5}{2}, \quad y > 0\right]$$ ET II 46(12)

7.183 $$\int_0^\infty x^{1-\mu}\left(1+a^2x^2\right)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax)\, J_\nu(xy)\,dx$$
$$= i(2\pi)^{1/2}e^{i\pi\left(\mu\mp\frac{1}{2}\nu\mp\frac{1}{4}\right)}a^{-1}y^{\mu-1}\, I_\nu\left(\tfrac{1}{2}a^{-1}y\right) K_\mu\left(\tfrac{1}{2}a^{-1}y\right)$$
$$\left[-\tfrac{3}{4}-\tfrac{1}{2}\operatorname{Re}\nu<\operatorname{Re}\mu<1+\operatorname{Re}\nu,\quad y>0,\quad \operatorname{Re}a>0\right]$$ ET II 46(11)

7.184

1. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}\left(x^{-1}\right) J_\nu(xa)\,dx = 2^{1/2}a^{-1-\mu}\pi^{-1/2}\cos\left[a+\tfrac{1}{2}(\nu-\mu)\pi\right]$$
$$\left[|\operatorname{Re}\mu|<\tfrac{1}{2},\quad \operatorname{Re}\nu>-1,\quad a>0\right]$$
ET II 44(2)a

2. $$\int_1^\infty x^{-\nu}\left(x^2-1\right)^{\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu-\frac{1}{2}}\left(2x^{-2}-1\right) K_\nu(ax)\,dx$$
$$= \pi^{1/2}2^{-\nu}a^{-2+\nu}\, W_{\mu+\frac{1}{2},\nu-\frac{1}{2}}(a)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$
$$\left[\operatorname{Re}\nu<\tfrac{3}{2},\quad a>0\right]$$ ET II 370(45)a

3. $$\int_0^\infty x^\nu\left(1+x^2\right)^{\frac{1}{4}+\frac{\nu}{2}} Q_\mu^{\nu+\frac{1}{2}}\left(1+\frac{2}{x^2}\right) J_\nu(ax)\,dx$$
$$= -ie^{i\pi\nu}\pi^{-\frac{1}{2}}2^\nu a^{-\nu-2}\left[\Gamma\left(\tfrac{3}{2}+\mu+\nu\right)\right]^2\Gamma\left(\tfrac{1}{2}+\nu-\mu\right)$$
$$\times W_{-\mu-\frac{1}{2},\nu+\frac{1}{2}}(a)\left[\frac{\cos(\mu\pi)}{\Gamma(2+2\nu)}\, M_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a)+\frac{\sin(\mu\pi)}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)}\, W_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a)\right]$$
$$\left[a>0,\quad \operatorname{Re}(\mu+\nu)>-\tfrac{3}{2},\quad \operatorname{Re}(\mu-\nu)<\tfrac{1}{2}\right]$$ ET II 46(14)

4. $$\int_0^1 x^\nu\left(1-x^2\right)^{\frac{1}{2}\nu+\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}\left(2x^{-2}-1\right) J_\nu(xy)\,dx$$
$$= 2^{\nu+\frac{1}{2}}y^\nu\frac{\Gamma\left(\frac{3}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}{(2\pi)^{1/2}\left[\Gamma\left(\frac{3}{2}+\nu\right)\right]^2}$$
$$\times\, {}_1F_1\left(\nu+\mu+\frac{3}{2};2\nu+2;iy\right)\, {}_1F_1\left(\nu+\mu+\frac{3}{2};2\nu+2;-iy\right)$$
$$\left[y>0,\quad -\tfrac{3}{2}-\operatorname{Re}\nu<\operatorname{Re}\mu<\operatorname{Re}\nu+\tfrac{1}{2}\right]$$ ET II 45(3)

5. $$\int_0^\infty x^{-\nu}\left(x^2+a^2\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}\left(1+2a^2x^{-2}\right) K_\nu(xy)\,dx$$
$$= ie^{-i\pi\nu}\pi^{1/2}2^{-\nu-1}a^{-\nu-\frac{1}{2}}y^{\nu-2}\left[\Gamma\left(\tfrac{3}{2}+\mu-\nu\right)\right]^2 W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(iay)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(-iay)$$
$$\left[\operatorname{Re}a>0,\quad \operatorname{Re}y>0,\quad \operatorname{Re}\mu>-\tfrac{3}{2},\quad \operatorname{Re}(\mu-\nu)>-\tfrac{3}{2}\right]$$ ET II 137(13)

6. $$\int_0^\infty x^{-\nu}\left(x^2+1\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}\left(1+2x^{-2}\right) J_\nu(ax)\,dx$$
$$= 2^{-\nu}a^{-\nu-2}\frac{ie^{-i\nu\pi}\pi^{1/2}\,\Gamma\left(\frac{3}{2}+\mu-\nu\right)}{\Gamma(2\nu)}\, M_{\mu+\frac{1}{2},\nu-\frac{1}{2}}(a)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$
$$\left[a>0,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu+\tfrac{3}{2}\right]$$ ET II 47(15)a

7. $$\int_0^\infty x^{-\nu}\left(x^2+a^2\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu}\left(1+2a^2x^{-2}\right) K_\nu(xy)\,dx$$
$$= ie^{-i\pi\nu}\pi^{3/2}2^{-\nu-3}a^{\frac{1}{2}-\nu}y^{\nu-1}\left[\Gamma(1-\nu)\right]^2\times\left\{\left[J_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2+\left[Y_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2\right\}$$
$[\operatorname{Re} a>0,\quad \operatorname{Re} y>0,\quad \operatorname{Re}\nu<1]$ ET II 136(12)

7.185 $$\int_0^\infty x^{1/2}\,Q_{\nu-\frac{1}{2}}\left[\left(a^2+x^2\right)x^{-1}\right] J_\nu(xy)\,dx = 2^{-1/2}\pi y^{-1}\exp\left[-\left(a^2-\tfrac{1}{4}\right)^{1/2}y\right] J_\nu\left(\tfrac{1}{2}y\right)$$
$\left[\operatorname{Re}\nu>-\frac{1}{2},\quad y>0\right]$ ET II 46(10)

7.186 $$\int_0^\infty x\left(1+x^2\right)^{-\nu-1} P_\nu\left(\frac{1-x^2}{1+x^2}\right) J_0(xy)\,dx = y^{2\nu}\left[2^\nu\,\Gamma(\nu+1)\right]^{-2} K_0(y)$$
$[\operatorname{Re}\nu>0]$ ET II 13(10)

7.187

1. $$\int_0^\infty x\,P_\mu^\nu\left(\sqrt{1+x^2}\right) K_\nu(xy)\,dx = y^{-3/2}\,S_{\nu+\frac{1}{2},\mu+\frac{1}{2}}(y)$$
$[\operatorname{Re}\nu<1,\quad \operatorname{Re} y>0]$ ET II 137(14)

2. $$\int_0^\infty x\left[P_{\lambda-\frac{1}{2}}\left(\sqrt{1+a^2x^2}\right)\right]^2 J_0(xy)\,dx = 2\pi^{-2}y^{-1}a^{-1}\cos(\lambda\pi)\left[K_\lambda\left(\frac{y}{2a}\right)\right]^2$$
$\left[\operatorname{Re} a>0,\quad |\operatorname{Re}\lambda|<\frac{1}{4},\quad y>0\right]$ ET II 13(11)

3. $$\int_0^\infty x\left(1+x^2\right)^{-1/2} P_\mu^\nu\left(\sqrt{1+x^2}\right) K_\nu(xy)\,dx = y^{-1/2}\,S_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(y)$$
$[\operatorname{Re}\nu<1,\quad \operatorname{Re} y>0]$ ET II 137(15)

4. $$\int_0^\infty x\,P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) Q_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx$$
$$= \frac{y^{-1}e^{-\frac{1}{2}\nu\pi i}\,\Gamma\left(1+\mu+\frac{1}{2}\nu\right)}{a\,\Gamma\left(1+\mu-\frac{1}{2}\nu\right)} I_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad \operatorname{Re}\mu>-\frac{3}{4},\quad \operatorname{Re}\nu>-1\right]$ ET II 47(16)

5. $$\int_0^\infty x\,P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) Q_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) J_0(xy)\,dx$$
$$= y^{-2}e^{\mu\pi i}\frac{\Gamma\left(\frac{1}{2}+\sigma-\mu\right)}{\Gamma(1+2\sigma)} W_{\mu,\sigma}\left(\frac{y}{a}\right) M_{-\mu,\sigma}\left(\frac{y}{a}\right)$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad \operatorname{Re}\sigma>-\frac{1}{4},\quad \operatorname{Re}\mu<1\right]$ ET II 14(15)

6. $$\int_0^\infty x\,P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) P_{\sigma-\frac{1}{2}}^{-\mu}\left(\sqrt{1+a^2x^2}\right) J_0(xy)\,dx$$
$$= 2\pi^{-1}y^{-2}\cos(\sigma\pi)\,W_{\mu,\sigma}\left(\frac{y}{a}\right) W_{-\mu,\sigma}\left(\frac{y}{a}\right)$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad |\operatorname{Re}\sigma|<\frac{1}{4}\right]$ ET II 14(14)

7. $$\int_0^\infty x\left\{P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right)\right\}^2 J_0(xy)\,dx = -i\pi^{-1}y^{-2}\,W_{\mu,\sigma}\left(\frac{y}{a}\right)\left[W_{\mu,\sigma}\left(e^{\pi i}\frac{y}{a}\right)-W_{\mu,\sigma}\left(e^{-\pi i}\frac{y}{a}\right)\right]$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad |\operatorname{Re}\sigma|<\frac{1}{4},\quad \operatorname{Re}\mu<1\right]$ ET II 14(13)

8. $$\int_0^\infty x\left(1+a^2x^2\right)^{-1/2} P_\mu^{-\frac{1}{2}-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) P_\mu^{\frac{1}{2}-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx = \frac{\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2}{\pi a^2\,\Gamma\left(\frac{\nu}{2}+\mu+\frac{3}{2}\right)\Gamma\left(\frac{\nu}{2}-\mu+\frac{1}{2}\right)}$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad -\frac{5}{4}<\operatorname{Re}\mu<\frac{1}{4}\right]$ ET II 46(9)

9. $$\int_0^\infty x\left\{P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right)\right\}^2 J_\nu(xy)\,dx = \frac{2\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2 y^{-1}}{\pi a\,\Gamma\left(1+\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)}$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad -\frac{3}{4}<\operatorname{Re}\mu<-\frac{1}{4},\quad \operatorname{Re}\nu>-1\right]$ ET II 45(7)

10. $$\int_0^\infty x\left(1+a^2x^2\right)^{-1/2} P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) P_{\mu+1}^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx = \frac{K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{3}{2}}\left(\frac{y}{2a}\right)}{\pi a^2\,\Gamma\left(2+\frac{1}{2}\nu+\mu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)}$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad -\frac{7}{4}<\operatorname{Re}\mu<-\frac{1}{4}\right]$ ET II 45(8)

7.188

1. $$\int_0^\infty x\left(a^2+x^2\right)^{-\frac{1}{2}\mu} P_{\mu-1}^{-\nu}\left[\frac{a}{\sqrt{a^2+x^2}}\right] J_\nu(xy)\,dx = \frac{y^{\mu-2}e^{-ay}}{\Gamma(\mu+\nu)}$$
$\left[\operatorname{Re} a>0,\quad y>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>\frac{1}{2}\right]$ ET II 45(4)

2. $$\int_0^\infty x^{\nu+1}\left(x^2+a^2\right)^{\frac{1}{2}\nu} P_\nu\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{\nu+1}y^{-\nu-1}}{\pi\,\Gamma(-\nu)}\left[K_{\nu+\frac{1}{2}}\left(\frac{ya}{2}\right)\right]^2$$
$[\operatorname{Re} a>0,\quad -1<\operatorname{Re}\nu<0,\quad y>0]$
ET II 45(5)

3. $$\int_0^\infty x^{1-\nu}\left(x^2+a^2\right)^{-\frac{1}{2}\nu} P_{\nu-1}\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{1-\nu}y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) K_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)$$
$[\operatorname{Re} a>0,\quad y>0,\quad 0<\operatorname{Re}\nu<1]$
ET II 45(6)

7.189

1. $$\int_0^\infty (a+x)^\mu e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = 0$$
$\left[-\frac{1}{2}<\operatorname{Re}\mu<0,\quad -\frac{1}{2}+\operatorname{Re}\mu<\operatorname{Re}\nu<-\frac{1}{2}-\operatorname{Re}\mu\right]$ ET II 366(18)

2. $$\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu-\frac{1}{2}\right)e^a}{\pi^{1/2}\,\Gamma(2\mu+\nu+1)\,\Gamma(2\mu-\nu)} W_{\frac{1}{2}-\mu,\frac{1}{2}+\nu}(2a)$$
$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>\left|\operatorname{Re}\nu+\frac{1}{2}\right|\right]$ ET II 367(19)

3. $$\int_0^\infty x^{-\mu}e^x\,P_\nu^{2\mu}\left(1+\frac{2x}{a}\right)K_\mu(x+a)\,dx$$
$$=\pi^{-1/2}2^{\mu-1}\cos(\mu\pi)\,\Gamma\left(\mu+\nu+\tfrac12\right)\Gamma\left(\mu-\nu+\tfrac12\right)\,W_{\frac12-\mu,\frac12+\nu}(2a)$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>\left|\operatorname{Re}\nu+\tfrac12\right|\right]$$ ET II 373(11)

4. $$\int_0^\infty x^{-\frac12\mu}(x+a)^{-1/2}e^{-x}\,P_{\nu-\frac12}^{\mu}\left(\frac{a-x}{a+x}\right)K_\nu(a+x)\,dx=\sqrt{\frac{\pi}{2}}a^{-\frac12\mu}\,\Gamma(\mu,2a)$$
$$[a>0,\quad \operatorname{Re}\mu<1]$$ ET II 374(12)

5. $$\int_0^\infty (\sinh x)^{\mu+1}\,(\cosh x)^{-2\mu-\frac32}\,P_\nu^{-\mu}\left[\cosh(2x)\right]I_{\mu-\frac12}\left(a\operatorname{sech}x\right)\,dx$$
$$=\frac{2^{\mu-\frac12}\,\Gamma(\mu-\nu)\,\Gamma\left(\mu+\nu+1\right)}{\pi^{1/2}a^{\mu+\frac32}\left[\Gamma\left(\mu+1\right)\right]^2}\,M_{\nu+\frac12,\mu}(a)\,M_{-\nu-\frac12,\mu}(a)$$
$$[\operatorname{Re}\mu>\operatorname{Re}\nu,\quad \operatorname{Re}\mu>-\operatorname{Re}\nu-1]$$ ET II 378(44)

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.191

1. $$\int_a^\infty x^{1/2}\left(x^2-a^2\right)^{-\frac14-\frac12\nu}P_\mu^{\nu+\frac12}\left(2x^2a^{-2}-1\right)\left[\mathbf{H}_\nu(x)-Y_\nu(x)\right]\,dx$$
$$=2^{-\nu-2}\pi^{1/2}a\operatorname{cosec}(\mu\pi)\cos(\nu\pi)\left\{\left[Y_\nu\left(\tfrac12a\right)\right]^2-\left[J_\nu\left(\tfrac12a\right)\right]^2\right\}$$
$$\left[-1<\operatorname{Re}\mu<0,\quad \operatorname{Re}\nu<\tfrac12\right]$$ ET II 384(6)

2. $$\int_0^\infty x^{1/2}\left(x^2-a^2\right)^{-1/4-\nu/2}P_\mu^{\nu+1/2}\left(2x^2a^{-2}-1\right)\left[I_{-\nu}(x)-\mathbf{L}_\nu(x)\right]\,dx$$
$$=2^{-\nu-1}\pi^{1/2}a\operatorname{cosec}(2\mu\pi)\cos(\nu\pi)\left\{\left[I_\nu\left(\tfrac12a\right)\right]^2-\left[I_{-\nu}\left(\tfrac12a\right)\right]^2\right\}$$
$$\left[-1<\operatorname{Re}\mu<0,\quad \operatorname{Re}\nu<\tfrac12\right]$$ ET II 385(15)

7.192

1. $$\int_0^1 x^{(\nu-\mu-1)/2}\left(1-x^2\right)^{(\nu-\mu-2)/4}P_{\nu-1/2}^{(\mu-\nu+2)/2}(x)\,S_{\mu,\nu}(ax)\,dx$$
$$=2^{\mu-3/2}\pi^{1/2}a^{-(\nu-\mu-1)/2}\,\Gamma\left(\frac{\mu+\nu+3}{4}\right)\Gamma\left(\frac{\mu-3\nu+3}{4}\right)\cos\left(\frac{\mu-\nu}{2}\pi\right)$$
$$\times\left[J_\nu\left(\tfrac12a\right)\,Y_{-(\mu-\nu+1)/2}\left(\tfrac12a\right)-Y_\nu\left(\tfrac12a\right)\,J_{-(\mu-\nu+1)/2}\left(\tfrac12a\right)\right]$$
$$[\operatorname{Re}(\mu-\nu)<0,\quad a>0,\quad |\operatorname{Re}(\mu+\nu)|<1,\quad \operatorname{Re}(\mu-3\nu)<1]$$ ET II 387(24)a

2. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{-\beta/2} P_\nu^\beta(x)\, S_{\mu,1/2}(ax)\,dx$$
$$= \frac{2^{-3/2+\beta-\mu} a^{\beta-1}\,\Gamma\left(\frac{\beta-\mu+\nu}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\beta-\mu-\nu}{2}-\frac{1}{4}\right)}{\pi^{1/2}\,\Gamma\left(\frac{1}{2}-\mu\right)}\, S_{\mu-\beta+1,\nu+1/2}(a)$$
$$\left[\operatorname{Re}\beta<1,\quad a>0,\quad \operatorname{Re}(\mu+\nu-\beta)<-\tfrac{1}{2},\quad \operatorname{Re}(\mu-\nu-\beta)<\tfrac{1}{2}\right]$$ ET II 387(25)a

7.193

1. $$\int_1^\infty x^{-\nu}\left(x^2-1\right)^{1/4-\nu/2} P_{\mu/2-\nu/2}^{\nu-1/2}\left(2x^{-2}-1\right) S_{\mu,\nu}(ax)\,dx$$
$$= \frac{2^{\mu-\nu} a^{\nu-2}\pi^{1/2}\,\Gamma\left(\frac{3\nu-\mu-1}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}\, W_{\rho,\sigma}\left(ae^{i\pi/2}\right) W_{\rho,\sigma}\left(ae^{-i\pi/2}\right)$$
$$\rho=\tfrac{1}{2}(\mu+1-\nu),\quad \sigma=\nu-\tfrac{1}{2},\qquad \left[\operatorname{Re}(\mu-\nu)<0,\quad a>0,\quad \operatorname{Re}\nu<\tfrac{3}{2},\quad \operatorname{Re}(3\nu-\mu)>1\right]$$
ET II 387(27)a

2. $$\int_1^\infty x\left(x^2-1\right)^{-\nu/2} P_\lambda^\nu\left(2x^2-1\right) S_{\mu,\nu}(ax)\,dx$$
$$= \frac{a^{\nu-1}\,\Gamma\left(\frac{\nu-\mu+1}{2}+\lambda\right)\Gamma\left(\frac{\nu-\mu-1}{2}-\lambda\right)}{2\,\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)}\, S_{\mu-\nu+1,2\lambda+1}(a)$$
$$\left[\operatorname{Re}\nu<1,\quad a>0,\quad \operatorname{Re}(\mu-\nu+\lambda)<-1,\quad \operatorname{Re}(\mu-\nu+\lambda)<0\right]$$ ET II 387(26)a

7.21 Integration of associated Legendre functions with respect to the order

7.211

1. $$\int_0^\infty P_{-x-\frac{1}{2}}(\cos\theta)\,dx = \frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\theta\right) \qquad [0<\theta<\pi]$$ ET II 329(19)

2. $$\int_{-\infty}^\infty P_x(\cos\theta)\,dx = \operatorname{cosec}\left(\frac{1}{2}\theta\right) \qquad [0<\theta<\pi]$$ ET II 329(20)

7.212 $$\int_0^\infty x^{-1}\tanh(\pi x)\, P_{-\frac{1}{2}+ix}(\cosh a)\,dx = 2e^{-\frac{1}{2}a}\,\boldsymbol{K}\left(e^{-a}\right)$$
$$[a>0]$$ ET II 330(22)

7.213 $$\int_0^\infty \frac{x\tanh(\pi x)}{a^2+x^2}\, P_{-\frac{1}{2}+ix}(\cosh b)\,dx = Q_{a-\frac{1}{2}}(\cosh b) \qquad [\operatorname{Re} a>0]$$ ET II 387(23)

7.214 $$\int_0^\infty \sinh(\pi x)\cos(ax)\, P_{-\frac{1}{2}+ix}(b)\,dx = \frac{1}{\sqrt{2\,(b+\cosh a)}}$$
$$[a>0,\quad |b|<1]$$ ET I 42(27)

7.215 $$\int_0^\infty \cos(bx)\, P^\mu_{-\frac{1}{2}+ix}(\cosh a)\,dx = 0 \qquad [0<a<b]$$
$$= \frac{\sqrt{\frac{\pi}{2}}\,(\sinh a)^\mu}{\Gamma\left(\frac{1}{2}-\mu\right)(\cosh a-\cosh b)^{\mu+\frac{1}{2}}} \qquad [0<b<a]$$
ET II 330(21)

7.216 $$\int_0^\infty \cos(bx)\,\Gamma(\mu+ix)\,\Gamma(\mu-ix)\,P^{\frac12-\mu}_{-\frac12+ix}(\cosh a)\,dx = \frac{\sqrt{\frac{\pi}{2}}\,\Gamma(\mu)\,(\sinh a)^{\mu-\frac12}}{(\cosh a+\cosh b)^{\mu}}$$
$$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$$
ET II 330(24)

7.217

1. $$\int_{-\infty}^\infty \left(\nu-\frac12+ix\right)\Gamma\left(\frac12-ix\right)\Gamma\left(2\nu-\frac12+ix\right)P^{\frac12-\nu}_{\nu+ix-1}(\cos\theta)\,I_{\nu-\frac12+ix}(a)\,K_{\nu-\frac12+ix}(b)\,dx$$
$$= \sqrt{2\pi}\,(\sin\theta)^{\nu-\frac12}\left(\frac{ab}{\omega}\right)^{\nu}K_\nu(\omega)$$
$$\left[\omega=\left(a^2+b^2+2ab\cos\theta\right)^{1/2}\right]$$ ET II 383(29)

2. $$\int_0^\infty xe^{\pi x}\tanh(\pi x)\,P_{-\frac12+ix}(-\cos\theta)\,H^{(2)}_{ix}(ka)\,H^{(2)}_{ix}(kb)\,dx = -\frac{2(ab)^{1/2}}{\pi R}e^{-ikR};$$
$$R=\left(a^2+b^2-2ab\cos\theta\right)^{1/2}\qquad [a>0,\quad b>0,\quad 0<\theta<\pi,\quad \operatorname{Im}k\le 0]$$ ET II 381(17)

3. $$\int_0^\infty xe^{\pi x}\sinh(\pi x)\,\Gamma(\nu+ix)\,\Gamma(\nu-ix)\,P^{\frac12-\nu}_{-\frac12+ix}(-\cos\theta)\,H^{(2)}_{ix}(a)\,H^{(2)}_{ix}(b)\,dx$$
$$= i(2\pi)^{1/2}(\sin\theta)^{\nu-\frac12}\left(\frac{ab}{R}\right)^{\nu}H^{(2)}_\nu(R)$$
$$R=\left(a^2+b^2-2ab\cos\theta\right)^{1/2}\qquad [a>0,\quad b>0,\quad 0<\theta<\pi,\quad \operatorname{Re}\nu>0]$$ ET II 381 (18)

4. $$\int_0^\infty x\sinh(\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,P^{\frac12-\lambda}_{-\frac12+ix}(\beta)\,dx = \frac{\pi^{1/2}}{\sqrt2}\left(\frac{ab}{z}\right)^{\lambda}\left(\beta^2-1\right)^{\frac12\lambda-\frac14}K_\lambda(z)$$
$$z=\sqrt{a^2+b^2+2ab\beta}\qquad \left[|\arg a|<\frac{\pi}{2},\quad |\arg(\beta-1)|<\pi,\quad \operatorname{Re}\lambda>0\right]$$ ET II 177(16)

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.221

1. $$\int_{-1}^1 P_n(x)\,P_m(x)\,dx = 0 \qquad [m\ne n]$$
$$= \frac{2}{2n+1}\qquad [m=n]$$ WH, EH I 170(8, 10)

2.[6] $$\int_0^1 P_n(x)\,P_m(x)\,dx = \frac{1}{2n+1}\qquad [m=n]$$
$$= 0\qquad [n-m \text{ is even},\quad m\ne n]$$
$$= \frac{(-1)^{\frac12(m+n-1)}m!n!}{2^{m+n-1}(m-n)\,(n+m+1)\left[\left(\frac{n}{2}\right)!\left(\frac{m-1}{2}\right)!\right]^2}\qquad [n \text{ is even, } m \text{ is odd}]$$
WH

3. $$\int_0^{2\pi} P_{2n}(\cos\varphi)\,d\varphi = 2\pi\left[\binom{2n}{n}2^{-2n}\right]^2.$$ MO 70, EH II 183(50)

7.222

1. $$\int_{-1}^{1} x^m\, P_n(x)\, dx = 0 \qquad [m < n]$$

2. $$\int_{-1}^{1} (1+x)^{m+n}\, P_m(x)\, P_n(x)\, dx = \frac{2^{m+n+1}\,[(m+n)!]^4}{(m!n!)^2\,(2m+2n+1)!}$$ ET II 277(15)

3. $$\int_{-1}^{1} (1+x)^{m-n-1}\, P_m(x)\, P_n(x)\, dx = 0 \qquad [m > n]$$ ET II 278(16)

4. $$\int_{-1}^{1} \left(1-x^2\right)^n P_{2m}(x)\, dx = \frac{2n^2}{(n-m)(2m+2n+1)} \int_{-1}^{1} \left(1-x^2\right)^{n-1} P_{2m}(x)\, dx$$
$$[m < n]$$ WH

5. $$\int_{0}^{1} x^2\, P_{n+1}(x)\, P_{n-1}(x)\, dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}$$ WH

7.223 $$\int_{-1}^{1} \frac{1}{z-x} \left\{P_n(x)\, P_{n-1}(x) - P_{n-1}(x)\, P_n(z)\right\} dx = -\frac{2}{n}$$ WH

7.224 [z belongs to the complex plane with a discontinuity along the interval from -1 to $+1$.]

1. $$\int_{-1}^{1} (z-x)^{-1}\, P_n(x)\, dx = 2\, Q_n(z)$$ ET II 277(7)

2. $$\int_{-1}^{1} x(z-x)^{-1}\, P_0(x)\, dx = 2\, Q_1(z)$$ ET II 277(8)

3. $$\int_{-1}^{1} x^{n+1}(z-x)^{-1}\, P_n(x)\, dx = 2z^{n+1}\, Q_n(z) - \frac{2^{n+1}\,(n!)^2}{(2n+1)!}$$ ET II 277(9)

4. $$\int_{-1}^{1} x^m(z-x)^{-1}\, P_n(x)\, dx = 2z^m\, Q_n(z) \qquad [m \le n]$$ ET II 277(10)a

5. $$\int_{-1}^{1} (z-x)^{-1}\, P_m(x)\, P_n(x)\, dx = 2\, P_m(z)\, Q_n(z) \qquad [m \le n]$$ ET II 278(18)a

6. $$\int_{-1}^{1} (z-x)^{-1}\, P_n(x)\, P_{n+1}(x)\, dx = 2\, P_{n+1}(z)\, Q_n(z) - \frac{2}{n+1}$$ ET II 278(19)

7. $$\int_{-1}^{1} x(z-x)^{-1}\, P_m(x)\, P_n(x)\, dx = 2z\, P_m(z)\, Q_n(z) \qquad [m < n]$$ ET II 278(21)

8. $$\int_{-1}^{1} x(z-x)^{-1}\, [P_n(x)]^2\, dx = 2z\, P_n(z)\, Q_n(z) - \frac{2}{2n+1}$$ ET II 278(20)

7.225

1. $$\int_{-1}^{x} (x-t)^{-1/2}\, P_n(t)\, dt = \left(n+\frac{1}{2}\right)^{-1} (1+x)^{-1/2}\, [T_n(x) + T_{n+1}(x)]$$ EH II 187(43)

2. $$\int_{x}^{1} (t-x)^{-1/2} P^{-1/2}\, P_n(t)\, dt = \left(n+\frac{1}{2}\right)^{-1} (1-x)^{-1/2}\, [T_n(x) - T_{n+1}(x)]$$ EH II 187(44)

3. $$\int_{-1}^{1}(1-x)^{-1/2}\,P_n(x)\,dx=\frac{2^{3/2}}{2n+1}$$ EH II 183(49)

4. $$\int_{-1}^{1}(\cosh 2p-x)^{-1/2}\,P_n(x)\,dx=\frac{2\sqrt{2}}{2n+1}\exp[-(2n+1)p]$$
$[p>0]$ WH

5.[10] $$\frac{1}{2}\int_{-1}^{1}\frac{P_\ell(z)\,dz}{\sqrt{(xy-z)^2-(x^2-1)\,(y^2-1)}}=P_\ell(x)\,Q_\ell(y)\qquad(1<x\le y)$$
$$=P_\ell(y)\,Q_\ell(x)\qquad(1<y\le x)$$

7.226

1. $$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}P_{2m}(x)\,dx=\left[\frac{\Gamma\left(\frac{1}{2}+m\right)}{m!}\right]^2$$ ET II 276(4)

2. $$\int_{-1}^{1}x\left(1-x^2\right)^{-1/2}P_{2m+1}(x)\,dx=\frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}$$ ET II 276(5)

3. $$\int_{-1}^{1}\left(1+px^2\right)^{-m-3/2}P_{2m}(x)\,dx=\frac{2}{2m+1}(-p)^m(1+p)^{-m-1/2}$$
$[|p|<1]$ MO 71

7.227 $$\int_0^1 x\left(a^2+x^2\right)^{-1/2}P_n\left(1-2x^2\right)\,dx=\frac{\left[a+\left(a^2+1\right)^{1/2}\right]^{-2n-1}}{2n+1}$$
$[\operatorname{Re}a>0]$ ET II 278(23)

7.228[6] $$\frac{1}{2}\Gamma(1+\mu)\int_{-1}^{1}P_l(x)(z-x)^{-\mu-1}\,dx=\left(z^2-1\right)^{-\mu/2}e^{-i\pi\mu}\,Q_l^{\mu}(z)$$
$[l=0,1,2,\ldots,\quad|\arg(z-1)|<\pi]$

7.23 Combinations of Legendre polynomials and powers

7.231

1. $$\int_0^1 x^\lambda\,P_{2m}(x)\,dx=\frac{(-1)^m\,\Gamma\left(m-\frac{1}{2}\lambda\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\lambda\right)}{2\,\Gamma\left(-\frac{1}{2}\lambda\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2}\lambda\right)}\qquad[\operatorname{Re}\lambda>-1]$$ EH II 183(51)

2.[6] $$\int_0^1 x^\lambda\,P_{2m+1}(x)\,dx=\frac{(-1)^m\,\Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)}$$
$[\operatorname{Re}\lambda>-2]$ EH II 183(52)

7.232

1. $$\int_{-1}^{1}(1-x)^{a-1}\,P_m(x)\,P_n(x)\,dx$$
$$=\frac{2^a\,\Gamma(a)\,\Gamma(n-a+1)}{\Gamma(1-a)\,\Gamma(n+a+1)}\;{}_4F_3\left(-m,m+1,a,a;1,a+n+1,a-n;1\right)$$
$[\operatorname{Re}a>0]$ ET II 278(17)

2. $$\int_{-1}^{1}(1-x)^{a-1}(1+x)^{b-1}\,P_n(x)\,dx=\frac{2^{a+b-1}\,\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\;{}_3F_2(-n,1+n,a;1,a+b;1)$$
$[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$ ET II 276(6)

3. $$\int_{0}^{1}(1-x)^{\mu-1}\,P_n(1-\gamma x)\,dx=\frac{\Gamma(\mu)n!}{\Gamma(\mu+n+1)}\,P_n^{(\mu,-\mu)}(1-\gamma)$$
$[\operatorname{Re}\mu>0]$ ET II 190(37)a

4. $$\int_{0}^{1}(1-x)^{\mu-1}x^{\nu-1}\,P_n(1-\gamma x)\,dx=\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\;{}_3F_2\left(-n,n+1,\nu;1,\mu+\nu;\frac{1}{2}\gamma\right)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ ET II 190(38)

7.233 $$\int_{0}^{1}x^{2\mu-1}\,P_n\left(1-2x^2\right)\,dx=\frac{(-1)^n\,[\Gamma(\mu)]^2}{2\,\Gamma(\mu+n+1)\,\Gamma(\mu-n)}$$
$[\operatorname{Re}\mu>0]$ ET II 278(22)

7.24 Combinations of Legendre polynomials and other elementary functions

7.241 $$\int_{0}^{\infty}P_n(1-x)e^{-ax}\,dx=e^{-a}a^n\left(\frac{1}{a}\frac{d}{da}\right)^n\left(\frac{e^a}{a}\right)$$
$$=a^n\left(1+\frac{1}{2}\frac{d}{da}\right)^n\left(\frac{1}{a^{n+1}}\right)$$
$[\operatorname{Re}a>0]$ ET I 171(2)

7.242 $$\int_{0}^{\infty}P_n\left(e^{-x}\right)e^{-ax}\,dx=\frac{(a-1)(a-2)\cdots(a-n+1)}{(a+n)(a+n-2)\cdots(a-n+2)}$$
$[n\geq 2,\quad \operatorname{Re}a>0]$ ET I 171(3)

7.243

1. $$\int_{0}^{\infty}P_{2n}(\cosh x)\,e^{-ax}\,dx=\frac{\left(a^2-1^2\right)\left(a^2-3^2\right)\cdots\left[a^2-(2n-1)^2\right]}{a\left(a^2-2^2\right)\left(a^2-4^2\right)\cdots\left[a^2-(2n)^2\right]}$$
$[\operatorname{Re}a>2n]$ ET I 171(6)

2. $$\int_{0}^{\infty}P_{2n+1}(\cosh x)\,e^{-ax}\,dx=\frac{a\left(a^2-2^2\right)\left(a^2-4^2\right)\cdots\left[a^2-(2n)^2\right]}{\left(a^2-1\right)\left(a^2-3^2\right)\cdots\left[a^2-(2n+1)^2\right]}$$
$[\operatorname{Re}a>2n+1]$ ET I 171(7)

3. $$\int_{0}^{\infty}P_{2n}(\cos x)\,e^{-ax}\,dx=\frac{\left(a^2+1^2\right)\left(a^2+3^2\right)\cdots\left[a^2+(2n-1)^2\right]}{a\left(a^2+2^2\right)\left(a^2+4^2\right)\cdots\left[a^2+(2n)^2\right]}$$
$[\operatorname{Re}a>0]$ ET I 171(4)

4. $$\int_{0}^{\infty}P_{2n+1}(\cos x)\,e^{-ax}\,dx=\frac{a\left(a^2+2^2\right)\left(a^2+4^2\right)\cdots\left[a^2+(2n)^2\right]}{\left(a^2+1^2\right)\left(a^2+3^2\right)\cdots\left[a^2+(2n+1)^2\right]}$$
$[\operatorname{Re}a>0]$ ET I 171(5)

5.[11] $$\int_{-1}^{1}e^{ix\alpha}\,P_n(x)\,dx=i^n\sqrt{\frac{2\pi}{\alpha}}\,J_{n+\frac{1}{2}}(\alpha)\qquad [n=0,1,2,\ldots,\quad a>0]$$
GH2 24 (171.10)

7.244

1. $$\int_0^1 P_n\left(1-2x^2\right)\sin ax\,dx = \frac{\pi}{2}\left[J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 \qquad [a>0]$$ ET I 94(2)

2. $$\int_0^1 P_n\left(1-2x^2\right)\cos ax\,dx = \frac{\pi}{2}(-1)^n\,J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right)$$
$$[a>0]$$ ET I 38(1)

7.245

1. $$\int_0^{2\pi} P_{2m+1}(\cos\theta)\cos\theta\,d\theta = \frac{\pi}{2^{4m+1}}\binom{2m}{m}\binom{2m+2}{m+1}$$ MO 70, EH II 183(5)

2. $$\int_0^{\pi} P_m(\cos\theta)\sin n\theta\,d\theta = \frac{2(n-m+1)(n-m+3)\cdots(n+m-1)}{(n-m)(n-m+2)\cdots(n+m)} \qquad [n>m \text{ and } n+m \text{ is odd}]$$
$$= 0 \qquad [n\le m \text{ or } n+m \text{ is even}]$$
MO 71

3.[10] $$\int_0^{2\pi} P_{2n+1}(\sin\alpha\sin\phi)\sin\phi\,d\phi = (-1)^{n+1}\frac{2\sqrt{\pi}\,\Gamma\left(n+\frac{3}{2}\right)}{(2n+1)\,\Gamma(n+2)}\,P^1_{2n+1}(\cos\alpha)$$
$$\left[\alpha\ne\tfrac{1}{2}(2n+1)\pi,\quad n \text{ an integer}\right]$$

4. $$\int_{-1}^{1}\cos(\alpha x)\,P_n(x)\,dx = 0 \qquad [n \text{ is odd}]$$
$$= (-1)^v\sqrt{\frac{2\pi}{\alpha}}\,J_{2v+\frac{1}{2}}(\alpha) \qquad [n=2v \text{ is even}]$$
GH2 24 (171.10a)

7.246 $$\int_0^{\pi} P_n\left(1-2\sin^2 x\sin^2\theta\right)\sin x\,dx = \frac{2\sin(2n+1)\theta}{(2n+1)\sin\theta}$$ MO 71

7.247 $$\int_0^1 P_{2n+1}(x)\sin ax\frac{dx}{\sqrt{x}} = (-1)^{n+1}\sqrt{\frac{\pi}{2a}}\,J_{2n+\frac{3}{2}}(a) \qquad [a>0]$$ ET I 94(1)

7.248

1. $$\int_{-1}^{1}\left(a^2+b^2-2abx\right)^{-1/2}\sin\left[\lambda\left(a^2+b^2-2abx\right)^{1/2}\right]P_n(x)\,dx = \pi(ab)^{-1/2}\,J_{n+\frac{1}{2}}(a\lambda)\,J_{n+\frac{1}{2}}(b\lambda)$$
$$[a>0,\quad b>0]$$ ET II 277(11)

2. $$\int_{-1}^{1}\left(a^2+b^2-2abx\right)^{-1/2}\cos\left[\lambda\left(a^2+b^2-2abx\right)^{1/2}\right]P_n(x)\,dx = -\pi(ab)^{-1/2}\,J_{n+\frac{1}{2}}(a\lambda)\,Y_{n+\frac{1}{2}}(b\lambda)$$
$$[0\le a\le b]$$ ET II 277(12)

7.249

1. $$\int_{-1}^{1}P_n(x)\arcsin x\,dx = 0 \qquad [n \text{ is even}]$$
$$= \pi\left\{\frac{(n-2)!!}{2^{\frac{1}{2}(n+1)}\left(\frac{n+1}{2}\right)!}\right\}^2 \qquad [n \text{ is odd}]$$
WH

2. $$P_n(x) = \frac{1}{t}\sum_{t=0}^{t-1}\left(x+\sqrt{x^2-1}\cos\frac{2\pi r}{t}\right)^n \qquad [t>n]$$

7.25 Combinations of Legendre polynomials and Bessel functions

7.251

1. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,Y_\nu(xy)\,dx = \pi^{-1}y^{-1}\left[S_{2n+1}(y)+\pi\,Y_{2n+1}(y)\right]$$
$$[n=0,1,\ldots;\quad y>0,\quad \nu>0]$$
ET II 108(1)

2. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,K_0(xy)\,dx = y^{-1}\left[(-1)^{n+1}\,K_{2n+1}(y)+\frac{i}{2}\,S_{2n+1}(iy)\right]$$
$$[y>0]$$
ET II 134(1)

3. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,J_0(xy)\,dx = y^{-1}\,J_{2n+1}(y) \qquad [y>0]$$
ET II 13(1)

4. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,[J_0(ax)]^2\,dx = \frac{1}{2(2n+1)}\left\{[J_n(a)]^2+[J_{n+1}(a)]^2\right\}$$
ET II 338(39)a

5. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,J_0(ax)\,Y_0(ax)\,dx = \frac{1}{2(2n+1)}\left[J_n(a)\,Y_n(a)+J_{n+1}(a)\,Y_{n+1}(a)\right]$$
ET II 339(48)a

6. $$\int_0^1 x^2\,P_n\left(1-2x^2\right)\,J_1(xy)\,dx = y^{-1}(2n+1)^{-1}\left[(n+1)\,J_{2n+2}(y)-n\,J_{2n}(y)\right]$$
$$[y>0]$$
ET II 20(23)

7. $$\int_0^1 x^{\mu-1}\,P_n\left(2x^2-1\right)\,J_\nu(ax)\,dx = \frac{2^{-\nu-1}a^\nu\left[\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu\right)\right]^2}{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+n+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-n\right)}$$
$$\times\,{}_2F_3\left(\frac{\mu+\nu}{2},\frac{\mu+\nu}{2};\nu+1,\frac{\mu+\nu}{2}+n+1,\frac{\mu+\nu}{2}-n;-\frac{a^2}{4}\right)$$
$$[a>0,\quad \mathrm{Re}(\mu+\nu)>0]$$
ET II 337(32)a

7.252 $$\int_0^1 e^{-ax}\,P_n(1-2x)\,I_0(ax)\,dx = \frac{e^{-a}}{2n+1}\left[I_n(a)+I_{n+1}(a)\right]$$
$$[a>0]$$
ET II 366(11)a

7.253 $$\int_0^{\pi/2}\sin(2x)\,P_n\left(\cos 2x\right)\,J_0\left(a\sin x\right)\,dx = a^{-1}\,J_{2n+1}(a)$$
ET II 361(20)

7.254 $$\int_0^1 x\,P_n\left(1-2x^2\right)\left[I_0(ax)-\mathbf{L}_0(ax)\right]\,dx = (-1)^n\left[I_{2n+1}(a)-\mathbf{L}_{2n+1}(a)\right]$$
$$[a>0]$$
ET II 385(14)a

7.3–7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.311

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx=0 \qquad \left[n>0,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 280(1)

2. $$\int_{0}^{1} x^{n+2\rho}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx=\frac{\Gamma(2\nu+n)\,\Gamma(2\rho+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\rho+\frac{1}{2}\right)}{2^{n+1}\,\Gamma(2\nu)\,\Gamma(2\rho+1)n!\,\Gamma(n+\nu+\rho+1)}$$
$$\left[\operatorname{Re}\rho>-\tfrac{1}{2},\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 280(2)

3. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{\beta}\, C_n^\nu(x)\,dx=\frac{2^{\beta+\nu+\frac{1}{2}}\,\Gamma(\beta+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)\,\Gamma\left(\beta-\nu+\frac{3}{2}\right)}{n!\,\Gamma(2\nu)\,\Gamma\left(\beta-\nu-n+\frac{3}{2}\right)\Gamma\left(\beta+\nu+n+\frac{3}{2}\right)}$$
$$\left[\operatorname{Re}\beta>-1,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 280(3)

4. $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}\, C_n^\nu(x)\,dx=\frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma\left(n+2\nu\right)}{n!\,\Gamma(2\nu)\,\Gamma(\alpha+\beta+2)}$$
$$\times\ {}_3F_2\left(-n,n+2\nu,\alpha+1;\nu+\frac{1}{2},\alpha+\beta+2;1\right)$$
$$\left[\operatorname{Re}\alpha>-1,\quad \operatorname{Re}\beta>-1\right]$$ ET II 281(4)

7.312 In the following integrals, z belongs to the complex plane with a cut along the interval of the real axis from -1 to 1.

1. $$\int_{-1}^{1} x^m(z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx=\frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)}e^{-\left(\nu-\frac{1}{2}\right)\pi i}z^m\left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$$
$$\left[m\le n,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 281(5)

2. $$\int_{-1}^{1} x^{n+1}(z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx=\frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)}e^{-\left(\nu-\frac{1}{2}\right)\pi i}z^{n+1}\left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$$
$$-\frac{\pi 2^{1-2\nu-n}n!}{\Gamma(\nu)\,\Gamma(\nu+n+1)}$$
$$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 281(6)

3.[6] $$\int_{-1}^{1}(z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_m^\nu(x)\, C_n^\nu(x)\,dx=\frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)}e^{-\left(\nu-\frac{1}{2}\right)\pi i}\left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z)\, Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$$
$$\left[m\le n,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 283(17)

7.313

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_m^\nu(x)\, C_n^\nu(x)\,dx=0 \qquad \left[m\ne n,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 282(12), MO 98a, EH I 177(16)

2. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}}\left[C_n^\nu(x)\right]^2\,dx=\frac{\pi 2^{1-2\nu}\,\Gamma(2\nu+n)}{n!(n+\nu)\left[\Gamma(\nu)\right]^2} \qquad \left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 281(8), MO 98a, EH I 177(17)

7.314

1. $$\int_{-1}^{1}(1-x)^{\nu-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(x)\right]^2\,dx=\frac{\pi^{1/2}\,\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!\,\Gamma(\nu)\,\Gamma(2\nu)}$$
$$\left[\operatorname{Re}\nu>\tfrac{1}{2}\right]$$ ET II 281(9)

2. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{2\nu-1}\left[C_n^{\nu}(x)\right]^2\,dx=\frac{2^{3\nu-\frac{1}{2}}\left[\Gamma(2\nu+n)\right]^2\Gamma\left(2n+\nu+\frac{1}{2}\right)}{(n!)^2\,\Gamma(2\nu)\,\Gamma\left(3\nu+2n+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\nu>0\right]$$ ET II 282(10)

3. $$\int_{-1}^{1}(1-x)^{3\nu+2n-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(x)\right]^2\,dx$$
$$=\frac{\pi^{1/2}\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2\Gamma\left(\nu+2n+\frac{1}{2}\right)\Gamma\left(2\nu+2n\right)\Gamma\left(3\nu+2n-\frac{1}{2}\right)}{2^{2\nu+2n}\left[n!\,\Gamma\left(\nu+n+\frac{1}{2}\right)\Gamma(2\nu)\right]^2\Gamma\left(2\nu+2n+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\nu>\tfrac{1}{6}\right]$$ ET II 282(11)

4. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{\nu+m-n-\frac{3}{2}}\,C_m^{\nu}(x)\,C_n^{\nu}(x)\,dx$$
$$=(-1)^m\frac{2^{2-2\nu-m+n}\pi^{3/2}\,\Gamma(2\nu+n)}{m!(n-m)!\left[\Gamma(\nu)\right]^2\Gamma\left(\frac{1}{2}+\nu+m\right)}\frac{\Gamma\left(\nu-\frac{1}{2}+m-n\right)\Gamma\left(\frac{1}{2}-\nu+m-n\right)}{\Gamma\left(\frac{1}{2}-\nu-n\right)\Gamma\left(\frac{1}{2}+m-n\right)}$$
$$\left[\operatorname{Re}\nu>-\tfrac{1}{2};\quad n\geq m\right]$$ ET II 282(13)a

5. $$\int_{-1}^{1}(1-x)^{2\nu-1}\,(1+x)^{\nu-\frac{1}{2}}\,C_m^{\nu}(x)\,C_n^{\nu}(x)\,dx$$
$$=\frac{2^{3\nu-\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+m\right)\Gamma(2\nu+n)}{m!n!\,\Gamma(2\nu)\,\Gamma\left(\frac{1}{2}-\nu\right)}\frac{\Gamma\left(\nu+\frac{1}{2}+m+n\right)\Gamma\left(\frac{1}{2}-\nu+n-m\right)}{\Gamma\left(\nu+\frac{1}{2}+n-m\right)\Gamma\left(3\nu+\frac{1}{2}+m+n\right)}$$
$$\left[\operatorname{Re}\nu>0\right]$$ ET II 282(14)

6. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{3\nu+m+n-\frac{3}{2}}\,C_m^{\nu}(x)\,C_n^{\nu}(x)\,dx$$
$$=\frac{2^{4\nu+m+n-1}\left[\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+m+n)\right]^2}{\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+n+\frac{1}{2}\right)\Gamma\left(2\nu+m\right)}\frac{\Gamma\left(\nu+m+n+\frac{1}{2}\right)\Gamma\left(3\nu+m+n-\frac{1}{2}\right)}{\Gamma(2\nu+n)\,\Gamma(4\nu+2m+2n)}$$
$$\left[\operatorname{Re}\nu>\tfrac{1}{6}\right]$$ ET II 282(15)

7. $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\nu-\frac{1}{2}}\,C_m^{\mu}(x)\,C_n^{\nu}(x)\,dx$$
$$=\frac{2^{\alpha+\nu+\frac{1}{2}}\,\Gamma(\alpha+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n-\frac{1}{2}\right)}{m!n!\,\Gamma\left(\nu-\alpha-\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n+\frac{3}{2}\right)}\frac{\Gamma(2\mu+m)\,\Gamma\left(2\nu+n\right)}{\Gamma(2\mu)\,\Gamma(2\nu)}$$
$$\times\,{}_4F_3\left(-m,m+2\mu,\alpha+1,\alpha-\nu+\frac{3}{2};\mu+\frac{1}{2},\nu+\alpha+n+\frac{3}{2},\alpha-\nu-n+\frac{3}{2};1\right)$$
$$\left[\operatorname{Re}\alpha>-1,\quad\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 283(16)

7.315 $$\int_{-1}^{1}\left(1-x^2\right)^{\frac{1}{2}\nu-1}\,C_{2n}^{\nu}(ax)\,dx=\frac{\pi^{1/2}\,\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)}\,C_n^{\frac{1}{2}\nu}\left(2a^2-1\right)$$
$$\left[\operatorname{Re}\nu>0\right]$$ ET II 283(19)

7.316 $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-1} C_n^\nu\left(\cos\alpha\cos\beta+x\sin\alpha\sin\beta\right)\,dx=\frac{2^{2\nu-1}n!\left[\Gamma(\nu)\right]^2}{\Gamma(2\nu+n)}\,C_n^\nu\left(\cos\alpha\right)C_n^\nu\left(\cos\beta\right)$$
$[\operatorname{Re}\nu>0]$ ET II 283(20)

7.317

1. $$\int_0^1(1-x)^{\mu-1}x^{\lambda-\frac12}\,C_n^\lambda\left(1-\gamma x\right)\,dx=\frac{\Gamma\left(2\lambda+n\right)\Gamma\left(\lambda+\frac12\right)\Gamma(\mu)}{\Gamma(2\lambda)\,\Gamma\left(\lambda+\mu+n+\frac12\right)}\,P_n^{(\alpha,\beta)}(1-\gamma)$$
$\alpha=\lambda+\mu-\frac12,\qquad \beta=\lambda-\mu-\frac12\qquad \left[\operatorname{Re}\lambda>-1,\quad \lambda\neq0,\quad -\frac12,\quad \operatorname{Re}\mu>0\right]$ ET II 190(39)a

2. $$\int_0^1(1-x)^{\mu-1}x^{\nu-1}\,C_n^\lambda(1-\gamma x)\,dx=\frac{\Gamma(2\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(2\lambda)\,\Gamma(\mu+\nu)}$$
$$\times\,{}_3F_2\left(-n,n+2\lambda,\nu;\lambda+\frac12,\mu+\nu;\frac{\gamma}{2}\right)$$
$[2\lambda\neq0,-1,-2,\ldots,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ ET II 191(40)a

7.318 $$\int_0^1 x^{2\nu}\left(1-x^2\right)^{\sigma-1}C_n^\nu\left(1-x^2y\right)\,dx=\frac{\Gamma(2\nu+n)\,\Gamma\left(\nu+\frac12\right)\Gamma(\sigma)}{2\,\Gamma(2\nu)\,\Gamma\left(n+\nu+\sigma+\frac12\right)}\,P_n^{(\alpha,\beta)}(1-y),$$
$\alpha=\nu+\sigma-\frac12,\qquad \beta=\nu-\sigma-\frac12\qquad \left[\operatorname{Re}\nu>-\frac12,\quad \operatorname{Re}\sigma>0\right]$ ET II 283(21)

7.319

1. $$\int_0^1(1-x)^{\mu-1}x^{\nu-1}\,C_{2n}^\lambda\left(\gamma x^{1/2}\right)\,dx=(-1)^n\frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\,{}_3F_2\left(-n,n+\lambda,\nu;\frac12,\mu+\nu;\gamma^2\right)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$ ET II 191(41)a

2. $$\int_0^1(1-x)^{\mu-1}x^{\nu-1}\,C_{2n+1}^\lambda\left(\gamma x^{1/2}\right)\,dx=\frac{(-1)^n2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac12\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac12\right)}$$
$$\times\,{}_3F_2\left(-n,n+\lambda+1,\nu+\tfrac12;\tfrac32,\mu+\nu+\frac12;\gamma^2\right)$$
$\left[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>-\frac12\right]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

7.321 $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac12}e^{iax}\,C_n^\nu(x)\,dx=\frac{\pi2^{1-\nu}i^n\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)}a^{-\nu}\,J_{\nu+n}(a)$$
$\left[\operatorname{Re}\nu>-\frac12\right]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a}\left[x(2a-x)\right]^{\nu-\frac12}C_n^\nu\left(\frac{x}{a}-1\right)e^{-bx}\,dx=(-1)^n\frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)}\left(\frac{a}{2b}\right)^\nu e^{-ab}\,I_{\nu+n}(ab)$$
$\left[\operatorname{Re}\nu>-\frac12\right]$ ET I 171(9)

7.323

1. $$\int_0^\pi C_n^\nu\left(\cos\varphi\right)\left(\sin\varphi\right)^{2\nu}\,d\varphi=0\qquad [n=1,2,3,\ldots]$$
$$=2^{-2\nu}\pi\,\Gamma(2\nu+1)\left[\Gamma(1+\nu)\right]^{-2}\qquad [n=0]$$
EH I 177(18)

2.[11] $$\int_0^\pi C_n^\nu(\cos\psi\cos\psi'+\sin\psi\sin\psi'\cos\varphi)(\sin\varphi)^{2\nu-1}\,d\varphi$$
$$=2^{2\nu-1}n!\,[\Gamma(\nu)]^2\,C_n^\nu(\cos\psi)\,C_n^\nu(\cos\psi')\,[\Gamma(2\nu+n)]^{-1}$$
$$[\operatorname{Re}\nu>0]$$ EH I 177(20)

7.324

1. $$\int_0^1 (1-x^2)^{\nu-\frac12}\,C_{2n+1}^\nu(x)\sin ax\,dx=(-1)^n\pi\frac{\Gamma(2n+2\nu+1)\,J_{2n+\nu+1}(a)}{(2n+1)!\,\Gamma(\nu)(2a)^\nu}$$
$$\left[\operatorname{Re}\nu>-\tfrac12,\quad a>0\right]$$ ET I 94(4)

2. $$\int_0^1 (1-x^2)^{\nu-\frac12}\,C_{2n}^\nu(x)\cos ax\,dx=\frac{(-1)^n\pi\,\Gamma(2n+2\nu)\,J_{\nu+2n}(a)}{(2n)!\,\Gamma(\nu)(2a)^\nu}$$
$$\left[\operatorname{Re}\nu>-\tfrac12,\quad a>0\right]$$ ET I 38(3)a

7.325* Complete System of Orthogonal Step Functions

Let $s_j(x)=(-1)^{\lfloor 2jx\rfloor}$ for $j\in\mathbb{N}$ and $c_j(x)=(-1)^{\lfloor 2jx+1/2\rfloor}$ for $j\in 0+\mathbb{N}$ where $\lfloor z\rfloor$ denotes the integer part of z. Thus, $c_j(z)$ and $s_j(z)$ have minimal period j^{-1} and manifest even and odd symmetry about $x=1/2$, respectively, and so are the discrete analogues of $\cos 2\pi jx$ and $\sin 2\pi jx$. Furthermore, for $j\in\mathbb{N}$ let $\underline{j}$ denote its odd part: the quotient of j by its highest power-of-two factor. Then for all j and $k\in\mathbb{N}$, if (j,k) denotes their highest common factor and $[j,k]$ denotes their lowest common multiple:

1. $$\int_0^1 s_j(x)s_k(x)\,dx=\begin{cases}\frac{(j,k)}{[j,k]} & \text{if } j/\underline{j}=k/\underline{k}\\ 0 & \text{otherwise}\end{cases}$$

2. $$\int_0^1 c_j(x)c_k(x)\,dx=\begin{cases}(-1)^{(j+k)/2+1}\frac{(j,k)}{[j,k]} & \text{if } j/\underline{j}=k/\underline{k}\\ 0 & \text{otherwise}\end{cases}$$

7.33 Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index

7.331

1. $$\int_1^\infty x^{2n+1-\nu}\left(x^2-1\right)^{\nu-2n-\frac12}C_{2n}^{\nu-2n}\left(\frac1x\right)J_\nu(xy)\,dx$$
$$=(-1)^n2^{2n-\nu+1}y^{-\nu+2n-1}\,[(2n)!]^{-1}\,\Gamma(2\nu-2n)\,[\Gamma(\nu-2n)]^{-1}\cos y$$
$$\left[y>0,\quad 2n-\tfrac12<\operatorname{Re}\nu<2n+\tfrac12\right]$$ ET II 44(10)a

7.332

1. $$\int_0^\infty x^{\nu+1}\left(x^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}}\left[\left(x^2+\beta^2\right)^{-1/2}\beta\right] J_{\nu+\frac{3}{2}+2n}\left[\left(x^2+\beta^2\right)^{1/2}a\right] J_\nu(xy)\,dx$$
$$= (-1)^n 2^{1/2}\pi^{-1/2}a^{\frac{1}{2}-\nu}y^\nu\left(a^2-y^2\right)^{-1/2}\sin\left[\beta\left(a^2-y^2\right)^{1/2}\right] C_{2n+1}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{1/2}\right] \qquad [0<y<a]$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-1]$$
ET II 59(23)

2. $$\int_0^\infty x^{\nu+1}\left(x^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}}\left[\beta\left(x^2+\beta^2\right)^{-1/2}\right] J_{\nu+\frac{1}{2}+2n}\left[\left(x^2+\beta^2\right)^{1/2}a\right] J_\nu(xy)\,dx$$
$$= (-1)^n 2^{1/2}\pi^{-1/2}a^{\frac{1}{2}-\nu}y^\nu\left(a^2-y^2\right)^{-1/2}\cos\left[\beta\left(a^2-y^2\right)^{1/2}\right] C_{2n}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{1/2}\right] \qquad [0<y<a]$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-1]$$
ET II 59(24)

7.333

1. $$\int_0^\pi (\sin x)^{\nu+1}\cos(a\cos\theta\cos x)\, C_n^{\nu+\frac{1}{2}}(\cos x)\, J_\nu(a\sin\theta\sin x)\,dx$$
$$= (-1)^{\frac{n}{2}}\left(\frac{2\pi}{a}\right)^{1/2}(\sin\theta)^\nu\, C_n^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+\frac{1}{2}+n}(a) \qquad [n=0,2,4,\ldots]$$
$$= 0 \qquad [n=1,3,5,\ldots]$$
$$[\operatorname{Re}\nu>-1]$$
WA 414(2)a

2. $$\int_0^\pi (\sin x)^{\nu+1}\sin(a\cos\theta\cos x)\, C_n^{\nu+\frac{1}{2}}(\cos x)\, J_\nu(a\sin\theta\sin x)\,dx$$
$$= 0 \qquad [n=0,2,4,\ldots]$$
$$= (-1)^{\frac{n-1}{2}}\left(\frac{2\pi}{a}\right)^{1/2}(\sin\theta)^\nu\, C_n^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+\frac{1}{2}+n}(a) \qquad [n=1,3,5,\ldots]$$
$$[\operatorname{Re}\nu>-1]$$
WA 414(3)a

7.334

1. $$\int_0^\pi (\sin x)^{2\nu}\, C_n^\nu(\cos x)\,\frac{J_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\,\Gamma(2\nu+n)}{2^{\nu-1}n!\,\Gamma(\nu)}\,\frac{J_{\nu+n}(\alpha)}{\alpha^\nu}\,\frac{J_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = \left(\alpha^2+\beta^2-2\alpha\beta\cos x\right)^{1/2} \qquad \left[n=0,1,2,\ldots;\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 362(29)

2. $$\int_0^\pi (\sin x)^{2\nu}\, C_n^\nu(\cos x)\,\frac{Y_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\,\Gamma(2\nu+n)}{2^{\nu-1}n!\,\Gamma(\nu)}\,\frac{J_{\nu+n}(\alpha)}{\alpha^\nu}\,\frac{Y_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = \left(\alpha^2+\beta^2-2\alpha\beta\cos x\right)^{1/2} \qquad \left[|\alpha|<|\beta|,\quad \operatorname{Re}\nu-\tfrac{1}{2}\right]$$
ET II 362(30)

Integration of Gegenbauer functions with respect to the index

7.335 $$\int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1}\, t^{\alpha}\, C_{\alpha}^{\nu}(z)\, d\alpha = -2i\left(1+2tz+t^2\right)^{-\nu}$$
$$[-2 < \operatorname{Re}\nu < c < 0, \quad |\arg(z\pm 1)| < \pi]$$
EH I 178(25)

7.336 $$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x)\left(\nu - \frac{1}{2} + ix\right) K_{\nu-\frac{1}{2}+ix}(a)\, I_{\nu-\frac{1}{2}+ix}(b)\, C_{-\frac{1}{2}+ix}^{\nu}(-\cos\varphi)\, dx$$
$$= \frac{2^{-\nu+1}(ab)^{\nu}}{\Gamma(\nu)}\omega^{-\nu}\, K_{\nu}(\omega)$$
$$\omega = \sqrt{a^2+b^2-2ab\cos\varphi}$$
EH II 55(45)

7.34 Combinations of Chebyshev polynomials and powers

7.341 $$\int_{-1}^{1} [T_n(x)]^2\, dx = 1 - \left(4n^2-1\right)^{-1}$$
ET II 271(6)

7.342 $$\int_{-1}^{1} U_n\left[x\left(1-y^2\right)^{1/2}\left(1-z^2\right)^{1/2} + yz\right] dx = \frac{2}{n+1}\, U_n(y)\, U_n(z)$$
$$[|y| < 1, \quad |z| < 1]$$
ET II 275(34)

7.343

1. $$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \qquad [m \neq n]$$
$$= \frac{\pi}{2} \qquad [m = n \neq 0]$$
$$= \pi \qquad [m = n = 0]$$
MO 104

2. $$\int_{-1}^{1} \sqrt{1-x^2}\, U_n(x)\, U_m(x)\, dx = 0 \qquad [m \neq n]$$
ET II 274(28)
$$= \frac{\pi}{2} \qquad [m = n]$$
ET II 274(27), MO 105a

7.344

1. $$\int_{-1}^{1} (y-x)^{-1}\left(1-y^2\right)^{-1/2} T_n(y)\, dy = \pi\, U_{n-1}(x) \qquad [n = 1, 2, \ldots]$$
EH II 187(47)

2. $$\int_{-1}^{1} (y-x)^{-1}\left(1-y^2\right)^{1/2} U_{n-1}(y)\, dy = -\pi\, T_n(x) \qquad [n = 1, 2, \ldots]$$
EH II 187(48)

7.345

1. $$\int_{-1}^{1} (1-x)^{-1/2}(1+x)^{m-n-\frac{3}{2}}\, T_m(x)\, T_n(x)\, dx = 0 \qquad [m > n]$$
ET II 272(10)

2. $$\int_{-1}^{1} (1-x)^{-1/2}(1+x)^{m+n-\frac{3}{2}}\, T_m(x)\, T_n(x)\, dx = \frac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!(2n-1)!}$$
$$[m+n \neq 0]$$
ET II 272(11)

3. $$\int_{-1}^{1}(1-x)^{1/2}(1+x)^{m+n+\frac{3}{2}}\,U_m(x)\,U_n(x)\,dx=\frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}$$ ET II 274(31)

4. $$\int_{-1}^{1}(1-x)^{1/2}(1+x)^{m-n-\frac{1}{2}}\,U_m(x)\,U_n(x)\,dx=0 \qquad [m>n]$$ ET II 274(30)

5. $$\int_{-1}^{1}(1-x)(1+x)^{1/2}\,U_m(x)\,U_n(x)\,dx=\frac{2^{5/2}(m+1)(n+1)}{\left(m+n+\frac{3}{2}\right)\left(m+n+\frac{5}{2}\right)\left[1-4(m-n)^2\right]}$$ ET II 274(29)

6. $$\int_{-1}^{1}(1+x)^{-1/2}\,(1-x)^{\alpha-1}\,T_m(x)\,T_n(x)\,dx$$
$$=\frac{\pi^{1/2}2^{\alpha-\frac{1}{2}}\,\Gamma(\alpha)\,\Gamma\left(n-\alpha+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\alpha+n+\frac{1}{2}\right)}\;{}_4F_3\left(-m,m,\alpha,\alpha+\frac{1}{2};\frac{1}{2},\alpha+n+\frac{1}{2},\alpha-n+\frac{1}{2};1\right)$$
$$[\operatorname{Re}\alpha>0]$$ ET II 272(12)

7. $$\int_{-1}^{1}(1+x)^{1/2}\,(1-x)^{\alpha-1}\,U_m(x)\,U_n(x)\,dx$$
$$=\frac{\pi^{1/2}2^{\alpha-\frac{1}{2}}(m+1)\,(n+1)\,\Gamma(\alpha)\,\Gamma\left(n-\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\alpha\right)\Gamma\left(\frac{3}{2}+\alpha+n\right)}$$
$$\times\;{}_4F_3\left(-m,m+2,\alpha,\alpha-\frac{1}{2};\frac{3}{2},\alpha+n+\frac{3}{2},\alpha-n-\frac{1}{2};1\right)$$
$$[\operatorname{Re}\alpha>0]$$ ET II 275(32)

7.346 $$\int_0^1 x^{s-1}\,T_n(x)\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{s2^s\,\mathrm{B}\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n,\frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)}$$
$$[\operatorname{Re}s>0]$$ ET II 324(2)

7.347

1. $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}\,T_n(x)\,dx=\frac{2^{\alpha+\beta+2n+1}\,(n!)^2\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(2n)!\,\Gamma(\alpha+\beta+2)}$$
$$\times\;{}_3F_2\left(-n,n,\alpha+1;\frac{1}{2},\alpha+\beta+2;1\right)$$
$$[\operatorname{Re}\alpha>-1,\quad \operatorname{Re}\beta>-1]$$ ET II 271(2)

2. $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}\,U_n(x)\,dx=\frac{2^{\alpha+\beta+2n+2}\,[(n+1)!]^2\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(2n+2)!\,\Gamma(\alpha+\beta+2)}$$
$$\times\;{}_3F_2\left(-n,n+1,\alpha+1;\frac{3}{2},\alpha+\beta+2;1\right)$$ ET II 273(22)

7.348 $$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}\,U_{2n}(xz)\,dx=\pi\,P_n\left(2z^2-1\right) \qquad [|z|<1]$$ ET II 275(33)

7.349 $$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}\,T_n\left(1-x^2y\right)\,dx=\frac{1}{2}\pi\left[P_n(1-y)+P_{n-1}(1-y)\right]$$ ET II 222(14)

7.35 Combinations of Chebyshev polynomials and elementary functions

7.351
$$\int_0^1 x^{-1/2}\left(1-x^2\right)^{-\frac{1}{2}} e^{-\frac{2a}{x}}\, T_n(x)\,dx = \pi^{1/2}\, D_{n-\frac{1}{2}}\left(2a^{1/2}\right) D_{-n-\frac{1}{2}}\left(2a^{1/2}\right)$$
$[\operatorname{Re} a > 0]$ ET II 272(13)

7.352

1. $$\int_0^\infty \frac{x\, U_n\left[a\left(a^2+x^2\right)^{-1/2}\right]}{\left(a^2+x^2\right)^{\frac{1}{2}n+1}\left(e^{\pi x}+1\right)}\,dx = \frac{a^{-n}}{2n} - 2^{-n-1}\,\zeta\left(n+1, \frac{a+1}{2}\right)$$
$[\operatorname{Re} a > 0]$ ET II 275(39)

2. $$\int_0^\infty \frac{x\, U_n\left[a\left(a^2+x^2\right)^{-1/2}\right]}{\left(a^2+x^2\right)^{\frac{1}{2}n+1}\left(e^{2\pi x}-1\right)}\,dx = \frac{1}{2}\,\zeta(n+1,a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n}$$
$[\operatorname{Re} a > 0]$ ET II 276(40)

7.353

1. $$\int_0^\infty \left(a^2+x^2\right)^{-\frac{1}{2}n} \operatorname{sech}\left(\frac{1}{2}\pi x\right) T_n\left[a\left(a^2+x^2\right)^{-1/2}\right] dx = 2^{1-2n}\left[\zeta\left(n, \frac{a+1}{4}\right) - \zeta\left(n, \frac{a+3}{4}\right)\right]$$
$$= 2^{1-n}\,\Phi\left(-1, n, \frac{a+1}{2}\right)$$
$[\operatorname{Re} a > 0]$ ET II 273(19)

2. $$\int_0^\infty \left(a^2+x^2\right)^{-\frac{1}{2}n}\left[\cosh\left(\frac{1}{2}\pi x\right)\right]^{-2} T_n\left[a\left(a^2+x^2\right)^{-1/2}\right] dx = \pi^{-1} n 2^{1-n}\,\zeta\left(n+1, \frac{a+1}{2}\right)$$
$[\operatorname{Re} a > 0]$ ET II 273(20)

7.354

1. $$\int_{-1}^1 \sin(xyz)\cos\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] T_{2n+1}(x)\,dx = (-1)^n \pi\, T_{2n+1}(y)\, J_{2n+1}(x)$$
ET II 271(4)

2. $$\int_{-1}^1 \sin(xyz)\sin\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] U_{2n+1}(x)\,dx = (-1)^n \pi\left(1-y^2\right)^{1/2} U_{2n+1}(y)\, J_{2n+2}(z)$$
ET II 274(25)

3. $$\int_{-1}^1 \cos(xyz)\cos\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] T_{2n}(x)\,dx = (-1)^n \pi\, T_{2n}(y)\, J_{2n}(z)$$
ET II 271(5)

4. $$\int_{-1}^1 \cos(xyz)\sin\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] U_{2n}(x)\,dx = (-1)^n \pi\left(1-y^2\right)^{1/2} U_{2n}(y)\, J_{2n+1}(z)$$
ET II 274(24)

7.355

1. $$\int_0^1 T_{2n+1}(x)\sin ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2}\, J_{2n+1}(a) \qquad [a > 0]$$
ET I 94(3)a

2. $$\int_0^1 T_{2n}(x)\cos ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2}\, J_{2n}(a) \qquad [a > 0]$$
ET I 38(2)a

7.36 Combinations of Chebyshev polynomials and Bessel functions

7.361 $$\int_0^1 \left(1-x^2\right)^{-1/2} T_n(x)\, J_\nu(xy)\, dx = \frac{1}{2}\pi\, J_{\frac{1}{2}(\nu+n)}\left(\frac{1}{2}y\right) J_{\frac{1}{2}(\nu-n)}\left(\frac{1}{2}y\right)$$
$[y>0, \quad \operatorname{Re}\nu > -n-1]$ ET II 42(1)

7.362 $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}} T_n\left(\frac{1}{x}\right) K_{2\mu}(ax)\, dx = \frac{\pi}{2a}\, W_{\frac{1}{2}n,\mu}(a)\, W_{-\frac{1}{2}n,\mu}(a)$$
$[\operatorname{Re} a > 0]$ ET II 366(17)a

7.37–7.38 Hermite polynomials

7.371 $$\int_0^x H_n(y)\, dy = [2(n+1)]^{-1}\left[H_{n+1}(x) - H_{n+1}(0)\right]$$ EH II 194(27)

7.372 $$\int_{-1}^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} H_{2n}\left(\sqrt{x}t\right)\, dx = \frac{(-1)^n \pi^{1/2}(2n)!\,\Gamma\left(\alpha+\frac{1}{2}\right) L_n^\alpha(x)}{\Gamma(n+\alpha+1)}$$
$\left[\operatorname{Re} a > -\frac{1}{2}\right]$ EH II 195(34)

7.373

1. $$\int_0^x e^{-y^2} H_n(y)\, dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$ [see **8.956**] EH II 194(26)

2. $$\int_{-\infty}^\infty e^{-x^2} H_{2m}(xy)\, dx = \sqrt{\pi}\frac{(2m)!}{m!}\left(y^2-1\right)^m$$ EH II 195(28)

7.374

1. $$\int_{-\infty}^\infty e^{-x^2} H_n(x)\, H_m(x)\, dx = 0 \qquad [m \neq n]$$
$$= 2^n \cdot n!\sqrt{\pi} \qquad [m = n]$$
SM II 567

SM II 568

2.[11] $$\int_{-\infty}^\infty e^{-2x^2} H_m(x)\, H_n(x)\, dx = (-1)^{\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor} 2^{\frac{m+n-1}{2}}\,\Gamma\left(\frac{m+n+1}{2}\right) \qquad [m+n \text{ is even}]$$
$$= 0 \qquad [m+n \text{ is odd}]$$
ET II 289(10)a

3. $$\int_{-\infty}^\infty e^{-x^2} H_m(ax)\, H_n(x)\, dx = 0 \qquad [m<n]$$ ET II 290(20)a

4. $$\int_{-\infty}^\infty e^{-x^2} H_{2m+n}(ax)\, H_n(x)\, dx = \sqrt{\pi}2^n\frac{(2m+n)!}{m!}\left(a^2-1\right)^m a^n$$ ET II 291(21)a

5. $$\int_{-\infty}^\infty e^{-2\alpha^2x^2} H_m(x)\, H_n(x)\, dx = 2^{\frac{m+n-1}{2}}\alpha^{-m-n-1}\left(1-2\alpha^2\right)^{\frac{m+n}{2}}\Gamma\left(\frac{m+n+1}{2}\right)$$
$$\times\, {}_2F_1\left(-m,n;\frac{1-m-n}{2};\frac{\alpha^2}{2\alpha^2-1}\right)$$
$\left[\operatorname{Re}\alpha^2 > 0, \quad \alpha^2 \neq \frac{1}{2}, \quad m+n \text{ is even}\right]$ ET II 289(12)a

6. $$\int_{-\infty}^\infty e^{-(x-y)^2} H_n(x)\, dx = \pi^{1/2}y^n 2^n$$ ET II 288(2)a, EH II 195(31)

7. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x)\, dx = 2^n \pi^{1/2} m! y^{n-m} L_m^{n-m}\left(-2y^2\right)$$
$[m \le n]$ BU 148(15), ET II 289(13)a

8. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x)\, dx = \pi^{1/2}\left(1-\alpha^2\right)^{\frac{n}{2}} H_n\left[\frac{\alpha y}{\left(1-\alpha^2\right)^{1/2}}\right]$$
ET II 290(17)a

9. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x)\, dx$$
$$= \pi^{1/2} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k}\binom{n}{k}\left(1-\alpha^2\right)^{\frac{m+n}{2}-k} H_{m+n-2k}\left[\frac{\alpha y}{\left(1-\alpha^2\right)^{1/2}}\right]$$
ET II 291(26)a

10. $$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x)\, dx = (2\pi u)^{1/2}(1-2u)^{\frac{n}{2}} H_n\left[y(1-2u)^{-1/2}\right]$$
$\left[0 \le u < \frac{1}{2}\right]$ EH II 195(30)

7.375

1. $$\int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x)\, dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)}\Gamma(s-k)\Gamma(s-m)\Gamma(s-n)$$
$2s = k+m+n+1 \qquad [k+m+n \text{ is even}]$ ET II 290(14)a

2. $$\int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x)\, dx = \frac{2^{\frac{m+n+k}{2}}\pi^{1/2} k! m! n!}{(s-k)!(s-m)!(s-n)!},$$
$2s = m+n+k \qquad [k+m+n \text{ is even}]$
ET II 290(15)a

7.376

1. $$\int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x)\, dx = (2\pi)^{1/2} e^{-\frac{y^2}{2}} H_n(y) i^n$$
MO 165a

2. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n}(x)\, dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}(\nu+1)}} F\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$ BU 150(18a)

3. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n+1}(x)\, dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}\nu+1}} F\left(-n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -2]$ BU 150(18b)

7.377[8] $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z)\, dx = 2^n \pi^{1/2} m! z^{n-m} L_m^{n-m}(-2yz)$$
$[m \le n]$ ET II 292(30)a

7.378 $$\int_0^{\infty} x^{\alpha-1} e^{-\beta x} H_n(x)\, dx = 2^n \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!\,\Gamma(\alpha+n-2m)}{m!(n-2m)!}(-1)^m 2^{-2m}\beta^{2m-\alpha-n}$$
$[\operatorname{Re}\alpha > 0, \text{ if } n \text{ is even}; \quad \operatorname{Re}\alpha > -1, \text{ if } n \text{ is odd}; \quad \operatorname{Re}\beta > 0]$ ET I 172(11)a

7.379

1. $$\int_{-\infty}^{\infty} xe^{-x^2}\, H_{2m+1}(xy)\, dx = \pi^{1/2}\frac{(2m+1)!}{m!} y\left(y^2-1\right)^m$$ EH II 195(28)

2. $$\int_{-\infty}^{\infty} x^n e^{-x^2}\, H_n(xy)\, dx = \pi^{1/2} n! P_n(y)$$ EH II 195(29)

7.381 $$\int_{-\infty}^{\infty} (x \pm ic)^\nu\, e^{-x^2}\, H_n(x)\, dx = 2^{n-1-\nu}\pi^{1/2}\frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma(-\nu)} \exp\left[\pm\tfrac{1}{2}\pi(\nu+n)i\right]$$
$[c > 0]$ ET II 288(3)a

7.382 $$\int_0^{\infty} x^{-1}\left(x^2+a^2\right)^{-1} e^{-x^2}\, H_{2n+1}(x)\, dx = (-2)^n \pi^{1/2} a^{-2}\left[2^\nu n! - (2n+1)! e^{\frac{1}{2}a^2}\, D_{-2n-2}\left(a\sqrt{2}\right)\right]$$
ET II 288(4)a

7.383

1. $$\int_0^{\infty} e^{-xp}\, H_{2n+1}\left(\sqrt{x}\right)\, dx = (-1)^n 2^n (2n+1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{3}{2}}$$
$[\operatorname{Re} p > 0]$ EF 151(261)a, ET I 172(12)a

2. $$\int_0^{\infty} e^{-(b-\beta x)}\, H_{2n+1}\left(\sqrt{(\alpha-\beta)x}\right)\, dx = (-1)^n \sqrt{\pi}\sqrt{\alpha-\beta}\frac{(2n+1)!}{n!}\frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}}$$
$[\operatorname{Re}(b-\beta) > 0]$ ET I 172(15)a

3. $$\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-(b-\beta)x}\, H_{2n}\left(\sqrt{(\alpha-\beta)x}\right)\, dx = (-1)^n \sqrt{\pi}\frac{(2n)!}{n!}\frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}}$$
$[\operatorname{Re}(b-\beta) > 0]$ ET I 172(16)a

4. $$\int_0^{\infty} x^{a-\frac{1}{2}n-1} e^{-bx}\, H_n\left(\sqrt{x}\right)\, dx = 2^n\, \Gamma(a) b^{-a}\; {}_2F_1\left(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; 1-a; b\right)$$
$$\left[\operatorname{Re} a > \tfrac{1}{2}n, \text{ if } n \text{ is even}, \quad \operatorname{Re} a > \tfrac{1}{2}n - \tfrac{1}{2}, \text{ if } n \text{ is odd}, \quad \operatorname{Re} b > 0,\right.$$
$$\left.\text{If } a \text{ is even, only the first } 1 + \left\lfloor \frac{n}{2} \right\rfloor \text{ terms are kept in the series for } {}_2F_1\right]$$
ET I 172(14)a

5. $$\int_0^{\infty} x^{-1/2} e^{-px}\, H_{2n}\left(\sqrt{x}\right)\, dx = (-1)^n 2^n (2n-1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{1}{2}}$$ MO 177a

7.384 $$\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-bx}\left[H_n\left(\frac{\alpha+\sqrt{x}}{\lambda}\right) + H_n\left(\frac{a-\sqrt{x}}{\lambda}\right)\right] dx = \sqrt{\frac{2\pi}{b}}\left(1-\lambda^{-2}b^{-1}\right)^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\lambda^2-\frac{1}{b}}}\right)$$
$[\operatorname{Re} b > 0]$ ET I 173(17)a

7.385

1. $$\int_0^{\infty} \frac{e^{-bx}}{\sqrt{e^x-1}}\, H_{2n}\left[\sqrt{s\left(1-e^{-x}\right)}\right]\, dx = (-1)^n 2^{2n}\sqrt{\pi}\frac{(2n)!\,\Gamma\left(b+\frac{1}{2}\right)}{\Gamma(n+b+1)}\, L_n^n(s)$$
$\left[\operatorname{Re} b > -\frac{1}{2}\right]$ ET I 174(23)a

2. $$\int_0^\infty e^{-bx}\, H_{2n+1}\left[\sqrt{s}\sqrt{1-e^{-x}}\right]\, dx = (-1)^n 2^{2n}\sqrt{\pi s}\frac{(2n+1)!\,\Gamma(b)}{\Gamma\left(n+b+\frac{3}{2}\right)}\, L_n^b(s)$$
$[\operatorname{Re} b > 0]$ ET I 174(24)a

7.386 $$\int_0^\infty x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}}\, H_n\left(\frac{q}{2\sqrt{x}}\right) e^{-px}\, dx = 2^n \pi^{1/2} p^{\frac{n-1}{2}} e^{-q\sqrt{p}}$$ EF 129(117)

7.387

1. $$\int_0^\infty e^{-x^2} \sinh\left(\sqrt{2}\beta x\right) H_{2n+1}(x)\, dx = 2^{n-\frac{1}{2}}\pi^{1/2}\beta^{2n+1} e^{\frac{1}{2}\beta^2}$$ ET II 289(7)a

2. $$\int_0^\infty e^{-x^2} \cosh\left(\sqrt{2}\beta x\right) H_{2n}(x)\, dx = 2^{n-1}\pi^{1/2}\beta^{2n} e^{\frac{1}{2}\beta^2}$$ ET II 289(8)a

7.388

1. $$\int_0^\infty e^{-x^2} \sin\left(\sqrt{2}\beta x\right) H_{2n+1}(x)\, dx = (-1)^n 2^{n-\frac{1}{2}}\pi^{1/2}\beta^{2n+1} e^{-\frac{1}{2}\beta^2}$$ ET II 288(5)a

2. $$\int_0^\infty e^{-x^2} \sin\left(\sqrt{2}\beta x\right) H_{2n+1}(ax)\, dx = (-1)^n 2^{-1}\pi^{1/2}\left(a^2-1\right)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2}\, H_{2n+1}\left(\frac{a\beta}{\sqrt{2}\left(a^2-1\right)^{1/2}}\right)$$
ET II 290(18)a

3. $$\int_0^\infty e^{-x^2} \cos\left(\sqrt{2}\beta x\right) H_{2n}(x)\, dx = (-1)^n 2^{n-1}\pi^{1/2}\beta^{2n} e^{-\frac{1}{2}\beta^2}$$ ET II 289(6)a

4. $$\int_0^\infty e^{-x^2} \cos\left(\sqrt{2}\beta x\right) H_{2n}(ax)\, dx = 2^{-1}\pi^{1/2}\left(1-a^2\right)^n e^{-\frac{1}{2}\beta^2}\, H_{2n}\left[\frac{a\beta}{\sqrt{2}\left(a^2-1\right)^{1/2}}\right]$$
ET II 290(19)a

5. $$\int_0^\infty e^{-y^2}\left[H_n(y)\right]^2 \cos\left(\sqrt{2}\beta y\right)\, dy = \pi^{1/2} 2^{n-1} n! e^{-\frac{\beta^2}{2}}\, L_n\left(\beta^2\right)$$ EH II 195(33)

6.[11] $$\int_0^\infty e^{-x^2} \sin(bx)\, H_n(x)\, H_{n+2m+1}(x)\, dx = 2^{n-1}(-1)^m\sqrt{\pi} n! b^{2m+1} e^{-\frac{b^2}{4}}\, L_n^{2m+1}\left(\frac{b^2}{2}\right)$$
$[b > 0]$ ET I 39(11)a

7. $$\int_0^\infty e^{-x^2} \cos(bx)\, H_n(x)\, H_{n+2m}(x)\, dx = 2^{n-\frac{1}{2}}\sqrt{\frac{\pi}{2}} n! (-1)^m b^{2m} e^{-\frac{b^2}{4}}\, L_n^{2m}\left(\frac{b^2}{2}\right)$$
$[b > 0]$ ET I 39(11)a

7.389 $$\int_0^\pi (\cos x)^n\, H_{2n}\left[a\left(1-\sec x\right)^{1/2}\right]\, dx = 2^{-n}(-1)^n \pi \frac{(2n)!}{(n!)^2}\left[H_n(a)\right]^2$$ ET II 292(31)

7.39 Jacobi polynomials

7.391

1. $$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta\, P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\, dx$$
$$= 0 \qquad [m \neq n, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$$
$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+1+2n)\,\Gamma(\alpha+\beta+n+1)} \qquad [m = n, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$$
ET II 285(5, 9)

2. $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\rho+\sigma+1}\,\Gamma(\rho+1)\,\Gamma(\sigma+1)\,\Gamma(n+1+\alpha)}{n!\,\Gamma(\rho+\sigma+2)\,\Gamma(1+\alpha)}$$
$$\times\ {}_3F_2\left(-n,\alpha+\beta+n+1,\rho+1;\alpha+1,\rho+\sigma+2;1\right)$$
$[\operatorname{Re}\rho>-1,\quad \operatorname{Re}\sigma>-1]$ ET II 284(3)

3.[6] $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\alpha+\sigma+1}\,\Gamma(\sigma+1)\,\Gamma(\alpha+1)\,\Gamma(\sigma-\beta+1)}{n!\,\Gamma(\sigma-\beta-n+1)\,\Gamma(\alpha+\sigma+n+2)}$$
$[\operatorname{Re}\alpha>-1,\quad \operatorname{Re}\sigma>-1]$ ET II 284(1)

4. $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\beta+\rho+1}\,\Gamma(\rho+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha-\rho+n)}{n!\,\Gamma(\alpha-\rho)\,\Gamma(\beta+\rho+n+2)}$$
$[\operatorname{Re}\rho>-1,\quad \operatorname{Re}\beta>-1]$ ET II 284(2)

5. $$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2\,dx=\frac{2^{\alpha+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\alpha\,\Gamma(\alpha+\beta+n+1)}$$
$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\beta>-1]$ ET II 285(6)

6. $$\int_{-1}^{1}(1-x)^{2\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2\,dx=\frac{2^{4\alpha+\beta+1}\,\Gamma\left(\alpha+\frac{1}{2}\right)\left[\Gamma(\alpha+n+1)\right]^2\Gamma(\beta+2n+1)}{\sqrt{\pi}\,(n!)^2\,\Gamma(\alpha+1)\,\Gamma(2\alpha+\beta+2n+2)}$$
$\left[\operatorname{Re}\alpha>-\frac{1}{2},\quad \operatorname{Re}\beta>-1\right]$ ET II 285(7)

7. $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\rho,\beta)}(x)\,dx$$
$$=\frac{2^{\rho+\beta+1}\,\Gamma(\rho+n+1)\,\Gamma(\beta+n+1)\,\Gamma\left(\alpha+\beta+2n+1\right)}{n!\,\Gamma(\beta+\rho+2n+2)\,\Gamma(\alpha+\beta+n+1)}$$
$[\operatorname{Re}\rho>-1,\quad \operatorname{Re}\beta>-1]$ ET II285(10)

8. $$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\rho,\beta)}(x)\,dx=\frac{2^{\rho+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\rho)}{n!\,\Gamma(\alpha+1)\,\Gamma(\rho+\beta+n+1)}$$
$[\operatorname{Re}\beta>-1,\quad \operatorname{Re}\rho>0]$ ET II 286(11)

9.[7] $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\sigma)}(x)\,dx$$
$$=\frac{2^{\alpha+\sigma+1}\,\Gamma(\alpha+n+1)\,\Gamma\left(\alpha+\beta+m+n+1\right)\Gamma(\sigma+m+1)\,\Gamma(\sigma-\beta+1)}{m!(n-m)!\,\Gamma\left(\alpha+\beta+n+1\right)\Gamma(\alpha+\sigma+m+n+2)\,\Gamma\left(\alpha-\beta+m-n+1\right)}$$
$[\operatorname{Re}\alpha>-1,\quad \operatorname{Re}\sigma>-1]$ ET II 286(12)

10.[6] $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_m^{(\rho,\beta)}(x)\,dx$$
$$=\frac{2^{\beta+\rho+1}\,\Gamma(\alpha+\beta+m+n+1)\,\Gamma\left(\beta+n+1\right)\Gamma(\rho+m+1)}{m!(n-m)!\,\Gamma(\alpha+\beta+n+1)\,\Gamma\left(\beta+\rho+m+n+2\right)}\,\frac{\Gamma\left(\alpha-\rho-m+n\right)}{\Gamma(\alpha-\rho)}$$
$[\operatorname{Re}\beta>-1,\quad \operatorname{Re}\rho>-1]$ ET II 287(16)

11. $$\int_{0}^{x}(1-y)^{\alpha}(1+y)^{\beta}\,P_n^{(\alpha,\beta)}(y)\,dy=\frac{1}{2n}\left[P_{n-1}^{(\alpha+1,\beta+1)}(0)-(1-x)^{\alpha+1}(1+x)^{\beta+1}\,P_{n-1}^{(\alpha+1,\beta+1)}(x)\right]$$
EH II 173(38)

7.392

1. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\, P_n^{(\alpha,\beta)}(1-\gamma x)\, dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\; {}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \frac{1}{2}\gamma\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(46)a

2. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\, P_n^{(\alpha,\beta)}(\gamma x-1)\, dx = (-1)^n \frac{\Gamma(\beta+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\beta+1)\,\Gamma(\lambda+\mu)}\; {}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \frac{1}{2}\gamma\right) a$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(47)a

3. $$\int_0^1 x^{\alpha}(1-x)^{\mu-1}\, P_n^{(\alpha,\beta)}(1-\gamma x)\, dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)}\, P_n^{(\alpha+\mu,\beta-\mu)}(1-\gamma)$$
$[\operatorname{Re} a > -1, \quad \operatorname{Re}\mu > 0]$ ET II 191(43)a

4. $$\int_0^1 x^{\beta}(1-x)^{\mu-1}\, P_n^{(\alpha,\beta)}(\gamma x-1)\, dx = \frac{\Gamma(\beta+n+1)\,\Gamma(\mu)}{\Gamma(\beta+\mu+n+1)}\, P_n^{(\alpha-\mu,\beta+\mu)}(\gamma-1)$$
$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\mu > 0]$ ET II 191(44)a

7.393

1. $$\int_0^1 \left(1-x^2\right)^{\nu} \sin bx\, P_{2n+1}^{(\nu,\nu)}(x)\, dx = \frac{(-1)^n \sqrt{\pi}\,\Gamma(2n+\nu+2)\, J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu}(2n+1)!\, b^{\nu+\frac{1}{2}}}$$
$[b > 0, \quad \operatorname{Re}\nu > -1]$ ET I 94(5)

2. $$\int_0^1 \left(1-x^2\right)^{\nu} \cos bx\, P_{2n}^{(\nu,\nu)}(x)\, dx = \frac{(-1)^n 2^{\nu-\frac{1}{2}} \sqrt{\pi}\,\Gamma(2n+\nu+1)\, J_{2n+\nu+\frac{1}{2}}(b)}{(2n)!\, b^{\nu+\frac{1}{2}}}$$
$[b > 0, \quad \operatorname{Re}\nu > -1]$ ET I 38(4)

7.41–7.42 Laguerre polynomials

7.411

1. $$\int_0^t L_n(x)\, dx = L_n(t) - L_{n+1}(t)/(n+1)$$ MO 110

2. $$\int_0^t L_n^{\alpha}(x)\, dx = L_n^{\alpha}(t) - L_{n+1}^{\alpha}(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}$$ EH II 189(16)a

3. $$\int_0^t L_{n-1}^{\alpha+1}(x)\, dx = -\, L_n^{\alpha}(t) + \binom{n+\alpha}{n}$$ EH II 189(15)a

4. $$\int_0^t L_m(x)\, L_n(t-x)\, dx = L_{m+n}(t) - L_{m+n+1}(t)$$ EH II 191(31)

5. $$\sum_{k=0}^{\infty} \left[\int_0^t \frac{L_k(x)}{k!}\, dx\right]^2 = e^t - 1 \qquad [t \geq 0]$$ MO 110

7.412

1. $$\int_0^1 (1-x)^{\mu-1} x^\alpha\, L_n^\alpha(ax)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)}\, L_n^{\alpha+\mu}(a)$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\mu > 0]$
EH II 191(30)a, BU 129(14c)

2. $$\int_0^1 (1-x)^{\mu-1} x^{\lambda-1}\, L_n^\alpha(\beta x)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\; {}_2F_2\left(-n,\lambda;\alpha+1,\lambda+\mu:\beta\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(50)a

7.413 $$\int_0^1 x^\alpha (1-x)^\beta\, L_m^\alpha(xy)\, L_n^\beta[(1-x)y]\,dx = \frac{(m+n)!\,\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{m!n!\,\Gamma(\alpha+\beta+m+n+2)}\, L_{m+n}^{\alpha+\beta+1}(y)$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$ ET II 293(7)

7.414

1.[11] $$\int_y^\infty e^{-x}\, L_n^\alpha(x)\,dx = e^{-y}\left[L_n^\alpha(y) - L_{n-1}^\alpha(y)\right]$$ EH II 191(29)

2. $$\int_0^\infty e^{-bx}\, L_n(\lambda x)\, L_n(\mu x)\,dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}}\, P_n\left[\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right]$$
$[\operatorname{Re} b > 0]$ ET I 175(34)

3.[8] $$\int_0^\infty e^{-x} x^\alpha\, L_n^\alpha(x)\, L_m^\alpha(x)\,dx = 0 \qquad [m \neq n, \quad \operatorname{Re}\alpha > -1]$$ BU 115(8), ET II 293(3)
$$= \frac{\Gamma(\alpha+n+1)}{n!} \qquad [m = n, \quad \operatorname{Re}\alpha > 0]$$ BU 115(8), ET II 292(2)

4. $$\int_0^\infty e^{-bx} x^\alpha\, L_n^\alpha(\lambda x)\, L_m^\alpha(\mu x)\,dx = \frac{\Gamma(m+n+\alpha+1)}{m!n!}\,\frac{(b-\lambda)^n(b-\mu)^m}{b^{m+n+\alpha+1}} \times F\left[-m,-n;-m-n-\alpha,\frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right]$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re} b > 0]$ ET I 175(35)

4(1)[9]. $$\int_0^\infty e^{-x} x^{\alpha+1/2}\, L_n^\alpha(x)\, L_m^\alpha(x)\,dx = \frac{\Gamma(\alpha+n+1)^2\,\Gamma(\alpha+m+1)\,\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(m-\frac{1}{2}\right)}{n!m!\,\Gamma(\alpha+1)\,\Gamma\left(-\frac{1}{2}\right)} \times {}_3F_2\left(-n,\alpha+\tfrac{3}{2},\tfrac{3}{2};\alpha+1,\tfrac{3}{2}-m;1\right)$$

5. $$\int_0^\infty e^{-bx}\, L_n^a(x)\,dx = \sum_{m=0}^n \binom{a+m-1}{m}\frac{(b-1)^{n-m}}{b^{n-m+1}} \qquad [\operatorname{Re} b > 0]$$ ET I 174(27)

6. $$\int_0^\infty e^{-bx}\, L_n(x)\,dx = (b-1)^n b^{-n-1} \qquad [\operatorname{Re} b > 0]$$ ET I 174(25)

7. $$\int_0^\infty e^{-st} t^\beta\, L_n^\alpha(t)\,dt = \frac{\Gamma(\beta+1)\,\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)} s^{-\beta-1}\, F\left(-n,\beta+1;\alpha+1;\frac{1}{s}\right)$$
$[\operatorname{Re}\beta > -1, \quad \operatorname{Re} s > 0]$
BU 119(4b), EH II 191(133)

8. $$\int_0^\infty e^{-st} t^\alpha\, L_n^\alpha(t)\,dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n!s^{\alpha+n+1}} \qquad [\operatorname{Re}\alpha > -1, \quad \operatorname{Re} s > 0]$$
EH II 191(32), MO 176a

9. $$\int_0^\infty e^{-x}x^{\alpha+\beta}\,L_m^\alpha(x)\,L_n^\beta(x)\,dx=(-1)^{m+n}(\alpha+\beta)!\binom{\alpha+m}{n}\binom{\beta+n}{m}$$
$[\mathrm{Re}(\alpha+\beta)>-1]$ ET II 293(4)

10.[6] $$\int_0^\infty e^{-bx}x^{2a}\left[L_n^a(x)\right]^2\,dx=\frac{2^{2a}\,\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\pi\,(n!)^2\,b^{2a+1}}\times F\left(-n,a+\frac{1}{2};\frac{1}{2}-n;\left(1-\frac{2}{b}\right)^2\right)\Gamma(a+n+1)$$
$\left[\mathrm{Re}\,a>-\frac{1}{2},\quad \mathrm{Re}\,b>0\right]$ ET I 174(30)

11. $$\int_0^\infty e^{-x}x^{\gamma-1}\,L_n^\mu(x)\,dx=\frac{\Gamma(\gamma)\,\Gamma(1+\mu+n-\gamma)}{n!\,\Gamma(1+\mu-\gamma)}\qquad[\mathrm{Re}\,\gamma>0]$$
BU 120(4b)

12. $$\int_0^\infty e^{-x\left(s+\frac{a_1+a_2}{2}\right)}x^{\mu+\beta}\,L_k^\mu\left(a_1x\right)L_k^\mu\left(a_2x\right)\,dx$$
$$=\frac{\Gamma(1+\mu+\beta)\,\Gamma(1+\mu+k)}{k!k!\,\Gamma\left(1+\mu\right)}\left\{\frac{d^k}{dh^k}\left[\frac{F\left(\frac{1+\mu+\beta}{2},1+\frac{\mu+\beta}{2};1+\mu;\frac{A^2}{B^2}\right)}{(1-h)^{1+\mu}B^{1+\mu+\beta}}\right]\right\}_{h=0}$$
$$A^2=\frac{4a_1a_2h}{(1-h)^2};\qquad B=s+\frac{a_1+a_2}{2}\frac{1+h}{1-h}$$
$$\left[\mathrm{Re}\left(s+\frac{a_1+a_2}{2}\right)>0,\quad a_1>0,\quad a_2>0,\quad \mathrm{Re}(\mu+\beta)>-1\right]$$
BU 142(19)

13. $$\int_0^\infty \exp\left[-x\left(s+\frac{a_1+a_2}{2}\right)\right]x^\mu\,L_k^\mu(a_1x)\,L_k^\mu(a_2x)\,dx=\frac{\Gamma(1+\mu+k)}{b_0^{1+\mu+k}}\cdot\frac{b_0^k}{k!}\cdot P_k^{(\mu,0)}\left(\frac{b_1^2}{b_0b_2}\right)$$
$$b_0=s+\frac{a_1+a_2}{2},\quad b_1^2=b_0b_2+2a_1a_2,\quad b_2=s-\frac{a_1+a_2}{2}$$
$$\left[\mathrm{Re}\,\mu>-1,\quad \mathrm{Re}\left(s+\frac{a_1+a_2}{2}\right)>0\right]$$
BU 144(22)

7.415 $$\int_0^1(1-x)^{\mu-1}x^{\lambda-1}e^{-\beta x}\,L_n^\alpha(\beta x)\,dx=\frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}\,\mathrm{B}(\lambda,\mu)\;{}_2F_2\left(\alpha+n+1,\lambda;\alpha+1,\lambda+\mu;-\beta\right)$$
$[\mathrm{Re}\,\lambda>0,\quad \mathrm{Re}\,\mu>0]$ ET II 193(51)a

7.416 $$\int_{-\infty}^\infty x^{m-n}\exp\left[-\frac{1}{2}(x-y)^2\right]L_n^{m-n}\left(x^2\right)\,dx=\frac{(2\pi)^{1/2}}{n!}i^{n-m}2^{-\frac{n+m}{2}}\,H_n\left(\frac{iy}{\sqrt{2}}\right)H_m\left(\frac{iy}{\sqrt{2}}\right)$$
BU 149(15b), ET II 293(8)a

7.417

1. $$\int_0^\infty x^{\nu-2n-1}e^{-ax}\sin(bx)\,L_{2n}^{\nu-2n-1}(ax)\,dx=(-1)^n i\,\Gamma(\nu)\frac{b^{2n}\left[(a-ib)^{-\nu}-(a+ib)^{-\nu}\right]}{2(2n)!}$$
$[b>0,\quad \mathrm{Re}\,a>0,\quad \mathrm{Re}\,\nu>2n]$
ET I 95(12)

2. $$\int_0^\infty x^{\nu-2n-2}e^{-ax}\sin(bx)\,L_{2n+1}^{\nu-2n-2}(ax)\,dx=(-1)^{n+1}\,\Gamma(\nu)\frac{b^{2n+1}\left[(a+ib)^{-\nu}+(a-ib)^{-\nu}\right]}{2(2n+1)!}$$
$[b>0,\quad \mathrm{Re}\,a>0,\quad \mathrm{Re}\,\nu>2n+1]$
ET I 95(13)

3. $$\int_0^\infty x^{\nu-2n}e^{-ax}\cos(bx)\,L_{\nu-2n}^{2n-1}(ax)\,dx = i(-1)^{n+1}\,\Gamma(\nu)\frac{b^{2n-1}\left[(a-ib)^{-\nu}-(a+ib)^{-\nu}\right]}{2(2n-1)!}$$
$$[b>0,\quad \operatorname{Re}a>0,\quad \operatorname{Re}\nu>2n-1]$$
ET I 39(12)

4. $$\int_0^\infty x^{\nu-2n-1}e^{-ax}\cos(bx)\,L_{2n}^{\nu-2n-1}(ax)\,dx = (-1)^n\,\Gamma(\nu)\frac{b^{2n}\left[(a+ib)^{-\nu}+(a-ib)^{-\nu}\right]}{2(2n)!}$$
$$[b>0,\quad \operatorname{Re}\nu>2n,\quad \operatorname{Re}a>0]$$
ET I 39(13)

7.418

1. $$\int_0^\infty e^{-\frac{1}{2}x^2}\sin(bx)\,L_n\left(x^2\right)\,dx = (-1)^n\frac{i}{2}n!\frac{1}{\sqrt{2\pi}}\left\{\left[D_{-n-1}(ib)\right]^2-\left[D_{-n-1}(-ib)\right]^2\right\}$$
$$[b>0]$$
ET I 95(14)

2. $$\int_0^\infty e^{-\frac{1}{2}x^2}\cos(bx)\,L_n\left(x^2\right)\,dx = \sqrt{\frac{\pi}{2}}\,(n!)^{-1}\,e^{-\frac{1}{2}b^2}2^{-n}\left[H_n\left(\frac{b}{\sqrt{2}}\right)\right]^2$$
$$[b>0]$$
ET I 39(14)

3. $$\int_0^\infty x^{2n+1}e^{-\frac{1}{2}x^2}\sin(bx)\,L_n^{n+\frac{1}{2}}\left(\frac{1}{2}x^2\right)\,dx = \sqrt{\frac{\pi}{2}}b^{2n+1}e^{-\frac{1}{2}b^2}\,L_n^{n+\frac{1}{2}}\left(\frac{b^2}{2}\right)$$
$$[b>0]$$
ET I 95(15)

4. $$\int_0^\infty x^{2n}e^{-\frac{1}{2}x^2}\cos(bx)\,L_n^{n-\frac{1}{2}}\left(\frac{1}{2}x^2\right)\,dx = \sqrt{\frac{\pi}{2}}b^{2n}e^{-\frac{1}{2}b^2}\,L_n^{n+\frac{1}{2}}\left(\frac{1}{2}b^2\right)$$
$$[b>0]$$
ET I 39(16)

5. $$\int_0^\infty xe^{-\frac{1}{2}x^2}\,L_n^{\alpha}\left(\frac{1}{2}x^2\right)L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2}ye^{-\frac{1}{2}y^2}\,L_n^{\alpha}\left(\frac{1}{2}y^2\right)L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}y^2\right)$$
ET II 294(11)

6. $$\int_0^\infty e^{-\frac{1}{2}x^2}\,L_n^{\alpha}\left(\frac{1}{2}x^2\right)L_n^{-\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right)\cos(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2}e^{-\frac{1}{2}y^2}\,L_n^{\alpha}\left(\frac{1}{2}y^2\right)L_n^{-\alpha-\frac{1}{2}}\left(\frac{1}{2}y^2\right)$$
ET II 294(12)

7.419 $$\int_0^\infty x^{n+2\nu-\frac{1}{2}}\exp[-(1+a)x]\,L_n^{2\nu}(ax)\,K_\nu(x)\,dx$$
$$=\frac{\pi^{1/2}\,\Gamma\left(n+\nu+\frac{1}{2}\right)\Gamma\left(n+3\nu+\frac{1}{2}\right)}{2^{n+2\nu+\frac{1}{2}}n!\,\Gamma(2\nu+1)}\,F\left(n+\nu+\frac{1}{2},n+3\nu+\frac{1}{2};2\nu+1;-\frac{1}{2}a\right)$$
$$\left[\operatorname{Re}a>-2,\quad \operatorname{Re}(n+\nu)>-\tfrac{1}{2},\quad \operatorname{Re}(n+3\nu)>-\tfrac{1}{2}\right]$$
ET II 370(44)

7.421

1. $$\int_0^\infty xe^{-\frac{1}{2}\alpha x^2}\,L_n\left(\frac{1}{2}\beta x^2\right)J_0(xy)\,dx = \frac{(\alpha-\beta)^n}{\alpha^{n+1}}e^{-\frac{1}{2\alpha}y^2}\,L_n\left[\frac{\beta y^2}{2\alpha(\beta-\alpha)}\right]$$
$$[y>0,\quad \operatorname{Re}\alpha>0]$$
ET II 13(4)a

2. $$\int_0^\infty xe^{-x^2}\,L_n\left(x^2\right)J_0(xy)\,dx = \frac{2^{-2n-1}}{n!}y^{2n}e^{-\frac{1}{4}y^2}$$
ET II 13(5)

3. $$\int_0^\infty x^{2n+\nu+1}e^{-\frac{1}{2}x^2}\,L_n^{\nu+n}\left(\frac{1}{2}x^2\right)J_\nu(xy)\,dx = y^{2n+\nu}e^{-\frac{1}{2}y^2}\,L_n^{\nu+n}\left(\frac{1}{2}y^2\right)$$
$[y>0,\quad \operatorname{Re}\nu>-1]$ MO 183

4. $$\int_0^\infty x^{\nu+1}e^{-\beta x^2}\,L_n^\nu\left(\alpha x^2\right)J_\nu(xy)\,dx = 2^{-\nu-1}\beta^{-\nu-n-1}(\beta-\alpha)^n y^\nu e^{-\frac{y^2}{4\beta}}\,L_n^\nu\left[\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right]$$
ET II 43(5)

5. $$\int_0^\infty e^{-\frac{1}{2q}x^2}x^{\nu+1}\,L_n^\nu\left[\frac{x^2}{2q(1-q)}\right]J_\nu(xy)\,dx = \frac{q^{n+\nu+1}}{(q-1)^n}e^{-\frac{qy^2}{2}}y^\nu\,L_n^\nu\left(\frac{y^2}{2}\right)$$
$[\nu>0]$ MO 183

6.* $$\int_0^\infty x^{\nu+1}e^{-x^2}\,L_n^\nu\left(x^2\right)J_\nu(xy)\,dx = \frac{1}{2n!}\left(\frac{y}{2}\right)^{2n+\nu}e^{-\frac{1}{4}y^2}$$

7.422

1. $$\int_0^\infty x^{\nu+1}e^{-\beta x^2}\left[L_n^{\frac{1}{2}\nu}\left(\alpha x^2\right)\right]^2 J_\nu(xy)\,dx$$
$$= \frac{y^\nu}{\pi n!}\Gamma\left(n+1+\tfrac{1}{2}\nu\right)(2\beta)^{-\nu-1}e^{-\frac{y^2}{4\beta}}$$
$$\times\sum_{l=0}^n \frac{(-1)^l\,\Gamma\left(n-l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l+1+\frac{1}{2}\nu\right)(n-l)!}\left(\frac{2\alpha-\beta}{\beta}\right)^{2l}L_{2l}^\nu\left[\frac{\alpha y^2}{2\beta(2\alpha-\beta)}\right]$$
$[y>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\nu>-1]$ ET II 43(7)

2.[9] $$\int_0^\infty x^{\nu+1}e^{-\alpha x^2}\,L_m^{\nu-\sigma}\left(\alpha x^2\right)L_n^\sigma\left(\alpha x^2\right)J_\nu(xy)\,dx$$
$$= (-1)^{m+n}(2\alpha)^{-\nu-1}y^\nu e^{-\frac{y^2}{4\alpha}}\,L_n^{m-n-\sigma}\left(\frac{y^2}{4\alpha}\right)L_m^{n-m+\sigma-\nu}\left(\frac{y^2}{4\alpha}\right)$$
$[y>0,\quad \operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-1,\quad n\neq 0,\quad \sigma\neq 0,\quad \alpha\neq 1]$ ET II 43(8)

7.423

1. $$\int_0^\infty e^{-\frac{1}{2}x^2}\,L_n\left(\frac{1}{2}x^2\right)H_{2n+1}\left(\frac{x}{2\sqrt{2}}\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2}e^{-\frac{1}{2}y^2}\,L_n\left(\frac{1}{2}y^2\right)H_{2n+1}\left(\frac{y}{2\sqrt{2}}\right)$$
ET II 294(13)a

2. $$\int_0^\infty e^{-\frac{1}{2}x^2}\,L_n\left(\frac{1}{2}x^2\right)H_{2n}\left(\frac{x}{2\sqrt{2}}\right)\cos(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2}e^{-\frac{1}{2}y^2}\,L_n\left(\frac{1}{2}y^2\right)H_{2n}\left(\frac{y}{2\sqrt{2}}\right)$$
ET II 294(14)a

7.5 Hypergeometric Functions

7.51 Combinations of hypergeometric functions and powers

7.511 $$\int_0^\infty F(a,b;c;-z)z^{-s-1}\,dx = \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c)\,\Gamma(-s)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+s)}$$
$[c\neq 0,-1,-2,\ldots,\quad \operatorname{Re}s<0,\quad \operatorname{Re}(a+s)>0,\quad \operatorname{Re}(b+s)>0]$ EH I 79(4)

7.512

1. $$\int_0^1 x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}\,F(\alpha,\beta;\gamma;x)\,dx=\frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma(\gamma)\,\Gamma\left(\alpha-\gamma+1\right)\Gamma\left(\gamma-\frac{\alpha}{2}-\beta\right)}{\Gamma\left(1+\alpha\right)\Gamma\left(1+\frac{\alpha}{2}-\beta\right)\Gamma\left(\gamma-\frac{\alpha}{2}\right)}$$
$$\left[\operatorname{Re}\alpha+1>\operatorname{Re}\gamma>\operatorname{Re}\beta,\quad \operatorname{Re}\left(\gamma-\frac{\alpha}{2}-\beta\right)>0\right]$$ ET II 398(1)

2. $$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n}\,F(-n,\beta;\gamma;x)\,dx=\frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\rho)\,\Gamma(\beta-\gamma+\rho+1)}$$
$$[n=0,1,2\ldots;\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\beta-\gamma)>n-1]$$ ET II 398(2)

3. $$\int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1}\,F(\alpha,\beta;\gamma;x)\,dx=\frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\rho)\,\Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\rho)}$$
$$[\operatorname{Re}\rho>0,\quad \operatorname{Re}(\beta-\rho)>0,\quad \operatorname{Re}(\gamma-\alpha-\rho)>0]$$ ET II 399(3)

4. $$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}\,F(\alpha,\beta;\gamma;x)\,dx=\frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0]$$ ET II 399(4)

5. $$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\,F(\alpha,\beta;\gamma;x)\,dx=\frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\;{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1)$$
$$[\operatorname{Re}\rho>0,\quad \operatorname{Re}\sigma>0,\quad \operatorname{Re}(\gamma+\sigma-\alpha-\beta)>0]$$ ET II 399(5)

6.[10] $$\int_0^1 x^{\lambda-1}(1-x)^{\beta-\lambda-1}\,F\left(\alpha,\beta;\lambda;\frac{zx}{b}\right)\,dx=\mathrm{B}(\lambda,\beta-\lambda)(1-z/b)^{-\alpha}$$ BU 9

7.[11] $$\int_0^1 x^{\gamma-1}(1-x)^{\delta-\gamma-1}\,F(\alpha,\beta;\gamma;xz)\,F\left(\delta-\alpha,\delta-\beta;\delta-\gamma;(1-x)\zeta\right)\,dx$$
$$=\frac{\Gamma(\gamma)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)}(1-\zeta)^{\alpha+\beta-\delta}\,F(\alpha,\beta;\delta;z+\zeta-z\zeta)$$
$$[0<\operatorname{Re}\gamma<\operatorname{Re}\delta,\quad |\arg(1-z)|<\pi,\quad |\arg(1-\zeta)|<\pi]$$ ET II 400(11)

8. $$\int_0^1 x^{\gamma-1}(1-x)^{\epsilon-1}(1-xz)^{-\delta}\,F(\alpha,\beta;\gamma;xz)\,F\left[\delta,\beta-\gamma;\epsilon;\frac{(1-x)z}{(1-xz)}\right]\,dx$$
$$=\frac{\Gamma(\gamma)\,\Gamma(\epsilon)}{\Gamma(\gamma+\epsilon)}\,F\left(\alpha+\delta,\beta;\gamma+\epsilon;z\right)$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\epsilon>0,\quad |\arg(z-1)|<\pi]$$ ET II 400(12), Eh I 78(3)

9. $$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma}\,F(\alpha,\beta;\gamma;x)\,dx$$
$$=\frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}(1-z)^{-\sigma}$$
$$\times\;{}_3F_2\left(\rho,\sigma,\gamma+\rho-\alpha-\beta;\gamma+\rho-\alpha,\gamma+\rho-\beta;\frac{z}{z-1}\right)$$
$$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}\left(\gamma+\rho-\alpha-\beta\right)>0,\quad |\arg(1-z)|<\pi]$$ ET II 399(6)

10. $$\int_0^\infty x^{\gamma-1}(x+z)^{-\sigma}\,F(\alpha,\beta;\gamma;-x)\,dx=\frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+\sigma)\,\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\,\Gamma(\alpha+\beta-\gamma+\sigma)}\times F(\alpha-\gamma+\sigma,\beta-\gamma+\sigma;\alpha+\beta-\gamma+\sigma;1-z)$$
$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}(\alpha-\gamma+\sigma)>0,\quad \operatorname{Re}(\beta-\gamma+\sigma)>0,\quad |\arg z|<\pi]$ ET II 400(10)

11. $$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\;{}_pF_q\,(a_1,\ldots,a_p;\nu,b_2,\ldots,b_q;ax)\;dx=\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\;{}_pF_q\,(a_1,\ldots,a_p;\mu+\nu,b_2,\ldots,b_q;a)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad p\le q+1;\ \text{if } p=q+1,\ \text{then } |a|<1]$ ET II 200(94)

12. $$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\;{}_pF_q\,(a_1,\ldots,a_p;b_1,\ldots,b_q;ax)\;dx=\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\;{}_{p+1}F_{q+1}\,(\nu,a_1,\ldots,a_p;\mu+\nu,b_1,\ldots,b_q;a)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad p\le q+1,\ \text{if } p=q+1,\ \text{then } |a|<1]$ ET II 200(95)

7.513 $$\int_0^1 x^{s-1}\left(1-x^2\right)^{\nu}F\left(-n,a;b;x^2\right)\,dx=\frac{1}{2}\,\mathrm{B}\left(\nu+1,\frac{s}{2}\right)\;{}_3F_2\left(-n,a,\frac{s}{2};b,\nu+1+\frac{s}{2};1\right)$$
$[\operatorname{Re}s>0,\quad \operatorname{Re}\nu>-1]$ ET I 336(4)

7.52 Combinations of hypergeometric functions and exponentials

7.521 $$\int_0^\infty e^{-st}\;{}_pF_q\,(a_1,\ldots,a_p;b_1,\ldots,b_q,t)\;dt=\frac{1}{s}\;{}_{p+1}F_q\left(1,a_1,\ldots,a_p;b_1,\ldots,b_q,s^{-1}\right)$$
$[p\le q]$ EH I 192

7.522

1.[11] $$\int_0^\infty e^{-\lambda x}x^{\gamma-1}\;{}_2F_1(\alpha,\beta;\delta;-x)\,dx=\frac{\Gamma(\delta)\lambda^{-\gamma}}{\Gamma(\alpha)\,\Gamma(\beta)}\,E(\alpha,\beta,\gamma:\delta:\lambda)$$
$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\gamma>0]$ EH I 205(10)

2.[6] $$\int_0^\infty e^{-bx}x^{a-1}\,F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;a;-\frac{x}{2}\right)\,dx=2^a e^b\frac{1}{\sqrt{\pi}}\,\Gamma(a)(2b)^{\frac{1}{2}-a}\,K_\nu(b)$$
$[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$ ET I 212(1)

3. $$\int_0^\infty e^{-bx}x^{\gamma-1}\,F(2\alpha,2\beta;\gamma;-\lambda x)\,dx=\Gamma(\gamma)b^{-\gamma}\left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}}e^{\frac{b}{2\lambda}}\;W_{\frac{1}{2}-\alpha-\beta,\alpha-\beta}\left(\frac{b}{\lambda}\right)$$
$[\operatorname{Re}b>0,\quad \operatorname{Re}\gamma>0,\quad |\arg\lambda|<\pi]$
BU 78(30), ET I 212(4)

4.[6] $$\int_0^\infty e^{-xt}t^{b-1}\,F(a,a-c+1;b;-t)\,dt=x^{a-b}\,\Gamma(b)\Psi(a,c;x)$$
$[\operatorname{Re}b>0,\quad \operatorname{Re}x>0]$ EH I 273(11)

5. $$\int_0^\infty e^{-x}x^{s-1}\;{}_pF_q\,(a_1,\ldots,a_p,b_1,\ldots,b_q;ax)\;dx=\Gamma(s)\;{}_{p+1}F_q\,(s,a_1,\ldots,a_p;b_1,\ldots,b_q;a)$$
$[p<q,\quad \operatorname{Re}s>0]$ ET I 337(11)

6. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\ {}_2F_2(-n,n+1;1,\beta;x)\,dx=\Gamma(\beta)\mu^{-\beta}\,P_n\left(1-\frac{2}{\mu}\right)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0]$ ET I 218(6)

7. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\ {}_2F_2\left(-n,n;\beta,\frac{1}{2};x\right)\,dx=\Gamma(\beta)\mu^{-\beta}\cos\left[2n\arcsin\left(\frac{1}{\sqrt{\mu}}\right)\right]$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0]$ ET I 218(7)

8. $$\int_0^\infty x^{\rho_n-1}e^{-\mu x}\ {}_mF_n\left(a_1,\ldots,a_m;\rho_1,\ldots,\rho_n;\lambda x\right)\,dx$$
$$=\Gamma\left(\rho_n\right)\mu^{-\rho_n}\ {}_mF_{n-1}\left(a_1,\ldots,a_m;\rho_1,\ldots,\rho_{n-1};\frac{\lambda}{\mu}\right)$$
$[m\le n;\quad \operatorname{Re}\rho_n>0,\quad \operatorname{Re}\mu>0,\ \text{if } m<n;\operatorname{Re}\mu>\operatorname{Re}\lambda,\ \text{if } m=n]$ ET I 219(16)a

9. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\ {}_mF_n\left(a_1,\ldots,a_m;\rho_1,\ldots,\rho_n;\lambda x\right)\,dx$$
$$=\Gamma(\sigma)\mu^{-\sigma}\ {}_{m+1}F_n\left(a_1,\ldots,a_m,\sigma;\rho_1,\ldots,\rho_n;\frac{\lambda}{\mu}\right)$$
$[m\le n,\quad \operatorname{Re}\sigma>0,\quad \operatorname{Re}\mu>0,\ \text{if } m<n;\operatorname{Re}\mu>\operatorname{Re}\lambda,\ \text{if } m=n]$ ET I 219(17)

7.523 $$\int_1^\infty (x-1)^{\mu-1}x^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}ax}\ W_{2\mu+\frac{1}{2},\lambda}(ax)\,dx=\Gamma(\mu)e^{-\frac{1}{2}a}\ W_{\mu+\frac{1}{2},\lambda}(a)$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}a>0]$

7.524

1. $$\int_0^\infty e^{-\lambda x}\,F\left(\alpha,\beta;\frac{1}{2};-x^2\right)\,dx=\lambda^{\alpha+\beta-1}\,S_{1-\alpha-\beta,\alpha-\beta}(\lambda)$$
$[\operatorname{Re}\lambda>0]$ ET II 401(13)

2. $$\int_0^\infty e^{-st}\ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;t^2\right)\,dx=s^{-1}\ {}_{p+2}F_q\left(a_1,\ldots,a_p,1,\frac{1}{2};b_1,\ldots,b_q;\frac{4}{s^2}\right)$$
$[p<q]$ MO 176

3. $$\int_0^\infty e^{-st}\ {}_0F_q\left(\frac{1}{q},\frac{2}{q},\ldots,\frac{q-1}{q},1;\frac{t^q}{q^q}\right)\,dt=s^{-1}\exp\left(s^{-q}\right)$$ MO 176

7.525

1. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\ {}_mF_n\left(a_1,\ldots,a_m;\rho_1,\ldots,\rho_n;(\lambda x)^k\right)\,dx$$
$$=\Gamma(\sigma)\mu^{-\sigma}\ {}_{m+k}F_n\left(a_1,\ldots,a_m,\frac{\sigma}{k},\frac{\sigma+1}{k},\ldots,\frac{\sigma+k-1}{k};\rho_1,\ldots,\rho_n;\left(\frac{k\lambda}{\mu}\right)^k\right)$$
$$\left[m+k\le n+1,\quad \operatorname{Re}\sigma>0;\quad \operatorname{Re}\mu>0,\ \text{if } m+k\le n;\right.$$
$$\left.\operatorname{Re}\left(\mu+k\lambda e^{\frac{2\pi i}{k}}\right)>0;\quad r=0,1,\ldots,k-1 \text{ for } m+k=n+1\right]$$
ET I 220(19)

2. $$\int_0^\infty xe^{-\lambda x}\, F\left(\alpha,\beta;\tfrac{3}{2};-x^2\right)\, dx = \lambda^{\alpha+\beta-2}\, S_{1-\alpha-\beta,\alpha-\beta}(\lambda)$$
$[\operatorname{Re}\lambda > 0]$ ET II 401(14)

7.526

1. $$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}s^{-b}\, F\left(a,b;a+b-c+1;1-\frac{1}{s}\right)\, dx = 2\pi i\frac{\Gamma(a+b-c+1)}{\Gamma(b)\,\Gamma(b-c+1)}t^{b-1}\,\Psi(a;c;t)$$
$\left[\operatorname{Re} b > 0,\quad \operatorname{Re}(b-c) > -1,\quad \gamma > \frac{1}{2}\right]$
EH I 273(12)

2. $$\int_0^\infty e^{-t}t^{\gamma-1}(x+t)^{-\alpha}(y+t)^{-a'}\, F\left[a,a';\gamma;\frac{t(x+y+t)}{(x+t)(y+t)}\right]\, dt = \Gamma(\gamma)\Psi(a,c;x)\Psi\left(a',c;y\right),$$
$\gamma = a+a'-c+1 \qquad [\operatorname{Re}\gamma > 0,\quad xy \neq 0]$ EH I 287(21)

3. $$\int_0^\infty x^{\gamma-1}(x+y)^{-\alpha}(x+z)^{-\beta}e^{-x}\, F\left[\alpha,\beta;\gamma;\frac{x(x+y+z)}{(x+y)(x+z)}\right]\, dx$$
$$= \Gamma(\gamma)(zy)^{-\frac{1}{2}-\mu}e^{\frac{y+z}{2}}\, W_{\nu,\mu}(y)\, W_{\lambda,\mu}(z)$$
$$2\nu = 1-\alpha+\beta-\gamma;\quad 2\lambda = 1+\alpha-\beta-\gamma;\quad 2\mu = \alpha+\beta-\gamma$$
$[\operatorname{Re}\gamma > 0,\quad |\arg y| < \pi,\quad |\arg z| < \pi]$
ET II 401(15)

7.527

1. $$\int_0^\infty \left(1-e^{-x}\right)^{\lambda-1} e^{-\mu x}\, F\left(\alpha,\beta;\gamma;\delta e^{-x}\right)\, dx = \mathrm{B}(\mu,\lambda)\; {}_3F_2(\alpha,\beta,\mu;\gamma,\mu+\lambda;\delta)$$
$[\operatorname{Re}\lambda > 0,\quad \operatorname{Re}\mu > 0,\quad |\arg(1-\delta)| < \pi]$ ET I 213(9)

2. $$\int_0^\infty \left(1-e^{-x}\right)^{\mu} e^{-\alpha x}\, F\left(-n,\mu+\beta+n;\beta;e^{-x}\right)\, dx = \frac{\mathrm{B}(\alpha,\mu+n+1)\,\mathrm{B}(\alpha,\beta+n-\alpha)}{\mathrm{B}(\alpha,\beta-\alpha)}$$
$[\operatorname{Re}\alpha > 0,\quad \operatorname{Re}\mu > -1]$ ET I 213(10)

3. $$\int_0^\infty \left(1-e^{-x}\right)^{\gamma-1} e^{-\mu x}\, F\left(\alpha,\beta;\gamma;1-e^{-x}\right)\, dx = \frac{\Gamma(\mu)\,\Gamma(\gamma-\alpha-\beta+\mu)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha+\mu)\,\Gamma(\gamma-\beta+\mu)}$$
$[\operatorname{Re}\mu > 0,\quad \operatorname{Re}\mu > \operatorname{Re}(\alpha+\beta-\gamma),\quad \operatorname{Re}\gamma > 0]$ ET I 213(11)

4. $$\int_0^\infty \left(1-e^{-x}\right)^{\gamma-1} e^{-\mu x}\, F\left[\alpha,\beta;\gamma;\delta\left(1-e^{-x}\right)\right]\, dx = \mathrm{B}(\mu,\gamma)\, F(\alpha,\beta;\mu+\gamma;\delta)$$
$[\operatorname{Re}\mu > 0,\quad \operatorname{Re}\gamma > 0,\quad |\arg(1-\delta)| < \pi]$ ET I 213(12)

7.53 Hypergeometric and trigonometric functions

7.531

1. $$\int_0^\infty x\sin\mu x\, F\left(\alpha,\beta;\frac{3}{2};-c^2x^2\right)\,dx = 2^{-\alpha-\beta+1}\pi c^{-\alpha-\beta}\mu^{\alpha+\beta-2}\frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\,\Gamma(\beta)}$$
$$\left[\mu>0,\quad \operatorname{Re}\alpha>\tfrac12,\quad \operatorname{Re}\beta>\tfrac12\right]$$
ET I 115(6)

2. $$\int_0^\infty \cos\mu x\, F\left(\alpha,\beta;\frac{1}{2};-c^2x^2\right)\,dx = 2^{-\alpha-\beta+1}\pi c^{-\alpha-\beta}\mu^{\alpha+\beta-1}\frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\,\Gamma(\beta)}$$
$$[\mu>0,\quad \operatorname{Re}\alpha>0,\quad \operatorname{Re}\beta>0,\quad c>0]$$
ET I 61(9)

7.54 Combinations of hypergeometric and Bessel functions

7.541 $$\int_0^\infty x^{\alpha+\beta-2\nu-1}(x+1)^{-\nu}e^{xz}\,K_\nu[(x+1)z]\,F\left(\alpha,\beta;\alpha+\beta-2\nu;-x\right)\,dx$$
$$=\pi^{-\frac12}\cos(\nu\pi)\,\Gamma\left(\tfrac12-\alpha+\nu\right)\Gamma\left(\tfrac12-\beta+\nu\right)\Gamma(\gamma)(2z)^{-\frac12-\frac12\gamma}\,W_{\frac12\gamma,\frac12(\beta-\alpha)}(2z)$$
$$\gamma=\alpha+\beta-2\nu\qquad\left[\operatorname{Re}(\alpha+\beta-2\nu)>0,\quad \operatorname{Re}\left(\tfrac12-\alpha+\nu\right)>0,\quad \operatorname{Re}\left(\tfrac12-\beta+\nu\right)>0,\quad |\arg z|<\tfrac32\pi\right]$$
ET II 401(16)

7.542

1. $$\int_0^\infty x^{\sigma-1}\,{}_pF_{p-1}\left(a_1,\dots,a_p;b_1,\dots,b_{p-1};-\lambda x^2\right)\,Y_\nu(xy)\,dx$$
$$=\frac{\Gamma(b_1)\dots\Gamma(b_{p-1})}{2\lambda^{\frac12\sigma}\,\Gamma(a_1)\dots\Gamma(a_p)}\,G^{p+2,1}_{p+2,p+3}\left(\frac{y^2}{4\lambda}\,\middle|\,\begin{matrix}b_0^*,\dots,b_{p+1}^*,\\ h,k,a_1^*,\dots,a_p^*,l\end{matrix}\right)$$
$$a_j^*=a_j-\frac{\sigma}{2},\quad j=1,\dots,p;\quad b_0^*=1-\frac{\sigma}{2};\quad b_j^*=b_j-\frac{\sigma}{2},$$
$$j=1,\dots,p-1;h=\frac{\nu}{2},\quad k=-\frac{\nu}{2},\quad l=-\frac{1+\nu}{2}$$
$$\left[|\arg\lambda|<\pi,\quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|,\quad \operatorname{Re}a_j>\tfrac12\operatorname{Re}\sigma-\tfrac34,\quad y>0\right]$$
ET II 118(53)

2. $$\int_0^\infty x^{\sigma-1}\,{}_pF_p\left(a_1,\dots,a_p;b_1,\dots,b_p;-\lambda x^2\right)\,Y_\nu(xy)\,dx$$
$$=\frac{\Gamma(b_1)\dots\Gamma(b_p)}{2\lambda^{\frac12\sigma}\,\Gamma(a_1)\dots\Gamma(a_p)}\,G^{p+2,1}_{p+2,p+3}\left(\frac{y^2}{4\lambda}\,\middle|\,\begin{matrix}b_0^*,\dots,b_p^*,l\\ h,k,a_1^*,\dots,a_p^*,l\end{matrix}\right)$$
$$b_0^*=1-\frac{\sigma}{2};\quad a_j^*=a_j-\frac{\sigma}{2},\quad b_j*=b_j-\frac{\sigma}{2};\quad j=1,\dots,p;\quad h=\frac{\nu}{2},\quad k=-\frac{\nu}{2},\quad l=-\frac{1+\nu}{2}$$
$$\left[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|,\quad \operatorname{Re}a_j>\tfrac12\operatorname{Re}\sigma-\tfrac34,\quad y>0\right]$$
ET II 119(54)

3. $$\int_0^\infty x^{\sigma-1}\ {}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;-\lambda x^2\right)\,Y_\nu(xy)\,dx$$
$$= -\pi^{-1}2^{\sigma-1}y^{-\sigma}\cos\left[\frac{\pi}{2}(\sigma-\nu)\right]\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right)$$
$$\times\ {}_{p+2}F_q\left(a_1,\dots,a_p,\frac{\sigma+\nu}{2},\frac{\sigma-\nu}{2};b_1,\dots,b_q;-\frac{4\lambda}{y^2}\right)$$
$$[y>0,\quad p\le q-1,\quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|]$$ ET II 119(55)

4. $$\int_0^\infty x^{\sigma-1}\ {}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;-\lambda x^2\right)K_\nu(xy)\,dx$$
$$= 2^{\sigma-2}y^{-\sigma}\,\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right)\ {}_{p+2}F_q\left(a_1,\dots,a_p,\frac{\sigma+\nu}{2},\frac{\sigma-\nu}{2};b_1,\dots,b_q;\frac{4\lambda}{y^2}\right)$$
$$[\operatorname{Re}y>0,\quad p\le q-1,\quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|]$$ ET II 153(88)

5. $$\int_0^\infty x^{2\rho}\ {}_pF_p\left(a_1,\dots,a_p;b_1,\dots,b_p;-\lambda x^2\right)J_\nu(xy)\,dx$$
$$= \frac{2^{2\rho}\,\Gamma(b_1)\dots\Gamma(b_p)}{y^{2\rho+1}\,\Gamma(a_1)\dots\Gamma(a_p)}\,G^{p+1,1}_{p+1,p+2}\left(\frac{y^2}{4\lambda}\middle|\begin{matrix}1, & b_1,\dots,b_p\\ h, & a_1,\dots,a_p,\quad k\end{matrix}\right)$$
$$h=\tfrac12+\rho+\tfrac12\nu,\qquad k=\tfrac12+\rho-\tfrac12\nu$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -1-\operatorname{Re}\nu<2\operatorname{Re}\rho<\tfrac12+2\operatorname{Re}a_r,\quad r=1,\dots,p\right]$$ ET II 91(18)

6. $$\int_0^\infty x^{2\rho}\ {}_{m+1}F_m\left(a_1,\dots,a_{m+1};b_1,\dots,b_m;-\lambda^2x^2\right)J_\nu(xy)\,dx$$
$$= \frac{2^{2\rho}\,\Gamma(b_1)\dots\Gamma(b_m)\,y^{-2\rho-1}}{\Gamma(a_1)\dots\Gamma(a_{m+1})}\,G^{m+2,1}_{m+1,m+3}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1, & b_1,\dots,b_m\\ h, & a_1,\dots,a_{m+1},\quad k\end{matrix}\right)$$
$$h=\tfrac12+\rho+\tfrac12\nu,\qquad k=\tfrac12+\rho-\tfrac12\nu,$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}(2\rho+\nu)>-1,\quad \operatorname{Re}(\rho-a_r)<\tfrac14;\quad r=1,\dots,m+1\right]$$ ET II 91(19)

7. $$\int_0^\infty x^\delta\,F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right)J_\nu(xy)\,dx$$
$$= \frac{2^\delta\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}y^{-\delta-1}\,G^{22}_{24}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1-\alpha, & 1-\beta\\ \dfrac{1+\delta+\nu}{2}, & 0,\quad 1-\gamma,\quad \dfrac{1+\delta-\nu}{2}\end{matrix}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -1-\operatorname{Re}\nu-2\min(\operatorname{Re}\alpha,\quad \operatorname{Re}\beta)<\operatorname{Re}\delta<-\tfrac12\right]$$ ET II 82(9)

8. $$\int_0^\infty x^\delta\,F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right)J_\nu(xy)\,dx = \frac{2^\delta y^{-\delta-1}\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,G^{31}_{24}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1, & \gamma\\ \dfrac{1+\delta+\nu}{2}, & \alpha,\beta,\ \dfrac{1+\delta-\nu}{2}\end{matrix}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -\operatorname{Re}\nu-1<\operatorname{Re}\delta<2\max(\operatorname{Re}\alpha,\quad \operatorname{Re}\beta)-\tfrac12\right]$$ ET II 81(6)

9. $$\int_0^\infty x^{\nu+1}\,F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right)J_\nu(xy)\,dx = \frac{2^{\nu+1}\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}y^{-\nu-2}\,G^{30}_{13}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}\gamma\\ \nu+1,\quad \alpha,\quad \beta\end{matrix}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -1<\operatorname{Re}\nu<2\max(\operatorname{Re}\alpha,\quad \operatorname{Re}\beta)-\tfrac32\right]$$ ET II 81(5)

10. $$\int_0^\infty x^{\nu+1}\, F\left(\alpha,\beta;\nu+1;-\lambda^2x^2\right) J_\nu(xy)\,dx = \frac{2^{\nu-\alpha-\beta+2}\,\Gamma(\nu+1)}{\lambda^{\alpha+\beta}\,\Gamma(\alpha)\,\Gamma(\beta)} y^{\alpha+\beta-\nu-2}\, K_{\alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -1<\operatorname{Re}\nu<2\max\left(\operatorname{Re}\alpha,\quad \operatorname{Re}\beta\right)-\tfrac{3}{2}\right]$$ ET II 81(3)

11. $$\int_0^\infty x^{\nu+1}\, F\left(\alpha,\beta;\nu+1;-\lambda^2x^2\right) K_\nu(xy)\,dx = 2^{\nu+1}\lambda^{-\alpha-\beta}y^{\alpha+\beta-\nu-2}\,\Gamma(\nu+1)\, S_{1-\alpha-\beta,\alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$\left[\operatorname{Re}y>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}\nu>-1\right]$$
ET II 152(86)

12. $$\int_0^\infty x^{\nu+1} F\left(\alpha,\beta;\frac{\beta+\nu}{2}+1;-\lambda^2x^2\right) J_\nu(xy)\,dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right) y^{\beta-1}\lambda^{-\nu-\beta-1}}{\pi^{\frac12}\,\Gamma(\alpha)\,\Gamma(\beta)2^{\beta-1}}\, K_{\frac12(\nu-\beta+1)}\left(\frac{y}{2\lambda}\right)^2$$
$$\left[y>0,\quad -1<\operatorname{Re}\nu<\left(2\max\left(\operatorname{Re}\alpha,\operatorname{Re}\beta\right)-\tfrac{3}{2}\right)\right]$$ ET II 81(4)

13. $$\int_0^\infty x^{\sigma+\frac12}\, F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{\lambda^{-\sigma-1}y^{-\frac12}\,\Gamma(\gamma)}{\sqrt{2}\,\Gamma(\alpha)\,\Gamma(\beta)}\, G^{41}_{35}\left(\frac{y^2}{4\lambda^2}\,\middle|\,\begin{matrix}1-p,\gamma-p,l\\ h,\quad k,\alpha-p,\beta-p,l\end{matrix}\right)$$
$$h=\tfrac14+\tfrac12\nu,\quad k=\tfrac14-\tfrac12\nu,\quad l=-\tfrac14-\tfrac12\nu,\quad p=\tfrac12+\tfrac12\sigma$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|-\tfrac32,\quad \operatorname{Re}\sigma<2\operatorname{Re}\alpha,\quad \operatorname{Re}\sigma<2\operatorname{Re}\beta\right]$$ ET II 118(52)

14. $$\int_0^\infty x^{\nu+2}\, F\left(\frac12,\frac12-\nu;\frac32;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{2^\nu y^{-\nu-1}}{\pi^{\frac12}\lambda^2\,\Gamma\left(\frac12-\nu\right)}\, K_\nu\left(\frac{y}{2\lambda}\right) K_{\nu+1}\left(\frac{y}{2\lambda}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -\tfrac32<\operatorname{Re}\nu<-\tfrac12\right]$$
ET II 117(49)

15. $$\int_0^\infty x^{\nu+2}\, F\left(1,2\nu+\frac32;\nu+2;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \pi^{-\frac12}2^{-\nu}\lambda^{-2\nu-3}\frac{\Gamma(\nu+2)}{\Gamma\left(2\nu+\frac32\right)}\left[K_\nu\left(\frac{y}{2\lambda}\right)\right]^2$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -\tfrac12<\operatorname{Re}\nu<\tfrac12\right]$$
ET II 117(50)

16. $$\int_0^\infty x^{\nu+2}\, F\left(1,\mu+\nu+\frac32;\frac32;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{\pi^{\frac12}2^{-\mu-\nu-1}\lambda^{-\mu-2\nu-3}y^{\mu+\nu}}{\Gamma\left(\mu+\nu+\frac32\right)}\, K_\mu\left(\frac{y}{\lambda}\right)$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad -\tfrac32<\operatorname{Re}\nu<\tfrac12,\quad \operatorname{Re}(2\mu+\nu)>-\tfrac32\right]$$ ET II 118(51)

17. $$\int_0^\infty x^{2\alpha+\nu}\, F\left(\alpha-\nu-\frac12,\alpha;2\alpha;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$=\frac{i\,\Gamma\left(\frac12+\alpha\right)\Gamma\left(\frac12+\alpha+\nu\right)}{\pi 2^{1-\nu-2\alpha}\lambda^{2\alpha-1}y^{\nu+2}}\, W_{\frac12-\alpha,-\frac12-\nu}\left(\frac{y}{\lambda}\right)\left[W_{\frac12-\alpha,-\frac12-\nu}\left(e^{-i\pi}\frac{y}{\lambda}\right)-W_{\frac12-\alpha,-\frac12-\nu}\left(e^{i\pi}\frac{y}{\lambda}\right)\right]$$
$$\left[y>0,\quad \operatorname{Re}\lambda>0,\quad \operatorname{Re}\nu<-\tfrac12,\quad \operatorname{Re}(\alpha+\nu)>-\tfrac12\right]$$ ET II 80(1)

18. $$\int_0^\infty x^{2\alpha-\nu}\, F\left(\nu+\alpha-\frac12,\alpha;2\alpha;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$=\frac{2^{2\alpha-\nu}\,\Gamma\left(\frac12+\alpha\right)y^{\nu-2}}{\lambda^{2\alpha-1}\,\Gamma(2\nu)}\, M_{\alpha-\frac12,\nu-\frac12}\left(\frac{y}{\lambda}\right) W_{\frac12-\alpha,\nu-\frac12}\left(\frac{y}{\lambda}\right)$$
ET II 80(2)

7.543

1. $$\int_0^\infty x^{-2\alpha-1}\, F\left(\frac{1}{2}+\alpha, 1+\alpha; 1+2\alpha; -\frac{4\lambda^2}{x^2}\right) J_\nu(xy)\, dx = \lambda^{-2\alpha}\, I_{\frac{1}{2}\nu+\alpha}(\lambda y)\, K_{\frac{1}{2}\nu-\alpha}(\lambda y)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\alpha>-\tfrac{1}{2}\right] \quad \text{ET II 81(7)}$$

2. $$\int_0^\infty x^{\nu+1-4\alpha}\, F\left(\alpha, \alpha+\frac{1}{2}; \nu+1; -\frac{\lambda^2}{x^2}\right) J_\nu(xy)\, dx$$
$$= \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1}\, I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\alpha-1<\operatorname{Re}\nu<4\operatorname{Re}\alpha-\tfrac{3}{2}\right] \quad \text{ET II 81(8)}$$

7.544 $$\int_0^\infty x^{\nu+1}(1+x)^{-2\alpha}\, F\left[\alpha, \nu+\frac{1}{2}; 2\nu+1; \frac{4x}{(1+x)^2}\right] J_\nu(xy)\, dx$$
$$= \frac{\Gamma(\nu+1)\,\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)}\, J_\nu(y)$$
$$\left[y>0, \quad -1<\operatorname{Re}\nu<2\operatorname{Re}\alpha-\tfrac{3}{2}\right] \quad \text{ET II 82(10)}$$

7.6 Confluent Hypergeometric Functions

7.61 Combinations of confluent hypergeometric functions and powers

7.611

1. $$\int_0^\infty x^{-1}\, W_{k,\mu}(x)\, dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma\left(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}\mu\right)}$$
$$\left[|\operatorname{Re}\mu|<\tfrac{1}{2}\right] \quad \text{ET II 406(22)}$$

2. $$\int_0^\infty x^{-1}\, M_{k,\mu}(x)\, W_{\lambda,\mu}(x)\, dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\,\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2}, \quad \operatorname{Re}(k-\lambda)>0\right]$$
BU 116(11), ET II 409(39)

3. $$\int_0^\infty x^{-1}\, W_{k,\mu}(x)\, W_{\lambda,\mu}(x)\, dx$$
$$= \frac{1}{(k-\lambda)\sin(2\mu\pi)}\left[\frac{1}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-\lambda-\mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2}-k-\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}\right]$$
$$\left[|\operatorname{Re}\mu|<\tfrac{1}{2}\right] \quad \text{BU 116(12), ET II 409(40)}$$

4. $$\int_0^\infty \{W_{\kappa,\mu}(z)\}^2\, \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu}\, \frac{\psi\left(\frac{1}{2}+\mu-\kappa\right)-\psi\left(\frac{1}{2}-\mu-\kappa\right)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}$$
$$\left[|\operatorname{Re}\mu|<\tfrac{1}{2}\right] \quad \text{BU 117(12a)}$$

5. $$\int_0^\infty \frac{1}{z}\,[W_{\kappa,0}(z)]^2\, dx = \frac{\psi'\left(\frac{1}{2}-\kappa\right)}{\left[\Gamma\left(\frac{1}{2}-\kappa\right)\right]^2} \quad \text{BU 117(12b)}$$

6. $$\int_0^\infty x^{\rho-1}\, W_{k,\mu}(x)\, W_{-k,\mu}(x)\, dx = \frac{\Gamma(\rho+1)\,\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}-\mu\right)}{2\,\Gamma\left(1+\frac{1}{2}\rho+k\right)\Gamma\left(1+\frac{1}{2}\rho-k\right)}$$
$[\operatorname{Re}\rho > 2|\operatorname{Re}\mu| - 1]$ ET II 409(41)

7.[11] $$\int_0^\infty x^{\rho-1}\, W_{k,\mu}(x)\, W_{\lambda,\nu}(x)\, dx$$
$$= \frac{\Gamma(1-\mu+\nu+\rho)\,\Gamma(1+\mu+\nu+\rho)\,\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\lambda-\nu\right)\Gamma\left(\frac{3}{2}-k+\nu+\rho\right)}$$
$$\times\ {}_3F_2\left(1-\mu+\nu+\rho, 1+\mu+\nu+\rho, \frac{1}{2}-\lambda+\nu; 1+2\nu, \frac{3}{2}-k+\nu+\rho; 1\right)$$
$$+\frac{\Gamma(1+\mu-\nu+\rho)\,\Gamma(1-\mu-\nu+\rho)\,\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right)\Gamma\left(\frac{3}{2}-k-\nu+\rho\right)}$$
$$\times\ {}_3F_2\left(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, \frac{1}{2}-\lambda-\nu; 1-2\nu, \frac{3}{2}-k-\nu+\rho; 1\right)$$
$[|\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\rho + 1]$ ET II 410(42)

7.612

1. $$\int_0^\infty t^{b-1}\ {}_1F_1(a;c;-t)\, dt = \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}$$
$[0 < \operatorname{Re} b < \operatorname{Re} a]$ EH I 285(10)

2. $$\int_0^\infty t^{b-1}\Psi(a,c;t)\, dt = \frac{\Gamma(b)\,\Gamma(a-b)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(a-c+1)}$$
$[0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b + 1]$
EH I 285(11)

7.613

1. $$\int_0^t x^{\gamma-1}(t-x)^{c-\gamma-1}\ {}_1F_1(a;\gamma;x)\, dx = t^{c-1}\frac{\Gamma(\gamma)\,\Gamma(c-\gamma)}{\Gamma(c)}\ {}_1F_1\,(a;c;t)$$
$[\operatorname{Re} c > \operatorname{Re}\gamma > 0]$
BU 9(16)a, EH I 271(16)

2. $$\int_0^t x^{\beta-1}(t-x)^{\gamma-1}\ {}_1F_1(t;\beta;x)\, dx = \frac{\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma)}t^{\beta+\gamma-1}\ {}_1F_1\,(t;\beta+\gamma;t)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ ET II 401(1)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{2\mu-\lambda}\ {}_1F_1\left(\frac{1}{2}+\mu-\nu;\lambda;xz\right)\, dx = \mathrm{B}(\lambda, 1+2\mu-\lambda)e^{\frac{1}{2}z}z^{-\frac{1}{2}-\mu}\, M_{\nu,\mu}(z)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(2\mu-\lambda) > -1]$
BU 14(14)

4. $$\int_0^t x^{\beta-1}(t-x)^{\delta-1}\ {}_1F_1(t;\beta;x)\ {}_1F_1\,(\gamma;\delta;t-x)\ dx = \frac{\Gamma(\beta)\,\Gamma(\delta)}{\Gamma(\beta+\delta)}t^{\beta+\delta-1}\ {}_1F_1\,(t+\gamma;\beta+\delta;t)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\delta > 0]$
ET II 402(2), EH I 271(15)

5. $$\int_0^t x^{\mu-\frac{1}{2}}(t-x)^{\nu-\frac{1}{2}}\, M_{k,\mu}(x)\, M_{\lambda,\nu}(t-x)\, dx = \frac{\Gamma(2\mu+1)\,\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)}t^{\mu+\nu}\, M_{k+\lambda,\mu+\nu+\frac{1}{2}}(t)$$
$$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$
BU 128(14), ET II 402(7)

6. $$\int_0^1 x^{\beta-1}(1-x)^{\sigma-\beta-1}\ {}_1F_1(\alpha;\beta;\lambda x)\ {}_1F_1[\sigma-\alpha;\sigma-\beta;\mu(1-x)]\,dx = \frac{\Gamma(\beta)\,\Gamma(\sigma-\beta)}{\Gamma(\sigma)}e^{\lambda}\ {}_1F_1\left(\alpha;\sigma;\mu-\lambda\right)$$
$[0<\operatorname{Re}\beta<\operatorname{Re}\sigma]$ ET II 402(3)

7.62–7.63 Combinations of confluent hypergeometric functions and exponentials

7.621

1. $$\int_0^\infty e^{-st}t^{\alpha}\,M_{\mu,\nu}(t)\,dt = \frac{\Gamma\left(\alpha+\nu+\frac{3}{2}\right)}{\left(\frac{1}{2}+s\right)^{\alpha+\nu+\frac{3}{2}}}\,F\left(\alpha+\nu+\frac{3}{2},-\mu+\nu+\frac{1}{2};2\nu+1;\frac{2}{2s+1}\right)$$
$\left[\operatorname{Re}\left(\alpha+\mu+\frac{3}{2}\right)>0,\quad \operatorname{Re}s>\frac{1}{2}\right]$
BU 118(1), MO 176a, EH I 270(12)a

2. $$\int_0^\infty e^{-st}t^{\mu-\frac{1}{2}}\,M_{\lambda,\mu}(qt)\,dt = q^{\mu+\frac{1}{2}}\,\Gamma(2\mu+1)\left(s-\tfrac{1}{2}q\right)^{\lambda-\mu-\frac{1}{2}}\left(s+\tfrac{1}{2}q\right)^{-\lambda-\mu-\frac{1}{2}}$$
$\left[\operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}s>\frac{|\operatorname{Re}q|}{2}\right]$
BU 119(4c), MO 176a, EH I 271(13)a

3. $$\int_0^\infty e^{-st}t^{\alpha}\,W_{\lambda,\mu}(qt)\,dt = \frac{\Gamma\left(\alpha+\mu+\frac{3}{2}\right)\Gamma\left(\alpha-\mu+\frac{3}{2}\right)q^{\mu+\frac{1}{2}}}{\Gamma(\alpha-\lambda+2)}\left(s+\frac{1}{2}q\right)^{-\alpha-\mu-\frac{3}{2}} \times F\left(\alpha+\mu+\frac{3}{2},\mu-\lambda+\frac{1}{2};\alpha-\lambda+2;\frac{2s-q}{2s+q}\right)$$
$\left[\operatorname{Re}\left(\alpha\pm\mu+\frac{3}{2}\right)>0,\quad \operatorname{Re}s>-\frac{q}{2},\quad q>0\right]$ EH I 271(14)a, BU 121(6), MO 176

4. $$\int_0^\infty e^{-st}t^{b-1}\ {}_1F_1(a;c;kt)\,dt = \Gamma(b)s^{-b}\,F\left(a,b;c;ks^{-1}\right) \qquad [|s|>|k|]$$
$$= \Gamma(b)(s-k)^{-b}\,F\left(c-a,b;c;\frac{k}{k-s}\right) \qquad [|s-k>|k||]$$
$[\operatorname{Re}b>0,\quad \operatorname{Re}s>\max(0,\operatorname{Re}k)]$ EH I 269(5)

5. $$\int_0^\infty t^{c-1}\ {}_1F_1(a;c;t)e^{-st}\,dt = \Gamma(c)s^{-c}\left(1-s^{-1}\right)^{-a} \qquad [\operatorname{Re}c>0,\quad \operatorname{Re}s>1]$$
EH I 270(6)

6. $$\int_0^\infty t^{b-1}\Psi\left(a,c;t\right)e^{-st}\,dt = \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}\,F\left(b,b-c+1;a+b-c+1;1-s\right)$$
$[\operatorname{Re}b>0,\quad \operatorname{Re}c<\operatorname{Re}b+1,\quad |1-s|<1]$
$$= \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}s^{-b}\,F\left(a,b;a+b-c+1;1-s^{-1}\right)$$
$\left[\operatorname{Re}s>\frac{1}{2}\right]$
EH I 270(7)

7. $$\int_0^\infty e^{-\frac{b}{2}x}x^{\nu-1}\,M_{\kappa,\mu}(bx)\,dx = \frac{\Gamma(1+2\mu)\,\Gamma\left(\kappa-\nu\right)\Gamma\left(\frac{1}{2}+\mu+\nu\right)}{\Gamma\left(\frac{1}{2}+\mu+\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}b^{\nu}$$
$\left[\operatorname{Re}\left(\nu+\frac{1}{2}+\mu\right)>0,\quad \operatorname{Re}\left(\kappa-\nu\right)>0\right]$
BU 119(3)a, ET I 215(11)a

8. $$\int_0^\infty e^{-sx}\, M_{\kappa,\mu}(x)\frac{dx}{x} = \frac{2\,\Gamma(1+2\mu)e^{-i\pi\kappa}}{\Gamma\left(\frac{1}{2}+\mu+\kappa\right)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} Q^{\kappa}_{\mu-\frac{1}{2}}(2s)$$
$$\left[\operatorname{Re}\left(\tfrac{1}{2}+\mu\right)>0, \quad \operatorname{Re} s>\tfrac{1}{2}\right]$$
BU 119(4a)

9. $$\int_0^\infty e^{-sx}\, W_{\kappa,\mu}(x)\frac{dx}{x} = \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} P^{\kappa}_{\mu-\frac{1}{2}}(2s)$$
$$\left[\operatorname{Re}\left(\tfrac{1}{2}\pm\mu\right)>0, \quad \operatorname{Re} s>-\tfrac{1}{2}\right]$$
BU 121(7)

10. $$\int_0^\infty x^{k+2\mu-1}e^{-\frac{3}{2}x}\, W_{k,\mu}(x)\,dx = \frac{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2k+6\mu+5)\right]}{\left(k+3\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2\mu-2k+3)\right]}$$
$$\left[\operatorname{Re}(k+\mu)>-\tfrac{1}{2}, \quad \operatorname{Re}(k+3\mu)>-\tfrac{1}{2}\right]$$
BU 122(8a), ET II 406(23)

11. $$\int_0^\infty e^{-\frac{1}{2}x}x^{\nu-1}\, W_{\kappa,\mu}(x)\,dx = \frac{\Gamma\left(\nu+\frac{1}{2}-\mu\right)\Gamma\left(\nu+\frac{1}{2}+\mu\right)}{\Gamma\left(\nu-\kappa+1\right)}$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\pm\mu\right)>0\right]$$
BU 122(8b)

12. $$\int_0^\infty e^{\frac{1}{2}x}x^{\nu-1}\, W_{\kappa,\mu}(x)\,dx = \Gamma\left(-\kappa-\mu\right)\frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\pm\mu\right)>0, \quad \operatorname{Re}\left(\kappa+\nu\right)<0\right]$$
BU 122(8c)a

7.622

1. $$\int_0^\infty e^{-st}t^{c-1}\ {}_1F_1(a;c;t)\ {}_1F_1\left(\alpha;c;\lambda t\right)\,dt$$
$$= \Gamma(c)(s-1)^{-a}(s-\lambda)^{-\alpha}s^{a+\alpha-c}\,F\left[a,\alpha;c;\lambda(s-1)^{-1}(s-\lambda)^{-1}\right]$$
$$\left[\operatorname{Re} c>0, \quad \operatorname{Re} s>\operatorname{Re}\lambda+1\right]$$
EH I 287(22)

2. $$\int_0^\infty e^{-t}t^{\rho}\ {}_1F_1(a;c;t)\Psi\left(a';c';\lambda t\right)\,dt$$
$$= C\frac{\Gamma(c)\,\Gamma(\beta)}{\Gamma(\gamma)}\lambda^{\sigma}\,F\left(c-a,\beta;\gamma;1-\lambda^{-1}\right),$$
$$\rho=c-1, \quad \sigma=-c, \quad \beta=c-c'+1, \quad \gamma=c-a+a'-c'+1, \quad C=\frac{\Gamma\left(a'-a\right)}{\Gamma\left(a'\right)}, \text{ or}$$

$$\rho=c+c'-2, \quad \sigma=1-c-c', \quad \beta=c+c'-1, \quad \gamma=a'-a+c, \quad C=\frac{\Gamma\left(a'-a-c'+1\right)}{\Gamma\left(a'-c'+1\right)}$$
EH I 287(24)

3. $$\int_0^\infty x^{\nu-1}e^{-bx}\,M_{\lambda_1,\mu_1-\frac12}(a_1x)\dots M_{\lambda_n,\mu_n-\frac12}(a_nx)\,dx$$
$$= a_1^{\mu_1}\dots a_n^{\mu_n}(b+A)^{-\nu-M}\,\Gamma(\nu+M)$$
$$\times F_A\left(\nu+M;\mu_1-\lambda_1,\dots,\mu_n-\lambda_n;2\mu_1,\dots,2\mu_n;\frac{a_1}{b+A},\dots,\frac{a_n}{b+A}\right),$$
$$M=\mu_1+\dots+\mu_n,\qquad A=\tfrac12(a_1+\dots+a_n)$$
$$\left[\operatorname{Re}(\nu+M)>0,\quad \operatorname{Re}\left(b\pm\tfrac12a_1\pm\dots+\tfrac12a_n\right)>0\right]$$ ET I 216(14)

7.623

1. $$\int_0^\infty e^{-x}x^{c+n-1}(x+y)^{-1}\;{}_1F_1(a;c;x)\,dx=(-1)^n\,\Gamma(c)\,\Gamma(1-a)y^{c+n-1}\Psi(c-a,c;y)$$
$$[-\operatorname{Re}c<n<1-\operatorname{Re}a,\quad n=0,1,2,\dots,\quad |\arg y|<\pi]$$ EH I 285(16)

2. $$\int_0^t x^{-1}(t-x)^{k-1}e^{\frac12(t-x)}\,M_{k,\mu}(x)\,dx=\frac{\Gamma(k)\,\Gamma(2\mu+1)}{\Gamma\left(k+\mu+\frac12\right)}\pi^{\frac12}t^{k-\frac12}l_\mu\left(\frac12t\right)$$
$$\left[\operatorname{Re}k>0,\quad \operatorname{Re}\mu>-\tfrac12\right]$$ ET II 402(5)

3. $$\int_0^t x^{k-1}(t-x)^{\lambda-1}e^{\frac12(t-x)}\,M_{k+\lambda,\mu}(x)\,dx=\frac{\Gamma(\lambda)\,\Gamma\left(k+\mu+\frac12\right)t^{k+\lambda-1}}{\Gamma\left(k+\lambda+\mu+\frac12\right)}\,M_{k,\mu}(t)$$
$$\left[\operatorname{Re}(k+\mu)>-\tfrac12,\quad \operatorname{Re}\lambda>0\right]$$
ET II 402(6)

4. $$\int_0^t x^{-k-\lambda-1}(t-x)^{\lambda-1}e^{\frac12x}\,W_{k,\mu}(x)\,dx=\frac{\Gamma(\lambda)\,\Gamma\left(\frac12-k-\lambda+\mu\right)\Gamma\left(\frac12-k-\lambda-\mu\right)}{t^{k+1}\,\Gamma\left(\frac12-k+\mu\right)\Gamma\left(\frac12-k-\mu\right)}\,W_{k+\lambda,\mu}(t)$$
$$\left[\operatorname{Re}\lambda>0,\quad \operatorname{Re}(k+\lambda)<\tfrac12-|\operatorname{Re}\mu|\right]$$
ET II 405(21)

5. $$\int_1^\infty (x-1)^{\mu-1}x^{\lambda-\frac12}e^{\frac12ax}\,W_{k,\lambda}(ax)\,dx=\frac{\Gamma(\mu)\,\Gamma\left(\frac12-k-\lambda-\mu\right)}{\Gamma\left(\frac12-k-\lambda\right)}a^{-\frac12\mu}e^{\frac12a}\,W_{k+\frac12\mu,\lambda+\frac12\mu}(a)$$
$$\left[|\arg(a)|<\tfrac32\pi,\quad 0<\operatorname{Re}\mu<\tfrac12-\operatorname{Re}(k+\lambda)\right]$$ ET II 211(72)a

6.[11] $$\int_1^\infty (x-1)^{\mu-1}x^{\mu-\frac12}e^{-\frac12ax}\,W_{2\mu+\frac12,\lambda}(ax)\,dx=\Gamma(\mu)e^{-\frac12a}\,W_{\mu+\frac12,\lambda}(a)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}a>0]$$ ET II 211(74)a

7. $$\int_1^\infty (x-1)^{\mu-1}x^{k-\mu-1}e^{-\frac12ax}\,W_{k,\lambda}(ax)\,dx=\Gamma(\mu)e^{-\frac12a}\,W_{k-\mu,\lambda}(a)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}a>0]$$ ET II 211(73)a

8. $$\int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax}\, W_{k,\lambda}(ax)\, dx$$
$$= \Gamma(\mu) e^{-\frac{1}{2}a} \sec[(k-\mu-\lambda)\pi]$$
$$\times \left\{ \sin(\mu\pi) \frac{\Gamma\left(k-\mu+\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda+1)} M_{k-\mu,\lambda}(a) + \cos[(k-\lambda)\pi]\, W_{k-\mu,\lambda}(a) \right\}$$
$$\left[0 < \operatorname{Re}\mu < \operatorname{Re}k - |\operatorname{Re}\lambda| + \tfrac{1}{2}\right] \quad \text{ET II 200(93)a}$$

7.624

1. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, M_{k,\mu}(x)\, dx$$
$$= \frac{-\sigma\,\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+k+\mu\right)} G^{23}_{34}\left(a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, -\sigma, \sigma, \frac{1}{2}-\mu+\rho \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\rho) > -\tfrac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > 0\right] \quad \text{ET II 403(8)}$$

2. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = -\pi^{-\frac{1}{2}} \sigma a^\sigma G^{32}_{34}\left(a \left| \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \tfrac{1}{2}\right] \quad \text{ET II 406(24)}$$

3. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx$$
$$= -\frac{\sigma \pi^{-\frac{1}{2}} a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G^{33}_{34}\left(a \left| \begin{matrix} \frac{1}{2}, 1, 1+k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \tfrac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < 0\right] \quad \text{ET II 406(25)}$$

4. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, M_{k,\mu}(x)\, dx$$
$$= \frac{\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+k+\mu\right)} G^{23}_{34}\left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k-\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}(\rho+\mu) > -\tfrac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > -\tfrac{1}{2}\right] \quad \text{ET II 403(9)}$$

5. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx$$
$$= \frac{\pi^{-\frac{1}{2}} a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G^{33}_{34}\left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}+k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \tfrac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < \tfrac{1}{2}\right] \quad \text{ET II 406(26)}$$

6. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = \pi^{-\frac{1}{2}} a^\sigma G^{32}_{34}\left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right.\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \tfrac{1}{2}\right] \quad \text{ET II 406(27)}$$

7.625

1. $$\int_0^\infty x^{\rho-1}\exp\left[-\tfrac{1}{2}(\alpha+\beta)x\right] M_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\,dx$$
$$= \frac{\Gamma(1+\mu+\nu+\rho)\,\Gamma(1+\mu-\nu+\rho)}{\Gamma\left(\frac{3}{2}-\lambda+\mu+\rho\right)}\alpha^{\mu+\frac{1}{2}}\beta^{-\mu-\rho-\frac{1}{2}}$$
$$\times\ {}_3F_2\left(\frac{1}{2}+k+\mu, 1+\mu+\nu+\rho, 1+\mu-\nu+\rho; 2\mu+1, \frac{3}{2}-\lambda+\mu+\rho; -\frac{\alpha}{\beta}\right)$$
$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}(\rho+\mu)>|\operatorname{Re}\nu|-1]$ ET II 410(43)

2. $$\int_0^\infty x^{\rho-1}\exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\,dx$$
$$=\beta^{-\rho}\left[\Gamma\left(\tfrac{1}{2}-k+\mu\right)\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-\lambda+\nu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\nu\right)\right]^{-1}$$
$$\times G^{33}_{33}\left(\frac{\beta}{\alpha}\,\middle|\,\begin{matrix}\frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho\\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, -k\end{matrix}\right)$$
$[|\operatorname{Re}\mu|+|\operatorname{Re}\nu|<\operatorname{Re}\rho+1,\quad \operatorname{Re}(k+\lambda+\rho)<0]$ ET II 410(44)a

3. $$\int_0^\infty x^{\rho-1}\exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\,dx = \beta^{-\rho}\, G^{22}_{33}\left(\frac{\beta}{\alpha}\,\middle|\,\begin{matrix}\frac{1}{2}+\mu, \frac{1}{2}-\nu, 1-\lambda+\rho\\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k\end{matrix}\right)$$
ET II 411(46)

4. $$\int_0^\infty x^{\rho-1}\exp\left[-\frac{1}{2}(\alpha-\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\,dx$$
$$=\beta^{-\rho}\left[\Gamma\left(\tfrac{1}{2}-\lambda+\nu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\nu\right)\right]^{-1} G^{23}_{33}\left(\frac{\beta}{\alpha}\,\middle|\,\begin{matrix}\frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho\\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k\end{matrix}\right)$$
$[\operatorname{Re}\alpha>0,\quad |\operatorname{Re}\mu|+|\operatorname{Re}\nu|<\operatorname{Re}\rho+1]$ ET II 411(45)

7.626

1. $$\int_0^1\left[\frac{k}{x}-\frac{1}{4}(\xi+\eta)\exp\left[-\frac{1}{2}(\xi+\eta)x\right]x^c\right]\ {}_1F_1(a;c;\xi x)\ {}_1F_1(a;c;\eta x)\ dx$$
$$=0 \qquad [\xi\neq\eta,\quad \operatorname{Re}c>0]$$
$$=\frac{a}{\xi}e^{-\xi}\left[\ {}_1F_1(a+1;c;\xi)\right]^2 \qquad [\xi=\eta,\quad \operatorname{Re}c>0]$$
[where ξ and η are any two zeros of the function ${}_1F_1(a;c;x)$] EH I 285

2. $$\int_1^\infty\left[\frac{k}{x}-\frac{1}{4}(\xi+\eta)\right]e^{-\frac{1}{2}(\xi+\eta)x}x^c\Psi(a,c;\xi x)\,\Psi(a,c;\eta x)\,dx=0 \qquad [\xi\neq\eta]\,;$$
$$=-\xi^{-1}e^{-\xi}[\Psi(a-1,c;\xi)]^2 \qquad [\xi=\eta]$$
[where ξ and η are any two zeros of the function $\Psi(a,c;x)$] EH I 286

7.627

1. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx=\frac{\Gamma(2\lambda)\,\Gamma\left(\frac{1}{2}-k+\mu-2\lambda\right)}{\Gamma\left(\frac{1}{2}-k+\mu\right)}a^{\lambda-\mu-\frac{1}{2}}\,W_{k+\lambda,\mu-\lambda}(a)$$
$$\left[|\arg a|<\pi,\quad 0<2\operatorname{Re}\lambda<\frac{1}{2}-\operatorname{Re}(k+\mu)\right]$$ ET II 411(50)

2. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\,M_{k,\mu}^{-\frac{1}{2}x}(a+x)\,dx$$
$$=\frac{\Gamma(2\lambda)\,\Gamma(2\mu+1)\,\Gamma\left(k+\mu-2\lambda+\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(1-2\lambda+2\mu)}a^{\lambda-\mu-\frac{1}{2}}\,M_{k-\lambda,\mu-\lambda}(a)$$
$$\left[\operatorname{Re}\lambda>0,\quad \operatorname{Re}(k+\mu-2\lambda)>-\tfrac{1}{2}\right]$$ ET II 405(20)

3. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx=\Gamma(2\lambda)a^{\lambda-\mu-\frac{1}{2}}\,W_{k-\lambda,\mu-\lambda}(a)$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\lambda>0]$$ ET II 411(47)

4. $$\int_0^\infty x^{\lambda-1}(a+x)^{k-\lambda-1}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx=\Gamma(\lambda)a^{k-1}\,W_{k-\lambda,\mu}(a)$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\lambda>0]$$ ET II 411(48)

5. $$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx=\Gamma(\rho)a^\rho e^{\frac{1}{2}a}\,G_{23}^{30}\left(a\left|\begin{matrix}0,1-k-\sigma\\-\rho,\frac{1}{2}+\mu-\sigma,\frac{1}{2}-\mu-\sigma\end{matrix}\right.\right)$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\rho>0]$$ ET II 411(49)

6. $$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx$$
$$=\frac{\Gamma(\rho)a^\rho e^{-\frac{1}{2}a}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)}\,G_{23}^{31}\left(a\left|\begin{matrix}k-\sigma+1,0\\-\rho,\frac{1}{2}+\mu-\sigma,\frac{1}{2}-\mu-\sigma\end{matrix}\right.\right)$$
$$[|\arg a|<\pi,\quad 0<\operatorname{Re}\rho<\operatorname{Re}(\sigma-k)]$$ ET II 412(51)

7. $$\int_0^\infty e^{-\frac{1}{2}(a+x)}\frac{(a+x)^{2\kappa-1}}{(ax)^\kappa}\,W_{\kappa,\mu}(x)\frac{dx}{x}=\frac{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}{a\,\Gamma(1-2\kappa)}\,W_{\kappa,\mu}(a)$$
$$\left[\operatorname{Re}\left(\tfrac{1}{2}\pm\mu-\kappa\right)>0\right]$$ BU 126(7a)

8. $$\int_0^\infty e^{-\frac{1}{2}x}x^{\gamma+\alpha-1}\,M_{\kappa,\mu}(x)\frac{dx}{(x+a)^\alpha}$$
$$=\frac{\Gamma(1+2\mu)\,\Gamma\left(\frac{1}{2}+\mu+\gamma\right)\Gamma(\kappa-\gamma)}{\Gamma\left(\frac{1}{2}+\mu-\gamma\right)\Gamma\left(\frac{1}{2}+\mu+\kappa\right)}\,{}_2F_2\left(\alpha,\kappa-\gamma;\frac{1}{2}+\mu-\gamma,\frac{1}{2}-\mu-\gamma;a\right)$$
$$+\frac{\Gamma\left(\alpha+\gamma+\frac{1}{2}+\mu\right)\Gamma\left(-\gamma-\frac{1}{2}-\mu\right)}{\Gamma(\alpha)}a^{\gamma+\frac{1}{2}+\mu}$$
$$\times\,{}_2F_2\left(\alpha+\gamma+\mu+\frac{1}{2},\kappa+\mu+\frac{1}{2};1+2\mu,\frac{3}{2}+\mu+\gamma;a\right)$$
$$\left[\operatorname{Re}\left(\gamma+\alpha+\tfrac{1}{2}+\mu\right)>0,\quad \operatorname{Re}(\gamma-\kappa)<0\right]$$ BU 126(8)a

9. $$\int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}}\, M_{\kappa,\mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a}\, \Gamma(1+2\mu)\, \Gamma\left(\frac{1}{2}-\mu+\kappa\right) W_{-\kappa,\mu}(a)$$
$$\left[n=0,1,2,\ldots, \quad \operatorname{Re}\left(\mu+1+\frac{n}{2}\right)>0, \quad \operatorname{Re}\left(\kappa-\mu-\frac{1}{2}\right)<n, \quad |\arg a|<\pi\right]$$ BU 127(10a)a

7.628

1. $$\int_0^\infty e^{-st} e^{-t^2} t^{2c-2}\, {}_1F_1\left(a;c;t^2\right)\, dt = 2^{1-2c}\, \Gamma(2c-1)\, \Psi\left(c-\frac{1}{2}, a+\frac{1}{2}; \frac{1}{4}s^2\right)$$
$$\left[\operatorname{Re} c > \tfrac{1}{2}, \quad \operatorname{Re} s > 0\right]$$ EH I 270(11)

2. $$\int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st}\, M_{-3\nu,\nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}}\, \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8}\, K_{2\nu}\left(\frac{as^2}{8}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{4}, \quad \operatorname{Re} s > 0\right]$$ ET I 215(12)

3. $$\int_0^\infty t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st}\, M_{\lambda,\mu}\left(\frac{t^2}{a}\right) dt$$
$$= 2^{-3\mu-\lambda}\, \Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}}\, W_{-\frac{1}{2}(\lambda+3\mu),\frac{1}{2}(\lambda-\mu)}\left(\frac{as^2}{4}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\tfrac{1}{4}, \quad \operatorname{Re} s > 0\right]$$ ET I 215(13)

7.629

1.[8] $$\int_0^\infty t^k \exp\left(\frac{a}{2t}\right) e^{-st}\, W_{k,\mu}\left(\frac{a}{t}\right) dt = 2^{1-2k} \sqrt{a} s^{-k-\frac{1}{2}}\, S_{2k,2\mu}\left(2\sqrt{a}s\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\tfrac{1}{2}, \quad \operatorname{Re} s > 0\right]$$ ET I 217(21)

2. $$\int_0^\infty t^{-k} \exp\left(-\frac{a}{2t}\right) e^{-st}\, W_{k,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{a} s^{k-\frac{1}{2}}\, K_{2\mu}\left(2\sqrt{as}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0\right]$$ ET I 217(22)

7.631

1. $$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\alpha^{-1}x - \beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right)\, W_{\lambda,\nu}\left(\beta x^{-1}\right)\, dx$$
$$= \beta^\rho \left[\Gamma\left(\tfrac{1}{2}-k+\mu\right)\Gamma\left(\tfrac{1}{2}-k-\mu\right)\right]^{-1}$$
$$\times G^{41}_{24}\left(\frac{\beta}{\alpha}\,\middle|\, \begin{matrix} 1+k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, \quad \frac{1}{2}+\nu-\rho, \quad \frac{1}{2}-\nu-\rho \end{matrix}\right)$$
$$\left[|\arg \alpha| < \tfrac{3}{2}\pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(k+\rho) < -|\operatorname{Re} \nu| - \tfrac{1}{2}\right]$$ ET II 412(55)

2. $$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\alpha^{-1}x - \beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right)\, W_{\lambda,\nu}\left(\beta x^{-1}\right)\, dx$$
$$= \beta^\rho \left[\Gamma\left(\tfrac{1}{2}-k+\mu\right)\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-\lambda+\nu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\nu\right)\right]^{-1}$$
$$\times G^{42}_{24}\left(\frac{\beta}{\alpha}\,\middle|\, \begin{matrix} 1+k, & 1+\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, \quad \frac{1}{2}+\nu-\rho, \quad \frac{1}{2}-\nu-\rho \end{matrix}\right)$$
$$\left[|\arg \alpha| < \tfrac{3}{2}\pi, \quad |\arg \beta| < \tfrac{3}{2}\pi, \quad \operatorname{Re}(\lambda-\rho) < \tfrac{1}{2} - |\operatorname{Re} \mu|, \quad \operatorname{Re}(k+\rho) < \tfrac{1}{2} - |\operatorname{Re} \nu|\right]$$
ET II 412(57)

3. $$\int_0^\infty x^{\rho-1}\exp\left[\frac{1}{2}\left(\alpha^{-1}x+\beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right) W_{\lambda,\nu}\left(\beta x^{-1}\right)\,dx$$
$$=\beta^\rho\, G^{40}_{24}\left(\frac{\beta}{\alpha}\,\middle|\,\begin{matrix}1-k, & 1-\lambda-\rho\\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, \quad \frac{1}{2}+\nu-\rho, \quad \frac{1}{2}-\nu-\rho\end{matrix}\right)$$
$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0]$ ET II 412(54)

7.632 $$\int_0^\infty e^{-st}\left(e^t-1\right)^{\mu-\frac{1}{2}}\exp\left(-\frac{1}{2}\lambda e^t\right) M_{k,\mu}\left(\lambda e^t-\lambda\right)\,dt$$
$$=\frac{\Gamma(2\mu+1)\,\Gamma\left(\frac{1}{2}+k-\mu+s\right)}{\Gamma(s+1)}\,W_{-k-\frac{1}{2}s,\mu-\frac{1}{2}s}(\lambda)$$
$\left[\operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}s>\operatorname{Re}(\mu-k)-\frac{1}{2}\right]$ ET I 216(15)

7.64 Combinations of confluent hypergeometric and trigonometric functions

7.641 $$\int_0^\infty \cos(ax)\;{}_1F_1(\nu+1;1;ix)\;{}_1F_1\left(\nu+1;1;-ix\right)\,dx$$
$$=-a^{-1}\sin(\nu\pi)\,P_\nu\left(2a^{-2}-1\right) \qquad [0<a<1]\,;$$
$$=0 \qquad [1<a<\infty]$$
$[-1<\operatorname{Re}\nu<0]$ ET II 402(4)

7.642[11] $$\int_0^\infty \cos(2xy)\;{}_1F_1\left(a;c;-x^2\right)\,dx=\frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma(c)}{\Gamma(a)}|y|^{2\alpha-1}e^{-y^2}\Psi\left(c-\tfrac{1}{2},a+\tfrac{1}{2};y^2\right)$$ EH I 285(12)

7.643

1. $$\int_0^\infty x^{4\nu}e^{-\frac{1}{2}x^2}\sin(bx)\;{}_1F_1\left(\frac{1}{2}-2\nu;2\nu+1;\frac{1}{2}x^2\right)\,dx=\sqrt{\frac{\pi}{2}}b^{4\nu}c^{-\frac{1}{2}b^2}\;{}_1F_1\left(\frac{1}{2}-2\nu;1+2\nu;\frac{1}{2}b^2\right)$$
$\left[b>0, \quad \operatorname{Re}\nu>-\frac{1}{4}\right]$ ET I 115(5)

2. $$\int_0^\infty x^{2\nu-1}e^{-\frac{1}{4}x^2}\sin(bx)\,M_{3\nu,\nu}\left(\frac{1}{2}x^2\right)\,dx=\sqrt{\frac{\pi}{2}}b^{2\nu-1}e^{-\frac{1}{4}b^2}\,M_{3\nu,\nu}\left(\frac{1}{2}b^2\right)$$
$\left[b>0, \quad \operatorname{Re}\nu>-\frac{1}{4}\right]$ ET I 116(10)

3. $$\int_0^\infty x^{-2\nu-1}e^{\frac{1}{4}x^2}\cos(bx)\,W_{3\nu,\nu}\left(\frac{1}{2}x^2\right)\,dx=\sqrt{\frac{\pi}{2}}b^{-2\nu-1}e^{\frac{1}{4}b^2}\,W_{3\nu,\nu}\left(\frac{1}{2}b^2\right)$$
$\left[\operatorname{Re}\nu<\frac{1}{4}, \quad b>0\right]$ ET I 61(7)

4. $$\int_0^\infty x^{-2\nu}e^{\frac{1}{4}x^2}\sin(bx)\,W_{3\nu-1,\nu}\left(\frac{1}{2}x^2\right)\,dx=\sqrt{\frac{\pi}{2}}b^{-2\nu}e^{\frac{1}{4}b^2}\,W_{3\nu-1,\nu}\left(\frac{1}{2}b^2\right)$$
$\left[\operatorname{Re}\nu<\frac{1}{2}, \quad b>0\right]$ ET I 116(9)

7.644

1.[11] $$\int_0^\infty x^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\sin\left(2ax^{\frac{1}{2}}\right)M_{k,\mu}(x)\,dx=\pi^{\frac{1}{2}}a^{k+\mu-1}\frac{\Gamma(3-2\mu)}{\Gamma\left(\frac{1}{2}+k+\mu\right)}\exp\left(-\frac{a^2}{2}\right)W_{\rho,\sigma}\left(a^2\right),$$
$2\rho=k-3\mu+1, \qquad 2\sigma=k+\mu-1 \qquad [a>0, \quad \operatorname{Re}(k+\mu)>0]$ ET II 403(10)

2. $$\int_0^\infty x^{\rho-1}\sin\left(cx^{\frac12}\right)e^{-\frac12 x}\,W_{k,\mu}(x)\,dx=\frac{c\,\Gamma(1+\mu+\rho)\,\Gamma(1-\mu+\rho)}{\Gamma\left(\frac32-k+\rho\right)}\times{}_2F_2\left(1+\mu+\rho,1-\mu+\rho;\frac32,\frac32-k+\rho;-\frac{c^2}{4}\right)$$
$[\operatorname{Re}\rho>|\operatorname{Re}\mu|-1]$ ET II 407(28)

3. $$\int_0^\infty x^{\rho-1}\sin\left(cx^{\frac12}\right)e^{\frac12 x}\,W_{k,\mu}(x)\,dx=\frac{\pi^{\frac12}}{\Gamma\left(\frac12-k+\mu\right)\Gamma\left(\frac12-k-\mu\right)}\,G^{22}_{23}\left(\frac{c^2}{4}\,\middle|\,\begin{matrix}\frac12+\mu-\rho,\frac12-\mu-\rho\\ \frac12,-k-\rho,0\end{matrix}\right)$$
$\left[c>0,\quad \operatorname{Re}\rho>|\operatorname{Re}\mu|-1,\quad \operatorname{Re}(k+\rho)<\frac12\right]$ ET II 407(29)

4. $$\int_0^\infty x^{\rho-1}\cos\left(cx^{\frac12}\right)e^{-\frac12 x}\,W_{k,\mu}(x)\,dx=\frac{\Gamma\left(\frac12+\mu+\rho\right)\Gamma\left(\frac12-\mu+\rho\right)}{\Gamma(1-k+\rho)}\times{}_2F_2\left(\frac12+\mu+\rho,\frac12-\mu+\rho;\frac12,1-k+\rho;-\frac{c^2}{4}\right)$$
$\left[\operatorname{Re}\rho>|\operatorname{Re}\mu|-\frac12\right]$ ET II 407(30)

5. $$\int_0^\infty x^{\rho-1}\cos\left(cx^{\frac12}\right)e^{\frac12 x}\,W_{k,\mu}(x)\,dx=\frac{\pi^{\frac12}}{\Gamma\left(\frac12-k+\mu\right)\Gamma\left(\frac12-k-\mu\right)}\,G^{22}_{23}\left(\frac{c^2}{4}\,\middle|\,\begin{matrix}\frac12+\mu-\rho,\frac12-\mu-\rho\\ 0,-k-\rho,\frac12\end{matrix}\right)$$
$\left[c>0,\quad \operatorname{Re}\rho>|\operatorname{Re}\mu|-\frac12,\quad \operatorname{Re}(k+\rho)<\frac12\right]$ ET II 407(31)

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.651

1. $$\int_0^\infty J_\nu(xy)\,M_{-\frac12\mu,\frac12\nu}(ax)\,W_{\frac12\mu,\frac12\nu}(ax)\,dx=ay^{-\mu-1}\frac{\Gamma(\nu+1)}{\Gamma\left(\frac12-\frac12\mu+\frac12\nu\right)}\left[a+\left(a^2+y^2\right)^{\frac12}\right]^{\mu}\left(a^2+y^2\right)^{-\frac12}$$
$\left[y>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu<\frac12,\quad \operatorname{Re}a>0\right]$ ET II 85(19)

2. $$\int_0^\infty M_{k,\frac12\nu}(-iax)\,M_{-k,\frac12\nu}(-iax)\,J_\nu(xy)\,dx$$
$$=\frac{ae^{-\frac12(\nu+1)\pi i}\left[\Gamma(1+\nu)\right]^2}{\Gamma\left(\frac12+k+\frac12\nu\right)\Gamma\left(\frac12-k+\frac12\nu\right)}y^{-1-2k}\times\left(a^2-y^2\right)^{-\frac12}\left\{\left[a+\left(a^2-y^2\right)^{\frac12}\right]^{2k}+\left[a-\left(a^2-y^2\right)^{\frac12}\right]^{2k}\right\}\qquad[0<y<a]\,;$$
$$=0\qquad[a<y<\infty]$$
$\left[a>0,\quad \operatorname{Re}\nu>-1,\quad |\operatorname{Re}k|<\frac14\right]$ ET II 85(18)

7.652 $$\int_0^\infty M_{-\mu,\frac{1}{2}\nu}\left\{a\left[\left(b^2+x^2\right)^{\frac{1}{2}}-b\right]\right\} W_{\mu,\frac{1}{2}\nu}\left\{a\left[\left(b^2+x^2\right)^{\frac{1}{2}}+b\right]\right\} J_\nu(xy)\,dx$$
$$=\frac{ay^{-2\mu-1}\,\Gamma(1+\nu)\left[\left(a^2+y^2\right)^{\frac{1}{2}}+a\right]^{2\mu}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\mu\right)\left(A^2+Y^2\right)^{\frac{1}{2}}}\exp\left[-b\left(a^2+y^2\right)^{\frac{1}{2}}\right]$$
$$\left[y>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu<\tfrac{1}{4},\quad \operatorname{Re}a>0,\quad \operatorname{Re}b>0\right]$$ ET II 87(29)

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

1. $$\int_0^\infty x^{-1}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_0(xy)\,dx$$
$$=e^{-ik\pi}\frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu+k\right)}\,P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]Q^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$$\left[y>0,\quad \operatorname{Re}a>0,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}k<\tfrac{3}{4}\right]$$ ET II 18(44)

2. $$\int_0^\infty x^{-1}\,W_{k,\mu}(ax)\,W_{-k,\mu}(ax)\,J_0(xy)\,dx=\frac{1}{2}\pi\cos(\mu\pi)\,P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]P^{-k}_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$$\left[y>0,\quad \operatorname{Re}a>0,\quad |\operatorname{Re}\mu|<\tfrac{1}{2}\right]$$
ET II 18(45)

3. $$\int_0^\infty x^{2\mu-\nu}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_\nu(xy)\,dx$$
$$=2^{2\mu-\nu+2k}a^{2k}y^{\nu-2\mu-2k-1}\frac{\Gamma(2\mu+1)}{\Gamma\left(\nu-k-\mu+\frac{1}{2}\right)}$$
$$\times\,{}_3F_2\left(\frac{1}{2}-k,1-k,\frac{1}{2}-k+\mu;1-2k,\frac{1}{2}-k-\mu+\nu;-\frac{y^2}{a^2}\right)$$
$$\left[y>0,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}a>0,\quad \operatorname{Re}(2\mu+2k-\nu)<\tfrac{1}{2}\right]$$ ET II 85(20)

4. $$\int_0^\infty x^{2\rho-\nu}\,W_{k,\mu}(iax)\,W_{k,\mu}(-iax)\,J_\nu(xy)\,dx$$
$$=2^{2\rho-\nu}y^{\nu-2\rho-1}\pi^{-\frac{1}{2}}\left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1}G^{24}_{44}\left(\frac{y^2}{a^2}\left|\begin{matrix}\frac{1}{2},0,\frac{1}{2}-\mu,\frac{1}{2}+\mu\\ \rho+\frac{1}{2},-k,k,\rho-\nu+\frac{1}{2}\end{matrix}\right.\right)$$
$$\left[y>0,\quad \operatorname{Re}a>0,\quad \operatorname{Re}\rho>|\operatorname{Re}\mu|-1,\quad \operatorname{Re}(2\rho+2k-\nu)<\tfrac{1}{2}\right]$$ ET II 86(23)a

5. $$\int_0^\infty x^{2\rho-\nu}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_\nu(xy)\,dx$$
$$=\frac{2^{2\rho-\nu}\,\Gamma(2\mu+1)}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}-k+\mu\right)}y^{\nu-2\rho-1}\,G^{23}_{44}\left(\frac{y^2}{a^2}\left|\begin{matrix}\frac{1}{2},0,\frac{1}{2}-\mu,\frac{1}{2}+\mu\\ \rho+\frac{1}{2},-k,k,\rho-\nu+\frac{1}{2}\end{matrix}\right.\right)$$
$$\left[y>0,\quad \operatorname{Re}a>0,\quad \operatorname{Re}\rho>-1,\quad \operatorname{Re}(\rho+\mu)>-1,\quad \operatorname{Re}(2e+2k+\nu)<\tfrac{1}{2}\right]$$ ET II 86(21)a

6. $$\int_0^\infty x^{2\rho-\nu}\, W_{k,\mu}(ax)\, W_{-k,\mu}(ax)\, J_\nu(xy)\, dx$$
$$= \frac{\Gamma(\rho+1+\mu)\,\Gamma(\rho+1-\mu)\,\Gamma(2\rho+2)}{\Gamma\left(\frac{3}{2}+k+\rho\right)\Gamma\left(\frac{3}{2}-k+\rho\right)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\rho-1}$$
$$\times\ {}_4F_3\left(\rho+1, \rho+\frac{3}{2}, \rho+1+\mu, \rho+1-\mu; \frac{3}{2}+k+\rho, \frac{3}{2}-k+\rho, 1+\nu; -\frac{y^2}{a^2}\right)$$
$[y>0, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu|-1, \quad \operatorname{Re} a > 0]$ ET II 86(22)a

7.662

1. $$\int_0^\infty x^{-1}\, M_{-\mu,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) W_{\mu,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\nu-\mu\right)} I_{\frac{1}{4}\nu-\mu}\left(\frac{1}{4}y^2\right) K_{\frac{1}{4}\nu+\mu}\left(\frac{1}{4}y^2\right)$$
$[y>0, \quad \operatorname{Re}\nu > -1]$ ET II 86(24)

2. $$\int_0^\infty x^{-1}\, M_{\alpha-\beta,\frac{1}{4}\nu-\gamma}\left(\frac{1}{2}x^2\right) W_{\alpha+\beta,\frac{1}{4}\nu+\gamma}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx$$
$$= \frac{\Gamma\left(1+\frac{1}{2}\nu-2\gamma\right)}{\Gamma\left(1+\frac{1}{2}\nu-2\beta\right)} y^{-2}\, M_{\alpha-\gamma,\frac{1}{4}\nu-\beta}\left(\frac{1}{2}y^2\right) W_{\alpha+\gamma,\frac{1}{4}\nu+\beta}\left(\frac{1}{2}y^2\right)$$
$\left[y>0, \quad \operatorname{Re}\beta < \frac{1}{8}, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu-4\gamma) > -2\right]$ ET II 86(25)

3. $$\int_0^\infty x^{-1}\, M_{k,0}\left(iax^2\right) M_{k,0}\left(-iax^2\right) K_0(xy)\, dx = \frac{\pi}{16}\left\{\left[J_k\left(\frac{y^2}{8a}\right)\right]^2 + \left[Y_k\left(\frac{y^2}{8a}\right)\right]^2\right\}$$
$[a>0]$ ET II 152(83)

4. $$\int_0^\infty x^{-1}\, M_{k,\mu}\left(iax^2\right) M_{k,\mu}\left(-iax^2\right) K_0(xy)\, dx = ay^{-2}\left[\Gamma(2\mu+1)\right]^2 W_{-\mu,k}\left(\frac{iy^2}{4a}\right) W_{-\mu,k}\left(-\frac{iy^2}{4a}\right)$$
$\left[a>0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET II 152(84)

7.663

1. $$\int_0^\infty x^{2\rho}\ {}_1F_1\left(a; b; -\lambda x^2\right) J_\nu(xy)\, dx = \frac{2^{2\rho}\,\Gamma(b)}{\Gamma(a)y^{2\rho+1}}\, G^{21}_{23}\left(\frac{y^2}{4\lambda}\,\middle|\, \begin{matrix} 1, b \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, a, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix}\right)$$
$\left[y>0, \quad -1-\operatorname{Re}\nu < 2\operatorname{Re}\rho < \frac{1}{2}+2\operatorname{Re} a, \quad \operatorname{Re}\lambda > 0\right]$ ET II 88(6)

2. $$\int_0^\infty x^{\nu+1}\ {}_1F_1\left(2a-\nu; a+1; -\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{2^{\nu-a+\frac{1}{2}}\,\Gamma(a+1)}{\pi^{\frac{1}{2}}\,\Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2}\, K_{a-\nu-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$\left[y>0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(4a-3\nu) > \frac{1}{2}\right]$ ET II 87(1)

3. $$\int_0^\infty x^a\ {}_1F_1\left(a; \frac{1+a+\nu}{2}; -\frac{1}{2}x^2\right) J_\nu(xy)\, dx = y^{a-1}\ {}_1F_1\left(a; \frac{1+a+\nu}{2}; -\frac{y^2}{2}\right)$$
$\left[y>0, \quad \operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a+\nu) > -1\right]$ ET II 87(2)

4. $$\int_0^\infty x^{\nu+1-2a}\ {}_1F_1\left(a;1+\nu-a;-\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{\pi^{\frac{1}{2}}\,\Gamma(1+\nu-a)}{\Gamma(a)} 2^{-2a+\nu+\frac{1}{2}} y^{2a-\nu-1} e^{-\frac{1}{4}y^2}\ I_{a-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$$\left[y>0,\quad \operatorname{Re} a-1<\operatorname{Re}\nu<4\operatorname{Re} a-\tfrac{1}{2}\right]$$ ET II 87(3)

5. $$\int_0^\infty x\ {}_1F_1\left(\lambda;1;-x^2\right) J_0(xy)\,dx = \left[2^{2\lambda-1}\,\Gamma(\lambda)\right]^{-1} y^{2\lambda-2} e^{-\frac{1}{4}y^2}$$
$$[y>0,\quad \operatorname{Re}\lambda>0]$$ ET II 18(46)

6. $$\int_0^\infty x^{\nu+1}\ {}_1F_1\left(a;b;-\lambda x^2\right) J_\nu(xy)\,dx = \frac{2^{1-a}\,\Gamma(b)}{\Gamma(a)\lambda^{\frac{1}{2}a+\frac{1}{2}\nu}} y^{a-2} e^{-\frac{y^2}{8\lambda}}\ W_{k,\mu}\left(\frac{y^2}{4\lambda}\right),\qquad 2k=a-2b+\nu+2,\qquad 2\mu=a-\nu-1$$
$$\left[y>0,\quad -1<\operatorname{Re}\nu<2\operatorname{Re} a-\tfrac{1}{2},\quad \operatorname{Re}\lambda>0\right]$$ ET II 88(4)

7. $$\int_0^\infty x^{2b-\nu-1}\ {}_1F_1\left(a;b;-\lambda x^2\right) J_\nu(xy)\,dx = \frac{2^{2b-2a-\nu-1}\,\Gamma(b)}{\Gamma(a-b+\nu+1)}\lambda^{-a} y^{2a-2b+\nu} \times {}_1F_1\left(a;1+a-b+\nu;-\frac{y^2}{4\lambda}\right)$$
$$\left[y>0,\quad 0<\operatorname{Re} b<\tfrac{3}{4}+\operatorname{Re}\left(a+\tfrac{1}{2}\nu\right),\quad \operatorname{Re}\lambda>0\right]$$ ET II 88(5)

7.664

1. $$\int_0^\infty x\ W_{\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right) K_\nu(xy)\,dx = 2ay^{-1}\,K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\right] K_{2\mu}\left[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}\right]$$
$$[\operatorname{Re} y>0,\quad \operatorname{Re} a>0]$$ ET II 152(85)

2. $$\int_0^\infty x\ W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy)\,dx = -4y^{-1}\left\{\sin\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right)+\cos\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]\ Y_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right)$$
$$[y>0,\quad \operatorname{Re}(\nu\pm 2\mu)>-1]$$ ET II 87(27)

3. $$\int_0^\infty x\ W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) Y_\nu(xy)\,dx = 4y^{-1}\left\{\left\{\cos\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right)-\sin\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]\ Y_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}$$
$$\left[y>0,\quad |\operatorname{Re}\mu|<\tfrac{1}{4}\right]$$ ET II 117(48)

4. $$\int_0^\infty x\ W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) M_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right) J_\nu(xy)\,dx = \frac{4\,\Gamma(1+2\mu)y^{-1}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\mu\right)}\ J_{2\mu}\left(2y^{\frac{1}{2}}\right) K_{2\mu}\left(2y^{\frac{1}{2}}\right)$$
$$\left[y>0,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}\mu>-\tfrac{1}{4}\right]$$ ET II 86(26)

5. $$\int_0^\infty x\, W_{-\frac{1}{2}\nu,\mu}\left(\frac{ia}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(-\frac{ia}{x}\right) J_\nu(xy)\,dx$$
$$= 4ay^{-1}\left[\Gamma\left(\tfrac{1}{2}+\mu+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}-\mu+\tfrac{1}{2}\nu\right)\right]^{-1} K_\mu\left[(2iay)^{\frac{1}{2}}\right] K_\mu\left[(-2iay)^{\frac{1}{2}}\right]$$
$$\left[y>0,\quad \operatorname{Re} a>0,\quad |\operatorname{Re}\mu|<\tfrac{1}{2},\quad \operatorname{Re}\nu>-1\right]$$ ET II 87(28)

7.665

1. $$\int_0^\infty x^{-\frac{1}{2}} J_\nu\left(ax^{\frac{1}{2}}\right) K_{\frac{1}{2}\nu-\mu}\left(\frac{1}{2}x\right) M_{k,\mu}(x)\,dx$$
$$= \frac{\Gamma(2\mu+1)}{a\,\Gamma\left(k+\frac{1}{2}\nu+1\right)}\, W_{\frac{1}{2}(k-\mu),\frac{1}{2}k-\frac{1}{4}\nu}\left(\frac{a^2}{2}\right) M_{\frac{1}{2}(k+\mu),\frac{1}{2}k+\frac{1}{4}\nu}\left(\frac{a^2}{2}\right)$$
$$\left[a>0,\quad \operatorname{Re} k>-\tfrac{1}{4},\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}\nu>-1\right]$$ ET II 405(18)

2. $$\int_0^\infty x^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi(a,c;x)\ {}_1F_1\left(a';c';-x\right) J_{c+c'-2}\left[2(xy)^{\frac{1}{2}}\right]\,dx$$
$$= \frac{\Gamma\left(c'\right)}{\Gamma\left(a+a'\right)} y^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi\left(c'-a',c+c'-a-a';y\right)\ {}_1F_1\left(a';a+a';-y\right)$$
$$\left[\operatorname{Re} c'>0,\quad 1<\operatorname{Re}\left(c+c'\right)<2\operatorname{Re}\left(a+a'\right)+\tfrac{1}{2}\right]$$ EH I 287(23)

7.666 $$\int_0^\infty x^{\frac{1}{2}c-\frac{1}{2}}\ {}_1F_1\left(a;c;-2x^{\frac{1}{2}}\right)\Psi\left(a,c;2x^{\frac{1}{2}}\right) J_{c-1}\left[2(xy)^{\frac{1}{2}}\right]\,dx$$
$$= 2^{-c}\frac{\Gamma(c)}{\Gamma(a)} y^{a-\frac{1}{2}c-\frac{1}{2}}\left[1+(1+y)^{\frac{1}{2}}\right]^{c-2a}(1+y)^{-\frac{1}{2}}$$
$$\left[\operatorname{Re} c>2,\quad \operatorname{Re}(c-2a)<\tfrac{1}{2}\right]$$ EH I 285(13)

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

1. $$\int_0^\infty x^{k-\frac{3}{2}}\exp\left[-\frac{1}{2}(a+1)x\right] K_\nu\left(\frac{1}{2}ax\right) M_{k,\nu}(x)\,dx$$
$$= \frac{\pi^{\frac{1}{2}}\,\Gamma(k)\,\Gamma(k+2\nu)}{a^{k+\nu}\,\Gamma\left(k+\nu+\frac{1}{2}\right)}\ {}_2F_1\left(k,k+2\nu;2\nu+1;-a^{-1}\right)$$
$$[\operatorname{Re} a>0,\quad \operatorname{Re} k>0,\quad \operatorname{Re}(k+2\nu)>0]$$ ET II 405(17)

2. $$\int_0^\infty x^{-k-\frac{3}{2}}\exp\left[-\frac{1}{2}(a-1)x\right] K_\mu\left(\frac{1}{2}ax\right) W_{k,\mu}(x)\,dx$$
$$= \frac{\pi\,\Gamma(-k)\,\Gamma(2\mu-k)\,\Gamma(-2\mu-k)}{\Gamma\left(\frac{1}{2}-k\right)\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} 2^{2k+1}a^{k-\nu}\ {}_2F_1\left(-k,2\mu-k;-2k;1-a^{-1}\right)$$
$$[\operatorname{Re} a>0,\quad \operatorname{Re} k<2\operatorname{Re}\mu<-\operatorname{Re} k]$$ ET II 408(36)

7.672

1. $$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} M_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} 2^{2\rho} y^{-2\rho-1} G^{21}_{23}\left(\frac{y^2}{4a}\middle|\begin{matrix}\frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, k, \frac{1}{2}+\rho-\frac{1}{2}\nu\end{matrix}\right)$$
$$\left[y>0, \quad -1-\operatorname{Re}\left(\tfrac{1}{2}\nu+\mu\right) < \operatorname{Re}\rho < \operatorname{Re}k - \tfrac{1}{4}, \quad \operatorname{Re}a>0\right]$$ ET II 83(10)

2. $$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} W_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{\Gamma\left(1+\mu+\frac{1}{2}\nu+\rho\right)\Gamma\left(1-\mu+\frac{1}{2}\nu+\rho\right)2^{-\nu-1}}{\Gamma(\nu+1)\,\Gamma\left(\frac{3}{2}-k+\frac{1}{2}\nu+\rho\right)} a^{-\frac{1}{2}\nu-\rho-\frac{1}{2}} y^\nu$$
$$\times\, {}_2F_2\left(\lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-k+\lambda; -\frac{y^2}{4a}\right),$$
$$\lambda = 1+\tfrac{1}{2}\nu+\rho \qquad \left[y>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\left(\rho\pm\mu+\tfrac{1}{2}\nu\right)>-1\right]$$ ET II 85(16)

3. $$\int_0^\infty x^{2\rho} e^{\frac{1}{2}ax^2} W_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{2^{2\rho} y^{-2\rho-1}}{\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)}$$
$$\times G^{22}_{23}\left(\frac{y^2}{4a}\middle|\begin{matrix}\frac{1}{2}-\mu, & \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, & -k, \quad \frac{1}{2}+\rho-\frac{1}{2}\nu\end{matrix}\right)$$
$$\left[y>0, \quad |\arg a|<\pi, \quad -1-\operatorname{Re}\left(\tfrac{1}{2}\nu\pm\mu\right) < \operatorname{Re}\rho < -\tfrac{1}{4}-\operatorname{Re}k\right]$$ ET II 85(17)

4. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx = \frac{2^\lambda y^{-1/2}\,\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} G^{31}_{34}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & \\ h, & \kappa, & -\lambda-\frac{1}{2}, \quad l\end{matrix}\right)$$
$$h=\tfrac{1}{4}+\tfrac{1}{2}\nu, \quad \kappa=\tfrac{1}{4}-\tfrac{1}{2}\nu, \quad l=-\tfrac{1}{4}-\tfrac{1}{2}\nu$$
$$\left[y>0, \quad \operatorname{Re}(k-\lambda)>0, \quad \operatorname{Re}\left(2\lambda+2\mu\pm\nu\right)>-\tfrac{5}{2}\right]$$ ET II 116(45)

5. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx$$
$$= 2^\lambda\left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} G^{32}_{34}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & l \\ h, & \kappa, & -\frac{1}{2}-k-\lambda, \quad l\end{matrix}\right) y^{-1/2},$$
$$h=\tfrac{1}{4}+\tfrac{1}{2}\nu, \quad \kappa=\tfrac{1}{4}-\tfrac{1}{2}\nu, \quad l=-\tfrac{1}{4}-\tfrac{1}{2}\nu$$
$$\left[y>0, \quad \operatorname{Re}(k+\lambda)<0, \quad \operatorname{Re}\left(2\lambda\pm2\mu\pm\nu\right)>-\tfrac{5}{2}\right]$$ ET II 117(47)

6. $$\int_0^\infty x^{-1/2} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}\left(x^2\right) J_\nu(xy)\,dx = (2\nu+1)2^{-\nu} y^{\nu-1}\left[1-\Phi\left(\frac{1}{2}y\right)\right]$$
$$\left[y>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ ET II 82(1)

7. $$\int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}\left(x^2\right) J_\nu(xy)dx = \frac{\Gamma(\nu+2)y^\nu}{\Gamma\left(\nu+\frac{3}{2}\right)2^\nu}\left[1-\Phi\left(\frac{1}{2}y\right)\right]$$
$$\left[y>0, \operatorname{Re}\nu>-1\right]$$ ET II 82(2)

8. $$\int_0^\infty e^{-\frac{1}{4}x^2}\, M_{k,\frac{1}{2}\nu}\left(\frac{1}{2}\right) x^2\, J_\nu(xy)\, dx = \frac{2^{-k}\,\Gamma(\nu+1)}{\Gamma\left(k+\frac{1}{2}\nu+\frac{1}{2}\right)} y^{2k-1} e^{-\frac{1}{2}y^2}$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}k<\tfrac{1}{2}\right]$$
ET II 83(7)

9. $$\int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2}\, M_{k,\mu}\left(\frac{1}{2}\right) x^2\, J_\nu(xy)\, dx$$
$$= 2^{\frac{1}{2}\left(\frac{1}{2}-k-3\mu+\nu\right)} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2}\, W_{\alpha,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = k-3\mu+\nu+\tfrac{1}{2}, \qquad 2\beta = k+\mu-\nu-\tfrac{1}{2}$$
$$\left[y>0, \quad -1<\operatorname{Re}\nu<2\operatorname{Re}(k+\mu)-\tfrac{1}{2}\right]$$ ET II 83(9)

10. $$\int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2}\, W_{k,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2}\, M_{\alpha,\beta}\left(\frac{1}{2}y^2\right)$$
$$2\alpha = \tfrac{1}{2}+k+\nu-3\mu, \quad 2\beta = \tfrac{1}{2}-k+\nu-\mu$$
$$[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-2\mu)>-1]$$
ET II 84(14)

11. $$\int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2}\, W_{k,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx$$
$$= \frac{\Gamma(1+\nu-2\mu)}{\Gamma\left(\frac{1}{2}+\mu-k\right)} 2^{\frac{1}{2}\left(\frac{1}{2}+k-3\mu+\nu\right)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2}\, W_{\alpha,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = k+3\mu-\nu-\tfrac{1}{2}, \qquad 2\beta = k-\mu+\nu+\tfrac{1}{2}$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-2\mu)>-1, \quad \operatorname{Re}\left(k-\mu+\tfrac{1}{2}\nu\right)<-\tfrac{1}{4}\right]$$ ET II 84(15)

12. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2}\, M_{k,\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx$$
$$= \frac{\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k-\mu+\nu\right)} 2^{\frac{1}{2}\left(\frac{1}{2}-k+3\mu-\nu\right)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2}\, M_{\alpha,\beta}\left(\frac{1}{2}y^2\right)$$
$$2\alpha = \tfrac{1}{2}+k+3\mu-\nu, \qquad 2\beta = -\tfrac{1}{2}+k-\mu+\nu$$
$$\left[y>0, \quad -\tfrac{1}{2}<\operatorname{Re}\mu<\operatorname{Re}\left(k+\tfrac{1}{2}\nu\right)-\tfrac{1}{4}\right]$$ ET II 83(8)

13. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2}\, M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\, dx$$
$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2}\, \Gamma(2\mu+1)$$
$$\times \Gamma\left(\tfrac{1}{2}-k-\mu\right) \left\{ \cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)}\, M_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right) - \sin[(\nu+k-\mu)\pi]\, W_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right) \right\}$$
$$2\alpha = 3\mu-\nu+k+\tfrac{1}{2}, \qquad 2\beta = \mu-\nu-k+\tfrac{1}{2}$$
$$\left[y>0, \quad -1<2\operatorname{Re}\mu<\operatorname{Re}(2k+\nu)+\tfrac{1}{2}, \quad \operatorname{Re}(2\mu-\nu)>-1\right]$$ ET II 116(44)

14. $$\int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx$$
$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} \Gamma(2\mu+1)$$
$$\times \Gamma\left(\tfrac{1}{2}-\mu-k\right) e^{-\frac{1}{4}y^2} \left\{\cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma\left(\mu+\nu-k+\frac{3}{2}\right)} M_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right)\right.$$
$$\left. + \sin[(\mu-k)\pi]\, W_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right)\right\}$$
$$2\alpha = 3\mu+\nu+k+\tfrac{1}{2}, \qquad 2\beta = \mu+\nu-k+\tfrac{1}{2}$$
$$\left[y>0, \quad -1<2\operatorname{Re}\mu<\operatorname{Re}(2k-\nu)+\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu)>-1\right]$$ ET II 116(43)

15. $$\int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\mu}\left(ax^2\right) K_\nu(xy)\,dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}}$$
$$\times \Gamma(2\mu+1)\,\Gamma(2\mu+\nu+1) \exp\left(\frac{y^2}{8a}\right) W_{\kappa,m}\left(\frac{y^2}{4a}\right),$$
$$2\kappa = -3\mu-\nu-k-\tfrac{1}{2}, \qquad 2m = \mu+\nu-k+\tfrac{1}{2}$$
$$\left[\operatorname{Re} y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\mu>-\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu)>-1\right]$$ ET II 152(82)

7.673

1.[10] $$\int_0^\infty e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\kappa,\frac{1}{2}\mu}(ax)\, J_\nu\left(2\sqrt{bx}\right) dx$$
$$= \left(\frac{b}{a}\right)^{\frac{\kappa-1}{2}-\frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)} \Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma\left(1+\dfrac{\kappa+\nu}{2}-\dfrac{1+\mu}{4}\right)}$$
$$\times M_{\frac{1}{2}(\kappa-\nu-1)+\frac{3}{4}(1+\mu),\frac{\kappa+\nu}{2}-\frac{1+\mu}{4}}\left(\frac{b}{a}\right)$$
$$\left[\operatorname{Re}(1+\mu)>0, \quad \operatorname{Re}\left(\kappa+\frac{\nu-\mu}{2}\right)>-\frac{3}{4}, \quad \operatorname{Im} b=0\right]$$ BU 128(12)a

2. $$\int_0^\infty e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\kappa,\frac{1}{2}\mu}(ax)\, J_\nu\left(2\sqrt{bx}\right) dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu)\, e^{\frac{b}{2a}}}{\Gamma\left(\frac{1\pm\mu}{2}-\kappa\right)} \left(\frac{a}{b}\right)^{\frac{1}{2}(\kappa+1)+\frac{1}{4}(1\mp\nu)}$$
$$\times W_{\frac{1}{2}(\kappa+1-\nu)-\frac{3}{4}(1\mp\mu),\frac{1}{2}(\kappa+\nu)+\frac{1}{4}(1\mp\mu)}\left(\frac{b}{a}\right)$$
$$\left[\operatorname{Re}\left(\frac{\nu\mp\mu}{2}+\kappa\right)<\frac{3}{4}, \quad \operatorname{Re}\nu>-1\right]$$ BU 128(13)

7.674

1. $$\int_0^\infty x^{\rho-1}e^{-\frac{1}{2}\kappa}\,J_{\lambda+\nu}\left(ax^{1/2}\right)J_{\lambda-\nu}\left(ax^{1/2}\right)W_{k,\mu}(x)\,dx$$
$$=\frac{\left(\frac{1}{2}a\right)^{2\lambda}\Gamma\left(\frac{1}{2}+\lambda+\mu+\rho\right)\Gamma\left(\frac{1}{2}+\lambda-\mu+\rho\right)}{\Gamma\left(1+\lambda+\nu\right)\Gamma(1+\lambda-\nu)\Gamma(1+\lambda-k+\rho)}$$
$$\times\,{}_4F_4\left(1+\lambda,\tfrac{1}{2}+\lambda,\tfrac{1}{2}+\lambda+\mu+\rho,\tfrac{1}{2}+\lambda-\mu+\rho;1+\lambda+\nu,\right.$$
$$\left.1+\lambda-\nu,1+2\lambda,1+\lambda-k+\rho;-a^2\right)$$
$$\left[|\operatorname{Re}\mu|<\operatorname{Re}(\lambda+\rho)+\tfrac{1}{2}\right]$$ ET II 409(37)

2. $$\int_0^\infty x^{\rho-1}e^{-\frac{1}{2}\kappa}\,I_{\lambda+\nu}\left(ax^{1/2}\right)K_{\lambda-\nu}\left(ax^{1/2}\right)W_{k,\mu}(x)\,dx$$
$$=\frac{\pi^{-1/2}}{2}\,G^{24}_{45}\left(a^2\left|\begin{matrix}0,\frac{1}{2},\frac{1}{2}+\mu-\rho,\frac{1}{2}-\mu-\rho\\ \lambda,\nu,-\lambda,-\nu,k-\rho\end{matrix}\right.\right)$$
$$\left[|\operatorname{Re}\mu|<\operatorname{Re}(\lambda+\rho)+\tfrac{1}{2},\quad |\operatorname{Re}\mu|<\operatorname{Re}(\nu+\rho)+\tfrac{1}{2}\right]$$ ET II 409(38)

Combinations of Struve functions and confluent hypergeometric functions

7.675

1. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{-\frac{1}{4}x^2}\,M_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx=\frac{2^{-\lambda}\,\Gamma(2\mu+1)}{y^{1/2}\,\Gamma\left(\frac{1}{2}+k+\mu\right)}\,G^{22}_{34}\left(\frac{y^2}{2}\left|\begin{matrix}l,-\mu-\lambda,mu-\lambda\\ l,k-\lambda-\frac{1}{2},h,\kappa\end{matrix}\right.\right)$$
$$h=\tfrac{1}{4}+\tfrac{1}{2}\nu,\quad \kappa=\tfrac{1}{4}-\tfrac{1}{2}\nu,\quad l=\tfrac{3}{4}+\tfrac{1}{2}\nu$$
$$\left[\operatorname{Re}(2\lambda+2\mu+\nu)>-\tfrac{7}{2},\quad \operatorname{Re}(k-\lambda)>0,\quad y>0,\quad \operatorname{Re}(2\lambda-2k+\nu)<-\tfrac{1}{2}\right]$$
ET II 171(42)

2. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{-\frac{1}{4}x^2}\,W_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx$$
$$=2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu}\pi^{-1/2}y^{\nu+1}\frac{\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda+\mu\right)\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda-\mu\right)}{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{9}{4}+\lambda-k-\frac{1}{2}\nu\right)}$$
$$\times\,{}_3F_3\left(1,\frac{7}{4}+\frac{\nu}{2}+\lambda+\mu,\frac{7}{4}+\frac{\nu}{2}+\lambda-\mu;\frac{3}{2},\nu+\frac{3}{2},\frac{9}{4}+\lambda-k+\frac{\nu}{2};-\frac{y^2}{2}\right)$$
$$\left[\operatorname{Re}(2\lambda+\nu)>2|\operatorname{Re}\mu|-\tfrac{7}{4},\quad y>0\right]$$ ET II 171(43)

3. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{\frac{1}{4}x^2}\,W_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx$$
$$=\left[2^\lambda\,\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1}y^{-1/2}\,G^{23}_{34}\left(\frac{y^2}{2}\left|\begin{matrix}l,-\mu-\lambda,\mu-\lambda\\ l,-k-\lambda-\frac{1}{2},h,\kappa\end{matrix}\right.\right)$$
$$h=\tfrac{1}{4}+\tfrac{1}{2}\nu,\quad \kappa=\tfrac{1}{4}-\tfrac{1}{2}\nu,\quad l=\tfrac{3}{4}+\tfrac{1}{2}\nu$$
$$\left[y>0,\quad \operatorname{Re}(2\lambda+\nu)>2|\operatorname{Re}\mu|-\tfrac{7}{2},\quad \operatorname{Re}\left(2k+2\lambda+\nu\right)<-\tfrac{1}{2},\quad \operatorname{Re}(k+\lambda)<0\right]$$ ET II 172(46)a

4. $$\int_0^\infty e^{\frac{1}{2}x^2}\,W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}\left(x^2\right)\mathbf{H}_\nu(xy)\,dx=2^{-\nu-1}y^\nu\pi e^{\frac{1}{4}y^2}\left[1-\Phi\left(\frac{y}{2}\right)\right]$$
$$\left[y>0,\quad \operatorname{Re}\nu>-1\right]$$ ET II 171(44)

7.68 Combinations of confluent hypergeometric functions and other special functions

Combinations of confluent hypergeometric functions and associated Legendre functions

7.681

1. $$\int_0^\infty x^{-1/2}(a+x)^\mu e^{-\frac{1}{2}x}\, P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right) M_{k,\mu}(x)\,dx$$
$$= -\frac{\sin(\nu\pi)}{\pi\,\Gamma(k)}\,\Gamma(2\mu+1)\,\Gamma\left(k-\mu+\nu+\tfrac{1}{2}\right)\Gamma\left(k-\mu-\nu-\tfrac{1}{2}\right)e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a),$$
$$\rho=\tfrac{1}{2}-k+\mu,\qquad \sigma=\tfrac{1}{2}+\nu$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(k-\mu)>\left|\operatorname{Re}\nu+\tfrac{1}{2}\right|\right]$$ ET II 403(11)

2. $$\int_0^\infty x^{-1/2}(a+x)^{-\mu}e^{-\frac{1}{2}x}\, P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right) M_{k,\mu}(x)\,dx$$
$$= \frac{\Gamma(2\mu+1)\,\Gamma\left(k+\mu+\nu+\frac{1}{2}\right)\Gamma\left(k+\mu-\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(2\mu+\nu+1)\,\Gamma(2\mu-\nu)}\,W_{\frac{1}{2}-k-\mu,\frac{1}{2}+\nu}(a)$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(k+\mu)>\left|\operatorname{Re}\nu+\tfrac{1}{2}\right|\right]$$ ET II 403(12)

3. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{\frac{1}{2}\mu}e^{-\frac{1}{2}x}\, P^{\mu}_{k+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right) W_{k,\nu}(x)\,dx$$
$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}+2\mu+\nu-k,\quad 2\sigma=k+3\nu-\tfrac{3}{2}$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+2\nu)<1]$$
ET II 407(32)

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-\frac{1}{2}\mu}e^{-\frac{1}{2}x}\, P^{\mu}_{k+\mu+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right) W_{k,\nu}(x)\,dx$$
$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}-k+\nu,\qquad 2\sigma=k+2\mu+3\nu-\tfrac{3}{2}$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+2\nu)<1]$$
ET II 408(33)

5. $$\int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}}(a+x)^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}\, Q^{\nu}_{\mu-k+\frac{3}{2}}\left(1+2\frac{x}{a}\right) M_{k,\nu}(x)\,dx$$
$$= \frac{e^{\nu\pi i}\,\Gamma(1+2\mu-\nu)\,\Gamma(1+2\mu)\,\Gamma\left(\frac{5}{2}-k+\mu+\nu\right)}{2\,\Gamma\left(\frac{1}{2}+k+\mu\right)}a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}-k-\mu+2\nu,\qquad 2\sigma=k-3\mu-\tfrac{3}{2}$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(2\mu-\nu)>-1\right]$$
ET II 404(14)

7.682

1. $$\int_0^\infty x^{-1/2}e^{-\frac{1}{2}x}\,P_\nu^{-2\mu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right]M_{k,\mu}(x)\,dx$$
$$=\frac{\Gamma(2\mu+1)\,\Gamma\left(k+\frac{1}{2}\nu\right)\Gamma\left(k-\frac{1}{2}\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{2^{2\mu}a^{1/4}\,\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{2}\nu\right)}\,W_{\frac{3}{4}-k,\frac{1}{4}+\frac{1}{2}\nu}(a)$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}k>\tfrac{1}{2}\operatorname{Re}\nu-\tfrac{1}{2},\quad \operatorname{Re}k>-\tfrac{1}{2}\operatorname{Re}\nu\right]$$ ET II 404(13)

2. $$\int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1}(a+x)^{-1/2}e^{-\frac{1}{2}x}\,Q_{k-\mu-\nu-1}^{1-k+\mu-\nu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right]M_{k,\mu}(x)\,dx$$
$$=e^{(1-k+\mu-\nu)\pi i}2^{\mu-k-\nu}a^{\frac{1}{2}(k+\mu-1)}\frac{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+2\mu)\,\Gamma(k+\mu+\nu)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a),$$
$$\rho=\tfrac{1}{2}-k-\tfrac{1}{2}\nu,\qquad \sigma=\mu+\tfrac{1}{2}\nu$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(k+\mu+\nu)>0\right]$$ ET II 404(15)

3. $$\int_0^\infty x^{\nu-\frac{1}{2}}e^{-\frac{1}{2}x}\,Q_{2k-2\nu-3}^{2\mu-2\nu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right]M_{k,\mu}(x)\,dx$$
$$=e^{2(\mu-\nu)\pi i}2^{2\mu-2\nu-1}a^{\frac{1}{2}(k+\mu-1)}e^{\frac{1}{2}a}\frac{\Gamma(2\mu+1)\,\Gamma(\nu+1)\,\Gamma\left(k+\mu-2\nu-\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}\,W_{\rho,\sigma}(a),$$
$$2\rho=1-k+\mu-2\nu,\qquad 2\sigma=k-\mu-2\nu-2$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}(k+\mu-2\nu)>\tfrac{1}{2}\right]$$ ET II 404(16)

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}e^{-\frac{1}{2}x}\,P_{2k+\mu+2\nu-3}^{\mu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]W_{k,\nu}(x)\,dx$$
$$=\frac{2^\mu\,\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a),$$
$$2\rho=1-k+\mu+\nu,\qquad 2\sigma=k+\mu+3\nu-2$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+2\nu)<1]$$
ET II 408(34)

5.[8] $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-1/2}e^{-\frac{1}{2}x}\,P_{2k+\mu+2\nu-2}^{\mu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right]W_{k,\nu}(x)\,dx$$
$$=\frac{2^\mu\,\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a),\qquad 2\rho=\mu+\nu-k,\qquad 2\sigma=k+\mu+3\nu-1$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$$ ET II 408(35)

A combination of confluent hypergeometric functions and orthogonal polynomials

7.683[8] $$\int_0^1 e^{-\frac{1}{2}ax}x^\alpha(1-x)^{\frac{\mu-\alpha}{2}-1}\,L_n^\alpha(ax)\,M_{\alpha-\frac{1+\alpha}{2},\frac{\mu-\alpha-1}{1}}\left[a(1-x)\right]dx$$
$$=\frac{\Gamma(\mu-\alpha)}{\Gamma(1+\mu)}\frac{\Gamma(1+n+\alpha)}{n!}a^{-\frac{1+\alpha}{2}}\,M_{\alpha+n,\frac{\mu}{2}}(a)$$
$$[\operatorname{Re}a>-1,\quad \operatorname{Re}(\mu-\alpha)>0,\quad n=0,1,2,\ldots]$$ BU 129(14b)

A combination of hypergeometric and confluent hypergeometric functions

7.684 $$\int_0^\infty x^{\rho-1}e^{-\frac{1}{2}x}\,M_{\gamma+\rho,\beta+\rho+\frac{1}{2}}(x)\;{}_2F_1\left(\alpha,\beta;\gamma;-\frac{\lambda}{x}\right)dx$$
$$=\frac{\Gamma(\alpha+\beta+2\rho)\,\Gamma(2\beta+2\rho)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta+\gamma+2\rho)}\lambda^{\frac{1}{2}\beta+\rho-\frac{1}{2}}e^{\frac{1}{2}\lambda}\;W_{k,\mu}(\lambda);$$
$$k=\tfrac{1}{2}-\alpha-\tfrac{1}{2}\beta-\rho,\qquad \mu=\tfrac{1}{2}\beta+\rho$$
$$[|\arg\lambda|<\pi,\quad \operatorname{Re}(\beta+\rho)>0,\quad \operatorname{Re}(\alpha+\beta+2\rho)>0,\quad \operatorname{Re}\gamma>0]$$
ET II 405(19)

7.69 Integration of confluent hypergeometric functions with respect to the index

7.691 $$\int_{-\infty}^{\infty}\operatorname{sech}(\pi x)\;W_{ix,0}(\alpha)\;W_{-ix,0}(\beta)\,dx=2\frac{(a\beta)^{1/2}}{\alpha+\beta}\exp\left[-\frac{1}{2}(\alpha+\beta)\right]$$ ET II 414(61)

7.692 $$\int_{-i\infty}^{i\infty}\Gamma(-a)\,\Gamma(c-a)\Psi(a,c;x)\Psi(c-a,c;y)\,da=2\pi i\,\Gamma(c)\Psi(c,2c;x+y)$$ EH I 285(15)

7.693

1. $$\int_{-\infty}^{\infty}\Gamma(ix)\,\Gamma(2k+ix)\;W_{k+ix,k-\frac{1}{2}}(\alpha)\;W_{-k-ix,k-\frac{1}{2}}(\beta)\,dx$$
$$=2\pi^{1/2}\,\Gamma(2k)(a\beta)^k(\alpha+\beta)^{\frac{1}{2}-2k}\,K_{2k-\frac{1}{2}}\left(\frac{a+\beta}{2}\right)$$
ET II 414(62)

2. $$\int_{-i\infty}^{i\infty}\Gamma\left(\tfrac{1}{2}+\nu+\mu+x\right)\Gamma\left(\tfrac{1}{2}+\nu+\mu-x\right)\Gamma\left(\tfrac{1}{2}+\nu-\mu+x\right)\Gamma\left(\tfrac{1}{2}+\nu-\mu-x\right)$$
$$\times M_{\mu+ix,\nu}(\alpha)\,M_{\mu-ix,\nu}(\beta)\,dx$$
$$=\frac{2\pi(a\beta)^{\nu+\frac{1}{2}}\left[\Gamma(2\nu+1)\right]^2\Gamma(2\nu+2\mu+1)\,\Gamma(2\nu-2\mu+1)}{(\alpha+\beta)^{2\nu+1}\,\Gamma(4\nu+2)}\,M_{2\mu,2\nu+\frac{1}{2}}(\alpha+\beta)$$
$$\left[\operatorname{Re}\nu>|\operatorname{Re}\mu|-\tfrac{1}{2}\right]$$ ET II 413(59)

7.694[11] $$\int_{-\infty}^{\infty}e^{-2\rho xi}\,\Gamma\left(\tfrac{1}{2}+\nu+ix\right)\Gamma\left(\tfrac{1}{2}+\nu-ix\right)M_{ix,\nu}(\alpha)\,M_{ix,\nu}(\beta)\,dx$$
$$=\pi\sqrt{\alpha\beta}\left[\Gamma(2\nu+1)\right]^2\operatorname{sech}\rho\exp\left[-\frac{1}{2}(\alpha+\beta)\tanh\rho\right]J_{2\nu}\left(\sqrt{\alpha\beta}\operatorname{sech}\rho\right)$$
$$\left[|\operatorname{Im}\rho|<\tfrac{1}{2}\pi,\quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$

7.7 Parabolic Cylinder Functions

7.71 Parabolic cylinder functions

7.711

1. $$\int_{-\infty}^{\infty}D_n(x)\,D_m(x)\,dx=0\qquad [m\neq n]$$
$$=n!(2\pi)^{1/2}\qquad [m=n]$$
WH

2. $$\int_0^\infty D_\mu(\pm t)\, D_\nu(t)\, dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu-\nu}\left[\frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu\right)\Gamma\left(-\frac{1}{2}\nu\right)} \mp \frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu\right)}\right]$$
[when the lower sign is taken, $\operatorname{Re}\mu > \operatorname{Re}\nu$] BU 11 117(13a), EH II 122(21)

3. $$\int_0^\infty [D_\nu(t)]^2\, dt = \pi^{1/2} 2^{-3/2} \frac{\psi\left(\frac{1}{2}-\frac{1}{2}\nu\right) - \psi\left(-\frac{1}{2}\nu\right)}{\Gamma(-\nu)}$$
BU 117(13b)a, EH II 122(22)a

7.72 Combinations of parabolic cylinder functions, powers, and exponentials

7.721

1. $$\int_{-\infty}^\infty e^{-\frac{1}{4}x^2}(x-z)^{-1}\, D_n(x)\, dx = \pm i e^{\mp n\pi i}(2\pi)^{1/2} n! e^{-\frac{1}{4}z^2}\, D_{-n-1}(\mp iz)$$
[The upper or lower sign is taken accordingly as the imaginary part of z is positive or negative.]
WH

2. $$\int_1^\infty x^\nu (x-1)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp\left[-\frac{(x-1)^2 a^2}{4}\right] D_\mu(ax)\, dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2}-\frac{\nu}{2}-1}\, \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$$
[$\operatorname{Re}(\mu-\nu) > 0$] ET II 395(4)a

7.722

1. $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu\, D_{\nu+1}(x)\, dx = 2^{-\frac{1}{2}-\frac{1}{2}\nu}\, \Gamma(\nu+1) \sin\frac{1}{4}(1-\nu)\pi$$
[$\operatorname{Re}\nu > -1$] WH

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\mu-1}\, D_{-\nu}(x)\, dx = \frac{\pi^{1/2} 2^{-\frac{1}{2}\mu-\frac{1}{2}\nu}\, \Gamma(\mu)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}$$
[$\operatorname{Re}\mu > 0$] EH II 122(20)

3.[11] $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu\, D_{\nu-1}(x)\, dx = 2^{-\frac{1}{2}\nu}\, \Gamma(\nu) \sin\left(\frac{1}{4}\pi\nu\right)$$
[$\operatorname{Re}\nu > -1$] ET II 395(2)

7.723

1. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^\nu \left(x^2+y^2\right)^{-1} D_\nu(x)\, dx = \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2}\, D_{-\nu-1}(y)$$
[$\operatorname{Re} y > 0, \quad \operatorname{Re}\nu > -1$]
EH II 121(18)a, ET II 396(6)a

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\nu-1} \left(x^2+y^2\right)^{-1/2} D_\nu(x)\, dx = y^{\nu-1}\, \Gamma(\nu) e^{\frac{1}{4}y^2}\, D_{-\nu}(y)$$
[$\operatorname{Re} y > 0, \quad \operatorname{Re}\nu > 0$] ET II 396(7)

3. $$\int_0^1 x^{2\nu-1} \left(1-x^2\right)^{\lambda-1} e^{\frac{a^2x^2}{4}}\, D_{-2\lambda-2\nu}(ax)\, dx = \frac{\Gamma(\lambda)\,\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} e^{\frac{a^2}{4}}\, D_{-2\nu}(a)$$
[$\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > 0$] ET II 395(3)a

7.724 $$\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2}\, D_\nu(x)\, dx = (2\pi\mu)^{1/2}(1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}}\, D_\nu\left[y(1-\mu)^{-1/2}\right]$$
[$0 < \operatorname{Re}\mu < 1$]
EH II 121(15)

7.725

1. $$\int_0^\infty e^{-pt}(2t)^{\frac{\nu-1}{2}}e^{-\frac{t}{2}}\,D_{-\nu-2}\left(\sqrt{2t}\right)\,dt=\left(\frac{\pi}{2}\right)^{1/2}\frac{\left(\sqrt{p+1}-1\right)^{\nu+1}}{(\nu+1)p^{\nu+1}}$$
$[\operatorname{Re}\nu>-1]$ MO 175

2. $$\int_0^\infty e^{-pt}(2t)^{\frac{\nu-1}{2}}e^{-\frac{t}{2}}\,D_{-\nu}\left(\sqrt{2t}\right)\,dt=\left(\frac{\pi}{2}\right)^{1/2}\frac{\left(\sqrt{p+1}-1\right)^{\nu}}{p^{\nu}\sqrt{p+1}}$$
$[\operatorname{Re}\nu>-1]$ MO 175

3. $$\int_0^\infty e^{-bx}\,D_{2n+1}\left(\sqrt{2x}\right)\,dx=(-2)^n\,\Gamma\left(n+\tfrac{3}{2}\right)\left(b-\tfrac{1}{2}\right)^n\left(b+\tfrac{1}{2}\right)^{-n-\frac{3}{2}}$$
$\left[\operatorname{Re}b>-\frac{1}{2}\right]$ ET I 210(3)

4. $$\int_0^\infty \left(\sqrt{x}\right)^{-1}e^{-bx}\,D_{2n}\left(\sqrt{2x}\right)\,dx=(-2)^n\,\Gamma\left(n+\frac{1}{2}\right)\left(b-\frac{1}{2}\right)^n\left(b+\frac{1}{2}\right)^{-n-\frac{1}{2}}$$
$\left[\operatorname{Re}b>-\frac{1}{2}\right]$ ET I 210(5)

5. $$\int_0^\infty x^{-\frac{1}{2}(\nu+1)}e^{-sx}\,D_\nu\left(\sqrt{x}\right)\,dx=\sqrt{\pi}\left(1+\sqrt{\tfrac{1}{2}+2s}\right)^{\nu}\frac{1}{\sqrt{\frac{1}{4}+s}}$$
$\left[\operatorname{Re}s>-\frac{1}{4},\quad \operatorname{Re}\nu<1\right]$ ET I 210(7)

6. $$\int_0^\infty e^{-zt}t^{-1+\frac{\beta}{2}}\,D_{-\nu}\left[2(kt)^{1/2}\right]\,dt=\frac{2^{1-\beta-\frac{\nu}{2}}\pi^{1/2}\,\Gamma(\beta)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}\right)}(z+k)^{-\frac{\beta}{2}}\,F\left(\frac{\nu}{2},\frac{\beta}{2};\frac{\nu+\beta+1}{2};\frac{z-k}{z+k}\right)$$
$\left[\operatorname{Re}(z+k)>0,\quad \operatorname{Re}\frac{z}{k}>0\right]$
EH II 121(11)

7.726 $$\int_{-\infty}^\infty e^{ixy-\frac{(1+\lambda)x^2}{4}}\,D_\nu\left[x(1-\lambda)^{1/2}\right]\,dx=(2\pi)^{1/2}\lambda^{\frac{1}{2}\nu}e^{-\frac{(1+\lambda)y^2}{4\lambda}}\,D_\nu\left[i\left(\lambda^{-1}-1\right)^{1/2}y\right]\qquad[\operatorname{Re}\lambda>0]$$
EH II 121(16)

7.727 $$\int_0^\infty \frac{e^{\frac{1}{2}x}e^{-bx}}{(e^x-1)^{\mu+\frac{1}{2}}}\exp\left(-\frac{a}{1-e^{-x}}\right)D_{2\mu}\left(\frac{2\sqrt{a}}{\sqrt{1-e^{-x}}}\right)\,dx=e^{-a}2^{b+\mu}\,\Gamma(b+\mu)\,D_{-2b}\left(2\sqrt{a}\right)$$
$[\operatorname{Re}a>0,\quad \operatorname{Re}b>-\operatorname{Re}\mu]$
ET I 211(13)

7.728 $$\int_0^\infty (2t)^{-\frac{\nu}{2}}e^{-pt}e^{-\frac{q^2}{8t}}\,D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right)\,dt=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}p^{\frac{1}{2}\nu-1}e^{-q\sqrt{p}}$$ MO 175

7.73 Combinations of parabolic cylinder and hyperbolic functions

7.731

1. $$\int_0^\infty \cosh(2\mu x)\exp\left[-(a\sinh x)^2\right]D_{2k}\left(2a\cosh x\right)\,dx=2^{k-\frac{3}{2}}\pi^{1/2}a^{-1}\,W_{k,\mu}\left(2a^2\right)$$
$\left[\operatorname{Re}^2 a>0\right]$ ET II 398(20)

2. $$\int_0^\infty \cosh(2\mu x)\exp\left[(a\sinh x)^2\right] D_{2k}\left(2a\cosh x\right)\,dx = \frac{\Gamma(\mu-k)\,\Gamma(-\mu-k)}{2^{k+\frac{5}{2}}\,a\,\Gamma(-2k)}\,W_{k+\frac{1}{2},\mu}\left(2a^2\right)$$
$$\left[|\arg a| < \frac{3\pi}{4},\quad \operatorname{Re} k + |\operatorname{Re}\mu| < 0\right]$$
ET II 398(21)

7.74 Combinations of parabolic cylinder and trigonometric functions

7.741

1. $$\int_0^\infty \sin(bx)\left\{\left[D_{-n-1}(ix)\right]^2 - \left[D_{-n-1}(-ix)\right]^2\right\}\,dx = (-1)^{n+1}\frac{i}{n!}\pi\sqrt{2\pi}e^{-\frac{1}{2}b^2}\,L_n\left(b^2\right)$$
$$[b>0]$$
ET I 115(3)

2. $$\int_0^\infty e^{-\frac{1}{4}x^2}\sin(bx)\,D_{2n+1}(x)\,dx = (-1)^n\sqrt{\frac{\pi}{2}}b^{2n+1}e^{-\frac{1}{2}b^2}$$
$$[b>0]$$
ET I 115(1)

3. $$\int_0^\infty e^{-\frac{1}{4}x^2}\cos(bx)\,D_{2n}(x)\,dx = (-1)^n\sqrt{\frac{\pi}{2}}b^{2n}e^{-\frac{1}{2}b^2}\qquad [b>0]$$
ET I 60(2)

4. $$\int_0^\infty e^{-\frac{1}{4}x^2}\sin(bx)\left[D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x)\right]\,dx = \sqrt{2\pi}\sin\left[\left(\nu-\tfrac{1}{4}\right)\pi\right]b^{2\nu-\frac{1}{2}}e^{-\frac{1}{2}b^2}$$
$$\left[\operatorname{Re}\nu > \tfrac{1}{4},\quad b>0\right]$$
ET I 115(2)

5. $$\int_0^\infty e^{-\frac{1}{2}x^2}\cos(bx)\left[D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x)\right]\,dx = \frac{2^{\frac{1}{4}-2\nu}\sqrt{\pi}b^{2\nu-\frac{1}{2}}e^{-\frac{1}{4}b^2}}{\operatorname{cosec}\left[\left(\nu+\frac{1}{4}\right)\pi\right]}$$
$$\left[\operatorname{Re}\nu > \tfrac{1}{4},\quad b>0\right]$$
ET I 61(4)

7.742

1. $$\int_0^\infty x^{2\rho-1}\sin(ax)e^{-\frac{x^2}{4}}\,D_{2\nu}(x)\,dx = 2^{\nu-\rho-\frac{1}{2}}\pi^{1/2}a\frac{\Gamma\left(2\rho+1\right)}{\Gamma(\rho-\nu+1)}$$
$$\times\,{}_2F_2\left(\rho+\frac{1}{2},\rho+1;\frac{3}{2},\rho-\nu+1;-\frac{a^2}{2}\right)$$
$$\left[\operatorname{Re}\rho > -\tfrac{1}{2}\right]$$
ET II 396(8)

2. $$\int_0^\infty x^{2\rho-1}\sin(ax)e^{\frac{x^2}{4}}\,D_{2\nu}(x)\,dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)}\,G^{22}_{23}\left(\frac{a^2}{2}\,\middle|\,\begin{matrix}\frac{1}{2}-\rho, 1-\rho\\ -\rho-\nu, \frac{1}{2}, 0\end{matrix}\right)$$
$$\left[a>0,\quad \operatorname{Re}\rho > -\tfrac{1}{2},\quad \operatorname{Re}(\rho+\nu) < \tfrac{1}{2}\right]$$
ET II 396(9)

3. $$\int_0^\infty x^{2\rho-1}\cos(ax)e^{-\frac{x^2}{4}}\,D_{2\nu}(x)\,dx = \frac{2^{\nu-\rho}\,\Gamma(2\rho)\pi^{1/2}}{\Gamma\left(\rho-\nu+\frac{1}{2}\right)}\,{}_2F_2\left(\rho,\rho+\frac{1}{2};\frac{1}{2},\rho-\nu+\frac{1}{2};-\frac{a^2}{2}\right)$$
$$[\operatorname{Re}\rho > 0]$$
ET II 396(10)a

4. $$\int_0^\infty x^{2\rho-1}\cos(ax)e^{\frac{x^2}{4}}\,D_{2\nu}(x)\,dx=\frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)}\,G^{22}_{23}\left(\frac{a^2}{2}\,\middle|\,\begin{matrix}\frac12-\rho,1-\rho\\-\rho-\nu,0,\frac12\end{matrix}\right)$$
$$\left[a>0,\quad \operatorname{Re}\rho>0,\quad \operatorname{Re}(\rho+\nu)<\tfrac12\right]$$
ET II 396(11)

7.743 $$\int_0^{\pi/2}(\cos x)^{-\mu-2}(\sin x)^{-\nu}\,D_\nu(a\sin x)\,D_\mu(a\cos x)\,dx=-\left(\tfrac12\pi\right)^{1/2}(1+\mu)^{-1}\,D_{\mu+\nu+1}(a)$$
$$[\operatorname{Re}\nu<1,\quad \operatorname{Re}\mu<-1]$$ ET II 397(19)

7.744

1. $$\int_0^\infty \sin(bx)\left[D_{-\nu-\frac12}\left(\sqrt{2x}\right)-D_{-\nu-\frac12}\left(-\sqrt{2x}\right)\right]D_{\nu-\frac12}\left(\sqrt{2x}\right)\,dx$$
$$=-\sqrt{2\pi}\sin\left[\left(\tfrac14+\tfrac12\nu\right)\pi\right]b^{-\nu-\frac12}\frac{\left(1+\sqrt{1+b^2}\right)^\nu}{\sqrt{1+b^2}}$$
$$[b>0]$$ ET I 115(4)

2. $$\int_0^\infty \cos(bx)\left[D_{-2\nu-\frac12}\left(\sqrt{2x}\right)+D_{-2\nu-\frac12}\left(-\sqrt{2x}\right)\right]D_{2\nu-\frac12}\left(\sqrt{2x}\right)\,dx$$
$$=-\frac{\sqrt{\pi}\sin\left[\left(\nu-\frac14\right)\pi\right]\left(1+\sqrt{1+b^2}\right)^{2\nu}}{\sqrt{1+b^2}\,b^{2\nu+\frac12}}$$
$$[b>0]$$ ET I 60(3)

7.75 Combinations of parabolic cylinder and Bessel functions

7.751

1. $$\int_0^\infty [D_n(ax)]^2\,J_1(xy)\,dx=(-1)^{n-1}y^{-1}\left[D_n\left(\frac{y}{a}\right)\right]^2\qquad [y>0]$$ ET II 20(24)

2. $$\int_0^\infty J_0(xy)\,D_n(ax)\,D_{n+1}(ax)\,dx=(-1)^n y^{-1}\,D_n\left(\frac{y}{a}\right)D_{n+1}\left(\frac{y}{a}\right)$$
$$\left[y>0,\quad |\arg a|<\tfrac14\pi\right]$$ ET II 17(42)

3. $$\int_0^\infty J_0(xy)\,D_\nu(x)\,D_{\nu+1}(x)\,dx=2^{-1}y^{-1}\left[D_\nu(-y)\,D_{\nu+1}(y)-D_{\nu+1}(-y)\,D_\nu(y)\right]$$ ET II 397(17)a

7.752

1. $$\int_0^\infty x^\nu e^{-\frac14x^2}\,D_{2\nu-1}(x)\,J_\nu(xy)\,dx=-\frac12\sec(\nu\pi)y^{\nu-1}e^{-\frac14y^2}\left[D_{2\nu-1}(y)-D_{2\nu-1}(-y)\right]$$
$$\left[y>0,\quad \operatorname{Re}\nu>-\tfrac12\right]$$
ET II 76(1), MO 183

2. $$\int_0^\infty x^\nu e^{\frac14x^2}\,D_{2\nu-1}(x)\,J_\nu(xy)\,dx=2^{\frac12-\nu}\pi\sin(\nu\pi)y^{-\nu}\,\Gamma(2\nu)e^{\frac14y^2}\,K_\nu\left(\tfrac14y^2\right)$$
$$\left[y>0,\quad -\tfrac12<\operatorname{Re}\nu<\tfrac12\right]$$ ET II 77(4)

3. $$\int_0^\infty x^{\nu+1}e^{-\frac14x^2}\,D_{2\nu}(x)\,J_\nu(xy)\,dx=\frac12\sec(\nu\pi)y^{\nu-1}e^{-\frac14y^2}\left[D_{2\nu+1}(y)-D_{2\nu+1}(-y)\right]$$
$$[y>0,\quad \operatorname{Re}\nu>-1]$$ ET II 78(13)

4. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\, D_{2\nu+1}(x)\, J_\nu(xy)\, dx = \frac{1}{2}\sec(\nu\pi) e^{-\frac{1}{4}y^2} y^\nu \left[D_{2\nu}(y) + D_{2\nu}(-y)\right]$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 77(5)

5. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2}\, D_{2\nu+2}(x)\, J_\nu(xy)\, dx = -\tfrac{1}{2}\sec(\nu\pi) y^\nu e^{-\frac{1}{4}y^2} \left[D_{2\nu+2}(y) + D_{2\nu+2}(-y)\right]$$
$$\left[\operatorname{Re}\nu > -1, \quad y > 0\right]$$ ET II 78(16)

6. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2}\, D_{2\nu+2}(x)\, J_\nu(xy)\, dx = \pi^{-1}\sin(\nu\pi)\,\Gamma(2\nu+3) y^{-\nu-2} e^{\frac{1}{4}y^2}\, K_{\nu+1}\left(\tfrac{1}{4}y^2\right)$$
$$\left[y > 0, \quad -1 < \operatorname{Re}\nu < -\tfrac{5}{6}\right]$$
ET II 78(19)

7. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\, D_{-2\nu}(x)\, J_\nu(xy)\, dx = 2^{-1/2}\pi^{1/2} y^{-\nu} e^{-\frac{1}{4}y^2}\, I_\nu\left(\tfrac{1}{4}y^2\right)$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 77(8)

8. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2}\, D_{-2\nu}(x)\, J_\nu(xy)\, dx = y^{\nu-1} e^{\frac{1}{4}y^2}\, D_{-2\nu}(y) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad y > 0\right]$$
ET II 77(9), EH II 121(17)

9. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2}\, D_{-2\nu-2}(x)\, J_\nu(xy)\, dx = (2\nu+1)^{-1} y^\nu e^{\frac{1}{4}y^2}\, D_{-2\nu-1}(y)$$
$$\left[y > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 77(10)

10. $$\int_0^\infty x^\nu e^{-\frac{1}{4}a^2x^2}\, D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{2^{\mu-\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right) y^\nu}{\Gamma(\nu-\mu+1) a^{1+2\nu}}\; {}_1F_1\left(\nu+\frac{1}{2}; \nu-\mu+1; -\frac{y^2}{2a^2}\right)$$
$$\left[y > 0, \quad |\arg a| < \tfrac{1}{4}\pi, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 77(11)

11. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2}\, D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{\Gamma\left(\frac{1}{2}+\nu\right) a^{2k} 2^{m+\mu}}{\Gamma\left(\frac{1}{2}-\mu\right) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}}\, W_{k,m}\left(\frac{y^2}{4a^2}\right)$$
$$2k = \tfrac{1}{2}+\mu-\nu, \quad 2m = \tfrac{1}{2}+\mu+\nu$$
$$\left[y > 0, \quad |\arg a| < \tfrac{1}{4}\pi, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \operatorname{Re}\left(\tfrac{1}{2}-2\mu\right)\right]$$
ET II 78(12)

12. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}a^2x^2}\, D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{2^\mu\,\Gamma\left(\nu+\frac{3}{2}\right) y^\nu}{\Gamma\left(\nu-\mu+\frac{3}{2}\right) a^{2\nu+2}}\; {}_1F_1\left(\nu+\frac{3}{2}; \nu-\mu+\frac{3}{2}; -\frac{y^2}{2a^2}\right)$$
$$\left[y > 0, \quad |\arg a| < \tfrac{1}{4}\pi, \quad \operatorname{Re}\nu > -1\right]$$
ET II 79(23)

13. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2x^2}\, D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{\Gamma\left(\frac{3}{2}+\nu\right) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu) y^{\mu+2}} e^{\frac{y^2}{4a^2}}\, W_{k,m}\left(\frac{y^2}{2a^2}\right)$$
$$2k = \mu-\nu-1, \quad 2m = \mu+\nu+1$$
$$\left[y > 0, \quad |\arg a| < \tfrac{3}{4}\pi, \quad -1 < \operatorname{Re}\nu < -\tfrac{1}{2} - 2\operatorname{Re}\mu\right]$$
ET II 79(24)

14. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2x^2}\, D_\mu(ax)\, J_\nu(xy)\, dx = \frac{2^{\lambda-\frac{1}{2}\mu}\pi^{-\frac{1}{2}}}{\Gamma(-\mu)y^{\lambda+\frac{3}{2}}}\, G^{22}_{23}\left(\frac{y^2}{2a^2}\,\middle|\,\begin{matrix}\frac{1}{2}, 1\\ \frac{3}{4}+\frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2}\end{matrix}\right)$$
$$\left[y>0, \quad |\arg a| < \tfrac{3}{4}\pi, \quad \operatorname{Re}\mu < -\operatorname{Re}\lambda < \operatorname{Re}\nu + \tfrac{3}{2}\right] \quad \text{ET II 80(26)}$$

15. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2}\, D_{-2\nu-1}(x)\, J_\nu(xy)\, dx = (2\nu+1)y^{\nu-1}e^{\frac{1}{4}y^2}\, D_{-2\nu-2}(y)$$
$$\left[y>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right] \quad \text{ET II 79(20)}$$

16. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2}\, D_{-2\nu-3}(x)\, J_\nu(xy)\, dx = 2^{-1/2}\pi^{1/2}y^{-\nu-2}e^{-\frac{1}{4}y^2}\, I_{\nu+1}\left(\tfrac{1}{4}y^2\right)$$
$$[y>0, \quad \operatorname{Re}\nu > -1] \quad \text{ET II 79(21)}$$

17. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2}\, D_{-2\nu-3}(x)\, J_\nu(xy)\, dx = y^\nu e^{\frac{1}{4}y^2}\, D_{-2\nu-3}(y)$$
$$[y>0, \quad \operatorname{Re}\nu > -1] \quad \text{ET II 79(22)}$$

18. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2}\, D_{\frac{1}{2}\nu-\frac{1}{2}}(ax)\, Y_\nu(xy)\, dx = -\pi^{-1}2^{\frac{3}{4}\nu+\frac{3}{4}}a^{-\nu}y^{-1}\,\Gamma(\nu+1)e^{\frac{y^2}{4a^2}}\, W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}\left(\frac{y^2}{2a^2}\right)$$
$$\left[y>0, \quad |\arg a| < \tfrac{3}{4}\pi, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{2}{3}\right] \quad \text{ET II 115(39)}$$

7.753

1. $$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2}\, I_{\nu-\frac{1}{2}}(2ax)\, D_\nu(2x)\, dx = \frac{1}{2}\pi^{-1/2}\,\Gamma(\nu)a^{\nu-\frac{1}{2}}\, D_{-\nu}(2a)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 0] \quad \text{ET II 397(12)}$$

2. $$\int_0^\infty x^{\nu-\frac{3}{2}} e^{-(x+a)^2}\, I_{\nu-\frac{3}{2}}(2ax)\, D_\nu(2x)\, dx = \frac{1}{2}\pi^{-1/2}\,\Gamma(\nu)a^{\nu-\frac{3}{2}}\, D_{-\nu}(2a)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 1] \quad \text{ET II 397(13)}$$

7.754

1. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu-1}(x) - D_{2\nu-1}(-x)\right\} J_\nu(xy)\, dx$$
$$= \pm y^{\nu-1}e^{-\frac{1}{4}y^2}\left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu-1}(y) - D_{2\nu-1}(-y)\right\}$$
$$\left[y>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right] \quad \text{ET II 76(2, 3)}$$

2. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu+1}(x) - D_{2\nu+1}(-x)\right\} J_\nu(xy)\, dx$$
$$= \mp y^\nu e^{-\frac{1}{4}y^2}\left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu}(y) + D_{2\nu}(-y)\right\}$$
$$\left[y>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right] \quad \text{ET II 77(6, 7)}$$

3. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2}\left\{[1 \pm 2\cos(\nu\pi)]\, D_{2\nu}(x) + D_{2\nu}(-x)\right\} J_\nu(xy)\, dx$$
$$= \pm y^{\nu-1}e^{-\frac{1}{4}y^2}\left\{[1 \pm 2\cos(\nu\pi)]\, D_{2\nu+1}(y) - D_{2\nu+1}(-y)\right\}$$
$$[y>0, \quad \operatorname{Re}\nu > -1] \quad \text{ET II 78(14, 15)}$$

4. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu+2}(x) + D_{2\nu+2}(-x)\right\} J_\nu(xy)\, dx$$
$$= \pm y^\nu e^{-\frac{1}{4}y^2} \left\{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu+2}(y) + D_{2\nu+2}(-y)\right\}$$
$$[y > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 78(17, 18)

7.755

1. $$\int_0^\infty x^{-1/2} D_\nu\left(\sqrt{ax}\right) D_{-\nu-1}\left(\sqrt{ax}\right) J_0(xy)\, dx$$
$$= 2^{-3/2}\pi a^{-1/2} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}}\left[\left(1+\frac{4y^2}{a^2}\right)^{1/2}\right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}}\left[\left(1+\frac{4y^2}{a^2}\right)^{1/2}\right]$$
$$[y > 0, \operatorname{Re} a > 0]$$ ET II 17(43)

2. $$\int_0^\infty x^{1/2} D_{-\frac{1}{2}-\nu}\left(ae^{\frac{1}{4}\pi i}x^{1/2}\right) D_{-\frac{1}{2}-\nu}\left(ae^{-\frac{1}{4}\pi i}x^{1/2}\right) J_\nu(xy)\, dx$$
$$= 2^{-\nu}\pi^{1/2} y^{-\nu-1}\left(a^2+2y\right)^{-1/2}\left[\Gamma\left(\nu+\tfrac{1}{2}\right)\right]^{-1}\left[\left(a^2+2y\right)^{1/2}-a\right]^{2\nu}$$
$$\left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 80(27)

3. $$a\int_0^\infty D_{-\frac{1}{2}-\nu}\left(ae^{\frac{1}{4}\pi i}x^{-1/2}\right) D_{-\frac{1}{2}-\nu}\left(ae^{-\frac{1}{4}\pi i}x^{-1/2}\right) J_\nu(xy)\, dx$$
$$= 2^{1/2}\pi^{1/2} y^{-1}\left[\Gamma\left(\nu+\tfrac{1}{2}\right)\right]^{-1}\exp\left[-a(2y)^{1/2}\right]$$
$$\left[y > 0, \quad \operatorname{Re} a > 0, \quad e\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 80(28)a

4. $$\int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}\left(ax^{-1/2}\right) D_{-\nu-\frac{1}{2}}\left(ax^{-1/2}\right) Y_\nu(xy)\, dx$$
$$= y^{-3/2}\exp\left(-ay^{1/2}\right)\sin\left[ay^{1/2}-\tfrac{1}{2}\left(\nu-\tfrac{1}{2}\right)\pi\right]$$
$$\left[y > 0, \quad |\arg a| < \tfrac{1}{4}\pi\right]$$ ET II 115(40)

5. $$\int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}\left(ax^{-1/2}\right) D_{-\nu-\frac{1}{2}}\left(ax^{-1/2}\right) K_\nu(xy)\, dx = 2^{-1}y^{-3/2}\pi\exp\left[-a(2y)^{1/2}\right]$$
$$\left[\operatorname{Re} y > 0, \quad |\arg a| < \tfrac{1}{4}\pi\right]$$ ET II 151(81)

Combinations of parabolic cylinder and Struve functions

7.756 $$\int_0^\infty x^{-\nu} e^{-\frac{1}{4}x^2}\left[D_\mu(x) - D_\mu(-x)\right] \mathbf{H}_\nu(xy)\, dx$$
$$= \frac{2^{3/2}\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu+\nu+1\right)} y^{\mu+\nu}\sin\left(\frac{1}{2}\mu\pi\right)\ {}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2};\frac{1}{2}\mu+\nu+1;-\frac{1}{2}y^2\right)$$
$$\left[y > 0, \quad \operatorname{Re}(\mu+\nu) > -\tfrac{3}{2}, \quad \operatorname{Re}\mu > -1\right]$$ ET II 171(41)

7.76 Combinations of parabolic cylinder functions and confluent hypergeometric functions

7.761

1. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1}\, D_{-\nu}(t)\; {}_1F_1\left(a;c;-\frac{1}{2}pt^2\right) dt$$
$$= \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c)\,\Gamma\left(\frac{1}{2}\nu-c+a\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}+\frac{1}{2}\nu\right)} F\left(a,c+\frac{1}{2};a+\frac{1}{2}+\frac{1}{2}\nu;1-p\right)$$
$$[|1-p|<1, \quad \operatorname{Re} c>0, \quad \operatorname{Re}\nu>2\operatorname{Re}(c-a)]$$ EH II 121(12)

2. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2}\, D_{-\nu}(t)\; {}_1F_1\left(a;c;-\frac{1}{2}pt^2\right) dt$$
$$= \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1)\,\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}-c+a\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}\nu\right)} F\left(a,c-\frac{1}{2};a+\frac{1}{2}\nu;1-p\right)$$
$$\left[|1-p|<1, \quad \operatorname{Re} c>\tfrac{1}{2}, \quad \operatorname{Re}\nu>2\operatorname{Re}(c-a)-1\right]$$ EH II 121(13)

7.77 Integration of a parabolic cylinder function with respect to the index

7.771 $$\int_0^\infty \cos(ax)\, D_{x-\frac{1}{2}}(\beta)\, D_{-x-\frac{1}{2}}(\beta)\, dx = \frac{1}{2}\left(\frac{\pi}{\cos a}\right)^{1/2} \exp\left(-\frac{\beta^2\cos a}{2}\right) \qquad \left[|a|<\tfrac{1}{2}\pi\right]$$
$$= 0 \qquad \left[|a|>\tfrac{1}{2}\pi\right]$$
ET II 298(22)

7.772

1. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left[\frac{\left(\tan\frac{1}{2}\varphi\right)^\nu}{\cos\frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\xi\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right)\right.$$
$$\left. + \frac{\left(\cot\frac{1}{2}\varphi\right)^\nu}{\sin\frac{1}{2}\varphi} D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\xi\right) D_\nu\left(-e^{\frac{1}{4}i\pi}\eta\right)\right] \frac{d\nu}{\sin\nu\pi}$$
$$= -2i(2\pi)^{1/2} \exp\left[-\tfrac{1}{4}i\left(\xi^2-\eta^2\right)\cos\varphi - \tfrac{1}{2}i\xi\eta\sin\varphi\right]$$
EH II 125(7)

2. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\left(\tan\frac{1}{2}\varphi\right)^\nu}{\cos\frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\zeta\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \frac{d\nu}{\sin\nu\pi}$$
$$= -2i\, D_0\left[e^{\frac{1}{4}i\pi}\left(\zeta\cos\tfrac{1}{2}\varphi+\eta\sin\tfrac{1}{2}\varphi\right)\right] D_{-1}\left[e^{\frac{1}{4}i\pi}\left(\eta\cos\tfrac{1}{2}\varphi-\zeta\sin\tfrac{1}{2}\varphi\right)\right]$$
EH II 125(8)

7.773

1. $$\int_{c-i\infty}^{c+i\infty} D_\nu(z) t^\nu\, \Gamma(-\nu)\, d\nu = 2\pi i e^{-\frac{1}{4}z^2-zt-\frac{1}{2}t^2} \qquad \left[c<0, \quad |\arg t|<\frac{\pi}{4}\right]$$ EH II 126(10)

2. $$\int_{c-i\infty}^{c+i\infty} \left[D_\nu(x)\, D_{-\nu-1}(iy) + D_\nu(-x)\, D_{-\nu-1}(iy)\right] \frac{t^{-\nu-1}\, d\nu}{\sin(-\nu\pi)} = \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{1/2}} \left(1+t^2\right)^{-\frac{1}{2}} \exp\left[\frac{1}{4}\frac{1-t^2}{1+t^2}\left(x^2+y^2\right) + i\frac{txy}{1+t^2}\right]$$
$$\left[-1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi\right]$$ EH II 126(11)

7.774 $$\int_{c-i\infty}^{c+i\infty} D_\nu\left[k^{\frac{1}{2}}(1+i)\xi\right] D_{-\nu-1}\left[k^{\frac{1}{2}}(1+i)\eta\right] \Gamma\left(-\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right)\, d\nu = 2^{1/2}\pi^2\, H_0^{(2)}\left[\tfrac{1}{2}k\left(\xi^2+\eta^2\right)\right]$$
$$[-1 < c < 0, \quad \operatorname{Re} ik \geq 0]$$ EH II 125(9)

7.8 Meijer's and MacRobert's Functions (G and E)

7.81 Combinations of the functions G and E and the elementary functions

7.811

1. $$\int_0^\infty G_{p,q}^{m,n}\left(\eta x \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) G_{\sigma,\tau}^{\mu,\nu}\left(\omega x \,\middle|\, \begin{matrix} c_1,\ldots,c_\sigma \\ d,\ldots,d_\tau \end{matrix}\right) dx = \frac{1}{\eta}\, G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}\left(\frac{\omega}{\eta} \,\middle|\, \begin{matrix} -b_1,\ldots,-b_m,c_1,\ldots,c_\sigma,-b_{m+1},\ldots,-b_q \\ -a_1,\ldots,-a_n,d_1,\ldots,d_\tau,-a_{n+1},\ldots,-a_p \end{matrix}\right)$$

subject to the following constraints

- $m,n,p,q,\mu,\nu,\sigma,\tau$ are integers;
- $1 \leq n \leq p < q < p+\tau-\sigma$
- $\frac{1}{2}p+\frac{1}{2}q-n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma+\frac{1}{2}\tau-\nu < \mu \leq \tau$
- $\operatorname{Re}(b_j+d_k) > -1 \qquad (j=1,\ldots,m; k=1,\ldots,\mu)$
- $\operatorname{Re}(a_j+c_k) < 1 \qquad (j=1,\ldots,n; k=1,\ldots,\tau)$
- $\omega \neq 0, \quad \eta \neq 0, \quad |\arg\eta| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg\omega| < \left(\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau\right)\pi$
- The following must not be integers:

$$\begin{aligned} &b_j - b_k \quad (j=1,\ldots,m; k=1,\ldots,m; j\neq k), \\ &a_j - a_k \quad (j=1,\ldots,n; k=1,\ldots,n; j\neq k), \\ &d_j - d_k \quad (j=1,\ldots,\mu; k=1,\ldots,\mu; j\neq k), \\ &a_j + d_k \quad (j=1,\ldots,n; k=1,\ldots,n); \end{aligned}$$

- The following must not be positive integers:

$$\begin{aligned} &a_j - b_k \quad (j=1,\ldots,n; k=1,\ldots,m) \\ &c_j - d_k \quad (j=1,\ldots,\nu; k=1,\ldots,\mu) \end{aligned}$$

Formula **7.811** 1 also holds for four sets of restrictions. See C. S. Meijer, Neue Integraldarstellungen für Whittakersche Funktionen, Nederl. Akad. Wetensch. Proc. **44** (1941), 82–92.

ET II 422(14)

Hereafter, $G_{p,q}^{m,n}$ will be written as G_{pq}^{mn}, and commas will only be inserted in entries like $G_{p+1,q+1}^{m,n+1}$, where their omission could cause ambiguity.

2. $$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\, G_{pq}^{\,mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx = \Gamma(\sigma)\, G_{p+1,q+1}^{\,m,n+1}\left(\alpha \left| \begin{matrix} 1-\rho, a_1,\ldots,a_p \\ b_1,\ldots,b_q, 1-\rho-\sigma \end{matrix}\right.\right)$$
where

- $(p+q) < 2(m+n)$
- $|\arg a| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $\operatorname{Re}(\rho+b_j) > 0,\ j=1,\ldots,m$
- $\operatorname{Re}\sigma > 0$
- either

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}\rho-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\rho+b_j) > 0; \quad j=1,\ldots,m; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (p-q)\left(\rho-\frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$p < q \quad (\text{or } p \le q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(p+b_j) > 0; \quad j=1,\ldots,m; \quad \operatorname{Re}\sigma > 0$$

ET II 417(1)

3. $$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\, G_{pq}^{\,mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx = \Gamma(\sigma)\, G_{p+1,q+1}^{\,m+1,n}\left(\alpha \left| \begin{matrix} a_1,\ldots,a_p, \rho \\ \rho-\sigma, b_1,\ldots,b_q \end{matrix}\right.\right)$$
where

- $p+q < 2(m+n)$
- $|\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $\operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\ldots,n$
- $\operatorname{Re}\sigma > 0$
- either

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\ldots,n; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (q-p)\left(\rho-\sigma+\frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$q < p \quad (\text{or } q \le p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\ldots,n; \quad \operatorname{Re}\sigma > 0$$

ET II 417(2)

4. $$\int_0^\infty x^{\rho-1}\, G_{pq}^{\,mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx = \frac{\prod_{j=1}^{m}\Gamma(b_j+\rho)\prod_{j=1}^{n}\Gamma(1-a_j-\rho)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\rho)\prod_{j=n+1}^{p}\Gamma(a_j+\rho)}\alpha^{-\rho}$$
$$p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad -\min_{1\le j\le m}\operatorname{Re} b_j < \operatorname{Re}\rho < 1-\max_{1\le j\le n}\operatorname{Re} a_j$$

ET II 418(3)a, ET I 337(14)

5. $$\int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma}\, G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx = \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)}\, G_{p+1,q+1}^{m+1,n+1}\left(\alpha\beta \left| \begin{matrix} 1-\rho, a_1,\ldots,a_p \\ \sigma-\rho, b_1,\ldots,b_q \end{matrix}\right.\right)$$

where

- $p+q<2(m+n)$
- $|\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $|\arg\beta|<\pi$
- $\operatorname{Re}(\rho+b_j)>0, \quad j=1,\ldots,m$
- $\operatorname{Re}(\rho-\sigma+a_j)<1, \quad j=1,\ldots,n$
- either

$$p\le q, \quad p+q\le 2(m+n), \quad |\arg\alpha|\le\left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad |\arg\beta|<\pi$$
$$\operatorname{Re}(\rho=b_j)>0, \quad j=1,\ldots,m, \quad \operatorname{Re}(\rho-\sigma+a_j)<1, \quad j=1,\ldots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^p a_j-\sum_{j=1}^q b_j-(q-p)\left(\rho-\sigma-\frac{1}{2}\right)\right]>1,$$

or

$$p\ge q, \quad p+q\le 2(m+n), \quad |\arg\alpha|\le\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg\beta|<\pi,$$
$$\operatorname{Re}(\rho+b_j)>0, \quad j=1,\ldots,m, \quad \operatorname{Re}(\rho-\sigma+a_j)<1, \quad j=1,\ldots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^p a_j-\sum_{j=1}^q b_j+(p-q)\left(\rho-\frac{1}{2}\right)\right]>1$$

ET II 418(4)

7.812

1. $$\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}\, E\left(a_1,\ldots,a_p:\rho_1,\ldots,\rho_q;\frac{z}{x^m}\right) dx$$
$$=\Gamma(\gamma-\beta)m^{\beta-\gamma}\, E\left(a_1,\ldots,a_{p+m}:\rho_1,\ldots,\rho_{q+m}:z\right)$$
$$a_{p+k}=\frac{\beta+k-1}{m}, \quad \rho_{q+k}=\frac{\gamma+k-1}{m}, \qquad k=1,\ldots,m$$
$$[\operatorname{Re}\gamma>\operatorname{Re}\beta>0, \quad m=1,2,\ldots]$$ ET II 414(2)

2. $$\int_0^\infty x^{\rho-1}(1+x)^{-\sigma}\, E\left[a_1,\ldots,a_p:\rho_1,\ldots,\rho_q:(1+x)z\right] dx$$
$$=\Gamma(\rho)\, E\left(a_1,\ldots,a_p,\sigma-\rho;\rho_1,\ldots,\rho_q,\sigma;z\right)$$
$$[\operatorname{Re}\sigma>\operatorname{Re}\rho>0]$$ ET II 415(3)

3. $$\int_0^\infty (1+x)^{-\beta} x^{s-1}\, G_{pq}^{mn}\left(\frac{ax}{1+x}\,\middle|\,\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right) dx = \Gamma(\beta-s)\, G_{p+1,q+1}^{m,n+1}\left(a\,\middle|\,\begin{matrix} 1-s, a_1,\dots,a_p \\ b_1,\dots,b_q, 1-\beta\end{matrix}\right)$$
$$\left[-\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re}\beta, \quad 1\le k\le m; \quad (p+q)<2(m+n),\right.$$
$$\left.|\arg a| < \left(m+n-\tfrac12 p-\tfrac12 q\right)\pi\right]$$
ET I 338(19)

7.813

1. $$\int_0^\infty x^{-\rho} e^{-\beta x}\, G_{pq}^{mn}\left(\alpha x\,\middle|\,\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right) dx = \beta^{\rho-1}\, G_{p+1,q}^{m,n+1}\left(\frac{\alpha}{\beta}\,\middle|\,\begin{matrix} \rho, a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac12 p-\tfrac12 q\right)\pi,\right.$$
$$\left.|\arg\beta| < \tfrac12\pi, \quad \operatorname{Re}(b_j-\rho) > -1, \quad j=1,\dots,m\right]$$
ET II 419(5)

2. $$\int_0^\infty e^{-\beta x}\, G_{pq}^{mn}\left(\alpha x^2\,\middle|\,\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right) dx = \pi^{-1/2}\beta^{-1}\, G_{p+2,q}^{m,n+2}\left(\frac{4\alpha}{\beta^2}\,\middle|\,\begin{matrix} 0, \frac12, a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac12 p-\tfrac12 q\right)\pi,\right.$$
$$\left.|\arg\beta| < \tfrac12\pi, \quad \operatorname{Re} b_j > -\tfrac12; \quad j=1,\dots,m\right]$$
ET II 419(6)

7.814

1. $$\int_0^\infty x^{\beta-1} e^{-x}\, E\left(a_1,\dots,a_p:\rho_1,\dots,\rho_q: xz\right) dx$$
$$= \pi \operatorname{cosec}(\beta\pi)\left[E\left(a_1,\dots,a_p: 1-\beta, \rho_1,\dots,\rho_q: e^{\pm i\pi} z\right)\right.$$
$$\left. - z^{-\beta} E\left(a_1+\beta,\dots,a_p+\beta: 1+\beta, \rho_1+\beta,\dots,\rho_l+\beta: e^{\pm i\pi} z\right)\right]$$
[$p\ge q+1$, $\operatorname{Re}(a_r+\beta)>0$, $r=1,\dots,p$, $|\arg z|<\pi$. The formula holds also for $p<q+1$, provided the integral converges.] ET II 415(4)

2. $$\int_0^\infty x^{\beta-1} e^{-x}\, E\left(a_1,\dots,a_p:\rho_1,\dots,\rho_q: x^{-m} z\right) dx$$
$$= (2\pi)^{\frac12-\frac12 m} m^{\beta-\frac12}\, E\left(a_1,\dots,a_{p+m}:\rho_1,\dots,\rho_q: m^{-m} z\right)$$
$$\left[\operatorname{Re}\beta>0, \quad a_{p+k} = \frac{\beta+k-1}{m}, \quad k=1,\dots,m; \quad m=1,2,\dots\right]$$ ET II 415(5)

7.815

1. $$\int_0^\infty \sin(cx)\, G_{pq}^{mn}\left(\alpha x^2\,\middle|\,\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right) dx = \sqrt{\pi} c^{-1}\, G_{p+2,q}^{m,n+1}\left(\frac{4\alpha}{c^2}\,\middle|\,\begin{matrix} 0, a_1,\dots,a_p, \frac12 \\ b_1,\dots,b_q\end{matrix}\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac12 p-\tfrac12 q\right)\pi,\right.$$
$$\left. c>0, \quad \operatorname{Re} b_j > -1, \quad j=1,2,\dots,m, \quad \operatorname{Re} a_j < \tfrac12, \quad j=1,\dots,n\right]$$
ET II 420(7)

2. $$\int_0^\infty \cos(cx)\, G_{pq}^{mn}\left(\alpha x^2 \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right) dx = \pi^{1/2} c^{-1}\, G_{p+2,q}^{m,n+1}\left(\frac{4\alpha}{c^2} \left| \begin{matrix} \frac{1}{2}, a_1,\ldots,a_p, 0 \\ b_1,\ldots,b_q \end{matrix} \right.\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \right.$$
$$\left. c>0, \quad \operatorname{Re} b_j > -\tfrac{1}{2}, \quad j=1,\ldots,m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j=1,\ldots,n\right]$$

ET II 420(8)

7.82 Combinations of the functions G and E and Bessel functions

7.821

1. $$\int_0^\infty x^{-\rho}\, J_\nu\left(2\sqrt{x}\right) G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right) dx = G_{p+2,q}^{m,n+1}\left(\alpha \left| \begin{matrix} \rho-\frac{1}{2}\nu, a_1,\ldots,a_p, \rho+\frac{1}{2}\nu \\ b_1,\ldots,b_q \end{matrix} \right.\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi \right.$$
$$\left. -\tfrac{3}{4} + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < 1 + \tfrac{1}{2}\operatorname{Re}\nu + \min_{1\le j\le m} \operatorname{Re} b_j \right]$$

ET II 420(9)

2. $$\int_0^\infty x^{-\rho}\, Y_\nu\left(2\sqrt{x}\right) G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right) dx$$
$$= G_{p+3,q+1}^{m,n+2}\left(\alpha \left| \begin{matrix} \rho-\frac{1}{2}\nu, \rho+\frac{1}{2}\nu, a_1,\ldots,a_p, \rho+\frac{1}{2}+\frac{1}{2}\nu \\ b_1,\ldots,b_q, \rho+\frac{1}{2}+\frac{1}{2}\nu \end{matrix} \right.\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \right.$$
$$\left. -\tfrac{3}{4} + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < \min_{1\le j\le m} \operatorname{Re} b_j + \tfrac{1}{2}|\operatorname{Re}\nu| + 1 \right]$$

ET II 420(10)

3. $$\int_0^\infty x^{-\rho}\, K_\nu\left(2\sqrt{x}\right) G_{pq}^{mn}\left(\alpha x \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right) dx = \frac{1}{2}\, G_{p+2,q}^{m,n+2}\left(\alpha \left| \begin{matrix} \rho-\frac{1}{2}\nu, \rho+\frac{1}{2}\nu, a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right)$$
$$\left[p+q<2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \right.$$
$$\left. \operatorname{Re}\rho < 1 - \tfrac{1}{2}|\operatorname{Re}\nu| + \min_{1\le j\le m} \operatorname{Re} b_j \right]$$

ET II 421(11)

7.822

1. $$\int_0^\infty x^{2\rho}\, J_\nu(xy)\, G_{pq}^{mn}\left(\lambda x^2 \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right.\right) dx = \frac{2^{2\rho}}{y^{2\rho+1}}\, G_{p+2,q}^{m,n+1}\left(\frac{4\lambda}{y^2} \left| \begin{matrix} h, a_1,\ldots,a_p, k \\ b_1,\ldots,b_q \end{matrix} \right.\right)$$
$$h = \tfrac{1}{2} - \rho - \tfrac{1}{2}\nu, \quad k = \tfrac{1}{2} - \rho + \tfrac{1}{2}\nu$$
$$\left[p+q<2(m+n), \quad |\arg\lambda| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad \operatorname{Re}\left(b_j+\rho+\tfrac{1}{2}\nu\right) > -\tfrac{1}{2}, \right.$$
$$\left. j=1,2,\ldots,m, \quad \operatorname{Re}(a_j+\rho) < \tfrac{3}{4}, \quad j=1,\ldots,n, \quad y>0 \right]$$

ET II 91(20)

2. $$\int_0^\infty x^{1/2}\, Y_\nu(xy)\, G_{pq}^{\,mn}\left(\lambda x^2 \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx$$
$$= (2\lambda)^{-1/2} y^{-1/2}\, G_{q+1,p+3}^{\,n+2,m}\left(\frac{y^2}{4\lambda}\left| \begin{matrix} \frac12-b_1,\ldots,\frac12-b_q,l \\ h,k,\frac12-a_1,\ldots,\frac12-a_p,l \end{matrix}\right.\right)$$
$$h=\tfrac14+\tfrac12\nu,\quad k=\tfrac14-\tfrac12\nu,\quad l=-\tfrac14-\tfrac12\nu$$
$$\Big[p+q<2(m+n),\quad |\arg\lambda|<\left(m+n-\tfrac12 p-\tfrac12 q\right)\pi,\quad y>0,$$
$$\operatorname{Re} a_j<1,\quad j=1,\ldots,n,\quad \operatorname{Re}\left(b_j\pm\tfrac12\nu\right)>-\frac34,\quad j=1,\ldots,m\Big]$$
ET II 119(56)

3. $$\int_0^\infty x^{1/2}\, K_\nu(xy)\, G_{pq}^{\,mn}\left(\lambda x^2 \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) dx$$
$$= 2^{-3/2}\lambda^{-1/2} y^{-1/2}\, G_{q,p+2}^{\,n+2,m}\left(\frac{y^2}{4\lambda}\left| \begin{matrix} \frac12-b_1,\ldots,\frac12-b_q \\ h,k,\frac12-a_1,\ldots,\frac12-a_p \end{matrix}\right.\right)$$
$$h=\tfrac14+\tfrac12\nu,\quad k=\tfrac14-\tfrac12\nu$$
$$\Big[\operatorname{Re} y>0,\quad p+q<2(m+n),\quad |\arg\lambda|<\left(m+n-\tfrac12 p-\tfrac12 q\right)\pi\,,$$
$$\operatorname{Re} b_j>\tfrac12|\operatorname{Re}\nu|-\tfrac34,\quad j=1,\ldots,m\Big]$$
ET II 153(90)

7.823

1. $$\int_0^\infty x^{\beta-1}\, J_\nu(x)\, E\left(a_1,\ldots,a_p:\rho_1,\ldots,\rho_q:x^{-2m}z\right) dx$$
$$= (2\pi)^{-m}(2m)^{\beta-1}\left\{\exp\left[\tfrac12\pi\left(\beta-\nu-1\right)i\right] E\left[a_1,\ldots,a_{p+2m}:\rho_1,\ldots,\rho_q:(2m)^{-2m}ze^{-m\pi i}\right]\right.$$
$$\left.+\exp\left[-\tfrac12\pi(\beta-\nu-1)i\right] E\left[a_1,\ldots,a_{p+2m}:\rho_1,\ldots,\rho_q:(2m)^{-2m}ze^{m\pi i}\right]\right\},$$
$$a_{p+k}=\frac{\beta+\nu+2k-2}{2m},\quad a_{p+m+k}=\frac{\beta-\nu+2k-2}{2m},\qquad m=1,2,\ldots,;\quad k=1,\ldots,m$$
$$\left[\operatorname{Re}(\beta+\nu)>0,\quad \operatorname{Re}\left(2a_r m-\beta\right)>-\tfrac32,\quad r=1,\ldots,p\right]$$
ET II 415(7)

2. $$\int_0^\infty x^{\beta-1}\, K_\nu(x)\, E\left(a_1,\ldots,a_p:\rho_1,\ldots,\rho_q:x^{-2m}z\right) dx$$
$$= (2\pi)^{1-m}2^{\beta-2}m^{\beta-1}$$
$$\times E\left[a_1,\ldots,a_{p+2m}:\rho_1,\ldots,\rho_q:(2m)^{-2m}z\right],$$
$$a_{p+k}=\frac{\beta+\nu+2k-2}{2m},\quad a_{p+m+k}=\frac{\beta-\nu+2k-2}{2m},\qquad k=1,2,\ldots,m$$
$$\left[\operatorname{Re}\beta>|\operatorname{Re}\nu|,\quad m=1,2,\ldots\right]$$
ET II 416(8)

7.824

1. $$\int_0^\infty x^{1/2}\,\mathbf{H}_\nu(xy)\,G_{pq}^{\,mn}\left(\lambda x^2\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)dx$$
$$=(2\lambda y)^{-1/2}\,G_{q+1,p+3}^{\,n+1,m+1}\left(\frac{y^2}{4\lambda}\left|\begin{matrix}l,\frac12-b_1,\dots,\frac12-b_q\\ l,\frac12-a_1,\dots,\frac12-a_p,h,k\end{matrix}\right.\right)$$
$$h=\frac14+\frac{\nu}{2},\quad k=\frac14-\frac{\nu}{2},\quad l=\frac34+\frac{\nu}{2}$$
$$\Big[p+q<2(m+n),\quad |\arg\lambda|<\left(m+n-\tfrac12p-\tfrac12q\right)\pi,\quad y>0,$$
$$\operatorname{Re}a_j<\min\left(1,\tfrac34-\tfrac12\nu\right),\quad j=1,\dots,n,\quad \operatorname{Re}(2b_j+\nu)>-\tfrac52,\quad j=1,\dots,m\Big]$$
ET II 172(47)

2. $$\int_0^\infty x^{-\rho}\,\mathbf{H}_\nu\left(2\sqrt{x}\right)G_{pq}^{\,mn}\left(\alpha x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)dx$$
$$=G_{p+3,q+1}^{\,m+1,n+1}\left(\alpha\left|\begin{matrix}\rho-\frac12-\frac12\nu,a_1,\dots,a_p,\rho+\frac12\nu,\rho-\frac12\nu\\ \rho-\frac12-\frac12\nu,b_1,\dots,b_q\end{matrix}\right.\right)$$
$$\Big[p+q<2(m+n),\quad |\arg\alpha|<\left(m+n-\tfrac12p-\tfrac12q\right)\pi\,,$$
$$\max\left(-\frac34,\operatorname{Re}\frac{\nu-1}{2}\right)+\max_{1\le j\le n}\operatorname{Re}a_j<\operatorname{Re}\rho<\min_{1\le j\le m}\operatorname{Re}b_j+\tfrac12\operatorname{Re}\nu+\tfrac32\Big]$$
ET II 421(12)

7.83 Combinations of the functions G and E and other special functions

7.831 $$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\,F(k+\sigma-\rho,\lambda+\sigma-\rho;\sigma;1-x)\,G_{pq}^{\,mn}\left(\alpha x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right)dx$$
$$=\Gamma(\sigma)\,G_{p+2,q+2}^{\,m+2,n}\left(\alpha\left|\begin{matrix}a_1,\dots,a_p,k+\lambda+\sigma-\rho,\rho\\ k,\lambda,b_1,\dots,b_q\end{matrix}\right.\right)$$

where ET II 421(13)

- $\operatorname{Re}\left[\sum_{j=1}^p a_j-\sum_{j=1}^q b_j+(q-p)\left(k+\frac12\right)\right]>-\frac12$
- $\operatorname{Re}\left[\sum_{j=1}^p a_j-\sum_{j=1}^q b_j+(q-p)\left(\lambda+\frac12\right)\right]>-\frac12$
- either

$$p+q<2(m+n),\quad |\arg\alpha|<\left(m+n-\tfrac12p-\tfrac12q\right)\pi,$$
$$\operatorname{Re}\sigma>0,\quad \operatorname{Re}k\ge\operatorname{Re}\lambda>\operatorname{Re}a_j-1,\quad j=1,\dots,n,$$

or

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re}k \ge \operatorname{Re}\lambda > \operatorname{Re}a_j - 1, \quad j=1,\dots,n,$$

7.832 $$\int_0^\infty x^{\beta-1}e^{-\frac{1}{2}x}\, W_{\kappa,\mu}(x)\, E\left(a_1,\dots,a_p:\rho_1,\dots,\rho_q:x^{-m}z\right)\, dx$$
$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta+\kappa-\frac{1}{2}}\, E\left(a_1,\dots,a_{p+2m}:\rho_1,\dots,\rho_{q+m}:m^{-m}z\right),$$
$$a_{p+k} = \frac{\beta+k+\mu-\frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta-\mu+k-\frac{1}{2}}{m}, \quad \rho_{q+k} = \frac{\beta-\kappa+k}{m}, \qquad k=1,\dots,m$$
$$\left[\operatorname{Re}\beta > |\operatorname{Re}\mu| - \tfrac{1}{2}, \quad m=1,2,\dots\right]$$ ET II 416(10)

This page intentionally left blank

8–9 Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.110

1. Every integral of the form $\int R\left(x, \sqrt{P(x)}\right) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx, \qquad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form.* The results of this reduction for the more frequently encountered integrals are given in formulas **3.13**–**3.17**. The number k is called the *modulus** of these integrals; the number $k' = \sqrt{1-k^2}$ is called the complementary modulus, and the number n is called the parameter of the integral of the third kind. BY (110.04)

2. By means of the substitution $x = \sin\varphi$, elliptic integrals can be reduced to the normal trigonometric forms

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad \int \sqrt{1-k^2\sin^2\varphi}\, d\varphi, \qquad \int \frac{d\varphi}{\left(1-n\sin^2\varphi\right)\sqrt{1-k^2\sin^2\varphi}}. \quad \text{BY (110.04)}$$

The results of reducing integrals of trigonometric functions to normal form are given in **2.58**–**2.62**.

3.[11] Elliptic integrals from 0 to 1 in the **8.110 1** formulation (or from 0 to $\frac{\pi}{2}$ in the **8.110 2** formulation) are called *complete elliptic integrals.*

4.* Take note that in mathematical software, and elsewhere, the notation for elliptic integrals is often modified by replacing the parameter k^2 that is used here with k.

8.111

Notations:

1. $\Delta\varphi = \sqrt{1-k^2\sin^2\varphi}; \quad k' = \sqrt{1-k^2}; \quad k^2 < 1$

*The quantity k is sometimes called the *module* of the functions.

2. The elliptic integral of the first kind:

$$F(\varphi,k)=\int_0^{\varphi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=\int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

3. The elliptic integral of the second kind:

$$E(\varphi,k)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\alpha}\,d\alpha=\int_0^{\sin\varphi}\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx$$ FI II 135

4.[11] The elliptic integral of the third kind:

$$\Pi(\varphi,n,k)=\int_0^{\varphi}\frac{d\alpha}{\left(1-n\sin^2\alpha\right)\sqrt{1-k^2\sin^2\alpha}}=\int_0^{\sin\varphi}\frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

BY (110.04)

5. $$D(\varphi,k)=\frac{F(\varphi,k)-E(\varphi,k)}{k^2}=\int_0^{\varphi}\frac{\sin^2\alpha\,d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=\int_0^{\sin\varphi}\frac{x^2\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

6.* $$\int_0^{\pi/2}\frac{dx}{\sqrt{a^2+\sin^2x}}\arctan\left(\frac{b}{\sqrt{a^2+\sin^2x}}\right)=\frac{\pi}{2|a|}F\left(\arcsin\left(\frac{b}{\sqrt{a^2+b^2+1}}\right),\frac{i}{a}\right)$$

[a and b are real]

8.112 Complete elliptic integrals

1. $\boldsymbol{K}(k)=F\left(\frac{\pi}{2},k\right)=\boldsymbol{K}'(k')$
2. $\boldsymbol{E}(k)=E\left(\frac{\pi}{2},k\right)=\boldsymbol{E}'(k')$
3. $\boldsymbol{K}'(k)=F\left(\frac{\pi}{2},k'\right)=\boldsymbol{K}(k')$
4. $\boldsymbol{E}'(k)=E\left(\frac{\pi}{2},k'\right)=\boldsymbol{E}(k')$
5. $\boldsymbol{D}=D\left(\frac{\pi}{2},k\right)=\frac{\boldsymbol{K}-\boldsymbol{E}}{k^2}$

In writing complete elliptic integrals, the modulus k, which acts as an independent variable, is often omitted, and we write

$$\boldsymbol{K}\,(\equiv\boldsymbol{K}(k)),\quad \boldsymbol{K}'\left(\equiv\boldsymbol{K}'(k)\right),\quad \boldsymbol{E}\,(\equiv\boldsymbol{E}(k)),\quad \boldsymbol{E}'\left(\equiv\boldsymbol{E}'(k)\right).$$

Series representations

8.113

1. $$\boldsymbol{K}=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2k^4+\cdots+\left(\frac{(2n-1)!!}{2^nn!}\right)^2k^{2n}+\ldots\right\}=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$$

FI II 487, WH 499

2. $$\boldsymbol{K}=\frac{\pi}{1+k'}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-k'}{1+k'}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-k'}{1+k'}\right)^4+\dots+\left(\frac{(2n-1)!!}{2^n n!}\right)^2\left(\frac{1-k'}{1+k'}\right)^{2n}+\dots\right\}$$

DW

3. $$\boldsymbol{K}=\ln\frac{4}{k'}+\left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\cdot 2}\right)k'^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\cdot 2}-\frac{2}{3\cdot 4}\right)k'^4$$
$$+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\ln\frac{4}{k'}-\frac{2}{1\cdot 2}-\frac{2}{3\cdot 4}-\frac{2}{5\cdot 6}\right)k'^6+\dots$$

DW

See also **8.197** 1 and **8.197** 2.

8.114

1.[6] $$\boldsymbol{E}=\frac{\pi}{2}\left\{1-\frac{1}{2^2}k^2-\frac{1^2\cdot 3}{2^2\cdot 4^2}k^4-\dots-\left(\frac{(2n-1)!!}{2^n n!}\right)^2\frac{k^{2n}}{2n-1}-\dots\right\}=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right)$$

WH 518, FI II 487

2. $$\boldsymbol{E}=\frac{(1+k')\pi}{4}\left\{1+\frac{1}{2^2}\left(\frac{1-k'}{1+k'}\right)^2+\frac{1^2}{2^2\cdot 4^2}\left(\frac{1-k'}{1+k'}\right)^4+\dots+\left(\frac{(2n-3)!!}{2^n n!}\right)^2\left(\frac{1-k'}{1+k'}\right)^{2n}+\dots\right\}$$

DW

3. $$\boldsymbol{E}=1+\frac{1}{2}\left(\ln\frac{4}{k'}-\frac{1}{1\cdot 2}\right)k'^2+\frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k'}-\frac{2}{1\cdot 2}-\frac{1}{3\cdot 4}\right)k'^4$$
$$+\frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'}-\frac{2}{1\cdot 2}-\frac{2}{3\cdot 4}-\frac{1}{5\cdot 6}\right)k'^6+\dots$$

DW

8.115 $$\boldsymbol{D}=\pi\left\{\frac{1}{1}\left(\frac{1}{2}\right)^2+\frac{2}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^2+\dots+\frac{n}{2n-1}\left[\frac{(2n-1)!!}{2^n n!}\right]^2k^{2(n-1)}+\dots\right\}$$ ZH 43(158)

8.116 $$\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-k^2\sin^2\varphi}}{1-n^2\sin^2\varphi}\,d\varphi=\sqrt{n'^2-k'^2}\left(\frac{\arccos\frac{1}{n'}}{n'\sqrt{n'^2-1}}+\mathbf{R}\right),\qquad\text{where}$$ ZH 44(163)

$$\mathbf{R}=\frac{k'^2}{2}\left(p+\frac{1}{2}\right)\frac{1}{n'^3}+\frac{k'^4}{16}\left[-1+\left(p+\frac{1}{4}\right)\frac{1}{n'^3}\left(1+\frac{6}{n'^2}\right)\right]$$
$$+\frac{k'^6}{16}\left[-\frac{7}{16}-\frac{1}{n'^2}+\left(p+\frac{1}{6}\right)\frac{1}{n'^3}\left(\frac{3}{8}+\frac{1}{n'^2}+\frac{5}{n'^4}\right)\right]$$
$$+\frac{15k'^8}{256}\left[-\frac{37}{144}-\frac{21}{40n'^2}-\frac{1}{n'^4}+\left(p+\frac{1}{8}\right)\frac{1}{n'^3}\left(\frac{5}{24}+\frac{9}{20n'^2}+\frac{1}{n'^4}+\frac{14}{3n'^6}\right)\right]+\dots,$$
$$p=\ln\frac{4}{k'},\quad k'=4e^{-p},\quad k'^2=1-k^2,\quad n'^2=1-n^2$$ ZH 44(163)

Trigonometric series

8.117 For *small* values of k and φ, we may use the series

1. $$F(\varphi,k)=\frac{2}{\pi}\boldsymbol{K}\varphi-\sin\varphi\cos\varphi\left(a_0+\frac{2}{3}a_1\sin^2\varphi+\frac{2\cdot 4}{3\cdot 5}a_2\sin^4\varphi+\dots\right),\qquad \text{where}$$

$$a_0=\frac{2}{\pi}\boldsymbol{K}-1;\quad a_n=a_{n-1}-\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}$$ ZH 10(19)

2. $$E(\varphi,k)=\frac{2}{\pi}\boldsymbol{E}\varphi+\sin\varphi\cos\varphi\left(b_0+\frac{2}{3}b_1\sin^2\varphi+\frac{2\cdot 4}{3\cdot 5}b_2\sin^4\varphi+\dots\right),\qquad \text{where}$$

$$b_0=1-\frac{2}{\pi}\boldsymbol{E},\quad b_n=b_{n-1}-\left[\frac{(2n-1)!!}{2^n n!}\right]^2\frac{k^{2n}}{2n-1}$$ ZH 27(86)

8.118 For k close to 1, we may use the series

1. $$F(\varphi,k)=\frac{2}{\pi}\boldsymbol{K}'\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)-\frac{\tan\varphi}{\cos\varphi}\left(a_0'-\frac{2}{3}a_1'\tan^2\varphi+\frac{2\cdot 4}{3\cdot 5}a_2'\tan^4\varphi-\dots\right),\qquad \text{where}$$

$$a_0'=\frac{2}{\pi}\boldsymbol{K}'-1;\quad a_n'=a_{n-1}-\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k'^{2n}$$ ZH 10(23)

2. $$E(\varphi,k)=\frac{2}{\pi}\left(\boldsymbol{K}'-\boldsymbol{E}'\right)\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{2}\right)$$
$$+\frac{\tan\varphi}{\cos\varphi}\left(b_1'-\frac{2}{3}b_2'\tan^2\varphi+\frac{2\cdot 4}{3\cdot 5}b_3'\tan^4\varphi-\dots\right)+\frac{1}{\sin\varphi}\left[1-\cos\varphi\sqrt{1-k^2\sin\varphi}\right],$$

where

$$b_0'=\frac{2}{\pi}\left(\boldsymbol{K}'-\boldsymbol{E}'\right),b_n'=b_{n-1}'-\left[\frac{(2n-3)!!}{2^{n-1}(n-1)!}\right]^2\left(\frac{2n-1}{2n}\right)k'^{2n}$$ ZH 27(90)

For the expansion of complete elliptic integrals in Legendre polynomials, see **8.928**.

8.119 Representation in the form of an infinite product:

1. $$\boldsymbol{K}(k)=\frac{\pi}{2}\prod_{n=1}^{\infty}\left(1+k_n\right),\qquad \text{where}$$

$$k_n=\frac{1-\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}};\qquad k_0=k$$ FI II 166

See also **8.197**.

8.12 Functional relations between elliptic integrals

8.121

1. $F(-\varphi,k) = -F(\varphi,k)$ JA
2. $E(-\varphi,k) = -E(\varphi,k)$ JA
3. $F(n\pi \pm \varphi,k) = 2n\,\boldsymbol{K}(k) \pm F(\varphi,k)$ JA
4. $E(n\pi \pm \varphi,k) = 2n\,\boldsymbol{E}(k) \pm E(\varphi,k)$ JA

8.122 $\boldsymbol{E}(k)\,\boldsymbol{K}'(k) + \boldsymbol{E}'(k)\,\boldsymbol{K}(k) - \boldsymbol{K}(k)\,\boldsymbol{K}'(k) = \dfrac{\pi}{2}$ FI II 691, 791

8.123

1. $\dfrac{\partial F}{\partial k} = \dfrac{1}{k'^2}\left(\dfrac{E - k'^2 F}{k} - \dfrac{k\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right)$ MO 138, BY (710.07)
2. $\dfrac{d\,\boldsymbol{K}(k)}{dk} = \dfrac{\boldsymbol{E}(k)}{kk'^2} - \dfrac{\boldsymbol{K}(k)}{k}$ FI II 691
3. $\dfrac{\partial E}{\partial k} = \dfrac{E-F}{k}$ MO 138
4. $\dfrac{d\,\boldsymbol{E}(k)}{dk} = \dfrac{\boldsymbol{E}(k) - \boldsymbol{K}(k)}{k}$ FI II 690

8.124

1. The functions $\boldsymbol{K}$ and $\boldsymbol{K}'$ satisfy the equation
$$\frac{d}{dk}\left\{kk'^2\frac{du}{dk}\right\} - ku = 0.$$
WH 499, WH 502

2. The functions $\boldsymbol{E}$ and $\boldsymbol{E}' - \boldsymbol{K}'$ satisfy the equation
$$k'^2\frac{d}{dk}\left(k\frac{du}{dk}\right) + ku = 0.$$
WH

8.125

1. $F\left(\psi, \dfrac{1-k'}{1+k'}\right) = (1+k')\,F(\varphi,k)$ $\quad[\tan(\psi-\varphi) = k'\tan\varphi]$ MO 130
2. $E\left(\psi, \dfrac{1-k'}{1+k'}\right) = \dfrac{2}{1+k'}\,[E(\varphi,k) + k'\,F(\varphi,k)] - \dfrac{1-k'}{1+k'}\sin\psi$
$[\tan(\psi-\varphi) = k'\tan\varphi]$ MO 131
3. $F\left(\psi, \dfrac{2\sqrt{k}}{1+k}\right) = (1+k)\,F(\varphi,k)$ $\quad\left[\sin\psi = \dfrac{(1+k)\sin\varphi}{1+k\sin^2\varphi}\right]$
4. $E\left(\psi, \dfrac{2\sqrt{k}}{1+k}\right) = \dfrac{1}{1+k}\left[2E(\varphi,k) - k'^2\,F(\varphi,k) + 2k\dfrac{\sin\varphi\cos\varphi}{1+k\sin^2\varphi}\sqrt{1-k^2\sin^2\varphi}\right]$
$\left[\sin\psi = \dfrac{(1+k)\sin\varphi}{1+k\sin^2\varphi}\right]$ MO 131

8.126 In particular,

1. $$\boldsymbol{K}\left(\frac{1-k'}{1+k'}\right)=\frac{1+k'}{2}\,\boldsymbol{K}(k)$$ MO 130

2. $$\boldsymbol{E}\left(\frac{1-k'}{1+k'}\right)=\frac{1}{1+k'}\left[\boldsymbol{E}(k)+k'\,\boldsymbol{K}(k)\right]$$ MO 130

3. $$\boldsymbol{K}\left(\frac{2\sqrt{k}}{1+k}\right)=(1+k)\,\boldsymbol{K}(k)$$ MO 130

4. $$\boldsymbol{E}\left(\frac{2\sqrt{k}}{1+k}\right)=\frac{1}{1+k}\left[2\,\boldsymbol{E}(k)-k'^2\,\boldsymbol{K}(k)\right]$$ MO 130

8.127[11]

k_1	$\sin\varphi_1$	$\cos\varphi_1$	$F(\varphi_1,k_1)$	$E(\varphi_1,k_1)$
$i\dfrac{k}{k'}$	$k'\dfrac{\sin\varphi}{\Delta\varphi}$	$\dfrac{\cos\varphi}{\Delta\varphi}$	$k'\,F(\varphi,k)$	$\dfrac{1}{k'}\left[E(\varphi,k)-\frac{k^2\sin\varphi\cos\varphi}{\Delta\varphi}\right]$
k'	$-i\tan\varphi$	$\sec\varphi$	$-i\,F(\varphi,k)$	$ia\left[E(\varphi,k)-F(\varphi,k)-\Delta\varphi\tan\varphi\right]$
$\dfrac{1}{k}$	$k\sin\varphi$	$\Delta\varphi$	$k\,F(\varphi,k)$	$\dfrac{1}{k}\left[E(\varphi,k)-k'^2\,F(\varphi,k)\right]$
$\dfrac{1}{k'}$	$-ik'\tan\varphi$	$\dfrac{\Delta\varphi}{\cos\varphi}$	$-ik'\,F(\varphi,k)$	$\dfrac{i}{k'}\left[E(\varphi,k)-k'^2\,F(\varphi,k)-\Delta\varphi\tan\varphi\right]$
$\dfrac{k'}{ik}$	$\dfrac{-ik\sin\varphi}{\Delta\varphi}$	$\dfrac{1}{\Delta\varphi}$	$-ik\,F(\varphi,k)$	$\dfrac{i}{k}\left[E(\varphi,k)-F(\varphi,k)-\frac{k^2\sin\varphi\cos\varphi}{\Delta\varphi}\right]$

(see **8.111** 1) MO 131

8.128 In particular,

1. $$\boldsymbol{K}\left(i\frac{k}{k'}\right)=k'\,\boldsymbol{K}(k)\qquad[\mathrm{Im}(k)<0]$$ MO 130

2. $$\boldsymbol{K}\left(i\frac{k}{k'}\right)=k'\left[\boldsymbol{K}'(k')-i\,\boldsymbol{K}(k)\right]\qquad[\mathrm{Im}(k)<0]$$ MO 130

3. $$\boldsymbol{K}\left(\frac{1}{k}\right)=k\left[\boldsymbol{K}(k)+i\,\boldsymbol{K}'(k)\right]\qquad[\mathrm{Im}(k)<0]$$ MO 130

For integrals of elliptic integrals, see **6.11**–**6.15**. For indefinite integrals of complete elliptic integrals, see **5.11**.

8.129 Special values:

1. $$\boldsymbol{K}\left(\sin\frac{\pi}{4}\right)=\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)=\boldsymbol{K}'\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^4}}=\frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$ MO 130

2. $$\boldsymbol{K}'\left(\sqrt{2}-1\right)=\sqrt{2}\,\boldsymbol{K}\left(\sqrt{2}-1\right)$$ MO 130

3. $\boldsymbol{K}'\left(\sin\frac{\pi}{12}\right) = \sqrt{3}\,\boldsymbol{K}\left(\sin\frac{\pi}{12}\right)$ MO 130

4. $\boldsymbol{K}'\left(\tan^2\frac{\pi}{8}\right) = \boldsymbol{K}'\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2\,\boldsymbol{K}\left(\tan^2\frac{\pi}{8}\right)$ MO 130

5.* $\boldsymbol{K}\left(\sin\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

6.* $\boldsymbol{E} = \frac{\pi\sqrt{3}}{12\boldsymbol{K}} + \sqrt{\frac{2}{3}}k'\boldsymbol{K}$

7.* $\boldsymbol{E}' = \frac{\pi\sqrt{3}}{4\boldsymbol{K}'} + \sqrt{\frac{2}{3}}k\boldsymbol{K}'$

8.13 Elliptic functions

8.130 Definition and general properties.

1. A single-valued function $f(z)$ of a complex variable, which is not a constant, is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$, that is

$$f\left(z + 2m\omega_1 + 2n\omega_2\right) = f(z) \qquad [m, n \text{ integers}]\,.$$

The ratio of the periods of an analytic function cannot be a real number. For an elliptic function $f(z)$, the z-plane can be partitioned into parallelograms—the period parallelograms—the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$. At corresponding points of these parallelograms, the function $f(z)$ has the same value. ZH 117, SI 299

2. Suppose that α is the angle between the sides a and b of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b}e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi\sin\alpha}\left[\cos\left(\frac{a}{b}\pi\cos\alpha\right) + i\sin\left(\frac{a}{b}\pi\cos\alpha\right)\right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods. SM III 598

4. A non-constant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points $a_1, a_2, \ldots, a_n$ and that their orders are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Suppose that the zeros of an analytic function that occur in a single parallelogram are $b_1, b_2, \ldots, b_m$ and that the orders of the zeros are $\beta_1, \beta_2, \ldots, \beta_m$, respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_m. \qquad \text{ZH 118}$$

The number γ representing this sum is called the *order* of the elliptic function.

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.

6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.

7. Every two elliptic functions with the same periods are related by an algebraic relationship. GO II 151

8.[7] A non-constant single-valued function which is not constant cannot have more than two periods. GO II 147

9. An elliptic function of order γ assumes *an arbitrary value* γ times in a period parallelogram. SM 601, SI 301

8.14 Jacobian elliptic functions

8.141 Consider the upper limit φ of the integral

$$u = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of u. Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude*. The quantity u is called the *argument*, and its dependence on φ is written

$$u = \arg \varphi.$$

8.142 The amplitude is an *infinitely many-valued* function of u and has a period of $4\boldsymbol{K}i$. The *branch points* of the amplitude correspond to the values of the argument

$$u = 2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i,$$ ZH 67–69

where m and n are arbitrary integers (see also **8.151**).

8.143 The first two of the following functions

$$\operatorname{sn} u = \sin \varphi = \sin \operatorname{am} u, \qquad \operatorname{cn} u = \cos \varphi = \cos \operatorname{am} u,$$
$$\operatorname{dn} u = \Delta\varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the *delta amplitude*. All these elliptic functions were exhibited by Jacobi and they bear his name. SI 16

The Jacobian elliptic functions are *doubly periodic* functions and have *two simple poles* in a period parallelogram. ZH 69

8.144

1. $$u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$ SI 21(23)

2. $$u = \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$$ SI 21(23)

3. $$u = \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}$$ SI 21(23)

8.145 Power series representations:

1.[11] $$\operatorname{sn} u = u - \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \frac{1+135k^2+135k^4+k^6}{7!}u^7 + \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}u^9 - \dots$$
$$[|u| < |\boldsymbol{K}'|]$$ ZH 81(97)

2. $$\operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \frac{1+408k^2+912k^4+64k^6}{8!}u^8 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ ZH 81(98)

3. $$\operatorname{dn} u = 1 - \frac{k^2}{2!}u^2 + \frac{k^2\left(4+k^2\right)}{4!}u^4 - \frac{k^2\left(16+44k^2+k^4\right)}{6!}u^6 + \frac{k^2\left(64+912k^2+408k^4+k^6\right)}{8!}u^8 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ ZH 81(99)

4. $$\operatorname{am} u = u - \frac{k^2}{3!}u^3 + \frac{k^2\left(4+k^2\right)}{5!}u^5 - \frac{k^2\left(16+44k^2+k^4\right)}{7!}u^7 + \frac{k^2\left(64+912k^2+408k^4+k^6\right)}{9!}u^9 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ LA 380(4)

8.146 Representation as a trigonometric series or a product $\left(q = e^{-\frac{\pi K'}{K}} = e^{\pi i\tau}\right)$*

1.[11] $$\operatorname{sn} u = \frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ WH 511a, ZH 84(108)

2.[11] $$\operatorname{cn} u = \frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ WH 511a, ZH 84(109)

3. $$\operatorname{dn} u = \frac{\pi}{2\boldsymbol{K}} + \frac{2\pi}{\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{\boldsymbol{K}}$$ WH 511a, ZH 84(110)

4.[11] $$\operatorname{am} u = \frac{\pi u}{2\boldsymbol{K}} + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\frac{n\pi u}{\boldsymbol{K}}$$ WH 511a

5. $$\frac{1}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(3)

6. $$\frac{1}{\operatorname{cn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[\frac{1}{\cos\frac{\pi u}{2\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n-1}}{1+q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(3)

7. $$\frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[1 + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(3)

8. $$\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1+q^{2n}}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(4)

9.[11] $$\frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'\boldsymbol{K}}\sum_{n=1}^{\infty}(-1)^n\frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(4)

10. $$\frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin\frac{\pi n u}{\boldsymbol{K}}\right]$$ LA 369(5)

---

*The expansions 1–22 are valid in every strip of the form $\left|\operatorname{Im}\frac{\pi u}{2\boldsymbol{K}}\right| < \frac{1}{2}\pi\operatorname{Im}\tau$. The expansions 23–25 are valid in an arbitrary bounded portion of u.

11. $$\frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}(-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(5)

12. $$\frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(6)

13. $$\frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\cos\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n-1}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(6)

14. $$\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(7)

15. $$\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left\{\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}\frac{q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right\}$$ LA 369(7)

16. $$\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}$$ LA 369(7)

17. $$\frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2\left(1-k^2\right)\boldsymbol{K}}\left[\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1-q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

18. $$\frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

19. $$\frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

20.[11] $$\ln\operatorname{sn} u = \ln\frac{2\boldsymbol{K}}{\pi} + \ln\sin\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}$$ LA 369(2)

21. $$\ln\operatorname{cn} u = \ln\cos\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}$$ LA 369(2)

22. $$\ln\operatorname{dn} u = -8\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{q^{2n-1}}{1-q^{2(2n-1)}}\sin^2(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(2)

23.[11] $$\operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}}\sin\frac{\pi u}{2\boldsymbol{K}}\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(145)

24. $$\operatorname{cn} u = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}}\cos\frac{\pi u}{2\boldsymbol{K}}\prod_{n=1}^{\infty}\frac{1+2q^{2n}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(146)

25. $$\operatorname{dn} u = \sqrt{k'}\prod_{n=1}^{\infty}\frac{1+2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(147)

26. $$\operatorname{sn}^3 u = \sum_{n=0}^{\infty}\left[\frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3}\frac{\pi^2}{4\boldsymbol{K}^2}\right]\frac{2\pi q^{n+\frac{1}{2}}\sin(2n+1)\frac{\pi u}{2\boldsymbol{K}}}{\boldsymbol{K}\left(1-q^{2n+1}\right)}$$
$$\left[\left|\operatorname{Im}\frac{u}{2\boldsymbol{K}}\right| < \operatorname{Im}\tau\right]$$ MO 147

27. $$\frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4\boldsymbol{K}^2}\operatorname{cosec}^2\frac{\pi u}{2\boldsymbol{K}} + \frac{\boldsymbol{K}-\boldsymbol{E}}{\boldsymbol{K}} - \frac{2\pi^2}{\boldsymbol{K}^2}\sum_{n=1}^{\infty}\frac{nq^{2n}\cos\frac{n\pi u}{\boldsymbol{K}}}{1-q^{2n}}$$
$$\left[\left|\operatorname{Im}\frac{u}{2\boldsymbol{K}}\right| < \frac{1}{2}\operatorname{Im}\tau\right]$$ MO 148

8.147

1. $$\operatorname{sn} u = \frac{\pi}{2k\boldsymbol{K}}\sum_{n=-\infty}^{\infty}\frac{1}{\sin\frac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}$$ MO 149

2. $$\operatorname{cn} u = \frac{\pi i}{2k\boldsymbol{K}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sin\frac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}$$ MO 150

3. $$\operatorname{dn} u = \frac{\pi i}{2\boldsymbol{K}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\tan\frac{\pi}{2\boldsymbol{K}}[u-(2n-1)i\boldsymbol{K}']}$$ MO150

8.148 The Weierstrass expansions of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$:
$$\operatorname{sn} u = \frac{B}{A}, \qquad \operatorname{cn} u = \frac{C}{A}, \qquad \operatorname{dn} u = \frac{D}{A},$$ ZH 82–83(105,106,107)

where

$$A = 1 - \sum_{n=1}^{\infty}(-1)^{n+1}a_{n+1}\frac{u^{2n+2}}{(2n+2)!} \qquad B = \sum_{n=0}^{\infty}(-1)^n b_n\frac{u^{2n+1}}{(2n+1)!}$$
$$C = \sum_{n=0}^{\infty}(-1)^n c_n\frac{u^{2n}}{(2n)!} \qquad D = \sum_{n=0}^{\infty}(-1)^n d_n\frac{u^{2n}}{(2n)!}$$

and

$$a_2 = 2k^2, \quad a_3 = 8\left(k^2+k^4\right), \quad a_4 = 32\left(k^2+k^6\right)+68k^4, \quad a_5 = 128\left(k^2+k^8\right)+480\left(k^4+k^6\right),$$
$$a_6 = 512\left(k^2+k^{10}\right)+3008\left(k^4+k^8\right)+5400k^6, \quad \ldots$$
$$b_0 = 1, \quad b_1 = 1+k^2, \quad b_2 = 1+k^4+4k^2, \quad b_3 = 1+k^6+9\left(k^2+k^4\right),$$
$$b_4 = 1+k^8+16\left(k^2+k^6\right)-6k^4, \quad b_5 = 1+k^{10}+25\left(k^2+k^8\right)-494\left(k^4+k^6\right),$$
$$b_6 = 1+k^{12}+36\left(k^2+k^{10}\right)-5781\left(k^4+k^8\right)-12184k^6, \quad \ldots$$
$$c_0 = 1, \quad c_1 = 1, \quad c_2 = 1+2k^2, \quad c_3 = 1+6k^2+8k^4, \quad c_4 = 1+12k^2+60k^4+32k^6,$$
$$c_5 = 1+20k^2+348k^4+448k^6+128k^8, \quad c_6 = 1+30k^2+2372k^4+4600k^6+2880k^8+512k^{10}, \quad \ldots$$
$$d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2+k^4, \quad d_3 = 8k^2+6k^4+k^6, \quad d_4 = 32k^2+60k^4+12k^4+k^8,$$
$$d_5 = 128k^2+448k^4+348k^6+20k^8+k^{10},$$
$$d_6 = 512k^2+2880k^4+4600k^6+2372k^8+30k^{10}+k^{12}, \quad \ldots$$

8.15 Properties of Jacobian elliptic functions and functional relationships between them

8.151 The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\boldsymbol{K}+2n\boldsymbol{K}'i$	$2m\boldsymbol{K}+2n\boldsymbol{K}'i$	$2m\boldsymbol{K}+(2n+1)\boldsymbol{K}'i$	$(-1)^m\dfrac{1}{k}$
$\operatorname{cn} u$	$4m\boldsymbol{K}+2n\left(\boldsymbol{K}+\boldsymbol{K}'i\right)$	$(2m+1)\,\boldsymbol{K}+2n\boldsymbol{K}'i$	$2m\boldsymbol{K}+(2n+1)\boldsymbol{K}'i$	$(-1)^{m-1}\frac{i}{k}$
$\operatorname{dn} u$	$2m\boldsymbol{K}+4n\boldsymbol{K}'i$	$(2m+1)\boldsymbol{K}+(2n+1)\boldsymbol{K}'i$	$2m\boldsymbol{K}+(2n+1)\boldsymbol{K}'i$	$(-1)^{n-1}i$

SM 630, ZH 69–72

2.

$u^*=u+\boldsymbol{K}$	$u+i\boldsymbol{K}$	$u+\boldsymbol{K}+i\boldsymbol{K}'$	$u+2\boldsymbol{K}$	$u+2i\boldsymbol{K}'$	$u+2\boldsymbol{K}+2i\boldsymbol{K}'$
$\operatorname{sn} u^*=\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$	$\dfrac{1}{k\operatorname{sn} u}$	$\dfrac{1}{k}\dfrac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\operatorname{sn} u$	$\operatorname{sn} u$	$-\operatorname{sn} u$
$\operatorname{cn} u^*=-k'\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$	$-\dfrac{i}{k}\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$	$-\dfrac{ik'}{k\operatorname{cn} u}$	$-\operatorname{cn} u$	$-\operatorname{cn} u$	$\operatorname{cn} u$
$\operatorname{dn} u^*=k'\dfrac{1}{\operatorname{dn} u}$	$-i\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$	$ik'\dfrac{\operatorname{sn} u}{\operatorname{cn} u}$	$\operatorname{dn} u$	$-\operatorname{dn} u$	$-\operatorname{dn} u$

SM 630

3.

$u^*=0$	$-u$	$\frac{1}{2}\boldsymbol{K}$	$\frac{1}{2}\left(\boldsymbol{K}+i\boldsymbol{K}'\right)$	$\frac{1}{2}i\boldsymbol{K}'$	$u+2m\boldsymbol{K}+2n\boldsymbol{K}'i$
$\operatorname{sn} u^*=0$	$-\operatorname{sn} u$	$\dfrac{1}{\sqrt{1+k'}}$	$\dfrac{\sqrt{1+k}+i\sqrt{1-k}}{\sqrt{2k}}$	$\dfrac{i}{\sqrt{k}}$	$(-1)^m\operatorname{sn} u$
$\operatorname{cn} u^*=1$	$\operatorname{cn} u$	$\dfrac{\sqrt{k'}}{\sqrt{1+k'}}$	$\dfrac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\dfrac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n}\operatorname{cn} u$
$\operatorname{dn} u^*=1$	$\operatorname{dn} u$	$\sqrt{k'}$	$\dfrac{\sqrt{k'}\left(\sqrt{1+k'}-i\sqrt{1-k'}\right)}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n\operatorname{dn} u$

SI 19, SI 18(13), WH, WH WH WH

8.152 Transformation formulas

u_1	l_1	$sn\,(u_1,k_1)$	$\mathrm{cn}\,(u_1,k_1)$	$\mathrm{dn}\,(u_1,k_1)$
ku	$\frac{1}{k}$	$k\,\mathrm{sn}(u,k)$	$\mathrm{dn}(u,k)$	$\mathrm{cn}(u,k)$
iu	k'	$i\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\frac{1}{\mathrm{cn}(u,k)}$	$\frac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$
$k'u$	$i\frac{k}{k'}$	$k'\frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\frac{1}{\mathrm{dn}(u,k)}$
iku	$i\frac{k'}{k}$	$ik\frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}$	$\frac{1}{\mathrm{dn}(u,k)}$	$\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$
$ik'u$	$\frac{1}{k'}$	$ik'\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)}$	$\frac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)}$	$\frac{1}{\mathrm{cn}(u,k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k)\,\mathrm{sn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\frac{\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$	$\frac{1-k\,\mathrm{sn}^2(u,k)}{1+k\,\mathrm{sn}^2(u,k)}$
$(1+k')\,u$	$\frac{1-k'}{1+k'}$	$(1+k')\,\frac{\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$	$\frac{1-(1+k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$	$\frac{1-(1-k')\,\mathrm{sn}^2(u,k)}{\mathrm{dn}(u,k)}$
$\frac{\left(1+\sqrt{k'}\right)^2}{2}u$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^2$	$\frac{k^2\,\mathrm{sn}(u,k)d\,\mathrm{cn}(u,k)}{\sqrt{k_1}\,[1+\mathrm{dn}(u,k)]\,[k'+\mathrm{dn}(u,k)]}$	$\frac{\mathrm{dn}(u,k)-\sqrt{k'}}{1-\sqrt{k'}}$ $\times\sqrt{\frac{2(1+k')}{[1+\mathrm{dn}(u,k)][k'+\mathrm{dn}(u,k)]}}$	$\frac{\sqrt{1+k_1}\left(\mathrm{dn}(u,k)+\sqrt{k'}\right)}{\sqrt{[1+\mathrm{dn}(u,k)]\,[k'+\mathrm{dn}(u,k)]}}$

8.153

1. $\operatorname{sn}(iu,k) = i\dfrac{\operatorname{sn}(u,k')}{\operatorname{cn}(u,k')}$ SI 50(64)

2. $\operatorname{cn}(iu,k) = \dfrac{1}{\operatorname{cn}(u,k')}$ SI 50(65)

3. $\operatorname{dn}(iu,k) = \dfrac{\operatorname{dn}(u,k')}{\operatorname{cn}(u,k')}$ SI 50(65)

4. $\operatorname{sn}(u,k) = k^{-1}\operatorname{sn}\left(ku,k^{-1}\right)$

5. $\operatorname{cn}(u,k) = \operatorname{dn}\left(ku,k^{-1}\right)$

6. $\operatorname{dn}(u,k) = \operatorname{cn}\left(ku,k^{-1}\right)$

7.[11] $\operatorname{sn}(u,ik) = \dfrac{1}{\sqrt{1+k^2}}\dfrac{\operatorname{sn}\left(u\sqrt{1+k^2},k\left(1+k^2\right)^{-1/2}\right)}{\operatorname{dn}\left(u\sqrt{1+k^2},k\left(1+k^2\right)^{-1/2}\right)}$

8.[11] $\operatorname{cn}(u,ik) = \dfrac{\operatorname{sn}\left(u\left(1+k^2\right)^{1/2},k\left(1+k^2\right)^{-1/2}\right)}{\operatorname{dn}\left(u\left(1+k^2\right)^{1/2},k\left(1+k^2\right)^{-1/2}\right)}$

9.[11] $\operatorname{dn}(u,ik) = \dfrac{1}{\operatorname{dn}\left(u\left(1+k^2\right)^{1/2},k\left(1+k^2\right)^{-1/2}\right)}$

Functional relations

8.154

1. $\operatorname{sn}^2 u = \dfrac{1-\operatorname{cn}2u}{1+\operatorname{dn}2u}$ MO 146

2. $\operatorname{cn}^2 u = \dfrac{\operatorname{cn}2u+\operatorname{dn}2u}{1+\operatorname{dn}2u}$ MO 146

3. $\operatorname{dn}^2 u = \dfrac{\operatorname{dn}2u+k^2\operatorname{cn}2u+k'^2}{1+\operatorname{dn}2u}$ MO 146

4. $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$ SI 16(9)

5. $\operatorname{dn}^2 u + k^2\operatorname{sn}^2 u = 1$ SI 16(9)

8.155

1. $\dfrac{1-\operatorname{dn}2u}{1+\operatorname{dn}2u} = k^2\dfrac{\operatorname{sn}^2 u\operatorname{cn}^2 u}{\operatorname{dn}^2 u}$ MO 146

2. $\dfrac{1-\operatorname{cn}2u}{1+\operatorname{cn}2u} = \dfrac{\operatorname{sn}^2 u\operatorname{dn}^2 u}{\operatorname{cn}^2 u}$ MO 146

8.156

1. $\operatorname{sn}(u\pm v) = \dfrac{\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v \pm \operatorname{sn}v\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 v}$ SI 46(56)

2. $$\operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$ SI 46(57)

3. $$\operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$ SI 46(58)

8.157

1. $$\operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}$$ SI 47(61), SU 67(15)

2. $$\operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}}$$ SI 48(62), SI 67(16)

3. $$\operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}}$$ SI 48(63), SI 67(17)

8.158

1. $$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$$ SI 21(21)

2. $$\frac{d}{du} \operatorname{cn} u = - \operatorname{sn} u \operatorname{dn} u$$ SI 21(21)

3.[8] $$\frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{dn} u \operatorname{cn} u$$ SI 21(21)

8.159 Jacobian elliptic functions are solutions of the following differential equations:

1. $$\frac{d}{du} \operatorname{sn} u = \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)}$$ SI 21(22)

2. $$\frac{d}{du} \operatorname{cn} u = -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)},$$ SI 21(22)

3. $$\frac{d}{du} \operatorname{dn} u = -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}$$ SI 21(22)

For the indefinite integrals of Jacobi's elliptic functions, see **5.13**.

8.16 The Weierstrass function $\wp(u)$

8.160 The Weierstrass elliptic function $\wp(u)$ is defined by

1. $$\wp(u) = \frac{1}{u^2} + \sum_{m,n}{}' \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},$$ SI 307(6)

   where the symbol $\sum'$ means that the summation is made over all combinations of integers m and n except for the combination $m = n = 0$; $2\omega_1$ and $2\omega_2$ are the periods of the function $\wp(u)$. Obviously,

2. $$\wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \text{ and } \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0,$$

3. $$\frac{d}{du}\,\wp(u) = -2\sum_{m,n}\frac{1}{(u-2m\omega_1-2n\omega_2)^3},$$

where the summation is made over all integral values of m and n.

The series **8.160** 1 and **8.160** 3 converge everywhere except at the poles, that is, at the points $2m\omega_1+2n\omega_2$ (where m and n are integers).

4. The function $\wp(u)$ is a *doubly periodic function* and has *one second-order pole* in a period parallelogram. SI 306

8.161 The function $\wp(u)$ satisfies the differential equation

1. $$\left[\frac{d\,\wp(u)}{du}\right]^2 = 4\,\wp^3(u) - g_2\,\wp(u) - g_3,$$ SI 142, 310, WH

where

2. $$g_2 = 60\sum_{m,n}{}'\,(m\omega_1+n\omega_2)^{-4}; \qquad g_3 = 140\sum_{m,n}{}'\,(m\omega_1+n\omega_2)^{-6}$$ WH, SI 310

The functions g_2 and g_3 are called the *invariants* of the function $\wp(u)$.

8.162 $$u = \int_{\wp(u)}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}} = \int_{\wp(u)}^{\infty}\frac{dz}{\sqrt{4\,(z-e_1)\,(z-e_2)\,(z-e_3)}},$$

where e_1, e_2, and e_3 are the roots of the equation $4z^3-g_2z-g_3=0$; that is,

$$e_1+e_2+e_3=0, \quad e_1e_2+e_2e_3+e_3e_1=-\frac{g_2}{4}, \quad e_1e_2e_3=\frac{g_3}{4}$$ SI 142, 143, 144

8.163 $\wp(\omega_1)=e_1$, $\wp(\omega_1)+\omega_2=e_2$, $\wp(\omega_2)=e_3$. Here, it is assumed that if e_1, e_2, and e_3 lie on a straight line in the complex plane, e_2 lies between e_1 and e_3.

8.164 The number $\Delta = g_2^3-27g_3^2$ is called the *discriminant* of the function $\wp(u)$. If $\Delta>0$, all roots e_1, e_2, and e_3 of the equation $4z^3-g_2z-g_3=0$ (where g_2 and g_3 are real numbers) are *real*. In this case, the roots e_1, e_2, and e_3 are numbered in such a way that $e_1>e_2>e_3$.

1. If $\Delta>0$, then

$$\omega_1 = \int_{e_1}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}}, \qquad \omega_2 = i\int_{-\infty}^{e_3}\frac{dz}{\sqrt{g_3+g_2z-4z^3}},$$

where ω_1 is real and ω_2 is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that ω_1 and $\dfrac{\omega_2}{i}$ will be positive.

2. If $\Delta<0$, the root e_2 of the equation $4z^3-g_2z-g_3=0$ is *real*, and the remaining two roots (e_1 and e_3) are *complex conjugates*. Suppose that $e_1=\alpha+i\beta$, and $e_3=\alpha-i\beta$. In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}} \quad\text{and}\quad \omega'' = \int_{e_3}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}}$$

as basic semiperiods.

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower half-plane. SI 151(21, 22)

8.165 Series representation:

1. $$\wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4\cdot 5} + \frac{g_3 u^4}{4\cdot 7} + \frac{g_2^2 u^6}{2^4\cdot 3\cdot 5^2} + \frac{3 g_2 g_3 u^8}{2^4\cdot 5\cdot 7\cdot 11} + \dots$$ WH

8.166 Functional relations

1. $$\wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u)$$

2. $$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$ SI 163(32)

8.167 $$\wp(u; g_2, g_3) = \mu^2\, \wp\left(\mu u; \frac{g_2}{\mu^4}, \quad \frac{g_3}{\mu^6}\right)$$ (the formula for homogeneity)

SI 149(13)

The special case: $\mu = i$.

1. $$\wp(u; g_2, g_3) = -\wp(iu; g_2, -g_3)$$

8.168 An arbitrary elliptic function can be expressed in terms of the elliptic function $\wp(u)$ having the same periods as the original function and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

8.169 A connection with the Jacobian elliptic functions. For $\Delta > 0$ (see **8.164** 1).

1. $$\begin{aligned}\wp\left(\frac{u}{\sqrt{e_1-e_2}}\right) &= e_1 + (e_1-e_3)\frac{\operatorname{cn}^2(u;k)}{\operatorname{sn}^2(u;k)}\\ &= e_2 + (e_1-e_3)\frac{\operatorname{dn}^2(u;k)}{\operatorname{sn}^2(u;k)}\\ &= e_3 + (e_1-e_3)\frac{1}{\operatorname{sn}^2(u;k)}\end{aligned}$$

SI 145(5), ZH 120(197–199)a

2. $$\omega_1 = \frac{\boldsymbol{K}}{\sqrt{e_1-e_3}}, \qquad \omega_2 = \frac{i\boldsymbol{K}'}{\sqrt{e_1-e_3}},$$ SI 154(29)

where

3. $$k = \sqrt{\frac{e_2-e_3}{e_1-e_3}}, \qquad k' = \sqrt{\frac{e_1-e_2}{e_1-e_3}}$$ SI 145(7)

For $\Delta < 0$ (see **8.164** 2)

4. $$\wp\left(\frac{u}{\sqrt[4]{9\alpha^2+\beta^2}}\right) = e_2 + \sqrt{9\alpha^2+\beta^2}\,\frac{1+\operatorname{cn}(2u;k)}{1-\operatorname{cn}(2u;k)};$$ SI 147(12)

5. $$\omega' = \frac{\boldsymbol{K}-i\boldsymbol{K}'}{2\sqrt{9\alpha^2+\beta^2}}, \qquad \omega'' = \frac{\boldsymbol{K}+i\boldsymbol{K}'}{\sqrt[4]{9\alpha^2+\beta^2}},$$ SI 153(28)

where

6.[11] $$k = \sqrt{\frac{1}{2} - \frac{3e_2}{4\sqrt{9\alpha^2+\beta^2}}}; \qquad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{4\sqrt{9\alpha^2+\beta^2}}}$$ SI 147

For $\Delta = 0$, all the roots e_1, e_2, and e_3 are real, and if $g_2 g_3 \neq 0$, two of them are equal to each other. If $e_1 = e_2 \neq e_3$, then

7. $\wp(u) = \dfrac{3g_3}{g_2} - \dfrac{9g_3}{2g_2}\coth^2\left(u\sqrt{-\dfrac{9g_3}{2g_2}}\right)$ SI 148

If $e_1 \neq e_2 = e_3$, then

8. $\wp(u) = -\dfrac{3g_3}{2g_2} + \dfrac{9g_3}{2g_2}\dfrac{1}{\sin^2\left(u\sqrt{\frac{9g_3}{2g_2}}\right)}$ SI 149

If $g_2 = g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and

9. $\wp(u) = \dfrac{1}{u^2}$ SI 149

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.171 Definitions:

1. $\zeta(u) = \dfrac{1}{u} - \displaystyle\int_0^u \left(\wp(z) - \frac{1}{z^2}\right) dz$ SI 181(45)

2. $\sigma(u) = u\exp\left\{\displaystyle\int_0^u \left(\wp(z) - \frac{1}{z^2}\right) dz\right\}$ SI 181(46)

8.172 Series and infinite-product representation

1. $\zeta(u) = \dfrac{1}{u} + \displaystyle\sum_{m,n}{}' \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 - 2n\omega_2)^2}\right)$ SI 307(8)

2. $\sigma(u) = u\displaystyle\prod_{mn,}{}' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2}\right)\exp\left\{\frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2\,(2m\omega_1 + 2n\omega_2)^2}\right\}$ SI 308(9)

8.173

1. $\zeta(u) = u - \dfrac{g_2u^3}{2^2\cdot 3\cdot 5} - \dfrac{g_3u^5}{2^2\cdot 5\cdot 7} - \dfrac{g_2^2u^7}{2^4\cdot 3\cdot 5^2\cdot 7} - \dfrac{3g_2g_3u^9}{2^4\cdot 5\cdot 7\cdot 9\cdot 11} - \cdots$ SI 181(49)

2. $\sigma(u) = u - \dfrac{g_2u^5}{2^4\cdot 3\cdot 5} - \dfrac{g_3u^7}{2^3\cdot 3\cdot 5\cdot 7} - \dfrac{g_2^2u^9}{2^9\cdot 3^2\cdot 5\cdot 7} - \dfrac{3g_2g_3u^{11}}{2^7\cdot 3^2\cdot 5^2\cdot 7\cdot 11} - \cdots$ SI 181(49)

8.174 $\zeta(u) = \dfrac{\zeta(\omega_1)}{\omega_1}u + \dfrac{\pi}{2\omega_1}\cot\dfrac{\pi u}{2\omega_1} + \dfrac{\pi}{2\omega_1}\displaystyle\sum_{n=1}^{\infty}\left\{\cot\left(\frac{\pi u}{2\omega_1} + n\pi\frac{\omega_2}{\omega_1}\right)\right.$

$\left. + \cot\left(\dfrac{\pi u}{2\omega_1} - n\pi\dfrac{\omega_2}{\omega_1}\right)\right\}$ MO 154

$= \dfrac{\zeta(\omega_1)}{\omega_1}u + \dfrac{\pi}{2\omega_1}\cot\dfrac{\pi u}{2\omega_1} + \dfrac{2\pi}{\omega_1}\displaystyle\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{\pi n u}{\omega_1}$ MO 155

Functional relations and properties

8.175 $\zeta(u) = -\zeta(-u), \quad \sigma(u) = -\sigma(-u)$ SI 181

8.176

1. $\zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1)$ SI 184(57)

2. $\zeta(u+2\omega_2) = \zeta(u) + 2\zeta(\omega_2)$ SI 184(57)

3. $\sigma(u+2\omega_1) = -\sigma(u)\exp\{2(u+\omega_1)\zeta(\omega_1)\}.$ SI 185(60)

4. $\sigma(u+2\omega_2) = -\sigma(u)\exp\{2(u+\omega_2)\zeta(\omega_2)\}.$ SI 185(60)

5. $\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{\pi}{2}i$ SI 186(62)

8.177

1. $\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$ SI 182(53)

2. $\wp(u) - \wp(v) = -\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}$ SI 183(54)

3. $\zeta(u-v) + \zeta(u+v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u)-\wp(v)}$ SI 182(51)

8.178

1. $\zeta(u;\omega_1,\omega_2) = t\,\zeta(tu;t\omega_1,t\omega_2)$ MO 154

2.[8] $\sigma(u;\omega_1,\omega_2) = t^{-1}\,\sigma(tu;t\omega_1,t\omega_2)$ MO 156

For the indefinite integrals of Weierstrass elliptic functions, see **5.14**.

8.18–8.19 Theta functions

8.180 *Theta functions* are defined as the sums (for $|q|<1$) of the following series:

1. $\vartheta_4(u) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2nui} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nu$ WH

2. $\vartheta_1(u) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)ui} = 2\sum_{n=1}^{\infty}(-1)^{n+1} q^{\left(n-\frac{1}{2}\right)^2}\sin(2n-1)u$ WH

3.[11] $\vartheta_2(u) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)ui} = 2\sum_{n=1}^{\infty} q^{\left(n-\frac{1}{2}\right)^2}\cos(2n-1)u$ WH

4. $\vartheta_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nu$ WH

The notations $\vartheta(u,q)$ and $\vartheta(u \mid \tau)$, where τ and q are related by $q = e^{i\pi\tau}$, are also used. Here, q is called the *nome* of the theta function and τ its *parameter*.

8.181 Representation of theta functions in terms of infinite products

1. $\vartheta_4(u) = \prod_{n=1}^{\infty}\left(1 - 2q^{2n-1}\cos 2u + q^{2(2n-1)}\right)\left(1-q^{2n}\right)$ SI 200(9), ZH 90(9)

2. $\vartheta_3(u) = \prod_{n=1}^{\infty}\left(1 + 2q^{2n-1}\cos 2u + q^{2(2n-1)}\right)\left(1-q^{2n}\right)$ SI 200(9), ZH 90(9)

3. $\vartheta_1(u) = 2\sqrt[4]{q}\sin u \prod_{n=1}^{\infty}\left(1 - 2q^{2n}\cos 2u + q^{4n}\right)\left(1 - q^{2n}\right)$ SI 200(9), ZH 90(9)

4.[8] $\vartheta_2(u) = 2\sqrt[4]{q}\cos u \prod_{n=1}^{\infty}\left(1 + 2q^{2n}\cos 2u + q^{4n}\right)\left(1 - q^{2n}\right)$ SI 200(0), ZH 90(9)

Functional relations and properties

8.182 Quasiperiodicity. Suppose that $q = e^{\pi\tau i}$ $(\operatorname{Im}\tau > 0)$. Then, theta functions that are periodic functions of u are called *quasiperiodic functions* of τ and u. This property follows from the equations

1. $\vartheta_4(u+\pi) = \vartheta_4(u)$ SI 200(10)
2. $\vartheta_4(u+\tau\pi) = -\dfrac{1}{q}e^{-2iu}\,\vartheta_4(u)$ SI 200(10)
3. $\vartheta_1(u+\pi) = -\,\vartheta_1(u)$ SI 200(10)
4. $\vartheta_1(u+\tau\pi) = -\dfrac{1}{q}e^{-2iu}\,\vartheta_1(u)$ SI 200(10)
5. $\vartheta_2(u+\pi) = -\,\vartheta_2(u)$ SI 200(10)
6. $\vartheta_2(u+\tau\pi) = \dfrac{1}{q}e^{-2iu}\,\vartheta_2(u)$ SI 200(10)
7. $\vartheta_3(u+\pi) = \vartheta_3(u)$ SI 200(10)
8. $\vartheta_3(u+\tau\pi) = \dfrac{1}{q}e^{-2iu}\,\vartheta_3(u)$ SI 200(10)

8.183

1. $\vartheta_4\left(u+\frac{1}{2}\pi\right) = \vartheta_3(u)$ WH
2. $\vartheta_1\left(u+\frac{1}{2}\pi\right) = \vartheta_2(u)$ WH
3. $\vartheta_2\left(u+\frac{1}{2}\pi\right) = -\,\vartheta_1(u)$ WH
4. $\vartheta_3\left(u+\frac{1}{2}\pi\right) = \vartheta_4(u)$ WH
5. $\vartheta_4\left(u+\frac{1}{2}\pi\tau\right) = iq^{-1/4}e^{-iu}\,\vartheta_1(u)$ WH
6. $\vartheta_1\left(u+\frac{1}{2}\pi\tau\right) = iq^{-1/4}e^{-iu}\,\vartheta_4(u)$ WH
7. $\vartheta_2\left(u+\frac{1}{2}\pi\tau\right) = q^{-1/4}e^{-iu}\,\vartheta_3(u)$ WH
8. $\vartheta_3\left(u+\frac{1}{2}\pi\tau\right) = q^{-1/4}e^{-iu}\,\vartheta_2(u)$ WH

8.184 Even and odd theta functions

1. $\vartheta_1(-u) = -\,\vartheta_1(u)$ WH
2. $\vartheta_2(-u) = \vartheta_2(u)$ WH
3. $\vartheta_3(-u) = \vartheta_3(u)$ WH
4. $\vartheta_4(-u) = \vartheta_4(u)$ WH

8.185 $\vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u)$ WH

8.186[7] Considering the theta functions as functions of two independent variables u and τ, we have

$$\pi i \frac{\partial^2 \vartheta_k(u \mid \tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u \mid \tau)}{\partial \tau} = 0 \qquad [k = 1, 2, 3, 4]$$ WH

8.187 We denote the partial derivatives of the theta functions with respect to u by a prime and consider them as functions of the single argument u. Then,

1. $\vartheta_1'(0) = \vartheta_2(0)\,\vartheta_3(0)\,\vartheta_4(0)$ WH

2. $\dfrac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \dfrac{\vartheta_2''(0)}{\vartheta_2(0)} + \dfrac{\vartheta_3''(0)}{\vartheta_3(0)} + \dfrac{\vartheta_4''(0)}{\vartheta_4(0)}$ WH

8.188 $\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)\,\vartheta_4(0) = \frac{1}{2}\,\vartheta_1(2u)\,\vartheta_2(0)\,\vartheta_3(0)\,\vartheta_4(0)$ WH

8.189 The zeros of the theta functions:

1.[8] $\vartheta_4(u) = 0$ for $u = 2m\dfrac{\pi}{2} + (2n-1)\dfrac{\pi\tau}{2}$ SI 201

2.[10] $\vartheta_1(u) = 0$ for $u = 2m\dfrac{\pi}{2} + 2n\dfrac{\pi\tau}{2}$ SI 201

3. $\vartheta_2(u) = 0$ for $u = (2m-1)\dfrac{\pi}{2} + 2n\dfrac{\pi\tau}{2}$ SI 201

4. $\vartheta_3(u) = 0$ for $u = (2m-1)\dfrac{\pi}{2} + (2n-1)\dfrac{\pi\tau}{2}$ [m and n are integers or zero] SI 201

For integrals of theta functions, see **6.16**.

8.191 Connections with the Jacobian elliptic functions:
For $\tau = i\frac{\boldsymbol{K}'}{\boldsymbol{K}}$, i.e. for $q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right)$,

1. $\operatorname{sn} u = \dfrac{1}{\sqrt{k}} \dfrac{\vartheta_1\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)} = \dfrac{1}{\sqrt{k}} \dfrac{H(u)}{\Theta(u)}$ SI 206(22), SI 209(35)

2. $\operatorname{cn} u = \sqrt{\dfrac{k'}{k}} \dfrac{\vartheta_2\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{\dfrac{k'}{k}} \dfrac{H_1(u)}{\Theta(u)}$ SI 207(23), SI 209(35)

3. $\operatorname{dn} u = \sqrt{k'} \dfrac{\vartheta_3\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{k'} \dfrac{\Theta_1(u)}{\Theta(u)}$ SI 207(24), SI 209(35)

8.192 Series representation of the functions H, H_1, Θ, Θ_1.
In these formulas, $q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right)$.

1. $\Theta(u) = \vartheta_4\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum\limits_{n=1}^{\infty} (-1)^n q^{n^2} \cos\dfrac{n\pi u}{\boldsymbol{K}}$ SI 207(25), SI 212(42)

2. $H(u) = \vartheta_1\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 2\sum\limits_{n=1}^{\infty} (-1)^{n+1} \sqrt[4]{q^{(2n+1)^2}} \sin(2n-1)\dfrac{\pi u}{2\boldsymbol{K}}$ SI 207(25), SI 212(43)

3. $\Theta_1(u) = \vartheta_3\left(\dfrac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum\limits_{n=1}^{\infty} q^{n^2} \cos\dfrac{n\pi u}{\boldsymbol{K}}$ SI 207(25), SI 212(45)

4. $$H_1(u) = \vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right) = 2\sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ SI 207(25), SI 212(44)

8.193 Connections with the Weierstrass elliptic functions

1. $$\wp(u) = e_1 + \left[\frac{H_1\left(u\sqrt{\lambda}\right) H'(0)}{H_1(0)\, H\left(u\sqrt{\lambda}\right)}\right]^2 \lambda = e_2 + \left[\frac{\Theta_1\left(u\sqrt{\lambda}\right) H'(0)}{\Theta_1(0)\, H'\left(u\sqrt{\lambda}\right)}\right]^2 \lambda = e_3 + \left[\frac{\Theta\left(u\sqrt{\lambda}\right) H'(0)}{\Theta(0)\, H'\left(u\sqrt{\lambda}\right)}\right]^2 \lambda$$ SI 235(77,78)

2. $$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda}\frac{H'\left(u\sqrt{\lambda}\right)}{H\left(u\sqrt{\lambda}\right)}$$ SI 234(73)

3. $$\sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H\left(u\sqrt{\lambda}\right)}{H'(0)}$$ SI 234(72)

where

$$\lambda = e_1 - e_3; \qquad \eta_1 = \zeta(\omega_1) = -\frac{\omega_1 \lambda}{3}\frac{H'''(0)}{H'(0)}$$ SI 236

8.194 The connection with elliptic integrals:

1. $$E(u,k) = u - u\frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)}$$ SI 228(65)

2.[11] $$\Pi\left(u, -k^2\sin^2 a, k\right) = \int_0^u \frac{d\varphi}{1 - k^2\sin^2 a \,\mathrm{sn}^2\varphi} = u + \frac{\mathrm{sn}\, a}{\mathrm{cn}\, a\, \mathrm{dn}\, a}\left[\frac{\Theta'(a)}{\Theta(a)}u + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}\right]$$ SI 228(65)

q-series and products, $q = \exp\left(-\pi\frac{\mathsf{K}'}{\mathsf{K}}\right)$

8.195 $$\frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2 = \boldsymbol{K} = \frac{\pi}{2}\,\Theta^2(\boldsymbol{K})$$ (cf. **8.197** 1) SI 219

8.196 $$\boldsymbol{E} = \boldsymbol{K} - \boldsymbol{K}\frac{\Theta''(0)}{\Theta(0)} = \boldsymbol{K} - \frac{2\pi^2}{\boldsymbol{K}}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}n^2 q^{n^2}}{1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}}$$ SI 230(67)

8.197

1. $$1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2\boldsymbol{K}}{\pi}} = \vartheta_3(0)$$ (cf. **8.195**) WH

2. $$\sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{k\boldsymbol{K}}{2\pi}} = \frac{1}{2}\,\vartheta_2(0)$$ WH

3. $$4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4 = k$$ SI 206(17, 18)

4. $$\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 = k'$$ SI 206(19, 20)

5. $$2\sqrt[4]{q}\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right)^2 = 2\sqrt{k}\frac{\boldsymbol{K}}{\pi}$$ WH

6. $$\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2 = 2\sqrt{k'}\frac{\boldsymbol{K}}{\pi}$$ WH

8.198

1. $$\lambda = \frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1+2\sum_{n=1}^{\infty} q^{4n^2}}$$ [for $0<k<1$, we have $0<\lambda<\frac{1}{2}$] WH

The series

2. $$q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \ldots$$ WH

is used to determine q from the given modulus k.

8.199[10] Identities involving products of theta functions

1. $$\vartheta_1(x,q)\,\vartheta_1(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right) - \vartheta_2\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right)$$ LW 7(1.4.7)
2. $$\vartheta_1(x,q)\,\vartheta_2(y,q) = \vartheta_1\left(x+y,q^2\right)\vartheta_4\left(x-y,q^2\right) + \vartheta_4\left(x+y,q^2\right)\vartheta_1\left(x-y,q^2\right)$$ LW 8(1.4.8)
3. $$\vartheta_2(x,q)\,\vartheta_2(y,q) = \vartheta_2\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) + \vartheta_3\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.9)
4. $$\vartheta_3(x,q)\,\vartheta_3(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) + \vartheta_2\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.10)
5. $$\vartheta_3(x,q)\,\vartheta_4(y,q) = \vartheta_4\left(x+y,q^2\right)\vartheta_4\left(x-y,q^2\right) - \vartheta_1\left(x+y,q^2\right)\vartheta_1\left(x-y,q^2\right)$$ LW 8(1.4.11)
6. $$\vartheta_4(x,q)\,\vartheta_4(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) - \vartheta_2\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.12)
7. $$\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_4^2(0) = \vartheta_3^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_3^2(y) = \vartheta_1^2(x)\,\vartheta_4^2(y) - \vartheta_4^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.16)
8. $$\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_3^2(y) = \vartheta_2^2(x)\,\vartheta_4^2(y) - \vartheta_3^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.17)
9. $$\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_3^2(y) - \vartheta_1^2(x)\,\vartheta_2^2(y) = \vartheta_3^2(x)\,\vartheta_4^2(y) - \vartheta_2^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.18)
10. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_4^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.15)
11. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_4^2(0) = \vartheta_3^2(x)\,\vartheta_3^2(y) - \vartheta_2^2(x)\,\vartheta_2^2(y) = \vartheta_4^2(x)\,\vartheta_4^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y)$$ LW 9(1.4.19)
12. $$\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_3^2(y) - \vartheta_3^2(x)\,\vartheta_1^2(y) = \vartheta_4^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.23)
13. $$\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_3^2(0) = \vartheta_2^2(x)\,\vartheta_3^2(y) - \vartheta_4^2(x)\,\vartheta_1^2(y) = \vartheta_3^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.24)
14. $$\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_1^2(y) + \vartheta_3^2(x)\,\vartheta_3^2(y) = \vartheta_2^2(x)\,\vartheta_2^2(y) + \vartheta_4^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.25)
15. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(y) + \vartheta_3^2(x)\,\vartheta_4^2(y) = \vartheta_2^2(x)\,\vartheta_1^2(y) + \vartheta_4^2(x)\,\vartheta_3^2(y)$$ LW 9(1.4.26)

16. $\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_1^2(y) = \vartheta_4^2(x)\,\vartheta_3^2(y) - \vartheta_3^2(x)\,\vartheta_4^2(y)$ LW 9(1.4.30)

17. $\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_2^2(0) = \vartheta_2^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y) = \vartheta_3^2(x)\,\vartheta_3^2(y) - \vartheta_4^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.31)

18. $\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_2^2(0) = \vartheta_3^2(x)\,\vartheta_2^2(y) + \vartheta_4^2(x)\,\vartheta_1^2(y) = \vartheta_2^2(x)\,\vartheta_3^2(y) + \vartheta_1^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.32)

19. $\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_2^2(0) = \vartheta_4^2(x)\,\vartheta_2^2(y) + \vartheta_3^2(x)\,\vartheta_1^2(y) = \vartheta_1^2(x)\,\vartheta_3^2(y) + \vartheta_2^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.33)

20. $\vartheta_3^2(x)\,\vartheta_3^2(0) = \vartheta_4^2(x)\,\vartheta_4^2(0) + \vartheta_2^2(x)\,\vartheta_2^2(0)$ LW 11(1.4.49)

21. $\vartheta_4^2(x)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(0) + \vartheta_3^2(x)\,\vartheta_4^2(0)$ LW 11(1.4.50)

22. $\vartheta_4^2(x)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_3^2(0) + \vartheta_2^2(x)\,\vartheta_4^2(0)$ LW 11(1.4.51)

23. $\vartheta_3^2(x)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_4^2(0) + \vartheta_2^2(x)\,\vartheta_3^2(0)$ LW 11(1.4.52)

24.[8] $\vartheta_3^4(x) = \vartheta_2^4(0) + \vartheta_4^4(0)$ LW 11(1.4.53)

8.199(2)[10] Derivatives of ratios of theta functions

1. $\dfrac{d}{dx}\,(\vartheta_1\,/\,\vartheta_4) = \vartheta_4^2(0)\,\vartheta_2(x)\,\vartheta_3(x)/\,\vartheta_4^2(x)$ LW 19(1.9.3)

2. $\dfrac{d}{dx}\,(\vartheta_2\,/\,\vartheta_4) = -\,\vartheta_3^2(0)\,\vartheta_1(x)\,\vartheta_3(x)/\,\vartheta_4^2(x)$ LW 19(1.9.6)

3. $\dfrac{d}{dx}\,(\vartheta_3\,/\,\vartheta_4) = -\,\vartheta_2^2(0)\,\vartheta_1(x)\,\vartheta_2(x)/\,\vartheta_4^2(x)$ LW 19(1.9.7)

4. $\dfrac{d}{dx}\,(\vartheta_1\,/\,\vartheta_3) = \vartheta_3^2(0)\,\vartheta_2(x)\,\vartheta_4(x)/\,\vartheta_3^2(x)$ LW 19(1.9.8)

5. $\dfrac{d}{dx}\,(\vartheta_2\,/\,\vartheta_3) = -\,\vartheta_4^2(0)\,\vartheta_1(x)\,\vartheta_4(x)/\,\vartheta_3^2(x)$ LW 19(1.9.9)

6. $\dfrac{d}{dx}\,(\vartheta_1\,/\,\vartheta_2) = \vartheta_2^2(0)\,\vartheta_3(x)\,\vartheta_4(x)/\,\vartheta_2^2(x)$ LW 19(1.9.10)

7. $\dfrac{d}{dx}\,(\vartheta_4\,/\,\vartheta_1) = -\,\vartheta_4^2(0)\,\vartheta_2(x)\,\vartheta_3(x)/\,\vartheta_1^2(x)$ LW 19(1.9.11)

8. $\dfrac{d}{dx}\,(\vartheta_4\,/\,\vartheta_2) = \vartheta_3^2(0)\,\vartheta_1(x)\,\vartheta_3(x)/\,\vartheta_2^2(x)$ LW 20(1.9.12)

9. $\dfrac{d}{dx}\,(\vartheta_4\,/\,\vartheta_3) = \vartheta_2^2(0)\,\vartheta_1(x)\,\vartheta_2(x)/\,\vartheta_3^2(x)$ LW 20(1.9.13)

10. $\dfrac{d}{dx}\,(\vartheta_3\,/\,\vartheta_1) = -\,\vartheta_3^2(0)\,\vartheta_2(x)\,\vartheta_4(x)/\,\vartheta_1^2(x)$ LW 20(1.9.14)

11. $\dfrac{d}{dx}\,(\vartheta_3\,/\,\vartheta_2) = \vartheta_4^2(0)\,\vartheta_1(x)\,\vartheta_4(x)/\,\vartheta_2^2(x)$ LW 20(1.9.15)

12. $\dfrac{d}{dx}\,(\vartheta_2\,/\,\vartheta_1) = -\,\vartheta_2^2(0)\,\vartheta_3(x)\,\vartheta_4(x)/\,\vartheta_1^2(x)$ LW 20(1.9.16)

8.199(3)[10] Derivatives of theta functions

1. $\dfrac{d}{du}\ln\vartheta_1(u) = \cot u + 4\sin 2u\sum\limits_{n=1}^{\infty}\dfrac{q^{2n}}{1-2q^{2n}\cos 2u + q^{4n}}$

2. $$\frac{d}{du}\ln\vartheta_2(u) = -\tan u - 4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n}}{1+2q^{2n}\cos 2u+q^{4n}}$$

3. $$\frac{d}{du}\ln\vartheta_3(u) = -4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+2q^{2n}\cos 2u+q^{4n-2}}$$

4. $$\frac{d}{du}\ln\vartheta_4(u) = 4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-2q^{2n}\cos 2u+q^{4n-2}}$$

5. $$\frac{d^2}{du^2}\ln\vartheta_2(u) = -\sum_{n=-\infty}^{\infty}\operatorname{sech}^2\{i(u+n\pi\tau)\}$$

8.2 The Exponential Integral Function and Functions Generated by It

8.21 The exponential integral function Ei(x)

8.211

1. $$\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}\,dt = \int_{-\infty}^{x}\frac{e^{t}}{t}\,dt = \mathrm{li}\,(e^x) \qquad [x<0]$$

2.[11] $$\mathrm{Ei}(x) = -\lim_{\varepsilon\to 0+}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}\,dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}\,dt\right] = \mathrm{PV}\int_{-\infty}^{x}\frac{e^{t}}{t}\,dt \qquad [x>0]$$

3.[7] $$\mathrm{Ei}(x) = \tfrac{1}{2}\{\mathrm{Ei}(x+i0)+\mathrm{Ei}(x-i0)\} \qquad [x>0]$$ ET I 386

8.212

1.[8] $$\mathrm{Ei}(-x) = \boldsymbol{C} + \ln x + \int_0^x \frac{e^{-t}-1}{t}\,dt \qquad [x>0]$$ NT 11(1)

$$= \boldsymbol{C} + e^{-x}\ln x + \int_0^x e^{-t}\ln t\,dt \qquad [x>0]$$ NT 11(10)

2.[7] $$\mathrm{Ei}(x) = e^x\left[\frac{1}{x} + \int_0^{\infty}\frac{e^{-t}\,dt}{(x-t)^2}\right] \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$

3. $$\mathrm{Ei}(-x) = e^{-x}\left[-\frac{1}{x} + \int_0^{\infty}\frac{e^{-t}\,dt}{(x+t)^2}\right] \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$ LA 281(28)

4. $$\mathrm{Ei}\,(\pm x) = \pm e^{\pm x}\int_0^1\frac{dt}{x\pm\ln t} \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$

5. $$\mathrm{Ei}\,(\pm xy) = \pm e^{\pm xy}\int_0^{\infty}\frac{e^{-xt}}{y\mp t}\,dt \qquad [\mathrm{Re}\,y>0,\quad x>0]$$ NT 19(11)

6. $$\mathrm{Ei}\,(\pm x) = -e^{\pm x}\int_0^{\infty}\frac{e^{-it}}{t\pm ix}\,dt \qquad [x>0]$$ NT 23(2, 3)

7.[8] $$\mathrm{Ei}(xy) = e^{xy}\int_0^1\frac{t^{y-1}}{x+\ln t}\,dt$$ LA 282(44)a

8. $\mathrm{Ei}(-xy) = -e^{-xy}\int_0^1 \frac{t^{y-1}}{x-\ln t}\,dt$ LA 282(45)a

$= x^{-1}e^{-xy}\left[\int_0^1 \frac{t^{x-1}}{(y-\ln t)^2}\,dt - y^{-1}\right]$ $[x>0, \quad y>0]$ LA 283(47)a

9. $\mathrm{Ei}(x) = e^x\int_1^\infty \frac{1}{x-\ln t}\frac{dt}{t^2}$ $[x>0]$ LA 283(48)

10. $\mathrm{Ei}(-x) = -e^{-x}\int_1^\infty \frac{1}{x+\ln t}\frac{dt}{t^2}$ $[x>0]$ LA 283(48)

11. $\mathrm{Ei}(-x) = -e^{-x}\int_0^\infty \frac{t\cos t + x\sin t}{t^2+x^2}\,dt$ $[x>0]$ NT 23(6)

12. $\mathrm{Ei}(-x) = -e^{-x}\int_0^\infty \frac{t\cos t - x\sin t}{t^2+x^2}\,dt$ $[x<0]$ NT 23(6)

13. $\mathrm{Ei}(-x) = \frac{2}{\pi}\int_0^\infty \frac{\cos t}{t}\arctan\frac{t}{x}\,dt$ $[\mathrm{Re}\,x>0]$ NT 25(13)

14. $\mathrm{Ei}(-x) = \frac{2e^{-x}}{\pi}\int_0^\infty \frac{x\cos t - t\sin t}{t^2+x^2}\ln t\,dt$ $[x>0]$ NT 26(7)

15. $\mathrm{Ei}(x) = 2\ln x - \frac{2e^{x}}{\pi}\int_0^\infty \frac{x\cos t + t\sin t}{t^2+x^2}\ln t\,dt$ $[x>0]$ NT 27(8)

16. $\mathrm{Ei}(-x) = -x\int_1^\infty e^{-tx}\ln t\,dt$ $[x>0]$ NT 32(12)

See also **3.327**, **3.881** 8, **3.916** 2 and 3, **4.326** 1, **4.326** 2, **4.331** 2, **4.351** 3, **4.425** 3, **4.581**. For integrals of the exponential integral function, see **6.22**–**6.23**, **6.78**.

Series and asymptotic representations

8.213

1. $\mathrm{li}(x) = \boldsymbol{C} + \ln(-\ln x) + \sum_{k=1}^\infty \frac{(\ln x)^k}{k\cdot k!}$ $[0<x<1]$ NT 3(9)

2. $\mathrm{li}(x) = \boldsymbol{C} + \ln\ln x + \sum_{k=1}^\infty \frac{(\ln x)^k}{k\cdot k!}$ $[x>1]$ NT 3(10)

8.214

1. $\mathrm{Ei}(x) = \boldsymbol{C} + \ln(-x) + \sum_{k=1}^\infty \frac{x^k}{k\cdot k!}$ $[x<0]$

2. $\mathrm{Ei}(x) = \boldsymbol{C} + \ln x + \sum_{k=1}^\infty \frac{x^k}{k\cdot k!}$ $[x>0]$

3. $\mathrm{Ei}(x) - \mathrm{Ei}(-x) = 2x\sum_{k=0}^\infty \frac{x^{2k}}{(2k+1)(2k+1)!}$ $[x>0]$ NT 39(13)

8.215[7] $\mathrm{Ei}(z) = \dfrac{e^z}{z}\left[\displaystyle\sum_{k=0}^{n}\frac{k!}{z^k} + R_n(z)\right] \qquad |R_n(z)| = O\left(|z|^{-n-1}\right)$

$[z \to \infty, \quad |\arg(-z)| \le \pi - \delta; \quad \delta > 0 \text{ small}], \qquad |R_n(z)| \le (n+1)!|z|^{-n-1} \quad [\operatorname{Re} z \le 0]$

8.216[7] $\mathrm{Ei}(nx) - \mathrm{Ei}(-nx) = e^{nx'}\left(\dfrac{1}{nx} + \dfrac{1}{n^2x^2} + \dfrac{k_n}{n^3x^3}\right),$

where $x' = x \operatorname{sign}\operatorname{Re}(x), \quad k_n = O(1),$ and $n \to \infty$ NT 39(15)

8.217 Functional relations:

1. $e^{x'}\,\mathrm{Ei}\,(-x') - e^{-x'}\,\mathrm{Ei}\,(x') = -2\displaystyle\int_0^\infty \frac{x'\sin t}{t^2+x^2}\,dt$ NT 24(11)

$= \dfrac{4}{\pi}\displaystyle\int_0^\infty \frac{x'\cos t}{t^2+x^2}\ln t\,dt - 2e^{-x'}\ln x' \qquad [x' = x\operatorname{sign}\operatorname{Re} x]$ NT 27(9)

2. $e^{x'}\,\mathrm{Ei}\,(-x') + e^{-x'}\,\mathrm{Ei}\,(x') = -2\displaystyle\int_0^\infty \frac{t\cos t}{t^2+x^2}\,dt = 2e^{-x'}\ln x' - \frac{4}{\pi}\int_0^\infty \frac{t\sin t}{t^2+x^2}\ln t\,dt$

$[x' = x\operatorname{sign}\operatorname{Re} x]$ NT 24(10), NT 27(10)

3. $\mathrm{Ei}(-x) - \mathrm{Ei}\left(-\dfrac{1}{x}\right) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos t}{t}\arctan\frac{t\left(x-\frac{1}{x}\right)}{1+t^2}\,dt$

$[\operatorname{Re} x > 0]$ NT 25(14)

4. $\mathrm{Ei}(-\alpha x)\,\mathrm{Ei}(-\beta x) - \ln(\alpha\beta)\,\mathrm{Ei}[-(\alpha+\beta)x] = e^{-(\alpha+\beta)x}\displaystyle\int_0^\infty \frac{e^{-tx}\ln[(\alpha+t)(\beta+t)]}{t+\alpha+\beta}\,dt$ NT 32(9)

See also **3.723** 1 and 5, **3.742** 2 and 4, **3.824** 4, **4.573** 2.

- For a connection with a confluent hypergeometric function, see **9.237**.
- For integrals of the exponential integral function, see **5.21**, **5.22**, **5.23**, **6.22**, and **6.23**.

8.218 Two numerical values:

1. $\mathrm{Ei}(-1) = -0.219\ 383\ 934\ 395\ 520\ 273\ 665\ldots$ NT 89
2. $\mathrm{Ei}(1) = 1.895\ 117\ 816\ 355\ 936\ 755\ 478\ldots$ NT 89

8.219* Definite integrals of exponential functions

1.* $\displaystyle\int_0^\infty \mathrm{Ei}^2(x)e^{-2x}\,dx = \frac{\pi^2}{4}$

2.* $\displaystyle\int_0^\infty \mathrm{Ei}^2(-x)e^{2x}\,dx = \frac{\pi^2}{4}$

3.* $\displaystyle\int_0^\infty \mathrm{Ei}(x)\,\mathrm{Ei}(-x)\,dx = 0$

8.22 The hyperbolic sine integral $\operatorname{shi} x$ and the hyperbolic cosine integral $\operatorname{chi} x$

8.221

1. $\operatorname{shi} x = \int_0^x \frac{\sinh t}{t}\, dt = -i\left[\frac{\pi}{2} + \operatorname{si}(ix)\right]$ (see **8.230** 1) EH II 146(17)

2.[11] $\operatorname{chi} x = \boldsymbol{C} + \ln x + \int_0^x \frac{\cosh t - 1}{t}\, dt$ EH II 146(18)

8.23 The sine integral and the cosine integral: $\operatorname{si} x$ and $\operatorname{ci} x$

8.230

1.[10] $\operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt = -\frac{\pi}{2} + \operatorname{Si}(x)$, where $\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$ NT 11(3)

2.[10] $\operatorname{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt = \boldsymbol{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\, dt$ [$\operatorname{ci}(x)$ is also written $\operatorname{Ci}(x)$] NT 11(2)

8.231

1. $\operatorname{si}(xy) = -\int_x^\infty \frac{\sin ty}{t}\, dt$ NT 18(7)

2. $\operatorname{ci}(xy) = -\int_x^\infty \frac{\cos ty}{t}\, dt$ NT 18(6)

3. $\operatorname{si}(x) = -\int_0^{\pi/2} e^{-x\cos t} \cos(x \sin t)\, dt$ NT 13(26)

8.232

1. $\operatorname{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^\infty \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!}$ NT 7(4)

2.[7] $\operatorname{ci}(x) = \boldsymbol{C} + \ln(x) + \sum_{k=1}^\infty (-1)^k \frac{x^{2k}}{2k(2k)!}$ NT 7(3)

8.233

1. $\operatorname{ci}(x) \pm i \operatorname{si}(x) = \operatorname{Ei}(\pm ix)$ NT 6a

2. $\operatorname{ci}(x) - \operatorname{ci}\left(xe^{\pm \pi i}\right) = \mp \pi i$ NT 7(5)

3. $\operatorname{si}(x) + \operatorname{si}(-x) = -\pi$ NT 7(7)

8.234

1.[7] $\operatorname{Ei}(-x) - \operatorname{ci}(x) = \int_0^{\pi/2} e^{-x\cos\varphi} \sin(s \sin\varphi)\, d\varphi$ NT 13(27)

2. $[\operatorname{ci}(x)]^2 + [\operatorname{si}(x)]^2 = -2\int_0^{\pi/2} \frac{\exp(-x\tan\varphi)\ln\cos\varphi}{\sin\varphi\cos\varphi}\, d\varphi$

[$\operatorname{Re} x > 0$] (see also **4.366**)

NT 32(11)

See also **3.341**, **3.351** 1 and 2, **3.354** 1 and 2, **3.721** 2 and 3, **3.722** 1, 3, 5 and 7, **3.723** 8 and 11, **4.338** 1, **4.366** 1.

8.235

1. $\lim\limits_{x\to+\infty}(x^{\varrho}\,\mathrm{si}(x))=0,\quad \lim\limits_{x\to+\infty}(x^{\varrho}\,\mathrm{ci}(x))=0 \qquad [\varrho<1]$ NT 38(5)

2. $\lim\limits_{x\to-\infty}\mathrm{si}(x)=-\pi,\quad \lim\limits_{x\to-\infty}\mathrm{ci}(x)=\pm\pi i$ NT 38(6)

- For integrals of the sine integral and cosine integral, see **6.24**–**6.26**, **6.781**, **6.782**, and **6.783**.
- For indefinite integrals of the sine integral and cosine integral, see **5.3**.

8.24 The logarithm integral $\mathrm{li}(x)$

8.240

1. $\mathrm{li}(x)=\int_0^x\frac{dt}{\ln t}=\mathrm{Ei}\,(\ln x) \qquad [x<1]$ JA

2. $\mathrm{li}(x)=\lim\limits_{\varepsilon\to0}\left[\int_0^{1-\varepsilon}\frac{dt}{\ln t}+\int_{1+\varepsilon}^x\frac{dt}{\ln t}\right]=\mathrm{Ei}\,(\ln x) \qquad [x>1]$ JA

3. $\mathrm{li}\left\{\exp\left(-xe^{\pm\pi i}\right)\right\}=\mathrm{Ei}\left(-xe^{\pm i\pi}\right)=\mathrm{Ei}\,(x\mp i0)=\mathrm{Ei}(x)\pm i\pi=\mathrm{li}\,(e^x)\pm i\pi$
$[x>0]$ JA, NT 2(6)

Integral representations

8.241

1. $\mathrm{li}(x)=\int_{-\infty}^{\ln x}\frac{e^t}{t}\,dt=x\ln\ln\frac{1}{x}-\int_{-\ln x}^{\infty}e^{-t}\ln t\,dt \qquad [x<1]$ LA 281(33)

2. $\mathrm{li}(x)=x\int_0^1\frac{dt}{\ln x+\ln t}$ LA 280(22)
$=\frac{x}{\ln x}+x\int_0^1\frac{dt}{(\ln x+\ln t)^2}$ LA 280(29)
$=x\int_1^{\infty}\frac{1}{\ln x-\ln t}\frac{dt}{t^2} \qquad [x<1]$ LA 280(30)

3. $\mathrm{li}\,(a^x)=\frac{1}{\ln a}\int_{-\infty}^x\frac{a^t}{t}\,dt \qquad [x>0]$

For integrals of the logarithm integral, see **6.21**

8.25 The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$ and $C(x)$, the error function $\mathrm{erf}(x)$, and the complementary error function $\mathrm{erfc}(x)$

8.250 Definition:

1.[11] $\Phi(x)=\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ (called the error function)

2. $S(x)=\frac{2}{\sqrt{2\pi}}\int_0^x\sin t^2\,dt$

3. $C(x) = \dfrac{2}{\sqrt{2\pi}} \displaystyle\int_0^x \cos t^2 \, dt$

4.[11] $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ (called the complementary error function)

5.* $$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin\left(a\sqrt{x}\right) \, dx = -\sinh\left(a\sqrt{p}\right) + \frac{1}{2} e^{-a\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} - \sqrt{py}\right) + \frac{1}{2} e^{a\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right)$$

6.* $$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos\left(a\sqrt{x}\right) \, dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \, \Phi\left(\frac{a}{a\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p} \cosh\left(a\sqrt{p}\right)$$
$[\operatorname{Re} p > 0, \quad a, b \text{ are real}]$

7.* $$\int_0^p \exp\left(-x^2\right) \Phi(p-x) \, dx = \int_0^p \exp\left(-x^2\right) \operatorname{erf}(p-x) \, dx = \frac{\sqrt{\pi}}{2} \left[\Phi\left(\frac{p}{\sqrt{2}}\right)\right]^2$$

8.* $$\int_0^p x^2 \exp\left(-x^2\right) \Phi(p-x) \, dx = \int_0^p x^2 \exp\left(-x^2\right) \operatorname{erf}(p-x) \, dx = \frac{\sqrt{\pi}}{4} \left[\Phi\left(\frac{p}{\sqrt{2}}\right)\right]^2 - \frac{p}{2\sqrt{2}} \, \Phi\left(-\frac{x^2}{2}\right) \operatorname{erf}\left(\frac{p}{\sqrt{2}}\right)$$

9.* $$\int_{(b-a)/\sqrt{2}}^{(b+a)/\sqrt{2}} \exp\left(-x^2\right) \Phi\left(b\sqrt{2} - x\right) \, dx + \int_{(a-b)/\sqrt{2}}^{(a+b)/\sqrt{2}} \exp\left(-x^2\right) \Phi\left(a\sqrt{2} - x\right) \, dx = \sqrt{\pi} \, \Phi(a) \, \Phi(b)$$

Integral representations

8.251

1. $\Phi(x) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} \, dt$ (see also **3.361** 1)

2. $S(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{x^2} \frac{\sin t}{\sqrt{t}} \, dt$

3. $C(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{x^2} \frac{\cos t}{\sqrt{t}} \, dt$

8.252

1. $\Phi(xy) = \dfrac{2y}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2 y^2} \, dt$ $\left[\operatorname{Re} y^2 > 0\right]$

2. $S(xy) = \dfrac{2y}{\sqrt{2\pi}} \displaystyle\int_0^x \sin\left(t^2 y^2\right) \, dt$

3. $C(xy) = \dfrac{2y}{\sqrt{2\pi}} \displaystyle\int_0^x \cos\left(t^2 y^2\right) \, dt$

4. $$\Phi(xy) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2} ty\, dt}{\sqrt{t^2+x^2}} \qquad [\operatorname{Re} y^2 > 0]$$ NT 19(11)a

$$= 1 - \frac{2x}{\pi} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2}\, dt}{t^2+x^2} \qquad [\operatorname{Re} y^2 > 0]$$ NT 19(13)a

5.[7] $$\Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2y^2} \sin(ty)\, dt \qquad [\operatorname{Re} x^2 > 0]$$ NT 28(3)a

6.[8] $$\Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} xe^{-\frac{y^2}{4}} \int_0^\infty e^{-t^2x^2-ty}\, dt \qquad [\operatorname{Re} x^2 > 0]$$ NT 27(1)a

See also **3.322**, **3.362** 2, **3.363**, **3.468**, **3.897**, **6.511** 4 and 5.

8.253[8] Series representations:

1.[11] $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} x\, F_1\left(1; \frac{3}{2}; x^2\right) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^\infty (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!}$$ NT 7(9)a

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^\infty \frac{2^k x^{2k+1}}{(2k+1)!!}$$ NT 10(11)a

2. $$S(x) = \frac{2}{\sqrt{2\pi}} \left(x \sin x^2\, F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) - \frac{2}{3} x^3 \cos x^2\, F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right)\right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)}$$ NT 8(14)a

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\}$$ NT 10(13)a

3. $$C(x) = \frac{2}{\sqrt{2\pi}} \left(\frac{2}{3} x^3 \sin x^2\, F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) - x \cos x^2\, F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right)\right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)}$$ NT 8(13)a

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\}$$ NT 10(12)a

For the expansions in Bessel functions, see **8.515** 2, **8.515** 3.

Asymptotic representations

8.254[8] $$\Phi(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left[\sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2z^2)^k} + O\left(|z|^{-2n-z}\right) \right],$$

$$[z \to \infty, \quad |\arg(-z)| \le \pi - \delta; \quad \delta > 0 \text{ small}]$$

where

$$|R_n| < \frac{\Gamma\left(n+\frac{1}{2}\right)}{|x|^{n+\frac{1}{2}}} \cos\frac{\varphi}{2}, \quad x = |x|e^{i\varphi} \text{ and } \varphi^2 < \pi^2$$ NT 37(10)a

8.255

1. $$S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi} x} \cos x^2 + O\left(\frac{1}{x^2}\right) \qquad [x \to \infty]$$ MO 127a

2. $C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}x} \sin x^2 + O\left(\frac{1}{x^2}\right)$ $[x \to \infty]$ MO 127a

8.256 Functional relations:

1. $C(z) + i\,S(z) = \sqrt{\frac{i}{2}}\,\Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{it^2}\,dt$

2. $C(z) - i\,S(z) = \frac{1}{\sqrt{2i}}\,\Phi\left(z\sqrt{i}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-it^2}\,dt$

3. $\left[\cos^2 u\,C(u) + \sin u^2\,S(u)\right] = \frac{1}{2}\left[\cos^2 u + \sin u^2\right] + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2\,dt$

$[\operatorname{Re} u \geq 0]$ NT 28(6)a

4. $\left[\cos^2 u\,S(u) - \sin u^2\,C(u)\right] = \frac{1}{2}\left[\cos^2 u - \sin u^2\right] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2\,dt$

$[\operatorname{Re} u \geq 0]$ NT 28(5)a

5.[11] $\left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 = \frac{2}{\pi} \int_0^{\pi/2} \frac{\exp\left(-x^2 \tan\varphi\right) \sin\frac{\varphi}{2}\sqrt{\cos\varphi}}{\sin 2\varphi}\,d\varphi$

(see also **6.322**) NT 33(18)a

- For a connection with a confluent hypergeometric function, see **9.236**.
- For a connection with a parabolic cylinder function, see **9.254**.

8.257

1. $\lim\limits_{x\to+\infty} \left(x^\varrho \left[S(x) - \frac{1}{2}\right]\right) = 0$ $[\varrho < 1]$ NT 38(11)

2. $\lim\limits_{x\to+\infty} \left(x^\varrho \left[C(x) - \frac{1}{2}\right]\right) = 0$ $[\varrho < 1]$ NT 38(11)

3. $\lim\limits_{x\to+\infty} S(x) = \frac{1}{2}$ NT 38(12)a

4. $\lim\limits_{x\to+\infty} C(x) = \frac{1}{2}$ NT 38(12)a

- For integrals of the probability integral, see **6.28**–**6.31**.
- For integrals of Fresnel's sine integral and cosine integral, see **6.32**.

8.258[10] Integrals involving the complementary error function

1. $\int_0^\infty \operatorname{erfc}^2(x) e^{-\beta x^2}\,dx = \frac{1}{\sqrt{\beta\pi}} \left(-\arccos\left(\frac{1}{1+\beta}\right) + 2\arctan\left(\sqrt{\beta}\right)\right)$

$[\beta > 0]$

2. $\int_0^\infty x \operatorname{erfc}^2(x) e^{-\beta x^2}\,dx = \frac{1}{2\beta} \left(1 - \frac{4}{\pi} \frac{\arctan\left(\sqrt{1+\beta}\right)}{\sqrt{1+\beta}}\right)$

$[\beta > 0]$

3. $$\int_0^\infty x^3 \operatorname{erfc}^2(x) e^{-\beta x^2}\, dx = \frac{1}{2\beta^2}\left(1 - \frac{4}{\pi}\frac{\arctan\left(\sqrt{1+\beta}\right)}{\sqrt{1+\beta}}\right) + \frac{1}{\beta\pi}\left(\frac{1}{(1+\beta)\left(\beta^2+2\beta+2\right)} - \frac{\arctan\left(\sqrt{1+\beta}\right)}{(1+\beta)^{\frac{3}{2}}}\right)$$
$[\beta > 0]$

4. $$\int_0^\infty x \operatorname{erfc}\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{1}{\beta^2}\left[1 - \frac{1+\frac{3}{2}\beta}{(1+\beta)^{\frac{3}{2}}}\right] \qquad [\beta > 0]$$

5.[11] $$\int_0^\infty \sqrt{x} \operatorname{erfc}\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\frac{\arctan\left(\sqrt{\beta}\right)}{\beta^{\frac{3}{2}}} - \frac{1}{2\beta(1+\beta)}\right)$$
$[\beta > 0]$

8.259* Integrals involving the error function and an exponential function

1. $$\int_{-\infty}^\infty e^{-px^2}\, \Phi(a+bx)\, dx = \sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) \qquad [\operatorname{Re} p > 0], \quad a, b \text{ real}$$

2. $$\int_{-\infty}^\infty x^2 e^{-px^2}\, \Phi(a+bx)\, dx = \frac{1}{2p}\sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) - \frac{ab^2}{p\left(b^2+p\right)^{3/2}} \exp\left(-\frac{a^2 p}{b^2+p}\right)$$
$[\operatorname{Re} p > 0, \quad a, b \text{ are real}]$

3. $$\int_{-\infty}^\infty x^{2n} e^{-px^2}\, \Phi(a+bx)\, dx = (-1)^n \frac{\partial^n}{\partial p^n}\left[\sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right)\right]$$
$[n = 0, 1, \ldots, \quad \operatorname{Re} p > 0, \quad a, b \text{ are real}]$

8.26 Lobachevskiy's function $L(x)$

8.260 Definition:

$$L(x) = -\int_0^x \ln\cos t\, dt \qquad \text{LO III 184(10)}$$

For integral representations of the function $L(x)$, see also **3.531** 8, **3.532** 2, **3.533**, and **4.224**.

8.261 Representation in the form of a series:

$$L(x) = x\ln 2 - \frac{1}{2}\sum_{k=1}^\infty (-1)^{k-1}\frac{\sin 2kx}{k^2} \qquad \text{LO III 185(11)}$$

8.262 Functional relationships:

1. $L(-x) = -L(x)$ $\quad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$ LO III 185(13)
2. $L(\pi - x) = \pi\ln 2 - L(x)$ LO III 286
3. $L(\pi + x) = \pi\ln 2 + L(x)$ LO III 286
4. $L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right)\ln 2 - \frac{1}{2}L\left(\frac{\pi}{2} - 2x\right)$ $\quad \left[0 \le x < \frac{\pi}{4}\right]$ LO III 186(14)

8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

8.310 Definition:

1. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ $\quad [\operatorname{Re} z > 0]$ (Euler) FI II 777(6)

Generalization:

2. $\Gamma(z) = -\dfrac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t}\, dt$

for z not an integer. The contour C is shown in the drawing: WH

$\Gamma(z)$ is an analytic function z with simple poles at the points $z = -l$ (for $l = 0, 1, 2, \ldots$) to which correspond to residues $\dfrac{(-1)^l}{l!}$. $\Gamma(z)$ satisfies the relation $\Gamma(1) = 1$. WH, MO 1

Integral representations

8.311 $\Gamma(z) = \dfrac{1}{e^{2\pi i z} - 1} \int_\infty^{(0+)} e^{-t} t^{z-1}\, dt$ MO 2

8.312

1. $\Gamma(z) = \int_0^1 \left(\ln \dfrac{1}{t}\right)^{z-1} dt$ $\quad [\operatorname{Re} z > 0]$ FI II 778

2. $\Gamma(z) = x^z \int_0^\infty e^{-xt} t^{z-1}\, dt$ $\quad [\operatorname{Re} z > 0, \quad \operatorname{Re} x > 0]$ FI II 779(8)

3. $\Gamma(z) = \dfrac{2a^z e^a}{\sin \pi z} \int_0^\infty e^{-at^2} \left(1 + t^2\right)^{z - \frac{1}{2}} \cos\left[2at + (2z-1) \arctan t\right] dt$

$[a > 0]$ WH

4. $\Gamma(z) = \dfrac{1}{2 \sin \pi z} \int_0^\infty e^{-t^2} t^{z-1} \left(1 + t^2\right)^{\frac{z}{2}} \left\{3 \sin\left[t + z \operatorname{arccot}(-t)\right] + \sin\left[t + (z-2) \operatorname{arccot}(-t)\right]\right\} dt$

[arccot denotes an obtuse angle] WH

5. $\Gamma(y) = x^y e^{-i\beta y} \int_0^\infty t^{y-1} \exp\left(-xte^{-i\beta}\right) dt$

$\left[x, y, \beta \text{ real}, \quad x > 0, \quad y > 0, \quad |\beta| < \dfrac{\pi}{2}\right]$ MO 8

6. $\Gamma(z) = \dfrac{b^z}{2 \sin \pi z} \int_{-\infty}^\infty e^{bti} (it)^{z-1}\, dt$ $\quad [b > 0, \quad 0 < \operatorname{Re} z < 1]$ NH 154(3)

7. $$\Gamma(z) = \frac{\left(\sqrt{a^2+b^2}\right)^z}{\cos\left(z\arctan\frac{b}{a}\right)}\int_0^\infty e^{-at}\cos(bt)t^{z-1}\,dt$$ NH 152(1)a

$$= \frac{\left(\sqrt{a^2+b^2}\right)^z}{\sin\left(z\arctan\frac{b}{a}\right)}\int_0^\infty e^{-at}\sin(bt)t^{z-1}\,dt$$ NH 152(2)

$[a>0, \quad b\geq 0, \quad \operatorname{Re} z>0]$

8. $$\Gamma(z) = \frac{b^z}{\cos\frac{\pi z}{2}}\int_0^\infty \cos(bt)t^{z-1}\,dt$$

$$= \frac{b^z}{\sin\frac{\pi z}{2}}\int_0^\infty \sin(bt)t^{z-1}\,dt$$

$[b>0, \quad 0<\operatorname{Re} z<1]$ NH 152(5)

9. $$\Gamma(z) = \int_0^\infty e^{-t}(t-z)t^{z-1}\ln t\,dt$$

$[\operatorname{Re} z>0]$ NH 173(7)

10. $$\Gamma(z) = \int_{-\infty}^\infty \exp\left(zt-e^t\right)\,dt$$

$[\operatorname{Re} z>0]$ NH 145(14)

11.[11] $$\Gamma(x)\cos\alpha x = \lambda^x\int_0^\infty t^{x-1}e^{-\lambda t\cos\alpha}\cos\left(\lambda t\sin\alpha\right)\,dt$$

$\left[\lambda>0, \quad x>0, \quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right]$ WH

12. $$\Gamma(x)\sin\alpha x = \lambda^x\int_0^\infty t^{x-1}e^{-\lambda t\cos\alpha}\sin\left(\lambda t\sin\alpha\right)\,dt$$

$\left[\lambda>0, \quad x>0, \quad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right]$ WH

13. $$\Gamma(-z) = \int_0^\infty \left[\frac{e^{-t}-\sum_{k=0}^n(-1)^k\frac{t^k}{k!}}{t^{z+1}}\right]dt$$ $[n=\lfloor\operatorname{Re} z\rfloor]$ MO 2

8.313 $$\Gamma\left(\frac{z+1}{v}\right) = vu^{\frac{z+1}{v}}\int_0^\infty \exp\left(-ut^v\right)t^z\,dt$$ $[\operatorname{Re} u>0, \quad \operatorname{Re} v>0, \quad \operatorname{Re} z>-1]$

JA, MO 7a

8.314* $$\Gamma(z) = \int_1^\infty e^{-t}t^{z-1}\,dt + \sum_{n=0}^\infty \frac{(-1)^k}{k!(z+k)}$$ $[z\to 0, \text{ in } |\arg z|<\pi]$

8.315

1.[11] $$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C(-t)^{-z}e^{-t}\,dt$$ [for the contour C, see **8.310** 2]

2.[8] $$\int_{-\infty}^\infty \frac{e^{bti}}{(a+it)^2}\,dt = \frac{2\pi e^{-ab}b^{z-1}}{\Gamma(z)}$$

$$\int_{-\infty}^\infty \frac{e^{-bti}}{(a+it)^z}\,dt = 0 \qquad \left[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re} z>0, \quad |\arg(a+it)|<\tfrac{1}{2}\pi\right]$$

3. $$\frac{1}{\Gamma(z)} = a^{1-z}\frac{e^a}{\pi}\int_0^{\pi/2} \cos(a\tan\theta - z\theta)\cos^{z-2}\theta\, d\theta \qquad [\operatorname{Re} z > 1]$$ NH 157(14)

See also **3.324** 2, **3.326**, **3.328**, **3.381** 4, **3.382** 2, **3.389** 2, **3.433**, **3.434**, **3.478** 1, **3.551** 1, 2, **3.827** 1, **4.267** 7, **4.272**, **4.353** 1, **4.369** 1, **6.214**, **6.223**, **6.246**, **6.281**.

8.32 Representation of the gamma function as series and products

8.321 Representation in the form of a series:

1.[6] $$\Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$
$$\left[c_0 = 1, \quad c_{n+1} = \frac{\sum_{k=0}^{n}(-1)^{k+1}s_{k+1}c_{n-k}}{n+1}; \quad s_1 = \boldsymbol{C}, \quad s_n = \zeta(n) \text{ for } n \geq 2, \quad |z| < 1\right]$$
NH 40(1, 3)

2.[11] $$\frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$
$$\left[d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^{n}(-1)^{k}s_{k+1}d_{n-k}}{n+1}; \quad s_1 = \boldsymbol{C}, \quad s_n = \zeta(n) \text{ for } n \geq 2\right]$$ NH 41(4, 6)

Infinite-product representation

8.322[11] $$\Gamma(z) = e^{-\boldsymbol{C}z}\frac{1}{z}\prod_{k=1}^{\infty}\frac{e^{z/k}}{1+\frac{z}{k}} \qquad [\operatorname{Re} z > 0]$$ SM 269

$$= \frac{1}{z}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}} \qquad [\operatorname{Re} z > 0]$$ WH

$$= \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k} \qquad [\operatorname{Re} z > 0]$$ SM 267(130)

8.323[7] $$\Gamma(z) = 2z^z e^{-z}\prod_{k=1}^{\infty}\sqrt[2^k]{\mathrm{B}\left(2^{k-1}z, \tfrac{1}{2}\right)}$$ NH 98(12)

8.324[7] $$\Gamma(1+z) = 4^z\prod_{k=1}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{z}{2^k}\right)}{\sqrt{\pi}}$$ MO 3

8.325

1. $$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{\gamma}{\alpha+k}\right)\left(1-\frac{\gamma}{\beta+k}\right)\right]$$ NH 62(2)

2.[11] $$\frac{e^{\boldsymbol{C}x}\Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty}\left[\left(1-\frac{x}{z+k}\right)e^{x/k}\right] \qquad [z \neq 0, -1, -2, \ldots; \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(z-x) > 0]$$

3.[7] $$\frac{\sqrt{\pi}}{\Gamma\left(1+\frac{z}{2}\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)} = \prod_{k=1}^{\infty}\left(1-\frac{z}{2k-1}\right)\left(1+\frac{z}{2k}\right)$$ MO 2

8.326

1. $$\frac{\dfrac{[\Gamma(x)]^2}{\Gamma(2x)}}{\mathrm{B}(x+iy,x-iy)} = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2 = \prod_{k=0}^{\infty}\left(1+\frac{y^2}{(x+k)^2}\right)$$
$[x, y$ are real, $\quad x \neq 0, -1, -2, \ldots]$ LO V, NH 63(4)

2.[11] $$\frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{xe^{-iCy}}{x+iy}\prod_{n=1}^{\infty}\frac{\exp\left(\frac{iy}{n}\right)}{1+\frac{iy}{x+n}}$$
$[x, y$ are real, $\quad x \neq 0, -1, -2, \ldots]$ MO 2

8.327 Asymptotic representation for large arguments:

1.* $$\Gamma(z) \sim z^{z-\frac{1}{2}}e^{-z}\sqrt{2\pi}\left\{1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+O\left(z^{-5}\right)\right\}$$
$[|\arg z| < \pi]$ WH

For z real and positive, the remainder of the series is less than the last term that is retained.

2.* $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ or equivalently } \Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$
[Stirling's asymptotic formula for $n \gg 0$] AS 6.1.38

3.* $$\ln\Gamma(z) \sim \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \ldots$$
$[z \to \infty, \quad |\arg z| < \pi]$ AS 6.1.38

8.328

1. $$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi}$$
$[x$ and y are real] MO 6

2. $$\lim_{|z|\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\ln z} = 1$$
MO 6

8.33 Functional relations involving the gamma function

8.331

1. $$\Gamma(x+1) = x\,\Gamma(x)$$

2.* $$\Gamma(x+a) = (x+a-1)\,\Gamma(x+a-1) = \frac{\Gamma(x+a+1)}{(x+a)}$$

3.* $$\Gamma(x-a) = (x-a-1)\,\Gamma(x-a-1) = \frac{\Gamma(x-a+1)}{(x-a)}$$

8.332

1. $|\Gamma(iy)|^2 = \dfrac{\pi}{y\sinh \pi y}$ [y is real] MO 3

2. $\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \dfrac{\pi}{\cosh \pi y}$ [y is real]

3. $\Gamma(1+ix)\,\Gamma(1-ix) = \dfrac{\pi x}{\sinh x\pi}$ [x is real] LO V

4. $\Gamma(1+x+iy)\,\Gamma(1-x+iy)\,\Gamma(1+x-iy)\,\Gamma(1-x-iy) = \dfrac{2\pi^2\left(x^2+y^2\right)}{\cosh 2y\pi - \cos 2x\pi}$
[x and y are real] LO V

8.333 $[\Gamma(n+1)]^n = G(n+1)\prod\limits_{k=1}^{n} k^k,$

where n is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left[-\frac{z(z+1)}{2} - \frac{\boldsymbol{C}}{2}z^2\right] \prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n \exp\left(-z+\frac{z^2}{2n}\right)\right\}$$ WH

8.334

1. $\displaystyle\prod_{k=1}^{n} \frac{1}{\Gamma\left(-z\exp\frac{2\pi k i}{n}\right)} = -z^n \prod_{k=1}^{\infty}\left[1-\left(\frac{z}{k}\right)^n\right]$ [$n = 2, 3, 3\ldots$] MO 2

2. $\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \dfrac{\pi}{\cos \pi x}$

3. $\Gamma(1-x)\,\Gamma(x) = \dfrac{\pi}{\sin \pi x}$ FI II 430

Special cases

8.335[7] $\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod\limits_{k=0}^{n-1} \Gamma\left(x+\dfrac{k}{n}\right)$ [product theorem] FI II 782a, WH

1. $\Gamma(2x) = \dfrac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)$ [doubling formula]

2. $\Gamma(3x) = \dfrac{3^{3x-\frac{1}{2}}}{2\pi}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right)$

3. $\displaystyle\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}$ WH

4.[10] $\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma^2\left(n-\frac{1}{2}\right)}{4\,(n!)^2\,\Gamma^2\left(-\frac{1}{2}\right)} = \frac{1}{4}+\frac{1}{16}+\frac{1}{256}+\frac{1}{1024}+\frac{25}{65536}+\cdots = \frac{1}{\pi}$

8.336 $\Gamma\left(-\dfrac{yz+xi}{2y}\right)\Gamma(1-z) = (2i)^{z+1}y\,\Gamma\left(1+\dfrac{yz-xi}{2y}\right)\displaystyle\int_0^{\infty} e^{-tx}\sin^z(ty)\,dt$

[$\mathrm{Re}(yi) > 0, \quad \mathrm{Re}(x-yzi) > 0$]

NH 133(10)

- For a connection with the psi function, see **8.361** 1.
- For a connection with the beta function, see **8.384** 1.
- For integrals of the gamma function, see **8.412** 4, **8.414**, **9.223**, **9.242** 3, **9.242** 4.

8.337

1. $\left[\Gamma'(x)\right]^2 < \Gamma(x)\,\Gamma''(x) \qquad [x > 0]$ MO 1
2. For $x > 0$, $\min \Gamma(1+x) = 0.88560\ldots$ is attained when $x = 0.46163\ldots$ JA

Particular values

8.338

1. $\Gamma(1) = \Gamma(2) = 1$
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
4. $\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2\left[(4k+1)^2-1\right]}{\left[(4k-1)^2-1\right](4k+1)^2}$ MO 1a
5. $\prod_{k=1}^{8} \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3$ WH

8.339 For n a natural number

1. $\Gamma(n) = (n-1)!$
2. $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!$
3. $\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}$
4. $\frac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)} = \frac{\left(4p^2-1^2\right)\left(4p^2-3^2\right)\ldots\left[4p^2-(2n-1)^2\right]}{2^{2n}}$ WA 221

5.* $\Gamma(n+k) = (n+k-1)!$
$$= \frac{\Gamma(n+k+1)}{(n+k)} \qquad [n+k \geq 0, 1, \ldots]$$

6.* $\Gamma(n-k) = (n-k-1)!$
$$= \frac{\Gamma(n-k+1)}{(n-k)} \qquad [n-k \geq 0, 1, \ldots]$$

8.34 The logarithm of the gamma function

8.341 Integral representation:

1. $$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-tz}}{t}\,dt$$
$[\operatorname{Re} z > 0]$ WH

2.[11] $$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + 2\int_0^\infty \frac{\arctan\frac{t}{z}}{e^{2\pi t} - 1}\,dt$$
$\left[\operatorname{Re} z > 0 \text{ and } \arctan w = \int_0^w \frac{du}{1+u^2} \text{ is taken over a rectangular path in the } w\text{-plane}\right]$ WH

3. $$\ln\Gamma(z) = \int_0^\infty \left\{\frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t}\right\}\frac{dt}{t}$$
$[\operatorname{Re} z > 0]$ WH

4. $$\ln\Gamma(z) = \int_0^\infty \left\{(z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)}\right\}\frac{dt}{t}$$
$[\operatorname{Re} z > 0]$ WH

5. $$\ln\Gamma(x) = \frac{\ln\pi - \ln\sin\pi x}{2} + \frac{1}{2}\int_0^\infty \left\{\frac{\sinh\left(\frac{1}{2} - x\right)t}{\sinh\frac{t}{2}} - (1-2x)e^{-t}\right\}\frac{dt}{t}$$
$[0 < x < 1]$ WH

6. $$\ln\Gamma(z) = \int_0^1 \left\{\frac{t^z - t}{t - 1} - t(z-1)\right\}\frac{dt}{t\ln t}$$
$[\operatorname{Re} z > 0]$ WH

7. $$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}}\right]\frac{dt}{t}$$
$[\operatorname{Re} z > 0]$ NH 187(7)

See also **3.427** 9, **3.554** 5.

8.342 Series representations:

1.[11] $\ln\Gamma(z+1)$

$$= \frac{1}{2}\left[\ln\left(\frac{\pi z}{\sin\pi z}\right) - \ln\frac{1+z}{1-z}\right] + (1 - \boldsymbol{C})\,z + \sum_{k=1}^\infty \frac{1 - \zeta(2k+1)}{2k+1} z^{2k+1}$$
$$= -\boldsymbol{C}z + \sum_{k=2}^\infty (-1)^k \frac{z^k}{k}\,\zeta(k)$$
$[|z| < 1]$ NH 38(16, 12)

2. $$\ln\Gamma(1+x) = \frac{1}{2}\ln\frac{\pi x}{\sin\pi x} - \boldsymbol{C}x - \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1}\,\zeta(2n+1)$$
$[|x| < 1]$ NH 38(14)

8.343

1. $$\ln\Gamma(x) = \ln\sqrt{2\pi} + \sum_{n=1}^\infty \left\{\frac{1}{2n}\cos 2n\pi x + \frac{1}{n\pi}\left(\boldsymbol{C} + \ln 2n\pi\right)\sin 2n\pi x\right\}$$
$[0 < x < 1]$ FI III 558

2. $$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}}$$
$[|\arg z| < \pi]$ MO 9

8.344[7] Asymptotic expansion for large values of $|z|$:

$$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \sum_{k=1}^{n-1}\frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1}\cos^{2n-1}\left(\frac{1}{2}\arg z\right)}$$ MO5

For integrals of $\ln\Gamma(x)$, see **6.44**.

8.35 The incomplete gamma function

8.350 Definition:

1. $\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}\,dt$ $[\operatorname{Re}\alpha > 0]$ EH II 133(1), NH 1(1)

2.[11] $\Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}\,dt$ EH II 133(2), NH 2(2), LE 339

3.* $\Gamma(z, 0) = \Gamma(z)$

4.* $\Gamma(a, \infty) = 0$

5.* $\gamma(a, 0) = 0$

8.351

1. $\gamma^*(\alpha, x) = \dfrac{x^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha, x)$ is an analytic function with respect to α and x EH II 133(5)

2. Another definition of $\Gamma(\alpha, x)$ that is also suitable for the case $\operatorname{Re}\alpha \le 0$:
$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha}e^{-x}\,\Phi(1, 1+\alpha; x) = \frac{x^\alpha}{\alpha}\,\Phi(a, 1+a; -x)$$ EH II 133(3)

3. For fixed x, $\Gamma(\alpha, x)$ is an entire function of α. For non-integral α, $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$.

4. A second definition of $\Gamma(\alpha, x)$:
$$\Gamma(\alpha, x) = x^\alpha e^{-x}\Psi(1, 1+\alpha; x) = e^{-x}\Psi(1-\alpha, 1-\alpha; x)$$ EH II 133(4)

8.352 Special cases:

1. $\gamma(1+n, x) = n!\left[1 - e^{-x}\left(\sum_{m=0}^{n}\frac{x^m}{m!}\right)\right]$ $[n = 0, 1, \ldots]$
EH II 136(17, 16), NH 6(11)

2. $\Gamma(1+n, x) = n!e^{-x}\sum_{m=0}^{n}\frac{x^m}{m!}$ $[n = 0, 1, \ldots]$ EH II 136(16, 18)

3.[11] $$\Gamma(-n,x)=\frac{(-1)^n}{n!}\left[\mathrm{Ei}(-z)-\frac{1}{2}\ln(-z)+\frac{1}{2}\ln\left(-\frac{1}{z}\right)-\ln z\right]-e^{-z}\sum_{k=1}^{n}\frac{z^{k-n-1}}{(-n)_k}$$
$[n=1,2,\ldots]$

4.* $$\Gamma(n,x)=(n-1)!e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}$$

5.* $$\Gamma(-n+1,x)=\frac{(-1)^{n+1}}{(n-1)!}\left[\Gamma(0,x)-e^{-z}\sum_{m=0}^{n-2}(-1)^m\frac{m!}{x^{m+1}}\right]$$
$[n=2,3,\ldots]$

6.* $$\gamma(n,x)=(n-1)!\left[1-e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}\right]$$ $[n=1,2,\ldots]$

7.* $$\Gamma(n,x)=(n-1)!e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}$$ $[n=1,2,\ldots]$

8.* $$\Gamma(-n+k,x)=\frac{(-1)^{n-k}}{(n-k)!}\left[\Gamma(0,x)-e^{-x}\sum_{m=0}^{n-k-1}(-1)^m\frac{m!}{x^{m+1}}\right]$$
$[n-k\geq 1,\quad k=0,1,\ldots]$

8.353 Integral representations:

1. $$\gamma(\alpha,x)=x^\alpha\operatorname{cosec}\pi\alpha\int_0^\pi e^x\cos\theta\cos(\alpha\theta+x\sin\theta)\,d\theta$$ $[x\neq 0,\quad \operatorname{Re}\alpha>0,\quad \alpha\neq 1,2,\ldots]$ EH II 137(2)

2. $$\gamma(\alpha,x)=x^{\frac{1}{2}\alpha}\int_0^\infty e^{-t}t^{\frac{1}{2}\alpha-1}J_\alpha\left(2\sqrt{xt}\right)dt$$ $[\operatorname{Re}\alpha>0]$ EH II 138(4)

3. $$\Gamma(\alpha,x)=\frac{\rho^{-x}x^\alpha}{\Gamma(1-\alpha)}\int_0^\infty\frac{e^{-t}t^{-\alpha}}{x+t}\,dt$$ $[\operatorname{Re}\alpha<1,\quad x>0]$ EH II 137(3), NH 19(12)

4. $$\Gamma(\alpha,x)=\frac{2x^{\frac{1}{2}\alpha}e^{-x}}{\Gamma(1-\alpha)}\int_0^\infty e^{-t}t^{-\frac{1}{2}\alpha}K_\alpha\left[2\sqrt{xt}\right]dt$$ $[\operatorname{Re}\alpha<1]$ EH II 138(5)

5. $$\Gamma(\alpha,xy)=y^\alpha e^{-xy}\int_0^\infty e^{-ty}(t+x)^{\alpha-1}\,dt$$
$[\operatorname{Re}y>0,\quad x>0,\quad \operatorname{Re}\alpha>1]$ (See also **3.936** 5, **3.944** 1–4) NH 19(10)

For integrals of the gamma function, see **6.45**.

8.354 Series representations:

1. $$\gamma(\alpha,x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}$$ EH II 135(4)

2. $$\Gamma(\alpha,x)=\Gamma(\alpha)-\sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \qquad [\alpha\neq 0,-1,-2,\ldots]$$

EH II 135(5), LE 340(2)

3. $$\Gamma(\alpha,x)-\Gamma(\alpha,x+y)=\gamma(\alpha,x+y)-\gamma(\alpha,x)$$
$$=e^{-x}x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k\left[1-e^{-y}e_k(y)\right]\Gamma(1-\alpha+k)}{x^k\,\Gamma(1-\alpha)}$$
$$e_k(x)=\sum_{m=0}^{k}\frac{x^m}{m!} \qquad [|y|<|x|]$$

EH II 139(2)

4. $$\gamma(\alpha,x)=\Gamma(\alpha)e^{-x}x^{\frac{1}{2}\alpha}\sum_{n=0}^{\infty}x^{\frac{1}{2}n}\,I_{n+\alpha}\left(2\sqrt{x}\right)\sum_{m=0}^{n}\frac{(-1)^m}{m!} \qquad [x\neq 0,\quad \alpha\neq 0,\quad -1,-2,\ldots]$$

EH II 139(3)

5. $$\Gamma(\alpha,x)=e^{-x}x^{\alpha}\sum_{n=0}^{\infty}\frac{L_n^{\alpha}(x)}{n+1} \qquad [x>0]$$

EH II 140(5)

8.355 $$\Gamma(\alpha,x)\,\gamma(\alpha,y)=e^{-x-y}(xy)^{\alpha}\sum_{n=0}^{\infty}\frac{n!\,\Gamma(\alpha)}{(n+1)\,\Gamma(\alpha+n+1)}\,L_n^{\alpha}(x)\,L_n^{\alpha}(y)$$
$$[y>0,\quad x\geq y,\quad \alpha\neq 0,-1,\ldots]$$

EH II 139(4)

8.356 Functional relations:

1.[11] $$\gamma(\alpha+1,x)=\alpha\,\gamma(\alpha,x)-x^{\alpha}e^{-x}$$ EH II 134(2)

2. $$\Gamma(\alpha+1,x)=\alpha\,\Gamma(\alpha,x)+x^{\alpha}e^{-x}$$ EH II 134(3)

3. $$\Gamma(\alpha,x)+\gamma(\alpha,x)=\Gamma(\alpha)$$ EH II 134(1)

4. $$\frac{d\,\gamma(\alpha,x)}{dx}=-\frac{d\,\Gamma(\alpha,x)}{dx}=x^{\alpha-1}e^{-x}$$ EH II 135(8)

5. $$\frac{\Gamma(\alpha+n,x)}{\Gamma(\alpha+n)}=\frac{\Gamma(\alpha,x)}{\Gamma(\alpha)}+e^{-x}\sum_{s=0}^{n-1}\frac{x^{\alpha+s}}{\Gamma(\alpha+s+1)}$$ NH 4(3)

6.[11] $$\Gamma(\alpha)\,\Gamma(\alpha+n,x)-\Gamma(\alpha+n)\,\Gamma(\alpha,x)=\Gamma(\alpha+n)\,\gamma(\alpha,x)-\Gamma(\alpha)\,\gamma(\alpha+n,x)$$ NH 5

7.* $$\Gamma(a+k,x)=(a+k-1)\,\Gamma(a+k-1,x)+x^{a+k-1}e^{-x}$$
$$=\frac{1}{a+k}\left[\Gamma(a+k+1,x)-x^{a+k}e^{-x}\right]$$

8.* $$\Gamma(a-k,x)=(a-k-1)\,\Gamma(a-k-1,x)+x^{a-k-1}e^{-x}$$
$$=\frac{1}{a-k}\left[\Gamma(a-k+1,x)-x^{a-k}e^{-x}\right]$$

9.* $$\gamma(a+k,x)=(a+k-1)\,\gamma(a+k-1,x)-x^{a+k-1}e^{-x}$$
$$=\frac{1}{a+k}\left[\Gamma(a+k+1,x)+x^{a+k}e^{-x}\right]$$

10.* $$\gamma(a-k,x) = (a-k-1)\,\gamma(a-k-1,x) - x^{a-k-1}e^{-x}$$
$$= \frac{1}{a-k}\left[\gamma(a-k+1,x) + x^{a-k}e^{-x}\right]$$

8.357 Asymptotic representation for large values of $|x|$:

1. $$\Gamma(\alpha,x) = x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(-1)^m\,\Gamma(1-\alpha+m)}{x^m\,\Gamma(1-\alpha)} + O\left(|x|^{-M}\right)\right]$$
$$\left[|x|\to\infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, \quad M = 1,2,\ldots\right]$$ EH II 135(6), NH 37(7), LE 340(3)

8.358 Representation as a continued fraction:

$$\Gamma(\alpha,x) = \cfrac{e^{-x}x^{\alpha}}{x + \cfrac{1-\alpha}{1 + \cfrac{1}{x + \cfrac{2-\alpha}{1 + \cfrac{2}{x + \cfrac{3-\alpha}{1+\ldots}}}}}}$$ EH II 136(13), NH 42(9)

8.359 Relationships with other functions:

1. $\Gamma(0,x) = -\operatorname{Ei}(-x)$ EH II 143(1)
2. $\Gamma\left(0, \ln\frac{1}{x}\right) = -\operatorname{li}(x)$ EH II 143(2)
3. $\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\,\Phi(x)$ EH II 147(2)
4.[11] $\gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\,\Phi(x)$ EH II 147(1)

8.36 The psi function $\psi(x)$

8.360 Definition:

1. $$\psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

8.361 Integral representations:

1.[8] $$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt$$ $[\operatorname{Re} z > 0]$ NH 183(1), WH

2. $$\psi(z) = \int_0^\infty\left\{e^{-t} - \frac{1}{(1+t)^z}\right\}\frac{dt}{t}$$ $[\operatorname{Re} z > 0]$ NH 184(7), WH

3. $$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty\frac{t\,dt}{(t^2+z^2)\,(e^{2\pi t}-1)}$$ $[\operatorname{Re} z > 0]$ WH

4. $$\psi(z) = \int_0^1\left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t}\right)dt$$ $[\operatorname{Re} z > 0]$ WH

5. $$\psi(z)=\int_0^\infty \frac{e^{-t}-e^{-zt}}{1-e^{-t}}\,dt-\boldsymbol{C},$$ WH

6. $$\psi(z)=\int_0^\infty \left\{(1+t)^{-1}-(1+t)^{-z}\right\}\frac{dt}{t}-\boldsymbol{C},\qquad [\operatorname{Re} z>0]$$ WH

7. $$\psi(z)=\int_0^1 \frac{t^{z-1}-1}{t-1}\,dt-\boldsymbol{C}$$ FI II 796, WH

8. $$\psi(z)=\ln z+\int_0^\infty e^{-tz}\left[\frac{1}{t}-\frac{1}{1-e^{-t}}\right]dt\qquad [\operatorname{Re} z>0]$$ MO 4

See also **3.244** 3, **3.311** 6, **3.317** 1, **3.457**, **3.458** 2, **3.471** 14, **4.253** 1 and 6, **4.275** 2, **4.281** 4, **4.482** 5.
For integrals of the psi function, see **6.46**, **6.47**.

Series representation

8.362

1. $$\psi(x)=-\boldsymbol{C}-\sum_{k=0}^\infty\left(\frac{1}{x+k}-\frac{1}{k+1}\right)$$ FI II 799(26), KU 26(1)

$$=-\boldsymbol{C}-\frac{1}{x}+x\sum_{k=1}^\infty\frac{1}{k(x+k)}$$ FI II 495

2. $$\psi(x)=\ln x-\sum_{k=0}^\infty\left[\frac{1}{x+k}-\ln\left(1+\frac{1}{x+k}\right)\right]$$ MO 4

3. $$\psi(x)=-\boldsymbol{C}+\frac{\pi^2}{6}(x-1)-(x-1)\sum_{k=1}^\infty\left(\frac{1}{k+1}-\frac{1}{x+k}\right)\sum_{n=0}^{k-1}\frac{1}{x+n}$$ NH 54(12)

8.363

1. $$\psi(x+1)=-\boldsymbol{C}+\sum_{k=2}^\infty(-1)^k\,\zeta(k)x^{k-1}$$ NH 37(5)

2. $$\psi(x+1)=\frac{1}{2x}-\frac{\pi}{2}\cot\pi x-\frac{x^2}{1-x^2}-\boldsymbol{C}+\sum_{k=1}^\infty\left[1-\zeta(2k+1)\right]x^{2k}$$ NH 38(10)

3. $$\psi(x)-\psi(y)=\sum_{k=0}^\infty\left(\frac{1}{y+k}-\frac{1}{x+k}\right)$$
(see also **3.219**, **3.231** 5, **3.311** 7, **3.688** 20, **4.253** 1, **4.295** 37) NH 99(3)

4. $$\psi(x+iy)-\psi(x-iy)=\sum_{k=0}^\infty\frac{2yi}{y^2+(x+k)^2}$$

5. $$\psi\left(\frac{p}{q}\right)=-\boldsymbol{C}+\sum_{k=0}^\infty\left(\frac{1}{k+1}-\frac{q}{p+kq}\right)$$ (see also **3.244** 3) NH 29(1)

6.[8] $$\psi\left(\frac{p}{q}\right) = -\boldsymbol{C} - \ln(2q) - \frac{\pi}{2}\cot\frac{p\pi}{q} + 2\sum_{k=1}^{\left[\frac{q+1}{2}\right]-1}\left[\cos\frac{2kp\pi}{q}\ln\sin\frac{k\pi}{q}\right]$$
$[q = 2, 3, \ldots, p = 1, 2, \ldots, q-1]$ MO 4, EH I 19(29)

7. $$\psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+kq)^n - 1}$$ NH 59(3)

8. $$\psi^{(n)}(x) = (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}} = (-1)^{n+1}n!\,\zeta(n+1, x)$$ NH 37(1)

Infinite-product representation

8.364

1. $$e^{\psi(x)} = x\prod_{k=0}^{\infty}\left(1 + \frac{1}{x+k}\right)e^{-\frac{1}{x+k}}$$ NH 65(12)

2. $$e^{y\,\psi(x)} = \frac{\Gamma(x+y)}{\Gamma(x)}\prod_{k=0}^{\infty}\left(1 + \frac{y}{x+k}\right)e^{-\frac{y}{x+k}}$$ NH 65(11)

See also **8.37**.

- For a connection with Riemann's zeta function, see **9.533** 2.
- For a connection with the gamma function, see **4.325** 12 and **4.352** 1.
- For a connection with the beta function, see **4.253** 1.
- For series of psi functions, see **8.403** 2, **8.446**, and **8.447** 3 (Bessel functions), **8.761** (derivatives of associated Legendre functions with respect to the degree), **9.153**, **9.154** (hypergeometric function), **9.237** (confluent hypergeometric function).
- For integrals containing psi functions, see **6.46**–**6.47**.

8.365 Functional relations:

1. $$\psi(x+1) = \psi(x) + \frac{1}{x}$$ JA

2. $$\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\,\beta(x)$$ (cf. **8.37** 0)

3. $$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1}\frac{1}{x+k}$$ GA 154(64)a

4. $$\psi(n+1) = -\boldsymbol{C} + \sum_{k=1}^{n}\frac{1}{k}$$ MO 4

5. $$\lim_{n\to\infty}\left[\psi(z+n) - \ln n\right] = 0$$ MO 3

6. $$\psi(nz) = \frac{1}{n}\sum_{k=0}^{n-1}\psi\left(z + \frac{k}{n}\right) + \ln n$$ $[n = 2, 3, 4, \ldots]$ MO 3

7. $\psi(x-n) = \psi(x) - \sum_{k=1}^{n} \frac{1}{x-k}$

8. $\psi(1-z) = \psi(z) + \pi \cot \pi z$ GA 155(68)a

9. $\psi\left(\frac{1}{2}+z\right) = \psi\left(\frac{1}{2}-z\right) + \pi \tan \pi z$ JA

10. $\psi\left(\frac{3}{4}-n\right) = \psi\left(\frac{1}{4}+n\right) + \pi$ $[n = 0, \quad \pm 1, \quad \pm 2, \ldots]$

8.366 Particular values

1. $\psi(1) = -\boldsymbol{C}$ (cf. **8.367** 1)

2. $\psi\left(\frac{1}{2}\right) = -\boldsymbol{C} - 2\ln 2 = -1.963\,510\,026\ldots$ GA 155a

3. $\psi\left(\frac{1}{2} \pm n\right) = -\boldsymbol{C} + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right]$ JA

4. $\psi\left(\frac{1}{4}\right) = -\boldsymbol{C} - \frac{\pi}{2} - 3\ln 2$ GA 157a

5. $\psi\left(\frac{3}{4}\right) = -\boldsymbol{C} + \frac{\pi}{2} - 3\ln 2$ GA 157a

6. $\psi\left(\frac{1}{3}\right) = -\boldsymbol{C} - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3$ GA 157a

7. $\psi\left(\frac{2}{3}\right) = -\boldsymbol{C} + \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3$ GA 157a

8. $\psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\ldots$ JA

9. $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\ldots$ JA

10. $\psi'(-n) = \infty$ [n is a natural number] JA

11. $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$ [n is a natural number] JA

12. $\psi'\left(\frac{1}{2}+n\right) = \frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$ [n is a natural number] JA

13. $\psi'\left(\frac{1}{2}-n\right) = \frac{\pi^2}{2} + 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$ [n is a natural number] JA

8.367 Euler's constant (also denoted by γ):

1. $\boldsymbol{C} = -\psi(1) = 0.577\ 215\ 664\ 90\ldots$ FI II 319, 795

2. $\boldsymbol{C} = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n\right]$ FI II 801a

3. $\boldsymbol{C} = \lim_{x\to 1+0}\left[\zeta(x) - \frac{1}{x-1}\right]$ FI II 804

Integral representations:

4. $\boldsymbol{C} = -\int_0^\infty e^{-t} \ln t \, dt$ FI II 807

5. $\boldsymbol{C} = -\int_0^1 \ln\left(\ln \frac{1}{t}\right) dt$ FI II 807

6. $\boldsymbol{C} = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t}\right] dt$ DW

7. $\boldsymbol{C} = -\int_0^\infty \left[\cos t - \frac{1}{1+t}\right] \frac{dt}{t}$ MO 10

8. $\boldsymbol{C} = 1 - \int_0^\infty \left[\frac{\sin t}{t} - \frac{1}{1+t}\right] \frac{dt}{t}$ MO 10

9. $\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t}\right] \frac{dt}{t}$ FI II 795, 802

10. $\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t^2}\right] \frac{dt}{t}$ DW, MO 10

11. $\boldsymbol{C} = \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{te^t}\right] dt$ DW

12. $\boldsymbol{C} = \int_0^1 \left(1 - e^{-t}\right) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} \, dt$ FI II 802

See also **8.361** 5–**8.361** 7, **3.311** 6, **3.435** 3 and 4, **3.476** 2, **3.481** 1 and 2, **3.951** 10, **4.283** 9, **4.331** 1, **4.421** 1, **4.424** 1, **4.553**, **4.572**, **6.234**, **6.264** 1, **6.468**.

13. Asymptotic expansions

$$\boldsymbol{C} = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$
$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)} \frac{\theta}{n^{2r+2}} \qquad [0 < \theta < 1]$$ FI II 827

8.37 The function $\beta(x)$

8.370 Definition:

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$$ NH 16(13)

8.371 Integral representations:

1.[3] $\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} \, dt$ $[\operatorname{Re} x > 0]$ WH

2. $\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} \, dt$ $[\operatorname{Re} x > 0]$ MO 4

3. $\beta\left(\frac{x+1}{2}\right) = \int_0^\infty \frac{e^{-xt}}{\cosh t} \, dt$ $[\operatorname{Re} x > -1]$

See also **3.241** 1, **3.251** 7, **3.522** 2 and 4, **3.623** 2 and 3, **4.282** 2, **4.389** 3, **4.532** 1 and 3.

Series representation

8.372

1.[7] $\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}$ $[-x \notin \mathbb{N}]$ NH 37, 101(1)

2.[7] $\beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)}$ $[-x \notin \mathbb{N}]$ NH 101(2)

3.[8] $\beta(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k}$ $[-x \notin \mathbb{N}]$

$[\beta$ has simple poles at $x = -n$ with residue $(-1)^n]$ NH 246(7)

8.373

1.[6] $\beta(x+1) = \ln 2 + \sum_{k=1}^{\infty} (-1)^k \left(1 - 2^{-k}\right) \zeta(k+1) x^k$ $[|x| < 1]$ NH 37(5)

2.[6] $\beta(x+1) = \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=1}^{\infty} \left[1 - \left(1 - 2^{-2k}\right) \zeta(2k+1)\right] x^{2k}$

$[0 < |x| < 2; \quad x \neq \pm 1]$ NH 38(11)

8.374 $\frac{d^n}{dx^n} \beta(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}}$ $[-x \in \mathbb{N}]$ NH 37(2)

8.375 Representation in the form of a finite sum:

1.[6] $\beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{\lfloor \frac{q-1}{2} \rfloor} \cos \frac{p(2k+1)\pi}{q} \ln \sin \frac{(2k+1)\pi}{2q}$

$[q = 2, 3, \dots, p = 1, 2, 3, \dots, q-1]$ (see also **8.362** 5–7) NH 23(9)

2. $\beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}$

Functional relations

8.376 $\sum_{k=0}^{2n} (-1)^k \beta\left(\frac{x+k}{2n+1}\right) = (2n+1)\, \beta(x)$ NH 19

8.377 $\sum_{k=1}^{n} \beta\left(2^k x\right) = \psi\left(2^n x\right) - \psi(x) - n \ln 2$ NH 20(10)

8.38 The beta function (Euler's integral of the first kind): $\mathrm{B}(x,y)$

Integral representation

8.380

1. $$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt^*$$
$$=2\int_0^1 t^{2x-1}\left(1-t^2\right)^{y-1}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ FI II 774(1)

2. $$\mathrm{B}(x,y)=2\int_0^{\pi/2}\sin^{2x-1}\varphi\cos^{2y-1}\varphi\,d\varphi \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ KU 10

3. $$\mathrm{B}(x,y)=\int_0^\infty\frac{t^{x-1}}{(1+t)^{x+y}}\,dt=2\int_0^\infty\frac{t^{2x-1}}{\left(1+t^2\right)^{x+y}}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ FI II 775

4. $$\mathrm{B}(x,y)=2^{2-y-x}\int_{-1}^1\frac{(1+t)^{2x-1}(1-t)^{2y-1}}{\left(1+t^2\right)^{x+y}}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ MO 7

5. $$\mathrm{B}(x,y)=\int_0^1\frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt=\int_1^\infty\frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt \qquad [\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ BI (1)(15)

6. $$\mathrm{B}(x,y)=\frac{1}{2^{x+y-1}}\int_0^1\left[(1+t)^{x-1}(1-t)^{y-1}+(1+t)^{y-1}(1-t)^{x-1}\right]dt$$
$$[\operatorname{Re}x>0,\quad \operatorname{Re}y>0]$$ BI (1)(15)

7. $$\mathrm{B}(x,y)=z^y(1+z)^x\int_0^1\frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\,dt$$
$$[\operatorname{Re}x>0,\quad \operatorname{Re}y>0,\quad 0>z>-1,\quad \operatorname{Re}(x+y)<1]$$ NH 163(8)

8. $$\mathrm{B}(x,y)=z^y(1+z)^x\int_0^{\pi/2}\frac{\cos^{2x-1}\varphi\sin^{2y-1}\varphi}{\left(z+\cos^2\varphi\right)^{x+y}}\,d\varphi$$
$$[\operatorname{Re}x>0,\quad \operatorname{Re}y>0,\quad 0>z>-1,\quad \operatorname{Re}(x+y)<1]$$ NH 163(8)

See also **3.196** 3, **3.198**, **3.199**, **3.215**, **3.238** 3, **3.251** 1–3, 11, **3.253**, **3.312** 1, **3.512** 1 and 2, **3.541** 1, **3.542** 1, **3.621** 5, **3.623** 1, **3.631** 1, 8, 9, **3.632** 2, **3.633** 1, 4, **3.634** 1, 2, **3.637**, **3.642** 1, **3.667** 8, **3.681** 2.

9. $$\mathrm{B}(x,x)=\frac{1}{2^{2x-2}}\int_0^1\left(1-t^2\right)^{x-1}\,dt=\frac{1}{2^{2x-1}}\int_0^1\frac{(1-t)^{x-1}}{\sqrt{t}}\,dt$$
See **8.384** 4, **8.382** 3, and also **3.621** 1, **3.642** 2, **3.665** 1, **3.821** 6, **3.839** 6.

10. $$\mathrm{B}(x+y,x-y)=4^{1-x}\int_0^\infty\frac{\cosh 2yt}{\cosh^{2x}t}\,dt \qquad [\operatorname{Re}x>|\operatorname{Re}y|,\quad \operatorname{Re}x>0]$$ MO 9

11. $$\mathrm{B}\left(x,\frac{y}{z}\right)=z\int_0^1\left(1-t^z\right)^{x-1}t^{y-1}\,dt \qquad \left[\operatorname{Re}z>0,\quad \operatorname{Re}\frac{y}{z}>0,\quad \operatorname{Re}x>0\right]$$
FI II 787a

*This equation is used as the definition of the function $\mathrm{B}(x,y)$.

8.381

1. $$\int_{-\infty}^{\infty}\frac{dt}{(a+it)^x(b-it)^y}=\frac{2\pi\,(a+b)^{1-x-y}}{(x+y-1)\,\mathrm{B}(x,y)}$$
$[a>0,\quad b>0;\quad x \text{ and } y \text{ are real},\quad x+y>1]$ MO 7

2. $$\int_{-\infty}^{\infty}\frac{dt}{(a-it)^x(b-it)^y}=0$$
$[a>0,\quad b>0;\quad x \text{ and } y \text{ are real},\quad x+y>1]$ MO 7

3. $$\mathrm{B}(x+iy,x-iy)=2^{1-2x}\alpha e^{-2i\gamma y}\int_{-\infty}^{\infty}\frac{e^{2i\alpha yt}\,dt}{\cosh^{2x}(\alpha t-\gamma)}$$
$[y,\alpha,\gamma \text{ are real},\quad \alpha>0;\quad \operatorname{Re}x>0]$
MI 8a

For an integral representation of $\ln\mathrm{B}(x,y)$, see **3.428** 7.

4. $$\frac{1}{\mathrm{B}\,(x,y)}=\frac{2^{x+y-1}\,(x+y-1)}{\pi}\int_0^{\pi/2}\cos[(x-y)t]\cos^{x+y-2}t\,dt$$ NH 158(5)a
$$=\frac{2^{x+y-2}(x+y-1)}{\pi\cos\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi}\cos[(x-y)t]\sin^{x+y-2}t\,dt$$ NH 159(8)a
$$=\frac{2^{x+y-2}(x+y-1)}{\pi\sin\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi}\sin[(x-y)t]\sin^{x+y-2}t\,dt$$ NH 159(9)a

Series representation

8.382

1. $$\mathrm{B}(x,y)=\frac{1}{y}\sum_{n=0}^{\infty}(-1)^n y\frac{(y-1)\dots(y-n)}{n!(x+n)}\qquad [y>0]$$ WH

2. $$\ln\mathrm{B}\left(\frac{1+x}{2},\frac{1}{2}\right)\ln\sqrt{2\pi}+\frac{1}{2}\left[\ln\left(\frac{\tan\frac{\pi x}{2}}{x}\right)-\ln\left(\frac{1+x}{1-x}\right)\right]+\sum_{k=0}^{\infty}\frac{1-\left(1-2^{-2k}\right)\zeta(2k+1)}{2k+1}x^{2k+1}$$
$[|x|<2]$ NH 39(17)

3. $$\mathrm{B}\left(z,\frac{1}{2}\right)=\sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k k!}\frac{1}{z+k}+\frac{1}{z}$$ (see also **8.384** and **8.380** 9) WH

8.383 Infinite-product representation:

$$(x+y+1)\,\mathrm{B}(x+1,y+1)=\prod_{k=1}^{\infty}\frac{k(x+y+k)}{(x+k)(y+k)}\qquad [x,\quad y\neq-1,\quad -2,\dots]$$ MO 2

8.384 Functional relations involving the beta function:

1. $$\mathrm{B}(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}=\mathrm{B}(y,x)$$ FI II 779

2. $$\mathrm{B}(x,y)\,\mathrm{B}(x+y,z)=\mathrm{B}(y,z)\,\mathrm{B}(y+z,x)$$ MO 6

3. $\displaystyle\sum_{k=0}^{\infty} \mathrm{B}(x, y+k) = \mathrm{B}(x-1, y)$ WH

4. $\mathrm{B}(x,x) = 2^{1-2x}\,\mathrm{B}\left(\frac{1}{2}, x\right)$ (see also **8.380** 9 and **8.382** 3) FI II 784

5. $\mathrm{B}(x,x)\,\mathrm{B}\left(x+\frac{1}{2}, x+\frac{1}{2}\right) = \dfrac{\pi}{2^{4x-1}x}$ WH

6. $\dfrac{1}{\mathrm{B}(n,m)} = m\dbinom{n+m-1}{n-1} = n\dbinom{n+m-1}{m-1}$ [m and n are natural numbers]

For a connection with the psi function, see **4.253** 1.

8.39 The incomplete beta function $\mathrm{B}_x(p,q)$

8.391[7] $\displaystyle \mathrm{B}_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{x^p}{p}\,{}_2F_1\left(p, 1-q; p+1; x\right)$ ET I 373

8.392 $\displaystyle I_x(p,q) = \frac{\mathrm{B}_x(p,q)}{\mathrm{B}(p,q)}$ ET II 429

8.4–8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.401 Bessel functions $Z_\nu(z)$ are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{d Z_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0 \qquad \text{KU 37(1)}$$

Special types of Bessel functions are what are called Bessel functions of the first kind $J_\nu(z)$, Bessel functions of the second kind $Y_\nu(z)$ (also called Neumann functions and often written $N_\nu(z)$), and Bessel functions of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (also called Hankel's functions).

8.402 $\displaystyle J_\nu(z) = \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(\nu+k+1)}$ $[|\arg z| < \pi]$ KU 55(1)

8.403

1. $\displaystyle Y_\nu(z) = \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_\nu(z) - J_{-\nu}(z)\right]$ [for non-integer ν, $|\arg z| < \pi$] KU 41(3)

2. $$\pi\, Y_n(z) = 2\, J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$

$$- \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)]$$

KU 43(10)

$$= 2\, J_n(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$

$$- \left(\frac{z}{2}\right)^n \frac{1}{n!} \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k!\,(k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m}\right]$$

$[n+1$ a natural number, $|\arg z| < \pi]$

KU 44, WA 75(3)a

8.404

1. $Y_{-n}(z) = (-1)^n\, Y_n(z)$ [n is a natural number] KU 41(2)
2. $J_{-n}(z) = (-1)^n\, J_n(z)$ [n is a natural number] KU 41(2)

8.405[7]

1. $H_\nu^{(1)}(z) = J_\nu(z) + i\, Y_\nu(z)$ KU 44(1)
2. $H_\nu^{(2)}(z) = J_\nu(z) - i\, Y_\nu(z)$ KU 44(1)

In all relationships that hold for an arbitrary Bessel function $Z_\nu(z)$, that is, for the functions $J_\nu(z)$, $Y_\nu(z)$, and linear combinations of them, for example, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, we shall write simply the letter Z instead of the letters J, Y, $H^{(1)}$, and $H^{(2)}$.

Modified Bessel functions of imaginary argument $I_\nu(z)$ and $K_\nu(z)$

8.406

1. $I_\nu(z) = e^{-\frac{\pi}{2}\nu i}\, J_\nu\left(e^{\frac{\pi}{2}i} z\right)$ $\left[-\pi < \arg z \le \frac{\pi}{2}\right]$ WA 92
2. $I_\nu(z) = e^{\frac{3}{2}\pi\nu i}\, J_\nu\left(e^{-\frac{3}{2}\pi i} z\right)$ $\left[\frac{\pi}{2} < \arg z \le \pi\right]$ WA 92

For integer ν,

3. $I_n(z) = i^{-n}\, J_n(iz)$ KU 46(1)

8.407

1.[8] $K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i}\, H_\nu^{(1)}\left(z e^{\frac{1}{2}\pi i}\right)$ $\left[-\pi < \arg z \le \frac{1}{2}\pi\right]$

2.[8] $K_\nu(z) = \frac{-\pi i}{2} e^{-\frac{\pi}{2}\nu i}\, H_{-\nu}^{(2)}\left(z e^{-\frac{1}{2}\pi i}\right)$ $\left[-\frac{1}{2}\pi < \arg z \le \pi\right]$ WA 92(8)

For the differential equation defining these functions, see **8.494**.

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

1.[11] $$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta \qquad [n = 0, 1, 2, \ldots]$$ WH

2. $$J_{2n}(z) = \frac{1}{\pi}\int_0^{\pi}\cos 2n\theta\cos(z\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos 2n\theta\cos(z\sin\theta)\,d\theta$$ [n an integer] WA 30(7)

3.[11] $$J_{2n+1}(z) = \frac{1}{\pi}\int_0^{\pi}\sin(2n+1)\theta\sin(z\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\sin(2n+1)\theta\sin(z\sin\theta)\,d\theta \qquad [n \text{ an integer}]$$ WA 30(6)

4. $$J_\nu(z) = 2\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi/2}\sin^{2\nu}\theta\cos(z\cos\theta)\,d\theta$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WH

5. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}\sin^{2\nu}\theta\cos(z\cos\theta)\,d\theta \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$

6. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2}\cos(z\sin\theta)\cos^{2\nu}\theta\,d\theta$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ KU 65(5), WA 35(4)a

7. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi}e^{\pm iz\cos\varphi}\sin^{2\nu}\varphi\,d\varphi \qquad \left[\operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right]$$ WH

8. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}\cos zt\,dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$ KU 65(6), WH

9. $$J_\nu(x) = 2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^{\infty}\frac{\sin xt}{\left(t^2-1\right)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2},\quad x > 0\right]$$ MO 37

10. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}e^{izt}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$ WA 34(3)

11. $$J_\nu(x) = \frac{2}{\pi}\int_0^{\infty}\sin\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh\nu t\,dt$$ WA 199(12)

12. $$J_\nu(z) = \frac{2^{\nu+1}z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi/2}\frac{\left(\cos^{\nu-\frac{1}{2}}\theta\right)\sin\left(z-\nu\theta+\frac{1}{2}\theta\right)}{\sin^{2\nu+1}\theta}e^{-2z\cot\theta}\,d\theta$$ $\left[|\arg z| < \frac{\pi}{2},\quad \operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right]$ WH

13.[10] $$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu\theta - z\sinh\theta}\,d\theta$$
$[\operatorname{Re} z > 0]$ WA 195(4)

14. $$J_\nu(z) = \frac{e^{\pm\nu\pi i}}{\pi}\left[\int_0^\pi \cos(\nu\theta + z\sin\theta)\,d\theta - \sin\nu\pi\int_0^\infty e^{-\nu\theta + z\sinh\theta}\,d\theta\right]$$
$\left[\text{for } \frac{\pi}{2} < |\arg z| < \pi, \text{ with the upper sign taken for } |\arg z| > \frac{\pi}{2}\right.$ $\left.\text{and the lower sign taken for } |\arg z| < -\frac{\pi}{2}\right]$
WH

8.412

1. $$J_\nu(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]dt$$
$\left[|\arg z| < \frac{\pi}{2}\right]$ WH, WA 195(2)

2. $$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left(t - \frac{z^2}{4t}\right)dt$$
WA 195(1)

3.[8] $$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i}\sum_{k=1}^\infty \frac{(-1)^k z^{2k}}{2^{2k}k!}\int_{-\infty}^{(0+)} e^t t^{-\nu-k-1}\,dt$$
WA 195(1)

4. $$J_\nu(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-t)}{\Gamma(\nu+t+1)}\left(\frac{x}{2}\right)^{\nu+2t}dt$$
$[\operatorname{Re}\nu > 0, \quad x > 0]$ WA 214(7)

5.[7] $$J_\nu(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{2\pi i\,\Gamma\left(\frac{1}{2}\right)}\int_A^{(1+,-1-)}\left(t^2-1\right)^{\nu-\frac{1}{2}}\cos(zt)\,dt$$
$\left[\nu \neq \frac{1}{2}, \frac{3}{2}, \ldots;\ \text{The point} A \text{ falls to the right of the point } t = 1,\right.$ $\left.\text{and } \arg(t-1) = \arg(t+1) = 0 \text{ at the point } A\right]$
WH

6.[8] $$J_\nu(z) = \frac{1}{2\pi}\int_{-\pi+\infty i}^{\pi+\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta$$
$[\operatorname{Re} z > 0]$

The path of integration being taken around the semi-infinite strip $y \geq 0, -\pi \leq x \leq \pi$.

8.413[8] $$\frac{J_\nu\left(\sqrt{z^2+\zeta^2}\right)}{(z^2-\zeta^2)^{\frac{\nu}{2}}} = \frac{1}{\pi(z+\zeta)^\nu}\left\{\int_0^\infty e^{\zeta\cos t}\cos(z\sin t - \nu t)\,dt - \sin\nu\pi\int_0^\infty \exp(-z\sinh t - \zeta\cosh t - \nu t)\,dt\right\}$$
$[\operatorname{Re}(z+\zeta) > 0]$ MO 40

8.414 $$\int_{2x}^\infty \frac{J_0(t)}{t}\,dt = \frac{1}{4\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{\Gamma(-t)}{t\,\Gamma(1+t)}x^{2t}\,dt$$
$[x > 0]$ MO 41

See **3.715** 2, 9, 10, 13, 14, 19–21, **3.865** 1, 2, 4, **3.996** 4.

- For an integral representation of $J_0(z)$, see **3.714** 2, **3.753** 2, 3, and **4.124**.
- For an integral representation of $J_1(z)$, see **3.697**, **3.711**, **3.752** 2, and **3.753** 5.

8.415

1. $$Y_0(x) = \frac{4}{\pi^2}\int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}}\sin(xt)\,dt - \frac{4}{\pi^2}\int_1^\infty \frac{\ln\left(t+\sqrt{t^2-1}\right)}{\sqrt{t^2-1}}\sin(xt)\,dt$$
$[x>0]$ MO 37

2. $$Y_\nu(x) = -2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^\infty \frac{\cos xt}{(t^2-1)^{\nu+\frac{1}{2}}}\,dt$$
$\left[-\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2},\quad x>0\right]$ KU 89(28)a, MO 38

3. $$Y_\nu(x) = -\frac{2}{\pi}\int_0^\infty \cos\left(x\cosh t-\frac{\nu\pi}{2}\right)\cosh\nu t\,dt$$
$[-1<\operatorname{Re}\nu<1,\quad x>0]$ WA 199(13)

4.[8] $$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin\left(z\sin\theta-\nu\theta\right)\,d\theta - \frac{1}{\pi}\int_0^\infty \left(e^{\nu t}+e^{-\nu t}\cos\nu\pi\right)e^{-z\sinh t}\,dt$$
$[\operatorname{Re} z>0]$ WA 197(1)

5. $$Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left[\int_0^{\pi/2}\sin\left(z\sin\theta\right)\cos^{2\nu}\theta\,d\theta - \int_0^\infty e^{-z\sinh\theta}\cosh^{2\nu}\theta\,d\theta\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2},\quad \operatorname{Re} z>0\right]$ WA 181(5)a

6. $$Y_\nu(z) = -\frac{2^{\nu+1}z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}}\frac{\cos^{\nu-\frac{1}{2}\theta}\cos\left(z-\nu\theta+\frac{1}{2}\theta\right)}{\sin^{2\nu+1}\theta}e^{-2z\cot\theta}\,d\theta$$
$\left[|\arg z|<\frac{\pi}{2},\quad \operatorname{Re}\left(\nu+\frac{1}{2}\right)>0\right]$ WA 186(8)

For an integral representation of $Y_0(z)$, see **3.714** 3, **3.753** 4, **3.864**. See also **3.865** 3.

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

8.421

1. $$H_\nu^{(1)}(x) = \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^\infty e^{ix\cosh t-\nu t}\,dt = \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_0^\infty e^{ix\cosh t}\cosh\nu t\,dt$$
$[-1<\operatorname{Re}\nu<1,\quad x>0]$ WA 199(10)

2. $$H_\nu^{(2)}(x) = -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^\infty e^{-ix\cosh t-\nu t}\,dt = -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i}\int_0^\infty e^{-ix\cosh t}\cosh\nu t\,dt$$
$[-1<\operatorname{Re}\nu<1,\quad x>0]$ WA 199(11)

3. $$H_\nu^{(1)}(z) = -\frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t \, e^{i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t} \exp\left(-2z\cot t\right) dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0\right]$$ WA 186(5)

4. $$H_\nu^{(2)}(z) = \frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{-i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t} \exp\left(-2z\cot t\right) dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0\right]$$ WA 186(6)

5. $$H_\nu^{(1)}(x) = -\frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{e^{ixt}}{\left(t^2-1\right)^{\nu+\frac{1}{2}}} dt \qquad \left[-\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}, \quad x > 0\right]$$ WA 87(1)

6. $$H_\nu^{(2)}(x) = \frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{e^{-ixt}}{\left(t^2-1\right)^{\nu+\frac{1}{2}}} dt \qquad \left[-\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}, \quad x > 0\right]$$ WA 187(2)

7. $$H_\nu^{(1)}(z) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} \int_0^\infty \exp\left[\frac{1}{2} i z \left(t+\frac{1}{t}\right)\right] t^{-\nu-1} dt$$
$$\left[0 < \arg z < \pi; \text{ or } \arg z = 0 \text{ and } -1 < \operatorname{Re}\nu < 1\right]$$ MO 38

8. $$H_\nu^{(1)}(xz) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} z^\nu \int_0^\infty \exp\left[\frac{1}{2} i x \left(t+\frac{z^2}{t}\right)\right] t^{-\nu-1} dt$$
$$\left[0 < \arg z < \frac{\pi}{2}, \quad x > 0, \quad \operatorname{Re}\nu > -1; \text{ or } \arg z = \frac{\pi}{2}, \quad x > 0 \text{ and } -1 < \operatorname{Re}\nu < 1\right]$$ MO 38

9. $$H_\nu^{(1)}(xz) = \sqrt{\frac{2}{\pi z}} \frac{x^\nu \exp\left[i\left(xz - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right]}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty \left(1+\frac{it}{2z}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}} e^{-xt} dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad -\tfrac{1}{2}\pi < \arg z < \tfrac{3}{2}\pi, \quad x > 0\right]$$ MO 39

10. $$H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi}\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty e^{iz\cosh t} \sinh^{2\nu} t \, dt$$
$$\left[0 < \arg z < \pi, \quad \operatorname{Re}\nu > -\tfrac{1}{2} \text{ or } \arg z = 0 \text{ and } -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ MO 38

11. $$H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^\infty \frac{\exp\left(i\sqrt{x^2+t^2}\right)}{\sqrt{x^2+t^2}} dt \qquad [x > 0]$$ MO 38

8.422

1. $$H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i\,\Gamma\left(\frac{1}{2}\right)} \int_{1+\infty i}^{(1+)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt \qquad [-\pi < \arg z < 2\pi]$$ WA 183(4)

2. $$H_\nu^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i\,\Gamma\left(\frac{1}{2}\right)} \int_{-1+\infty i}^{(-1-)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt$$
$$[-2\pi < \arg z < \pi]$$

The paths of integration are shown in the drawing.

8.423

1. $$H_\nu^{(1)}(z) = -\frac{1}{\pi}\int_{-\infty i}^{-\pi+\infty i} e^{-iz\sin\theta+i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0]$$ WA 197(2)a

2. $$H_\nu^{(2)}(z) = -\frac{1}{\pi}\int_{\pi+\infty i}^{-\infty i} e^{-iz\sin\theta+i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0]$$ WA 197(3)a

The path of integration for **8.423** 1 is shown in the left-hand drawing and for **8.423** 2 in the right-hand drawing.

8.424

1. $$H_\nu^{(1)}(z)\,J_\nu(\zeta) = \frac{1}{\pi i}\int_0^{\gamma+i\infty} \exp\left[\frac{1}{2}\left(t-\frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t}$$
$$[\gamma > 0, \quad \operatorname{Re}\nu > -1, \quad |\zeta| < |z|]$$ MO 45

2. $$H_\nu^{(2)}(z)\,J_\nu(\zeta) = \frac{i}{\pi}\int_0^{\gamma-i\infty} \exp\left[\frac{1}{2}\left(t-\frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t}$$
$$[\gamma > 0, \quad \operatorname{Re}\nu > -1, \quad |\zeta| < |z|]$$ MO 45

8.43 Integral representations of the functions $I_\nu(z)$ and $K_\nu(z)$

The function $I_\nu(z)$

8.431

1. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}} e^{\pm zt}\,dt \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

2. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}} \cosh zt\,dt \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

3. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

4. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} \cosh(z\cos\theta)\sin^{2\nu}\theta\,d\theta \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

5. $$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\nu\theta\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}\,dt$$
$$\left[|\arg z| \le \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right]$$ WA 201(4)

See also **3.383** 2, **3.387** 1, **3.471** 6, **3.714** 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see **3.366** 1, **3.534 3.856** 6.

The function $K_\nu(z)$

8.432

1. $$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt \qquad \left[|\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$$
MO 39

2. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t\,dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ WA 190(5), WH

3. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\,dt$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$$ WA 190(4)

4. $$K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty \cos(x\sinh t)\cosh\nu t\,dt \qquad [x > 0, \quad -1 < \operatorname{Re}\nu < 1]$$ WA 202(13)

5. $$K_\nu(xz) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu}{x^\nu\,\Gamma\left(\frac{1}{2}\right)}\int_0^\infty \frac{\cos xt\,dt}{\left(t^2+z^2\right)^{\nu+\frac{1}{2}}} \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) \ge 0, \quad x > 0, \quad |\arg z| < \frac{\pi}{2}\right]$$
WA 191(1)

6.[11] $$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-z^2/4t}\,dt}{t^{\nu+1}} \qquad \left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0\right]$$ WA 203(15)

7.[7] $$K_\nu(xz) = \frac{z^\nu}{2}\int_0^\infty \exp\left[-\frac{x}{2}\left(t+\frac{z^2}{t}\right)\right]t^{-\nu-1}\,dt$$
$$\left[|\arg z| < \frac{\pi}{4} \text{ or } |\arg z| = \frac{\pi}{4} \text{ and } \operatorname{Re}\nu < 1\right]$$ MO 39

8. $$K_\nu(xz) = \sqrt{\frac{\pi}{2z}}\,\frac{x^\nu e^{-xz}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{\nu-\frac{1}{2}}\left(1+\frac{t}{2z}\right)^{\nu-\frac{1}{2}}\,dt$$
$$\left[|\arg z| < \pi, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, x > 0\right]$$
MO 39

9. $$K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2z}\right)^\nu \int_0^\infty \frac{\exp\left(-x\sqrt{t^2+z^2}\right)}{\sqrt{t^2+z^2}}t^{2\nu}\,dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re}\sqrt{t^2+z^2} > 0, \quad x > 0\right]$$ MO 39

See also **3.383** 3, **3.387** 3, 6, **3.388** 2, **3.389** 4, **3.391**, **3.395** 1, **3.471** 9, **3.483**, **3.547** 2, **3.856**, **3.871** 3, 4, **7.141** 5.

8.433 $K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}}\int_0^\infty \cos\left(t^3 + xt\right)\, dt$ KU 98(31), WA 211(2)

For an integral representation of $K_0(z)$, see **3.754** 2, **3.864**, **4.343**, **4.356**, **4.367**.

8.44 Series representation

The function $J_\nu(z)$

8.440 $J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}$ $[|\arg z| < \pi]$ WH 358 a

8.441 Special cases:

1. $J_0(z) = \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}\,(k!)^2}$

2. $J_1(z) = -\,J_0'(z) = \frac{z}{2}\sum_{k=0}^\infty \frac{(-1)^k z^{2k}}{2^{2k}k!(k+1)!}$

3. $J_{\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{4}{3}\right)}\sqrt[3]{\frac{z}{2}}\sum_{k=0}^\infty (-1)^k \frac{\left(z\sqrt{3}\right)^{2k}}{2^{2k}k!\cdot 1\cdot 4\cdot 7\cdot\cdots\cdot(3k+1)}$

4. $J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\sqrt[3]{\frac{2}{z}}\left\{1 + \sum_{k=1}^\infty (-1)^k \frac{\left(z\sqrt{3}\right)^{2k}}{2^{2k}k!\cdot 2\cdot 5\cdot 8\cdot\cdots\cdot(3k-1)}\right\}$

For the expansion of $J_\nu(z)$ in Laguerre polynomials, see **8.975** 3.

8.442

1.[7] $J_\nu(z)\,J_\mu(z) = \sum_{m=0}^\infty \frac{(-1)^m\left(\frac{1}{2}z\right)^{\mu+\nu+2m}(\mu+\nu+m+1)_m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}$

2.[8] $J_\nu(az)\,J_\mu(bz) = \frac{\left(\frac{az}{2}\right)^\nu\left(\frac{bz}{2}\right)^\mu}{\Gamma(\mu+1)}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu-1; \frac{b^2}{a^2}\right)}{k!\,\Gamma(\nu+k+1)}$ MO 28

The function $Y_\nu(z)$

8.443[11] $Y_\nu(z) = \frac{1}{\sin\nu\pi}\left\{\cos\nu\pi\left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(\nu+k+1)} - \left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(k-\nu+1)}\right\}$

$[\nu \neq \text{an integer}]$ (cf. **8.403** 1)

For $\nu+1$ a natural number, see **8.403** 2.; for ν a negative integer, see **8.404** 1

8.444 Special cases,

1. $$\pi\, Y_0(z) = 2\, J_0(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^{k} \frac{1}{m}$$ KU 44

2.[11] $$\pi\, Y_1(z) = 2\, J_1(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - \frac{2}{z} - \frac{z}{2} - \sum_{k=2}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2}\right)^{2k-1}}{k!(k-1)!} \left(\frac{1}{k} + 2 \sum_{m=1}^{k-1} \frac{1}{m}\right)$$

The functions $I_\nu(z)$ and $K_n(z)$

8.445 $$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$ WH 372a

8.446[8] $$K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!\left(\frac{z}{2}\right)^{n-2k}}$$

$$+(-1)^{n+1} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!} \left[\ln \frac{z}{2} - \frac{1}{2}\,\psi(k+1) - \frac{1}{2}\,\psi(n+k+1)\right]$$

WA 95(15)

$$= (-1)^{n+1}\, I_n(z) \left(\ln \frac{1}{2} z + \boldsymbol{C}\right) + \frac{1}{2} (-1)^n \sum_{l=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2l}}{l!(n+l)!} \left(\sum_{k=1}^{l} \frac{1}{k} + \sum_{k=1}^{n+l} \frac{1}{k}\right)$$

$$+\frac{1}{2} \sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!} \left(\frac{z}{2}\right)^{2l-n}$$

[$n+1$ is a natural number]

MO 29

8.447 Special cases:

1. $$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}$$

2. $$I_1(z) = I_0'(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!}$$

3. $$K_0(z) = -\ln \frac{z}{2}\, I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\,(k!)^2}\, \psi(k+1)$$ WA 95(14)

8.45 Asymptotic expansions of Bessel functions

8.451 For large values of $|z|$ *

1. $$J_{\pm\nu}(z)=\sqrt{\frac{2}{\pi z}}\left\{\cos\left(z\mp\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma\left(\nu+2k+\frac{1}{2}\right)}{(2k)!\,\Gamma\left(\nu-2k+\frac{1}{2}\right)}+R_1\right]\right.$$
$$\left.-\sin\left(z\mp\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma\left(\nu+2k+\frac{3}{2}\right)}{(2k+1)!\,\Gamma\left(\nu-2k-\frac{1}{2}\right)}+R_2\right]\right\}$$
$[|\arg z|<\pi]$ (see **8.339** 4) WA 222(1, 3)

2.[11] $$Y_{\pm\nu}(z)=\sqrt{\frac{2}{\pi z}}\left\{\sin\left(z\mp\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma\left(\nu+2k+\frac{1}{2}\right)}{(2k)!\,\Gamma\left(\nu-2k+\frac{1}{2}\right)}+R_1\right]\right.$$
$$\left.+\cos\left(z\mp\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma\left(\nu+2k+\frac{3}{2}\right)}{(2k+1)!\,\Gamma\left(\nu-2k-\frac{1}{2}\right)}+R_2\right]\right\}$$
$[|\arg z|<\pi]$ (see **8.339** 4) WA 222(2, 4, 5)

3.[11] $$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi z}}e^{i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}+\theta_1\frac{(-1)^n}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2},\quad |\arg z|<\pi\right]$ (see **8.339** 4) WA 221(5)

4.[11] $$H_\nu^{(2)}(z)=\sqrt{\frac{2}{\pi z}}e^{-i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{1}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}+\theta_2\frac{1}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
$\left[\operatorname{Re}\nu>-\frac{1}{2},\quad |\arg z|<\pi\right]$ (see **8.339** 4) WA 221(6)

For indices of the form $\nu=\frac{2n-1}{2}$ (where n is a natural number), the series **8.451** terminate. In this case, the closed formulas **8.46** are valid for all values.

5. $$I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}$$
$$+\frac{\exp\left[-z\pm\left(\nu+\frac{1}{2}\right)\pi i\right]}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{1}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}$$
[The + sign is taken for $-\frac{1}{2}\pi<\arg z<\frac{3}{2}\pi$, the − sign for $-\frac{3}{2}\pi<\arg z<\frac{1}{2}\pi$]* (see **8.339** 4)]
WA 226(2,3)

6.[11] $$K_\nu(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\left[\sum_{k=0}^{n-1}\frac{1}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}+\theta_3\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{(2z)^n n!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
(see **8.339** 4) WA 231, 245(9)

An estimate of the remainders of the asymptotic series in formulas **8.451**:

*An estimate of the remainders in formulas **8.451** is given in **8.451** 7 and **8.451** 8.

*The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

7. $$|R_1| < \left| \frac{\Gamma\left(\nu + 2n + \frac{1}{2}\right)}{(2z)^{2n}(2n)!\,\Gamma\left(\nu - 2n + \frac{1}{2}\right)} \right| \qquad \left[n > \frac{\nu}{2} - \frac{1}{4}\right]$$ WA 231

8. $$|R_2| < \left| \frac{\Gamma\left(\nu + 2n + \frac{3}{2}\right)}{(2z)^{2n+1}(2n+1)!\,\Gamma\left(\nu - 2n - \frac{1}{2}\right)} \right| \qquad \left[n \geq \frac{\nu}{2} - \frac{3}{4}\right]$$ WA 231

For $-\dfrac{\pi}{2} < \arg z < \dfrac{3}{2}\pi$, ν real, and $n + \frac{1}{2} > |\nu|$ WA 245

$$|\theta_1| < \begin{cases} 1, & \text{if } \operatorname{Im} z \geq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \leq 0 \end{cases}$$

For $-\dfrac{3}{2}\pi < \arg z < \dfrac{\pi}{2}$, ν real, and $n + \frac{1}{2} > |\nu|$ WA 246

$$|\theta_2| < \begin{cases} 1, & \text{if } \operatorname{Im} z \leq 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \geq 0 \end{cases}$$

For ν real, WA 245

$$|\theta_3| < \begin{cases} 1 & \text{if } \operatorname{Re} z \geq 0 \\ |\operatorname{cosec}(\arg z)|, & \text{if } \operatorname{Re} z < 0 \end{cases}$$
$$\operatorname{Re}\theta_3 \geq 0, \qquad \text{if } \operatorname{Re} z \geq 0$$

For ν and z real and $n \geq \nu - \frac{1}{2}$, WA 231

$$0 \leq |\theta_3| \leq 1$$

In particular, it follows from **8.451** 7 and **8.451** 8 that for real positive values of z and ν, the errors $|R_1|$ and $|R_2|$ are less than the absolute value of the first discarded term. For values of $|\arg z|$ close to π, the series **8.451** 1 and **8.451** 2 may not be suitable for calculations. In particular, the error for $|\arg z| > \pi$ can be greater in absolute value than the first discarded term.

"Approximation by tangents"

8.452[11] For large values of the index (where the argument is less than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cosh\alpha$. Then, for large values of ν, the following expansions are valid:

1. $$J_\nu\left(\frac{\nu}{\cosh\alpha}\right) \sim \frac{\exp(\nu\tanh\alpha - \nu\alpha)}{\sqrt{2\nu\pi\tanh\alpha}} \left\{1 + \frac{1}{\nu}\left(\frac{1}{8}\coth\alpha - \frac{5}{24}\coth^3\alpha\right) + \frac{1}{\nu^2}\left(\frac{9}{128}\coth^2\alpha - \frac{231}{576}\coth^4\alpha + \frac{1155}{3456}\coth^6\alpha\right) + \dots\right\}$$

WA 269(3)

2. $$Y_\nu\left(\frac{\nu}{\cosh\alpha}\right)\sim-\frac{\exp\left(\nu\alpha-\nu\tanh\alpha\right)}{\sqrt{\frac{\pi}{2}\nu\tanh\alpha}}\left\{1-\frac{1}{\nu}\left(\frac{1}{8}\coth\alpha-\frac{5}{24}\coth^3\alpha\right)\right.$$
$$\left.+\frac{1}{\nu^2}\left(\frac{9}{128}\coth^2\alpha-\frac{231}{576}\coth^4\alpha+\frac{1155}{3456}\coth^6\alpha\right)+\ldots\right\}$$

WA 270(5)

8.453 For large values of the index (where the argument is greater than the index).

Suppose that $x>0$ and $\nu>0$. Let us set $\nu/x=\cos\beta$. Then, for large values of ν, the following expansions are valid:

1. $$J_\nu\left(\nu\sec\beta\right)\sim\sqrt{\frac{2}{\nu\pi\tan\beta}}\left\{\left[1-\frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta+\frac{231}{576}\cot^4\beta\right.\right.\right.$$
$$\left.\left.+\frac{1155}{3456}\cot^6\beta\right)+\ldots\right]\cos\left(\nu\tan\beta-\nu\beta-\frac{\pi}{4}\right)$$
$$\left.+\left[\frac{1}{\nu}\left(\frac{1}{8}\cot\beta+\frac{5}{24}\cot^3\beta\right)-\ldots\right]\sin\left(\nu\tan\beta-\nu\beta-\frac{\pi}{4}\right)\right]$$

WA 271(4)

2. $$Y_\nu\left(\nu\sec\beta\right)\sim\sqrt{\frac{2}{\nu\pi\tan\beta}}\left\{\left[1-\frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta+\frac{231}{576}\cot^4\beta\right.\right.\right.$$
$$\left.\left.+\frac{1155}{3456}\cot^6\beta\right)+\ldots\right]\sin\left(\nu\tan\beta-\nu\beta-\frac{\pi}{4}\right)$$
$$\left.-\frac{1}{\nu}\left(\frac{1}{8}\cot\beta+\frac{5}{24}\cot^3\beta\right)-\ldots\right]\cos\left(\nu\tan\beta-\nu\beta-\frac{\pi}{4}\right)\right]$$

WA 271(5)

3. $$H_\nu^{(1)}\left(\nu\sec\beta\right)\sim\frac{\exp\left[\nu i\left(\tan\beta-\beta\right)-\frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\tan\beta}}\left\{1-\frac{i}{\nu}\left(\frac{1}{8}\cot\beta+\frac{5}{24}\cot^3\beta\right)\right.$$
$$\left.-\frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta+\frac{231}{576}\cot^4\beta+\frac{1155}{3456}\cot^6\beta\right)+\ldots\right\}$$

WA 271(1)

4. $$H_\nu^{(2)}\left(\nu\sec\beta\right)\sim\frac{\exp\left[-\nu i\left(\tan\beta-\beta\right)+\frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\tan\beta}}\left\{1+\frac{i}{\nu}\left(\frac{1}{8}\cot\beta+\frac{5}{24}\cot^3\beta\right)\right.$$
$$\left.-\frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta+\frac{231}{576}\cot^4\beta+\frac{1155}{3456}\cot^6\beta\right)+\ldots\right\}$$

WA 271(2)

Formulas **8.453** are not valid when $|x-\nu|$ is of a size comparable to $x^{\frac{1}{3}}$. For arbitrary small (and also large) values of $|x-\nu|$, we may use the following formulas:

8.454 Suppose that $x > 0$ and $\nu > 0$, we set

$$w = \sqrt{\frac{x^2}{\nu^2} - 1};$$

Then,

1. $$H_\nu^{(1)}(x) = \frac{w}{\sqrt{3}} \exp\left\{\left[\frac{\pi}{6} + \nu\left(w - \frac{w^3}{3} - \arctan w\right)\right] i\right\} H_{\frac{1}{3}}^{(1)}\left(\frac{\nu}{3} w^3\right) + O\left(\frac{1}{|\nu|}\right)$$

2. $$H_\nu^{(2)}(x) = \frac{w}{\sqrt{3}} \exp\left\{\left[-\frac{\pi}{6} - \nu\left(w - \frac{w^3}{3} - \arctan w\right)\right] i\right\} H_{\frac{1}{3}}^{(2)}\left(\frac{\nu}{3} w^3\right) + O\left(\frac{1}{|\nu|}\right)$$ MO 34

The absolute value of the error $O\left(\frac{1}{|\nu|}\right)$ is then less than $24\sqrt{2}\left|\frac{1}{\nu}\right|$.

8.455 For x real and ν a natural number ($\nu = n$), if $n \gg 1$, the following approximations are valid:

1.[7] $$J_n(x) \approx \frac{1}{\pi}\sqrt{\frac{2(n-x)}{3x}}\, K_{\frac{1}{3}}\left\{\frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}}\right\}$$

$[n > x]$ (see also **8.433**)

WA 276(1)

$$\approx \frac{1}{2} e^{\frac{2}{3}\pi i}\sqrt{\frac{2\,(n-x)}{3x}}\, H_{\frac{1}{3}}^{(1)}\left\{\frac{i}{3}\frac{[2(n-x)]^{\frac{3}{2}}}{\sqrt{x}}\right\}$$

$[n > x]$

MO 34

$$\approx \frac{1}{\sqrt{3}}\sqrt{\frac{2(x-n)}{3x}}\left\{J_{\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right] + J_{-\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]\right\}$$

(see also **8.441** 3, **8.441** 4)

WA 276(2)

2. $$Y_n(x) \approx \sqrt{\frac{2(x-n)}{3x}}\left\{J_{-\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right] - J_{\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]\right\}$$

$[x > n]$ WA 276(3)

An estimate of the error in formulas **8.455** has not yet been achieved.

8.456[11] $$J_\nu^2(z) + Y_\nu^2(z) \approx \frac{2}{\pi z}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{2^k z^{2k}}\frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k!\,\Gamma\left(\nu - k + \frac{1}{2}\right)}\ [|\arg z| < \pi]$$ (see also **8.479** 1)

WA 250(5)

8.457 $$J_\nu^2(x) + J_{\nu+1}^2(x) \approx \frac{2}{\pi x}$$ $[x \gg |\nu|]$ WA 223

8.46 Bessel functions of order equal to an integer plus one-half

The function $J_\nu(z)$

8.461

1.[11] $$J_{n+\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\left\{\sin\left(z-\frac{\pi}{2}n\right)\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(n+2k)!}{(2k)!(n-2k)!}(2z)^{-2k}\right.$$
$$\left.+\cos\left(z-\frac{\pi}{2}n\right)\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(-1)^k(n+2k+1)!}{(2k+1)!(n-2k-1)!}(2z)^{-(2k+1)}\right\}$$
[$n+1$ is a natural number] (cf. **8.451** 1) KU 59(6), WA 66(2)

2. $$J_{-n-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\left\{\cos\left(z+\frac{\pi}{2}n\right)\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k\,(n+2k)!}{(2k)!(n-2k)!(2z)^{2k}}\right.$$
$$\left.-\sin\left(z+\frac{\pi}{2}n\right)\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(-1)^k(n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}}\right\}$$
[$n+1$ is a natural number] (cf. **8.451** 1) KU 58(7), WA 67(5)

8.462

1. $$J_{n+\frac{1}{2}}(z)=\frac{1}{\sqrt{2\pi z}}\left\{e^{iz}\sum_{k=0}^{n}\frac{i^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k}+e^{-iz}\sum_{k=0}^{n}\frac{(-i)^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k}\right\}$$
[$n+1$ is a natural number]
KU 59(6), WA 66(1)

2. $$J_{-n-\frac{1}{2}}(z)=\frac{1}{\sqrt{2\pi z}}\left\{e^{iz}\sum_{k=0}^{n}\frac{i^{n+k}(n+k)!}{k!(n-k)!(2z)^k}+e^{-iz}\sum_{k=0}^{n}\frac{(-i)^{n+k}(n+k)!}{k!(n-k)!(2z)^k}\right\}$$
[$n+1$ is a natural number]
KU 59(7), WA 67(4)

8.463

1. $$J_{n+\frac{1}{2}}(z)=(-1)^n z^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{d^n}{(z\,dz)^n}\left(\frac{\sin z}{z}\right)$$ KU 58(4)

2. $$J_{-n-\frac{1}{2}}(z)=z^{n+\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{d^n}{(z\,dz)^n}\left(\frac{\cos z}{z}\right)$$ KU 58(5)

8.464 Special cases:

1. $$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z$$ DW

2. $$J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\cos z$$ DW

3. $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right)$ DW

4. $J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(-\sin z - \frac{\cos z}{z}\right)$ DW

5.[8] $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left\{\left(\frac{3}{z^2} - 1\right)\sin z - \frac{3}{z}\cos z\right\}$ DW

6. $J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left\{\frac{3}{z}\sin z + \left(\frac{3}{z^2} - 1\right)\cos z\right\}$ DW

The function $Y_{n+\frac{1}{2}}(z)$

8.465

1. $Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z)$ JA

2. $Y_{-n-\frac{1}{2}}(z) = (-1)^{n} J_{n+\frac{1}{2}}(z)$ JA

The functions $H^{(1,2)}_{n+\frac{1}{2}}(z)$, $I_{n+\frac{1}{2}}(z)$, $K_{n+\frac{1}{2}}(z)$

8.466

1. $H^{(1)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$

(cf. **8.451** 3)

2. $H^{(2)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{n} e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$ (cf. **8.451** 4)

8.467 $I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^{z}\sum_{k=0}^{n}\frac{(-1)^k(n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!(2z)^k}\right]$

(cf. **8.451** 5) KU 60a

8.468 $K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k}$ (cf. **8.451** 6) KU 60

8.469 Special cases:

1. $Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}}\cos z$

2. $Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z$

3. $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ WA 95(13)

4. $H^{(1)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{e^{iz}}{i}$ MO 27

5. $H^{(2)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{-iz}}{-i}$ MO 27

6. $H^{(1)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,e^{iz}$ MO 27

7. $H^{(2)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,e^{-iz}$ MO 27

8.47–8.48 Functional relations

8.471[8] Recursion formulas:

1. $z\,Z_{\nu-1}(z) + z\,Z_{\nu+1}(z) = 2\nu\,Z_\nu(z)$ KU 56(13), WA 56(1), WA 79(1), WA 88(3)

2. $Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2\frac{d}{dz}\,Z_\nu(z)$ KU 56(12), WA 56(2), WA 79(2), We 88(4)

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of z that satisfy the recursion relations **8.471**. Z denotes J, N, $H^{(1)}$, $H^{(2)}$ or any linear combination of these functions, the coefficients of which are independent of z and ν.

8.472 Consequences of the recursion formulas for Z defined as above:

1. $z\frac{d}{dz}\,Z_\nu(z) + \nu\,Z_\nu(z) = z\,Z_{\nu-1}(z)$ KU 56(11), WA 56(3), WA 79(3), WA 88(5)

2. $z\frac{d}{dz}\,Z_\nu(z) - \nu\,Z_\nu(z) = -z\,Z_{\nu+1}(z)$ KU 56(10), WA 56(4), WA 79(4), WA 88(6)

3. $\left(\frac{d}{z\,dz}\right)^m (z^\nu\,Z_\nu(z)) = z^{\nu-m}\,Z_{\nu-m}(z)$ KU 56(8), WA 57(5), WA 89(9)

4. $\left(\frac{d}{z\,dz}\right)^m \left(z^{-\nu}\,Z_\nu(z)\right) = (-1)^m z^{-\nu-m}\,Z_{\nu+m}(z)$ WA 89(10), Ku 55(5), WA 57(6)

5. $Z_{-n}(z) = (-1)^n\,Z_n(z)$ [n is a natural number] (cf. **8.404**)

8.473 Special cases:

1. $J_2(z) = \frac{2}{z}\,J_1(z) - J_0(z)$

2. $Y_2(z) = \frac{2}{z}\,Y_1(z) - Y_0(z)$

3. $H^{(1,2)}_2(z) = \frac{2}{z}\,H^{(1,2)}_1(z) - H^{(1,2)}_0(z)$

4. $\frac{d}{dz}\,J_0(z) = -\,J_1(z)$

5. $\frac{d}{dz}\,Y_0(z) = -\,Y_1(z)$

6. $\frac{d}{dz}\,H^{(1,2)}_0(z) = -\,H^{(1,2)}_1(z)$

8.474[8] Each of the pairs of functions $J_\nu(z)$ and $J_{-\nu}(z)$ (for $\nu \neq 0, \pm1, \pm2, \ldots$), $J_\nu(z)$ and $Y_\nu(z)$, and $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$, which are solutions of equation **8.401**, and also the pair $I_\nu(z)$ and $K_\nu(z)$ is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z}\sin\nu\pi, \quad \frac{2}{\pi z}, \quad -\frac{4i}{\pi z}, \quad -\frac{1}{z}$$ KU 52(10, 11, 12), WA 90(1, 4)

8.475[6] The functions $J_\nu(z)$, and $Y_\nu(z)$, $H_\nu^{(1,2)}(z)$, $I_\nu(z)$, $K_\nu(z)$, with the exception of $J_n(z)$ and $I_n(z)$, for n an integer are *non-single-valued*: $z=0$ is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut $(-\infty, 0)$ are connected by the relations

8.476

1. $J_\nu\left(e^{m\pi i}z\right) = e^{m\nu\pi i}\, J_\nu(z)$ WA 90(1)
2. $Y_\nu\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, Y_\nu(z) + 2i\sin m\nu\pi\cot\nu\pi\, J_\nu(z)$ WA 90(3)
3. $Y_{-\nu}\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, Y_{-\nu}(z) + 2i\sin m\nu\pi\operatorname{cosec}\nu\pi\, J_\nu(z)$ WA 90(4)
4. $I_\nu\left(e^{m\pi i}z\right) = e^{m\nu\pi i}\, I_\nu(z)$ WA 95(17)
5. $K_\nu\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, K_\nu(z) - i\pi\dfrac{\sin m\nu\pi}{\sin\nu\pi}\, I_\nu(z)$ [ν not an integer] WA 95(18)
6. $$\begin{aligned} H_\nu^{(1)}\left(e^{m\pi i}z\right) &= e^{-m\nu\pi i}\, H_\nu^{(1)}(z) - 2e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, J_\nu(z) \\ &= \frac{\sin(1-m)\nu\pi}{\sin\nu\pi}\, H_\nu^{(1)}(z) - e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, H_\nu^{(2)}(z) \end{aligned}$$ WA 95(5)
7. $$\begin{aligned} H_\nu^{(2)}\left(e^{m\pi i}z\right) &= e^{-m\nu\pi i}\, H_\nu^{(2)}(z) + 2e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, J_\nu(z) \\ &= \frac{\sin(1+m)\nu\pi}{\sin\nu\pi}\, H_\nu^{(2)}(z) + e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, H_\nu^{(1)}(z) \end{aligned}$$ [m an integer] WA 90(6)
8. $H_\nu^{(1)}\left(e^{i\pi}z\right) = -\,H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu}\, H_\nu^{(2)}(z)$ MO 26
9. $H_\nu^{(2)}\left(e^{-i\pi}z\right) = -\,H_{-\nu}^{(1)}(z) = -e^{i\pi\nu}\, H_\nu^{(1)}(z)$ MO 26
10.[8] $\overline{H}_\nu^{(2)}(z) = H_{\overline{\nu}}^{(1)}\left(\overline{z}\right)$ MO 26

8.477

1. $J_\nu(z)\, Y_{\nu+1}(z) - J_{\nu+1}(z)\, Y_\nu(z) = -\dfrac{2}{\pi z}$ WA 91(12)
2. $I_\nu(z)\, K_{\nu+1}(z) + I_{\nu+1}(z)\, K_\nu(z) = \dfrac{1}{z}$ WA 95(20)

See also **3.863**.

- For a connection with Legendre functions, see **8.722**.
- For a connection with the polynomials $C_n^\lambda(t)$, see **8.936** 4.
- For a connection with a confluent hypergeometric function, see **9.235**.

8.478 For $\nu > 0$ and $x > 0$, the product

$$x\left[J_\nu^2(x) + Y_\nu^2(x)\right],$$

considered as a function of x, decreases monotonically, if $\nu > \frac{1}{2}$ and increases monotonically if $0 < \nu < \frac{1}{2}$. MO 35

8.479

1.[11] $\dfrac{1}{\sqrt{x^2-\nu^2}} > \dfrac{\pi}{2}\left[J_\nu^2(x)+Y_\nu^2(x)\right] \geq \dfrac{1}{x}$ $\quad\left[x \geq \nu \geq \frac{1}{2}\right]$ MO 35

2. $|J_n(nz)| \leq 1$

$$\left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1, n \text{ a natural number}\right]$$ MO 35

Relations between Bessel functions of the first, second, and third kinds

8.481 $J_\nu(z) = \dfrac{Y_{-\nu}(z) - Y_\nu(z)\cos\nu\pi}{\sin\nu\pi} = H_\nu^{(1)}(z) - i\,Y_\nu(z)$

$= H_\nu^{(2)}(z) + i\,Y_\nu(z) = \dfrac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right)$

(cf. **8.403** 1, **8.405**) WA 89(1), JA

8.482 $Y_\nu(z) = \dfrac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} = i\,J_\nu(z) - i\,H_\nu^{(1)}(z)$

$= i\,H_\nu^{(2)}(z) - i\,J_\nu(z) = \dfrac{i}{2}\left(H_\nu^{(2)}(z) - H_\nu^{(1)}(z)\right)$

(cf. **8.403** 1, **8.405**) WA 89(3), JA

8.483

1. $H_\nu^{(1)}(z) = \dfrac{J_{-\nu}(z) - e^{-\nu\pi i}\,J_\nu(z)}{i\sin\nu\pi} = \dfrac{Y_{-\nu}(z) - e^{-\nu\pi i}\,Y_\nu(z)}{\sin\nu\pi} = J_\nu(z) + i\,Y_\nu(z)$ WA 89(5)

2. $H_\nu^{(2)}(z) = \dfrac{e^{\nu\pi i}\,J_\nu(z) - J_{-\nu}(z)}{i\sin\nu\pi} = \dfrac{Y_{-\nu}(z) - e^{\nu\pi i}\,Y_\nu(z)}{\sin\nu\pi} = J_\nu(z) - i\,Y_\nu(z)$

(cf. **8.405**) WA 89(6)

8.484

1. $H_{-\nu}^{(1)}(z) = e^{\nu\pi i}\,H_\nu^{(1)}(z)$ WA 89(7)

2. $H_{-\nu}^{(2)}(z) = e^{-\nu\pi i}\,H_\nu^{(2)}(z)$ WA 89(7)

8.485[7] $K_\nu(z) = \dfrac{\pi}{2}\dfrac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}$ $\quad[\nu \text{ not an integer}]$ (see also **8.407**)

WA 92(6)

8.486 Recursion formulas for the functions $I_\nu(z)$ and $K_\nu(z)$ and their consequences:

1. $z\,I_{\nu-1}(z) - z\,I_{\nu+1}(z) = 2\nu\,I_\nu(z)$ WA 93(1)

2. $I_{\nu-1}(z) + I_{\nu+1}(z) = 2\dfrac{d}{dz}\,I_\nu(z)$ WA 93(2)

3. $z\dfrac{d}{dz}\,I_\nu(z) + \nu\,I_\nu(z) = z\,I_{\nu-1}(z)$ WA 93(3)

4. $z\dfrac{d}{dz}\,I_\nu(z) - \nu\,I_\nu(z) = z\,I_{\nu+1}(z)$ WA 93(4)

5. $\left(\dfrac{d}{z\,dz}\right)^m \{z^\nu\,I_\nu(z)\} = z^{\nu-m}\,I_{\nu-m}(z)$ WA 93(5)

6. $\left(\frac{d}{z\,dz}\right)^m \left\{z^{-\nu}\, I_\nu(z)\right\} = z^{-\nu-m}\, I_{\nu+m}(z)$ WA 93(6)

7. $I_{-n}(z) = l_n(z)$ [n a natural number] WA 93(8)

8. $I_2(z) = -\frac{2}{z} l_1(z) + I_0(z)$

9. $\frac{d}{dz}\, I_0(z) = I_1(z)$ WA 93(7)

10. $z\, K_{\nu-1}(z) - z\, K_{\nu+1}(z) = -2\nu\, K_\nu(z)$ WA 93(1)

11. $K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz}\, K_\nu(z)$ WA 93(2)

12. $z\frac{d}{dz}\, K_\nu(z) + \nu\, K_\nu(z) = -z\, K_{\nu-1}(z)$ WA 93(3)

13. $z\frac{d}{dz}\, K_\nu(z) - \nu\, K_\nu(z) = -z\, K_{\nu+1}(z)$ WA 93(4)

14. $\left(\frac{d}{z\,dz}\right)^m \left\{z^{\nu}\, K_\nu(z)\right\} = (-1)^m z^{\nu-m}\, K_{\nu-m}(z)$ WA 93(5)

15. $\left(\frac{d}{z\,dz}\right)^m \left\{z^{-\nu}\, K_\nu(z)\right\} = (-1)^m z^{-\nu-m}\, K_{\nu+m}(z)$ WA 93(6)

16. $K_{-\nu}(z) = K_\nu(z)$ WA 93(8)

17. $K_2(z) = \frac{2}{z}\, K_1(z) + K_0(z)$

18. $\frac{d}{dz}\, K_0(z) = -\, K_1(z)$ WA 93(7)

19. $\frac{\partial\, J_\nu(z)}{\partial\nu} = \left[\ln\frac{z}{2} - \psi(\nu+1)\right] J_\nu(z) + \frac{(z/2)^{\nu+1}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(z/2)^n\, J_{n+1}(z)}{n!(\nu+n+1)^2}$ LUKE 360

8.486(1)[7] Differentiation with respect to order

1. $\frac{\partial\, J_\nu(z)}{\partial\nu} = J_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}$

$\left[\nu \neq n \text{ or } n+\frac{1}{2}, \quad n \text{ integer}\right]$ MS 3.1.3

2. $\frac{\partial\, J_{-\nu}(z)}{\partial\nu} = -\, J_{-\nu}(z) \ln\left(\frac{1}{2}z\right) + \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{-\nu+2k} \frac{\psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)}$

$\left[\nu \neq n \text{ or } n+\frac{1}{2}, \quad n \text{ integer}\right]$ MS 3.1.3

3. $\frac{\partial\, Y_\nu(z)}{\partial\nu} = \cot\pi\nu \frac{\partial\, J_\nu(z)}{\partial\nu} - \operatorname{cosec}\pi\nu \frac{\partial\, J_{-\nu}(z)}{\partial\nu} - \pi \operatorname{cosec}\pi\nu\, Y_\nu(z)$

$\left[\nu \neq n \text{ or } n+\frac{1}{2}, \quad n \text{ integer}\right]$ MS 3.1.3

4. $\frac{\partial\, I_\nu(z)}{\partial\nu} = I_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}$ $\left[\nu \neq n \text{ or } n+\frac{1}{2}, \quad n \text{ integer}\right]$

MS 3.1.3

5. $$\frac{\partial K_\nu(z)}{\partial \nu} = -\pi \cot \pi\nu \, K_\nu(z) + \frac{1}{2}\pi \operatorname{cosec} \pi\nu \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]$$
$$\left[\nu \neq n \text{ or } n+\tfrac{1}{2}, \quad n \text{ integer}\right]$$ MS 3.1.3

6. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \frac{1}{2}\pi\,(\pm 1)^n\, Y_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} J_k(z)}{k!(n-k)} \qquad [n=0,1,\ldots]$$ MS 3.2.3

7. $$\left[\frac{\partial Y_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = -\frac{1}{2}\pi\,(\pm 1)^n\, J_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} Y_k(z)}{k!(n-k)} \qquad [n=0,1,\ldots]$$
MS 3.2.3

8. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm (-1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)} \qquad [n=0,1,\ldots]$$
MS 3.2.3

9. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \pm\frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)} \qquad [n=0,1,\ldots]$$ MS 3.2.3

10. $$(-1)^n \left[\frac{\partial}{\partial \nu} I_\nu(z)\right]_{\nu=n} = -K_n(z) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)}$$
$$[n=0,1,\ldots]$$ AS 9.6.44

11.[11] $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=n} = \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)} \qquad [n=0,1,\ldots]$$ AS 9.6.45

Special cases

12. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=0} = \tfrac{1}{2}\pi\, Y_0(z)$$ MS 3.2.3

13. $$\left[\frac{\partial Y_\nu(z)}{\partial \nu}\right]_{\nu=0} = -\tfrac{1}{2}\pi\, J_0(z)$$ MS 3.2.3

14. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=0} = -K_0(z)$$ MS 3.2.3

15. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=0} = 0$$ MS 3.2.3

16. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left[\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)\right]$$ MS 3.3.3

17. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left[\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)\right]$$ MS 3.3.3

18. $$\left[\frac{\partial Y_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left\{\cos x \operatorname{Ci}(2x) + \sin x \left[\operatorname{Si}(2x) - \pi\right]\right\}$$ MS 3.3.3

19. $$\left[\frac{\partial Y_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = -\left(\tfrac{1}{2}\pi x\right)^{-1/2} \left\{\sin x \operatorname{Ci}(2x) - \cos x \left[\operatorname{Si}(2x) - \pi\right]\right\}$$ MS 3.3.3

20. $\left[\dfrac{\partial I_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-1/2}\left[e^x \operatorname{Ei}(-2x) \mp e^{-x}\overline{\operatorname{Ei}}(2x)\right]$ MS 3.3.3

21. $\left[\dfrac{\partial K_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = \mp\left(\dfrac{\pi}{2x}\right)^{\frac{1}{2}} e^x \operatorname{Ei}(-2x)$ MS 3.3.3

8.487 Continuity with respect to the order*:

1. $\lim\limits_{\nu\to n} Y_\nu(z) = Y_n(z)$ [n an integer] WA 76

2. $\lim\limits_{\nu\to n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z)$ [n an integer] WA 183

3. $\lim\limits_{\nu\to n} K_\nu(z) = K_n(z)$ [n an integer] WA 92

8.49 Differential equations leading to Bessel functions

See also **8.401**

8.491

1. $\dfrac{1}{z}\dfrac{d}{dz}(zu') + \left(\beta^2 - \dfrac{\nu^2}{z^2}\right)u = 0$ $u = Z_\nu(\beta z)$ JA

2. $\dfrac{1}{z}\dfrac{d}{dz}(zu') + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 - \left(\dfrac{\nu\gamma}{z}\right)^2\right]u = 0$ $u = Z_\nu\left(\beta z^\gamma\right)$ JA

3. $u'' + \dfrac{1-2\alpha}{z}u' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 + \dfrac{\alpha^2-\nu^2\gamma^2}{z^2}\right]u = 0$ $u = z^\alpha Z_\nu\left(\beta z^\gamma\right)$ JA

4. $u'' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 - \dfrac{4\nu^2\gamma^2-1}{4z^2}\right]u = 0$ $u = \sqrt{z}\, Z_\nu\left(\beta z^\gamma\right)$ JA

5. $u'' + \left(\beta^2 - \dfrac{4\nu^2-1}{4z^2}\right)u = 0$ $u = \sqrt{z}\, Z_\nu(\beta z)$ JA

6. $u'' + \dfrac{1-2\alpha}{z}u' + \left(\beta^2 + \dfrac{\alpha^2-\nu^2}{z^2}\right)u = 0$ $u = z^\alpha Z_\nu(\beta z)$ JA

7. $u'' + bz^m u = 0$ $u = \sqrt{z}\, Z_{\frac{1}{m+2}}\left(\dfrac{2\sqrt{b}}{m+2}z^{\frac{m+2}{2}}\right)$ JA 111(5)

8. $u'' + \dfrac{1}{z}u' + 4\left(z^2 - \dfrac{\nu^2}{z^2}\right)u = 0$ $u = Z_\nu\left(z^2\right)$ WA 111(6)

9. $u'' + \dfrac{1}{z}u' + \dfrac{1}{4z}\left(1 - \dfrac{\nu^2}{z}\right)u = 0$ $u = Z_\nu\left(\sqrt{z}\right)$ WA 111(7)

10. $u'' + \dfrac{1-\nu}{z}u' + \dfrac{1}{4}\dfrac{u}{z} = 0$ $u = z^{\frac{\nu}{2}} Z_\nu\left(\sqrt{z}\right)$ WA 111(9)a

11. $u'' + \beta^2\gamma^2 z^{2\beta-2}u = 0$ $u = z^{1/2} Z_{\frac{1}{2\beta}}\left(\gamma z^\beta\right)$ WA 110(3)

*The continuity of the functions $J_\nu(z)$ and $I_\nu(z)$ follows directly from the series representations of these functions.

12. $z^2u'' + (2\alpha - 2\beta\nu + 1)zu' + \left[\beta^2\gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta\nu)\right] u = 0$

$u = z^{\beta\nu-\alpha}\, Z_\nu\left(\gamma z^\beta\right)$ WA 112(21)

8.492

1. $u'' + \left(e^{2z} - \nu^2\right) u = 0$ $\quad u = Z_\nu\left(e^z\right)$ WA 112(22)

2. $u'' + \dfrac{e^{2/z} - \nu^2}{z^4} u = 0$ $\quad u = z\, Z_\nu\left(e^{1/z}\right)$ WA 112(22)

8.493

1. $u'' + \left(\dfrac{1}{z} - 2\tan z\right) u' - \left(\dfrac{\nu^2}{z^2} + \dfrac{\tan z}{z}\right) u = 0$ $\quad u = \sec z\, Z_\nu(z)$ JA

2. $u'' + \left(\dfrac{1}{z} + 2\cot z\right) u' - \left(\dfrac{\nu^2}{z^2} - \dfrac{\cot z}{z}\right) u = 0$ $\quad u = \operatorname{cosec} z\, Z_\nu(z)$ JA

8.494

1. $u'' + \dfrac{1}{z}u' - \left(1 + \dfrac{\nu^2}{z^2}\right) u = 0$ $\quad u = Z_\nu(iz) = C_1\, I_\nu(z) + C_2\, K_\nu(z)$ JA

2. $u'' + \dfrac{1}{z}u' - \left[\dfrac{1}{z} + \left(\dfrac{\nu}{2z}\right)^2\right] u = 0$ $\quad u = Z_\nu\left(2i\sqrt{z}\right)$ JA

3. $u'' + u' + \dfrac{1}{z^2}\left(\dfrac{1}{4} - \nu^2\right) u = 0$ $\quad u = \sqrt{z} e^{-\frac{z}{2}}\, Z_\nu\left(\dfrac{iz}{2}\right)$ JA

4.[10] $u'' + \left(\dfrac{2\nu+1}{z} - k\right) u' - \dfrac{2\nu+1}{2z} ku = 0$ $\quad u = z^{-\nu} e^{\frac{1}{2}kx}\, Z_\nu\left(\dfrac{ikz}{2}\right)$ JA

5. $u'' + \dfrac{1-\nu}{z}u' - \dfrac{1}{4}\dfrac{u}{z} = 0$ $\quad u = z^{\frac{\nu}{2}}\, Z_\nu\left(i\sqrt{z}\right)$ WA 111(8)

6. $u'' \pm \dfrac{u}{\sqrt{z}} = 0$

$u = \sqrt{z}\, Z_{\frac{2}{3}}\left(\frac{4}{3}z^{\frac{3}{4}}\right), \qquad u = \sqrt{z}\, Z_{\frac{2}{3}}\left(\frac{4}{3}iz^{\frac{3}{4}}\right)$ WA 111(10)

7. $u'' \pm zu = 0$

$u = \sqrt{z}\, Z_{\frac{1}{3}}\left(\frac{2}{3}z^{\frac{3}{2}}\right), \qquad u = \sqrt{z}\, Z_{\frac{1}{3}}\left(\frac{2}{3}iz^{\frac{3}{2}}\right)$ WA 111(10)

8. $u'' - \left(c^2 + \dfrac{\nu(\nu+1)}{z^2}\right) u = 0$ $\quad u = \sqrt{z}\, Z_{\nu+\frac{1}{2}}(icz)$ WA 108(1)

9. $u'' - \dfrac{2\nu}{z}u' - c^2 u = 0$ $\quad u = z^{\nu+\frac{1}{2}}\, Z_{\nu+\frac{1}{2}}(icz)$ WA 109(3, 4)

10. $u'' - c^2 z^{2\nu-2} u = 0$ $\quad u = \sqrt{z}\, Z_{\frac{1}{2\nu}}\left(i\dfrac{c}{\nu}z^\nu\right)$ WA 109(5, 6)

8.495

1. $u'' + \dfrac{1}{z}u' + \left(i - \dfrac{\nu^2}{z^2}\right) u = 0$ $\quad u = Z_\nu\left(z\sqrt{i}\right)$ JA

2. $u'' + \left(\frac{1}{z} \mp 2i\right) u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right) u = 0$ $\qquad u = e^{\pm iz}\, Z_\nu(z)$ JA

3. $u'' + \frac{1}{z} u' + s e^{i\alpha} u = 0$ $\qquad u = Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right)$ JA

4. $u'' + \left(s e^{i\alpha} + \frac{1}{4z^2}\right) u = 0$ $\qquad u = \sqrt{z}\, Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right)$ JA

8.496

1. $\frac{d^2}{dz^2}\left(z^4 \frac{d^2 u}{dz^2}\right) - z^2 u = 0$ $\qquad u = \frac{1}{z}\left\{Z_2\left(2\sqrt{z}\right) + \overline{Z_2\left(2i\sqrt{z}\right)}\right\}$ WA 122(7)

2. $\frac{d^2}{dz^2}\left(z^{\frac{16}{5}} \frac{d^2 u}{dz^2}\right) - z^{\frac{8}{5}} u = 0$ $\qquad u = z^{-7/10}\left\{Z_{\frac{5}{6}}\left(\frac{5}{3} z^{\frac{3}{5}}\right) + \overline{Z_{\frac{5}{6}}\left(\frac{5}{3} i z^{\frac{3}{5}}\right)}\right\}$ WA 122(8)

3. $\frac{d^2}{dz^2}\left(z^{12} \frac{d^2 u}{dz^2}\right) - z^6 u = 0$

$$u = z^{-4}\left\{Z_{10}\left(2z^{-1/2}\right) + \overline{Z_{10}\left(2iz^{-1/2}\right)}\right\} \quad \text{WA 122(9)}$$

4. $\frac{d^4 u}{dz^4} + \frac{2}{z}\frac{d^3 u}{dz^3} - \frac{2\nu^2+1}{z^2}\frac{d^2 u}{dz^2} + \frac{2\nu^2+1}{z^3}\frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1\right) u = 0,$

$u = A_1\, J_\nu(z) + A_2\, Y_\nu(z) + A_3\, I_\nu(z) + A_4\, K_\nu(z),$ where $A_1,\, A_2,\, A_3,\, A_4$ are constants MO 29

8.51–8.52 Series of Bessel functions

8.511 Generating functions for Bessel functions:

1. $\exp\frac{1}{2}\left(t - \frac{1}{t}\right) z = J_0(z) + \sum_{k=1}^{\infty}\left[t^k + (-t)^{-k}\right] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k$

$[|z| < |t|]$ KU 119(12)

2. $\exp\left(t - \frac{1}{t}\right) z = \left\{\sum_{k=-\infty}^{\infty} t^k\, J_k(z)\right\}\left\{\sum_{m=-\infty}^{\infty} t^m\, J_m(z)\right\}$ WA 40

3. $\exp\left(\pm iz \sin\varphi\right) = J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos 2k\varphi \pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z)\sin(2k+1)\varphi$ KU 120(13)

4. $\exp\left(iz\cos\varphi\right) = \sqrt{\frac{\pi}{2z}}\sum_{k=0}^{\infty}(2k+1) i^k\, J_{k+\frac{1}{2}}(z)\, P_k\left(\cos\varphi\right)$ WA 401(1)

$= \sum_{k=-\infty}^{\infty} i^k\, J_k(z) e^{ik\varphi}$ MO 27

$= J_0(z) + 2\sum_{k=1}^{\infty} i^k\, J_k(z)\cos k\varphi$ MO 27

5. $$\sqrt{\frac{i}{\pi}}e^{iz\cos 2\varphi}\int_{-\infty}^{\sqrt{2z}\cos\varphi} e^{-it^2}\,dt = \frac{1}{2}\,J_0(z) + \sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i}\,J_{\frac{k}{2}}(z)\cos k\varphi$$ MO 28

The series $\sum J_k(z)$

8.512

1. $$J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) = 1$$ WA 44

2. $$\sum_{k=0}^{\infty}\frac{(n+2k)(n+k-1)!}{k!}\,J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \qquad [n=1,2,\ldots]$$ WA 45

3. $$\sum_{k=0}^{\infty}\frac{(4k+1)(2k-1)!!}{2^k k!}\,J_{2k+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}}$$

8.513

Notation: In formulas **8.513** $$Q_k^{(p)} = \sum_{m=0}^{\lfloor\frac{k-1}{2}\rfloor}\frac{(-1)^m\binom{k}{m}(k-2m)^p}{2^k k!}$$

1. $$\sum_{k=1}^{\infty}(2k)^{2p}\,J_{2k}(z) = \sum_{k=0}^{p} Q_{2k}^{(2p)} z^{2k} \qquad [p=1,2,3,\ldots]$$ WA 46(1)

2. $$\sum_{k=0}^{\infty}(2k+1)^{2p+1}\,J_{2k+1}(z) = \sum_{k=0}^{p} Q_{2k+1}^{(2p+1)} z^{2k+1} \qquad [p=0,1,2,3,\ldots]$$ WA 46(2)

In particular:

3. $$\sum_{k=0}^{\infty}(2k+1)^3\,J_{2k+1}(z) = \frac{1}{2}\left(z+z^3\right)$$ WA 47(4)

4. $$\sum_{k=1}^{\infty}(2k)^2\,J_{2k}(z) = \frac{1}{2}z^2$$ WA 47(4)

5. $$\sum_{k=1}^{\infty}2k(2k+1)(2k+2)\,J_{2k+1}(z) = \frac{1}{2}z^3$$ WA 47(4)

8.514

1. $$\sum_{k=0}^{\infty}(-1)^k\,J_{2k+1}(z) = \frac{\sin z}{2}$$ WH

2. $$J_0(z) + 2\sum_{k=1}^{\infty}(-1)^k\,J_{2k}(z) = \cos z$$ WH

3. $$\sum_{k=1}^{\infty}(-1)^{k+1}(2k)^2\,J_{2k}(z) = \frac{z\sin z}{2}$$ WA 32(9)

4. $$\sum_{k=0}^{\infty}(-1)^k(2k+1)^2\,J_{2k+1}(z)=\frac{z\cos z}{2}$$ WA 32(10)

5. $$J_0(z)+2\sum_{k=1}^{\infty}J_{2k}(z)\cos 2k\theta=\cos\left(z\sin\theta\right)$$ KU 120(14), WA 32

6. $$\sum_{k=0}^{\infty}J_{2k+1}(z)\sin(2k+1)\theta=\frac{\sin\left(z\sin\theta\right)}{2}$$ KU 120(15), WA 32

7. $$\sum_{k=0}^{\infty}J_{2k+1}(x)=\frac{1}{2}\int_0^x J_0(t)\,dt \qquad [x \text{ is real}]$$ WA 638

8.515

1. $$\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\left(\frac{2z+t}{2z}\right)^k J_{\nu+k}(z)=\left(\frac{z}{z+t}\right)^{\nu}J_{\nu}(z+t)$$ AD (9140)

2. $$\sum_{k=1}^{\infty}J_{2k-\frac{1}{2}}\left(x^2\right)=S(x)$$ MO 127a

3. $$\sum_{k=0}^{\infty}J_{2k+\frac{1}{2}}\left(x^2\right)=C(x)$$ MO 127a

8.516 $$\sum_{k=0}^{\infty}\frac{(2n+2k)(2n+k-1)!}{k!}\,J_{2n+2k}\left(2z\sin\theta\right)=\left(z\sin\theta\right)^{2n}$$ WA 47

The series $\sum A_k\,J_k(kx)$ and $\sum A_k\,J'_k(kx)$

8.517

1. $$\sum_{k=1}^{\infty}J_k(kz)=\frac{z}{2(1-z)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right|<1\right]$$ WA 615(1)

2. $$\sum_{k=1}^{\infty}(-1)^k\,J_k(kz)=-\frac{z}{2(1+z)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right|<1\right]$$ WA 622(1)

3. $$\sum_{k=1}^{\infty}J_{2k}(2kz)=\frac{z^2}{2\left(1-z^2\right)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right|<1\right]$$ MO 58

8.518

1.[11] $$\sum_{k=1}^{\infty}\frac{J'_k(kx)}{k}=\frac{1}{2}+\frac{x}{4} \qquad [0\le x<1]$$ MO 58

2.[11] $$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{J'_k(kx)}{k}=\frac{1}{2}-\frac{x}{4} \qquad [0\le x<1]$$ MO 58

3. $$\sum_{k=1}^{\infty}k\,J'_k(kx)=\frac{1}{2(1-x)^2} \qquad [0\le x<1]$$ MO 58

4. $$\sum_{k=1}^{\infty}(-1)^{k-1}\, J_k'(kx)k = \frac{1}{2(1+x)^2} \qquad [0 \le x < 1]$$ MO 58

The series $\sum A_k\, J_0(kx)$

8.519 If, on the interval $[0 \le x \le \pi]$, a function $f(x)$ possesses a continuous derivative with respect to x that is of bounded variation, then

1. $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, J_0(kx) \qquad [0 < x < \pi]$$

where

2. $$a_0 = 2f(0) + \frac{2}{\pi}\int_0^{\pi} du \int_0^{\pi/2} uf'\,(u\sin\varphi)\; d\varphi$$

3. $$a_n = \frac{2}{\pi}\int_0^{\pi} du \int_0^{\pi/2} uf'\,(u\sin\varphi)\cos nu\, d\varphi$$ WH

8.521 Examples:

1. $$\sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2\sum_{m=1}^{n}\frac{1}{\sqrt{x^2-4m^2\pi^2}} \qquad [2n\pi < x < 2(n+1)\pi]$$ MO 59

2. $$\sum_{k=1}^{\infty}(-1)^{k+1}\, J_0(kx) = \frac{1}{2} \qquad [0 < x < \pi]$$ KU 124(12)

3. $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\, J_0\left\{(2k-1)x\right\} \quad \frac{\pi^2}{8} - \frac{|x|}{2} \qquad [-\pi < x < \pi]$$ KU 124

$$= \frac{\pi^2}{8} + \sqrt{x^2-\pi^2} - \frac{x}{2} - \pi\arccos\frac{\pi}{x} \qquad [\pi < x < 2\pi]$$ MO 59

4. $$\sum_{k=1}^{\infty} e^{-kz}\, J_0\left(k\sqrt{x^2+y^2}\right)$$

$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty}\left\{\frac{1}{\sqrt{(2ki\pi+z)^2+x^2+y^2}} - \frac{1}{\sqrt{(2ki\pi-z)^2+x^2+y^2}}\right\}$$

$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{(2k)!}B_{2k}r^{2k-1}\, P_{2k-1}\left(\frac{z}{r}\right) \qquad [0 < r < 2\pi]$$ MO 59

where $r = \sqrt{x^2+y^2+z^2}$ and where the radical indicates the square root with a positive real part. In formula **8.521** 4, the first equation holds when x and y are real and $\operatorname{Re} z > 0$; the second equation holds when x, y, and z are all real.

The series $\sum A_k\, Z_0(kx)\sin kx$ and $\sum A_k\, Z_0(kx)\cos kx$

8.522

1. $$\sum_{k=1}^{\infty} J_0(kx)\cos kxt = -\frac{1}{2} + \sum_{l=1}^{m} \frac{1}{\sqrt{x^2-(2\pi l+tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^{n} \frac{1}{\sqrt{x^2-(2\pi l-tx)^2}}$$

MO 59

2. $$\sum_{k=1}^{\infty} J_0(kx)\sin kxt = \frac{1}{2\pi}\left\{\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{l}\right\} + \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l+tx)^2-x^2}} - \frac{1}{2\pi l}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l-tx)^2-x^2}} - \frac{1}{2\pi l}\right\}$$

MO 59

3. $$\sum_{k=1}^{\infty} Y_0(kx)\cos kxt = -\frac{1}{\pi}\left(C + \ln\frac{x}{4\pi}\right) + \frac{1}{2\pi}\left\{\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{n}\frac{1}{l}\right\} - \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l+tx)^2-x^2}} - \frac{1}{2\pi l}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2\pi l-tx)^2-x^2}} - \frac{1}{2\pi l}\right\}$$

MO 60

In formulas **8.522**, $x>0, 0\le t<1, 2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.523

1. $$\sum_{k=1}^{\infty}(-1)^k J_0(kx)\cos kxt = -\frac{1}{2} + \sum_{l=1}^{m}\frac{1}{\sqrt{x^2-[(2l-1)\pi+tx]^2}} + \sum_{l=1}^{n}\frac{1}{\sqrt{x^2-[(2l-1)\pi-tx]^2}}$$

MO 60

2. $$\sum_{k=1}^{\infty}(-1)^k J_0(kx)\sin kxt \quad \frac{1}{2\pi}\left\{\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{l}\right\} + \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi+tx]^2-x^2}} - \frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}} - \frac{1}{2l\pi}\right\}$$

MO 60

3. $$\sum_{k=1}^{\infty}(-1)^k\, Y_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)+\frac{1}{2\pi}\left\{\sum_{l=1}^{m}\frac{1}{l}+\sum_{l=1}^{n}\frac{1}{l}\right\} - \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi+tx]^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}}-\frac{1}{2l\pi}\right\}$$

MO 60

In formulas **8.523**, $x>0, 0\le t<1$, $(2m-1)\pi<x(1-t)<(2m+1)\pi$, $(2n-1)\pi<x(1+t)<(2n+1)\pi$, m and n are natural numbers.

8.524

1. $$\sum_{k=1}^{\infty} J_0(kx)\cos kxt = -\frac{1}{2}+\sum_{l=m+1}^{n}\frac{1}{\sqrt{x^2-(2l\pi-tx)^2}}$$ MO 60

2. $$\sum_{k=1}^{\infty} J_0(kx)\sin kxt \sum_{l=0}^{m}\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}+\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi+tx)^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}-\frac{1}{2l\pi}\right\}+\frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l}$$

MO 60

3.[6] $$\sum_{k=1}^{\infty} Y_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)-\sum_{l=0}^{m}\frac{1}{\sqrt{(2\pi l-tx)^2-x^2}}+\frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi+tx)^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}-\frac{1}{2l\pi}\right\}$$

MO 61

In formulas **8.524**, $x>0, t>1, 2m\pi<x(t-1)<2(m+1)\pi$, $2n\pi<x(t+1)<2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.525

1. $$\sum_{k=1}^{\infty}(-1)^k\, J_0(kx)\cos kxt = -\frac{1}{2}+\sum_{l=m+1}^{n}\frac{1}{\sqrt{x^2-[(2l-1)\pi-tx]^2}}$$ MO 61

2. $$\sum_{k=1}^{\infty}(-1)^k\, J_0(kx)\sin kxt = \sum_{l=1}^{m}\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} + \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

3. $$\sum_{k=1}^{\infty}(-1)^k\, Y_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

In formulas **8.525**, $x > 0, t > 1$, $(2m-1)\pi < x(t-1) < (2m+1)\pi$, $(2n-1)\pi < x(t+1) < (2n+1)\pi$, m and n are natural numbers.

8.526

1. $$\sum_{k=1}^{\infty} K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi}\right\} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

2. $$\sum_{k=1}^{\infty}(-1)^k\, K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi}\right\} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi}\right\}$$

$[x > 0, \quad t \text{ real}]$ (see also **8.66**) MO 62

8.53 Expansion in products of Bessel functions

"Summation theorems"

8.530 Suppose that $r > 0, \varrho > 0, \varphi > 0$, and $R = \sqrt{r^2 + \varrho^2 - 2r\varrho\cos\varphi}$; that is, suppose that r, ϱ, and R are the sides of a triangle such that the angle between the sides r and ϱ is equal to φ. Suppose also that $\varrho <$r and that ψ is the angle opposite the side ϱ, so that

1. $$0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}$$

When these conditions are satisfied, we have the "summation theorem" for Bessel functions:

1. $$e^{i\nu\psi}\, Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho)\, Z_{\nu+k}(mr) e^{ik\varphi} \qquad [m \text{ is an arbitrary complex number}]$$

WA 394(6)

For $Z_\nu = J_\nu$ and ν an integer, the restriction $\varrho <$r is superfluous. MO 31

8.531 Special cases:

1. $$J_0(mR) = J_0(m\varrho)\, J_0(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)\, J_k(mr)\cos k\varphi$$ WA 391(1)

2. $$H_0^{(1,2)}(mR) = J_0(m\varrho)\, H_0^{(1,2)}(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)\, H_k^{(1,2)}(mr)\cos k\varphi$$ MO 31

3. $$J_0\left(z\sin\alpha\right) = J_0^2\left(\frac{z}{2}\right) + 2\sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right)\cos 2k\alpha$$
$$= \sqrt{\frac{2\pi}{z}}\sum_{k=0}^{\infty}\left(2k + \frac{1}{2}\right)\frac{(2k-1)!!}{2^k k!}\, J_{2k+\frac{1}{2}}(z)\, P_{2k}\left(\cos\alpha\right)$$ MO 31

8.532 The term "summation theorem" is also applied to the formula

1. $$\frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu}\,\Gamma(\nu)\sum_{k=0}^{\infty}(\nu + k)\frac{J_{\nu+k}(m\varrho)}{\varrho^\nu}\frac{Z_{\nu+k}(mr)}{r^\nu}\, C_k^\nu\left(\cos\varphi\right)$$

[$\nu \neq -1, -2, -3, \ldots$; the conditions on r, ϱ, R, φ, and m are the same as in formula **8.530**; for $Z_\nu = J_\nu$ and ν an integer, formula **8.532** 1 is valid for arbitrary r, ϱ, and φ]. WA 398(4)

8.533 Special cases:

1. $$\frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}}\sum_{k=0}^{\infty}(2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, H_{k+\frac{1}{2}}^{(1)}(mr)\, P_k\left(\cos\varphi\right)$$ MO 31

2. $$\frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}}\sum_{k=0}^{\infty}(2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, H_{k+\frac{1}{2}}^{(2)}(mr)\, P_k\left(\cos\varphi\right)$$ MO 31

8.534 A degenerate addition theorem ($r \to \infty$):

$$e^{im\varrho\cos\varphi} = \sqrt{\frac{\pi}{2m\varrho}} \sum_{k=0}^{\infty} i^k (2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, P_k(\cos\varphi) \qquad \text{WA 401(1)}$$

$$= 2^\nu\, \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) i^k (m\varrho)^{-\nu}\, J_{\nu+k}(m\varrho)\, C_k^\nu(\cos\varphi) \qquad [\nu \neq 0, -1, -2, \ldots] \qquad \text{WA 401(2)}$$

8.535 The term "product theorem" is also applied to the formula

$$Z_\nu(\lambda z) = \lambda^\nu \sum_{k=0}^{\infty} \frac{1}{k!}\, Z_{\nu+k}(z) \left(\frac{1-\lambda^2}{2} z\right)^k \qquad \left[|1-\lambda|^2 < 1\right]$$

For $Z_\nu = J_\nu$, it is valid for all values of λ and z. MO 32

8.536

1. $$\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0] \qquad \text{WA 47(1)}$$

2. $$2\sum_{k=n}^{\infty} \frac{k\,\Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0] \qquad \text{WA 47(2)}$$

3. $$J_0^2(z) + 2\sum_{k=1}^{\infty} J_k^2(z) = 1 \qquad \text{WA 41(3)}$$

8.537

1. $$\sum_{k=-\infty}^{\infty} Z_{\nu-k}(t)\, J_k(z) = Z_\nu(z+t) \qquad [|z| < |t|] \qquad \text{WA 158(2)}$$

2. $$\sum_{k=-\infty}^{\infty} J_k(z)\, J_{n-k}(z) = J_n(2z) \qquad \text{WA 41}$$

8.538

1. $$\sum_{k=-\infty}^{\infty} (-1)^k\, J_{-\nu+k}(t)\, J_k(z) = J_{-\nu}(z+t) \qquad [|z| < |t|] \qquad \text{WA 159}$$

2. $$\sum_{k=-\infty}^{\infty} Z_{\nu+k}(t)\, J_k(z) = Z_\nu(t-z) \qquad [|z| < |t|] \qquad \text{WA 159(5)}$$

8.54 The zeros of Bessel functions

8.541 For arbitrary real ν, the function $J_\nu(z)$ has infinitely many real zeros. For $\nu > -1$, all its zeros are real. WA 526, 530

A Bessel function $Z_\nu(z)$ has no multiple zeros except possibly the coordinate origin. WA 528

8.542 All zeros of the function $Y_0(z)$ with positive real parts are real. WA 531

8.543 If $-(2s+2) < \nu < -(2s+1)$, where s is a natural number or 0, then $J_\nu(z)$ has exactly $4s+2$ complex roots, two of which are purely imaginary. If $-(2s+1) < \nu < -2s$, where s is a natural number, then the function $J_\nu(z)$ has exactly $4s$ complex zeros, none of which are purely imaginary. WA 532

8.544 If x_ν and x'_ν are, respectively, the smallest positive zeros of the functions $J_\nu(z)$ and $J'_\nu(z)$ for $\nu > 0$, then $x_\nu > \nu$ and $x'_\nu > \nu$. Suppose also that y_ν is the smallest positive zero of the function $Y_\nu(z)$. Then, $x_\nu < y_\nu < x'_\nu$. WA 534, 536

Suppose that $z_{\nu,m}$ (for $m = 1, 2, 3, \dots$) are the zeros of the function $z^{-\nu} J_\nu(z)$, numbered in order of the absolute value of their real parts. Here, we assume that $\nu \neq -1, -2, -3, \dots$. Then, for arbitrary z

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right). \qquad \text{WA 550}$$

8.545[8] The number of zeros of the function $z^{-\nu} J_\nu(z)$ that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + \tfrac{1}{2}\operatorname{Re}\nu + \tfrac{1}{4}\right)\pi, \qquad \text{WA 497}$$

is exactly m.

8.546 For $\nu \geq 0$, the number of zeros of the function $K_\nu(z)$ that occur in the region $\operatorname{Re} z < 0$, $|\arg z| < \pi$ is equal to the even number closest to $\nu - \frac{1}{2}$. WA 562

8.547 Large zeros of the functions $J_\nu(z)\cos\alpha - Y_\nu(z)\sin\alpha$, where ν and α are real numbers, are given by the asymptotic expansion

$$x_{\nu,m} \sim \left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]} - \frac{\left(4\nu^2 - 1\right)\left(28\nu^2 - 31\right)}{384\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]^3} - \dots \qquad \text{KU 109(24), WA 558}$$

8.548 In particular, large zeros of the function $J_0(z)$ are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m-1) + \frac{1}{2\pi(4m-1)} - \frac{31}{6\pi^3(4m-1)^3} + \frac{3779}{15\pi^5(4m-1)^5} - \dots \qquad \text{KU 109(25), WA 556}$$

This series is suitable for calculating all (except the smallest x_{01}) zeros of the function $J_0(z)$ correctly to at least five digits.

8.549 To calculate the roots $x_{\nu,m}$ of the function $J_\nu(z)$ of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}. \qquad \text{KU 112(27)a, WA 554}$$

8.55 Struve functions

8.550 Definitions:

1. $$\mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)} \qquad \text{WA 358(2)}$$

2. $$\mathbf{L}_\nu(z) = -ie^{-i\nu\frac{\pi}{2}}\,\mathbf{H}_\nu\left(ze^{i\frac{\pi}{2}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)} \qquad \text{WA 360(11)}$$

8.551 Integral representations:

1. $$\mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 \left(1 - t^2\right)^{\nu - \frac{1}{2}} \sin zt\, dt = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sin\left(z\cos\varphi\right)(\sin\varphi)^{2\nu}\, d\varphi$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right] \qquad \text{WA 358(1)}$$

2. $$\mathbf{L}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\pi/2} \sinh\left(z\cos\varphi\right)\left(\sin\varphi\right)^{2\nu}\,d\varphi$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 360(11)

8.552 Special cases:

1.[6] $$\mathbf{H}_n(z) = \frac{1}{\pi}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma\left(n+\frac{1}{2}-m\right)} - \mathbf{E}_n(z)$$
$[n=1,2,\ldots]$ EH II 40(66), WA 337(1)

2.[6] $$\mathbf{H}_{-n}(z) = (-1)^{n+1}\frac{1}{\pi}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{\Gamma\left(n-m-\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)} - \mathbf{E}_{-n}(z)$$
$[n=1,2,\ldots]$ EH II 40(67), WA 337(2)

3. $$\mathbf{H}_{n+\frac{1}{2}}(z) = Y_{n+\frac{1}{2}}(z) + \frac{1}{\pi}\sum_{m=0}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)}$$
$[n=0,1,\ldots]$ EH II 39(64)

4. $\mathbf{H}_{-\left(n+\frac{1}{2}\right)}(z) = (-1)^n\,J_{n+\frac{1}{2}}(z)$ $[n=0,1,\ldots]$ EH II 39(65)

5. $\mathbf{L}_{-\left(n+\frac{1}{2}\right)}(z) = I_{n+\frac{1}{2}}(z)$ $[n=0,1,\ldots]$ EH II 39(65)

6. $\mathbf{H}_{\frac{1}{2}}(z) = \dfrac{\sqrt{2}}{\sqrt{\pi z}}\left(1-\cos z\right)$ EH II 39, WA 364(3)

7. $\mathbf{H}_{\frac{3}{2}}(z) = \left(\dfrac{z}{2\pi}\right)^{1/2}\left(1+\dfrac{2}{z^2}\right) - \left(\dfrac{2}{\pi z}\right)^{1/2}\left(\sin z + \dfrac{\cos z}{z}\right)$ WA 364(3)

8.553 Functional relations:

1. $\mathbf{H}_\nu\left(ze^{im\pi}\right) = e^{i\pi(\nu+1)m}\,\mathbf{H}_\nu(z)$ $[m=1,2,3,\ldots]$ WA 362(5)

2. $\dfrac{d}{dz}\left[z^\nu\,\mathbf{H}_\nu(z)\right] = z^\nu\,\mathbf{H}_{\nu-1}(z)$ WA 358

3. $\dfrac{d}{dz}\left[z^{-\nu}\,\mathbf{H}_\nu(z)\right] = 2^{-\nu}\pi^{-1/2}\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1} - z^{-\nu}\,\mathbf{H}_{\nu+1}(z)$ WA 359

4. $\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{H}_\nu(z) + \pi^{-1/2}\left(\dfrac{z}{2}\right)^\nu\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}$ WA 359(5)

5. $\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\,\mathbf{H}'_\nu(z) - \pi^{-1/2}\left(\dfrac{z}{2}\right)^\nu\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}$ WA 359(6)

8.554 Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = Y_\nu(\xi) + \frac{1}{\pi}\sum_{m=0}^{p-1}\frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu+\frac{1}{2}-m\right)} + O\left(|\xi|^{\nu-2p-1}\right)$$
$[|\arg\xi| < \pi]$ EH II 39(63), WA 363(2)

For the asymptotic representation of $Y_\nu(\xi)$, see **8.451** 2.

8.555 The differential equation for Struve functions:

$$z^2y'' + zy' + \left(z^2 - \nu^2\right) y = \frac{1}{\sqrt{\pi}} \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\Gamma\left(\nu + \frac{1}{2}\right)} \qquad \text{WA 359(10)}$$

8.56 Thomson functions and their generalizations

$\mathrm{ber}_\nu(z)$, $\mathrm{bei}_\nu(z)$, $\mathrm{her}_\nu(z)$, $\mathrm{hei}_\nu(z)$, $\mathrm{ker}_\nu(z)$, $\mathrm{kei}_\nu(z)$

8.561

1. $\mathrm{ber}_\nu(z) + i\,\mathrm{bei}_\nu(z) = J_\nu\left(ze^{\frac{3}{4}\pi i}\right)$ WA 96(6)

2. $\mathrm{ber}_\nu(z) - i\,\mathrm{bei}_\nu(z) = J_\nu\left(ze^{-\frac{3}{4}\pi i}\right).$ WA 96(6)

8.562

1. $\mathrm{her}_\nu(z) + i\,\mathrm{hei}_\nu(z) = H^\nu_{(1)}\left(ze^{\frac{3}{4}\pi i}\right)$ (see also **8.567**) WA 96(7)

2. $\mathrm{her}_\nu(z) - i\,\mathrm{hei}_\nu(z) = H^\nu_{(1)}\left(ze^{-\frac{3}{4}\pi i}\right)$ (see also **8.567**) WA 96(7)

8.563

1. $\mathrm{ber}_0(z) \equiv \mathrm{ber}(z); \quad \mathrm{bei}_0(z) \equiv \mathrm{bei}(z)$ WA 96(8)

2. $\mathrm{ker}(z) \equiv -\frac{\pi}{2}\,\mathrm{hei}_0(z); \quad \mathrm{kei}(z) \equiv \frac{\pi}{2}\,\mathrm{hei}_0(z)$ WA 96(8)

For integral representations, see **6.251**, **6.536**, **6.537**, **6.772** 4, **6.777**.

Series representation

8.564

1. $$\mathrm{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k}\left[(2k)!\right]^2} \qquad \text{WA 96(3)}$$

2. $$\mathrm{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2}\left[(2k+1)!\right]^2} \qquad \text{WA 96(4)}$$

3. $$\mathrm{ker}(z) = \left(\ln\frac{2}{z} - \boldsymbol{C}\right)\mathrm{ber}(z) + \frac{\pi}{4}\,\mathrm{bei}(z) + \sum_{k=1}^{\infty}(-1)^k \frac{z^{4k}}{2^{4k}\left[(2k)!\right]^2} \sum_{m=1}^{2k} \frac{1}{m} \qquad \text{WA 96(9)a, DW}$$

4. $$\mathrm{kei}(z) = \left(\ln\frac{2}{z} - \boldsymbol{C}\right)\mathrm{bei}(z) - \frac{\pi}{4}\,\mathrm{ber}(z) + \sum_{k=0}^{\infty}(-1)^k \frac{z^{4k+2}}{2^{4k+2}\left[(2k+1)!\right]^2} \sum_{m=1}^{2k+1} \frac{1}{m} \qquad \text{WA 96(10)a, DW}$$

8.565 $$\mathrm{ber}_\nu^2(z) + \mathrm{bei}_\nu^2(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2\nu+4k}}{k!\,\Gamma(\nu+k+1)\,\Gamma(\nu+2k+1)} \qquad \text{WA 163(6)}$$

Asymptotic representation

8.566

1. $\operatorname{ber}(z) = \dfrac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos\beta(z)$ $\left[|\arg z| < \dfrac{\pi}{4}\right]$ WA 227(1)

2. $\operatorname{bei}(z) = \dfrac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin\beta(z)$ $\left[|\arg z| < \dfrac{\pi}{4}\right]$ WA 227(1)

3. $\ker(z) = \sqrt{\dfrac{\pi}{2z}} e^{\alpha(-z)} \cos\beta(-z)$ $\left[|\arg z| < \dfrac{5}{4}\pi\right]$ WA 227(2)

4. $\operatorname{kei}(z) = \sqrt{\dfrac{\pi}{2z}} e^{\alpha(-z)} \sin\beta(-z)$ $\left[|\arg z| < \dfrac{5}{4}\pi\right]$, WA 227(2)

where

$$\alpha(z) \sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots,$$
$$\beta(z) \sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots$$

8.567 Functional relations

1. $\ker(z) + i\operatorname{kei}(z) = K_0\left(z\sqrt{i}\right)$ (see **8.562**) WA 96(5), DW

2. $\ker(z) - i\operatorname{kei}(z) = K_0\left(z\sqrt{-i}\right)$ (see **8.562**) WA 96(5), DW

For integrals of Thomson's functions, see **6.87**.

8.57 Lommel functions

8.570 Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$:

1. $$s_{\mu,\nu}(z) = \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2-\nu^2]\,[(\mu+3)^2-\nu^2]\dots\left[(\mu+2m+1)^2-\nu^2\right]}$$
$$= z^{\mu-1}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+2}\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+m+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+m+\frac{3}{2}\right)}$$
$[\mu \pm \nu$ is not a negative odd integer$]$ EH II 40(69), WA 377(2)

2.[11] $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}\right)$$
$$\times\frac{\cos\left[\frac{1}{2}(\mu-\nu)\pi\right]J_{-\nu}(z)-\cos\left[\frac{1}{2}(\mu+\nu)\pi\right]J_\nu(z)}{\sin\nu\pi}$$ EH II 40(71), WA 379(2)
$$= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}\right)$$
$$\times\left\{\sin\left[\tfrac{1}{2}(\mu-\nu)\pi\right]J_\nu(z)-\cos\left[\tfrac{1}{2}(\mu-\nu)\pi\right]Y_\nu(z)\right\}$$ EH II 41(71), WA 379(3)

Integral representations

8.571 $$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z z^\mu\, J_\nu(z)\,dz - J_\nu(z)\int_0^z z^\mu\, Y_\nu(z)\,dz\right]$$ WA 378(9)

8.572 $$s_{\mu,\nu}(z) = 2^\mu\left(\frac{z}{2}\right)^{\frac{1}{2}(1+\nu+\mu)}\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu\right)\int_0^{\pi/2} J_{\frac{1}{2}(1+\mu-\nu)}\left(z\sin\theta\right)(\sin\theta)^{\frac{1}{2}(1+\nu-\mu)}(\cos\theta)^{\nu+\mu}\,d\theta$$
$[\mathrm{Re}(\nu+\mu+1)>0]$ EH II 42(86)

8.573 Special cases:

1. $S_{1,2n}(z) = zO_{2n}(z)$ WA 382(1)
2. $S_{0,2n+1}(z) = \dfrac{z}{2n+1}\,O_{2n+1}(z)$ WA 382(1)
3. $S_{-1,2n}(z) = \dfrac{1}{4n}\,S_{2n}(z)$ WA 382(2)
4. $S_{0,2n+1}(z) = \dfrac{1}{2}\,S_{2n+1}(z)$ WA 382(2)
5. $S_{\nu,\nu}(z) = \Gamma\left(\nu+\dfrac{1}{2}\right)\sqrt{\pi}2^{\nu-1}\,\mathbf{H}_\nu(z)$ EH II 42(84)
6. $S_{\nu,\nu}(z) = [\mathbf{H}_\nu(z) - Y_\nu(z)]\,2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)$ EH II 42(84)

8.574 Connections with other special functions:

1. $\mathbf{J}_\nu(z) = \dfrac{1}{\pi}\sin(\nu\pi)\,[s_{0,\nu}(z) - \nu\, s_{-1,\nu}(z)]$ EH II 41(82)
2. $\mathbf{E}_\nu(z) = -\dfrac{1}{\pi}\,[(1+\cos\nu\pi)\,s_{0,\nu}(z) + \nu\,(1-\cos\nu\pi)\,s_{-1,\nu}(z)]$ EH II 42(83)

A connection with a hypergeometric function

3. $$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\;{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{z^2}{4}\right)$$
EH II 40(69), WA 378(10)

8.575 Functional relations:

1. $s_{\mu+2,\nu}(z) = z^{\mu+1} - \left[(\mu+1)^2-\nu^2\right]s_{\mu,\nu}(z)$ EH II 41(73), WA 380(1)
2.[8] $s'_{\mu,\nu}(z) + \left(\dfrac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z)$ EH II 41(74), WA 380(2)
3. $s'_{\mu,\nu}(z) - \left(\dfrac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(75), WA 380(3)
4. $\left(2\dfrac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(76), WA 380(4)
5.[8] $2\,s'_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(77), WA 380(5)

In formulas **8.575** 1–5, $s_{\mu,\nu}(z)$ can be replaced with $S_{\mu,\nu}(z)$.

8.576 Asymptotic expansion of $S_{\mu,\nu}(z)$.

In the case in which $\mu\pm\nu$ is not a positive odd integer, $S_{\mu,\nu}(z)$ has the following asymptotic expansion:

$$S_{\mu,\nu}(z) \sim z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1-\mu+\nu}{2}\right)_m \left(\frac{1-\mu-\nu}{2}\right)_m \left(\frac{z}{2}\right)^{-2m}$$

$$[|z| \to \infty, \quad |\arg z| < \pi]$$ WA 347, 352

The series terminates and is equal to $S_{\mu,\nu}(z)$ when $\mu \pm \nu$ is a positive odd integer.

8.577 Lommel functions satisfy the following differential equation:

$$z^2 w'' + z w' + \left(z^2 - \nu^2\right) w = z^{\mu+1}$$ WA 377(1), EH II 40(68)

8.578 Lommel functions of two variables $U_\nu(w,z)$ and $V_\nu(w,z)$:

Definition

1. $$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z)$$ EH II 42(87), WA 591(5)

2. $$V_\nu(w,z) = \cos\left[\frac{1}{2}\left(w + \frac{z^2}{w} + \nu\pi\right)\right] + U_{-\nu+2}(w,z)$$ EH II 42(88), WA 591(6)

Particular values:

3. $$U_0(z,z) = V_0(z,z) = \tfrac{1}{2}\left\{J_0(z) + \cos z\right\}$$ WA 591(9)

4. $$U_1(z,z) = -V_1(z,z) = \tfrac{1}{2}\sin z$$ WA 591(10)

5. $$U_{2n}(z,z) = \frac{(-1)^n}{2}\left\{\cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z)\right\}$$

$$[n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$$ WA 591(11)

6. $$U_{2n+1}(z,z) = \frac{(-1)^n}{2}\left\{\sin z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m+1} J_{2m+1}(z)\right\}$$

$$[n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$$ WA 591(12)

7. $$V_n(w,z) = (-1)^n U_n\left(\frac{z^2}{w}, z\right)$$

8. $$U_\nu(w,0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right)$$ WA 593(9)

9. $$V_{-\nu+2}(w,0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right)$$ WA 593(10)

8.579 Functional relations:

1. $$2\frac{\partial}{\partial w} U_\nu(w,z) = U_{\nu-1}(w,z) + \left(\frac{z}{w}\right)^2 U_{\nu+1}(w,z)$$ WA 593(2)

2. $$2\frac{\partial}{\partial w} V_\nu(w,z) = V_{\nu+1}(w,z) + \left(\frac{z}{w}\right)^2 V_{\nu-1}(w,z)$$ WA 593(4)

3. The function $U_\nu(w,z)$ is a particular solution of the differential equation

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z}\frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z)$$ WA 592(2)

4. The function $V_\nu(w,z)$ is a particular solution of the differential equation

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z}\frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z)$$ WA 592(3)

8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.580 Definitions:

1. The Anger function $\mathbf{J}_\nu(z)$:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\, d\theta$$ WA 336(1), EH II 35(32)

2. The Weber function $\mathbf{E}_\nu(z)$:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)\, d\theta$$ WA 336(2), EH II 35(32)

8.581 Series representations:

1. $$\mathbf{J}_\nu(z) = \cos\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} + \sin\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}$$

EH II 36(36), WA 337(3)

2. $$\mathbf{E}_\nu(z) = \sin\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} - \cos\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}$$

EH II 36(37), WA 338(4)

8.582 Functional relations:

1.[6] $2\,\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z)$ EH II 36(40), WA 340(2)

2.[6] $2\,\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z)$ EH II 36(41), WA 340(6)

3.[6] $\mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{J}_\nu(z) - 2(\pi z)^{-1}\sin(\nu\pi)$ EH II 36(42), WA 340(1)

4.[6] $\mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{E}_\nu(z) - 2(\pi z)^{-1}(1-\cos\nu\pi)$ EH II 36(43), WA 340(5)

8.583 Asymptotic expansions:

1.[6] $$\mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin\nu\pi}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)}\frac{\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n}\right.$$
$$\left. + O\left(|z|^{-2p}\right) - \nu\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1} + \nu\, O\left(|z|^{-2p-1}\right)\right]$$
$[|\arg z| < \pi]$ EH II 37(47), WA 344(1)

2. $$\mathbf{E}_\nu(z) = -\,Y_\nu(z)$$
$$-\frac{1+\cos(\nu\pi)}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n} + O\left(|z|^{-2p}\right)\right]$$
$$-\frac{\nu\,(1-\cos\nu\pi)}{z\pi}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1} + O\left(|z|^{-2p-1}\right)\right]$$
WA344(2), EH II 37(48)

For the asymptotic expansion of $J_\nu(z)$ and $Y_\nu(z)$, see **8.451**.

8.584 The Anger and Weber functions satisfy the differential equation
$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = f(\nu, z),$$
where $f(\nu, z) = \dfrac{z-\nu}{\pi z^2}\sin\nu\pi$ for $\mathbf{J}_\nu(z)$ WA 341(9), EH II 37(44)

and $f(\nu, z) = -\dfrac{1}{\pi z^2}\left[z + \nu + (z-\nu)\cos\nu\pi\right]$ for $\mathbf{E}_\nu(z)$ EH II 37(45), WA 341(10)

8.59 Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$

8.590 Definition of Neumann's polynomials

1. $O_n(z) = \dfrac{1}{4}\displaystyle\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n-1}$ $[n \geq 1]$ WA 299(2), EH II 33(6)

2. $O_{-n}(z) = (-1)^n\, O_n(z)$ $[n \geq 1]$ WA 303(8)

3. $O_0(z) = \dfrac{1}{z}$ WA 299(3), EH II 33(7)

4. $O_1(z) = \dfrac{1}{z^2}$ EH II 33(7)

5. $O_2(z) = \dfrac{1}{z} + \dfrac{4}{z^3}$ EH II 33(7)

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$.

8.591 Functional relations:

1. $O_0'(z) = -\,O_1(z)$ EH II 33(9), WA 301(3)

2. $2\,O_n'(z) = O_{n-1}(z) - O_{n+1}(z)$ $[n \geq 1]$ EH II 33(10), WA 301(2)

3. $(n-1)\,O_{n+1}(z)+(n+1)\,O_{n-1}(z)-2z^{-1}\left(n^2-1\right)O_n(z)=2nz^{-1}\left(\sin n\frac{\pi}{2}\right)^2$ $[n\geq 1]$ EH II 33(11), WA 301(1)

4. $nz\,O_{n-2}(z)-\left(n^2-1\right)O_n(z)=(n-1)z\,O_n'(z)+n\left(\sin n\frac{\pi}{2}\right)^2$ EH II 33(12), WA 303(4)

5. $nz\,O_{n+1}(z)-\left(n^2-1\right)O_n(z)=-(n+1)z\,O_n'(z)+n\left(\sin n\frac{\pi}{2}\right)^2$ EH II 33(13), WA 303(5)a

8.592 The generating function:

$$\frac{1}{z-\xi}=J_0(\xi)z^{-1}+2\sum_{n=1}^{\infty}J_n(\xi)\,O_n(z)\qquad[|\xi|<|z|]$$ EH II 32(1), WA 298(1)

8.593 The integral representation:

$$O_n(z)=\int_0^{\infty}\frac{\left[u+\sqrt{u^2+z^2}\right]^n+\left[u-\sqrt{u^2+z^2}\right]^n}{2z^{n+1}}e^{-u}\,du$$

See also **3.547** 6, 8, **3.549** 1, 2. EH II 32(3), WA 305(1)

8.594 The inequality

$$|O_n(z)|\leq 2^{n-1}n!|z|^{-n-1}e^{\frac{1}{4}|z|^2}\qquad[n>1]$$ EH II 33(8), WA 300(8)

8.595 Neumann's polynomial $O_n(z)$ satisfies the differential equation

$$z^2\frac{d^2y}{dz^2}+3z\frac{dy}{dz}+\left(z^2+1-n^2\right)y=z\left(\cos n\frac{\pi}{2}\right)^2+n\left(\sin n\frac{\pi}{2}\right)^2$$ EH II 33(14), WA 303(1)

8.596 Schläfli's polynomials $S_n(z)$. These are the functions that satisfy the formulas

1. $S_0(z)=0$ EH II 34(18), WA 312(2)

2. $S_n(z)=\frac{1}{n}\left[2zO_n(z)-2\left(\cos n\frac{\pi}{2}\right)^2\right]\qquad[n\geq 1]$ EH II 34(19), WA 312(3)

$$=\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n}\qquad[n\geq 1]$$ EH II 34(18)

3. $S_{-n}(z)=(-1)^{n+1}\,S_n(z)$ WA 313(6)

8.597 Functional relations:

1. $S_{n-1}(z)+S_{n+1}(z)=4\,O_n(z)$ WA 313(7)

Other functional relations may be obtained from **8.591** by replacing $O_n(z)$ with the expression for $S_n(z)$ given by **8.596** 2.

8.6 Mathieu Functions

8.60 Mathieu's equation

$$\frac{d^2y}{dz^2}+\left(a-2k^2\cos 2z\right)y=0,\quad k^2=q$$ MA

8.61 Periodic Mathieu functions

8.610 In general, Mathieu's equation **8.60** does not have periodic solutions. If k is a real number, there exist infinitely many *eigenvalues* a, not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z).$$

If k is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions*.

8.611 Mathieu's equation has four series of distinct periodic solutions:

1. $\mathrm{ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz$ MA
2. $\mathrm{ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z$ MA
3. $\mathrm{se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z$ MA
4. $\mathrm{se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z$ MA
5. The coefficients A and B depend on q. The eigenvalues a of the functions ce_{2n}, ce_{2n+1}, se_{2n}, se_{2n+1} are denoted by a_{2n}, a_{2n+1}, b_{2n}, b_{2n+1}.

8.612 The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2\,dx = \pi$$ MO 65

8.613

1. $\lim_{q\to 0} \mathrm{ce}_0(x) = \frac{1}{\sqrt{2}}$
2. $\lim_{q\to 0} \mathrm{ce}_n(x) = \cos nx$ $[n \neq 0]$
3. $\lim_{q\to 0} \mathrm{se}_n(x) = \sin nx$ MO 65

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$

8.621

1. $aA_0^{(2n)} - qA_2^{(2n)} = 0$ MA
2. $(a-4)A_2^{(2n)} - q\left(A_4^{(2n)} + 2A_0^{(2n)}\right) = 0$ MA
3. $\left(a - 4r^2\right) A_{2r}^{(2n)} - q\left(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}\right) = 0$ $[r \geq 2]$ MA

8.622

1. $(a-1-q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0$ MA

2. $\left[a-(2r+1)^2\right] A_{2r+1}^{(2n+1)} - q\left(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)}\right) = 0 \qquad [r \geq 1]$ MA

8.623

1. $(a-1+q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0$ MA

2. $\left[a-(2r+1)^2\right] B_{2r+1}^{(2n+1)} - q\left(B_{2r+3}^{(2n+1)} + B_{2r-1}^{(2n+1)}\right) = 0$

$[r \geq 1]$ MA

8.624

1. $(a-4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0$ MA

2.[11] $\left(a-4r^2\right) B_{2r}^{(2n+2)} - q\left(B_{2r+2}^{(2n+2)} + B_{2r-2}^{(2n+2)}\right) = 0 \qquad [r \geq 2]$ MA

8.625 We can determine the coefficients A and B from equations **8.612**, **8.613** and **8.621-8.624** provided a is known. Suppose, for example, that we need to determine the coefficients $A_{2r}^{(2n)}$ for the function $\mathrm{ce}_{2n}(z,q)$. From the recursion formulas, we have

1. $$\begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a-4 & -q & 0 & 0 & \dots \\ 0 & -q & a-16 & -q & 0 & \dots \\ 0 & 0 & -q & a-36 & -q & \\ 0 & 0 & 0 & -q & a-64 & \\ \vdots & \vdots & \vdots & & & \ddots \end{vmatrix} = 0$$ ST

For given q in equation **8.625** 1, we may determine the eigenvalues

2. $a = A_0, A_2, A_4, \dots \qquad [|A_0| \leq |A_2| \leq |A_4| \leq \dots]$

If we now set $a = A_{2n}$, we can determine the coefficients $A_{2r}^{(2n)}$ from the recursion formulas **8.621** up to a proportionality coefficient. This coefficient is determined from the formula

3. $$2\left[A_0^{(2n)}\right]^2 + \sum_{r=1}^{\infty}\left[A_{2r}^{(2n)}\right]^2 = 1,$$ MA

which follows from the conditions of normalization.

8.63 Mathieu functions with a purely imaginary argument

8.630 If, in equation **8.60**, we replace z with iz, we arrive at the differential equation

1.[11] $$\frac{d^2y}{dz^2} + (-a + 2q\cosh 2z)\, y = 0$$

We can find the solutions of this equation if we replace the argument z with iz in the functions $\mathrm{ce}_n(z,q)$ and $\mathrm{se}_n(z,q)$. The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

1. $\mathrm{Ce}_{2n}(z,q), \quad \mathrm{Ce}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+2}(z,q)$

8.631

1. $\mathrm{Ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2rz$ MA

2. $\mathrm{Ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)z$ MA

3. $\mathrm{Se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)z$ MA

4. $\mathrm{Se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)z$ MA

8.64 Non-periodic solutions of Mathieu's equation

Along with each periodic solution of equation **8.60**, there exists a second non-periodic solution that is linearly independent. The non-periodic solutions are denoted as follows:

$$\mathrm{fe}_{2n}(z,q), \quad \mathrm{fe}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+2}(z,q).$$

Analogously, the second solutions of equation **8.630** 1 are denoted by

$$\mathrm{Fe}_{2n}(z,q), \quad \mathrm{Fe}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+2}(z,q).$$

8.65 Mathieu functions for negative q

8.651 If we replace the argument z in equation **8.60** with$\pm\left(\frac{\pi}{2} \pm z\right)$, we get the equation

$$\frac{d^2y}{dz^2} + (a + 2q\cos 2z)\, y = 0. \qquad \text{MA}$$

This equation has the following solutions:

8.652

1. $\mathrm{ce}_{2n}(z,-q) = (-1)^n \,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi - z, q\right)$ MA

2. $\mathrm{ce}_{2n+1}(z,-q) = (-1)^n \,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA

3. $\mathrm{se}_{2n+1}(z,-q) = (-1)^n \,\mathrm{ce}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA

4. $\mathrm{se}_{2n+2}(z,-q) = (-1)^n \,\mathrm{se}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ MA

5. $\mathrm{fe}_{2n}(z,-q) = (-1)^{n+1} \,\mathrm{fe}_{2n}\left(\frac{1}{2}\pi - z, q\right)$ MA

6. $\mathrm{fe}_{2n+1}(z,-q) = (-1)^n \,\mathrm{ge}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA

7. $\mathrm{ge}_{2n+1}(z,-q) = (-1)^n \,\mathrm{fe}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA

8. $\mathrm{ge}_{2n+2}(z,-q) = (-1)^n \,\mathrm{ge}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ MA

8.653 Analogously, if we replace z with $\frac{\pi}{2}i + z$ in equation **8.630** 1, we get the equation

$$\frac{d^2y}{dz^2} - (a + 2q\cosh z)\, y = 0.$$

It has the following solutions:

8.654

1. $\mathrm{Ce}_{2n}(z,-q) = (-1)^n \,\mathrm{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right)$ MA
2. $\mathrm{Ce}_{2n+1}(z,-q) = (-1)^{n+1} i\,\mathrm{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
3. $\mathrm{Se}_{2n+1}(z,-q) = (-1)^{n+1} i\,\mathrm{Ce}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
4. $\mathrm{Se}_{2n+2}(z,-q) = (-1)^{n+1}\,\mathrm{Se}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA
5. $\mathrm{Fe}_{2n}(z,-q) = (-1)^{n}\,\mathrm{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right)$ MA
6.[11] $\mathrm{Fe}_{2n+1}(z,-q) = (-1)^{n+1} i\,\mathrm{Ge}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
7.[11] $\mathrm{Ge}_{2n+1}(z,-q) = (-1)^{n+1} i\,\mathrm{Fe}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA
8.[11] $\mathrm{Ge}_{2n+2}(z,-q) = (-1)^{n+1}\,\mathrm{Ge}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA

8.66 Representation of Mathieu functions as series of Bessel functions

8.661

1. $$\mathrm{ce}_{2n}(z,q) = \frac{\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, J_{2r}\left(2k\cos z\right)$$ MA
$$= \frac{\mathrm{ce}_{2n}(0,q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, I_{2r}\left(2k\sin z\right)$$ MA

2. $$\mathrm{ce}_{2n+1}(z,q) = -\frac{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)}\, J_{2r+1}\left(2k\cos z\right)$$ MA
$$= \frac{\mathrm{ce}_{2n+1}(0,q)}{kA_1(2n+1)} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)}\, I_{2r+1}\left(2k\sin z\right)$$ MA

3. $$\mathrm{se}_{2n+1}(z,q) = \frac{\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{kB_1^{(2n+1)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)}\, J_{2r+1}\left(2k\cos z\right)$$ MA
$$= \frac{\mathrm{se}'_{2n+1}(0,q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)}\, I_{2r+1}\left(2k\sin z\right)$$ MA

4. $$\mathrm{se}_{2n+2}(z,q) = \frac{-\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{k^2 B_2^{(2n+2)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)}\, J_{2r+2}(2k\ \cos_z)$$ MA
$$= \frac{\mathrm{se}'_{2n+2}(0,q)}{k^2 B_2^{(2n+2)}} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)}\, I_{2r+2}(2k\sin z)$$ MA

8.662

1. $$\mathrm{fe}_{2n}(z,q) = -\frac{\pi\,\mathrm{fe}'_{2n}(0,q)}{2\,\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, \mathrm{Im}\left[J_r\left(ke^{iz}\right)\, Y_r\left(ke^{-iz}\right)\right]$$ MA

2. $$\mathrm{fe}_{2n+1}(z,q)=\frac{\pi k\,\mathrm{fe}'_{2n+1}(0,q)}{2\,\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}\times\sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\,\mathrm{Im}\left[J_r\left(ke^{iz}\right)\,Y_{r+1}\left(ke^{-iz}\right)+J_{r+1}\left(ke^{iz}\right)\,Y_r\left(ke^{-iz}\right)\right]$$

MA

3. $$\mathrm{ge}_{2n+1}(z,q)=-\frac{\pi k\,\mathrm{ge}_{2n+1}(0,q)}{2\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}\times\sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}\,\mathrm{Re}\left[J_r\left(ke^{iz}\right)\,Y_{r+1}\left(ke^{-iz}\right)-J_{r+1}\left(ke^{iz}\right)\,Y_r\left(ke^{-iz}\right)\right]$$

MA

4. $$\mathrm{ge}_{2n+2}(z,q)=-\frac{\pi k^2\,\mathrm{ge}_{2n+2}(0,q)}{2\,\mathrm{se}'_{2n+2}\left(\frac{1}{2}\pi,q\right)}\times\sum_{r=0}^{\infty}(-1)^r\,\mathrm{Re}\left[J_k\left(ke^{iz}\right)\,Y_{r+2}\left(ke^{-iz}\right)-J_{r+2}\left(ke^{iz}\right)\,Y_r\left(ke^{-iz}\right)\right]$$

MA

The expansions of the functions Fe_n and Ge_n as series of the functions Y_ν are denoted, respectively, by Fey_n and Gey_n, and the expansions of these functions as series of the functions K_ν are denoted, respectively, by Fek_n and Gek_n.

8.663

1. $$\mathrm{Fey}_{2n}(z,q)=\frac{\mathrm{ce}_{2n}(0,q)}{A_0^{(2n)}}\sum_{r=0}^{\infty}A_{2r}^{(2n)}\,Y_{2r}(2k\sinh z)$$

$$k^2=q\,[|\sinh z|>1,\quad \mathrm{Re}\,z>0]$$

MA

$$=\frac{\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}{A_0^{(2n)}}\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}\,Y_{2r}(2k\cosh z)$$

$$[|\cosh z|>1]$$

MA

$$=\frac{\mathrm{ce}_{2n}(0,q)\,\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}{\left[A_0^{(2n)}\right]^2}\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}\,J_r\left(ke^{-z}\right)\,Y_r\left(ke^{z}\right)$$

MA

2. $$\mathrm{Fey}_{2n+1}(z,q) = \frac{\mathrm{ce}_{2n+1}(0,q)\coth z}{kA_1(2n+1)} \sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(2n+1)}\ Y_{2r+1}(2k\sinh z),$$

$$k^2 = q, \quad [|\sinh z| > 1, \quad \mathrm{Re}\, z > 0]$$

MA

$$= -\frac{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\ Y_{2r+1}(2k\cosh z)$$

$$[|\cosh z| > 1]$$

MA

$$= -\frac{\mathrm{ce}_{2n+1}(0,q)\,\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{k\left[A_1^{(2n+1)}\right]^2}$$

$$\times \sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\left[J_r\left(ke^{-z}\right)Y_{r+1}(ke^z) + J_{r+1}\left(ke^{-z}\right)Y_r(ke^z)\right]$$

MA

3. $$\mathrm{Gey}_{2n+1}(z,q) = \frac{\mathrm{se}'_{2n+1}(0,q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}\ Y_{2r+1}(2k\sinh z)$$

$$[|\sinh z| > 1, \quad \mathrm{Re}\, z > 0]$$

MA

$$= \frac{\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{kB_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty}(-1)^r(2r+1)B_{2r+1}^{(2n+1)}\ Y_{2r+1}(2k\cosh z)$$

$$[|\cosh z| > 1]$$

MA

$$= \frac{\mathrm{se}_{2n+1}(0,q)\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{k\left[B_1^{(2n+1)}\right]^2} \sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}$$

$$\times \left[J_r\left(ke^{-z}\right)Y_{r+1}(ke^z)\right] J_{r+1}\left(ke^{-z}\right)Y_r(ke^z)$$

MA

4. $$\mathrm{Gey}_{2n+2}(z,q)=\frac{\mathrm{se}'_{2n+2}(0,q)}{k^2B_2^{(2n+2)}}\coth z\sum_{r=0}^{\infty}(2r+2)B_{2r+2}^{(2n+2)}\,Y_{2r+2}(2k\sinh z)$$
$$[|\sinh z|>1,\quad \operatorname{Re} z>0]$$
MA
$$=-\frac{\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{k^2B_2^{(2n+2)}}\tanh z\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}^{(2n+2)}\,Y_{2r+2}(2k\cosh z)$$
$$[|\cosh z|>1]$$
MA
$$=\frac{\mathrm{se}'_{2n+2}(0,q)\,\mathrm{se}'_{2n+2\left(\frac{\pi}{2},q\right)}}{k^2\left[B_2^{(2n+2)}\right]^2}\sum_{r=0}^{\infty}(-1)^rB_{2r+2}^{(2n+2)}$$
$$\times\left[J_r\left(ke^{-z}\right)Y_{r+2}\left(ke^{z}\right)\right]-J_{r+2}\left(ke^{-z}\right)Y_r\left(ke^{z}\right)$$
MA

8.664

1. $$\mathrm{Fek}_{2n}(z,q)=\frac{\mathrm{ce}_{2n}(0,q)}{\pi A_0^{(2n)}}\sum_{r=0}^{\infty}(-1)^rA_{2r}^{(2n)}\,K_{2r}(-2ik\sinh z)$$
$$k^2=q,\qquad [|\sinh z|>1,\quad \operatorname{Re} z>0]$$
MA

2. $$\mathrm{Fek}_{2n+1}(z,q)=\frac{\mathrm{ce}_{2n+1}(0,q)}{\pi kA_1^{(2n+1)}}\coth z\sum_{r=0}^{\infty}(-1)^r(2r+1)A_{2r+1}^{(2n+1)}\,K_{2r+1}(-2ik\sinh z)$$
$$k^2=q\qquad [|\sinh z|>1,\quad \operatorname{Re} z>0]$$
MA

3. $$\mathrm{Gek}_{2n+1}(z,q)=\frac{\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{\pi kB_1^{(2n+1)}}\tanh z\sum_{r=0}^{\infty}(2r+1)B_{2r+1}^{(2n+1)}\,K_{2r+1}(-2ik\cosh z)$$ MA

4. $$\mathrm{Gek}_{2n+2}(z,q)=\frac{\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{\pi k^2B_2^{(2n+2)}}\tanh z\sum_{r=0}^{\infty}(2r+2)B_{2r+2}^{(2n+2)}\,K_{2r+2}(-2ik\cosh z)$$ MA

8.67 The general theory

If $i\mu$ is not an integer, the general solution of equation **8.60** can be found in the form

8.671

1. $$y=Ae^{\mu z}\sum_{r=-\infty}^{\infty}c_{2r}e^{2rzi}+Be^{-\mu z}\sum_{r=-\infty}^{\infty}c_{2r}e^{-2rzi}$$ MA

The coefficients c_{2r} can be determined from the homogeneous system of linear algebraic equations

2.[11] $$c_{2r}+\xi_{2r}\left(c_{2r+2}+c_{2r-2}\right)=0,\qquad r=\ldots,-2,-1,0,1,2,\ldots,$$ MA

where

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}$$

The condition that this system be compatible yields an equation that μ must satisfy:

3.[7] $$\Delta(i\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$ MA

This equation can also be written in the form

4. $\cosh \mu\pi = 1 - 2\Delta(0)\sin^2\left(\frac{\pi\sqrt{a}}{2}\right)$, where $\Delta(0)$ is the value that is assumed by the determinant of the preceding article if we set $\mu = 0$ in the expressions for ξ_{2r}.

5. If the pair (a, q) is such that $|\cosh \mu\pi| < 1$, then $\mu = i\beta, \operatorname{Im}\beta = 0$, and the solution **8.671** 1 is bounded on the real axis.

6. If $|\cosh \mu\pi| > 1, \mu$ may be real or complex, and the solution **8.671** 1 will not be bounded on the real axis.

7. If $\cosh \mu\pi = \pm 1$, then $i\mu$ will be an integer. In this case, one of the solutions will be of period π or 2π (depending on whether n is even or odd). The second solution is non-periodic (see **8.61** and **8.64**).

8.7–8.8 Associated Legendre Functions

8.70 Introduction

8.700 An *associated Legendre function* is a solution of the differential equation

1. $$\left(1 - z^2\right)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1 - z^2}\right]u = 0,$$

in which ν and μ are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see **9.151**). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable z that lie in the interval $[-1, 1]$ and, on the other hand, in solutions corresponding to an arbitrary complex number z such that $\operatorname{Re} z > 1$. These are multiple-valued in the z-plane. To separate these functions into single-valued branches, we make a cut along the real axis from $-\infty$ to $+1$. We are also interested in those solutions of equation **8.700** 1 for which ν or μ or both are integers. Of special significance is the case in which $\mu = 0$.

8.701 In connection with this, we shall use the following notations:

The letter z will denote *an arbitrary complex variable*; the letter x will denote a *real* variable that varies over the interval $[-1, +1]$. We shall sometimes set $x = \cos\varphi$, where φ is a real number.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular for $|z| < 1$ and, in particular, uniquely determined for $z = x$.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular forRe $z > 1$. When these functions cannot be unrestrictedly extended without violating their single-valuedness, we make a cut along the real axis to the left of the point $z = 1$. The values of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ on the upper and lower boundaries of that portion of the cuts lying between the points -1 and $+1$ are denoted, respectively, by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters n and m denote natural numbers or zero. The letters ν and μ denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z)$$

The *linearly independent* functions

8.702 $$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1-\mu; \quad \frac{1-z}{2}\right)$$

$$\left[\arg\frac{z+1}{z-1} = 0, \text{ if } z \text{ is real and greater than 1 and}\right]$$ MO 80, WH

8.703 $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)\,\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(z^2-1\right)^{\frac{\mu}{2}} z^{-\nu-\mu-1}\, F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

[$\arg\left(z^2-1\right) = 0$ when z is real and greater than 1; $\arg z = 0$ when z is real and greater than zero] which are solutions of the differential equation **8.700** 1, are called *associated Legendre functions* (or *spherical functions*) of *the first* and *second kinds*, respectively. They are uniquely defined, respectively, in the intervals $|1-z| < 2$ and $|z| > 1$, with the portion of the real axis that lies between $-\infty$ and $+1$ excluded. They can be extended by means of hypergeometric series to the entire z-plane where the above-mentioned cut was made. These expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ lose their meaning when $1-\mu$ and $\nu+\frac{3}{2}$ are non-positive integers, respectively. MO 80

When z is a real number lying on the interval $[-1, +1]$, so that $(z = x = \cos\varphi)$, we take the following functions as linearly independent solutions of the equation:

8.704 $$P_\nu^\mu(x) = \tfrac{1}{2}\left[e^{\frac{1}{2}\mu\pi i}\, P_\nu^\mu(\cos\varphi + i0) + e^{-\frac{1}{2}\mu\pi i}\, P_\nu^\mu(\cos\varphi - i0)\right]$$ EH I 143(1)

$$= \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$ EH I 143(6)

8.705 $$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-\mu\pi i}\left[e^{-\frac{1}{2}\mu\pi i}\, Q_\nu^\mu(x+i0) + e^{\frac{1}{2}\mu\pi i}\, Q_\nu^\mu(x-i0)\right]$$ EH I 143(2)

$$= \frac{\pi}{2\sin\mu\pi}\left[P_\nu^\mu(x)\cos\mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\, P_\nu^{-\mu}(x)\right] \qquad \text{(cf. } \mathbf{8.732}\ 5)$$

If $\mu = \pm m$ is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

8.706

1. $$Q_\nu^m(x) = (-1)^m\left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m}\, Q_\nu(x) \qquad \text{(cf. } \mathbf{8.752}\ 1)$$ EH I 149(7)

2.[11] $$Q_\nu^{-m}(x) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\, Q_\nu^m(x)$$ EH I 144(18)

The functions $Q_\nu^\mu(z)$ are not defined when $\nu+\mu$ is equal to a negative integer. Therefore, we must exclude the cases when $\nu+\mu = -1, -2, -3, \ldots$ for these formulas.

The functions

$$P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

are *linearly independent solutions* of the differential equation for $\nu+\mu \neq 0, \pm 1, \pm 2, \ldots$.

8.707 Nonetheless, two linearly independent solutions can always be found. Specifically, for $\nu \pm \mu$ not an integer, the differential equation **8.700** 1 has the following solutions:

1. $P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$

   respectively, for $z = x = \cos\varphi$,

2. $P_\nu^{\pm\mu}(\pm x), \quad Q_\nu^{\pm\mu}(\pm x), \quad P_{-\nu-1}^{\pm\mu}(\pm x), \quad Q_{-\nu-1}^{\pm\mu}(\pm x)$.

   If $\nu \pm \mu$ is not an integer, the solutions

3. $P_\nu^\mu(z), \quad Q_\nu^\mu(z)$, respectively, and $P_\nu^\mu(x), \quad Q_\nu^\mu(x)$

   are linearly independent. If $\nu \pm \mu$ is an integer but μ itself is not an integer, the following functions are linearly independent solutions of equation **8.700** 1:

4. $P_\nu^\mu(z), \quad P_\nu^{-\mu}(z)$, respectively, and $P_\nu^\mu(x), \quad P_\nu^{-\mu}(x)$.

   If $\mu = \pm m, \nu = n$, or $\nu = -n-1$, the following functions are linearly independent solutions of equation **8.700** 1 for $n \geq$m:

5. $P_n^m(z), \quad Q_n^m(z)$, respectively, and $P_n^m(x), \quad Q_n^m(x)$,

   and for $n <$m, the following functions will be linearly independent solutions

6. $P_n^{-m}(z), \quad Q_n^m(z)$, respectively, and $P_n^{-m}(x), \quad Q_n^m(x)$.

8.71 Integral representations

8.711

1. $$P_\nu^{-\mu}(z) = \frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{2^\mu\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)} \int_{-1}^{1} \frac{\left(1-t^2\right)^{\mu-\frac{1}{2}}}{\left(z+t\sqrt{z^2-1}\right)^{\mu-\nu}}\,dt \qquad \left[\operatorname{Re}\mu > -\tfrac{1}{2}, \quad |\arg(z\pm 1)| < \pi\right]$$

   MO 88

2. $$P_\nu^m(z) = \frac{(\nu+1)(\nu+2)\ldots(\nu+m)}{\pi}\int_0^\pi \left[z+\sqrt{z^2-1}\cos\varphi\right]^\nu \cos m\varphi\,d\varphi$$
   $$= (-1)^m \frac{\nu(\nu-1)\ldots(\nu-m+1)}{\pi}\int_0^\pi \frac{\cos m\varphi\,d\varphi}{\left[z+\sqrt{z^2-1}\cos\varphi\right]^{\nu+1}}$$
   $$\left[|\arg z| < \frac{\pi}{2}, \quad \arg\left(z+\sqrt{z^2-1}\cos\varphi\right) = \arg z \text{ for } \varphi = \frac{\pi}{2}\right] \quad \text{(cf. } \mathbf{8.822}\ 1)$$

   SM 483(15), WH

3. $$Q_\nu^\mu(z) = \sqrt{\pi}\,\frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu+1)}\left(z^2-1\right)^{\frac{\mu}{2}}\int_0^\infty \frac{\sinh^{2\mu} t\,dt}{\left(z+\sqrt{z^2-1}\cosh t\right)^{\nu+\mu+1}}$$
   $$[\operatorname{Re}(\nu\pm\mu) > -1, \quad |\arg(z\pm 1)| < \pi] \qquad \text{(cf. } \mathbf{8.822}\ 2)$$

   MO 88

4. $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty \frac{\cosh\mu t\,dt}{\left(z+\sqrt{z^2-1}\cosh t\right)^{\nu+1}}$$
   $$[\operatorname{Re}(\nu+\mu) > -1, \nu \neq -1, -2, -3, \ldots, \quad |\arg(z\pm 1)| < \pi]$$

   WH, MO 88

5. $$\int_{-1}^{1} P_l^2(x)\, P_l^0(x)\, dx = -\frac{l!}{(l-2)!}\frac{1}{2l+1} = -\frac{l(l-1)}{2l+1}$$

8.712 $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\,\Gamma(\nu+1)}\left(z^2-1\right)^{-\frac{\mu}{2}}\int_{-1}^{1}\left(1-t^2\right)^\nu (z-t)^{-\nu-\mu-1}\,dt$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}\mu > -1, \quad |\arg(z\pm 1)| < \pi]$ (cf. **8.821** 2) MO 88a, EH I 155(5)a

8.713

1. $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(z^2-1\right)^{\frac{\mu}{2}}\left\{\int_0^\pi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(z-\cos t)^{\mu+\frac{1}{2}}} - \cos\nu\pi\int_0^\infty \frac{e^{-\left(\nu+\frac{1}{2}\right)t}\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}\right\}$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(\nu+\mu) > -1, \quad |\arg(z\pm 1)| < \pi\right]$ MO 89

2. $$P_\nu^{-\mu}(z) = \frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{2^\nu\,\Gamma(\mu-\nu)\,\Gamma(\nu+1)}\int_0^\infty \frac{\sinh^{2\nu+1} t}{(z+\cosh t)^{\nu+\mu+1}}\,dt$$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+1) > 0, \quad \operatorname{Re}(\mu-\nu) > 0]$ MO 89

3. $$P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}\int_0^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}$$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$ MO 89

8.714

1. $$P_\nu^\mu(\cos\varphi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\varphi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(\cos t-\cos\varphi)^{\mu+\frac{1}{2}}}$$
$\left[0 < \varphi < \pi, \quad \operatorname{Re}\mu < \frac{1}{2}\right]$; (cf. **8.823**)
MO 87

2. $$P_\nu^{-\mu}(\cos\varphi) = \frac{\Gamma(2\mu+1)\sin^\mu\varphi}{2^\mu\,\Gamma(\mu+1)\,\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}\int_0^\infty \frac{t^{\nu+\mu}\,dt}{\left(1+2t\cos\varphi+t^2\right)^{\mu+\frac{1}{2}}}$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$
MO 89

3. $$Q_\nu^\mu(\cos\varphi) = \frac{1}{2^{\mu+1}}\,\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\,\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)} \times\int_0^\infty\left[\frac{\sinh^{2\mu} t}{(\cos\varphi+i\sin\varphi\cosh t)^{\nu+\mu+1}} + \frac{\sinh^{2\mu} t}{(\cos\varphi-i\sin\varphi\cosh t)^{\nu+\mu+1}}\right]dt$$
$\left[\operatorname{Re}(\nu+\mu+1) > 0, \quad \operatorname{Re}(\nu-\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ MO 89

4. $$P_\nu^\mu(\cos\varphi) = \frac{i}{2^\mu}\,\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\,\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)} \times\int_0^\infty\left[\frac{\sinh^{2\mu} t}{(\cos\varphi+i\sin\varphi\cosh t)^{\nu+\mu+1}} - \frac{\sinh^{2\mu} t}{(\cos\varphi-i\sin\varphi\cosh t)^{\nu+\mu+1}}\right]dt$$
$\left[\operatorname{Re}(\nu\pm\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ MO 89

8.715

1. $$P_\nu^\mu(\cosh\alpha) = \frac{\sqrt{2}\sinh^\mu\alpha}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\alpha \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(\cosh\alpha-\cosh t)^{\mu+\frac{1}{2}}}$$
$$\left[\alpha>0,\quad \operatorname{Re}\mu<\tfrac{1}{2}\right]$$ MO 87

2. $$Q_\nu^\mu(\cosh\alpha) = \sqrt{\frac{\pi}{2}}\frac{e^{\mu\pi i}\sinh^\mu\alpha}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\alpha^\infty \frac{e^{-\left(\nu+\frac{1}{2}\right)t}\,dt}{(\cosh t-\cosh\alpha)^{\mu+\frac{1}{2}}}$$
$$\left[\alpha>0,\quad \operatorname{Re}\mu<\tfrac{1}{2},\quad \operatorname{Re}(\nu+\mu)>-1\right]$$ MO 87

See also **3.277** 1, 4, 5, 7, **3.318**, **3.516** 3, **3.518** 1, 2, **3.542** 2, **3.663** 1, **3.894**, **3.988** 3, **6.622** 3, **6.628** 1, 4–7, and also **8.742**.

8.72 Asymptotic series for large values of $|\nu|$

8.721[6] For real values of $\mu, |\nu|\gg 1, |\nu|\gg|\mu|, |\arg\nu|<\pi$, we have:

1. $$P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}}\Gamma(\nu+\mu+1)\sum_{k=0}^\infty \frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)}\frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi+\frac{\pi}{4}(2k-1)+\frac{\mu\pi}{2}\right]}{k!\,\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$$
$$\Big[\nu+\mu\neq-1,-2,-3,\ldots;\quad \nu\neq-\tfrac{3}{2},-\tfrac{5}{2},\tfrac{7}{2}\ldots;\ \text{for } \frac{\pi}{6}<\varphi<\frac{5\pi}{6}$$
This series also converges for complex values of ν and μ. In the remaining cases, it is an asymptotic expansion for $|\nu|\gg|\mu|, |\nu|\gg 1$, if $\nu>0, \mu>0$ and $0<\varepsilon\le\varphi\le\pi-\varepsilon\Big]$

MO 92

2.[6] $$Q_\nu^\mu(\cos\varphi) = \sqrt{\pi}\,\Gamma(\nu+\mu+1)$$
$$\times\sum_{k=0}^\infty(-1)^k\frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)}\frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi-\frac{\pi}{4}(2k-1)+\frac{\mu\pi}{2}\right]}{k!\,\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$$
$$\Big[\nu+\mu\neq-1,-2,-3,\ldots;\quad \nu\neq-\tfrac{3}{2},-\tfrac{5}{2},-\tfrac{7}{2},\ldots;\ \text{for } \frac{\pi}{6}<\varphi<\frac{5}{6}\pi$$
This series also converges for complex values of ν and μ. In the remaining cases, it is an asymptotic expansion for $|\nu|\gg|\mu|,\quad |\nu|\gg 1$, if $\nu>0,\quad \mu>0,\quad 0<\varepsilon\le\varphi\le\pi-\varphi\Big]$

EH I 147(6), MO 92

3. $$P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{\cos\left[\left(\nu+\frac{1}{2}\right)\varphi-\frac{\pi}{4}+\frac{\mu\pi}{2}\right]}{\sqrt{2\sin\varphi}}\left[1+O\left(\frac{1}{\nu}\right)\right]$$
$$\left[0<\varepsilon\le\varphi\le\pi-\varepsilon,\quad |\nu|\gg\frac{1}{\varepsilon}\right]$$ MO 92

For $\nu>0, \mu>0$ and $\nu>\mu$, it follows from formulas **8.721** 1 and **8.721** 2 that

4. $$\nu^{-\mu}\, P_\nu^\mu\,(\cos\varphi) = \sqrt{\frac{2}{\nu\pi\sin\varphi}}\cos\left[\left(\nu+\frac{1}{2}\right)\varphi - \frac{\pi}{4} + \frac{\mu\pi}{2}\right] + O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

5. $$\nu^{-\mu}\, Q_\nu^\mu\,(\cos\varphi) = \sqrt{\frac{\pi}{2\nu\sin\varphi}}\cos\left[\left(\nu+\frac{1}{2}\right)\varphi + \frac{\pi}{4} + \frac{\mu\pi}{2}\right] O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

$$\left[0<\varepsilon\le\varphi\le\pi-\varepsilon;\quad \nu\gg\frac{1}{\varepsilon}\right]$$ MO 92

8.722 If φ is sufficiently close to 0 or π that $\nu\varphi$ or $\nu(\pi-\varphi)$ is small in comparison with 1, the asymptotic formulas **8.721** become unsuitable. In this case, the following asymptotic representation is applicable for $\mu\le 0, \nu\gg 1$, and *small* values of φ:

1. $$\left[\left(\nu+\frac{1}{2}\right)\cos\frac{\varphi}{2}\right]^\mu P_\nu^{-\mu}\,(\cos\varphi) = J_\mu(\eta) + \sin^2\frac{\varphi}{2}\left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6}\,J_{\mu+3}(\eta)\right] + O\left(\sin^4\frac{\varphi}{2}\right)$$

where $\eta = (2\nu+1)\sin\frac{\varphi}{2}$. In particular, it follows that

1. $$\lim_{\nu\to\infty}\nu^\mu\, P_\nu^{-\mu}\left(\cos\frac{x}{\nu}\right) = J_\mu(x) \qquad [x\ge 0, \mu\ge 0]$$ MO 93

8.723 We can see how the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ behave for large $|\nu|$ and real values of $z > \frac{3}{2\sqrt{2}}$:

1. $$P_\nu^\mu\,(\cosh\alpha) = \frac{2^\mu}{\sqrt{\pi}}\left\{\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}\frac{e^{(\mu-\nu)\alpha}\sinh^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}}\,F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};\nu+\frac{3}{2};\frac{1}{1-e^{2\alpha}}\right)\right.$$
$$\left. + \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{e^{(\nu+\mu+1)\alpha}\sinh^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}}\,F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};-\nu+\frac{1}{2};\frac{1}{1-e^{2\alpha}}\right)\right\}$$
$$\left[\nu\ne\pm\tfrac{1}{2},\pm\tfrac{3}{2},\pm\tfrac{5}{2},\ldots;\quad a>\tfrac{1}{2}\ln 2\right]$$ MO 94

2. $$Q_\nu^\mu\,(\cosh\alpha) = e^{\mu\pi i}2^\mu\sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{e^{-(\nu+\mu+1)\alpha}}{(1-e^{-2\alpha})^{\mu+\frac{1}{2}}}\sinh^\mu\alpha$$
$$\times F\left(\mu+\frac{1}{2},-\mu+\frac{1}{2};\nu+\frac{3}{2};\frac{1}{1-e^{2\alpha}}\right)$$
$$\left[\mu+\nu+1\ne 0,-1,-2,\ldots;\quad \alpha>\tfrac{1}{2}\ln 2\right]$$ MO 94

See also **8.776**.

8.724 For the inequalities in **8.776** 1–4, ν and μ are arbitrary real numbers satisfying the inequalities $\nu\ge 1$, $\nu-\mu+1>0$, and $\mu\ge 0$:

1. $$\left|P_\nu^{\pm\mu}\,(\cos\varphi)\right| < \sqrt{\frac{8}{\nu\pi}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

2. $$\left|Q_\nu^{\pm\mu}\,(\cos\varphi)\right| < \sqrt{\frac{2\pi}{\nu}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

3. $$\left|P_\nu^{\pm\mu}\,(\cos\varphi)\right| < \frac{2}{\sqrt{\nu\pi}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

4. $$\left|Q_\nu^{\pm\mu}(\cos\varphi)\right| < \sqrt{\frac{\pi}{\nu}}\,\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\,\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

5.[8] $$\left|\sqrt{\sin\varphi}\,P_n^m(\cos\varphi)\right| < \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n-m+1)}2^{(m+n)^2/n}\sup_{0<t<\infty}\left|\sqrt{t}\,J_m(t)\right|$$

[uniformly $0\le m\le n$]

8.725[10] For fixed z and ν and $\operatorname{Re}\mu\to\infty$, with z not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$, the following are asymptotic expansions in which the upper and lower signs are taken according to whether $\operatorname{Im} z$ is greater than or less than 0:

1. $$P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}{\pi\,\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\sin\mu\pi\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\right.$$
$$\left.-\frac{\sin\nu\pi}{\sin\mu\pi}e^{\mp i\mu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$
AS 8.10.1

2. $$Q_\nu^\mu(z) = \frac{1}{2}e^{i\mu\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\Gamma(\mu-\nu)\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\right.$$
$$\left.-e^{\mp i\nu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$
AS 8.10.2

3. $$Q_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi}\operatorname{cosec}[\pi(\nu-\mu)]}{2\pi\,\Gamma(1+\mu)}\left[e^{\mp i\nu\pi}\left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\,\nu+1;\,1+\mu;\,\frac{1}{2}-\frac{1}{2}z\right)\right.$$
$$\left.-\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\,\nu+1;\,1+\mu;\,\frac{1}{2}+\frac{1}{2}z\right)\right]$$
AS 8.10.3

8.73–8.74 Functional relations

8.731

1. $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = (\nu-\mu+1)\,P_{\nu+1}^\mu(z) - (\nu+1)z\,P_\nu^\mu(z)$$
(cf. **8.832** 1, **8.914** 2)
EH I 161(10), MO 81

1(1)[9] $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = \nu z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)$$ AS 8.5.4

1(2) $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)\sqrt{z^2-1}\,P_\nu^{\mu-1}(z) - \mu z\,P_\nu^\mu(z)$$ AS 8.5.2

2. $(2\nu+1)z\,P_\nu^\mu(z) = (\nu-\mu+1)\,P_{\nu+1}^\mu(z) + (\nu+\mu)\,P_{\nu-1}^\mu(z)$ (cf. **8.832** 2, **8.914** 1) EH I 160(2), MO 81

3. $P_\nu^{\mu+2}(z) + 2(\mu+1)\dfrac{z}{\sqrt{z^2-1}}\,P_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)\,P_\nu^\mu(z)$ MO 82, EH I 160(1)

3(1)[9] $P_\nu^{\mu+1}(z) = \left(z^2-1\right)^{-1/2}\left[(\nu-\mu)z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)\right]$ AS 8.5.1

4. $P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu+1)\sqrt{z^2-1}\,P_\nu^{\mu-1}(z)$ EH I 160(3), MO 82

4(1)[9] $(\nu-\mu+1)\,P_{\nu+1}^\mu(z) = (2\nu+1)z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)$ AS 334(8.5.3)

4(2)[9] $P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)\left(z^2-1\right)^{1/2}P_\nu^{\mu-1}(z)$ AS 334(8.5.5)

5. $P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$ (cf. **8.820**, **8.832** 4) EH I 140(1), MO 82

8.732

1. $\left(z^2-1\right)\dfrac{d\,Q_\nu^\mu(z)}{dz} = (\nu-\mu+1)\,Q_{\nu+1}^\mu(z) - (\nu+1)z\,Q_\nu^\mu(z)$ (cf. **8.832** 3) MO 82

2.[10] $(2\nu+1)z\,Q_\nu^\mu(z) = (\nu-\mu+1)\,Q_{\nu+1}^\mu(z) + (\nu+\mu)\,Q_{\nu-1}^\mu(z)$ (cf. **8.832** 4) MO 82

3. $Q_\nu^{\mu+2}(z) + 2(\mu+1)\dfrac{z}{\sqrt{z^2-1}}\,Q_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)\,Q_\nu^\mu(z)$ MO 82

4. $Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu+1)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$ MO 82a

5. $e^{-\mu\pi i}\,Q_\nu^\mu(x\pm i0) = e^{\pm\frac{1}{2}\mu\pi i}\left[Q_\nu^\mu(x) \mp i\dfrac{\pi}{2}\,P_\nu^\mu(x)\right]$ MO 83

8.733

1. $$\begin{aligned}\left(1-x^2\right)\frac{d\,P_\nu^\mu(x)}{dx} &= P_\nu^\mu(x) - (\nu-\mu+1)\,P_{\nu+1}^\mu(x) \qquad \text{(cf. } \mathbf{8.731}\ 1)\\ &= -\nu x\,P_\nu^\mu(x) + (\nu+\mu)\,P_{\nu-1}^\mu(x)\\ &= -\sqrt{1-x^2}\,P_\nu^{\mu+1}(x) - \mu x\,P_\nu^\mu(x);\\ &= (\nu-\mu+1)(\nu+\mu)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x) + \mu x\,P_\nu^\mu(x)\end{aligned}$$ MO 82

2. $(2\nu+1)x\,P_\nu^\mu(x) = (\nu-\mu+1)\,P_{\nu+1}^\mu(x) + (\nu+\mu)\,P_{\nu-1}^\mu(x)$ (cf. **8.731** 2) MO 82

3.[11] $P_\nu^{\mu+2}(x) + 2(\mu+1)\dfrac{x}{\sqrt{1-x^2}}\,P_\nu^{\mu+1}(x) + (\nu-\mu)(\nu+\mu+1)\,P_\nu^\mu(x) = 0$ (cf. **8.731** 3) MO 82

4. $P_{\nu-1}^\mu(x) - P_{\nu+1}^\mu(x) = (2\nu+1)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$ (cf. **8.731** 4) MO 82

5. $P_{-\nu-1}^\mu(x) = P_\nu^\mu(x)$ (cf. **8.731** 5)

8.734

1. $(\nu+\mu+1)z\,Q_\mu^\nu(z)+\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu+1)\,Q_{\nu+1}^\mu(z)$ MO 82

2. $(\nu+\mu)\,Q_{\nu-1}^\mu(z)+\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu)z\,Q_\nu^\mu(z)$ MO 82

3. $Q_{\nu-1}^\mu(z)-z\,Q_\nu^\mu(z)=-(\nu-\mu+1)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$ MO 82

4. $z\,Q_\nu^\mu(z)-Q_{\nu+1}^\mu(z)=-(\nu+\mu)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$ MO 82

5. $(\nu+\mu)(\nu+\mu+1)\,Q_{\nu-1}^\mu(z)+(2\nu+1)\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu)(\nu-\mu+1)\,Q_{\nu+1}^\mu(z)$ MO 82

8.735

1. $(\nu+\mu+1)x\,P_\nu^\mu(x)+\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)=(\nu-\mu+1)\,P_{\nu+1}^\mu(x)$ MO 83

2. $(\nu-\mu)x\,P_\nu^\mu(x)-(\nu+\mu)\,P_{\nu-1}^\mu(x)=\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)$ MO 83

3. $P_{\nu-1}^\mu(x)-x\,P_\nu^\mu(x)=(\nu-\mu+1)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$ MO 83

4. $x\,P_\nu^\mu(x)-P_{\nu+1}^\mu(x)=(\nu+\mu)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$ MO 83

5. $(\nu-\mu)(\nu-\mu+1)\,P_{\nu+1}^\mu(x)=(\nu+\mu)(\nu+\mu+1)\,P_{\nu-1}^\mu(x)+(2\nu+1)\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)$ MO 83

8.736

1. $P_\nu^{-\mu}(z)=\dfrac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z)-\dfrac{2}{\pi}e^{-\mu\pi i}\sin\mu\pi\,Q_\nu^\mu(z)\right]$ MO 83

2. $P_\nu^\mu(-z)=e^{\nu\pi i}\,P_\nu^\mu(z)-\dfrac{2}{\pi}\sin[(\nu+\mu)\pi]e^{-\mu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im}z<0]$ (cf. **8.833** 1) MO 83

3. $P_\nu^\mu(-z)=e^{-\nu\pi i}\,P_\nu^\mu(z)-\dfrac{2}{\pi}\sin[(\nu+\mu)\pi]e^{-\mu\pi i}\,Q_\nu^\mu(z)$
$[\operatorname{Im}z>0]$ (cf. **8.833** 2) MO 83

4. $Q_\nu^{-\mu}(z)=e^{-2\mu\pi i}\dfrac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,Q_\nu^\mu(z)$ MO 82

5. $Q_\nu^\mu(-z)=-e^{-\nu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im}z<0]$ MO 82

6. $Q_\nu^\mu(-z)=-e^{\nu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im}z>0]$ MO 82

7.[6] $Q_\nu^\mu(z)\sin[(\nu+\mu)\pi]-Q_{-\nu-1}^\mu(z)\sin[(\nu-\mu)\pi]=\pi e^{\mu\pi i}\cos\nu\pi\,P_\nu^\mu(z)$ MO 83

8.737

1. $P_\nu^{-\mu}(x)=\dfrac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[\cos\mu\pi\,P_\nu^\mu(x)-\dfrac{2}{\pi}\sin(\mu\pi)\,Q_\nu^\mu(x)\right]$ MO 84

2. $P_\nu^\mu(-x)=\cos[(\nu+\mu)\pi]\,P_\nu^\mu(x)-\dfrac{2}{\pi}\sin[(\nu+\mu)\pi]\,Q_\nu^\mu(x)$ MO 84

3. $Q_\nu^\mu(-x)=-\cos[(\nu+\mu)\pi]\,Q_\nu^\mu(x)-\dfrac{\pi}{2}\sin[(\nu+\mu)\pi]\,P_\nu^\mu(x)$ MO 83, EH I 144(15)

4. $Q_{-\nu-1}^\mu(x)=\dfrac{\sin[(\nu+\mu)\pi]}{\sin[(\nu-\mu)\pi]}\,Q_\nu^\mu(x)-\dfrac{\pi\cos\nu\pi\cos\mu\pi}{\sin[(\nu-\mu)\pi]}\,P_\nu^\mu(x)$ MO 84

8.738

1.[11] $$Q_\nu^\mu(i\cot\varphi)=\exp\left[i\pi\left(\mu-\frac{\nu+1}{2}\right)\right]\sqrt{\pi}\,\Gamma(\nu+\mu+1)\sqrt{\frac{1}{2}\sin\varphi}\,P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi)$$
$$\left[0<\varphi<\frac{\pi}{2}\right]$$ MO 83

2.[6] $$P_\nu^\mu(i\cot\varphi)=\sqrt{\frac{2}{\pi}}\exp\left[i\pi\left(\nu+\frac{1}{4}\right)\right]\frac{\sqrt{\sin\varphi}}{\Gamma(-\nu-\mu)}\,Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi-i0)$$
$$\left[0<\varphi<\frac{\pi}{2}\right]$$ MO 83

8.739 $$e^{-\mu\pi i}\,Q_\nu^\mu(\cosh\alpha)=\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{\sqrt{2\sinh\alpha}}\,P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\coth\alpha)\qquad[\operatorname{Re}(\cosh\alpha)>0]$$ MO 83

8.741

1. $$P_\nu^{-\mu}(x)\frac{d\,P_\nu^\mu(x)}{dx}-P_\nu^\mu(x)\frac{d\,P_\nu^{-\mu}(x)}{dx}=\frac{2\sin\mu\pi}{\pi\,(1-x^2)}$$ MO 83

2. $$P_\nu^\mu(x)\frac{d\,Q_\nu^\mu(x)}{dx}-Q_\nu^\mu(x)\frac{d\,P_\nu^\mu(x)}{dx}=\frac{2^{2\mu}}{1-x^2}\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)}$$ MO 83

8.742

1. $$\frac{\Gamma(\nu-\mu-1)}{\Gamma(\nu+\mu+1)}\left\{\cos\mu\pi\,P_\nu^\mu(\cos\varphi)-\frac{2}{\pi}\sin\mu\pi\,Q_\nu^\mu(\cos\varphi)\right\}=\sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\varphi\frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(\cos t-\cos\varphi)^{\frac{1}{2}-\mu}}$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2}\right]$$ MO 88

2. $$\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left\{\cos\nu\pi\,P_\nu^\mu(\cos\varphi)-\frac{2}{\pi}\sin\nu\pi\,Q_\nu^\mu(\cos\varphi)\right\}$$
$$=\sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_\varphi^\pi\frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]\,dt}{(\cos\varphi-\cos t)^{\frac{1}{2}-\mu}}$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2}\right]$$ MO 88

3. $$P_\nu^\mu(\cos\varphi)\cos(\nu+\mu)\pi-\frac{2}{\pi}\,Q_\nu^\mu(\cos\varphi)\sin(\nu+\mu)\pi=\sqrt{\frac{2}{\pi}}\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\varphi^\pi\frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]\,dt}{(\cos\varphi-\cos t)^{\mu+\frac{1}{2}}}$$
$$\left[\operatorname{Re}\mu<\tfrac{1}{2}\right]$$ MO 88

4. $$\cos\mu\pi\,P_\nu^\mu(\cos\varphi)-\frac{2}{\pi}\sin\mu\pi\,Q_\nu^\mu(\cos\varphi)$$
$$=\frac{1}{2^\mu\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\pi\frac{\sin^{2\mu}t\,dt}{(\cos\varphi\pm i\sin\varphi\cos t)^{\nu-\mu}}$$
$$\left[\operatorname{Re}\mu>-\tfrac{1}{2},\quad 0<\varphi<\pi\right]$$ MO 38

For integrals of Legendre functions, see **7.11**–**7.21**.

8.75 Special cases and particular values

8.751

1. $$P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu+m+1)\left(1-x^2\right)^{\frac{m}{2}}}{2^m\,\Gamma(\nu-m+1)m!}\,F\left(m-\nu, m+\nu+1; m+1; \frac{1-x}{2}\right)$$ MO 84

2. $$P_\nu^m(z) = \frac{\Gamma(\nu+m+1)\left(z^2-1\right)^{\frac{m}{2}}}{2^m m!\,\Gamma(\nu-m+1)}\,F\left(m-\nu, m+\nu+1; m+1; \frac{1-z}{2}\right)$$ MO 84

3.[8] $$Q_{n+\frac{1}{2}}^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma\left(\mu+n+\frac{3}{2}\right)}{2^{n+\frac{3}{2}}(n+1)!}\left(z^2-1\right)^{\frac{\mu}{2}}\pi^{1/2}z^{-n-\mu-3/2}\,F\left(\frac{\mu+n+\frac{5}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}; n+2; \frac{1}{z^2}\right)$$ MO 84

8.752

1. $$P_\nu^m(x) = (-1)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_\nu(x)$$ WH, MO 84, EH I 148(6)

2. $$P_\nu^{-m}(x) = (-1)^m\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}P_\nu^m(x) = \left(1-x^2\right)^{-\frac{m}{2}}\int_x^1\cdots\int_x^1 P_\nu(x)(dx)^m$$ $[m \geq 1]$ HO 99a, MO 85, EH I 149(10)a

3. $$P_\nu^{-m}(z) = \left(z^2-1\right)^{-\frac{m}{2}}\int_1^z\cdots\int_1^z P_\nu(z)(dz)^m$$ $[m \geq 1]$ MO 85, EH I 149(8)

4. $$Q_\nu^m(z) = \left(z^2-1\right)^{\frac{m}{2}}\frac{d^m}{dz^m}Q_\nu(z)$$ WH, MO 85, EH I 148(5)

5. $$Q_\nu^{-m}(z) = (-1)^m\left(z^2-1\right)^{-\frac{m}{2}}\int_z^\infty\cdots\int_z^\infty Q_\nu(z)(dz)^m$$ $[m \geq 1]$ MO 85, EH I 149(9)

Special values of the indices

8.753

1. $$P_0^\mu(\cos\varphi) = \frac{1}{\Gamma(1-\mu)}\cot^\mu\frac{\varphi}{2}$$ MO 84

2. $$P_\nu^{-1}(\cos\varphi) = -\frac{1}{\nu(\nu+1)}\frac{d\,P_\nu(\cos\varphi)}{d\varphi}$$ MO 84

3. $$P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0 \qquad \text{for } m > n$$ MO 85

8.754

1. $$P_{\nu-\frac{1}{2}}^{1/2}(\cosh\alpha) = \sqrt{\frac{2}{\pi\sinh\alpha}}\cosh\nu\alpha$$ MO 85

2. $$P_{\nu-\frac{1}{2}}^{1/2}(\cos\varphi) = \sqrt{\frac{2}{\pi\sin\varphi}}\cos\nu\varphi$$ MO 85

3. $$P_{\nu-\frac{1}{2}}^{-1/2}(\cos\varphi) = \sqrt{\frac{2}{\pi\sin\varphi}}\frac{\sin\nu\varphi}{\nu}$$ MO 85

4. $$Q^{1/2}_{\nu-\frac{1}{2}}(\cosh\alpha) = i\sqrt{\frac{\pi}{2\sinh\alpha}}e^{-\nu\alpha}$$ MO 85

8.755

1. $$P^{-\nu}_{\nu}(\cos\varphi) = \frac{1}{\Gamma(1+\nu)}\left(\frac{\sin\varphi}{2}\right)^{\nu}$$ MO 85

2. $$P^{-\nu}_{\nu}(\cosh\alpha) = \frac{1}{\Gamma(1+\nu)}\left(\frac{\sinh\alpha}{2}\right)^{\nu}$$ MO 85

Special values of Legendre functions

8.756

1. $$P^{\mu}_{\nu}(0) = \frac{2^{\mu}\sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)}$$ MO 84

2. $$\frac{d\,P^{\mu}_{\nu}(0)}{dx} = \frac{2^{\mu+1}\sin\frac{1}{2}(\nu+\mu)\pi\,\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$ MO 84

3. $$Q^{\mu}_{\nu}(0) = -2^{\mu-1}\sqrt{\pi}\sin\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}$$ MO84

4. $$\frac{d\,Q^{\mu}_{\nu}(0)}{dx} = 2^{\mu}\sqrt{\pi}\cos\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$ MO 84

8.76 Derivatives with respect to the order

8.761 $$\frac{\partial P^{-\mu}_{\nu}(x)}{\partial\nu} = \frac{1}{\Gamma(\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}}\sum_{n=1}^{\infty}\frac{(-\nu)(1-\nu)\ldots(n-1-\nu)(\nu+1)(\nu+2)\ldots(\nu+n)}{(\mu+1)(\mu+2)\ldots(\mu+n)1\cdot 2\ldots n}$$
$$\times\left[\psi(\nu+n+1)-\psi(\nu-n+1)\right]\left(\frac{1-x}{2}\right)^{n}$$
$$[\nu\neq 0,\pm1,\pm2,\ldots;\quad \operatorname{Re}\mu>-1]$$ MO 94

8.762

1. $$\left[\frac{\partial\,P_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = 2\ln\cos\frac{\varphi}{2}$$ MO 94

2. $$\left[\frac{\partial P^{-1}_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = -\tan\frac{\varphi}{2} - 2\cot\frac{\varphi}{2}\ln\cos\frac{\varphi}{2}$$ MO 94

3. $$\left[\frac{\partial P^{-1}_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=1} = -\frac{1}{2}\tan\frac{\varphi}{2}\sin^2\frac{\varphi}{2} + \sin\varphi\ln\cos\frac{\varphi}{2}$$ MO 94

- For a connection with the polynomials $C^{\lambda}_{n}(x)$, see **8.936**.
- For a connection with a hypergeometric function, see **8.77**.

8.77 Series representation

For a representation in the form of a series, see **8.721**. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

8.771

1. $$P_\nu^\mu(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$ MO 15

2.[8] $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{z^{\nu+\mu+1}} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$ MO 15

See also **8.702**, **8.703**, **8.704**, **8.723**, **8.751**, **8.772**.

The analytic continuation for $|z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see **9.154** and **9.155**):

8.772

1. $$P_\nu^\mu(z) = \frac{\sin(\nu+\mu)\pi\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi}\cos\nu\pi\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(z^2-1\right)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$
$$+\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)} \left(z^2-1\right)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$
$$[2\nu \neq \pm1, \pm3, \pm5, \ldots; \quad |z| > 1; \quad |\arg(z\pm1)| < \pi]$$ MO 85

2. $$P_\nu^\mu(z) = \frac{\Gamma\left(-\nu-\frac{1}{2}\right)\left(z^2-1\right)^{-\frac{\nu+1}{2}}}{2^{\nu+1}\sqrt{\pi}\,\Gamma(-\nu-\mu)} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)$$
$$+\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)} \left(z^2-1\right)^{\frac{\nu}{2}} F\left(\frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}; \frac{1}{2}-\nu; \frac{1}{1-z^2}\right)$$
$$\left[2\nu \neq \pm1, \pm3, \pm5; \ldots; \quad \left|1-z^2\right| > 1; \quad |\arg(z\pm1)| < \pi\right]$$ MO 85

3. $$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^\nu F\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right)$$
$$\left[\left|\frac{z-1}{z+1}\right| < 1\right]$$ MO 86

8.773

1. $$Q_\nu^\mu(z) = e^{\mu\pi i} \frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(z^2-1\right)^{-\frac{\nu+1}{2}} F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)$$
$$\left[\nu+\mu \neq -1, -2, -3, \ldots; \quad |\arg(z\pm1)| < \pi; \quad \left|1-z^2\right| > 1\right]$$ MO 86

2. $$Q_\nu^\mu(z) = \frac{1}{2} e^{\mu\pi i} \left\{\Gamma(\mu) \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)\right.$$
$$\left. + \frac{\Gamma(-\mu)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1+\mu; \quad \frac{1-z}{2}\right)\right\}$$
$$[|\arg(z\pm1)| < \pi, \quad |1-z| < 2]$$ MO 86

8.774 $$P_\nu^\mu(i\cot\varphi)=\sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}e^{-i(\nu+1)\frac{\pi}{2}}\left(\tan\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}}F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\nu+\frac{3}{2};\sin^2\frac{\varphi}{2}\right)$$
$$+\sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}e^{i\nu\frac{\pi}{2}}\left(\cot\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}}F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\frac{1}{2}-\nu;\sin^2\frac{\varphi}{2}\right)$$
$$\left[2\nu\neq\pm1,\pm3,\pm5,\ldots,\quad 0<\varphi<\frac{\pi}{2}\right]$$ MO 86

8.775

1.[6] $$P_\nu^\mu(x)=\frac{2^\mu\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\left(1-x^2\right)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right)$$
$$+\frac{2^{\mu+1}}{\sqrt{\pi}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}x\left(1-x^2\right)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu}{2}+1,\frac{-\nu+\mu+1}{2};\frac{3}{2};x^2\right)$$
MO 87

2.[6] $$Q_\nu^\mu(x)=-\frac{\sqrt{\pi}}{2^{1-\mu}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\left(1-x^2\right)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right)$$
$$+2^\mu\sqrt{\pi}\frac{\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}x\left(1-x^2\right)^{\frac{\mu}{2}}F\left(\frac{\nu+\mu}{2}+1,\frac{\mu-\nu+1}{2};\frac{3}{2};x^2\right)$$
MO 87

8.776 For $|z|\gg1$

1. $$P_\nu^\mu(z)=\left\{\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}z^\nu+\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\,\Gamma(-\nu-\mu)}z^{-\nu-1}\right\}\left(1+O\left(\frac{1}{z^2}\right)\right)$$
$$[2\nu\neq\pm1,\pm3,\pm5,\ldots,\quad|\arg z|<\pi]$$
MO 87

2. $$Q_\nu^\mu(z)=\sqrt{\pi}\frac{e^{\mu\pi i}}{2^{\nu+1}}\frac{\Gamma(\mu+\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}\left(1+O\left(\frac{1}{z^2}\right)\right)$$
$$[2\nu\neq-3,-5,-7,\ldots;\quad|\arg z|<\pi]$$
MO 87

8.777 Set $\zeta=z+\sqrt{z^2-1}$. The variable ζ is uniquely defined by this equation on the entire z-plane in which a cut is made from $-\infty$ to $+1$. Here, we are considering that branch of the variable ζ for which values of ζ exceeding 1 correspond to real values of z exceeding 1. In this case,

1. $$P_\nu^\mu(z)=\frac{2^\mu\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}}F\left(\frac{1}{2}+\mu,\nu+\mu+1;\nu+\frac{3}{2};\frac{1}{\zeta^2}\right)$$
$$+\frac{2^\mu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\mu-\nu}}F\left(\frac{1}{2}+\mu,\mu-\nu;\frac{1}{2}-\nu;\frac{1}{\zeta^2}\right)$$
$$[2\nu\neq\pm1,\pm3,\pm5,\ldots;\quad|\arg(z-1)|<\pi]$$ MO 86

2. $$Q_\nu^\mu(z)=2^\mu e^{\mu\pi i}\sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}}F\left(\frac{1}{2}+\mu,\nu+\mu+1;\nu+\frac{3}{2};\frac{1}{\zeta^2}\right)$$
$$[|\arg(z-1)|<\pi]$$ MO 86

8.78 The zeros of associated Legendre functions

8.781 The function $P_\nu^{-\mu}(\cos\varphi)$, considered as a function of ν, has infinitely many zeros for $\mu \geq 0$. These are all simple and real. If a number ν_0 is a zero of the function $P_\nu^{-\mu}(\cos\varphi)$, the number $-\nu_0 - 1$ is also a zero of this function. MO 91

8.782 If ν and μ are both real and $\mu \leq 0$, or if ν and μ are integers, the function $P_\nu^\mu(t)$ has no *real* zeros exceeding 1. If ν and μ are both real with $\nu < \mu < 0$, the function $P_\nu^\mu(t)$ has no real zeros exceeding 1 when $\sin\mu\pi\sin(\mu-\nu)\pi > 0$, but does have one such zero when $\sin\mu\pi\sin(\mu-\nu)\pi < 0$. Finally, if $\mu \leq \nu$, the function $P_\nu^\mu(t)$ has no zeros exceeding 1 for $\lfloor\mu\rfloor$ even but does have one zero for $\lfloor\mu\rfloor$ odd.

8.783 If $\nu > -\frac{3}{2}$ and $\nu + \mu + 1 > 0$, the function $Q_\nu^\mu(t)$ has no real zeros exceeding 1. MO 91

8.784 The function $P_{-\frac{1}{2}+i\lambda}(z)$ has infinitely many zeros for real λ. All these zeros are *real* and *greater than unity.*

8.785 For n a natural number, the function $P_n(x)$ has exactly n real zeros which lie in the closed interval $-1, +1$.

8.786 The function $Q_n(z)$ has no zeros for which $|\arg(z-1)| < \pi$ if n is a natural number. The function $Q_n(\cos\varphi)$ has exactly $n+1$ zeros in the interval $0 \leq \varphi \leq \pi$. MO 91

8.787 The following approximate formula can be used to calculate the values of ν for which the equation $P_\nu^{-\mu}(\cos\varphi) = 0$ holds for given small values of φ:

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2\sin\frac{\varphi}{2}}\left\{1 - \frac{\sin^2\frac{\varphi}{2}}{6}\left(1 - \frac{4\mu^2-1}{j_\mu^2}\right) + O\left(\sin^4\frac{\varphi}{2}\right)\right\}. \qquad \text{MO 93}$$

Here, j_μ denotes an arbitrary nonzero root of the equation $J_\mu(z) = 0$ (for $\mu \geq 0$). If φ is close to π then, instead of this formula, we can use the following formulas:

1. $$\nu \approx \mu + k + \frac{\Gamma(2\mu+k+1)}{\Gamma(\mu)\,\Gamma(\mu+1)\,\Gamma(k+1)}\left(\frac{\pi-\varphi}{3}\right)^{2\mu} \qquad [\mu > 0, \quad k = 0, 1, 2, \ldots] \qquad \text{MO 93}$$

2. $$\nu \approx k + \frac{1}{2\ln\left(\frac{2}{\pi-\varphi}\right)} \qquad [\mu = 0, \quad k = 0, 1, 2, \ldots] \qquad \text{MO 93}$$

8.79 Series of associated Legendre functions

8.791

1. $$\frac{1}{z-t} = \sum_{k=0}^{\infty}(2k+1)\,P_k(t)\,Q_k(z) \qquad \left[\left|t+\sqrt{t^2-1}\right| < \left|z+\sqrt{z^2-1}\right|\right]$$

   Here, t must lie inside an ellipse passing through the point z with foci at the points ± 1.

2. $$\frac{1}{\sqrt{1-2tz+t^2}}\ln\frac{z-t+\sqrt{1-2tz+t^2}}{\sqrt{z^2-1}} = \sum_{k=0}^{\infty} t^k\,Q_k(z)$$

   $$[\operatorname{Re} z > 1, \quad |t| < 1] \qquad \text{MO 78}$$

8.792 $$P_\nu^{-\alpha}(\cos\varphi)\,P_\nu^{-\beta}(\cos\psi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right]P_k^{-\alpha}(\cos\varphi)\,P_k^{-\beta}(\cos\psi)$$

$$[a \geq 0, \quad \beta \geq 0, \quad \nu \text{ real}, \quad -\pi < \varphi \pm \psi < \pi] \qquad \text{MO 94}$$

8.793 $$P_\nu^{-\mu}(\cos\varphi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k}-\frac{1}{\nu+k+1}\right)P_k^{-\mu}(\cos\varphi) \qquad [\mu\geq 0,\quad 0<\varphi<\pi]$$

MO 94

Addition theorems

8.794

1.[11] $$P_\nu(\cos\psi_1\cos\psi_2+\sin\psi_1\sin\psi_2\cos\varphi)$$
$$= P_\nu(\cos\psi_1)\,P_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)\,P_\nu^k(\cos\psi_2)\cos k\varphi$$
$$= P_\nu(\cos\psi_1)\,P_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}P_\nu^k(\cos\psi_1)\,P_\nu^k(\cos\psi_2)\cos k\varphi$$
$$[0\leq\psi_1<\pi,\quad 0\leq\psi_2<\pi,\quad \psi_1+\psi_2<\pi,\quad \varphi\text{ real}] \qquad (\text{cf. }\mathbf{8.814},\ \mathbf{8.844}\ 1)$$

MO 90

2. $$Q_\nu(\cos\psi_1)\cos\psi_2+\sin\psi_1\sin\psi_2\cos\varphi$$
$$= P_\nu(\cos\psi_1)\,Q_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)\,Q_\nu\,k(\cos\psi_2)\cos k\varphi$$
$$\left[0<\psi_1<\frac{\pi}{2},\quad 0<\psi_2<\pi,\quad 0<\psi_1+\psi_2<\pi;\quad \varphi\text{ real}\right] \qquad (\text{cf. }\mathbf{8.844}\ 3)$$

MO 90

8.795

1. $$P_\nu\left(z_1z_2-\sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\varphi\right) = P_\nu(z_1)\,P_\nu(z_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^k(z_1)\,P_\nu^{-k}(z_2)\cos k\varphi$$
$$[\operatorname{Re}z_1>0,\quad \operatorname{Re}z_2>0,\quad |\arg(z_1-1)|<\pi,\quad |\arg(z_2-1)|<\pi]$$

MO 91

2. $$Q_\nu\left(x_1x_2-\sqrt{x_1^2-1}\sqrt{x_2^2-1}\cos\varphi\right) = P_\nu(x_1)\,Q_\nu(x_2)+2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(x_1)\,Q_\nu^k(x_2)\cos k\varphi$$
$$[1<x_1<x_2,\quad \nu\neq -1,-2,-3,\ldots,\quad \varphi\text{ real}]$$

MO 91

3. $$Q_n\left(x_1x_2+\sqrt{x_1^2+1}\sqrt{x_2^2+1}\cosh\alpha\right) = \sum_{k=n+1}^{\infty}\frac{1}{(k-n-1)!(k+n)!}\,Q_n^k(ix_1)\,Q_n^k(ix_2)\,e^{-k\alpha}$$
$$[x_1>0,\quad x_2>0,\quad \alpha>0]$$

MO 91

8.796 $$P_\nu(-\cos\psi_1\cos\psi_2-\sin\psi_1\sin\psi_2\cos\varphi) = P_\nu(-\cos\psi_1)\,P_\nu(\cos\psi_2)+2\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)}$$
$$\times P_\nu^{-k}(-\cos\psi_1)\,P_\nu^{-k}(\cos\psi_2)\cos k\varphi$$
$$[0<\psi_2<\psi_1<\pi,\quad \varphi\text{ real}] \qquad (\text{cf. }\mathbf{8.844}\ 2)$$

MO 91

See also **8.934** 3.

8.81 Associated Legendre functions with integer indices

8.810 For *integer* values of ν and μ, the differential equation **8.700** 1. (with $|\nu| > |\mu|$) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions $P_n^m(x)$ are called *associated Legendre functions* (or *spherical functions*) *of the first kind.* The number n is called the *degree*, and the number m is called the *order* of the function $P_n^m(x)$. The functions $\{\cos m\vartheta\, P_n^m(\cos\varphi)\,, \quad \sin m\vartheta\, P_n^m(\cos\varphi)\}$, which depend on the angles φ and ϑ, are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for $m < n$ and *sectoral harmonics* for $m = n$. These last functions are periodic with respect to the angles φ and ϑ. Their periods are, respectively, π and 2π. They are single-valued and continuous everywhere on the surface of the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (where $x_1 = \sin\varphi\cos\vartheta$, $x_2 = \sin\varphi\sin\vartheta$, $x_3 = \cos\varphi$), and they are solutions of the differential equation

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\vartheta^2} + n(n+1)Y = 0.$$

8.811 The integral representation

$$P_n^m(\cos\varphi) = \frac{(-1)^m(n+m)!}{\Gamma\left(m+\frac{1}{2}\right)(n-m)!}\sqrt{\frac{2}{\pi}}\sin^{-m}\varphi\int_0^\varphi (\cos t - \cos\varphi)^{m-\frac{1}{2}}\cos\left(n+\tfrac{1}{2}\right)t\,dt \qquad \text{MO 75}$$

8.812 The series representation:

$$\begin{aligned} P_n^m(x) &= \frac{(-1)^m(n+m)!}{2^m m!(n-m)!}\left(1-x^2\right)^{\frac{m}{2}}\left\{1 - \frac{(n-m)(m+n+1)}{1!(m+1)}\frac{1-x}{2}\right. \\ &\quad \left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)}\left(\frac{1-x}{2}\right)^2 - \dots\right\} && \text{MO 73} \\ &= \frac{(-1)^m(2n-1)!!}{(n-m)!}\left(1-x^2\right)^{\frac{m}{2}}\left\{x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}x^{n-m-2}\right. \\ &\quad \left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-m-4} - \dots\right\} && \text{MO 73} \\ &= \frac{(-1)^m(2n-1)!!}{(n-m)!}\left(1-x^2\right)^{\frac{m}{2}} x^{n-m}\, F\left(\frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right) && \text{MO 73} \end{aligned}$$

8.813 Special cases:

1. $P_1^1(x) = -\left(1-x^2\right)^{1/2} = -\sin\varphi$ MO 73
2. $P_2^1(x) = -3\left(1-x^2\right)^{1/2} x = -\frac{3}{2}\sin 2\varphi$ MO 73
3. $P_2^2(x) = 3\left(1-x^2\right) = \frac{3}{2}(1-\cos 2\varphi)$ MO 73
4. $P_3^1(x) = -\frac{3}{2}\left(1-x^2\right)^{1/2}\left(5x^2-1\right) = -\frac{3}{8}(\sin\varphi + 5\sin 3\varphi)$ MO 73
5. $P_3^2(x) = 15\left(1-x^2\right)x = \frac{15}{4}(\cos\varphi - \cos 3\varphi)$ MO 73
6. $P_3^3(x) = -15\left(1-x^2\right)^{3/2} = -\frac{15}{4}(3\sin\varphi - \sin 3\varphi)$ MO 73

Functional relations

For recursion formulas, see **8.731**.

8.814 $P_n(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2\cos\Theta)$

$$= P_n(\cos\varphi_1)\,P_n(\cos\varphi_2)+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\varphi_1)\,P_n^m(\cos\varphi_2)\cos m\Theta$$

$$[0\le\varphi_1\le\pi,\quad 0\le\varphi_2\le\pi]\qquad(\text{“addition theorem”})$$ MO 74

8.815 If

$$Y_{n_1}(\varphi,\vartheta)=A_0\,P_{n_1}(\cos\varphi)+\sum_{m=1}^{n_1}(a_m\cos m\vartheta+b_m\sin m\vartheta)\,P_{n_1}^m(\cos\varphi)\,,$$

$$Z_{n_2}(\varphi,\vartheta)=\alpha_0\,P_{n_2}(\cos\varphi)+\sum_{m=1}^{n_2}(\alpha_m\cos m\vartheta+\beta_m\sin m\vartheta)\,P_{n_2}^m(\cos\varphi)\,,$$

then

$$\int_0^{2\pi}d\vartheta\int_0^{\pi}\sin\varphi\,d\varphi\;Y_{n_1}(\varphi,\vartheta)\;Y_{n_2}(\varphi,\vartheta)=0,$$

$$\int_0^{2\pi}d\vartheta\int_0^{\pi}\sin\varphi\,d\varphi\;Y_n(\varphi,\vartheta)\,P_n\left[\cos\varphi\cos\psi+\sin\varphi\sin\psi\cos(\vartheta-\theta)\right]=\frac{4\pi}{2n+1}\;Y_n(\psi,\theta)$$ MO 75

8.816 $(\cos\varphi+i\sin\varphi\cos\vartheta)^n=P_n(\cos\varphi)+2\sum_{m=1}^{n}(-1)^m\frac{n!}{(n+m)!}\cos m\vartheta\,P_n^m(\cos\varphi)$ MO 75

For integrals of the functions, $P_n^m(x)$, see **7.112** 1, **7.122** 1.

8.82–8.83 Legendre functions

8.820 The differential equation

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{du}{dz}\right]+\nu(\nu+1)u=0\qquad(\text{cf. }\mathbf{8.700}\ 1),$$

where the parameter ν can be an arbitrary number, has the following two linearly independent solutions:

1. $P_\nu(z)=F\left(-\nu,\nu+1;1;\dfrac{1-z}{2}\right)$

2. $Q_\nu(z)=\dfrac{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}\,F\left(\dfrac{\nu+2}{2},\dfrac{\nu+1}{2};\dfrac{2\nu+3}{2};\dfrac{1}{z^2}\right)$ SM 518(137)

   The functions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first* and *second kind* respectively. If ν is not an integer, the function $P_\nu(z)$ has *singularities* at $z=-1$ and $z=\infty$. However, if $\nu=n=0,1,2,\ldots$, the function $P_\nu(z)$ becomes the *Legendre polynomial* $P_n(z)$ (see **8.91**) For $\nu=-n=-1,-2,\ldots$, we have

   $$P_{-n-1}(z)=P_n(z).$$

3. If $\nu\ne0,1,2,\ldots$, the function $Q_\nu(z)$ has singularities at the points $z=\pm1$ and $z=\infty$. These points are branch points of the function. On the other hand, if $\nu=n=0,1,2,\ldots$, the function $Q_n(z)$ is single-valued for $|z|>1$ and regular for $z=\infty$.

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \qquad [\operatorname{Re} z > 0]$$

5. The function $P_\nu(z)$ is uniquely determined by equations **8.820** 1 and **8.820** 4 within a circle of radius 2 with its center at the point $z = 1$ in the right half-plane.

For $z = x = \cos\varphi$, a solution of equation **8.820** is the function

6. $P_\nu(x) = P_\nu(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\frac{\varphi}{2}\right);$

In general,

7. $P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x),$ for $z = x$

8. The function $Q_\nu(z)$ for $|z| > 1$ is uniquely determined by equation **8.820** 2 everywhere in the z-plane in which a cut is made from the point $z = -\infty$ to the point $z = 1$. By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut $(-1 \le x \le +1)$ of the real axis, the function $Q_\nu(x)$ is determined by the equation

9. $Q_\nu(x) = \frac{1}{2}\left[Q_\nu(x+i0) + Q_\nu(x-i0)\right]$ HO 52(53), WH

Integral representations

8.821

1. $$P_\nu(z) = \frac{1}{2\pi i}\int_A^{(1+,z+)} \frac{\left(t^2-1\right)^\nu}{2^\nu (t-z)^{\nu+1}}\,dt$$

Here, A is a point on the real axis to the right of the point $t = 1$ and to the right of z if z is real. At the point A, we set

$$\arg(t-1) = \arg(t+1) = 0 \text{ and } [|\arg(t-z)| < \pi]$$ WH

2. $$Q_\nu(z) = \frac{1}{4i\sin\nu\pi}\int_A^{(1-,1+)} \frac{\left(t^2-1\right)^\nu}{2^\nu (z-t)^{\nu+1}}\,dt$$

[ν is not an integer; the point A is at the end of the major axis of an ellipse to the right of $t = 1$ drawn in the t-plane with foci at the points ± 1 and with a minor axis sufficiently small that the point z lies outside it. The contour begins at the point A, follows the path $(1-, -1+)$, and returns to A; $|\arg z| \le \pi$ and $|\arg(z-t)| \to \arg z$ as $t \to 0$ on the contour; $\arg(t+1) = \arg(t-1) = 0$ at the point A; z does not lie on the real axis between -1 and 1.]

For $\nu = n$ an integer,

3. $$Q_n(z) = \frac{1}{2^{n+1}}\int_{-1}^{1} \left(1-t^2\right)^n (z-t)^{-n-1}\,dt$$ SM 517(134), WH

8.822

1. $$P_\nu(z) = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{\left(z+\sqrt{z^2-1}\cos\varphi\right)^{\nu+1}} = \frac{1}{\pi}\int_0^\pi \left(z+\sqrt{z^2-1}\cos\varphi\right)^\nu d\varphi$$

$$\left[\operatorname{Re} z > 0 \text{ and } \arg\left\{z+\sqrt{z^2-1}\cos\varphi\right\} = \arg z \text{ for } \varphi = \frac{\pi}{2}\right]$$ WH

2. $$Q_\nu(z) = \int_0^\infty \frac{d\varphi}{\left(z+\sqrt{z^2-1}\cosh\varphi\right)^{\nu+1}},$$
$$\left[\operatorname{Re}\nu > -1; \quad \text{if } \nu \text{ is not an integer}, \left\{\left(z+\sqrt{z^2-1}\right)\cosh\varphi\right\} \text{ for } \varphi = 0 \text{ has its principal value}\right]$$
WH

8.823 $$P_\nu(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos\left(\nu+\frac{1}{2}\right)\varphi}{\sqrt{2(\cos\varphi-\cos\theta)}}\,d\varphi$$ WH

8.824 $$Q_n(z) = 2^n n!\int_z^\infty \ldots \int_z^\infty \frac{(dz)^{n+1}}{(z^2-1)^{n+1}} = 2^n\int_z^\infty \frac{(t-z)^n}{(t^2-1)^{n+1}}\,dt$$
$$= \frac{(-1)^n}{(2n-1)!!}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\int_z^\infty \frac{dt}{(t^2-1)^{n+1}}\right] \qquad [\operatorname{Re} z > 1]$$
WH, MO 78

8.825 $$Q_n(z) = \frac{1}{2}\int_{-1}^1 \frac{P_n(t)}{z-t}\,dt \qquad [|\arg(z-1)| < \pi]$$ WH, MO 78

See also **6.622** 3, **8.842**.

8.826 Fourier series:

1. $$P_n(\cos\varphi) = \frac{2^{n+2}}{\pi}\frac{n!}{(2n+1)!!}\left[\sin(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\sin(n+3)\varphi + \frac{1\cdot 3(n+1)(n+2)}{1\cdot 2(2n+3)(2n+5)}\sin(n+5)\varphi + \ldots\right]$$
$[0 < \varphi < \pi]$ MO 79

2. $$Q_n(\cos\varphi) = 2^{n+1}\frac{n!}{(2n+1)!!}\left[\cos(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\cos(n+3)\varphi + \frac{1\cdot 3}{1\cdot 2}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\cos(n+5)\varphi + \ldots\right]$$
$[0 < \varphi < \pi]$ MO 79

The expressions for Legendre functions in terms of a hypergeometric function (see **8.820**) provide other series representations of these functions.

Special cases and particular values

8.827

1. $$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x} = \operatorname{arctanh} x$$ JA

2. $$Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1$$ JA

3. $$Q_2(x) = \frac{1}{4}\left(3x^2-1\right)\ln\frac{1+x}{1-x} - \frac{3}{2}x$$ JA

4. $$Q_3(x) = \frac{1}{4}\left(5x^3-3x\right)\ln\frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}$$ JA

5. $Q_4(x) = \frac{1}{16}\left(35x^4 - 30x^2 + 3\right) \ln \frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x$ JA

6. $Q_5(x) = \frac{1}{16}\left(63x^5 - 70x^3 + 15x\right) \ln \frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}$ JA

8.828

1. $P_\nu(1) = 1$ MO 79

2. $P_\nu(0) = -\frac{1}{2}\frac{\sin \nu\pi}{\sqrt{\pi^3}}\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)$ MO 79

8.829 $Q_\nu(0) = \frac{1}{4\sqrt{\pi}}\,(1-\cos\nu\pi)\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)$ MO 79

Functional relationships

8.831

1. $Q_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left[\cos\nu\pi\, P_\nu(x) - P_\nu(-x)\right]$ $[\nu \neq 0, \quad \pm1, \pm2, \ldots]$ MO 76

2. $Q_n(x) = \frac{1}{2}\,P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x)$ $[n = 0, 1, 2, \ldots]$,

where

3. $W_{n-1}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2(n-2k)-1}{(2k+1)(n-k)}\,P_{n-2k-1}(x) = \sum_{k=1}^{n} \frac{1}{k}\,P_{k-1}(x)\,P_{n-k}(x)$

and

4. $W_{-1}(x) \equiv 0$ (see also **8.839**) SM 516(131), MO 76

5. $\sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right) P_k(\cos\varphi) = \frac{\pi}{\sin\nu\pi}\,P_\nu(\cos\varphi)$

$[\nu$ not an integer; $\quad 0 \le \varphi < \pi]$ MO 77

6. $\sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right) P_k(\cos\varphi)\,P_k(\cos\psi) = \frac{\pi}{\sin\nu\pi}\,P_\nu(\cos\varphi)\,P_\nu(\cos\psi)$

$[\nu$ not an integer, $\quad -\pi < \varphi+\psi < \pi, \quad -\pi < \varphi-\psi < \pi]$ MO 77

See also **8.521** 4.

8.832

1. $\left(z^2-1\right)\frac{d}{dz}\,P_\nu(z) = (\nu+1)\left[P_{\nu+1}(z) - z\,P_\nu(z)\right]$ WH

2. $(2\nu+1)z\,P_\nu(z) = (\nu+1)\,P_{\nu+1}(z) + \nu\,P_{\nu-1}(z)$ WH

3. $\left(z^2-1\right)\frac{d}{dz}\,Q_\nu(z) = (\nu+1)\left[Q_{\nu+1}(z) - z\,Q_\nu(z)\right]$ WH

4. $(2\nu+1)z\,Q_\nu(z)\,Q_{\nu+1}(z) + \nu\,Q_{\nu-1}(z)$ WH

8.833

1. $P_\nu(-z) = e^{\nu\pi i}\, P_\nu(z) - \dfrac{2}{\pi}\sin\nu\pi\, Q_\nu(z)$ $[\operatorname{Im} z < 0]$ MO 77

2. $P_\nu(-z) = e^{-\nu\pi i}\, P_\nu(z) - \dfrac{2}{\pi}\sin\nu\pi\, Q_\nu(z)$ $[\operatorname{Im} z > 0]$ MO 77

3. $Q_\nu(-z) = -e^{-\nu\pi i}\, Q_\nu(z)$ $[\operatorname{Im} z < 0]$ MO 77

4. $Q_\nu(-z) = -e^{\nu\pi i}\, Q_\nu(z)$ $[\operatorname{Im} z > 0]$ MO 77

8.834

1. $Q_\nu\,(x \pm i0) = Q_\nu(x) \mp \dfrac{\pi i}{2}\, P_\nu(x)$ MO 77

2. $Q_n(z) = \dfrac{1}{2}\, P_n(z) \ln\dfrac{z+1}{z-1} - W_{n-1}(z)$ (see **8.831** 3) MO 77

8.835

1. $Q_\nu(z) - Q_{-\nu-1}(z) = \pi\cot\nu\pi\, P_\nu(z)$ $[\sin\nu\pi \neq 0]$ MO 77

2. $Q_{-\nu-1}\,(\cos\varphi) = Q_\nu\,(\cos\varphi) - \pi\cot\nu\pi\, P_\nu\,(\cos\varphi)$ $[\sin\nu\pi \neq 0]$ MO 77

3. $Q_\nu\,(-\cos\varphi) = -\cos\nu\pi\, Q_\nu\,(\cos\varphi) - \dfrac{\pi}{2}\sin\nu\pi\, P_\nu\,(\cos\varphi)$ MO 77

8.836

1. $Q_n(z) = \dfrac{1}{2^n n!}\dfrac{d^n}{dz^n}\left[\left(z^2-1\right)^n \ln\dfrac{z+1}{z-1}\right] - \dfrac{1}{2}\, P_n(z)\ln\dfrac{z+1}{z-1}$ MO 79

2. $Q_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[\left(x^2-1\right)^n \ln\dfrac{1+x}{1-x}\right] - \dfrac{1}{2}\, P_n(x)\ln\dfrac{1+x}{1-x}$ MO 79

8.837

1. $P_\nu(x) = P_\nu\,(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\dfrac{\varphi}{2}\right)$ (cf. **8.820** 6) MO 76

2. $$P_\nu(z) = \frac{\tan\nu\pi}{2^{\nu+1}\sqrt{\pi}}\frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1}\, F\left(\frac{\nu}{2}+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) + \frac{2^\nu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} z^\nu\, F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$ MO 78

See also **8.820**.

For integrals of Legendre functions, see **7.1**–**7.2**.

8.838 Inequalities ($0 \le \varphi \le \pi$, $\nu > 1$, and C_0 is a number that does not depend on the values of ν or φ):

1. $\left|P_\nu\,(\cos\varphi) - P_{\nu+2}\,(\cos\varphi)\right| \le 2C_0\sqrt{\dfrac{1}{\nu\pi}}$ MO 78

2. $\left|Q_\nu\,(\cos\varphi) - Q_{\nu+2}\,(\cos\varphi)\right| < C_0\sqrt{\dfrac{\pi}{\nu}}$ MO 78

With regard to the zeros of Legendre functions of the second kind, see **8.784**, **8.785**, and **8.786**. For the expansion of Legendre functions in series of associated Legendre functions, see **8.794**, **8.795**, and **8.796**.

8.839 A differential equation leading to the functions W_{n-1} (see **8.831** 3):

$$\left(1-x^2\right)\frac{d^2W_{n-1}}{dx^2}-2x\frac{dW_{n-1}}{dx}+(n+1)nW_{n-1}=2\frac{d\,P_\nu}{dx}$$ MO 76

8.84 Conical functions

8.840 Let us set

$$\nu=-\tfrac{1}{2}+i\lambda,$$

where λ is a real parameter, in the defining differential equation **8.700** 1 for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x),\quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class—the class of conical functions. The most important of these peculiarities is the following

8.841 The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos\varphi)=1+\frac{4\lambda^2+1^2}{2^2}\sin^2\frac{\varphi}{2}+\frac{\left(4\lambda^2+1^2\right)\left(4\lambda^2+3^2\right)}{2^2 4^2}\sin^4\frac{\varphi}{2}+\dots$$

are real for real values of φ. Also,

$$P_{-\frac{1}{2}+i\lambda}(x)\equiv P_{-\frac{1}{2}-i\lambda}(x)$$ MO 95

8.842 Integral representations:

1. $$P_{-\frac{1}{2}+i\lambda}(\cos\varphi)=\frac{2}{\pi}\int_0^\varphi\frac{\cosh\lambda u\,du}{\sqrt{2\left(\cos u-\cos\varphi\right)}}=\frac{2}{\pi}\cosh\lambda\pi\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cos\varphi+\cosh u\right)}}$$ MO 95

2.[6] $$Q_{-\frac{1}{2}\mp\lambda i}(\cos\varphi)=\pm i\sinh\lambda\pi\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cosh u+\cos\varphi\right)}}+\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cosh u-\cos\varphi\right)}}$$ MO 95

Functional relations

(See also **8.73**)

8.843 $$P_{-\frac{1}{2}+i\lambda}(-\cos\varphi)=\frac{\cosh\lambda\pi}{\pi}\left[Q_{-\frac{1}{2}+i\lambda}(\cos\varphi)+Q_{-\frac{1}{2}-i\lambda}(\cos\varphi)\right]$$ MO 95

8.844

1. $$P_{-\frac{1}{2}+i\lambda}(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi)$$
$$=P_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P_{-\frac{1}{2}+i\lambda}(\cos\vartheta)+2\sum_{k=1}^\infty\frac{(-1)^k2^{2k}\,P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P^k_{-\frac{1}{2}+i\lambda}(\cos\vartheta)\cos k\varphi}{(4\lambda^2+1^2)(4\lambda^2+3^2)\cdots[4\lambda^2+(2k-1)^2]}$$
$$\left[0<\vartheta<\frac{\pi}{2},\quad 0<\psi<\pi,\quad 0<\psi+\vartheta<\pi\right]\qquad\text{(cf. }\mathbf{8.794}\text{ 1)}$$ MO 95

2. $$P_{-\frac{1}{2}+i\lambda}(-\cos\psi\cos\vartheta-\sin\psi\sin\vartheta\cos\varphi)$$
$$=P_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P_{-\frac{1}{2}+i\lambda}(-\cos\vartheta)+2\sum_{k=1}^\infty\frac{(-1)^k2^{2k}\,P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P^k_{-\frac{1}{2}+i\lambda}(-\cos\vartheta)\cos k\varphi}{(4\lambda^2+1)(4\lambda^2+3^2)\cdots[4\lambda^2+(2k-1)^2]}$$
$$\left[0<\psi<\frac{\pi}{2}<\vartheta,\quad \psi+\vartheta<\pi\right]\qquad\text{(cf. }\mathbf{8.796}\text{)}$$ MO 95

3. $$Q_{-\frac{1}{2}+i\lambda}\left(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi\right)$$
$$=P_{-\frac{1}{2}+i\lambda}\left(\cos\psi\right)Q_{-\frac{1}{2}+i\lambda}\left(\cos\vartheta\right)+2\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k}\,P^k_{-\frac{1}{2}+i\lambda}\left(\cos\psi\right)Q^k_{-\frac{1}{2}+i\lambda}\left(\cos\vartheta\right)\cos k\varphi}{\left(4\lambda^2+1\right)\left(4\lambda^2+3^2\right)\cdots\left[4\lambda^2+\left(2k-1\right)^2\right]}$$
$$\left[0<\psi<\frac{\pi}{2}<\vartheta,\quad \psi+\vartheta<\pi\right]\qquad \text{(cf. } \mathbf{8.794}\ 2)$$ MO 96

Regarding the zeros of conical functions, see **8.784**.

8.85 Toroidal functions

8.850 Solutions of the differential equation

1. $$\frac{d^2u}{d\eta^2}+\frac{\cosh\eta}{\sinh\eta}\frac{du}{d\eta}-\left(n^2-\frac{1}{4}+\frac{m^2}{\sinh^2\eta}\right)u=0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions

$$P^m_{n-\frac{1}{2}}\left(\cosh\eta\right),\qquad Q^m_{n-\frac{1}{2}}\left(\sinh\eta\right)$$ MO 96

are solutions of equation **8.850** 1.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

8.851 Integral representations:

1. $$P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=\frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\frac{\left(\sinh\eta\right)^m}{2^m\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)}\int_0^{\pi}\frac{\sin^{2m}\varphi\,d\varphi}{\left(\cosh\eta+\sinh\eta\cos\varphi\right)^{n+m+\frac{1}{2}}}$$
$$=\frac{(-1)^m}{2\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\int_0^{2\pi}\frac{\cos m\varphi\,d\varphi}{\left(\cosh\eta+\sinh\eta\cos\varphi\right)^{n+\frac{1}{2}}}$$
MO 96

2. $$Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=(-1)^m\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\int_0^{\infty}\frac{\cosh mt\,dt}{\left(\cosh\eta+\sinh\eta\cosh t\right)^{n+\frac{1}{2}}}\qquad [n\geq m]$$
$$=(-1)^m\frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\ln\coth\frac{\eta}{2}}\left(\cosh\eta-\sinh\eta\cosh t\right)^{n-\frac{1}{2}}\cosh mt\,dt$$
MO 96

8.852 Functional relations:

1. $$Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=(-1)^m\frac{2^m\,\Gamma\left(n+m+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)}\sinh^m\left(\eta e^{-\left(n+m+\frac{1}{2}\right)\eta}\right)$$
$$\times F\left(m+\tfrac{1}{2},n+m+\tfrac{1}{2};n+1;e^{-2\eta}\right)$$
MO 96

*Sometimes called *torus functions*

2. $$P_{n-\frac{1}{2}}^{-m}(\cosh\eta) = \frac{2^{-2m}}{\Gamma(m+1)}\left(1-e^{-2\eta}\right)^m e^{-\left(n+\frac{1}{2}\right)\eta} F\left(m+\tfrac{1}{2}, n+m+\tfrac{1}{2}; 2m+1; 1-e^{-2\eta}\right)$$

MO 96

8.853 An asymptotic representation $P_{n-\frac{1}{2}}(\cosh\eta)$ for large values of n:

$$P_{n-\frac{1}{2}}(\cosh\eta) = \frac{\Gamma(n)e^{\left(n-\frac{1}{2}\right)\eta}}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \times \left[\frac{2\,\Gamma^2\left(n+\frac{1}{2}\right)}{\pi n!\,\Gamma(n)}\ln\left(4e^{\eta}\right)e^{-2n\eta}F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; e^{-2\eta}\right) + A + B\right],$$

where

$$A = 1 + \frac{1}{2^2}\frac{1\cdot(2n-1)}{1\cdot(n-1)}e^{-2\eta} + \frac{1}{2^4}\frac{1\cdot 3\cdot(2n-1)(2n-3)}{1\cdot 2\cdot(n-1)(n-2)}e^{-4\eta} + \cdots + \frac{1}{2^{2n-2}}\left(\frac{(2n-1)!!}{(n-1)!}\right)^2 e^{-2(n-1)\eta}$$

$$B = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi^3}\,\Gamma(n)}\sum_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(n+k+\frac{1}{2}\right)}{\Gamma(n+k+1)\,\Gamma(k+1)}\left(u_{n+k}+u_k-v_{n+k-\frac{1}{2}}-v_{k-\frac{1}{2}}\right)e^{-2(n+k)\eta}$$

Here,

$$u_r = \sum_{s=1}^{r}\frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^{r}\frac{2}{2s-1} \qquad [r \text{ is a natural number}]$$

MO 97

8.9 Orthogonal Polynomials

8.90 Introduction

8.901 Suppose that $w(x)$ is a nonnegative real function of a real variable x. Let (a,b) be a fixed interval on the x-axis. Let us suppose further that, for $n = 0, 1, 2, \ldots$, the integral

$$\int_a^b x^n w(x)\,dx$$

exists and that the integral

$$\int_a^b w(x)\,dx$$

is positive. In this case, there exists a sequence of polynomials $p_0(x), p_1(x), \ldots, p_n(x), \ldots$, that is uniquely determined by the following conditions:

1. $p_n(x)$ is a polynomial of degree n and the coefficient of x^n in this polynomial is positive.
2. The polynomials $p_0(x), p_1(x), \ldots$ are orthonormal; that is,

$$\int_a^b p_n(x)p_m(x)w(x)\,dx = \begin{cases} 0 & \text{for } n \neq m, \\ 1 & \text{for } n = m. \end{cases}$$

We say that the polynomials $p_n(x)$ constitute *a system of orthogonal polynomials on the interval* (a,b) *with the weight function* $w(x)$.

8.902 If q_n is the coefficient of x^n in the polynomial $p_n(x)$, then

1. $$\sum_{k=0}^{n} p_k(x)p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \qquad \text{(Darboux–Christoffel formula)}$$

EH II 159(10)

2.[11] $$\sum_{k=0}^{n} [p_k(x)]^2 = \frac{q_n}{q_{n+1}} \left[p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)\right]$$ EH II 159(11)

8.903 Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n)\, p_{n-1}(x) - C_n p_{n-2}(x) \qquad [n = 2, 3, 4, \ldots]$$

In this formula, A_n, B_n, and C_n are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2}$$ MO 102

8.904 Examples of normalized systems of orthogonal polynomials:

Notation and name		Interval	Weight
$\left(n+\frac{1}{2}\right)^{1/2} P_n(x)$	see **8.91**	$(-1,+1)$	1
$2^\lambda\,\Gamma(\lambda)\left[\dfrac{(n+\lambda)\,n!}{2\pi\,\Gamma(2\lambda+n)}\right]^{1/2} C_n^\lambda(x)$	see **8.93**	$(-1,+1)$	$\left(1-x^2\right)^{\lambda-\frac{1}{2}}$
$\sqrt{\dfrac{\varepsilon_n}{\pi}}\, T_n(x), \quad \varepsilon_0 = 1, \varepsilon_n = 2$ for $n = 1, 2, 3, \ldots$	see **8.94**	$(-1,+1)$	$\left(1-x^2\right)^{-1/2}$
$2^{-\frac{n}{2}}\pi^{-1/4}\,(n!)^{-1/2}\, H_n(x)$	see **8.95**	$(-\infty,\infty)$	e^{-x^2}
$\left[\dfrac{\Gamma(n+1)\,\Gamma(\alpha+\beta+1+n)(\alpha+\beta+1+2n)}{\Gamma(\alpha+1+n)\,\Gamma(\beta+1+n)2^{\alpha+\beta+1}}\right]^{1/2} P_n^{(\alpha,\beta)}(x)$	see **8.96**	$(-1,+1)$	$(1-x)^\alpha(1+x)^\beta$
$\left[\dfrac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right]^{1/2} (-1)^n\, L_n^\alpha(x)$	see **8.97**	$(0,\infty)$	$x^\alpha e^{-x}$

Cf. **7.221** 1, **7.313**, **7.343**, **7.374** 1, **7.391** 1, **7.414** 3.

8.91 Legendre polynomials

8.910 Definition. The Legendre polynomials $P_n(z)$ are polynomials satisfying equation **8.700** 1 with $\mu = 0$ and $\nu = n$: that is, they satisfy the differential equation

1. $$\left(1-z^2\right)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + n(n+1)u = 0$$

This equation has a polynomial solution if, and only if, n is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree n are of the form

2. $$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left(z^2-1\right)^n$$

8.911 Legendre polynomials written in expanded form:

1. $$P_n(z) = \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k}$$
$$= \frac{(2n)!}{2^n\,(n!)^2}\left(z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} z^{n-4} - \dots\right)$$
$$= \frac{(2n-1)!!}{n!} z^n\, F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right)$$

HO 13, AD (9001), MO 69

2. $$P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n n!}\left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots\right)$$
$$= (-1)^n \frac{(2n-1)!!}{2^n n!}\, F\left(-n, n+\frac{1}{2}; \frac{1}{2}; z^2\right)$$

AD (9002), MO 69

3. $$P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n n!}\left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots\right)$$
$$= (-1)^n \frac{(2n+1)!!}{2^n n!}\, z\, F\left(-n, n+\frac{3}{2}; \frac{3}{2}; z^2\right)$$

AD (9002), MO 69

4. $$P_n(\cos\varphi) = \frac{(2n-1)!!}{2^n n!}\left(\cos n\varphi + \frac{1}{1}\frac{n}{2n-1}\cos(n-2)\varphi \right.$$
$$+ \frac{1\cdot 3}{1\cdot 2}\frac{n(n-1)}{(2n-1)(2n-3)}\cos(n-4)\varphi$$
$$\left. + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos(n-6)\varphi - \dots\right)$$

WH

5. $$P_{2n}(\cos\varphi) = (-1)^n \frac{(2n-1)!!}{2^n n!}$$
$$\times\left\{\sin^{2n}\varphi - \frac{(2n)^2}{2!}\sin^{2n-2}\varphi\cos^2\varphi + \dots + (-1)^n \frac{2^n n!}{(2n-1)!!}\cos^{2n}\varphi\right\}$$

AD (9011)

6. $$P_{2n+1}(\cos\varphi) = (-1)^n \frac{(2n+1)!!}{2^n n!}\cos\varphi$$
$$\times\left\{\sin^{2n}\varphi - \frac{(2n)^2}{3!}\sin^{2n-2}\varphi\cos^2\varphi + \dots + (-1)^n \frac{2^n n!}{(2n+1)!!}\cos^{2n}\varphi\right\}$$

AD (9012)

7. $$P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{(n-k)!\,(k!)^2\, 2^{k+1}}\left[(1-z)^k + (-1)^n (1+z)^k\right]$$ WH

8.912 Special cases:

1. $P_0(x) = 1$ JA
2. $P_1(x) = x = \cos\varphi$ JA
3. $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) = \frac{1}{4}\left(3\cos 2\varphi + 1\right)$ JA
4. $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) = \frac{1}{8}\left(5\cos 3\varphi + 3\cos\varphi\right)$ JA
5. $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) = \frac{1}{64}\left(35\cos 4\varphi + 20\cos 2\varphi + 9\right)$ JA
6. $P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) = \frac{1}{128}\left(63\cos 5\varphi + 35\cos 3\varphi + 30\cos\varphi\right)$ JA
7.[10] $P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right) = \frac{1}{512}\left(231\cos 6\varphi + 126\cos 4\varphi + 105\cos 2\varphi + 50\right)$
8. $P_7(x) = \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)$
$= \frac{1}{1024}\left(429\cos 7\varphi + 231\cos 5\varphi + 189\cos 3\varphi + 175\cos\varphi\right)$
9. $P_8(x) = \frac{1}{128}\left(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35\right)$
$= \frac{1}{16384}\left(6435\cos 8\varphi - 3432\cos 6\varphi + 2772\cos 4\varphi - 2520\cos 2\varphi + 1225\right)$

8.913 Integral representations:

1. $P_n(\cos\varphi) = \frac{2}{\pi}\int_\varphi^\pi \frac{\sin\left(n + \frac{1}{2}\right)t}{\sqrt{2\left(\cos\varphi - \cos t\right)}}\,dt$ WH

   See also **3.611** 3, **3.661** 3, 4.

2.[7] Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{\left(t^2 - 1\right)^n}{2^n (t - z)^{n+1}}\,dt,$$

   with C a simple contour containing z. SA 175(9)

3.[10] Laplace integral formula:

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \left[x + \left(x^2 - 1\right)^{1/2}\cos\varphi\right]^n d\varphi \qquad [|x| \le 1]$$ SA 180(19)

Functional relations

8.914 Recurrence formulas:

1. $(n+1)\,P_{n+1}(z) - (2n+1)z\,P_n(z) + n\,P_{n-1}(z) = 0$ WH

2. $\left(z^2-1\right)\dfrac{d\,P_n}{dz}=n\left[z\,P_n(z)-P_{n-1}(z)\right]=\dfrac{n(n+1)}{2n+1}\left[P_{n+1}(z)-P_{n-1}(z)\right]$ WH

8.915

1.[10] $\displaystyle\sum_{k=0}^{n}(2k+1)\,P_k(x)\,P_k(y)=(n+1)\frac{P_n(x)\,P_{n+1}(y)-P_n(y)\,P_{n+1}(x)}{y-x}$

(Christoffel summation formula)

MO 70

1(1)[10]. $\displaystyle(y-x)\sum_{k=0}^{n}(2k+1)\,P_k(x)\,Q_k(y)=1-(n+1)\left[P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\right]$

AS 335(8.9.2)

2.[7] $\displaystyle\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(2n-4k-1)\,P_{n-2k-1}(z)=P_n'(z)$ (summation theorem) MO 70

3.[7] $\displaystyle\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}(2n-4k-3)\,P_{n-2k-2}(z)=z\,P_n'(z)-n\,P_n(z)$ SM 491(42), WH

4.[10] $\displaystyle\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}(2n-4k+1)[k(2n-2k+1)-2]\,P_{n-2k}(z)=z^2\,P_n''(z)-n(n-1)\,P_n(z)$ WH

5.[11] $\displaystyle\sum_{k=0}^{m}\frac{a_{m-k}a_ka_{n-k}}{a_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right)P_{n+m-2k}(z)=P_n(z)\,P_m(z)$

$\left[a_k=\dfrac{(2k-1)!!}{k!},\quad m\le n\right]$ AD (9036)

8.916

1. $P_n(\cos\varphi)=\dfrac{(2n-1)!!}{2^n n!}e^{\mp in\varphi}\,F\left(\dfrac{1}{2},-n;\dfrac{1}{2}-n;e^{\pm 2i\varphi}\right)$ MO 69

2. $P_n(\cos\varphi)=F\left(n+1,-n;1;\sin^2\dfrac{\varphi}{2}\right)$ MO 69

3. $P_n(\cos\varphi)=(-1)^n\,F\left(n+1,-n;1;\cos^2\dfrac{\varphi}{2}\right)$ WH

4. $P_n(\cos\varphi)=\cos^n\varphi\,F\left(-\dfrac{1}{2}n,\dfrac{1}{2}-\dfrac{1}{2}n;1;-\tan^2\varphi\right)$ HO 23

5. $P_n(\cos\varphi)=\cos^{2n}\dfrac{\varphi}{2}\,F\left(-n,-n;1;-\tan^2\dfrac{\varphi}{2}\right)$ HO 23, 29, WH

See also **8.911** 1, **8.911** 2, **8.911** 3. For a connection with other functions, see **8.936** 3, **8.836**, **8.962** 2.

- For integrals of Legendre polynomials, see **7.22**–**7.25**.
- For the zeros of Legendre polynomials, see **8.785**.

8.917 Inequalities:

1. $P_0(x) < P_1(x) < P_2(x) < \cdots < P_n(x) < \ldots \qquad [x > 1]$ MO 71
2. For $x > -1$, $P_0(x) + P_1(x) + \cdots + P_n(x) > 0$. MO 71
3. $[P_n(\cos\varphi)]^2 > \dfrac{\sin(2n+1)\varphi}{(2n+1)\sin\varphi}$ MO 71
4. $\sqrt{n\sin\varphi}|P_n(\cos\varphi)| \le 1$. MO 71
5. $|P_n(\cos\varphi)| \le 1$. WH

6.[10] Let $n \ge 2$. The successive relative maxima of $|P_n(x)|$, when x decreases from 1 to 0, form a decreasing sequence. More precisely, if $\mu_1, \mu_2, \ldots, \mu_{\lfloor n/2 \rfloor}$ denote these maxima corresponding to decreasing values of x, we have

$$1 > \mu_1 > \mu_2 > \cdots > \mu_{\lfloor n/2 \rfloor}$$ SZ 162(7.3.1)

7.[10] Let $n \ge 2$. The successive relative maxima of $(\sin\theta)^{1/2}|P_n(\cos\theta)|$ when θ increases from 0 to $\pi/2$, form an increasing sequence. SZ 163(7.3.2)

8.[10] We have

$$(\sin\theta)^{1/2}|P_n(\cos\theta)| < (2/\pi)^{1/2} n^{-1/2} \qquad [0 \le q\theta \le q\pi]$$ SZ 163(7.3.8)

Here the constant $(2/\pi)^{1/2}$ cannot be replaced by a smaller one.

9.[10] $\max\limits_{0 \le q\theta \le q\pi} (\sin\theta)^{1/2}|P_n(\cos\theta)| \cong (2/\pi)^{1/2} n^{-\frac{1}{2}} \qquad [n \to \infty]$ SZ 164(7.3.12)

10.[10] Stieltjes' first theorem:

$$|P_n(\cos\theta)| \le \left(\frac{2}{\pi}\right)^{1/2} \frac{4}{\sqrt{n\sin\theta}} \qquad [n = 1, 2, \ldots, 0 < \theta < \pi]$$ SA 197(8)

11.[10] Stieltjes' second theorem:

$$|P_n(x) - P_{n+2}(x)| < \frac{4}{\sqrt{\pi}\sqrt{n+2}} \qquad [|x| \le 1]$$ SA 199(15)

12.[10] $\left|\dfrac{d\,P_n(x)}{dx}\right| < \dfrac{2}{\sqrt{\pi}} \dfrac{\sqrt{n}}{1-x^2} \qquad [|x| < 1, \quad n = 1, 2, \ldots]$ SA 201(18)

13.[10] $|P_{n+1}(x) + P_n(x)| < 6\left(\dfrac{2}{\pi n}\right)^{\frac{1}{2}} (1-x)^{-1/2} \qquad [|x| < 1, \quad n = 0, 1, \ldots]$ SA 201(19)

8.918[10] Asymptotic approximations:

1. $P_n(\cos\theta) = \left(\dfrac{2}{\pi n \sin\varphi}\right)^{1/2} \cos\left[\left(n + \dfrac{1}{2}\right)\theta - \dfrac{\pi}{4}\right] + O\left(n^{-3/2}\right)$
$[\varepsilon \le \theta \le \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2m]$ (Laplace's formula) SA 208(1)

2. $$P_n(\cos\theta) = \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \left\{\left(1 - \frac{1}{4n}\right)\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + \frac{1}{8n}\cos\theta\sin\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right]\right\} + O\left(n^{-5/2}\right)$$
$[\varepsilon \le \theta \le \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2]$ (Bonnet–Heine formula) SA 208(2)

8.919[10] Series of products of Legendre and Chebyshev polynomials

1. $$2\int_{-1}^{1} T_n(x)\, P_n(x)\, dx = \sum_{i,j=0}^{i+j=n} \int_{-1}^{1} P_i(x)\, P_j(x)\, P_n(x)\, dx$$

8.92 Series of Legendre polynomials

8.921 The generating function:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k\, P_k(z) \qquad \left[|t| < \min\left|z \pm \sqrt{z^2-1}\right|\right]$$ SM 489(31), WH

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}}\, P_k(z) \qquad \left[|t| > \max\left|z \pm \sqrt{z^2-1}\right|\right]$$ MO 70

8.922

1. $$z^{2n} = \frac{1}{2n+1}\, P_0(z) + \sum_{k=1}^{\infty}(4k+1)\frac{2n(2n-2)\ldots(2n-2k+2)}{(2n+1)(2n+3)\ldots(2n+2k+1)}\, P_{2k}(z)$$ MO 72

2. $$z^{2n+1} = \frac{3}{2n+3}\, P_1(z) + \sum_{k=1}^{\infty}(4k+3)\frac{2n(2n-2)\ldots(2n-2k+2)}{(2n+3)(2n+5)\ldots(2n+2k+3)}\, P_{2k+1}(z)$$ MO 72

3. $$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{k=0}^{\infty}(4k+1)\left\{\frac{(2k-1)!!}{2^k k!}\right\}^2 P_{2k}(x) \qquad [|x| < 1, \quad (-1)!! \equiv 1]$$
MO 72, LA 385(15)

4. $$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{k=0}^{\infty}(4k+3)\frac{(2k-1)!!(2k+1)!!}{2^{2k+1}k!(k+1)!}\, P_{2k+1}(x)$$
$[|x| < 1, \quad (-1)!! \equiv 1]$ LA 385(17)

5. $$\sqrt{1-x^2} = \frac{\pi}{2}\left\{\frac{1}{2} - \sum_{k=1}^{\infty}(4k+1)\frac{(2k-3)!!(2k-1)!!}{2^{2k+1}k!(k+1)!}\, P_{2k}(x)\right\}$$
$[|x| < 1, \quad (-1)!! \equiv 1]$ LA 385(18)

6.[10] $$\sqrt{\frac{1-x}{2}} = \frac{2}{3}\, P_0(x) - 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+3)}\, P_n(x) \qquad [-1 \le x \le 1]$$

7.[10] $$\frac{1-\rho^2}{(1-2\rho x+\rho^2)^{1/2}} = 1+\sum_{n=0}^{\infty}(2n+1)\rho^n\, P_n(x), \qquad [|\rho|<1, \quad |x|\le 1]$$ SA 170(4)

8.923 $$\arcsin x = \frac{\pi}{2}\sum_{k=1}^{\infty}\left\{\frac{(2k-1)!!}{2^k k!}\right\}^2 [P_{2k+1}(x)-P_{2k-1}(x)] + \pi x/2$$
$$[|x|<1, \quad (-1)!!\equiv 1]$$ WH

8.924

1. $$-\frac{1+\cos n\pi}{2\left(n^2-1\right)}P_0(\cos\theta) - \frac{1+\cos n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2\left(n^2-2^2\right)\ldots\left[n^2-(2k)^2\right]}{\left(n^2-1^2\right)\left(n^2-3^2\right)\ldots\left[n^2-(2k+3)^2\right]}P_{2k+2}(\cos\theta)$$
$$-\frac{3\left(1-\cos n\pi\right)}{2\left(n^2-2^2\right)}P_1(\cos\theta)$$
$$-\frac{1-\cos n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)\left(n^2-1^2\right)\ldots\left[n^2-(2k-1)^2\right]}{\left(n^2-2^2\right)\left(n^2-4^2\right)\ldots\left[n^2-(2k+2)^2\right]}P_{2k+1}(\cos\theta) = \cos n\theta$$
AD (9060.1)

2. $$\frac{-\sin n\pi}{2\left(n^2-1\right)}P_0(\cos\theta) - \frac{\sin n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2\left(n^2-2^2\right)\ldots\left[n^2-(2k)^2\right]}{\left(n^2-1^2\right)\left(n^2-3^2\right)\ldots\left[n^2-(2k+3)^2\right]}P_{2k+2}(\cos\theta)$$
$$+\frac{3\sin n\pi}{2\left(n^2-2^2\right)}P_1(\cos\theta)$$
$$+\frac{\sin n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)\left(n^2-1^2\right)\left(n^2-3^2\right)\ldots\left[n^2-(2k-1)^2\right]}{\left(n^2-2^2\right)\left(n^2-4^2\right)\ldots\left[n^2-(2k+2)^2\right]}P_{2k+1}(\cos\theta) = \sin n\theta$$
AD (9060.2)

3.[3] $$\frac{2^{n-1}n!}{(2n-1)!!}P_n(\cos\theta) - n\sum_{k=1}^{\lfloor n/2\rfloor}(2n-4k+1)\frac{2^{n-2k-1}(n-k-1)!(2k-3)!!}{(2n-2k+1)!!k!}P_{n-2k}(\cos\theta)$$
$$= \cos n\theta$$
AD (9061.1)

4. $$\frac{(2n-1)!!P_{n-1}(\cos\theta)}{2^{n-1}(n-1)!} - \frac{n}{2^{n+1}}\sum_{k=0}^{\infty}\frac{(2n+2k-1)!!(2k-1)!!\,(2n+4k+3)}{2^{2k}(n+k+1)!(k+1)!}P_{n+2k+1}(\cos\theta)$$
$$= \frac{4\sin n\theta}{\pi}$$
AD (9061.2)

8.925

1. $$\sum_{k=1}^{\infty}\frac{4k-1}{2^{2k}(2k-1)^2}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k-1}(\cos\theta) = 1-\frac{2\theta}{\pi}$$

2. $$\sum_{k=1}^{\infty}\frac{4k+1}{2^{2k+1}(2k-1)(k+1)}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k}(\cos\theta) = \frac{1}{2}-\frac{2\sin\theta}{\pi}$$ AD (9062.2)

3. $$\sum_{k=1}^{\infty}\frac{k(4k-1)}{2^{2k-1}(2k-1)}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k-1}(\cos\theta) = \frac{2\cot\theta}{\pi}$$ AD (9062.3)

4. $$\sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k}(\cos\theta) = \frac{2}{\pi \sin\theta} - 1$$ AD (9062.4)

8.926

1. $$\sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos\theta) = \ln \frac{2\tan\frac{\pi-\theta}{4}}{\sin\theta} = -\ln\sin\frac{\theta}{2} - \ln\left(1+\sin\frac{\theta}{2}\right)$$ AD (9063.2)

2. $$\sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos\theta) = \ln \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} - 1$$ AD (9063.1)

8.927 $$\sum_{k=0}^{\infty} \cos\left(k+\tfrac{1}{2}\right)\beta\, P_k(\cos\varphi) = \frac{1}{\sqrt{2(\cos\beta - \cos\varphi)}} \qquad [0 \le \beta < \varphi < \pi]$$
$$= 0 \qquad [0 < \varphi < \beta < \pi]$$

MO 72

8.928

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n (4k+1)\left[(2n-1)!!\right]^3}{2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\,\boldsymbol{K}(\sin\theta)}{\pi^2} - 1$$ AD (9064.1)

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1)\left[(2n-1)!!\right]^3}{(2n-1)(2n+2)2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\,\boldsymbol{E}(\sin\theta)}{\pi^2} - \frac{1}{2}$$ AD (9064.2)

- For series of products of Bessel functions and Legendre polynomials, see **8.511** 4, **8.531** 3, **8.533** 1, **8.533** 2, and **8.534**.
- For series of products of Legendre and Chebyshev polynomials, see **8.919**.

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.930 Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$\left(1 - 2t\alpha + \alpha^2\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t)\alpha^n$$ WH

Thus, the polynomials $C_n^\lambda(t)$ are a *generalization of the Legendre polynomials.*

1.[10] $C_0^\lambda(t) = 1$

2.[10] $C_1^\lambda(t) = 2\lambda t$

3.[10] $C_2^\lambda(t) = 2\lambda(\lambda+1)t^2 - \lambda$

4.[10] $C_3^\lambda(t) = \frac{1}{3}\lambda\left(4\lambda^2 + 12\lambda + 8\right)t^3 - 2\lambda(\lambda+1)t$

5.[11] $C_4^\lambda(t) = \frac{2}{3}\lambda\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)t^4 - 2\lambda\left(\lambda^2 + 3\lambda + 2\right)t^2 + \frac{1}{2}\lambda(\lambda+1)$

6.[10] $C_5^\lambda(t) = \frac{1}{15}\lambda\left(4\lambda^4 + 40\lambda^3 + 140\lambda^2 + 200\lambda + 96\right)t^5$
$\qquad - \frac{1}{3}\lambda\left(4\lambda^3 + 24\lambda^2 + 44\lambda + 24\right)t^3 + \lambda\left(\lambda^2 + 3\lambda + 2\right)t$

7.[10] $$C_6^\lambda(t) = \tfrac{1}{45}\lambda\left(\lambda^5 + 60\lambda^4 + 340\lambda^3 + 900\lambda^2 + 1096\lambda + 480\right)t^6 - \tfrac{1}{3}\lambda\left(2\lambda^4 + 20\lambda^3 + 70\lambda^2 + 100\lambda + 48\right)t^4 + \lambda\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)t^2 + \tfrac{1}{6}\lambda\left(\lambda^2 + 3\lambda + 2\right)$$

8.931 Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)}\frac{\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(\lambda)}\int_0^\pi \left(t + \sqrt{t^2-1}\cos\varphi\right)^n \sin^{2\lambda-1}\varphi\, d\varphi \qquad \text{MO 99}$$

See also **3.252** 11, **3.663** 2, **3.664** 4.

Functional relations

8.932 Expressions in terms of hypergeometric functions:

1. $$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(2\lambda)}\, F\left(2\lambda+n, -n; \lambda+\frac{1}{2}; \frac{1-t}{2}\right)^* \qquad \text{MO 97}$$
$$= \frac{2^n\,\Gamma(\lambda+n)}{n!\,\Gamma(\lambda)} t^n\, F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-\lambda-n; \frac{1}{t^2}\right) \qquad \text{MO 99}$$

2. $$C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda+n)\,\mathrm{B}(\lambda, n+1)}\, F\left(-n, n+\lambda; \frac{1}{2}; t^2\right) \qquad \text{MO 99}$$

3. $$C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{\mathrm{B}(\lambda, n+1)}\, F\left(-n, n+\lambda+1; \frac{3}{2}; t^2\right) \qquad \text{MO 99}$$

8.933 Recursion formulas:

1. $$(n+2)\, C_{n+2}^\lambda(t) = 2(\lambda+n+1)t\, C_{n+1}^\lambda(t) - (2\lambda+n)\, C_n^\lambda(t) \qquad \text{Mo 98}$$

2. $$n\, C_n^\lambda(t) = 2\lambda\left[t\, C_{n-1}^{\lambda+1}(t) - C_{n-2}^{\lambda+1}(t)\right] \qquad \text{WH}$$

3. $$(2\lambda+n)\, C_n^\lambda(t) = 2\lambda\left[C_n^{\lambda+1}(t) - t\, C_{n-1}^{\lambda+1}(t)\right] \qquad \text{WH}$$

4. $$n\, C_n^\lambda(t) = (2\lambda+n-1)t\, C_{n-1}^\lambda(t) - 2\lambda\left(1-t^2\right) C_{n-2}^{\lambda+1}(t) \qquad \text{WH}$$

8.934

1. $$C_n^\lambda(t) = \frac{(-1)^n}{2^n}\frac{\Gamma(2\lambda+n)\,\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(2\lambda)\,\Gamma\left(\frac{2\lambda+1}{2}+n\right)}\frac{\left(1-t^2\right)^{\frac{1}{2}-\lambda}}{n!}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{\lambda+n-\frac{1}{2}}\right] \qquad \text{WH}$$

2. $$C_n^\lambda(\cos\varphi) = \sum_{\substack{k,l=0\\k+l=n}}^{n} \frac{\Gamma(\lambda+k)\,\Gamma(\lambda+l)}{k!l!\,[\Gamma(\lambda)]^2}\cos(k-l)\varphi \qquad \text{MO 99}$$

---

*Equation 8.932.1 defines the generalized functions $C_n^\lambda(t)$, where the subscript n can be an arbitrary number.

3. $$C_n^\lambda(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi) = \frac{\Gamma(2\lambda-1)}{[\Gamma(\lambda)]^2}\sum_{k=0}^{n}\frac{2^{2k}(n-k)!\,[\Gamma(\lambda+k)]^2}{\Gamma(2\lambda+n+k)}(2\lambda+2k-1)\sin^k\psi\,\mathrm{si}\,n^k\vartheta \times C_{n-k}^{\lambda+k}(\cos\psi)\,C_{n-k}^{\lambda+k}(\cos\vartheta)\,C_k^{\lambda-\frac12}(\cos\varphi)$$
$\left[\psi,\vartheta,\varphi \text{ real};\quad \lambda\neq\frac12\right]$ ["summation theorem"] (see also **8.794–8.796**) WH

4. $$\lim_{\lambda\to0}\Gamma(\lambda)\,C_n^\lambda(\cos\varphi)=\frac{2\cos n\varphi}{n}$$ MO 98

For orthogonality, see **8.904, 7.313**.

8.935 Derivatives:

1. $$\frac{d^k}{dt^k}\,C_n^\lambda(t)=2^k\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}\,C_{n-k}^{\lambda+k}(t)$$ MO 99

In particular,

2.[11] $$\frac{d\,C_n^\lambda(t)}{dt}=2\lambda\,C_{n-1}^{\lambda+1}(t)$$ WH

For integrals of the polynomials $C_n^\lambda(x)$ see **7.31–7.33**.

8.936 Connections with other functions:

1. $$C_n^\lambda(t)=\frac{\Gamma(2\lambda+n)\,\Gamma\left(\lambda+\frac12\right)}{\Gamma(2\lambda)\,\Gamma(n+1)}\left\{\frac14\left(t^2-1\right)\right\}^{\frac14-\frac{\lambda}{2}}P_{\lambda+n-\frac12}^{\frac12-\lambda}(t)$$ MO 98

2. $$C_{n-m}^{m+\frac12}(t)=\frac{1}{(2m-1)!!}\frac{d^m\,P_n(t)}{dt^m}=(-1)^m\frac{\left(1-t^2\right)^{-\frac{m}{2}}\,m!2^m}{(2m)!}\,P_n^m(t)$$
$[m+1 \text{ a natural number}]$ MO 98, WH

3. $$C_n^{1/2}(t)=P_n(t)$$

4. $$J_{\lambda-\frac12}(r\sin\vartheta\sin\alpha)\,(r\sin\vartheta\sin\alpha)^{-\lambda+\frac12}\,e^{-ir\cos\vartheta\cos\alpha} = \sqrt2\frac{\Gamma(\lambda)}{\Gamma\left(\lambda+\frac12\right)}\sum_{k=0}^{\infty}(\lambda+k)i^{-k}\frac{\mathbf{J}_{\lambda+k}(r)\,C_k^\lambda(\cos\vartheta)\,C_k^\lambda(\cos\alpha)}{r^\lambda\,C_k^\lambda(1)}$$
MO 99

5. $$\lim_{\lambda\to\infty}\lambda^{-\frac{n}{2}}\,C_n^{\frac{\lambda}{2}}\left(t\sqrt{\frac{2}{\lambda}}\right)=\frac{2^{-\frac{n}{2}}}{n!}\,H_n(t)$$ MO 99a

See also **8.932**.

8.937 Special cases and particular values:

1. $$C_n^1(\cos\varphi)=\frac{\sin(n+1)\varphi}{\sin\varphi}$$ MO 99

2. $$C_0^0(\cos\varphi)=1$$ MO 98

3. $$C_0^\lambda(t)\equiv1$$ MO 98

4. $C_n^\lambda(1) \equiv \binom{2\lambda+n-1}{n}$ MO 98

8.938 A differential equation leading to the polynomials $C_n^\lambda(t)$:

$$y'' + \frac{(2\lambda+1)t}{t^2-1}y' - \frac{n(2\lambda+n)}{t^2-1}y = 0 \qquad \text{(cf. } \mathbf{9.174}\text{)}$$ WH

For series of products of Bessel functions and the polynomials $C_n^\lambda(x)$, see **8.532, 8.534**.

8.939[10] Differentiation and Rodrigues' formulas and orthogonality relation

1. $\frac{d}{dt}\, C_n^\lambda(t) = 2\lambda\, C_{n-1}^{\lambda+1}(t)$ MS 5.3.2

2. $\frac{d^m}{dt^m}\, C_n^\lambda(t) = 2^m\lambda(\lambda+1)(\lambda+2)\dots(\lambda+m-1)\, C_{n-m}^{\lambda+m}(t)$ MS 5.3.2

3. $\frac{d}{dt}\, C_{n-1}^\lambda(t) = t\frac{d}{dt}\, C_n^\lambda(t) - n\, C_n^\lambda(t)$ MS 5.3.2

4. $\frac{d}{dt}\, C_{n+1}^\lambda(t) = t\frac{d}{dt}\, C_n^\lambda(t) + (2\lambda+n)\, C_n^\lambda(t)$ MS 5.3.2

5. $$\left(1-t^2\right)\frac{d}{dt}\, C_n^\lambda(t) = (n+2\lambda-1)\, C_{n-1}^\lambda(t) - nt\, C_n^\lambda(t) = (n+2\lambda)t\, C_n^\lambda(t) - (n+1)\, C_{n+1}^\lambda(t)$$
$$= 2\lambda\left(1-t^2\right) C_{n-1}^{\lambda+1}(t)$$
MS 5.3.2

6. $\frac{d}{dt}\left[C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)\right] = 2(n+\lambda)\, C_n^\lambda(t)$ MS 5.3.2

7. $$C_n^\lambda(t) = \frac{(-1)^n 2\lambda(2\lambda+1)(2\lambda+2)\dots(2\lambda+n-1)\left(1-t^2\right)^{\frac12-\lambda}}{2^n n!\left(\lambda+\frac12\right)\left(\lambda+\frac32\right)\dots\left(\lambda+n-\frac12\right)}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{n+\lambda-\frac12}\right]$$
$$= \frac{(-1)^n\,\Gamma\left(\lambda+\frac12\right)\Gamma(n+2\lambda)\left(1-t^2\right)^{\frac12-\lambda}}{2^n n!\,\Gamma(2\lambda)\,\Gamma\left(n+\lambda+\frac12\right)}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{n+\lambda-\frac12}\right]$$
[Rodrigues' formula] MS 5.3.2

8. $$\int_{-1}^{1} C_n^\lambda(t)\, C_m^\lambda(t)\left(1-t^2\right)^{\lambda-\frac12} dt = 0 \qquad n \neq m$$
$$= \frac{\pi 2^{1-2\lambda}\,\Gamma(n+2\lambda)}{n!(\lambda+n)\left[\Gamma(\lambda)\right]^2} \qquad n = m$$
$[\lambda \neq 0]$ [Orthogonality relation] MS 5.3.2

8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

8.940 Definition

1. Chebyshev's polynomials of the first kind
$$T_n(x) = \cos(n\arccos x) = \frac12\left[\left(x+i\sqrt{1-x^2}\right)^n + \left(x-i\sqrt{1-x^2}\right)^n\right]$$
$$= x^n - \binom{n}{2}x^{n-2}\left(1-x^2\right) + \binom{n}{4}x^{n-4}\left(1-x^2\right)^2 - \binom{n}{6}x^{n-6}\left(1-x^2\right)^3 + \dots$$
NA 66, 71

2. Chebyshev's polynomials of the second kind:

$$U_n(x) = \frac{\sin\left[(n+1)\arccos x\right]}{\sin\left[\arccos x\right]} = \frac{1}{2i\sqrt{1-x^2}}\left[\left(x+i\sqrt{1-x^2}\right)^{n+1} - \left(x-i\sqrt{1-x^2}\right)^{n+1}\right]$$
$$= \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}\left(1-x^2\right) + \binom{n+1}{5}x^{n-4}\left(1-x^2\right)^2 - \ldots$$

Functional relations

8.941 Recursion formulas:

1. $T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0$ NA 358
2. $U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0$
3. $T_n(x) = U_n(x) - x\,U_{n-1}(x)$ EH II 184(3)
4. $\left(1-x^2\right)U_{n-1}(x) = x\,T_n(x) - T_{n+1}(x)$ EH II 184(4)

For the orthogonality, see **7.343** and **8.904**.

8.942 Relations with other functions:

1. $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right)$ MO 104
2. $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!}\frac{d^n}{dx^n}\left(1-x^2\right)^{n-\frac{1}{2}}$ MO 104
3. $U_n(x) = \frac{(-1)^n(n+1)}{\sqrt{1-x^2}(2n+1)!!}\frac{d^n}{dx^n}\left(1-x^2\right)^{n+\frac{1}{2}}$ EH II 185(15)

See also **8.962** 3.

8.943[10] Special cases

1. $T_0(x) = 1$
2. $T_1(x) = x$
3. $T_2(x) = 2x^2 - 1$
4. $T_3(x) = 4x^3 - 3x$
5. $T_4(x) = 8x^4 - 8x^2 + 1$
6. $T_5(x) = 16x^5 - 20x^3 + 5x$
7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
10. $U_0(x) = 1$
11. $U_1(x) = 2x$
12. $U_2(x) = 4x^2 - 1$
13. $U_3(x) = 8x^3 - 4x$
14. $U_4(x) = 16x^4 - 12x^2 + 1$
15. $U_5(x) = 32x^5 - 32x^3 + 6x$
16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$
17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$
18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

8.944 Particular values:

1. $T_n(1) = 1$
2. $T_n(-1) = (-1)^n$
3. $T_{2n}(0) = (-1)^n$
4. $T_{2n+1}(0) = 0$
5. $U_{2n+1}(0) = 0$
6. $U_{2n}(0) = (-1)^n$

8.945 The generating function:

1.[11] $$\frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2\sum_{k=1}^{\infty} T_k(x)t^k \qquad [|t|<1]$$ MO 104

2.[11] $$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k \qquad [|t|<1]$$ MO 104a, EH II 186(31)

8.946 Zeros. The polynomials $T_n(x)$ and $U_n(x)$ only have real simple zeros. All these zeros lie in the interval $(-1,+1)$.

8.947 The functions $T_n(x)$ and $\sqrt{1-x^2}\,U_{n-1}(x)$ are two linearly independent solutions of the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$$ NA 69(58)

8.948 Of all polynomials of degree n with leading coefficient equal to 1, the one that deviates the least from zero on the interval $[-1,+1]$ is the polynomial $2^{-n+1}\,T_n(x)$.

8.949[10] Differentiation and Rodrigues' formulas and orthogonality relations

1. $$\frac{d}{dx}\,T_n(x) = n\,U_{n-1}(x)$$ MS 5.7.2
2. $$\frac{d^m}{dx^m}\,T_n(x) = 2^{m-1}\,\Gamma(m)n\,C_{n-m}^m(x)$$ MS 5.7.2
3. $$\left(1-x^2\right)\frac{d}{dx}\,T_n(x) = n\left[T_{n-1}(x) - x\,T_n(x)\right] = n\left[x\,T_n(x) - T_{n+1}(x)\right]$$ MS 5.7.2
4. $$\frac{d}{dx}\,U_n(x) = 2\,C_{n-1}^2(x)$$ MS 5.7.2
5. $$\frac{d^m}{dx^m}\,U_n(x) = 2^m m! C_{n-m}^{m+1}(x)$$ MS 5.7.2
6. $$\left(1-x^2\right)\frac{d}{dx}\,U_n(x) = (n+1)\,U_{n-1}(x) - nx\,U_n(x) = (n+2)x\,U_n(x) - (n+1)\,U_{n+1}(x)$$ MS 5.7.2
7. $$T_n(x) = \frac{(-1)^n\pi^{1/2}\left(1-x^2\right)^{c\frac{1}{2}}}{2^{n+1}\,\Gamma\left(n+\frac{1}{2}\right)}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n-\frac{1}{2}}\right] \qquad \text{[Rodrigues' formula]}$$ MS 5.7.2
8. $$U_n(x) = \frac{(-1)^n\pi^{1/2}(n+1)\left(1-x^2\right)^{-1/2}}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+\frac{1}{2}}\right]$$ [Rodrigues' formula] MS 5.7.2

9. $$\int_{-1}^{1} T_m(x)\, T_n(x) \left(1-x^2\right)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

[Orthogonality relation] MS 5.7.2

10. $$\int_{-1}^{1} U_m(x)\, U_n(x) \left(1-x^2\right)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n \end{cases}$$

[Orthogonality relation] MS 5.7.2

8.95 The Hermite polynomials $H_n(x)$

8.950 Definition

1. $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$ SM 567(14)

or

2. $H_n(x) = 2^n x^n - 2^{n-1}\binom{n}{2} x^{n-2} + 2^{n-2}\cdot 1\cdot 3\cdot \binom{n}{4} x^{n-4} - 2^{n-3}\cdot 1\cdot 3\cdot 5\cdot \binom{n}{6} x^{n-6} + \ldots$ MO 105a

3.[10] $H_0(x) = 1$

4.[10] $H_1(x) = 2x$

5.[10] $H_2(x) = 4x^2 - 2$

6.[10] $H_3(x) = 8x^3 - 12x$

7.[10] $H_4(x) = 16x^4 - 48x^2 + 12$

8.[10] $H_5(x) = 32x^5 - 160x^3 + 120x$

9.[10] $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

10.[10] $H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

11.[10] $H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$

8.951 The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2}\, dt$$ MO 106a

Functional relations

8.952 Recursion formulas:

1. $\dfrac{d\, H_n(x)}{dx} = 2n\, H_{n-1}(x)$ SM 569(22)

2. $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$ SM 570(23)

For the orthogonality, see **7.374** 1 and **8.904**.

3.[10] $n\, H_n(x) = -n\, H'_{n-1}(x) + x\, H'_n(x)$ MS 5.6.2

4.[10] $H_n(x) = 2x\, H_{n-1}(x) - H'_{n-1}(x)$ MS 5.6.2

8.953 The connection with other functions:

1. $H_{2n}(x) = (-1)^n \dfrac{(2n)!}{n!}\, \Phi\left(-n, \frac{1}{2}; x^2\right)$ MO 106a

2. $H_{2n+1}(x) = (-1)^n 2 \dfrac{(2n+1)!}{n!} x\, \Phi\left(-n, \frac{3}{2}; x^2\right)$ MO 106a

- For a connection with the polynomials $C_n^\lambda(x)$, see **8.936** 5.
- For a connection with the Laguerre polynomials, see **8.972** 2 and **8.972** 3.
- For a connection with functions of a parabolic cylinder, see **9.253**.

8.954 Inequalities:

1.[10] $|H_n(x)| \le 2^{\frac{n}{2} - \lfloor \frac{n}{2} \rfloor} \dfrac{n!}{\lfloor n/2 \rfloor!} e^{2x\sqrt{\lfloor n/2 \rfloor}}$ MO 106a

2.[10] $|H_n(x)| < k\sqrt{n!}\, 2^{n/2} e^{x^2/2}, \quad k \approx 1.086435$ SA 324

8.955 Asymptotic representation:

1. $H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[\cos\left(\sqrt{4n+1}\, x\right) + O\left(\dfrac{1}{\sqrt[4]{n}}\right)\right]$ SM 579

2. $H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1}\, e^{x^2/2} \left[\sin\left(\sqrt{4n+3}\, x\right) + O\left(\dfrac{1}{\sqrt[4]{n}}\right)\right]$ SM 579

8.956 Special cases and particular values:

1. $H_0(x) = 1$
2. $H_1(x) = 2x$
3. $H_2(x) = 4x^2 - 2$
4. $H_3(x) = 8x^3 - 12x$
5. $H_4(x) = 16x^4 - 48x^2 + 12$
6. $H_{2n}(0) = (-1)^n 2^n (2n-1)!!$ SM 570(24)
7. $H_{2n+1}(0) = 0$

Series of Hermite polynomials

8.957 The generating function:

1. $\exp\left(-t^2 + 2tx\right) = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x)$ SM 569(21)

2. $\dfrac{1}{e} \sinh 2x = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x)$ MO 106a

3. $\frac{1}{e}\cosh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x)$ MO 106a

4. $e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x)$ MO 106a

5. $e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x)$ MO 106a

8.958 "The summation theorem":

1.[11] $$\frac{\left(\sum_{k=1}^{r} a_k^2\right)^{\frac{n}{2}}}{n!} H_n\left(\frac{\sum_{k=1}^{r} a_k x_k}{\sqrt{\sum a_k^2}}\right) = \sum_{m_1+m_2+\cdots+m_r=n} \prod_{k=1}^{r} \left\{\frac{a_k^{m_k}}{m_k!} H_{m_k}(x_k)\right\}$$ MO 106a

2. A special case:

$$2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\left(x\sqrt{2}\right) H_k\left(y\sqrt{2}\right)$$ MO 107a

8.959 Hermite polynomials satisfy the differential equation

1. $\frac{d^2 u_n}{dx^2} - 2x\frac{du_n}{dx} + 2nu_n = 0;$ SM 566(9)

A second solution of this differential equation is provided by the functions (A and B are arbitrary constants):

2. $u_{2n} = Ax\,\Phi\left(\frac{1}{2} - n; \frac{3}{2}; x^2\right),$

3. $u_{2n+1} = B\,\Phi\left(-\frac{1}{2} - n; \frac{1}{2}; x^2\right)$ MO 107

8.959(1)[10] Rodrigues' formula and orthogonality relation

1. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right]$ [Rodrigues' formula] MS 5.6.2

2. $\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi^{1/2} 2^n n! & \text{for } m = n \end{cases}$ MS 5.6.2

8.96 Jacobi's polynomials

8.960 Definition

1. $P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$ EH II 169(10), CO

$= \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m$ EH II 169(2)

8.961 Functional relations:

1.[11] $P_n^{(\alpha,\alpha)}(-x) = (-1)^n\, P_n^{(\alpha,\alpha)}(x)$ EH II 169(13)

2. $$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\, P_{n+1}^{(\alpha,\beta)}(x) = (2n+\alpha+\beta+1)\left[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+\alpha^2-\beta^2\right] P_n^{(\alpha,\beta)}(x) -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\, P_{n-1}^{(\alpha,\beta)}(x)$$ EH II 169(11)

3. $$(2n+\alpha+\beta)\left(1-x^2\right)\frac{d}{dx}\, P_n^{(\alpha,\beta)}(x) = n[(\alpha-\beta)-(2n+\alpha+\beta)x]\, P_n^{(\alpha,\beta)}(x) +2(n+\alpha)(n+\beta)\, P_{n-1}^{(\alpha,\beta)}(x)$$ EH II 170(15)

4.[11] $$\frac{d^m}{dx^m}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{2^m}\frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\, P_{n-m}^{(\alpha+m,\beta+m)}(x) \qquad [m=1,2,\ldots,n]$$ EH II 170(17)

5. $\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1-x)\, P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)\, P_n^{(\alpha,\beta)}(x) - (n+1)\, P_{n+1}^{(\alpha,\beta)}(x)$ EH II 173(32)

6. $\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1+x)\, P_n^{(\alpha,\beta+1)}(x) = (n+\beta+1)\, P_n^{(\alpha,\beta)}(x) + (n+1)\, P_{n+1}^{(\alpha,\beta)}(x)$ EH II 173(33)

7. $(1-x)\, P_n^{(\alpha+1,\beta)}(x) + (1+x)\, P_n^{(\alpha,\beta+1)}(x) = 2\, P_n^{(\alpha,\beta)}(x)$ EH II 173(34)

8. $(2n+\alpha+\beta)\, P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)\, P_n^{(\alpha,\beta)}(x) - (n+\beta)\, P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(35)

9. $(2n+\alpha+\beta)\, P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)\, P_n^{(\alpha,\beta)}(x) + (n+\alpha)\, P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(36)

10. $P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(37)

8.962 Connections with other functions:

1. $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\,\Gamma(n+1+\beta)}{n!\,\Gamma(1+\beta)}\, F\left(n+\alpha+\beta+1,-n;1+\beta;\frac{1+x}{2}\right)$$ CO, EH II 170(16)
$$= \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)}\, F\left(n+\alpha+\beta+1,-n;1+\alpha;\frac{1-x}{2}\right)$$ EH II 170(16)
$$= \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)}\left(\frac{1+x}{2}\right)^n F\left(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}\right)$$ EH II 170(16)
$$= \frac{\Gamma(n+1+\beta)}{n!\,\Gamma(1+\beta)}\left(\frac{x-1}{2}\right)^n F\left(-n,-n-\alpha;\beta+1;\frac{x+1}{x-1}\right)$$ EH II 170(16)

2. $P_n(x) = P_n^{(0,0)}(x)$ CO, EH II 179(3)

3. $$T_n(x) = \frac{2^{2n}\,(n!)^2}{(2n)!}\, P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)$$ CO, EH II 184(5)a

4. $$C_n^{\nu}(x) = \frac{\Gamma(n+2\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\,\Gamma\left(n+\nu+\frac{1}{2}\right)}\, P_n^{(\nu-1/2,\nu-1/2)}(x)$$ MO 108a, EH II 174(4)

8.963 The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

$$R = \sqrt{1-2xz+z^2} \qquad [|z|<1]$$

EH II 172(29)

8.964 The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$\left(1-x^2\right) y'' + [\beta - \alpha - (\alpha+\beta+2)x] y' + n(n+\alpha+\beta+1) y = 0.$$

EH II 169(14)

8.965 Asymptotic representation

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos\left\{\left[n+\frac{1}{2}(\alpha+\beta+1)\right]\theta - \left(\frac{1}{2}\alpha+\frac{1}{4}\right)\pi\right\}}{\sqrt{\pi n}\left(\sin\frac{1}{2}\theta\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{1}{2}\theta\right)^{\beta+\frac{1}{2}}} + O\left(n^{-3/2}\right) \qquad [\operatorname{Im}\alpha = \operatorname{Im}\beta = 0, \quad 0<\theta<\pi]$$

EH II 198(10)

8.966 A limit relationship:

$$\lim_{n\to\infty}\left[n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)\right] = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z)$$

EH II 173(41)

8.967 If $\alpha > -1$ and $\beta > -1$, all the zeros of the polynomial $P_n^{(\alpha,\beta)}(x)$ are simple, and they lie in the interval $(-1,1)$.

8.97 The Laguerre polynomials

8.970 Definition.

1. $$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right) \qquad \text{[Rodrigues' formula]}$$ EH II 188(5), MO 108

$$= \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}$$ MO 109, EH II 188(7)

2. $L_n^0(x) = L_n(x)$ ET I 369

3.[10] $L_0^\alpha(x) = 1$

4.[10] $L_1^\alpha(x) = -x + \alpha + 1$

5.[10] $L_2^\alpha(x) = \frac{1}{2}\left[x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2)\right]$

6.[10] $L_3^\alpha(x) = -\frac{1}{6}\left[x^3 - 3(\alpha+3)x^2 + 3(\alpha+2)(\alpha+3)x - (\alpha+1)(\alpha+2)(\alpha+3)\right]$

7.[10] $$L_4^\alpha(x) = \frac{1}{24}\Big[x^4 - 4(\alpha+4)x^3 + 6(\alpha+3)(\alpha+4)x^2 - 4(\alpha+2)(\alpha+3)(\alpha+4)x + (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\Big]$$

8.[10] $$L_5^\alpha(x) = -\frac{1}{120}\Big[x^5 - 5(\alpha+5)x^4 + 10(\alpha+4)(\alpha+5)x^3 - 10(\alpha+3)(\alpha+4)(\alpha+5)x^2 + 5(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)x - (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)\Big]$$

8.971 Functional relations:

1. $\frac{d}{dx}\left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] = L_n^\alpha(x)$ EH II 189(16)

2.[11] $\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{n\, L_n^\alpha(x) - (n+\alpha)\, L_{n-1}^\alpha(x)}{x}$ EH II 189(15), SM 575(42)a

3. $x\frac{d}{dx} L_n^\alpha(x) = n\, L_n^\alpha(x) - (n+\alpha)\, L_{n-1}^\alpha(x)$
$= (n+1)\, L_{n+1}^\alpha(x) - (n+\alpha+1-x)\, L_n^\alpha(x)$
EH II 189(12), MO 109

4. $x\, L_n^{\alpha+1}(x) = (n+\alpha+1)\ L_n^\alpha(x) - (n+1)\, L_{n+1}^\alpha(x)$
$= (n+\alpha)\, L_{n-1}^\alpha(x) - (n-x)\, L_n^\alpha(x)$
SM 575(43)a, EH II 190(23)

5. $L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x)$ SM 575(44)a, EH II 190(24)

6. $(n+1)\, L_{n+1}^\alpha(x) - (2n+\alpha+1-x)\, L_n^\alpha(x) + (n+\alpha)\, L_{n-1}^\alpha(x) = 0$
$[n = 1, 2, \ldots]$ MO 109, EH II 190(25, 24)

7.[10] $(n+\alpha)\, L_n^{\alpha-1}(x) = (n+1)\, L_{n+1}^\alpha(x) - (n+1-x)\, L_n^\alpha(x)$ MS 5.5.2

8.[10] $n\, L_n^\alpha(x) = (2n+\alpha-1-x)\, L_{n-1}^\alpha(x) - (n+\alpha-1)\, L_{n-2}^\alpha(x)$
$[n = 2, 3, \ldots]$ MS 5.5.2

8.972 Connections with other functions:

1. $L_n^\alpha(x) = \binom{n+\alpha}{n}\, \Phi(-n, \alpha+1; x)$ MO 109, FI II 189(14)

2. $H_{2n}(x) = (-1)^n 2^{2n} n!\, L_n^{-1/2}\left(x^2\right)$ EH II 193(2), SM 576(47)

3. $H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x\, L_n^{1/2}\left(x^2\right)$ EH II 193(3), SM 577(48)

8.973 Special cases:

1. $L_0^\alpha(x) = 1$ EH II 188(6)

2. $L_1^\alpha(x) = \alpha + 1 - x$ EH II 188(6)

3. $L_n^\alpha(0) = \binom{n+\alpha}{n}$ EH II 189(13)

4. $L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}$ MO 109

5. $L_1(x) = 1 - x$

6. $L_2(x) = 1 - 2x + \frac{x^2}{2}$ MO 109

8.974 Finite sums:

1. $$\sum_{m=0}^{n} \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x)\, L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} \left[L_n^\alpha(x)\, L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x)\, L_n^\alpha(y)\right]$$
EH II 188(9)

2.[11] $$\sum_{m=0}^{n} \frac{\Gamma(\alpha-\beta+m)}{\Gamma(\alpha-\beta)m!} L_{n-m}^\beta(x) = L_n^\beta(x)$$
MO 110, EH II 192(39)

3. $$\sum_{m=0}^{n} L_m^\alpha(x) = L_n^{\alpha+1}(x)$$
EH II 192(38)

4.[11] $$\sum_{m=0}^{n} L_m^\alpha(x)\, L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y)$$
EH II 192(41)

8.975 Arbitrary functions:

1. $$(1-z)^{-\alpha-1} \exp\frac{xz}{z-1} = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n \qquad [|z|<1]$$
EH II 189(17), MO 109

2. $$e^{-xz}(1+z)^\alpha = \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n \qquad [|z|<1]$$
MO 110, EH II 189(19)

3. $$J_\alpha\left(2\sqrt{xz}\right) e^z (xz)^{-\frac{1}{2}\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \qquad [\alpha>-1]$$
EH II 189(18), MO 109

8.976 Other series of Laguerre polynomials:

1. $$\sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x)\, L_n^\alpha(y) z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z} \exp\left(-z\frac{x+y}{1-z}\right) I_\alpha\left(2\frac{\sqrt{xyz}}{1-z}\right)$$
$[|z|<1]$ EH II 189(20)

2. $$\sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+1} = e^x x^{-\alpha}\, \Gamma(\alpha, x) \qquad [\alpha>-1, \quad x>0]$$
EH II 215(19)

3.[6] $$L_n^\alpha(x)^2 = \frac{\Gamma(n+\alpha+1)}{2^{2n} n!} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(2k)!}{k!} \frac{1}{\Gamma(\alpha+k+1)} L_{2k}^{2\alpha}(2x)$$
MO 110

4.[6] $$L_n^\alpha(x)\, L_n^\alpha(y) = \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^{n} \frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)} \frac{(xy)^k}{k!}$$
MO 110, EH II 192(42)

8.977 Summation theorems:

1. $$L_n^{\alpha_1+\alpha_2+\cdots+\alpha_k+k-1}(x_1+x_2+\cdots+x_k) = \sum_{i_1+i_2+\cdots+i_2=n} L_{i_1}^{\alpha_1}(x_1)\, L_{i_2}^{\alpha_2}(x_2) \cdots L_{i_k}^{\alpha_k}(x_k)$$
MO 110

2. $$L_n^\alpha(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k\, L_n^{\alpha+k}(x)$$
MO 110

8.978 Limit relations and asymptotic behavior:

1. $$L_n^\alpha(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)$$ EH II 191(35)

2. $$\lim_{n\to\infty}\left[n^{-\alpha}\, L_n^\alpha\left(\frac{x}{n}\right)\right] = x^{-\frac{1}{2}\alpha}\, J_\alpha\left(2\sqrt{x}\right)$$ EH II 191(36)

3. $$L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha-\frac{1}{4}} n^{\frac{1}{2}\alpha-\frac{1}{4}} \cos\left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O\left(n^{\frac{1}{2}\alpha-\frac{3}{4}}\right)$$
$$[\operatorname{Im}\alpha = 0, \quad x > 0]$$ EH II 199(1)

8.979 Laguerre polynomials satisfy the following differential equation:
$$x\frac{d^2u}{dx^2} + (\alpha - x + 1)\frac{du}{dx} + nu = 0$$ EH II 188(10), SM 574(34)

8.980[11] Orthogonality relation
$$\int_0^\infty e^{-x} x^\alpha\, L_n^\alpha(x)\, L_m^\alpha(x)\, dx = \begin{cases} 0, & m \neq n \\ \Gamma(1+\alpha)\binom{n+\alpha}{n}, & m = n \end{cases}$$ MS 5.5.2

8.981[10] Behavior of relative maxima of $|L_n^\alpha(x)|$

1. Let α be arbitrary and real. The sequence formed by the relative maxima of $|L_n^\alpha(x)|$ and by the value of this function at $x = 0$, is decreasing for $x < \alpha + \frac{1}{2}$, and increasing for $x > \alpha + \frac{1}{2}$. The successive relative maxima of $|L_n^\alpha(x)|$ form a decreasing sequence for $x \leq 0$, and an increasing sequence for $x \geq 0$. SZ 174(7.6.1)

2. Let α be an arbitrary real number. The successive relative maxima of
$$e^{-x/2} x^{(\alpha+1)/2} |L_n^\alpha(x)| \text{ and } e^{-x/2} x^{\alpha/2+\frac{1}{4}} |L_n^\alpha(x)|$$
form an increasing sequence, provided $x > x_0$. In the first case
$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \dfrac{\alpha^2 - 1}{2n + \alpha + 1} & \text{if } \alpha^2 > 1 \end{cases}$$
In the second case,
$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq q\frac{1}{4}, \\ \left(\alpha^2 - \frac{1}{4}\right)^{\frac{1}{2}} & \text{if } \alpha^2 > \frac{1}{4} \end{cases}$$ SZ 174(7.6.2)
In the first case, we take n so large that $2n + \alpha + 1 > 0$.

8.982[10] Asymptotic and limiting behavior of $L_n^\alpha(x)$

1. Let α be arbitrary and real, c and w fixed positive constants, and let $n \to \infty$. Then
$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}}\, O\left(n^{\alpha/2-\frac{1}{4}}\right) & \text{if } cn^{-1} \leq qx \leq q\omega \\ O\left(n^\alpha\right) & \text{if } 0 \leq qx \leq qcn^{-1} \end{cases}$$
These bounds are precise as regards their orders in n. For $\alpha \geq q - \frac{1}{2}$, both bounds hold in both intervals, that is,

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}} O\left(n^{\alpha/2-\frac{1}{4}}\right), & 0 < x \le q\omega, \quad \alpha \ge q - \frac{1}{2} \\ O\left(n^\alpha\right), \end{cases}$$ SZ 175(7.6.4)

2. Let α be arbitrary and real. Then for an arbitrary complex z

$$\lim_{n\to\infty} n^{-\alpha}\, L_n^\alpha(x) = z^{-\alpha/2}\, J_\alpha\left(2z^{1/2}\right),$$ SZ 191(8.1.3)

uniformly if z is bounded.

9.1 Hypergeometric Functions

9.10 Definition

9.100 A *hypergeometric series* is a series of the form

$$F(\alpha,\beta;\gamma;z)=1+\frac{\alpha\cdot\beta}{\gamma\cdot 1}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}z^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 1\cdot 2\cdot 3}z^3+\dots$$

9.101 A hypergeometric series terminates if α or β is equal to a negative integer or to zero. For $\gamma=-n\,(n=0,1,2,\dots)$, the hypergeometric series is indeterminate if neither α nor β is equal to $-m$ (where $m<n$ and m is a natural number). However,

1. $$\lim_{\gamma\to -n}\frac{F(\alpha,\beta;\gamma;z)}{\Gamma(\gamma)}=\frac{\alpha(\alpha+1)\dots(\alpha+n)\beta(\beta+1)\dots(\beta+n)}{(n+1)!}\times z^{n+1}\,F(\alpha+n+1,\beta+n+1;n+2;z)$$

EH I 62(16)

9.102 If we exclude these values of the parameters α,β,γ, a hypergeometric series converges in the unit circle $|z|<1$. F then has a branch point at $z=1$. Then we have the following conditions for convergence on the unit circle:

1. $1>\mathrm{Re}(\alpha+\beta-\gamma)\geq 0$. The series converges throughout the entire unit circle, except at the point $z=1$.
2. $\mathrm{Re}(\alpha+\beta-\gamma)<0$. The series converges (absolutely) throughout the entire unit circle.
3. $\mathrm{Re}(\alpha+\beta-\gamma)\geq 1$. The series diverges on the entire unit circle. FI II 410, WH

9.11 Integral representations

9.111 $$F(\alpha,\beta;\gamma;z)=\frac{1}{\mathrm{B}(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt\qquad[\mathrm{Re}\,\gamma>\mathrm{Re}\,\beta>0]$$ WH

9.112[8] $$F\left(p,n+p;n+1;z^2\right)=\frac{z^{-n}}{2\pi}\frac{\Gamma(p)n!}{\Gamma(p+n)}\int_0^{2\pi}\frac{\cos nt\,dt}{(1-2z\cos t+z^2)^p}$$
$[n=0,1,2,\dots;\quad p\neq 0,-1,-2,\dots;\quad |z|<1]$ WH, MO 16

9.113 $$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+t)\,\Gamma(\beta+t)\,\Gamma(-t)}{\Gamma(\gamma+t)}(-z)^t\,dt$$

Here, $|\arg(-z)|<\pi$ and the path of integration are chosen in such a way that the poles of the functions $\Gamma(\alpha+t)$ and $\Gamma(\beta+t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of it.

9.114 $$F\left(-m,-\frac{p+m}{2};1-\frac{p+m}{2};-1\right)=\frac{(-2)^m(p+m)}{\sin p\pi}\int_0^{\pi}\cos^m\varphi\cos p\varphi\,d\varphi$$
$[m+1$ is a natural number; $\quad p\neq 0,\quad \pm 1,\dots\,]$ EH I 80(8), MO 16

See also **3.194** 1, 2, 5, **3.196** 1, **3.197** 6, 9, **3.259** 3, **3.312** 3, **3.518** 4–6, **3.665** 2, **3.671** 1, 2, **3.681** 1, **3.984** 7.

9.12 Representation of elementary functions in terms of a hypergeometric functions

9.121

1.[8] $F(-n,\beta;\beta;-z)=(1+z)^n$ EH I 101(4), GA 127 Ia

2. $F\left(-\frac{n}{2},-\frac{n-1}{2};\frac{1}{2};\frac{z^2}{t^2}\right)=\frac{(t+z)^n+(t-z)^n}{2t^n}$ GA 127 II

3. $\lim\limits_{\omega\to\infty}F\left(-n,\omega;2\omega;-\frac{z}{t}\right)=\left(1+\frac{z}{2t}\right)^n$ GA 127 IIIa

4. $F\left(-\frac{n-1}{2},-\frac{n-2}{2};\frac{3}{2};\frac{z^2}{t^2}\right)=\frac{(t+z)^n-(t-z)^n}{2nzt^{n-1}}$ GA 127 IV

5. $F\left(1-n,1;2;-\frac{z}{t}\right)=\frac{(t+z)^n-t^n}{nzt^{n-1}}$ GA 127 V

6. $F(1,1;2;-z)=\frac{\ln(1+z)}{z}$ GA 127 VI

7. $F\left(\frac{1}{2},1;\frac{3}{2};z^2\right)=\frac{\ln\frac{1+z}{1-z}}{2z}$ GA 127 VII

8. $$\lim_{k\to\infty}F\left(1,k;1;\frac{z}{k}\right)=1+z\lim_{k\to\infty}F\left(1,k;2;\frac{z}{k}\right)$$
$$=1+z+\frac{z^2}{2}\lim_{k\to\infty}F\left(1,k;3;\frac{z}{k}\right)=\cdots=e^z$$
GA 127 VIII

9. $\lim\limits_{\substack{k\to\infty\\k'\to\infty}}F\left(k,k';\frac{1}{2};\frac{z^2}{4kk'}\right)=\frac{e^z+e^{-z}}{2}=\cosh z$ GA 127 IX

10. $\lim\limits_{\substack{k\to\infty\\k'\to\infty}}F\left(k,k';\frac{3}{2};\frac{z^2}{4kk'}\right)=\frac{e^z-e^{-z}}{2z}=\frac{\sinh z}{z}$ GA 127 X

11. $\lim\limits_{\substack{k\to\infty\\k'\to\infty}}F\left(k,k';\frac{3}{2};-\frac{z^2}{4kk'}\right)=\frac{\sin z}{z}$ GA 127 XI

12. $\lim\limits_{\substack{k\to\infty\\k'\to\infty}}F\left(k,k';\frac{1}{2};-\frac{z^2}{4kk'}\right)=\cos z$ GA 127 XII

13. $F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\sin^2 z\right)=\frac{z}{\sin z}$ GA 127 XIII

14. $F\left(1,1;\frac{3}{2};\sin^2 z\right)=\frac{z}{\sin z\cos z}$ GA 127 XIV

15. $F\left(\frac{1}{2},1;\frac{3}{2};-\tan^2 z\right)=\frac{z}{\tan z}$ GA 127 XV

16. $F\left(\frac{n+1}{2},-\frac{n-1}{2};\frac{3}{2};\sin^2 z\right)=\frac{\sin nz}{n\sin z}$ GA 127 XVI

17. $F\left(\frac{n+2}{2},-\frac{n-2}{2};\frac{3}{2};\sin^2 z\right)=\frac{\sin nz}{n\sin z\cos z}$ GA 127 XVII

18. $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}$ GA 127 XVIII

19. $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}$ GA 127 XIX

20. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz$ EH I 101(11), GA 127 XX

21. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}$ EH I 101(11), GA 127 XXI

22. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\tan^2 z\right) = \frac{\cos nz}{\cos^n z}$ EH I 101(11), GA 127 XXII

23. $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\tan^2 z\right) = \cos nz \cos^n z$ GA 127 XXIII

24. $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z}$ $\left[|z| \le \frac{1}{2}; \quad |z(1-z)| \le \frac{1}{4}\right]$

25. $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z$

26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$ (cf. **9.121** 13)

27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}$ (cf. **9.121** 15)

28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arcsinh} z}{z}$ (cf. **9.121** 26)

29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$ (cf. **9.121** 16)

30. $F\left(1+\frac{n}{2}, 1-\frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$ (cf. **9.121** 17)

31. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z)$ (cf. **9.121** 20)

32. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}$ (cf. **9.121** 21)

The representation of special functions in terms of a hypergeometric function:

- for complete elliptic integrals, see **8.113** 1 and **8.114** 1;
- for integrals of Bessel functions, see **6.574** 1, 3, **6.576** 2–5, **6.621** 1–3;
- for Legendre polynomials, see **8.911** and **8.916**. (All these hypergeometric series terminate; that is, these series are finite sums);
- for Legendre functions, see **8.820** and **8.837**;
- for associated Legendre functions, see **8.702, 8.703, 8.751, 8.77, 8.852**, and **8.853**;
- for Chebyshev polynomials, see **8.942** 1;
- for Jacobi's polynomials, see **8.962**;

- for Gegenbauer polynomials, see **8.932**;
- for integrals of parabolic cylinder functions, see **7.725** 6.

9.122 Particular values:

1. $$F(\alpha,\beta;\gamma;1)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad [\operatorname{Re}\gamma>\operatorname{Re}(\alpha+\beta)]$$

GA 147(48), FI II 793

2. $$F(\alpha,\beta;\gamma;1)=F\left(-\alpha,-\beta;\gamma-\alpha-\beta;1\right) \qquad [\operatorname{Re}\gamma>\operatorname{Re}(\alpha+\beta)]$$ GA 148(49)

$$=\frac{1}{F(-\alpha,\beta;\gamma-\alpha;1)} \qquad [\operatorname{Re}\gamma>\operatorname{Re}(\alpha+\beta)]$$ GA 148(50)

$$=\frac{1}{F(\alpha,-\beta;\gamma-\beta;1)} \qquad [\operatorname{Re}\gamma>\operatorname{Re}(\alpha+\beta)]$$ GA 148(51)

3. $$F\left(1,1;\frac{3}{2};\frac{1}{2}\right)=\frac{\pi}{2}$$ (cf. **9.121** 14)

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.130 The series $F(\alpha,\beta;\gamma;z)$ defines an analytic function that, speaking generally, has singularities at the points $z=0$, 1, and ∞. (In the general case, there are branch points.) We make a cut in the z-plane along the real axis from $z=1$ to $z=\infty$; that is, we require that $|\arg(-z)|<\pi$ for $|z|\geq 1$. Then, the series $f(\alpha,\beta;\gamma;z)$ will, in the cut plane, yield a single-valued analytic continuation, which we can obtain by means of the formulas below (provided $\gamma+1$ is not a natural number and $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers). These formulas make it possible to calculate the values of F in the given region, even in the case in which $|z|>1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α,β,γ.

Transformation formulas

9.131

1.[11] $$F(\alpha,\beta;\gamma;z)=(1-z)^{-\alpha}\,F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right)$$ GA 218(91)

$$=(1-z)^{-\beta}\,F\left(\beta,\gamma-\alpha;\gamma;\frac{z}{z-1}\right)$$ GA 218(92)

$$=(1-z)^{\gamma-\alpha-\beta}\,F\left(\gamma-\alpha,\gamma-\beta;\gamma;z\right)$$

2. $$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,F\left(\alpha,\beta;\alpha+\beta-\gamma+1;1-z\right)$$
$$+(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,F\left(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z\right)$$

EH I 94, MO 13

9.132

1. $$F(\alpha,\beta;\gamma;z)=\frac{(1-z)^{-\alpha}\,\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\,F\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right)$$
$$+(1-z)^{-\beta}\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}\,F\left(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right)$$
MO 13

2.[11] $$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\,F\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}(-z)^{-\beta}\,F\left(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}\right)$$
$$[|\arg z|<\pi,\quad \alpha-\beta\neq\pm m,\quad m=0,1,2,\ldots]$$ GA 220(93)

9.133 $$F\left(2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};z\right)=F\left(\alpha,\beta;\alpha+\beta+\tfrac{1}{2};4z(1-z)\right)$$
$$\left[|z|\le\tfrac{1}{2},\quad |z(1-z)|\le\tfrac{1}{4}\right]$$ WH

9.134

1. $$F(\alpha,\beta;2\beta;z)=\left(1-\frac{z}{2}\right)^{-\alpha}F\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\beta+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right)$$ MO 13, EH I 111(4)

2. $$F(2\alpha,2\alpha+1-\gamma;\gamma;z)=(1+z)^{-2\alpha}\,F\left(\alpha,\alpha+\frac{1}{2};\gamma;\frac{4z}{(1+z)^2}\right)$$ GA 225(100)

3. $$F\left(\alpha,\alpha+\frac{1}{2}-\beta;\beta+\frac{1}{2};z^2\right)=(1+z)^{-2\alpha}\,F\left(\alpha,\beta;2\beta;\frac{4z}{(1+z)^2}\right)$$ GA 225(101)

9.135 $$F\left(\alpha,\beta;\alpha+\beta+\frac{1}{2};\sin^2\varphi\right)=F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\sin^2\frac{\varphi}{2}\right)$$
$$\left[x=\sin^2\frac{\varphi}{2}\ \text{real};\quad \frac{1-\sqrt{2}}{2}<x<\frac{1}{2}\right]$$
MO 13

9.136[8] We set
$$A=\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)},\qquad B=\frac{-\Gamma\left(\alpha+\beta+\frac{1}{2}\right)2\sqrt{\pi}}{\Gamma(\alpha)\,\Gamma(\beta)};$$
then

1. $$F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{1-\sqrt{z}}{2}\right)=AF\left(\alpha,\beta;\frac{1}{2};z\right)+B\sqrt{z}\,F\left(\alpha+\frac{1}{2},\beta+\frac{1}{2};\frac{3}{2};z\right)$$
GA 227(106)

2. $$F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{1+\sqrt{z}}{2}\right)=AF\left(\alpha,\beta;\frac{1}{2};z\right)-B\sqrt{z}\,F\left(\alpha+\frac{1}{2},\beta+\frac{1}{2};\frac{3}{2};z\right)$$
GA 227(107)

3. $$\frac{\left(\alpha-\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)}{\alpha+\beta-\frac{1}{2}}A\sqrt{z}\,F\left(\alpha,\beta;\frac{3}{2};z\right)=F\left(2\alpha-1,2\beta-1;\alpha+\beta-\frac{1}{2};\frac{1+\sqrt{z}}{2}\right)$$
$$-F\left(2\alpha-1,2\beta-1;\alpha+\beta-\frac{1}{2};\frac{1-\sqrt{z}}{2}\right)$$
GA 229(110)

9.137[7] Gauss' recursion functions:

1. $\gamma[\gamma-1-(2\gamma-\alpha-\beta-1)z]\,F(\alpha,\beta;\gamma;z)+(\gamma-\alpha)(\gamma-\beta)z\,F(\alpha,\beta;\gamma+1;z)+\gamma(\gamma-1)(z-1)\,F(\alpha,\beta;\gamma-1;z)=0$
2. $(2\alpha-\gamma-\alpha z+\beta z)\,F(\alpha,\beta;\gamma;z)+(\gamma-\alpha)\,F(\alpha-1,\beta;\gamma;z)+\alpha(z-1)\,F(\alpha+1,\beta;\gamma;z)=0$
3. $(2\beta-\gamma-\beta z+\alpha z)\,F(\alpha,\beta;\gamma;z)+(\gamma-\beta)\,F(\alpha,\beta-1;\gamma;z)+\beta(z-1)\,F(\alpha,\beta+1;\gamma;z)=0$
4. $\gamma\,F(\alpha,\beta-1;\gamma;z)-\gamma\,F(\alpha-1,\beta;\gamma;z)+(\alpha-\beta)z\,F(\alpha,\beta;\gamma+1;z)=0$
5.[8] $\gamma(\alpha-\beta)\,F(\alpha,\beta;\gamma;z)-\alpha(\gamma-\beta)\,F(\alpha+1,\beta;\gamma+1;z)+\beta(\gamma-\alpha)\,F(\alpha,\beta+1;\gamma+1;z)=0$
6. $\gamma(\gamma+1)\,F(\alpha,\beta;\gamma;z)-\gamma(\gamma+1)\,F(\alpha,\beta;\gamma+1;z)-\alpha\beta z\,F(\alpha+1,\beta+1;\gamma+2;z)=0$
7. $\gamma\,F(\alpha,\beta;\gamma;z)-(\gamma-\alpha)\,F(\alpha,\beta+1;\gamma+1;z)-\alpha(1-z)\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
8. $\gamma\,F(\alpha,\beta;\gamma;z)+(\beta-\gamma)\,F(\alpha+1,\beta;\gamma+1;z)-\beta(1-z)\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
9. $\gamma(\gamma-\beta z-\alpha)\,F(\alpha,\beta;\gamma;z)-\gamma(\gamma-\alpha)\,F(\alpha-1,\beta;\gamma;z)+\alpha\beta z(1-z)\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
10. $\gamma(\gamma-\alpha z-\beta)\,F(\alpha,\beta;\gamma;z)-\gamma(\gamma-\beta)\,F(\alpha,\beta-1;\gamma;z)+\alpha\beta z(1-z)\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
11. $\gamma\,F(\alpha,\beta;\gamma;z)-\gamma\,F(\alpha,\beta+1;\gamma;z)+\alpha z\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
12.[8] $\gamma\,F(\alpha,\beta;\gamma;z)-\gamma\,F(\alpha+1,\beta;\gamma;z)+\beta z\,F(\alpha+1,\beta+1;\gamma+1;z)=0$
13. $\gamma[\alpha-(\gamma-\beta)z]\,F(\alpha,\beta;\gamma;z)-\alpha\gamma(1-z)\,F(\alpha+1,\beta;\gamma;z)+(\gamma-\alpha)(\gamma-\beta)z\,F(\alpha,\beta;\gamma+1;z)=0$
14. $\gamma[\beta-(\gamma-\alpha)z]\,F(\alpha,\beta;\gamma;z)-\beta\gamma(1-z)\,F(\alpha,\beta+1;\gamma;z)+(\gamma-\alpha)(\gamma-\beta)z\,F(\alpha,\beta;\gamma+1;z)=0$
15.[8] $\gamma(\gamma+1)\,F(\alpha,\beta;\gamma;z)-\gamma(\gamma+1)\,F(\alpha,\beta+1;\gamma+1;z)+\alpha(\gamma-\beta)z\,F(\alpha+1,\beta+1;\gamma+2;z)=0$
16. $\gamma(\gamma+1)\,F(\alpha,\beta;\gamma;z)-\gamma(\gamma+1)\,F(\alpha+1,\beta;\gamma+1;z)+\beta(\gamma-\alpha)z\,F(\alpha+1,\beta+1;\gamma+2;z)=0$
17. $\gamma\,F(\alpha,\beta;\gamma;z)-(\gamma-\beta)\,F(\alpha,\beta;\gamma+1;z)-\beta\,F(\alpha,\beta+1;\gamma+1;z)=0$
18.[8] $\gamma\,F(\alpha,\beta;\gamma;z)-(\gamma-\alpha)\,F(\alpha,\beta;\gamma+1;z)-\alpha\,F(\alpha+1,\beta;\gamma+1;z)=0$ MO 13–14

9.14 A generalized hypergeometric series

The series

1. $${}_pF_q\,(\alpha_1,\alpha_2,\ldots,\alpha_p;\ \beta_1,\beta_2,\ldots,\beta_q;\ z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\,(\alpha_2)_k\ldots(\alpha_p)_k}{(\beta_1)_k\,(\beta_2)_k\ldots(\beta_q)_k}\frac{z^k}{k!}$$ MO 14

   is called a *generalized hypergeometric series* (see also 9.210).

2. ${}_2F_1(\alpha,\beta;\gamma;z)\equiv F(\alpha,\beta;\gamma;z)$ MO 15

For integral representations, see **3.254** 2, **3.259** 2, and **3.478** 3.

9.15 The hypergeometric differential equation

9.151 A hypergeometric series is one of the solutions of the differential equation

$$z(1-z)\frac{d^2u}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{du}{dz}-\alpha\beta u=0,$$ WH

which is called the *hypergeometric equation*.

The solution of the hypergeometric differential equation

9.152 The hypergeometric differential equation **9.151** possesses *two linearly independent solutions.* These solutions have analytic continuations to the entire z-plane, except possibly for the three points 0, 1, and ∞. Generally speaking, the points $z = 0, 1, \infty$ are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio $w(z)$ of two linearly independent solutions satisfies the differential equation

$$2\frac{w'''}{w'} - 3\left(\frac{w''}{w'}\right)^2 = \frac{1-a_1^2}{z^2} + \frac{1-a_2^2}{(z-1)^2} + \frac{a_1^2+a_2^2-a_3^2-1}{z(z-1)},$$

where

$$a_1^2 = (1-\gamma)^2, \quad a_2^2 = (\gamma-\alpha-\beta)^2, \quad a_3^2 = (\alpha-\beta)^2.$$

If α, β, γ are real, the function $w(z)$ maps the upper (Im $z > 0$) or the lower (Im $z < 0$) half-plane onto a curvilinear triangle whose angles are $\pi a_1, \pi a_2, \pi a_3$. The vertices of this triangle are the images of the points $z = 0, z = 1$, and $z = \infty$.

9.153 Within the unit circle $|z| < 1$, the linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeometric differential equation are given by the following formulas:

1. If γ is not an integer,

$$u_1 = F(\alpha, \beta; \gamma; z),$$
$$u_2 = z^{1-\gamma} e\, F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z)$$

2. If $\gamma = 1$, then

$$u_1 = F(\alpha, \beta; 1; z),$$
$$u_2 = F(\alpha, \beta; 1; z)\ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k(\beta)_k}{(k!)^2}$$
$$\times \{\psi(\alpha+k) - \psi(\alpha) + \psi(\beta+k) - \psi(\beta) - 2\,\psi(k+1) + 2\,\psi(1)\}$$

(see **9.14** 2)

3. If $\gamma = m+1$ (where m is a natural number), and if neither α nor β is a positive number not exceeding m, then

$$u_1 = F(\alpha, \beta; m+1; z),$$
$$u_2 = F(\alpha, \beta; m+1; z)\ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k(\beta)_k}{(1+m)_k}\{h(k) - h(0)\} - \sum_{k=1}^{m} \frac{(k-1)!(-m)_k}{(1-\alpha)_k(1-\beta)_k} z^{-k}$$

(see **9.14** 2)

where

$$h(n) = \psi(\alpha+n) + \psi(\beta+n) - \psi(m+1+n) - \psi(n+1) \qquad [n+1 \text{ is a natural number}]$$

4.[11] Suppose that $\gamma = m+1$ (where m is a natural number) and that α or β is equal to $m'+1$, where $0 \le m' < m$. Then, for example, for $\alpha = m'+1$, we obtain

$$u_1 = F(1+m', \beta; 1+m; z),$$
$$u_2 = z^{-m} F(1+m'-m, \beta-m; 1-m; z)$$

In this case, u_2 is a polynomial in z^{-1}.

5. If $\gamma = 1 - m$ (where m is a natural number) and if α and β are both different from the numbers $0, -1, -2, \ldots, 1 - m$, then

$$u_1 = z^m\, F(\alpha + m, \beta + m; 1 + m; z),$$

$$u_2 = z^m\, F(\alpha + m, \beta + m; 1 + m; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha + m)_k (\beta + m)_k}{(1 + m)_k k!} \{h^*(k) - h^*(0)\}$$

$$- \sum_{k=1}^{\infty} \frac{(k-1)!(-m)_k}{(1 - \alpha - m)_k (1 - \beta - m)_k} z^{m-n}$$

(see **9.14** 2)

where

$$h^*(n) = \psi(\alpha + m + n) + \psi(\beta + m + n) - \psi(1 + m + n) - \psi(1 + n)$$

We note that

$$\psi(\alpha + n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha + 1} + \cdots + \frac{1}{\alpha + n - 1}$$ (cf. **8.365** 3)

and that, for $\alpha = -\lambda$, where λ is a natural number or zero and $n = \lambda + 1, \lambda + 2, \ldots$ the expression

$$(\alpha)_k \left[\psi(\alpha + n) - \psi(\alpha)\right]$$

in formulas **9.153** 2–5 should be replaced with the expression

$$(-1)^\lambda \lambda!(n - \lambda - 1)!$$

6. Suppose that $\gamma = 1 - m$ (where m is a natural number) and that α or β is an integer $(-m')$, where m' is one of the following numbers: $0, 1, \ldots, m - 1$. Suppose, for example, that $\alpha = -m'$. Then,

$$u_1 = F\left(-m', \beta; 1 - m; z\right),$$
$$u_2 = F\left(-m' + m, \beta + m; 1 + m; z\right)$$

MO 18

7. For $\gamma = \frac{1}{2}(\alpha + \beta + 1)$

$$u_1 = F\left(\alpha, \beta; \tfrac{1}{2}(\alpha + \beta + 1); z\right),$$
$$u_2 = F\left(\alpha, \beta; \tfrac{1}{2}(\alpha + \beta + 1); 1 - z\right)$$

are two linearly independent solutions of the hypergeometric differential equation, provided α, β, and γ are not zero or negative integers. MO 17–19

The analytic continuation of a solution that is regular at the point $z = 0$

9.154 Formulas **9.153** make possible the analytic continuation, by means of the hypergeometric series, of the function $F(\alpha, \beta; \gamma; z)$ defined inside the circle $|z| < 1$ to the region $|z| > 1$, and $|\arg(-z)| < \pi$. Here, it is assumed that $\alpha - \beta$ is not an integer. In the event that $\alpha - \beta$ is an integer (for example, if $\beta = \alpha + m$, where m is a natural number), then, for $|z| > 1$, and $|\arg(-z)| < \pi$ we have:

1. $$\frac{\Gamma(\alpha)\,\Gamma(\alpha+m)}{\Gamma(\gamma)}\,F(\alpha,\alpha+m;\gamma;z)$$
$$=\frac{\sin\pi(\gamma-\alpha)}{\pi}\left\{\sum_{k=0}^{m-1}\frac{\Gamma(\alpha+k)\,\Gamma(1-\gamma+\alpha+k)\,\Gamma(m-k)}{k!}(-z)^{-\alpha-k}\right.$$
$$\left.+(-z)^{-\alpha-m}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+m+k)\,\Gamma(1-\gamma+\alpha+m+k)}{k!(k+m)!}g(k)z^{-k}\right\}$$
where

2. $$g(n)=\ln(-z)+\pi\cot\pi(\gamma-\alpha)+\psi(n+1)+\psi(n+m+1)$$
$$-\psi(\alpha+m+n)-\psi(1-\gamma+\alpha+m+n)$$
For $m=0$, we should set $\sum\limits_{k=0}^{m-1}=0$.

9.155 This formula loses its meaning when α,γ, or $\alpha-\gamma+1$ is equal to one of the numbers $0,-1,-2,\ldots$. In this last case, we have

1. If α is a non-positive integer and γ is not an integer, $F(\alpha,\alpha+m;\gamma;z)$ is a polynomial in z.
2. Suppose that γ is a non-positive integer and that α is not an integer. We then set $\gamma=-\lambda$, where $\lambda=0,1,2,\ldots$. Then,
$$\frac{\Gamma(\alpha+\lambda+1)\,\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)}z^{\lambda+1}\,F(\alpha+\lambda+1,\alpha+\lambda+m+1;\lambda+2;z)$$
is a solution of the hypergeometric equation that is regular at the point $z=0$. This solution is equal to the right-hand member of formula **9.154** 1 if we replace γ with λ in this equation and in formula **9.154** 2.
3. If $\alpha-\gamma+1$ is a non-positive integer and if α and γ are not themselves integers, we may use the formula
$$F(\alpha,\alpha+m;\gamma;z)=(1-z)^{\gamma-2\alpha-m}\,F(\gamma-\alpha-m,\gamma-\alpha;\gamma;z)$$
and apply formula **9.154** 1 to its right-hand member, provided $\gamma-\alpha-m>0$. However, if $\alpha-\gamma-m\leq 0$, the right member of this expression is a polynomial taken to the $(1-z)^{\text{th}}$ power.
4. If α,β, and γ are integers, the hypergeometric differential equation always has a solution that is regular for $z=0$ and that is of the form
$$R_1(z)+\ln(1-z)R_2(z),$$
where $R_1(z)$ and $R_2(z)$ are rational functions of z. To get a solution of this form, we need to apply formulas **9.137** 1–**9.137** 3 to the function $F(\alpha,\beta;\gamma;z)$. However, if $\gamma=-\lambda$, where $\lambda+1$ is a natural number, formulas **9.137** 1 and **9.137** 2 should be applied not to $F(\alpha,\beta;\gamma;z)$ but to the function $z^{\lambda+1}\,F(\alpha+\lambda+1,\beta+\lambda+1;\lambda+2,z)$.

   By successive applications of these formulas, we can reduce the positive values of the parameters to the pair, unity and zero. Furthermore, we can obtain the desired form of the solution from the formulas
$$F(1,1;2;z)=-z^{-1}\ln(1-z),$$
$$F(0,\beta;\gamma;z)=F(\alpha,0;\gamma;z)=1$$

9.16 Riemann's differential equation

9.160 The hypergeometric differential equation is a particular case of Riemann's differential equation

1.[11]
$$\frac{d^2u}{dz^2}+\left[\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right]\frac{du}{dz}$$
$$+\left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}\ \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)}=0$$
WH

The coefficients of this equation have poles at the points a, b, and c, and the numbers α,α'; β,β'; γ,γ' are called the indices corresponding to these poles. The indices α,α'; β,β'; γ,γ' are related by the following equation:

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'-1=0$$ WH

2. The differential equations **9.160** 1 are written diagramatically as follows:

3. $$u=P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column. WH

9.161 The two following transformation formulas are valid for Riemann's P-equation:

1. $$\left(\frac{z-a}{z-b}\right)^k\left(\frac{z-c}{z-b}\right)^l P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta-k-1 & \gamma+l & z \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{Bmatrix}$$ WH

2. $$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$ WH

The first of these formulas means that if

$$u=P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

then the function

$$u_1=\left(\frac{z-a}{z-b}\right)^k\left(\frac{z-c}{z-b}\right)^l u$$

satisfies a second-order differential equation having the same singular points as equation **9.161** 2 and indices equal to $\alpha+k,\alpha'+k;\beta-k-l,\beta'-k-l;\gamma+l,\gamma'+l$. The second transformation formula converts a differential equation with singularities at the points a,b, and c, indices $\alpha,\alpha';\beta,\beta';\gamma,\gamma'$, and an independent variable z into a differential equation with the same indices, singular points a_1,b_1, and c_1, and independent variable z_1. The variable z_1 is connected with the variable z by the fractional transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \qquad [AD - BC \neq 0]$$

The same transformation connects the points a_1, b_1, and c_1 with the points a, b, and c.

WH, MO 20

9.162 By the successive application of the two transformation formulas **9.161** 1 and **9.161** 2, we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

For $k = -\alpha, l = -\gamma$, and $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$, we have

1. $$u = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$
$$= \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

MO 23

Thus, this solution can be expressed as a hypergeometric series as follows:

2. $$u = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)}\right)$$

If the constants a, b, c; α, α'; β, β'; γ, γ' are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ is an integer): WH, MO 23

9.163

1. $$u_1 = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

2. $$u_2 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha'+\beta+\gamma, \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

3. $$u_3 = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha+\beta+\gamma', \alpha+\beta'+\gamma'; 1+\alpha-\alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

4. $$u_4 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha'+\beta+\gamma', \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

9.164

1.[10] $$u_5 = \left(\frac{z-b}{z-c}\right)^{\beta}\left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta+\gamma+\alpha, \beta+\gamma'+\alpha; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

2. $$u_6 = \left(\frac{z-b}{z-c}\right)^{\beta'}\left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta'+\gamma+\alpha, \beta'+\gamma'+\alpha; 1+\beta'-\beta; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

3. $$u_7 = \left(\frac{z-b}{z-c}\right)^{\beta}\left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta+\gamma+\alpha', \beta+\gamma'+\alpha'; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

4. $$u_8 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta'+\gamma+\alpha', \beta'+\alpha'+\gamma'; 1+\beta'-\beta; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

9.165

1. $$u_9 = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma+\alpha+\beta, \gamma+\alpha'+\beta; 1+\gamma-\gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

2. $$u_{10} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma'+\alpha+\beta, \gamma'+\alpha'+\beta; 1+\gamma'-\gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

3. $$u_{11} = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma+\alpha+\beta', \gamma+\alpha'+\beta'; 1+\gamma-\gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

4. $$u_{12} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma'+\alpha+\beta', \gamma'+\alpha'+\beta'; 1+\gamma'-\gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

9.166

1. $$u_{13} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha+\gamma+\beta, \alpha+\gamma'+\beta; 1+\alpha-\alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

2. $$u_{14} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha'+\gamma+\beta, \alpha'+\gamma'+\beta; 1+\alpha'-\alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

3. $$u_{15} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha+\gamma+\beta', \alpha+\gamma'+\beta'; 1+\alpha-\alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

4. $$u_{16} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha'+\gamma+\beta', \alpha'+\gamma'+\beta'; 1+\alpha'-\alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

9.167

1. $$u_{17} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma+\beta+\alpha, \gamma+\beta'+\alpha; 1+\gamma-\gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

2. $$u_{18} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma'+\beta+\alpha, \gamma'+\beta'+\alpha; 1+\gamma'-\gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

3. $$u_{19} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma+\beta+\alpha', \gamma+\beta'+\alpha'; 1+\gamma-\gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

4. $$u_{20} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma'+\beta+\alpha', \gamma'+\beta'+\alpha'; 1+\gamma'-\gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

9.168

1. $$u_{21} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta+\alpha+\gamma, \beta+\alpha'+\gamma; 1+\beta-\beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

2. $$u_{22} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta'+\alpha+\gamma, \beta'+\alpha'+\gamma; 1+\beta'-\beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

3. $$u_{23} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta+\alpha+\gamma', \beta+\alpha'+\gamma'; 1+\beta-\beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

4. $$u_{24} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta'+\alpha+\gamma', \beta'+\alpha'+\gamma'; 1+\beta'-\beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$ WH

9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme

9.171 The hypergeometric equation (see **9.151**):

$$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & z \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{matrix}\right\}$$ WH

9.172 The associated Legendre's equation defining the functions $P_n^m(z)$ for n and m integers (see **8.700** 1):

1. $$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & \dfrac{1-z}{2} \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}$$ WH

2. $$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ -\frac{1}{2}n & \frac{1}{2}m & 0 & \dfrac{1}{1-z^2} \\ \dfrac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} & \end{matrix}\right\}$$ WH

9.173 The function $P_n^m\left(1-\dfrac{z^2}{2n^2}\right)$ satisfies the equation

$$u = P\left\{\begin{matrix} 4n^2 & \infty & 0 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & z^2 \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}$$ WH

The function $J_m(z)$ satisfies the limiting form of this equation obtained as $n \to \infty$.

9.174 The equation defining the Gegenbauer polynomials $C_n^{\lambda}(z)$ (see **8.938**):

$$u = P\left\{\begin{matrix} -1 & \infty & 1 & \\ \frac{1}{2}-\lambda & n+2\lambda & \frac{1}{2}-\lambda & z \\ 0 & -n & 0 & \end{matrix}\right\}$$ WH

9.175 Bessel's equation (see **8.401**) is the limiting form of the equations:

1. $$u = P\left\{\begin{matrix} 0 & \infty & c & \\ n & ic & \frac{1}{2}+ic & z \\ -n & -ic & \frac{1}{2}-ic & \end{matrix}\right\}$$ WH

2. $$u = e^{iz}P\left\{\begin{matrix} 0 & \infty & c & \\ n & \frac{1}{2} & 0 & z \\ -n & \frac{3}{2}-2ic & 2ic-1 & \end{matrix}\right\}$$ WH

3. $$u = P\left\{\begin{matrix} 0 & \infty & c^2 & \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 & z^2 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 & \end{matrix}\right\}$$ WH

as $c \to \infty$.

9.18 Hypergeometric functions of two variables

9.180

1. $$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^my^n \qquad [|x|<1,\quad |y|<1]$$
EH I 224(6), AK 14(11)

2. $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n\,m!n!}x^my^n \qquad [|x|+|y|<1]$$
EH I 224(7), AK 14(12)

3. $$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^my^n \qquad [|x|<1,\quad |y|<1]$$
EH I 224(8), AK 14(13)

4. $$F_4(\alpha,\beta,\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n\,m!n!}x^my^n \qquad \left[\left|\sqrt{x}\right|+|\sqrt{y}|<1\right]$$
EH I 224(9), AK 14(14)

9.181 The functions F_1, F_2, F_3, and F_4 satisfy the following systems of partial differential equations for z:

1. System of equations for $z=F_1$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2}+y(1-x)\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma-(\alpha+\beta+1)x]\frac{\partial z}{\partial x}-\beta y\frac{\partial z}{\partial y}-\alpha\beta z=0,$$
EH I 233(9)
$$y(1-y)\frac{\partial^2 z}{\partial y^2}+x(1-y)\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma-(\alpha+\beta'+1)\,y]\frac{\partial z}{\partial x}-\beta' x\frac{\partial z}{\partial x}-\alpha\beta' z=0$$

2. System of equations for $z=F_2$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2}-xy\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma-(\alpha+\beta+1)x]\frac{\partial z}{\partial x}-\beta y\frac{\partial z}{\partial y}-\alpha\beta z=0,$$
EH I 234(10)
$$y(1-y)\frac{\partial^2 z}{\partial y^2}-xy\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma'-(\alpha+\beta'+1)\,y]\frac{\partial z}{\partial y}-\beta' x\frac{\partial z}{\partial x}-\alpha\beta' z=0$$

3. System of equations for $z=F_3$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma-(\alpha+\beta+1)x]\frac{\partial z}{\partial x}-\alpha\beta z=0,$$
$$y(1-y)\frac{\partial^2 z}{\partial y^2}+x\frac{\partial^2 z}{\partial x\,\partial y}+[\gamma-(\alpha'+\beta'+1)\,y]\frac{\partial z}{\partial y}-\alpha'\beta' z=0$$
EH I 234(11)

4. System of equations for $z = F_4$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + [\gamma - (\alpha+\beta+1)x]\,\frac{\partial z}{\partial x} - (\alpha+\beta+1)y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(12)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + [\gamma' - (\alpha+\beta+1)y]\,\frac{\partial z}{\partial y} - (\alpha+\beta+1)x\frac{\partial z}{\partial x} - \alpha\beta z = 0$$

AK 44

9.182 For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

1. $F_1\left(\alpha,\beta,\beta',\beta+\beta';x,y\right) = (1-y)^{-\alpha}\,F\left(\alpha,\beta;\beta+\beta';\dfrac{x-y}{1-y}\right)$ EH I 238(1), AK 24(28)

2. $F_2\left(\alpha,\beta,\beta',\beta,\gamma';x,y\right) = (1-x)^{-\alpha}\,F\left(\alpha,\beta';\gamma';\dfrac{y}{1-x}\right)$ EH I 238(2), AK 23

3. $F_2\left(\alpha,\beta,\beta',\alpha,\alpha;x,y\right) = (1-x)^{-\beta}(1-y)^{-\beta'}\,F\left(\beta,\beta';\alpha;\dfrac{xy}{(1-x)(1-y)}\right)$ EH I 238(3)

4. $F_3\left(\alpha,\gamma-\alpha,\beta,\gamma-\beta,\gamma;x,y\right) = (1-y)^{\alpha+\beta-\gamma}\,F(\alpha,\beta;\gamma;x+y-xy)$ EH I 238(4), AK 25(35)

5. $F_4\left(\alpha,\gamma+\gamma'-\alpha-1,\gamma,\gamma';x(1-y),y(1-x)\right)$
$$= F\left(\alpha,\gamma+\gamma'-\alpha-1;\gamma;x\right)F\left(\alpha,\gamma+\gamma'-\alpha-1;\gamma';y\right)$$
EH I 238(5)

6. $F_4\left(\alpha,\beta,\alpha,\beta;-\dfrac{x}{(1-x)(1-y)},\dfrac{-y}{(1-x)(1-y)}\right) = \dfrac{(1-x)^{\beta}(1-y)^{\alpha}}{(1-xy)}$ EH I 238(6)

7. $F_4\left(\alpha,\beta,\beta,\beta;-\dfrac{x}{(1-x)(1-y)},-\dfrac{y}{(1-x)(1-y)}\right) = (1-x)^{\alpha}(1-y)^{\alpha}\,F\left(\alpha,1+\alpha-\beta;\beta;xy\right)$
EH I 238(7)

8. $F_4\left(\alpha,\beta,1+\alpha-\beta,\beta;-\dfrac{x}{(1-x)(1-y)},-\dfrac{y}{(1-x)(1-y)}\right)$
$$= (1-y)^{\alpha}\,F\left[\alpha,\beta;1+\alpha-\beta;-\frac{x(1-y)}{1-x}\right]$$
EH I 238(8)

9. $F_4\left(\alpha,\alpha+\dfrac{1}{2},\gamma,\dfrac{1}{2};x,y\right) = \dfrac{1}{2}\left(1+\sqrt{y}\right)^{-2\alpha}F\left(\alpha,\alpha+\dfrac{1}{2};\gamma;\dfrac{x}{\left(1+\sqrt{y}\right)^2}\right)$
$$+\frac{1}{2}\left(1-\sqrt{y}\right)^{-2\alpha}F\left(\alpha,\alpha+\frac{1}{2};\gamma;\frac{x}{\left(1-\sqrt{y}\right)^2}\right)$$
AK 23

10. $F_1\left(\alpha,\beta,\beta',\gamma;x,1\right) = \dfrac{\Gamma(\gamma)\,\Gamma\left(\gamma-\alpha-\beta'\right)}{\Gamma(\gamma-\alpha)\,\Gamma\left(\gamma-\beta'\right)}\,F\left(\alpha,\beta:\gamma-\beta';x\right)$ EH I 239(10), AK 22(23)

11. $F_1(\alpha,\beta,\beta',\gamma;x,x) = F(\alpha,\beta+\beta';\gamma;x)$ EH I 239(11), AK 23(25)

9.183 Functional relations between hypergeometric functions of two variables:

1. $$F_1(\alpha,\beta,\beta',\gamma;x,y) = (1-x)^{-\beta}(1-y)^{-\beta}\,F_1\left(\gamma-\alpha,\beta,\beta',\gamma;\frac{x}{x-1},\frac{y}{y-1}\right)$$

EH I 239(1)

$$= (1-x)^{-\alpha}\,F_1\left(\alpha,\gamma-\beta-\beta',\beta',\gamma;\frac{x}{x-1},\frac{y-x}{1-x}\right)$$

EH I 239(2)

$$= (1-y)^{-\alpha}\,F_1\left(\alpha,\beta,\gamma-\beta-\beta',\gamma;\frac{y-x}{y-1},\frac{y}{y-1}\right)$$

EH I 239(3)

$$= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'}\,F_1\left(\gamma-\alpha,\gamma-\beta-\beta',\beta',\gamma;x,\frac{x-y}{1-y}\right)$$

EH I 240(4)

$$= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}\,F_1\left(\gamma-\alpha,\beta,\gamma-\beta-\beta',\gamma;\frac{x-y}{x-1},y\right)$$

EH I 240(5), AK 30(5)

2.[8] $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = (1-x)^{-\alpha}\,F_2\left(\alpha,\gamma-\beta,\beta',\gamma,\gamma';\frac{x}{x-1},\frac{y}{1-x}\right)$$

EH I 240(6)

$$= (1-y)^{-\alpha}\,F_2\left(\alpha,\beta,\gamma'-\beta',\gamma,\gamma';\frac{x}{1-y},\frac{y}{y-1}\right)$$

EH I 240(7)

$$= (1-x-y)^{-\alpha}\,F_2\left(\alpha,\gamma-\beta,\gamma'-\beta',\gamma,\gamma';\frac{x}{x+y-1},\frac{y}{x+y-1}\right)$$

EH I 240(8), AK 32(6)

3.[7] $$F_4(\alpha,\beta,\gamma,\gamma';x,y) = \frac{\Gamma(\gamma')\,\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\,\Gamma(\beta)}(-y)^{-\alpha}\,F_4\left(\alpha,\alpha+1-\gamma',\gamma,\alpha+1-\beta;\frac{x}{y},\frac{1}{y}\right)$$
$$+\frac{\Gamma(\gamma'(\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\,\Gamma(\alpha)}(-y)^{\beta}\,F_4\left(\beta+1-\gamma',\beta,\gamma,\beta+1-\alpha;\frac{x}{y},\frac{1}{y}\right)$$

EH I 240(9), AK 26(37)

9.184 Integral representations: Double integrals of the Euler type

1. $$F_1(\alpha,\beta,\beta',\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')}\times\iint\limits_{\substack{u\geq 0,v\geq 0\\ u+v\leq 1}} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-\alpha}\,du\,dv$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\beta'>0,\quad \operatorname{Re}(\gamma-\beta-\beta')>0]$$ EH I 230(1), AK 28(1)

2. $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=\frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta)\,\Gamma(\gamma'-\beta')}\times\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}\,du\,dv$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\beta'>0,\quad \operatorname{Re}(\gamma-\beta)>0,\quad \operatorname{Re}(\gamma'-\beta')>0]$$ EH I 230(2), AK 28(2)

3. $$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')}\times\iint\limits_{\substack{u\geq 0,v\geq 0\\ u+v\leq 1}} u^{\beta-1}v^{\beta'-1}(1-u-v)^{-\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}\,du\,dv$$
$$[\operatorname{Re}\beta>0,\quad \operatorname{Re}\beta'>0,\quad \operatorname{Re}(\gamma-\beta-\beta')>0]$$ EH I 230(3), AK 28(3)

4. $$F_4(\alpha,\beta,\gamma,\gamma';x(1-y),y(1-x))=\frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma'-\beta)}\int_0^1\int_0^1 u^{\alpha-1}v^{\beta-1}(1-u)^{\gamma-\alpha-1}(1-v)^{\gamma'-\beta-1}\times(1-ux)^{\alpha-\gamma-\gamma'+1}(1-vy)^{\beta-\gamma-\gamma'+1}(1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1}\,du\,dv$$
$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}(\gamma-\alpha)>0,\quad \operatorname{Re}(\gamma'-\beta)>0]$$ EH I 230(4)

9.185 Integral representations: Integrals of the Mellin–Barnes type

The functions F_1, F_2, F_3, and F_4 can be represented by means of double integrals of the following form:

$$F(x,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\Psi(s,t)\,\Gamma(-s)\,\Gamma(-t)(-x)^s(-y)^t\,ds\,dt$$

$\Psi(s,t)$	$F(x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)}{\Gamma(\beta')\,\Gamma(\gamma+s+t)}$	$F_1(\alpha,\beta,\beta',\gamma;x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)\,\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma+s)\,\Gamma(\gamma'+t)}$	$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)$
$\dfrac{\Gamma(\alpha+s)\,\Gamma(\alpha'+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)}{\Gamma(\alpha')\,\Gamma(\beta')\,\Gamma(\gamma+s+t)}$	$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(\gamma')}{\Gamma(\gamma+s)\,\Gamma(\gamma'+t)}$	$F_4(\alpha,\beta,\gamma,\gamma';x,y)$
$[\alpha,\alpha',\beta,\beta'$ may not be negative integers]	EH I 232(9–13), AK 41(33)

9.19 A hypergeometric function of several variables

$$F_A(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;z_1,\ldots,z_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}\,m_1!\cdots m_n!}z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$$

ET I 385

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.201[10] A *confluent hypergeometric function* is obtained by taking the limit as $c\to\infty$ in the solution of Riemann's differential equation

$$u = P\left\{\begin{matrix} 0 & \infty & c & \\ \frac{1}{2}+\mu & -c & c-\lambda & z \\ \frac{1}{2}-\mu & 0 & \lambda & \end{matrix}\right\}$$

WH

9.202 The equation obtained by means of this limiting process is of the form

1. $$\frac{d^2u}{dz^2}+\frac{du}{dz}+\left(\frac{\lambda}{z}+\frac{\frac{1}{4}-\mu^2}{z^2}\right)u=0$$ WH

Equation **9.202** 1 has the following two linearly independent solutions:

2. $z^{\frac{1}{2}+\mu}e^{-z}\,\Phi\left(\frac{1}{2}+\mu-\lambda,2\mu+1;z\right)$
3. $z^{\frac{1}{2}-\mu}e^{-z}\,\Phi\left(\frac{1}{2}-\mu-\lambda,-2\mu+1;z\right)$

which are defined for all values of $\mu\neq\pm\frac{1}{2},\pm\frac{2}{2},\pm\frac{3}{2},\ldots$ MO 111

9.21 The functions $\Phi(\alpha,\gamma;z)$ and $\Psi(\alpha,\gamma;z)$

9.210[10] The series

1. $$\Phi(\alpha,\gamma;z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \dots$$
   is also called a *confluent hypergeometric function.*
   A second notation: $\Phi(\alpha,\gamma;z) = {}_1F_1(\alpha;\gamma;z)$.
2. $$\Psi(\alpha,\gamma;z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha,\gamma;z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(\alpha-\gamma+1,2-\gamma;z)$$ EH I 257(7)
3. Bateman's function $\mathrm{k}_\nu(x)$ is defined by
   $$\mathrm{k}_\nu(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\tan\theta - \nu\theta)\,d\theta \qquad [x,\ \nu \text{ real}]$$ EH I 267

9.211 Integral representation:

1. $$\Phi(\alpha,\gamma;z) = \frac{2^{1-\gamma}e^{\frac{1}{2}z}}{\mathrm{B}(\alpha,\gamma-\alpha)}\int_{-1}^{1}(1-t)^{\gamma-\alpha-1}(1+t)^{\alpha-1}e^{\frac{1}{2}zt}\,dt$$
   $[0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma]$ MO 114
2. $$\Phi(\alpha,\gamma;z) = \frac{1}{\mathrm{B}(\alpha,\gamma-\alpha)}z^{1-\gamma}\int_0^z e^t t^{\alpha-1}(z-t)^{\gamma-\alpha-1\,dt}$$
   $[0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma]$ MO 114
3. $$\Phi(-\nu,\alpha+1;z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}e^z z^{-\frac{\alpha}{2}}\int_0^\infty e^{-t}t^{\nu+\frac{\alpha}{2}}\,J_\alpha\left(2\sqrt{zt}\right)\,dt$$
   $\left[\operatorname{Re}(\alpha+\nu+1) > 0, \quad |\arg z| < \frac{\pi}{2}\right]$
   MO 115
4.[8] $$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}\,dt \qquad [\operatorname{Re}\alpha > 0, \quad \operatorname{Re} z > 0]$$ EH I 255(2)

Functional relations

9.212

1. $\Phi(\alpha,\gamma;z) = e^z\,\Phi(\gamma-\alpha,\gamma;-z)$ MO 112
2. $\frac{z}{\gamma}\,\Phi(\alpha+1,\gamma+1;z) = \Phi(\alpha+1,\gamma;z) - \Phi(\alpha,\gamma;z)$ MO 112
3. $\alpha\,\Phi(\alpha+1,\gamma+1;z) = (\alpha-\gamma)\,\Phi(\alpha,\gamma+1;z) + \gamma\,\Phi(\alpha,\gamma;z)$ MO 112
4. $\alpha\,\Phi(\alpha+1,\gamma;z) = (z+2a-\gamma)\,\Phi(\alpha,\gamma;z) + (\gamma-\alpha)\,\Phi(\alpha-1,\gamma;z)$ MO 112

9.213 $\frac{d\Phi}{dz} = \frac{\alpha}{\gamma}\,\Phi(\alpha+1,\gamma+1;z)$ MO 112

9.214 $\lim\limits_{\gamma\to-n}\frac{1}{\Gamma(\gamma)}\,\Phi(\alpha,\gamma;z) = z^{n+1}\binom{\alpha+n}{n+1}\Phi(\alpha+n+1,n+2;z) \qquad [n = 0,1,2,\dots]$ MO 112

9.215[10]

1. $\Phi(\alpha,\alpha;z)=e^z$ MO 15

2. $\Phi(\alpha,2\alpha;2z)=2^{\alpha-\frac{1}{2}}\exp\left[\frac{1}{4}(1-2\alpha)\pi i\right]\Gamma\left(\alpha+\frac{1}{2}\right)e^z z^{\frac{1}{2}-\alpha}\,J_{\alpha-\frac{1}{2}}\left(ze^{\frac{\pi}{2}i}\right)$ MO 112

3. $\Phi\left(p+\frac{1}{2},2p+1;2iz\right)=\Gamma(p+1)\left(\frac{z}{2}\right)^{-p}e^{iz}\,J_p(z)$ MO 15

For a representation of special functions in terms of a confluent hypergeometric function $\Phi(\alpha,\gamma;z)$, see:

- for the probability integral, **9.236**;
- for integrals of Bessel functions, **6.631** 1;
- for Hermite polynomials, **8.953** and **8.959**;
- for Laguerre polynomials, **8.972** 1;
- for parabolic cylinder functions, **9.240**;
- for the Whittaker functions $M_{\lambda,\mu}(z)$, **9.220** 2 and **9.220** 3.

9.216 The function $\Phi(\alpha,\gamma;z)$ is a solution of the differential equation

1. $z\dfrac{d^2F}{dz^2}+(\gamma-z)\dfrac{dF}{dz}-\alpha F=0$ MO 111

This equation has two linearly independent solutions:

2. $\Phi(\alpha,\gamma;z)$

3. $z^{1-\gamma}\,\Phi(\alpha-\gamma+1,2-\gamma;z)$ MO 112

9.22–9.23 The Whittaker functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$

9.220 If we make the change of variable $u=e^{-\frac{z}{2}}W$ in equation **9.202** 1, we obtain the equation

1. $\dfrac{d^2W}{dz^2}+\left(-\dfrac{1}{4}+\dfrac{\lambda}{z}+\dfrac{\frac{1}{4}-\mu^2}{z^2}\right)W=0$ MO 115

Equation **9.220** 1 has the following two linearly independent solutions:

2. $M_{\lambda,\mu}(z)=z^{\mu+\frac{1}{2}}e^{-z/2}\,\Phi\left(\mu-\lambda+\frac{1}{2},2\mu+1;z\right)$

3.[11] $M_{\lambda,-\mu}(z)=z^{-\mu+\frac{1}{2}}e^{-z/2}\,\Phi\left(-\mu-\lambda+\frac{1}{2},-2\mu+1;z\right)$ MO 115

To obtain solutions that are also suitable for $2\mu=\pm1,\pm2,\ldots$, we introduce Whittaker's function

4. $W_{\lambda,\mu}(z)=\dfrac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)}M_{\lambda,\mu}(z)+\dfrac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}M_{\lambda,-\mu}(z)$ WH

which, for 2μ approaching an integer, is also a solution of equation **9.220** 1.

For the functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$, $z=0$ is a branch point and $z=\infty$ is an essential singular point. Therefore, we shall examine these functions only for $|\arg z|<\pi$.

These functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are linearly independent solutions of equation **9.220** 1.

Integral representations

9.221 $$M_{\lambda,\mu}(z)=\frac{z^{\mu+\frac{1}{2}}}{2^{2\mu}\,\mathrm{B}\left(\mu+\lambda+\frac{1}{2},\mu-\lambda+\frac{1}{2}\right)}\int_{-1}^{1}(1+t)^{\mu-\lambda-\frac{1}{2}}(1-t)^{\mu+\lambda-\frac{1}{2}}e^{\frac{1}{2}zt}\,dt,$$ WH

if the integral converges. See also **6.631** 1 and **7.623** 3.

9.222

1.[11] $$W_{\lambda,\mu}(z)=\frac{z^{\mu+\frac{1}{2}}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\int_0^\infty e^{-zt}t^{\mu-\lambda-\frac{1}{2}}(1+t)^{\mu+\lambda-\frac{1}{2}}\,dt$$
$$\left[\mathrm{Re}(\mu-\lambda)>-\tfrac{1}{2},\quad |\arg z|<\frac{\pi}{2}\right]$$ MO 118

2. $$W_{\lambda,\mu}(z)=\frac{z^{\lambda}e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)}\int_0^\infty t^{\mu-\lambda-\frac{1}{2}}e^{-t}\left(1+\frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}}\,dt$$
$$\left[\mathrm{Re}(\mu-\lambda)>-\tfrac{1}{2},\quad |\arg z|<\pi\right]$$ WH

9.223 $$W_{\lambda,\mu}(z)=\frac{e^{-\frac{z}{2}}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(u-\lambda)\,\Gamma\left(-u-\mu+\frac{1}{2}\right)\Gamma\left(-u+\mu+\frac{1}{2}\right)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right)\Gamma\left(-\lambda-\mu+\frac{1}{2}\right)}z^u\,du$$
[the path of integration is chosen in such a way that the poles of the function $\Gamma(u-\lambda)$ are separated from the poles of the functions $\Gamma\left(-u-\mu+\frac{1}{2}\right)$ and $\Gamma\left(-u+\mu+\frac{1}{2}\right)$.] See also **7.142**. MO 118

9.224 $$W_{\mu,\frac{1}{2}+\mu}(z)=z^{\mu+1}e^{-\frac{1}{2}z}\int_0^\infty(1+t)^{2\mu}e^{-zt}\,dt=z^{-\mu}e^{\frac{1}{2}z}\int_z^\infty t^{2\mu}e^{-t}\,dt\qquad[\mathrm{Re}\,z>0]$$ WH

9.225

1. $$W_{\lambda,\mu}(x)\;W_{-\lambda,\mu}(x)=-x\int_0^\infty\tanh^{2\lambda}\frac{t}{2}\left\{J_{2\mu}\left(x\sinh t\right)\sin(\mu-\lambda)\pi\right.$$
$$\left.+\;Y_{2\mu}\left(x\sinh t\right)\cos(\mu-\lambda)\pi\right\}\,dt$$
$$\left[|\mathrm{Re}\,\mu|-\mathrm{Re}\,\lambda<\tfrac{1}{2};\quad x>0\right]$$ MO 119

2. $$W_{\kappa,\mu}\left(z_1\right)\,W_{\lambda,\mu}\left(z_2\right)=\frac{\left(z_1z_2\right)^{\mu+\frac{1}{2}}\exp\left[-\frac{1}{2}\left(z_1+z_2\right)\right]}{\Gamma(1-\kappa-\lambda)}$$
$$\times\int_0^\infty e^{-t}t^{-\kappa-\lambda}\left(z_1+t\right)^{-\frac{1}{2}+\kappa-\mu}\left(z_2+t\right)^{-\frac{1}{2}+\lambda-\mu}$$
$$\times F\left(\tfrac{1}{2}-\kappa+\mu,\tfrac{1}{2}-\lambda+\mu;1-\kappa-\lambda;\Theta\right)\,dt$$
$$\Theta=\frac{t\left(z_1+z_2+t\right)}{\left(z_1+t\right)\left(z_2+t\right)},\qquad[z_1\neq0,\quad z_2\neq0,\quad|\arg z_1|<\pi,\quad|\arg z_2|<\pi,\quad\mathrm{Re}(\kappa+\lambda)<1]$$
MO 119

See also **3.334**, **3.381** 6, **3.382** 3, **3.383** 4, 8, **3.384** 3, **3.471** 2.

9.226 Series representations

$$M_{0,\mu}(z)=z^{\frac{1}{2}+\mu}\left\{1+\sum_{k=1}^{\infty}\frac{z^{2k}}{2^{4k}k!(\mu+1)(\mu+2)\ldots(\mu+k)}\right\}$$ WH

Asymptotic representations

9.227[7] For large values of $|z|$

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^{\lambda} \left(1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] \left[\mu^2 - \left(\lambda - \frac{3}{2}\right)^2\right] \dots \left[\mu^2 - \left(\lambda - k + \frac{1}{2}\right)^2\right]}{k! z^k} \right)$$

$[|\arg z| \le \pi - \alpha < \pi]$ WH

9.228 For large values of $|\lambda|$

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu+1) \lambda^{-\mu-\frac{1}{4}} z^{1/4} \cos\left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi\right)$$ MO 118

9.229

1. $$W_{\lambda,\mu} \sim -\left(\frac{4z}{\lambda}\right)^{\frac{1}{4}} e^{-\lambda+\lambda\ln\lambda} \sin\left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4}\right)$$ MO 118

2. $$W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda}\right)^{\frac{1}{4}} e^{\lambda-\lambda\ln\lambda-2\sqrt{\lambda z}}$$ MO 118

Formulas **9.228** and **9.229** are applicable for
$|\lambda| \gg 1, \quad |\lambda| \gg |z|, \quad |\lambda| \gg |\mu|, \quad z \ne 0, \quad |\arg\sqrt{z}| < \frac{3\pi}{4}$ and $|\arg\lambda| < \frac{\pi}{2}$. MO 118

Functional relations

9.231

1. $$M_{n+\mu+\frac{1}{2},\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{1}{2}z}}{(2\mu+1)(2\mu+2)\dots(2\mu+n)} \frac{d^n}{dz^n}\left(z^{n+2\mu} e^{-z}\right)$$

$[n = 0, 1, 2, \dots; \quad 2\mu \ne -1, -2, -3, \dots]$ MO 117

2. $$z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z)$$ $[2\mu \ne -1, -2, -3, \dots]$ WH

9.232

1. $$W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z)$$ MO 116

2. $$W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu+\lambda\right)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu+\lambda\right)} M_{-\lambda,-\mu}(-z)$$

$\left[|\arg(-z)| < \frac{3}{2}\pi\right]$ WH

9.233

1. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{i\pi\lambda} W_{-\lambda,\mu}\left(e^{i\pi} z\right) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda,\mu}(z)$$

$\left[-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi; \quad 2\mu \ne -1, -2, \dots\right]$ MO 117

2. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} e^{-i\pi\lambda} W_{-\lambda,\mu}\left(e^{-i\pi} z\right) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+\lambda+\frac{1}{2}\right)} \exp\left[-i\pi\left(\lambda-\mu-\frac{1}{2}\right)\right] W_{\lambda,\mu}(z)$$

$\left[-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi; \quad 2\mu \ne -1, -2, \dots\right]$ MO 117

9.234 Recursion formulas

1. $$W_{\lambda,\mu}(z) = \sqrt{z}\, W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(z) + \left(\tfrac{1}{2}+\mu-\lambda\right) W_{\lambda-1,\mu}(z)$$ WH

2.[11] $$W_{\lambda,\mu}(z) = \sqrt{z}\, W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(z) + \left(\tfrac{1}{2}-\mu-\lambda\right) W_{\lambda-1,\mu}(z)$$ WH

3. $$z\frac{d}{dz}\, W_{\lambda,\mu}(z) = \left(\lambda-\tfrac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda-\tfrac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z)$$ WH

4. $$\left[\left(\mu+\frac{1-z}{2}\right) W_{\lambda,\mu}(z) - z\frac{d}{dz}\, W_{\lambda,\mu}(z)\right]\left(\mu+\tfrac{1}{2}+\lambda\right) = \left[\left(\mu+\frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z\frac{d}{dz}\, W_{\lambda,\mu+1}(z)\right]\left(\mu+\tfrac{1}{2}-\lambda\right)$$ MO 117

5. $$\left(\tfrac{3}{2}+\lambda+\mu\right)\left(\tfrac{1}{2}+\lambda+\mu\right) z\, W_{\lambda,\mu}(z) = z(z+2\mu+1)\frac{d}{dz}\, W_{\lambda+1,\mu+1}(z) + \left[\tfrac{1}{2}z^2 + \left(\mu-\lambda-\tfrac{1}{2}\right)z + 2\mu^2 + 2\mu + \tfrac{1}{2}\right] W_{\lambda+1,\mu+1}(z)$$ MO 117

Connections with other functions

9.235

1. $$M_{0,\mu}(z) = 2^{2\mu}\,\Gamma(\mu+1)\sqrt{z}\, I_\mu\left(\frac{z}{2}\right)$$ MO 125a

2. $$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}\, K_\mu\left(\frac{z}{2}\right)$$ MO 125

9.236

1. $$\Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}}\, W_{-\frac{1}{4},\frac{1}{4}}\left(x^2\right) = \frac{2x}{\sqrt{\pi}}\,\Phi\left(\tfrac{1}{2},\tfrac{3}{2};-x^2\right)$$ WH, MO 126

2. $$\mathrm{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln\frac{1}{2}}}\, W_{-\frac{1}{2},0}(-\ln z)$$ WH

3. $$\Gamma(\alpha,x) = e^{-x}\,\Psi(1-\alpha,1-\alpha;x)$$ EH I 266(21)

4. $$\gamma(\alpha,x) = \frac{x^\alpha}{\alpha}\,\Phi(\alpha,\alpha+1;-x)$$ EH I 266(22)

9.237

1. $$W_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)\Gamma\left(\frac{1}{2}+\mu-\lambda\right)} \times \left\{\sum_{k=0}^{\infty}\frac{\Gamma\left(\mu+k-\lambda+\frac{1}{2}\right)}{k!(2\mu+k)!} z^k \left[\Psi(k+1)+\Psi(2\mu+k+1)-\Psi\left(\mu+k-\lambda+\tfrac{1}{2}\right)-\ln z\right] + (-z)^{-2\mu}\sum_{k=0}^{2\mu-1}\frac{\Gamma(2\mu-k)\,\Gamma\left(k-\mu-\lambda+\frac{1}{2}\right)}{k!}(-z)^k\right\}$$
$$\left[|\arg z| < \tfrac{3}{2}\pi;\quad 2\mu+1 \text{ is a natural number}\right]$$ MO 116

2. Set $\lambda - \mu - \frac{1}{2} = l$, where $l+1$ is a natural number. Then

3. $$W_{l+\mu+\frac{1}{2},\mu}(z) = (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}(2\mu+1)(2\mu+2)\cdots(2\mu+l)\,\Phi(-l,2\mu+1;z)$$
$$= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}\, L_l^{2\mu}(z)$$

MO 116

9.238

1. $$J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2ix\right)$$ EH I 265(9)

2. $$I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(10)

3. $$K_\nu(x) = \sqrt{\pi} e^{-x}(2x)^\nu\,\Psi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(13)

9.24–9.25 Parabolic cylinder functions $D_p(z)$

9.240 $$D_p(z) = 2^{\frac{1}{4}+\frac{p}{2}}\, W_{\frac{1}{4}+\frac{p}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-1/2}$$
$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)}\,\Phi\left(-\frac{p}{2},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma\left(-\frac{p}{2}\right)}\,\Phi\left(\frac{1-p}{2},\frac{3}{2};\frac{z^2}{2}\right)\right\}$$

MO 120a

are called *parabolic cylinder functions*.

Integral representations

9.241

1. $$D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz}\,dx \qquad [\operatorname{Re} p > -1; \quad \text{for } x<0, \quad \arg x^p = p\pi i]$$

MO 122

2. $$D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^\infty e^{-xz-\frac{x^2}{2}} x^{-p-1}\,dx \qquad [\operatorname{Re} p < 0] \qquad (\text{cf. } \mathbf{3.462}\ 1)$$ MO 122

9.242

1.[10] $$D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-p-1}\,dt \qquad [|\arg(-t)| \le \pi]$$ WH

2. $$D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t}(1+t)^{-\frac{1}{2}p-1}(1-t)^{\frac{1}{2}(p-1)}\,dt$$
$$\left[|\arg z| < \frac{\pi}{4}; \quad |\arg(1+t)| \le \pi\right]$$ WH

3. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t\,dt$$
$$\left[|\arg z| < \tfrac{3}{4}\pi; \quad p \text{ is not a positive integer}\right]$$ WH

4. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0-)} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t \, dt$$

[for all values of $\arg z$; also, the contours encircle the poles of the function $\Gamma(-t)$, but they do not encircle the poles of the function $\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)$]. WH

9.243

1. $$D_n(z) = (-1)^\mu \left(\frac{\pi}{2}\right)^{-1/2} \left(\sqrt{n}\right)^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt \right.$$
$$\left. + \int_0^\infty \left[e^{\frac{1}{2}n\left(1-t^2\right)} t^n - e^{-n(t-1)^2}\right] \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt - \int_{-\infty}^0 e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt \right\}$$
[n is a natural number] WH

2. $$D_n(z) = (-1)^\mu 2^{n+2} (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \int_0^\infty t^n e^{-2t^2} \begin{matrix}\cos\\ \sin\end{matrix} (2zt)\, dt$$
[n is a natural number, $\mu = \left\lfloor \frac{n}{2} \right\rfloor$, and the cosine or sine is chosen accordingly as n is even or odd]

WH

9.244

1. $$D_{-p-1}[(1+i)z] = \frac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}}\,\Gamma\left(\frac{p+1}{2}\right)} \int_0^\infty \frac{e^{-ix^2z^2} x^p}{(1+x^2)^{1+\frac{p}{2}}}\, dx \qquad \left[\operatorname{Re} p > -1, \quad \operatorname{Re}\left(iz^2\right) \geq 0\right]$$ MO 122

2. $$D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma\left(-\frac{p}{2}\right)} \int_1^\infty e^{-\frac{i}{2}z^2 x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}}\, dx \qquad \left[\operatorname{Re} p < 0; \quad \operatorname{Re}\left(iz^2\right) \geq 0\right]$$ MO 122

See also **3.383** 6, 7, **3.384** 2, 6, **3.966** 5, 6.

9.245

1.[10] $$D_p(x)\, D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \coth^{p+\frac{1}{2}}\left(\frac{t}{2}\right) \frac{1}{\sqrt{\sinh t}} \sin\left(\frac{x^2 \sinh t + p\pi}{2}\right) dt$$
[x is real, $\operatorname{Re} p < 0$] MO 122

2. $$D_p\left(ze^{\frac{\pi}{4}i}\right) D_p\left(ze^{-\frac{\pi}{4}i}\right) = \frac{1}{\Gamma(-p)} \int_0^\infty \coth^p t \exp\left(-\frac{z^2}{2}\sinh 2t\right) \frac{dt}{\sinh t}$$
$\left[|\arg z| < \frac{\pi}{4}; \quad \operatorname{Re} p < 0\right]$ MO 122

See also **6.613**.

9.246 Asymptotic expansions. If $|z| \gg 1$ and $|z| \gg |p|$, then

1. $$D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$
$\left[|\arg z| < \frac{3}{4}\pi\right]$ MO 121

2.[11] $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$
$$- \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right)$$
$\left[\frac{1}{4}\pi < \arg z < \frac{5}{4}\pi\right]$ MO 121

3.[11] $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4z^4} - \dots\right)$$
$$-\frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2\cdot 4z^4} + \dots\right)$$
$$\left[-\tfrac{1}{4}\pi > \arg z > -\tfrac{5}{4}\pi\right]$$ MO 121

Functional relations

9.247 Recursion formulas:

1. $$D_{p+1}(z) - z\, D_p(z) + p\, D_{p-1}(z) = 0$$ WH

2. $$\frac{d}{dz} D_p(z) + \frac{1}{2} z\, D_p(z) - p\, D_{p-1}(z) = 0$$ WH

3. $$\frac{d}{dz} D_p(z) - \frac{1}{2} z\, D_p(z) + D_{p+1}(z) = 0$$ MO 121

9.248 Linear relations:

1. $$D_p(z) = \frac{\Gamma(p+1)}{\sqrt{2\pi}} \left[e^{\pi/2} D_{-p-1}(iz) + e^{-\pi p i/2} D_{-p-1}(-iz)\right]$$
$$= e^{-p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-\pi(p+1)i/2} D_{-p-1}(iz)$$
$$= e^{p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{\pi(p+1)i/2} D_{-p-1}(-iz)$$
MO 121

9.249[10] $$D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp\left[-\frac{i}{2}\left(x^2 + p\frac{\pi}{2}\right)\right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it^2/4}\, dt$$
$$[x \text{ real}; \quad -1 < \operatorname{Re} p < 0]$$ MO 122

9.251[10] $$D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n}\left(e^{-z^2/2}\right) \qquad [n = 0, 1, 2, \dots]$$ WH

9.252 $$D_p(ax+by) = \exp\frac{(bx-ay)^2}{4} \left(\frac{a}{\sqrt{a^2+b^2}}\right)^p \sum_{k=0}^\infty \binom{p}{k} D_{p-k}\left(\sqrt{a^2+b^2}\,x\right) D_k\left(\sqrt{a^2+b^2}\,y\right) \left(\frac{b}{a}\right)^k$$
$$[a > b > 0, \quad x > 0, \quad y > 0, \quad \operatorname{Re} p \geq 0] \qquad \text{“summation theorem”}$$ MO 124

Connections with other functions

9.253[11] $$D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right)$$ MO 123a

9.254

1. $$D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right]$$ MO 123

2.[11] $$D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{\sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} - z\left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right]\right\}$$ MO 123

9.255 Differential equations leading to parabolic cylinder functions:

1. $$\frac{d^2u}{dz^2}+\left(p+\frac{1}{2}-\frac{z^2}{4}\right)u=0$$
The solutions are $u=D_p(z)$, $D_p(-z)$, $D_{-p-1}(iz)$, and $D_{-p-1}(-iz)$.
(These four solutions are linearly dependent. See **9.248**.)

2. $$\frac{d^2u}{dz^2}+\left(z^2+\lambda\right)u=0, \qquad u=D_{-\frac{1+i\lambda}{2}}\left[\pm(1+i)z\right]$$
EH II 118(12,13)a, MO 123

3.[7] $$\frac{d^2u}{dz^2}+z\frac{du}{dz}+(p+1)u=0, \qquad u=e^{-\frac{z^2}{4}}\,D_p(z)$$
MO 123

9.26 Confluent hypergeometric series of two variables

9.261

1.[6] $$\Phi_1(\alpha,\beta,\gamma,x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}m!n!}x^my^n \qquad [|x|<1]$$
EH I 225(20)

2. $$\Phi_2\left(\beta,\beta',\gamma,x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m\left(\beta'\right)_m}{(\gamma)_{m+n}m!n!}x^my^n$$
EH I 225(21)a, ET I 385

3. $$\Phi_3(\beta,\gamma,x,y)=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}m!n!}x^my^n$$
EH I 225(22)

The functions Φ_1, Φ_2, Φ_3 satisfy the following systems of partial differential equations:

9.262

1. $z=\Phi_1\left(\alpha,\beta,\gamma,x,y\right)$ EH I 235(23)

$$x(1-x)\frac{\partial^2z}{\partial x^2}+y(1-x)\frac{\partial^2z}{\partial x\,\partial y}+[\gamma-(\alpha+\beta+1)x]\frac{\partial z}{\partial x}-\beta y\frac{\partial z}{\partial y}-\alpha\beta z=0,$$
$$y\frac{\partial^2z}{\partial y^2}+x\frac{\partial^2z}{\partial x\,\partial y}+(\gamma-y)\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial x}-\alpha z=0$$

2. $z=\Phi_2\left(\beta,\beta',\gamma,x,y\right)$ EH I 235(24)

$$x\frac{\partial^2z}{\partial x^2}+y\frac{\partial^2z}{\partial x\,\partial y}+(\gamma-x)\frac{\partial z}{\partial x}-\beta z=0,$$
$$y\frac{\partial^2z}{\partial y^2}+x\frac{\partial^2z}{\partial x\,\partial y}+(\gamma-y)\frac{\partial z}{\partial y}-\beta' z=0$$

3. $z = \Phi_3(\beta, \gamma, x, y)$ EH I 235(25)

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} + (\gamma - x)\frac{\partial z}{\partial x} - \beta z = 0,$$
$$y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\,\partial y} + \gamma\frac{\partial z}{\partial y} - z = 0$$

9.3 Meijer's G-Function

9.30 Definition

9.301 $$G_{p,q}^{m,n}\left(x\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int \frac{\prod\limits_{j=1}^{m}\Gamma(b_j - s)\prod\limits_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod\limits_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod\limits_{j=n+1}^{p}\Gamma(a_j - s)}x^s\,ds$$

$[0 \le m \le q, \quad 0 \le n \le p$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_k + s)$ for any j and k (where $j = 1, \ldots, m; \quad k = 1, \ldots, n]$). Besides **9.301**, the following notations are also used:

$$G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right), \qquad G_{pq}^{mn}(x), \qquad G(x)$$ EH I 207(1)

9.302 Three types of integration paths L in the right member of **9.301** can be exhibited:

1. The path L runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1 - a_k + s)$ lie to the left, and the poles of the functions $\Gamma(b_j - s)$ lie to the right of L (for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$). In this case, the conditions under which the integral **9.301** converges are of the form

$$p + q < 2(m + n), \quad |\arg x| < \left(m + n - \tfrac{1}{2}p - \tfrac{1}{2}q\right)\pi.$$ EH I 207(2)

2. L is a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \ldots, m$) once in the negative direction. All the poles of the functions $\Gamma(1 - a_k + s)$ must remain outside this loop. Then, the conditions under which the integral **9.301** converges are:

$$q \ge 1 \text{ and either } p < q \text{ or } p = q \text{ and } |x| < 1.$$ EH I 207(3)

3. L is a loop, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1 - a_k + s)$ (for $k = 1, 2, \ldots, n$) once in the positive direction. All the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \ldots, m$) must remain outside this loop.

   The conditions under which the integral in **9.301** converges are

$$p \ge 1 \text{ and either } p > q \text{ or } p = q \text{ and } |x| > 1.$$ EH I 207(4)

The function $G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right)$ is analytic with respect to x; it is symmetric with respect to the parameters $a_1, \ldots, a_n$ and also with respect to $a_{n+1}, \ldots, a_p$; $b_1, \ldots, b_m$; $b_{m+1}, \ldots, b_q$.

EH I 208

9.303[11] If no two b_j (for $j = 1, 2, \ldots, n$) differ by an integer, then, under the conditions that either $p < q$ or $p = q$ and $|x| < 1$,

$$G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right) = \sum_{h=1}^{m} \frac{\prod_{j=1}^{m}\Gamma(b_j - b_h)\prod_{j=1}^{n}\Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^{q}\Gamma(1 + b_h - b_j)\prod_{j=n+1}^{p}\Gamma(a_j - b_h)} x^{b_h}$$
$$\times\ {}_pF_{q-1}\left[1 + b_h - a_1, \ldots, 1 + b_h - a_p;\quad 1 + b_h - b_1, \ldots \right.$$
$$\left. \ldots, *, \ldots, 1 + b_h - b_q;\quad (-1)^{p-m-n}x\right]$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when $j = h$. The asterisk in the function ${}_pF_{q-1}$ denotes the omission of the hth parameter.

9.304[7] If no two a_k (for $k = 1, 2, \ldots, n$) differ by an integer then, under the conditions that $q < p$ or $q = p$ and $|x| > 1$,

$$G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right) = \sum_{h=1}^{n} \frac{\prod_{j=1}^{n}{}'\Gamma(a_h - a_j)\prod_{j=1}^{m}\Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^{p}\Gamma(a_j - a_h + 1)\prod_{j=m+1}^{q}\Gamma(a_h - b_j)} x^{a_h - 1}$$
$$\times\ {}_qF_{p-1}\left[1 + b_1 - a_h, \ldots, 1 + b_q - a_h;\quad 1 + a_1 - a_h, \ldots \right.$$
$$\left. \ldots, *, \ldots, 1 + a_p - a_h;\quad (-1)^{q-m-n}x^{-1}\right]$$

EH I 208(6)

9.31 Functional relations

If one of the parameters a_j (for $j = 1, 2, \ldots, n$) coincides with one of the parameters b_j (for $j = m + 1, \quad m + 2, \ldots, q$), the order of the G-function decreases. For example,

1. $$G_{pq}^{mn}\left(x\left|\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_{q-1}, a_1\end{matrix}\right.\right) = G_{p-1,q-1}^{m,n-1}\left(x\left|\begin{matrix}a_2, \ldots, a_p\\ b_1, \ldots, b_{q-1}\end{matrix}\right.\right)$$
$$[n, p, q \geq 1]$$

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \ldots, m$) coincides with one of the a_j (for $j = n + 1, \ldots, p$). In this case, it is m and not n that decreases by one unit.

The G-function with $p > q$ can be transformed into the G-function with $p < q$ by means of the relationships:

2. $$G_{pq}^{mn}\left(x^{-1}\left|\begin{matrix}a_r\\ b_s\end{matrix}\right.\right) = G_{qp}^{nm}\left(x\left|\begin{matrix}1 - b_s\\ 1 - a_r\end{matrix}\right.\right)$$ EH I 209(9)

3. $$x\frac{d}{dx}G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\b_s\end{matrix}\right.\right)=G_{pq}^{mn}\left(x\left|\begin{matrix}a_1-1,a_2,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right)+(a_1-1)\,G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\b_s\end{matrix}\right.\right)$$
$[n\geq 1]$ EH I 210(13)

4. $$G_{p+1,q+1}^{m+1,n}\left(z\left|\begin{matrix}\mathbf{a}_p,1-r\\0,\mathbf{b}_q\end{matrix}\right.\right)=(-1)^r\,G_{p+1,q+1}^{m,n+1}\left(z\left|\begin{matrix}1-r,\mathbf{a}_p\\\mathbf{b}_q,1\end{matrix}\right.\right)$$
$[r=0,1,2,\ldots]$ MS2 6 (1.2.2)

5. $$z^k\,G_{pq}^{mn}\left(z\left|\begin{matrix}\mathbf{a}_p\\\mathbf{b}_q\end{matrix}\right.\right)=G_{pq}^{mn}\left(z\left|\begin{matrix}\mathbf{a}_p+k\\\mathbf{b}_q+k\end{matrix}\right.\right)$$
MS2 7 (1.2.7)

9.32 A differential equation for the G-function

$G_{pq}^{mn}\left(x\left|\begin{matrix}a_r\\b_s\end{matrix}\right.\right)$ satisfies the following linear q^{th}-order differential equation:

$$\left[(-1)^{p-m-n}x\prod_{j=1}^{p}\left(x\frac{d}{dx}-a_j+1\right)-\prod_{j=1}^{q}\left(x\frac{d}{dx}-b_j\right)\right]y=0\qquad[p\leq q]$$
EH I 210(1)

9.33 Series of G-functions

$$G_{pq}^{mn}\left(\lambda x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right)=\lambda^{b_1}\sum_{r=0}^{\infty}\frac{1}{r!}(1-\lambda)^r\,G_{pq}^{mn}\left(x\left|\begin{matrix}a_1,\ldots,a_p\\b_1+r,b_2,\ldots,b_q\end{matrix}\right.\right)$$
$[|\lambda-1|<1,\quad m\geq 1,\quad$ if $m=1$ and $p<q$, λ may be arbitrary$]$
EH I 213(1)

$$=\lambda^{b_q}\sum_{r=0}^{\infty}\frac{1}{r!}(\lambda-1)^r\,G_{pq}^{mn}\left(x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_{q-1},b_q+r\end{matrix}\right.\right)$$
$[m<q,\quad |\lambda-1|<1]$
EH I 213(2)

$$=\lambda^{a_1-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\lambda-\frac{1}{\lambda}\right)^r G_{pq}^{mn}\left(x\left|\begin{matrix}a_1-r,a_2,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right)$$
$\left[n\geq 1,\quad \operatorname{Re}\lambda>\frac{1}{2},\quad\right.$ (if $n=1$ and $p>q$, then λ may be arbitrary)$\left.\right]$
EH I 213(3)

$$=\lambda^{a_p-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{1}{\lambda}-1\right)^r G_{pq}^{mn}\left(x\left|\begin{matrix}a_1,\ldots,a_{p-1},a_p-r\\b_1,\ldots,b_q\end{matrix}\right.\right)$$
$\left[n<p,\quad \operatorname{Re}\gamma>\frac{1}{2}\right]$
EH I 213(4)

For integrals of the G-function, see **7.8**.

9.34 Connections with other special functions

1. $$J_\nu(x)x^\mu=2^\mu\,G_{02}^{10}\left(\frac{1}{4}x^2\left|\begin{matrix}\\\frac{1}{2}\nu+\frac{1}{2}\mu,\frac{1}{2}\mu-\frac{1}{2}\nu\end{matrix}\right.\right)$$
EH I 219(44)

2. $$Y_\nu(x)x^\mu=2^\mu\,G_{13}^{20}\left(\frac{1}{4}x^2\left|\begin{matrix}\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\\\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\end{matrix}\right.\right)$$
EH I 219(46)

3. $$K_\nu(x)x^\mu = 2^{\mu-1}\, G^{\,20}_{\,02}\left(\frac{1}{4}x^2\,\middle|\,{}_{\frac{1}{2}\mu+\frac{1}{2}\nu,\,\frac{1}{2}\mu-\frac{1}{2}\nu}\right)$$ EH I 219(47)

4. $$K_\nu(x) = e^x\sqrt{\pi}\, G^{\,20}_{\,12}\left(2x\,\middle|\,{}^{\frac{1}{2}}_{\nu,\,-\nu}\right)$$ EH I 219(49)

5. $$\mathbf{H}_\nu(x)x^\mu = 2^\mu\, G^{\,11}_{\,13}\left(\frac{1}{4}x^2\,\middle|\,{}^{\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu}_{\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu,\,\frac{1}{2}\mu-\frac{1}{2}\nu,\,\frac{1}{2}\mu+\frac{1}{2}\nu}\right)$$ EH I 220(51)

6. $$S_{\mu,\nu}(x) = 2^{\mu-1}\frac{1}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)}\, G^{\,31}_{\,13}\left(\frac{1}{4}x^2\,\middle|\,{}^{\frac{1}{2}+\frac{1}{2}\mu}_{\frac{1}{2}+\frac{1}{2}\mu,\,\frac{1}{2}\nu,\,-\frac{1}{2}\nu}\right)$$ EH I 220(55)

7.[7] $${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)x}{\Gamma(a)\,\Gamma(b)}\, G^{\,12}_{\,22}\left(x\,\middle|\,{}^{-a,\,-b}_{-1,\,-c}\right)$$ EH I 222(74)a

8. $$\begin{aligned}{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right) &= \frac{\prod_{j=1}^q\Gamma\left(b_j\right)}{\prod_{j=1}^p\Gamma\left(a_j\right)}\, G^{\,1,p}_{\,p,q+1}\left(-x\,\middle|\,{}^{1-a_1,\ldots,1-a_p}_{0,1-b_1,\ldots,1-b_q}\right)\\ &= \frac{\prod_{j=1}^q\Gamma\left(b_j\right)}{\prod_{j=1}^p\Gamma\left(a_j\right)}\, G^{\,p,1}_{\,q+1,p}\left(-\frac{1}{x}\,\middle|\,{}^{1,b_1,\ldots,b_q}_{a_1,\ldots,a_p}\right)\end{aligned}$$ EH I 215(1)

9. $$W_{k,m}(x) = \frac{2^k\sqrt{x}e^{\frac{1}{2}x}}{\sqrt{2\pi}}\, G^{\,40}_{\,24}\left(\frac{x^2}{4}\,\middle|\,{}^{\frac{1}{4}-\frac{1}{2}k,\,\frac{3}{4}-\frac{1}{2}k}_{\frac{1}{2}+\frac{1}{2}m,\,\frac{1}{2}-\frac{1}{2}m,\,\frac{1}{2}m,\,-\frac{1}{2}m}\right)$$ EH I 221(70)

9.4 MacRobert's E-Function

9.41 Representation by means of multiple integrals

$$\begin{aligned}E\left(p;\alpha_r:q;\varrho_s:x\right) = {}&\frac{\Gamma\left(\alpha_{q+1}\right)}{\Gamma\left(\varrho_1-\alpha_1\right)\Gamma\left(\varrho_2-\alpha_2\right)\cdots\Gamma\left(\varrho_q-\alpha_q\right)}\\ &\times\prod_{\mu=1}^q\int_0^\infty\lambda_\mu^{\varrho_\mu-\alpha_\mu-1}\left(1-\lambda_\mu\right)^{-\varrho_\mu}\,d\lambda_\mu\prod_{\nu=2}^{p-q-1}\int_0^\infty e^{-\lambda_{q+\nu}}\lambda_{q+\nu}^{\alpha_{q+\nu}-1}\,d\lambda_{q+\nu}\\ &\times\int_0^\infty e^{-\lambda_p}\lambda_p^{\alpha_p-1}\left[1+\frac{\lambda_{q+2}\lambda_{q+3}\cdots\lambda_p}{\left(1+\lambda_1\right)\cdots\left(1+\lambda_q\right)x}\right]^{-\alpha_{q+1}}d\lambda_p\end{aligned}$$

[$|\arg x| < \pi$, $p \geq q+1$, α_r and ϱ_s are bounded by the condition that the integrals on the right be convergent.] EH I 204(3)

9.42 Functional relations

1. $$\begin{aligned}\alpha_1 x\, E\left(\alpha_1,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right) &= x\, E\left(\alpha_1+1,\alpha_2,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right)\\ &\quad+E\left(\alpha_1+1,\alpha_2+1,\ldots,\alpha_p+1:\varrho_1+1,\ldots,\varrho_q+1:x\right)\end{aligned}$$ EH I 205(7)

2. $$(\varrho_1-1)\,x\,E\left(\alpha_1,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right)=x\,E\left(\alpha_1,\ldots,\alpha_p:\varrho_1-1,\varrho_2,\ldots,\varrho_q:x\right)$$
$$+E\left(\alpha_1+1,\ldots,\alpha_p+1:\varrho_1+1,\ldots,\varrho_q+1:x\right)$$
EH I 205(9)

3. $$\frac{d}{dx}E\left(\alpha_1,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right)=x^{-2}\,E\left(\alpha_1+1,\ldots,\alpha_p+1:\varrho_1+1,\ldots,\varrho_q+1:x\right)$$
EH I 205(8)

9.5 Riemann's Zeta Functions $\zeta(z,q)$ and $\zeta(z)$, and the Functions $\Phi(z,s,v)$ and $\xi(s)$

9.51 Definition and integral representations

9.511 $$\zeta(z,q)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}e^{-qt}}{1-e^{-t}}\,dt;$$
WH
$$=\frac{1}{2}q^{-z}+\frac{q^{1-z}}{z-1}+2\int_0^\infty\left(q^2+t^2\right)^{-\frac{z}{2}}\left[\sin\left(z\arctan\frac{t}{q}\right)\right]\frac{dt}{e^{2\pi t}-1}$$
$[0<q<1,\quad \operatorname{Re}z>1]$
WH

9.512 $$\zeta(z,q)=-\frac{\Gamma(1-z)}{2\pi i}\int_\infty^{(0+)}\frac{(-\theta)^{z-1}e^{-q\theta}}{1-e^{-\theta}}\,d\theta$$
This equation is valid for all values of z, except for $z=1,2,3,\ldots$. It is assumed that the path of integration (see drawing below) does not pass through the points $2n\pi i$ (where n is a natural number).

See also **4.251** 4, **4.271** 1, 4, 8, **4.272** 9, 12, **4.294** 11.

9.513

1. $$\zeta(z)=\frac{1}{\left(1-2^{1-z}\right)\Gamma(z)}\int_0^\infty\frac{t^{z-1}}{e^t+1}\,dt \qquad [\operatorname{Re}z>0]$$ WH

2. $$\zeta(z)=\frac{2^z}{\left(2^z-1\right)\Gamma(z)}\int_0^\infty\frac{t^{z-1}e^t}{e^{2t}-1}\,dt \qquad [\operatorname{Re}z>1]$$ WH

3.[11] $$\zeta(z)=\frac{\pi^{\frac{z}{2}}}{\Gamma\left(\frac{z}{2}\right)}\left[\frac{1}{z(z-1)}+\int_1^\infty\left(t^{\frac{1-z}{2}}+t^{\frac{z}{2}}\right)t^{-1}\sum_{k=1}^\infty e^{-k^2\pi t}\,dt\right]$$ WH

4. $$\zeta(z)=\frac{2^{z-1}}{z-1}-2^z\int_0^\infty\left(1+t^2\right)^{-\frac{z}{2}}\sin\left(z\arctan t\right)\frac{dt}{e^{\pi t}+1}$$ WH

5. $$\zeta(z)=\frac{2^{z-1}}{2^z-1}\frac{z}{z-1}+\frac{2}{2^z-1}\int_0^\infty\left(\frac{1}{4}+t^2\right)^{-z/2}\sin\left(z\arctan 2t\right)\frac{dt}{e^{2\pi t}-1}$$ WH

See also **3.411** 1, **3.523** 1, **3.527** 1, 3, **4.271** 8.

9.52 Representation as a series or as an infinite product

9.521

1. $$\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^z} \qquad [\operatorname{Re} z>1, \quad q\neq 0,-1,-2,\ldots]$$ WH

2. $$\zeta(z,q)=\frac{2\,\Gamma(1-z)}{(2\pi)^{1-z}}\left[\sin\frac{z\pi}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi qn}{n^{1-z}}+\cos\frac{z\pi}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi qn}{n^{1-z}}\right]$$
$$[\operatorname{Re} z<0, \quad 0<q\leq 1]$$ WH

3.[8] $$\zeta(z,q)=\sum_{n=0}^{N}\frac{1}{(q+n)^z}-\frac{1}{(1-z)(N+q)^{z-1}}-\sum_{n=N}^{\infty}F_n(z),$$

where

$$\begin{aligned}F_n(z)&=\frac{1}{1-z}\left(\frac{1}{(n+1+q)^{z-1}}-\frac{1}{(n+q)^{z-1}}\right)-\frac{1}{(n+1+q)^z}\\&=z\int_n^{n+1}\frac{(t-n)\,dt}{(t+q)^{z+1}}\end{aligned}$$ WH

9.522

1. $$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z} \qquad [\operatorname{Re} z>1]$$ WH

2. $$\zeta(z)=\frac{1}{1-2^{1-z}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^z} \qquad [\operatorname{Re} z>0]$$ WH

9.523 The following product and summation are taken over all primes p:

1.[7] $$\zeta(z)=\prod_p\frac{1}{1-p^{-z}} \qquad [\operatorname{Re} z>1]$$ WH

2. $$\ln\zeta(z)=\sum_p\sum_{k=1}^{\infty}\frac{1}{kp^{kz}} \qquad [\operatorname{Re} z>1]$$ WH

9.524[11] $$\frac{\zeta'(z)}{\zeta(z)}=-\sum_{k=1}^{\infty}\frac{\Lambda(k)}{k^z}, \qquad [\operatorname{Re} z>1]$$
where $\Lambda(k)=0$ when k is not a power of a prime and $\Lambda(k)=\ln p$ when k is a power of a prime p. WH

9.53 Functional relations

9.531 $$\zeta(-n,q)=-\frac{B'_{n+2}(q)}{(n+1)(n+2)}=\frac{-B_{n+1}(q)}{n+1}$$
[n is a nonnegative integer] see EH I 27 (11) WH

9.532 $$\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}z^k\,\zeta(k,q)=\ln\frac{e^{-Cz}\,\Gamma(q)}{\Gamma(z+q)}-\frac{z}{q}+\sum_{k=1}^{\infty}\frac{qz}{k(q+k)} \qquad [|z|<q]$$ WH

9.533

1. $\lim\limits_{z\to 1}\dfrac{\zeta(z,q)}{\Gamma(1-z)} = -1$ WH

2. $\lim\limits_{z\to 1}\left\{\zeta(z,q) - \dfrac{1}{z-1}\right\} = -\Psi(q)$ WH

3. $\left\{\dfrac{d}{dz}\,\zeta(z,q)\right\}_{z=0} = \ln\Gamma(q) - \dfrac{1}{2}\ln 2\pi$ WH

9.534 $\zeta(z,1) = \zeta(z)$

9.535

1. $\zeta(z) = \dfrac{1}{2^z-1}\,\zeta\left(z,\tfrac{1}{2}\right)$ $[\operatorname{Re} z > 1]$ WH

2.[11] $2^z\,\Gamma(1-z)\,\zeta(1-z)\sin\left(\dfrac{z\pi}{2}\right) = \pi^{1-z}\,\zeta(z)$ WH

3. $2^{1-z}\,\Gamma(z)\,\zeta(z)\cos\dfrac{z\pi}{2} = \pi^z\,\zeta(1-z)$ WH

4. $\Gamma\left(\dfrac{z}{2}\right)\pi^{-\frac{z}{2}}\,\zeta(z) = \Gamma\left(\dfrac{1-z}{2}\right)\pi^{\frac{z-1}{2}}\,\zeta(1-z)$ WH

9.536 $\lim\limits_{z\to 1}\left\{\zeta(z) - \dfrac{1}{z-1}\right\} = \boldsymbol{C}$

9.537 Set $z = \frac{1}{2} + it$. Then, $\Xi(t) = \dfrac{(z-1)\,\Gamma\left(\frac{z}{2}+1\right)}{\sqrt{\pi^z}}\,\zeta(z) = \Xi(-t)$ is an even function of t with real coefficients in its expansion in powers of t^2. JA

9.54 Singular points and zeros

9.541[7]

1. $z = 1$ is the only singular point of the function $\zeta(z)$ WH

2. The function $\zeta(z)$ has simple zeros at the points $-2n$, where n is a natural number. All other zeros of the function $\zeta(z)$ lie in the strip $0 \le \operatorname{Re} z < 1$.

3.[8] Riemann's hypothesis: All zeros of the function $\zeta(z)$ lie on the straight line $\operatorname{Re} z = \frac{1}{2}$. It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first 1,500,000,001 zeros lying in $0 < \operatorname{Im} z < 545,439,823.215$ are known to have $\operatorname{Re} z = \frac{1}{2}$. WH

9.542 Particular values:

1. $\zeta(2m) = \dfrac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!}$ [m is a natural number] WH

2. $\zeta(1-2m) = -\dfrac{B_{2m}}{2m}$ [m is a natural number] WH

3. $\zeta(-2m) = 0$ [m is a natural number] WH

4. $\zeta'(0) = -\frac{1}{2}\ln 2\pi$ WH

9.55 The Lerch function $\Phi(z,s,v)$

9.550 Definition:

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \qquad [|z|<1, \quad v\neq 0,-1,\ldots]$$ EH I 27(1)

Functional relations

9.551 $$\Phi(z,s,v)=z^m\,\Phi(z,s,m+v)+\sum_{n=0}^{m-1}(v+n)^{-s}z^n \qquad [m=1,2,3,\ldots, \quad v\neq 0,-1,-2,\ldots]$$ EH I 27(1)

9.552 $$\Phi(z,s,v)$$
$$=iz^{-v}(2\pi)^{s-1}\,\Gamma(1-s)\left[e^{-i\pi\frac{s}{2}}\,\Phi\left(e^{-2\pi iv},1-s,\frac{\ln z}{2\pi i}\right)-e^{i\pi\left(\frac{s}{2}-2v\right)}\,\Phi\left(e^{2\pi iv},1-s,1-\frac{\ln z}{2\pi i}\right)\right]$$ EH I 29(7)

Series representation

9.553 $$\Phi(z,s,v)=z^{-v}\,\Gamma(1-s)\sum_{n=-\infty}^{\infty}(-\ln z+2\pi ni)^{s-1}\,e^{2\pi nvi}$$
$$[0<v\leq 1, \quad \operatorname{Re}s<0, \quad |\arg(-\ln z+2\pi ni)|\leq\pi]$$ EH I 28(6)

9.554 $$\Phi(z,m,v)=z^{-v}\left\{\sum_{n=0}^{\infty}{}'\,\zeta(m-n,v)\frac{(\ln z)^n}{n!}+\frac{(\ln z)^{m-1}}{(m-1)!}\left[\Psi(m)-\Psi(v)-\ln\left(\ln\frac{1}{z}\right)\right]\right\}^{*}$$
$$[m=2,3,4,\ldots, \quad |\ln z|<2\pi, \quad v\neq 0,-1,-2,\ldots]$$ EH I 30(9)

9.555 $$\Phi(z,-m,v)=\frac{m!}{z^v}\left(\ln\frac{1}{z}\right)^{-m-1}-\frac{1}{z^v}\sum_{r=0}^{\infty}\frac{B_{m+r+1}(v)\,(\ln z)^r}{r!(m+r+1)} \qquad [|\ln z|<2\pi]$$ EH I 30(11)

Integral representation

9.556 $$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}\,dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}\,dt}{e^t-z}$$
$$[\operatorname{Re}v>0, \text{ or } |z|\leq 1, \quad z\neq 1, \quad \operatorname{Re}s>0, \text{ or } z=1, \quad \operatorname{Re}s>1]$$ EH I 27(3)

Limit relationships

9.557 $$\lim_{z\to 1}(1-z)^{1-s}\,\Phi(z,s,v)=\Gamma(1-s) \qquad [\operatorname{Re}s<1]$$ EH I 30(12)

9.558 $$\lim_{z\to 1-}\frac{\Phi(z,1,v)}{-\ln(1-z)}=1$$ EH I 30(13)

A connection with a hypergeometric function

9.559 $$\Phi(z,1,v)=v^{-1}\;{}_2F_1(1,v;1+v;z) \qquad [|z|<1]$$ EH I 30(10)

*In 9.554 the prime on the symbol $\sum$ means that the term corresponding to $n=m-1$ is omitted.

9.56 The function $\xi(s)$

9.561 $\xi(s) = \frac{1}{2}s(s-1)\frac{\Gamma\left(\frac{1}{2}s\right)}{\pi^{\frac{1}{2}s}}\zeta(s)$ EH III 190(10)

9.562 $\xi(1-s) = \xi(s)$ EH III 190(11)

9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions $\nu(x)$, $\nu(x,\alpha)$, $\mu(x,\beta)$, $\mu(x,\beta,\alpha)$, $\lambda(x,y)$ and Euler Polynomials

9.61 Bernoulli numbers

9.610 The numbers B_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad [0 < |t| < 2\pi],$$

are called *Bernoulli* numbers. Thus, the function $\frac{t}{e^t-1}$ is a generating function for the Bernoulli numbers. GE 48(57), FI II 520

9.611 Integral representations

1. $B_{2n} = (-1)^{n-1}4n\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx$ $[n = 1, 2, \ldots]$ (cf. **3.411** 2, 4)

FI II 721a

2. $B_{2n} = (-1)^{n-1}\pi^{-2n}\int_0^\infty \frac{x^{2n}}{\sinh^2 x}\,dx$ $[n = 1, 2, \ldots]$

3. $B_{2n} = (-1)^{n-1}\frac{2n(1-2n)}{\pi}\int_0^\infty x^{2n-2}\ln\left(1-e^{-2\pi x}\right)\,dx$

$[n = 1, 2, \ldots]$

4.* $B_n = \lim_{x\to 0}\frac{d^n}{dx^n}\left(\frac{x}{e^x-1}\right)$

See also **3.523** 2, **4.271** 3.

Properties and functional relations

9.612[8] A symbolic notation:

$$(B+\alpha)^{[n]} = \sum_{k=0}^{n}\binom{n}{k}B_k\alpha^{n-k} \qquad [n \geq 2]$$

in particular

$$B_n = (B+1)^{[n]} = \sum_{k=0}^{n}\binom{n}{k}B_k \qquad [n \geq 2]$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!} \qquad [n \geq 2]$$

9.613 All the Bernoulli numbers are rational numbers.

9.614 Every number B_n can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where C_n is an integer and the sum is taken over all $k > 0$ such that $k+1$ is a prime and k is a divisor of n. GE 64

9.615[11] All the Bernoulli numbers with odd index are equal to zero, except that $B_1 = -\frac{1}{2}$; that is, $B_{2n+1} = 0$ for n a natural number. GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1;\ k \text{ even}}^{n-1} \frac{2n(2n-1)\dots(2n-2k+2)}{(2k)!} B_{k/2} \qquad [n \geq 1]$$

9.616 $B_{2n} = \dfrac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \qquad [n \geq 0]$ (cf. **9.542**) GE 56(79), FI II 721a

9.617[7] $B_{2n} = (-1)^{n-1} \dfrac{2(2n)!}{(2\pi)^{2n}} \dfrac{1}{\prod_{p=2}^{\infty} \left(1 - \dfrac{1}{p^{2n}}\right)} \qquad [n \geq 1]$ (cf. **9.523**)

(where the product is taken over all primes p).

- For a connection with Riemann's zeta function, see **9.542**.
- For a connection with the Euler numbers, see **9.635**.
- For a table of values of the Bernoulli numbers, see **9.71**

9.619 An inequality

$$\left|(B-\theta)^{[n]}\right| \leq |B_n| \qquad [0 < \theta < 1]$$

9.62 Bernoulli polynomials

9.620 The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$ GE 51(62)

or symbolically, $B_n(x) = (B+x)^{[n]}$. GE 52(68)

9.621 The generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \qquad [0 < |t| < 2\pi] \qquad \text{(cf. 1.213)}$$ GE 65(89)a

9.622 Series representation

1.[7] $B_n(x) = -2 \dfrac{n!}{(2\pi)^n} \displaystyle\sum_{k=1}^{\infty} \frac{\cos\left(2\pi k x - \frac{1}{2}\pi n\right)}{k^n}$

$[n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0]$ AS 805(23.1.16)

2.[7] $B_{2n-1}(x) = 2\dfrac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \dfrac{\sin 2k\pi x}{k^{2n-1}}$

$[n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0]$ AS 805(23.1.17)

3.[10] $B_{2n}(x) = \dfrac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \dfrac{\cos 2k\pi x}{k^{2n}}$ $[0 \leq x \leq 1, \quad n = 1, 2, \ldots]$ GE 71

9.623 Functional relations and properties:

1. $B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m$

[n and m are natural numbers] (see also **0.121**) GE 51(65)

2. $B_n(x+1) - B_n(x) = nx^{n-1}$ GE 65(90)
3. $B'_n(x) = n\, B_{n-1}(x)$ $[n = 1, 2, \ldots]$ GE 66
4. $B_n(1-x) = (-1)^n\, B_n(x)$ GE 66
5.[10] $(-1)^n\, B_n(-x) = B_n(x) + nx^{n-1}$ $[n = 0, 1, \ldots]$ AS 804(23.1.9)

9.624[7] $B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \dfrac{k}{m}\right)$

$[m = 1, 2, \ldots n = 0, 1, \ldots]$; "summation theorem" GE 67

9.625 For n odd, the differences

$$B_n(x) - B_n$$

vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1. They change sign at the point $x = \frac{1}{2}$. For n even, these differences vanish at the end points of the interval $[0, 1]$. Within this interval, they do not change sign, and their greatest absolute value occurs at the point $x = \frac{1}{2}$.

9.626 The polynomials

$$B_{2n}(x) - B_{2n} \text{ and } B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval $(0, 1)$. GE 87

9.627 Special cases:

1. $B_1(x) = x - \frac{1}{2}$ GE 70
2. $B_2(x) = x^2 - x + \frac{1}{6}$ GE 70
3. $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ GE 70
4. $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ GE 70
5. $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$ GE 70

9.628 Particular values:

1. $B_n(0) = B_n$
2. $B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n$ $[n \neq 1]$ GE 76

9.63 Euler numbers

9.630 The numbers E_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \left[|t| < \frac{\pi}{2}\right],$$

are known as the *Euler numbers*. Thus, the function $\frac{1}{\cosh t}$ is a generating function for the Euler numbers. CE 330

9.631 A recursion formula

$$(E+1)^{[n]} + (E-1)^{[n]} = 0 \qquad [n \geq 1], \qquad E_0 = 1$$ CE 329

Properties of the Euler numbers

9.632 The Euler numbers are integers.

9.633 The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0.$$ CE 329

9.634 If $\alpha, \beta\gamma, \ldots$ are the divisors of the number $n-m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha+1, 2\beta+1, 2\gamma+1, \ldots$ that are primes.

9.635 A connection with the Bernoulli numbers (symbolic notation):

1.[11] $$E_{n-1} + 4(-1)^n \left(3^{n-1}-1\right) B_1 = \frac{(4B-1)^{[n]} - (4B-3)^{[n]}}{2n} + 4(-1)^{n+1}\left(3^{n-1}-1\right) B_1$$ CE 330

2. $$B_n = \frac{n(E+1)^{[n-1]}}{2^n\left(2^n-1\right)} \qquad [n \geq 2]$$ CE 330

3.[6] $$\left(B+\tfrac{1}{4}\right)^{[2n+1]} = -4^{-2n-1}(2n+1)E_{2n} \qquad [n \geq 0]$$ CE 341

4. $$E_{n-1} = \frac{(4B+3)^{[n]} - (4B+1)^{[n]}}{2n} \qquad [n \geq 1]$$

For a table of values of the Euler numbers, see **9.72**.

9.64 The functions $\nu(x)$, $\nu(x,\alpha)$, $\mu(x,\beta)$, $\mu(x,\beta,\alpha)$, and $\lambda(x,y)$

9.640

1. $$\nu(x) = \int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}$$ EH III 217(1)

2. $$\nu(x,\alpha) = \int_0^\infty \frac{x^{\alpha+t}\,dt}{\Gamma(\alpha+t+1)}$$ EH III 217(1)

3. $$\mu(x,\beta) = \int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(t+1)}$$ EH III 217(2)

4. $$\mu(x,\beta,\alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(\alpha+t+1)}$$ EH III 217(2)

5. $$\lambda(x,y) = \int_0^y \frac{\Gamma(u+1)\,du}{x^u}$$ MI 9

9.65[10] Euler polynomials

9.650 The Euler polynomials are defined by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$$ AS 804 (23.1.7)

9.651 The generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$ AS 804 (23.1.1)

9.652 Series representation:

1. $$E_n(x) = 4\frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)\pi x - \frac{1}{2}\pi n\right)}{(2k+1)^{n+1}}$$
$[n > 0, \quad 1 \geq x \geq 0, \quad n = 1, \quad 1 > x > 0]$ AS 804 (23.1.16)

2.[10] $$E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}}$$ $[n = 1, 2, \ldots, \quad 1 \geq x \geq 0]$
AS 804 (23.1.17)

3. $$E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}}$$
$[n > 0, \quad 1 \geq x \geq 0, \quad n = 0, \quad 1 > x > 0]$ AS 804 (23.1.18)

9.653 Functional relations and properties:

1. $$E_m(n+1) = 2\sum_{k=1}^{n} (-1)^{n-k} k^m + (-1)^{n+1} E_m(0),$$ [m and n are natural numbers]
AS 804 (23.1.4)

2. $E_n'(x) = nE_{n-1}(x).$ $[n = 1, 2, \ldots]$ AS 804 (23.1.5)

3. $E_n(x+1) + E_n(x) = 2x^n$ $[n = 0, 1, \ldots]$ AS 804 (23.1.6)

4.[8] $$E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x - \frac{k}{m}\right)$$ $[n = 0, 1, \ldots, m = 1, 3, \ldots]$
AS 804 (23.1.10)

5. $$E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right)$$ $[n = 0, 1, \ldots, m = 2, 4, \ldots]$
AS 804 (23.1.10)

9.654 Special cases:

1. $E_1(x) = x - \frac{1}{2}$
2. $E_2(x) = x^2 - x$

3. $E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$
4. $E_4(x) = x^4 - 2x^3 + x$
5. $E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$

9.655 Particular values:

1.	$E_{2n+1} = 0.$	$[n = 0, 1, \ldots]$	AS 805 (23.1.19)
2.	$E_n(0) = -E_n(1) = -2(n+1)^{-1}\left(2^{n+1} - 1\right) B_{n+1}$	$[n = 1, 2, \ldots]$	AS 805 (23.1.20)
3.	$E_n\left(\frac{1}{2}\right) = 2^{-n} E_n$	$[n = 0, 1, \ldots]$	AS 805 (23.1.21)
4.	$E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -(2n)^{-1}\left(1 - 3^{1-2n}\right)\left(2^{2n} - 1\right) B_{2n}$	$[n = 1, 2, \ldots]$	AS 806 (23.1.22)

9.7 Constants

9.71 Bernoulli numbers

- $B_0 = 1$
- $B_1 = -1/2$
- $B_2 = 1/6$
- $B_4 = -1/30$
- $B_6 = 1/42$
- $B_8 = -1/30$
- $B_{10} = 5/66$
- $B_{12} = -691/2730$
- $B_{14} = 7/6$
- $B_{16} = -3617/510$
- $B_{18} = 43\,867/798$
- $B_{20} = -174\,611/330$
- $B_{22} = 854\,513/138$
- $B_{24} = -236\,364\,091/2730$
- $B_{26} = 8\,553\,103/6$
- $B_{28} = -23\,749\,461\,029/870$
- $B_{30} = 8\,615\,841\,276\,005/14\,322$
- $B_{32} = -7\,709\,321\,041\,217/510$
- $B_{34} = 2\,577\,687\,858\,367/6$

9.72 Euler numbers

- $E_0 = 1$
- $E_2 = -1$
- $E_4 = 5$
- $E_6 = -61$
- $E_8 = 1385$
- $E_{10} = -50\,521$
- $E_{12} = 2\,702\,765$
- $E_{14} = -199\,360\,981$
- $E_{16} = 19\,391\,512\,145$
- $E_{18} = -2\,404\,879\,675\,441$
- $E_{20} = 370\,371\,188\,237\,525$

The Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

9.73 Euler's and Catalan's constants

Euler's constant

$\boldsymbol{C} = 0.577\,215\,664\,901\,532\,860\,606\,512\ldots$ (cf. **8.367**)

Catalan's constant

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\,965\,594\ldots$$

9.74[10] Stirling numbers

9.740 The **Stirling number of the first kind** $S_n^{(m)}$ is defined by the requirement that $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles. AS 824 (23.1.3)

9.741 Generating functions:

1. $$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m$$ AS 824 (24.1.3)

2. $$\{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \qquad [|x| < 1]$$ AS 824 (24.1.3)

9.742 Recurrence relations:

1.[8] $$S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}; \quad S_n^{(0)} = \delta_{0n}; \quad S_n^{(1)} = (-1)^{n-1}(n-1)!; \quad S_n^{(n)} = 1 \qquad [n \geq m \geq 1]$$ AS 824 (24.1.3)

2. $$\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)} \qquad [n \geq m \geq r]$$ AS 824 (24.1.3)

9.743 Functional relations and properties

1. $$x(x-h)(x-2h)\cdots(x-mh+h) = \frac{h^m\,\Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}-m+1\right)} = h^m \sum_{k=1}^{m} \left(\frac{x}{h}\right)^k S_k^{(m)}$$

2. $$[(x+1)(x+2)\cdots(x+m)]^{-1} = \left[\binom{x+m}{m} m!\right]^{-1} = \left[\sum_{k=1}^{p} (x+m)^k S_k^{(m)}\right]^{-1}$$

3. $$[(x+h)(x+2h)\cdots(x+mh)]^{-1} = \frac{\Gamma\left(\frac{x}{h}+1\right)}{h^m\,\Gamma\left(\frac{x}{h}+m+1\right)} = \left[h^m \sum_{k=1}^{m} \left(\frac{x}{h}+m\right)^k S_k^{(m)}\right]^{-1}$$

9.744 The Stirling number of the second kind $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

9.745 Generating functions:

1. $$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1)$$ AS 824 (24.1.4)

2. $$(e^x-1)^m = m! \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}$$ AS 824 (24.1.4)

3. $$[(1-x)(1-2x)\cdots(1-mx)]^{-1} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} x^{n-m} \qquad \left[|x| < m^{-1}\right]$$ AS 824 (24.1.4)

9.746 Closed form expression:

1. $$\mathfrak{S}_n^{(m)} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n$$ AS 824 (24.1.4)

9.747 Recurrence relations:

1.[8] $$\mathfrak{S}_{n+1}^{(m)} = m\mathfrak{S}_n^{(m)} + \mathfrak{S}_n^{(m-1)}, \quad \mathfrak{S}_n^{(0)} = \delta_{0n}, \quad \mathfrak{S}_n^{(1)} = \mathfrak{S}_n^{(n)} = 1 \qquad [n \geq m \geq 1]$$ AS 825(24.1.4)

2. $$\binom{m}{r} \mathfrak{S}_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} \mathfrak{S}_{n-k}^{(r)} \mathfrak{S}_k^{(m-r)} \qquad [n \geq m \geq r]$$ AS 825 (24.1.4)

3. $$S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathfrak{S}_{n-m+k}^{(k)}$$ AS 824 (24.1.3)

9.748[7] Particular values:

Stirling numbers of the first kind $S_n^{(m)}$

m	$S_1^{(m)}$	$S_2^{(m)}$	$S_3^{(m)}$	$S_4^{(m)}$	$S_5^{(m)}$	$S_6^{(m)}$	$S_7^{(m)}$	$S_8^{(m)}$	$S_9^{(m)}$
1	1	-1	2	-6	24	-120	720	-5040	40320
2		1	-3	11	-50	274	-1764	13068	-109584
3			1	-6	35	-225	1624	-13132	118121
4				1	-10	85	-735	6769	-67284
5					1	-15	175	-1960	22449
6						1	-21	332	-4536
7							1	-28	546
8								1	-36
9									1

Stirling numbers of the second kind $\mathfrak{S}_n^{(m)}$

m	$\mathfrak{S}_1^{(m)}$	$\mathfrak{S}_2^{(m)}$	$\mathfrak{S}_3^{(m)}$	$\mathfrak{S}_4^{(m)}$	$\mathfrak{S}_5^{(m)}$	$\mathfrak{S}_6^{(m)}$	$\mathfrak{S}_7^{(m)}$	$\mathfrak{S}_8^{(m)}$	$\mathfrak{S}_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

9.749[8] Relationship between Stirling numbers of the first kind and derivatives of $(\ln x)^{-m}$:

1. $$\frac{d^n}{dx^n}\left(\frac{1}{\ln^m x}\right) = \frac{1}{\ln^m x}\sum_{k=1}^{n}\frac{(-1)^k (m)_k S_n^{(k)}}{x^n \ln^k x}$$

where $(m)_k = \Gamma(m+k)/\Gamma(m)$, $\qquad$ [m, n are positive integers]

10 Vector Field Theory

10.1–10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_2)$ be arbitrary vectors, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the set of orthogonal unit vectors in terms of which the components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written $\mathbf{a} \cdot \mathbf{b}$, and the vector product, written either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$. Their properties are as follows:

1. $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (scalar product)
2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ (vector product)
3. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (triple scalar product)
4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$ (triple vector product)

10.12 Properties of scalar product

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative)
2. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}$.

   Note: $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is also written $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; thus (2) may also be written
3. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

10.13 Properties of vector product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative)
2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$
3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

10.14 Differentiation of vectors

If $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$, $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, $\phi(t)$ is a scalar and all functions of t are differentiable, then

1. $$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$$
2. $$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$
3. $$\frac{d}{dt}(\phi\mathbf{a}) = \frac{d\phi}{dt}\mathbf{a} + \phi\frac{d\mathbf{a}}{dt}$$
4. $$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$$
5. $$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$
6. $$\frac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \frac{d\mathbf{c}}{dt}$$
7. $$\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right)$$

10.21 Operators grad, div, and curl

In cartesian coordinates $O\{x_1, x_2, x_3\}$, in which system it is convenient to denote the triad of unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the vector operator ∇, called either "del" or "nabla," has the form

1. $$\nabla \equiv \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}$$

   If $\Phi(x, y, z)$ is any differentiable scalar function, the gradient of Φ, written grad Φ, is

2. $$\operatorname{grad}\Phi \equiv \nabla\,\Phi = \frac{\partial\Phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\Phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\Phi}{\partial x_3}\mathbf{e}_3$$

   The divergence of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written $\operatorname{div}\mathbf{f}$, is

3. $$\operatorname{div} f \equiv \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

   The curl, or rotation, of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written either curl $\mathbf{f}$ or rot $\mathbf{f}$, is

4. $$\operatorname{curl}\mathbf{f} \equiv \operatorname{rot}\mathbf{f} \equiv \nabla \times \mathbf{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{e}_3,$$

   or equivalently,

$$\operatorname{curl}\mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

10.31 Properties of the operator ∇

Let $\Phi(x_1, x_2, x_3)$, $\Psi(x_1, x_2, x_3)$ be any two differentiable scalar functions, $\mathbf{f}(x_1, x_2, x_3)$, $\mathbf{g}(x_1, x_2, x_3)$ any two differentiable vector functions, and $\mathbf{a}$ an arbitrary vector. Define the scalar operator ∇^2, called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Then, in terms of the operator ∇, we have the following: MF I 114

1. $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$
2. $\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$
3. $\nabla(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\nabla)\mathbf{g} + (\mathbf{g}\cdot\nabla)\mathbf{f} + \mathbf{f}\times(\nabla\times\mathbf{g}) + \mathbf{g}\times(\nabla\times\mathbf{f})$
4. $\nabla\cdot(\Phi\mathbf{f}) = \Phi(\nabla\cdot\mathbf{f}) + \mathbf{f}\cdot\nabla\Phi$
5. $\nabla\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\nabla\times\mathbf{f}) - \mathbf{f}\cdot(\nabla\times\mathbf{g})$
6. $\nabla\times(\Phi\mathbf{f}) = \Phi(\nabla\times\mathbf{f}) + (\nabla\Phi)\times\mathbf{f}$
7. $\nabla\times(\mathbf{f}\times\mathbf{g}) = \mathbf{f}(\nabla\cdot\mathbf{g}) - \mathbf{g}(\nabla\cdot\mathbf{f}) + (\mathbf{g}\cdot\nabla)\mathbf{f} - (\mathbf{f}\cdot\nabla)\mathbf{g}$
8. $\nabla\times(\nabla\times\mathbf{f}) = \nabla(\nabla\cdot\mathbf{f}) - \nabla^2\mathbf{f}$
9. $\nabla\times(\nabla\Phi) \equiv \mathbf{0}$
10. $\nabla\cdot(\nabla\times\mathbf{f}) \equiv 0$
11.[10] $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2(\nabla\Phi)\cdot(\nabla\Psi) + \Psi\nabla^2\Phi$

The equivalent results in terms of grad, div, and curl are as follows:

1. $\operatorname{grad}(\Phi + \Psi) = \operatorname{grad}\Phi + \operatorname{grad}\Psi$
2. $\operatorname{grad}(\Phi\Psi) = \Phi\operatorname{grad}\Psi + \Psi\operatorname{grad}\Phi$
3. $\operatorname{grad}(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\operatorname{grad})\mathbf{g} + (\mathbf{g}\cdot\operatorname{grad})\mathbf{f} + \mathbf{f}\times\operatorname{curl}\mathbf{g} + \mathbf{g}\times\operatorname{curl}\mathbf{f}$
4. $\operatorname{div}(\Phi\mathbf{f}) = \Phi\operatorname{div}\mathbf{f} + \mathbf{f}\cdot\operatorname{grad}\Phi$
5. $\operatorname{div}(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot\operatorname{curl}\mathbf{f} - \mathbf{f}\cdot\operatorname{curl}\mathbf{g}$
6. $\operatorname{curl}(\Phi\mathbf{f}) = \Phi\operatorname{curl}\mathbf{f} + \operatorname{grad}\Phi\times\mathbf{f}$
7. $\operatorname{curl}(\mathbf{f}\times\mathbf{g}) = \mathbf{f}\operatorname{div}\mathbf{g} - \mathbf{g}\operatorname{div}\mathbf{f} + (\mathbf{g}\cdot\operatorname{grad})\mathbf{f} - (\mathbf{f}\cdot\operatorname{grad})\mathbf{g}$
8. $\operatorname{curl}(\operatorname{curl}\mathbf{f}) = \operatorname{grad}(\operatorname{div}\mathbf{f}) - \nabla^2\mathbf{f}$
9. $\operatorname{curl}(\operatorname{grad}\Phi) \equiv \mathbf{0}$
10. $\operatorname{div}(\operatorname{curl}\mathbf{f}) \equiv 0$
11. $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2\operatorname{grad}\Phi\cdot\operatorname{grad}\Psi + \Psi\nabla^2\Phi$

The expression $(\mathbf{a}\cdot\nabla)$ or, equivalently $(\mathbf{a}\cdot\operatorname{grad})$, defined by

$$(\mathbf{a}\cdot\nabla) \equiv a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector $\mathbf{a}$.

10.41 Solenoidal fields

A vector field $\mathbf{f}$ is said to be solenoidal if $\operatorname{div}\mathbf{f} \equiv 0$. We have the following representation:

10.411 *Representation theorem for vector Helmholtz equation.* If u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and $\mathbf{m}$ is a constant unit vector, then the vectors

$$\mathbf{X} = \operatorname{curl}(\mathbf{m}u), \qquad \mathbf{Y} = \frac{1}{\lambda}\operatorname{curl}\mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector $\mathbf{H}$. The general solution of the equation is

$$\mathbf{H} = \operatorname{curl}(\mathbf{m}u) + \frac{1}{\lambda}\operatorname{curl}\operatorname{curl}(\mathbf{m}u).$$

10.51–10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates $O\{x_1, x_2, x_3\}$ to the general orthogonal curvilinear coordinates $O\{u_1, u_2, u_3\}$:

$$x_1 = x_1(u_1, u_2, u_3), \qquad x_2 = x_2(u_1, u_2, u_3), \qquad x_3 = x_3(u_1, u_2, u_3)$$

Then,

1. $$dx_i = \frac{\partial x_i}{\partial u_1}du_1 + \frac{\partial x_i}{\partial u_2}du_2 + \frac{\partial x_i}{\partial u_3}du_3 \qquad (i = 1, 2, 3),$$

   and the length element dl may be determined from

2. $$dl^2 = g_{11}\,du_1^2 + g_{22}\,du_2^2 + g_{33}\,du_3^2 + 2g_{23}\,du_2\,du_3 + 2g_{31}\,du_3\,du_1 + 2g_{12}\,du_1\,du_2,$$

   where

3.[3] $$g_{ij} = \frac{\partial x_1}{\partial u_i}\frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i}\frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i}\frac{\partial x_3}{\partial u_j} = g_{ji}, \qquad g_{ij} = 0, \quad i \neq j,$$

   provided the Jacobian of the transformation

4. $$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

   does not vanish (see **14.313**).

   Define the metrical coefficients

5. $$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

   then the volume element dV in orthogonal curvilinear coordinates is

6. $$dV = h_1 h_2 h_3\,du_1\,du_2\,du_3,$$

   and the surface elements of area ds_i on the surfaces $u_i =$ constant, for $i = 1, 2, 3$, are

7. $$ds_1 = h_2 h_3\,du_2\,du_3, \quad ds_2 = h_1 h_3\,du_1\,du_3, \quad ds_3 = h_1 h_2\,du_1\,du_2$$

   Denote by $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ the triad of orthogonal unit vectors that are tangent to the u_1, u_2, and u_3 coordinate lines through any given point P, and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components f_{u_1}, f_{u_2}, and f_{u_3} of $\mathbf{f}$ along the coordinate line,

8. $\mathbf{f} = f_{u_1}\mathbf{e}_1 + f_{u_2}\mathbf{e}_2 + f_{u_3}\mathbf{e}_3$ MF I 115

10.611 $\nabla\Phi$, $\operatorname{div}\mathbf{f}$, $\operatorname{curl}\mathbf{f}$, *and* ∇^2 *in general orthogonal curvilinear coordinates.*

1. $$\operatorname{grad}\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

2.[3] $$\operatorname{div}\mathbf{f} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3f_{u_1}) + \frac{\partial}{\partial u_2}(h_3h_1f_{u_2}) + \frac{\partial}{\partial u_3}(h_1h_2f_{u_3})\right)$$

3. $$\operatorname{curl}\mathbf{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1f_{u_1} & h_2f_{u_2} & h_3f_{u_3}\end{vmatrix}$$

4. $$\nabla^2 \equiv \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right)$$ MF I 21-31

10.612 *Cylindrical polar coordinates.* In terms of the coordinates $O\{r,\phi,z\}$, that is, $u_1 = r$, $u_2 = \phi$, $u_3 = z$, where $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $x_3 = z$ for $-\pi < \phi \le \pi$, it follows that

1. $h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$

   and

2. $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z,$$

3. $$\operatorname{div}\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_z}{\partial z},$$

4. $$\operatorname{curl}\mathbf{f} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ f_r & rf_\phi & f_z\end{vmatrix},$$

5. $$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$ MF I 116

10.613 *Spherical polar coordinates.* In terms of the coordinates $O\{r,\theta,\phi\}$, that is, $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, where $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$, for $0 \le \theta \le \pi, -\pi < \phi \le \pi$, we have

1. $h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$

   and

2.[10] $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi,$$

3. $$\operatorname{div}\mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi},$$

4. $$\operatorname{curl}\mathbf{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_r & rf_\theta & r\sin\theta f_\phi\end{vmatrix},$$

5. $$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$ MF I 116

Special Orthogonal Curvilinear Coordinates and their Metrical Coefficients h_1, h_2, h_3

10.614 *Elliptic cylinder coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_1 u_2, \qquad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \qquad x_3 = u_3$

2. $h_1 = \sqrt{\dfrac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = 1$ MF I 657

10.615 *Parabolic cylinder coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \dfrac{1}{2}\left(u_1^2 - u_2^2\right), \qquad x_2 = u_1 u_2, \qquad x_3 = u_3$

2. $h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = 1$ MF I 658

10.616 *Conical coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \dfrac{u_1}{a}\sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \qquad x_2 = \dfrac{u_1}{b}\sqrt{(b^2 + u_2^2)(b^2 - u_3^2)}, \qquad x_3 = \dfrac{u_1 u_2 u_3}{ab}$

   with $a^2 + b^2 = 1$

2. $h_1 = 1, \qquad h_2 = u_1\sqrt{\dfrac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \qquad h_3 = u_1\sqrt{\dfrac{u_2^2 + u_3^2}{(a^2 + u_3^2)(b^2 - u_3^2)}}$ MF I 659

10.617 *Rotational parabolic coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_1 u_2 u_3, \qquad x_2 = u_1 u_2\sqrt{1 - u_3^2}, \qquad x_3 = \dfrac{1}{2}\left(u_1^2 - u_2^2\right)$

2. $h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = \dfrac{u_1 u_2}{\sqrt{1 - u_3^2}}$ MF I 660

10.618 *Rotational prolate spheroidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)(1 - u_3^2)}, \qquad x_3 = u_1 u_2$

2. $h_1 = \sqrt{\dfrac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 - a^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{\dfrac{(u_1^2 - a^2)(1 - u_2^2)}{1 - u_3^2}}$ MF I 661

10.619 *Rotational oblate spheroidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_3\sqrt{(u_1^2 + a^2)(1 - u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 + a^2)(1 - u_2^2)(1 - u_3^2)}, \qquad x_3 = u_1 u_2$

2. $h_1 = \sqrt{\dfrac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 + a^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{\dfrac{(u_1^2 + a^2)(1 - u_2^2)}{1 - u_3^2}}$ MF I 662

10.620 *Ellipsoidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}}, \quad x_3 = \frac{u_1 u_2 u_3}{ab}$$

2. $$h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \qquad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 663

10.621 *Paraboloidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \qquad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}},$$
$$x_3 = \tfrac{1}{2}\left(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2\right)$$

2. $$h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = u_2\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = u_3\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 664

10.622 *Bispherical coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = a u_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \quad x_2 = a\frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \qquad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2}$$

2. $$h_1 = \frac{a}{(u_1 - u_2)\sqrt{u_1^2 - 1}},$$
$$h_2 = \frac{a}{(u_1 - u_2)\sqrt{1 - u_2^2}}, \qquad h_3 = \left(\frac{a}{u_1 - u_2}\right)\sqrt{\frac{1 - u_2^2}{1 - u_3^2}}$$ MF I 665

10.71–10.72 Vector integral theorems

10.711 *Gauss's divergence theorem.* Let V be a volume bounded by a simple closed surface S and let $\mathbf{f}$ be a continuously differentiable vector field defined in V and on S. Then, if $d\mathbf{S}$ is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f}\, dV$$ KE 39

10.712 *Green's theorems.* Let Φ and Ψ be scalar fields which, together with $\nabla^2 \Phi$ and $\nabla^2 \Psi$, are defined both in a volume V and on its surface S, which we assume to be simple and closed. Then, if $\partial/\partial \mathrm{n}$ denotes differentiation along the outward drawn normal to S, we have

10.713 *Green's first theorem*

$$\int_S \Phi \frac{\partial \Psi}{\partial n}\, dS = \int_V \left(\Phi\, \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi\right)\, dV$$ KE 212

10.714 *Green's second theorem*

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}\right) dS = \int_V \left(\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi\right) dV$$ KE 215

10.715 *Special cases*

1. $$\int_S (\Phi \operatorname{grad} \Phi) \cdot d\mathbf{S} = \int_V \left(\Phi \nabla^2 \Phi + (\operatorname{grad} \Phi)^2\right) dV$$

2. $$\int_S \frac{\partial \Phi}{\partial n} dS = \int_V \nabla^2 \Phi \, dV$$ MV 81

10.716 *Green's reciprocal theorem.* If Φ and Ψ are harmonic, so that $\nabla^2 \Phi = \nabla^2 \Psi = 0$, then

3. $$\int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_S \Psi \frac{\partial \Phi}{\partial n} dS$$ MM 105

10.717 *Green's representation theorem.* If Φ and $\nabla^2 \Phi$ are defined within a volume V bounded by a simple closed surface S, and P is an interior point of V, then in three dimensions

4. $$\Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \Phi \, dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r}\right) dS$$ KE 219

If Φ is harmonic within V, so that $\nabla^2 \Phi = 0$, then the previous result becomes

5. $$\Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r}\right) dS$$

In the case of two dimensions, result (4) takes the form

6. $$\Phi(p) = \frac{1}{2\pi} \int_S \nabla^2 \Phi(q) \ln |p-q| \, dS + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p-q| \, dq - \frac{1}{2\pi} \int \ln |p-q| \frac{\partial}{\partial n_q} \Phi(q) \, dq$$ MM 116

where C is the boundary of the planar region S, and result (5) takes the form

7. $$\Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p-q| \, dq - \frac{1}{2\pi} \int_C \ln |p-q| \frac{\partial}{\partial n_q} \Phi(q) \, dq$$ VL 280

10.718 *Green's representation theorem in R^n.* If Φ is twice differentiable within a region Ω in R^n bounded by the surface Σ with outward drawn unit normal $\mathbf{n}$, then for $p \notin \Sigma$ and $n > 3$

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_\Omega \frac{\nabla^2 \Phi(q)}{|p-q|^{n-2}} d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_\Sigma \left(\frac{1}{|p-q|^{n-2}} \frac{\partial \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p-q|^{n-2}}\right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$ VL 279

is the area of the unit sphere in R^n.

10.719 *Green's theorem of the arithmetic mean.* If Φ is harmonic in a sphere, then the value of Φ at the center of the sphere is the arithmetic mean of its value on the surface. KE 223

10.720 *Poisson's integral in three dimensions.* If Φ is harmonic in the interior of a spherical volume V of radius R and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates (r, θ, ϕ), it satisfies the boundary condition $\Phi(R, \theta, \phi) = f(\theta, \phi)$, then

$$\Phi(r,\theta,\phi) = \frac{R\left(R^2 - r^2\right)}{4\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{f\left(\theta',\phi'\right)\sin\theta'\,d\theta'\,d\phi'}{\left(r^2 + R^2 - 2rR\cos\gamma\right)^{3/2}},$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi - \phi'\right). \qquad \text{KE 241}$$

10.721 *Poisson's integral in two dimensions.* If Φ is harmonic in the interior of a circular disk S of radius R and is continuous on the boundary of the disk on which, in terms of the polar coordinates (r,θ), it satisfies the boundary condition $\Phi(R,\theta) = f(\theta)$, then

$$\Phi(r,\theta) = \frac{\left(R^2 - r^2\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,d\phi}{r^2 + R^2 - 2rR\cos(\theta - \phi)}.$$

10.722 *Stokes' theorem.* Let a simple closed curve C be spanned by a surface S. Define the positive normal $\mathbf{n}$ to S, and the positive sense of description of the curve C with line element $d\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $\mathbf{S} = \mathbf{n}\,dS$,

$$\oint_C \mathbf{f}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{f}\cdot d\mathbf{S}, \qquad \text{MM 143}$$

where the line integral around C is taken in the positive sense.

10.723 *Planar case of Stokes' theorem.* If a region R in the (x,y)-plane is bounded by a simple closed curve C, and $f_1(x,y), f_2(x,y)$ are any two functions having continuous first derivatives in R and on C, then

$$\oint_C \left(f_1\,dx + f_2\,dy\right) = \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\,dy, \qquad \text{MM 143}$$

where the line integral is taken in the counterclockwise sense.

10.81 Integral rate of change theorems

10.811 *Rate of change of volume integral bounded by a moving closed surface.* Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt}\int_{V(t)} f\,dV = \int_{V(t)}\frac{\partial f}{\partial t}\,dV + \int_{S(t)} f\mathbf{v}\cdot d\mathbf{S},$$

where $d\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\,.$$

By virtue of Gauss's theorem, this also takes the form

$$\frac{D}{Dt}\int_{V(t)} f\,dV = \int_{V(t)}\left(\frac{Df}{Dt} + f\operatorname{div}\mathbf{v}\right)dV. \qquad \text{MV 88}$$

10.812 *Rate of change of flux through a surface.* Let $\mathbf{q}$ be a vector function that may also depend on the time t, and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \int_S \mathbf{q} \cdot \mathbf{n}\, dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v}\, \mathrm{div}\, \mathbf{q} + \mathrm{curl}\, (\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n}\, dS. \qquad \text{MV 90}$$

10.813 *Rate of change of the circulation around a given moving curve.* Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \int_C \mathbf{q} \cdot\, d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathrm{curl}\, \mathbf{q}) \times \mathbf{v} \right) \cdot\, d\mathbf{r}. \qquad \text{MV 94}$$

11 Algebraic Inequalities

11.1–11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.111 *Lagrange's identity.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum (a_k b_j - a_j b_k)^2$$

BB 3

11.112 *Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional.

MT 30

11.113 *Minkowski's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers, and let $p > 1$; then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional.

MT 55

11.114 *Hölder's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers, and let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$; then

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \ge \sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, the sequences $a_1^p, a_2^p, \ldots, a_n^p$ and $b_1^q, b_2^q, \ldots, b_n^q$ are proportional.

MT 50

11.115 *Chebyshev's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers such that either $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, or $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$; then

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \le \frac{1}{n}\sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

11.116 *Arithmetic-geometric inequality.* Let $a_1, a_2, \ldots, a_n$ be any set of positive numbers, with arithmetic mean

$$A_n = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

and geometric mean

$$G_n = (a_1 a_2 \ldots a_n)^{1/n};$$

then $A_n \geq G_n$ or, equivalently,

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq (a_1 a_2 \ldots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers a_i are equal. BB 4

11.117 *Carleman's inequality.* If $a_1, a_2, \ldots, a_n$ is any finite set of non-negative numbers, then

$$\sum_{r=1}^{n} (a_1 a_2 \ldots a_r)^{1/r} \leq e\,(a_1 + a_2 + \cdots + a_n),$$

where e is the best possible constant in this inequality. The inequality is strict except for the trivial case when $a_r = 0$ for $r = 1, 2, \ldots, n$. MT 131

11.118 *An inequality involving absolute values.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^{n} \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \qquad 0 < p \leq 2.$$

11.21 Algebraic inequalities involving complex numbers

If α, β are any two real numbers, the complex number $z = \alpha + i\beta$ with real part α and imaginary part β has for its modulus $|z|$ the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude)$\arg z$ the angle $\arg z = \theta$ such that

$$\cos\theta = \frac{\alpha}{|z|} \text{ and } \sin\theta = \frac{\beta}{|z|},$$

where $-\pi < \theta \leq \pi$. The complex number $\overline{z} = \alpha - i\beta$ is said to be the **complex conjugate** of $z = \alpha + i\beta$.

$$\text{If } z = re^{i\theta} = r\,(\cos\theta + i\sin\theta),$$

then

$$z^n = r^n e^{in\theta} = r^n\,(\cos n\theta + i\sin n\theta),$$

and, setting $r = 1$, we have **de Moivre's theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It follows directly that, if $z = e^{i\theta}$, then

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\alpha = -\frac{i}{2}\left(z - \frac{1}{z}\right),$$

and

$$\cos r\theta = \frac{1}{2}\left(z^r + \frac{1}{z^r}\right), \qquad \sin r\theta = -\frac{i}{2}\left(z^r - \frac{1}{z^r}\right).$$

If $w = z^{p/q}$ with p, q integral, and $z = re^{i\theta}$, then the q roots of $w_0, w_1, \ldots, w_{q-1}$ of z are

$$w_k = r^{p/q}\left[\cos\left(\frac{p\theta+2k\pi}{q}\right)+i\sin\left(\frac{p\theta+2k\pi}{q}\right)\right],$$

with $k=0,1,2,\ldots,q-1$.

11.211[7] *Simple properties and inequalities involving the modulus and the complex conjugate.* If the real part of z is denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$, then

$$\begin{aligned}
z+\overline{z} &= 2\operatorname{Re} z = 2\alpha,\\
z-\overline{z} &= 2\operatorname{Im} z = 2i\beta,\\
z &= \overline{(\overline{z})},\\
\frac{1}{\overline{z}} &= \overline{\left(\frac{1}{z}\right)},\\
\overline{(z^n)} &= (\overline{z})^n,\\
\left|\frac{\overline{z_1}}{\overline{z_2}}\right| &= \frac{|\overline{z_1}|}{|\overline{z_2}|},\\
\overline{(z_1+z_2+\cdots+z_n)} &= \overline{z_1}+\overline{z_2}+\cdots+\overline{z_n},\\
\overline{z_1 z_2\cdots z_n} &= \overline{z_1}\,\overline{z_2}\cdots\overline{z_n}.
\end{aligned}$$

11.212 *Inequalities for pairs of complex numbers.* If a,b are any two complex numbers, then

(i) $|a+b|\le|a|+|b|$ (triangle inequality),

(ii) $|a-b|\ge||a|-|b||$.

11.31 Inequalities for sets of complex numbers

11.311 *Complex Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be any two arbitrary sets of complex numbers; then

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \le \left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right).$$

The equality holds if, and only if, the sequences $\overline{a_1},\overline{a_2},\ldots,\overline{a_n}$ and $b_1,b_2,\ldots,b_n$ are proportional.

MT 42

11.312 *Complex Minkowski inequality.* Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be any two arbitrary sets of complex numbers, and let the real number p be such that $p>1$; then

$$\left(\sum_{k=1}^{n}|a_k+b_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n}|b_k|^p\right)^{1/p}.$$

MT 56

11.313 *Complex Hölder inequality.* Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be any two arbitrary sets of complex numbers, and let the real numbers p,q be such that $p>1$ and $\dfrac{1}{p}+\frac{1}{q}=1$; then

$$\left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|b_k|^q\right)^{1/p} \ge \left|\sum_{k=1}^{n} a_k b_k\right|.$$

The equality holds if, and only if, the sequences

$$|a_1|^p,\quad |a_2|^p,\ldots,|a_n|^p \text{ and } |b_1|^p,|b_2|^p,\ldots|b_n|^p,$$

are proportional and $\arg a_k b_k$ is independent of k for $k=1,2,\ldots,n$.

MT 53

This page intentionally left blank

12 Integral Inequalities

12.11 Mean Value Theorems

12.111 First mean value theorem

Let $f(x)$ and $g(x)$ be two bounded functions integrable in $[a, b]$, and let $g(x)$ be of one sign in this interval. Then

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx, \qquad \text{CA 105}$$

with $a \le \xi \le b$.

12.112 Second mean value theorem

(i) Let $f(x)$ be a bounded, monotonic decreasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx,$$

with $a \le \xi \le b$.

(ii) Let $f(x)$ be a bounded, monotonic increasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx,$$

with $a \le \eta \le b$.

(iii) Let $f(x)$ be bounded and monotonic in $[a, b]$, and let $g(x)$ be a bounded integrable function which experiences only a finite number of sign changes in $[a, b]$. Then,

$$\int_a^b f(x)g(x)\,dx = f(a+0)\int_a^\xi g(x)\,dx + f(b-0)\int_\xi^b g(x)\,dx, \qquad \text{CA 107}$$

with $a \le \xi \le b$.

12.113 First mean value theorem for infinite integrals

Let $f(x)$ be bounded for $x \ge a$, and integrable in the arbitrary interval $[a, b]$, and let $g(x)$ be of one sign in $x \ge a$ and such that $\int_a^\infty g(x)\,dx$ is finite. Then,

$$\int_a^\infty f(x)g(x)\,dx = \mu\int_a^\infty g(x)\,dx, \qquad \text{CA 123}$$

where $m \le \mu \le M$ and m, M are, respectively, the lower and upper bounds of $f(x)$ for $x \ge a$.

12.114 Second mean value theorem for infinite integrals

Let $f(x)$ be bounded and monotonic when $x \ge a$, and $g(x)$ be bounded and integrable in the arbitrary interval $[a, b]$ in which it experiences only a finite number of changes of sign. Then, provided $\int_a^\infty g(x)\,dx$ is finite,

$$\int_a^\infty f(x)g(x)\,dx = f(a+0)\int_a^\xi g(x)\,dx + f(\infty)\int_\xi^\infty g(x)\,dx, \qquad \text{CA 123}$$

with $a \le \xi \le \infty$.

12.21 Differentiation of Definite Integral Containing a Parameter

12.211 Differentiation when limits are finite

Let $\phi(\alpha)$ and $\psi(\alpha)$ be twice differentiable functions in some interval $c \le \alpha \le d$, and let $f(x, \alpha)$ be both integrable with respect to x over the interval $\phi(\alpha) \le x \le \psi(\alpha)$ and differentiable with respect to α. Then,

$$\frac{d}{d\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx = \left(\frac{d\psi}{d\alpha}\right) f\left(\psi(\alpha),\alpha\right) - \left(\frac{d\phi}{d\alpha}\right) f\left(\phi(\alpha),\alpha\right) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial \alpha}\,dx. \qquad \text{FI II 680}$$

12.212 Differentiation when a limit is infinite

Let $f(x, \alpha)$ and $\partial f/\partial\alpha$ both be integrable with respect to x over the semi-infinite region $x \ge a$, $b \le \alpha < c$. Then, if the integral

$$f(\alpha) = \int_a^\infty f(x,\alpha)\,dx$$

exists for all $b \le \alpha \le c$, and if $\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx$ is uniformly convergent for α in $[b, c]$, it follows that

$$\frac{d}{d\alpha}\int_a^\infty f(x,\alpha)\,dx = \int_a^\infty \frac{\partial f}{\partial \alpha}\,dx$$

12.31 Integral Inequalities

12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals

Let $f(x)$ and $g(x)$ be any two real integrable functions on $[a, b]$. Then,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right),$$

and the equality will hold if, and only if, $f(x) = kg(x)$, with k real. BB 21

12.312 Hölder's inequality for integrals

Let $f(x)$ and$g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^q$ are integrable on $[a, b]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\int_a^b f(x)g(x)\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(\int_a^b |g(x)|^q\,dx\right)^{1/q}.$$

The equality holds if, and only if, $\alpha|f(x)|^p = \beta|g(x)|^q$, where α and β are positive constants. BB 21

12.313 Minkowski's inequality for integrals

Let $f(x)$ and $g(x)$be any two real functions for which $|f(x)|^p$ and $|g(x)|^p$ are integrable on $[a,b]$ for $p>0$; then

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

The equality holds if, and only if, $f(x) = kg(x)$ for some real $k \ge 0$. BB 21

12.314 Chebyshev's inequality for integrals

Let $f_1, f_2, \ldots, f_n$ be nonnegative integrable functions on $[a,b]$ which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x)\,dx \int_a^b f_2(x)\,dx \ldots \int_a^b f_n(x)\,dx \le (b-a)^{n-1}\int_a^b f_1(x)f_2(x)\ldots f_n(x)\,dx \qquad \text{MT 39}$$

12.315 Young's inequality for integrals

Let $f(x)$ be a real-valued continuous strictly monotonic increasing function on the interval $[0,a]$, with $f(0)=0$ and $b \le f(a)$. Then

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

where $f^{-1}(y)$ denotes the function inverse to $f(x)$. The equality holds if, and only if, $b = f(a)$. BB 15

12.316 Steffensen's inequality for integrals

Let $f(x)$ be nonnegative and monotonic decreasing in $[a,b]$, and $g(x)$ be such that $0 \le g(x) \le 1$ in $[a,b]$. Then

$$\int_{b-k}^b f(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^{a+k} f(x)\,dx,$$

where $k = \int_a^b g(x)\,dx$. MT 107

12.317 Gram's inequality for integrals

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be real square integrable functions on $[a,b]$; then

$$\begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x)f_2(x)\,dx & \cdots & \int_a^b f_1(x)f_n(x)\,dx \\ \int_a^b f_2(x)f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x)f_n(x)\,dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x)\,dx & \int_a^b f_n(x)f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix} \ge 0. \qquad \text{MT 47}$$

12.318 Ostrowski's inequality for integrals

Let $f(x)$ be a monotonic function integrable on $[a,b]$, and let $f(a)f(b) \geq 0, |f(a)| \geq |f(b)|$. Then, if g is a real function integrable on $[a,b]$,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq |f(a)| \max_{a\leq\xi\leq b}\left|\int_a^\xi g(x)\,dx\right|.$$

12.41 Convexity and Jensen's Inequality

A function $f(x)$ is said to be **convex** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}.$$

A function $f(x)$ is said to be **concave** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$ the function $-f(x)$ is convex in that interval.

If the function $f(x)$ possesses a second derivative in the interval $[a,b]$, then a necessary and sufficient condition for it to be convex on that interval is that $f''(x) \geq 0$ for all x in $[a,b]$.

A function $f(x)$ is said to be **logarithmically convex** on the interval $[a,b]$ if $f > 0$ and $\log f(x)$ is concave on $[a,b]$.

If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a,b]$, then the functions $f(x)+g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a,b]$. MT 17

12.411 Jensen's inequality

Let $f(x), p(x)$ be two functions defined for $a \leq x \leq b$ such that $\alpha \leq f(x) \leq \beta$ and $p(x) \geq 0$, with $p(x) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then

$$\phi\left(\frac{\int_a^b f(x)p(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{\int_a^b \phi(f)\,p(x)\,dx}{\int_a^b p(x)\,dx}.$$ HL 151

12.412 Carleman's inequality for integrals

If $f(x) \geq 0$ and the integrals exist, then

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\,dx \leq e\int_0^\infty f(x)\,dx.$$

12.51 Fourier Series and Related Inequalities

The trigonometric **Fourier series** representation of the function $f(x)$ integrable on $[-\pi,\pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients** a_n and b_n of $f(x)$ are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^\pi f(x)\cos nx\,dx, \qquad b_n = \frac{1}{2\pi}\int_{-\pi}^\pi f(x)\sin nx\,dx.$$

(See **0.320–0.328** for convergence of Fourier series on $(-l,l)$.) TF 1

12.511 Riemann-Lebesgue lemma

If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\sin tx\,dx \to 0$$

and

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\cos tx\,dx \to 0. \qquad \text{TF 11}$$

12.512 Dirichlet lemma

$$\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}\,dx = \frac{\pi}{2},$$

in which $\sin\left(n+\frac{1}{2}\right)x \,/\, 2\sin\frac{1}{2}x$ is called the **Dirichlet kernel**. ZY 21

12.513 Parseval's theorem for trigonometric Fourier series

If $f(x)$ is square integrable on $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty}\left(a_r^2 + b_r^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx. \qquad \text{Y 10}$$

12.514 Integral representation of the nth partial sum

If $f(x)$ is integrable on $[-\pi, \pi]$, then the nth partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^{n}\left(a_r\cos rx + b_r\sin rx\right)$$

has the following integral representation in terms of the Dirichlet kernel:

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\,dt. \qquad \text{Y 20}$$

12.515 Generalized Fourier series

Let the set of functions $\{\phi_n\}_{n=0}^{\infty}$ form an **orthonormal set** over $[a, b]$, so that

$$\int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 1 & \text{for} \quad m = n, \\ 0 & \text{for} \quad m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function $f(x)$ on $[a, b]$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n\phi_n(x),$$

where the generalized Fourier coefficients of $f(x)$ are given by

$$c_n = \int_a^b f(x)\phi_n(x)\,dx.$$

12.516 Bessel's inequality for generalized Fourier series

For any square integrable function defined on $[a, b]$,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x)\,dx,$$

where the c_n are the generalized Fourier coefficients of $f(x)$.

12.517 Parseval's theorem for generalized Fourier series

If $f(x)$ is a square integrable function defined on $[a, b]$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ is a **complete orthonormal** set of continuous functions defined on $[a, b]$, then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x)\,dx,$$

where the c_n are generalized Fourier coefficients of $f(x)$.

13 Matrices and Related Results

13.11–13.12 Special Matrices

13.111 Diagonal matrix

A square matrix $\mathbf{A}$ of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

13.112 Identity matrix and null matrix

The **identity matrix** is a diagonal matrix $\mathbf{I}$ in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

13.113 Reducible and irreducible matrices

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is said to be **reducible**, if the indices $1, 2, \dots, n$ can be divided into two disjoint non-empty sets $i_1, i_2, \dots, i_\mu; j_1, j_2, \dots, j_\nu$ with $(\mu + \nu = n)$, such that

$$a_{i_\alpha j_\beta} = 0 \qquad (\alpha = 1, 2, \dots, \mu; \quad \beta = 1, 2, \dots, \nu)\,.$$

Otherwise, $\mathbf{A}$ will be said to be irreducible. GA 61

13.114 Equivalent matrices

An $m \times n$ matrix $\mathbf{A}$ is **equivalent** to an $m \times n$ matrix $\mathbf{B}$ if, and only if, $\mathbf{B} = \mathbf{PAQ}$ for suitable non-singular $m \times m$ and $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$, respectively.

13.115 Transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix with element a_{ij} in the ith row and the jth column, then the transpose $\mathbf{A}^{\mathrm{T}}$ of $\mathbf{A}$ is the $n \times m$ matrix

$$\mathbf{A}^{\mathrm{T}} = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from $\mathbf{A}$ by interchanging rows and columns.

13.116 Adjoint matrix

If $\mathbf{A}$ is an $n \times n$ matrix, then its **adjoint**, denoted by $\operatorname{adj} \mathbf{A}$, is the transpose of the matrix of cofactors A_{ij} of $\mathbf{A}$, so that

$$\operatorname{adj} \mathbf{A} = [A_{ij}]^{\mathrm{T}} \qquad \text{(see } \mathbf{14.13}\text{)}.$$

13.117 Inverse matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with a nonsingular determinant $|\mathbf{A}|$, then its **inverse** $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj} \mathbf{A}}{|\mathbf{A}|}.$$

13.118 Trace of a matrix

The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, written $\operatorname{tr} \mathbf{A}$, is defined to be the sum of the terms on the leading diagonal, so that

$$\operatorname{tr} \mathbf{A} = a_{11} + a_{22} + \ldots + a_{nn}.$$

13.119 Symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **symmetric** if $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, n$.

13.120 Skew-symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for $i, j = 1, 2, \ldots, n$.

13.121 Triangular matrices

An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is of **upper triangular type** if $a_{ij} = 0$ for $i > j$ and of **lower triangular type** if $a_{ij} = 0$ for $j > i$.

13.122 Orthogonal matrices

A real $n \times n$ matrix $\mathbf{A}$ is **orthogonal** if, and only if, $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$.

13.123 Hermitian transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with complex elements, then its **hermitian transpose** $\mathbf{A}^{\mathrm{H}}$ is defined to be

$$\mathbf{A}^{\mathrm{H}} = [\overline{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

13.124 Hermitian matrix

An $n \times n$ matrix $\mathbf{A}$ is **hermitian** if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$, or equivalently, if $\mathbf{A} = \overline{\mathbf{A}}^{\mathrm{T}}$, with the bar denoting the complex conjugate operation.

13.125 Unitary matrix

An $n \times n$ matrix $\mathbf{A}$ is **unitary** if $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$.

13.126 Eigenvalues and eigenvectors

If $\mathbf{A}$ is an $n \times n$ matrix, each eigenvector $\mathbf{x}$ corresponding to λ satisfies the equation

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{x},$$

while the **eigenvalues** λ satisfy the **characteristic equation**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \qquad \text{(see } \mathbf{15.61}\text{)}.$$

13.127 Nilpotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **nilpotent** if $\mathbf{A}^k = \mathbf{0}$ for some k.

13.128 Idempotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

13.129 Positive definite

An $n \times n$ matrix $\mathbf{A}$ is **positive definite** if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.130 Non-negative definite

An $n \times n$ matrix $\mathbf{A}$ is **non-negative definite** if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.131 Diagonally dominant

An $n \times n$ matrix $\mathbf{A}$ is **diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i.

13.21 Quadratic Forms

A **quadratic form** involving the n real variables $x_1, x_2, \ldots, x_n$ that are associated with the real $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar expression

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

In terms of matrix notation, if $\mathbf{x}$ is the $n \times 1$ column vector with real elements $x_1, x_2, \ldots, x_n$, and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, then

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x}, \mathbf{A}\mathbf{x}).$$

If the $n \times n$ matrix $\mathbf{A}$ is hermitian, so that $\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}$, where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix $\mathbf{A}$ and the vector $\mathbf{x}$, which may have complex elements, is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x}).$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x}),$$

where $\mathbf{A} = [a_{ij}]$ is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \qquad \text{for } i = 1, 2, \ldots, n$$

and

$$a_{ij} = \tfrac{1}{2}(\alpha_{ij} + \alpha_{ji}) \qquad \text{for } i, j = 1, 2, \ldots, n \quad \text{and} \quad i \neq j.$$

13.211 Sylvester's law of inertia

When a quadratic form Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2,$$

the number p of positive squares appearing in the reduction is an invariant of the quadratic form Q, and it does not depend on the method of reduction itself. ML 377

13.212 Rank

The **rank** of the quadratic form Q in the above canonical form is the total number r of squared terms (both positive and negative) appearing in its reduced form. ML 360

13.213 Signature

The **signature** of the quadratic form Q above is the number s of positive squared terms appearing in its reduced form. It is sometimes also defined to be $2s - r$. ML 378

13.214 Positive definite and semidefinite quadratic form

The quadratic form $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is said to be **positive definite** when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. It is said to be **positive semidefinite** if $Q(x) \geq 0$ for $x \neq 0$. ML 394

13.215 Basic theorems on quadratic forms

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.
2. A real quadratic form in n variables is positive definite if, and only if, its canonical form is
$$Q = z_1^2 + z_2^2. + \ldots + z_n^2.$$
3. A real symmetric matrix $\mathbf{A}$ is positive definite if, and only if, there exists a real nonsingular matrix $\mathbf{M}$ such that $\mathbf{A} = \mathbf{M}\mathbf{M}^{\mathrm{T}}$.
4. Any real quadratic form in n variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \ldots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is positive definite if, and only if, every eigenvalue of $\mathbf{A}$ is positive; it is positive semidefinite if, and only if, all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is indefinite if the eigenvalues of $\mathbf{A}$ are of both signs.

6. The necessary conditions for an hermitian matrix $\mathbf{A}$ to be positive definite are

   (i) $a_{ii} > 0$ for all i,
   (ii) $a_{ii}a_{ij} > |a_{ij}|^2$ for $i \neq j$,
   (iii) the element of largest modulus must lie on the leading diagonal,
   (iv) $|\mathbf{A}| > 0$.

7. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ with $\mathbf{A}$ hermitian will be positive definite if all the principal minors in the top left-hand corner of $\mathbf{A}$ are positive, so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots .$$

ML 353-379

13.31 Differentiation of Matrices

If the $n \times m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have elements that are differentiable functions of t, so that

$$\mathbf{A}(t) = [a_{ij}(t)], \qquad \mathbf{B}(t) = [b_{ij}(t)]$$

then

1. $$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{d}{dt}a_{ij}(t)\right]$$

2. $$\frac{d}{dt}[\mathbf{A}(t) \pm \mathbf{B}(t)] = \left[\frac{d}{dt}a_{ij}(t) \pm \frac{d}{dt}b_{ij}(t)\right] = \frac{d}{dt}\mathbf{A}(t) \pm \frac{d}{dt}\mathbf{B}(t).$$

3. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt}[\mathbf{A}(t)\mathbf{B}(t)] = \left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{B}(t) + \mathbf{A}(t)\left(\frac{d}{dt}\mathbf{B}(t)\right).$$

4. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt}[\mathbf{A}(t)\mathbf{B}(t)]^{\mathrm{T}} = \left(\frac{d}{dt}\mathbf{B}(t)\right)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(t) + \mathbf{B}^{\mathrm{T}}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)^{\mathrm{T}}.$$

5. If the square matrix $\mathbf{A}$ is nonsingular, so that $|\mathbf{A}| \neq 0$, then

$$\frac{d}{dt}\left[\mathbf{A}^{-1}\right] = -\mathbf{A}^{-1}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{A}^{-1}(t)$$

6. $$\int_{t_0}^{T}\mathbf{A}(\tau)\,d\tau = \left[\int_{t_0}^{T}a_{ij}(\tau)\,d\tau\right]$$

13.41 The Matrix Exponential

If $\mathbf{A}$ is a square matrix, and z is any complex number, then the matrix exponential e^{Az} is defined to be

$$e^{Az} = \mathbf{I} + \mathbf{A}z + \ldots + \frac{\mathbf{A}^n z^n}{n!} + \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r z^r.$$

3.411 Basic properties

1. $e^0 = \mathbf{I}, \quad e^{Iz} = \mathbf{I}e^z, \quad e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2},$ [when $\mathbf{A}+\mathbf{B}$ is defined and $\mathbf{AB} = \mathbf{BA}$]
   $e^{-\mathbf{A}z} = \left(e^{\mathbf{A}z}\right)^{-1}, \quad e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$
2. $\dfrac{d^r}{dz^r}\left(e^{\mathbf{A}z}\right) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z}\mathbf{A}^r.$

ML 340

3. If the square matrix $\mathbf{A}$ can be expressed in the form $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$, with $\mathbf{B}$ and $\mathbf{C}$ square matrices, then
$$e^{\mathbf{A}z} = \begin{bmatrix} e^{\mathbf{B}z} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{C}z} \end{bmatrix}.$$

14 Determinants

14.11 Expansion of Second- and Third-Order Determinants

1. $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

2. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

14.12 Basic Properties

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices. Then the following results are true:

1. If any two adjacent rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.
2. If any two rows (or columns) of a determinant are identical, the determinant is zero.
3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).
4. $|k\mathbf{A}| = k^n|\mathbf{A}|$ for any scalar k.
5. $\left|\mathbf{A}^{\mathrm{T}}\right| = |\mathbf{A}|$ where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.
6. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.
7. $\left|\mathbf{A}^{-1}\right| = \frac{1}{|\mathbf{A}|}$ when the inverse exists.
8. If the elements a_{ij} of $\mathbf{A}$ are functions of x, then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,j=1}^{n} \frac{da_{ij}}{dx} A_{ij} \qquad \text{(see } \mathbf{14.13}\text{)}.$$

14.13 Minors and Cofactors of a Determinant

The **minor** M_{ij} of the element a_{ij} in the nth-order determinant $|\mathbf{A}|$ associated with the square $n \times n$ matrix $\mathbf{A}$ is the $(n-1)^{\text{th}}$-order determinant derived from $\mathbf{A}$ by deletion of the ith row and j^{th} column. The cofactor A_{ij} of the element a_{ij} is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}. \qquad \text{ML 20}$$

14.14 Principal Minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of $\mathbf{A}$. ML 197

14.15* Laplace Expansion of a Determinant

The nth-order determinant denoted by $|\mathbf{A}|$, or det $\mathbf{A}$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ may be expanded either by elements of the ith row as

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

or by elements of the jth column as

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{ij} A_{ij},$$

where A_{ij} is the cofactor of element a_{ij}. The cofactors A_{ij} satisfy the following n linear equations:

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \qquad \sum_{i=1}^{n} a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$

ML 21

$$\text{for } i, j, k = 1, 2, \ldots, n \text{ and } \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

14.16 Jacobi's Theorem

Let M_r be an r-rowed minor of the nth-order determinant $|\mathbf{A}|$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, in which the rows $i_1, i_2, \ldots, i_r$ are represented together with the columns $k_1, k_2, \ldots, k_r$.

Define the **complementary minor** to M_r to be the $(n-k)$-rowed minor obtained from $|\mathbf{A}|$ by deleting all the rows and columns associated with M_r, and the **signed complementary minor** $M^{(r)}$ to M_r to be

$$M^{(r)} = (-1)^{i_1+i_2+\cdots+i_r+k_1+k_2+\cdots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if Δ is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and M_r and M_r' are corresponding r-rowed minors of $|\mathbf{A}|$ and Δ, it follows that

$$M_r' = |\mathbf{A}|^{r-1} M^{(r)}.$$

ML 25

Corollary. If $|\mathbf{A}| = 0$, then

$$A_{pk} A_{nq} = A_{nk} A_{pq}.$$

14.17 Hadamard's Theorem

If $|\mathbf{A}|$ is an $n \times n$ determinant with elements a_{ij} that may be complex, then $|\mathbf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| .$$

14.18 Hadamard's Inequality

Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $n \times n$ nonsingular matrix with real elements and determinant $|\mathbf{A}|$. Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}^2 \right) .$$

This result is also true when $\mathbf{A}$ is hermitian. ML 418

Deductions.

1. If $M = \max |a_{ij}|$, then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$ ML 419

2. If the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is positive definite, then

$$|\mathbf{A}| \leq a_{11} a_{22} \ldots a_{nn}.$$ BL 126

3. If the real $n \times n$ matrix $\mathbf{A}$ is diagonally dominant, so that $\sum_{j\neq 1}^{n} |a_{ij}| < |a_{ii}|$ for $i = 1, 2, \ldots, n$, then $|\mathbf{A}| \neq 0$.

14.21 Cramer's Rule

If the n linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n, \end{array}$$

have a nonsingular coefficient matrix $\mathbf{A} = [a_{ij}]$, so that $|\mathbf{A}| \neq 0$, then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_j}{|\mathbf{A}|}$$

for $j = 1, 2, \ldots, n$, where A_{ij} is the cofactor of element a_{ij} in the coefficient matrix $\mathbf{A}$. ML 134

14.31 Some Special Determinants

14.311 Vandermonde's determinant (alternant)

Third order.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the nth-order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}n(n-1)$ pairs of numbers taken from $x_1, x_2, \ldots, x_n$, with the order of the differences taken in the reverse order of the suffixes that are involved. ML 17

14.312 Circulants

Second order.

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

Third order.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)\left(x_1 + \omega x_2 + \omega^2 x_3\right)\left(x_1 + \omega^2 x_2 + \omega x_3\right),$$

where ω and ω^2 are the complex cube roots of 1. In general, the nth-order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1}^{n} \left(x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1}\right),$$

where ω_j is an nth root of 1. The eigenvalues λ (see **15.61**) of an $n \times n$ circulant matrix are

$$\lambda_j = x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1},$$

where ω_j is again an nth root of 1. ML 36

14.313 Jacobian determinant

If $f_1, f_2, \ldots, f_n$ are n real-valued functions which are differentiable with respect to $x_1, x_2, \ldots, x_n$, then the Jacobian $J_f(x)$ of the f_i with respect to the x_j is the determinant

$$J_f(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

The notation

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

is also used to denote the Jacobian $J_f(x)$.

14.314 Hessian determinants

The Jacobian of the derivatives $\dfrac{\partial\phi}{\partial x_1}, \dfrac{\partial\phi}{\partial x_2}, \ldots, \dfrac{\partial\phi}{\partial x_n}$ of a function $\phi(x_1, x_2, \ldots, x_n)$ with respect to x_1, x_2, $\ldots$, x_n is called the Hessian H of ϕ, so that

$$H = \begin{vmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\,\partial x_2} & \frac{\partial^2\phi}{\partial x_1\,\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_1\,\partial x_n} \\ \frac{\partial^2\phi}{\partial x_2\,\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \frac{\partial^2\phi}{\partial x_2\,\partial x_3} & \cdots & \frac{\partial^2 d2\phi}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial x_n\,\partial x_1} & \frac{\partial^2\phi}{\partial x_n\,\partial x_2} & \frac{\partial^2\phi}{\partial x_n\,\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{vmatrix}.$$

14.315 Wronskian determinants

Let f_1, f_2, $\ldots$, f_n be n functions each n times differentiable with respect to x in some open interval (a, b). Then the Wronskian $W(x)$ of $f_1, f_2, \ldots, f_n$ is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where $f_i^{(r)} = \dfrac{d^r f_i}{dx^r}$.

14.316 Properties

1. $\dfrac{dW}{dx}$ follows from $W(x)$ by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}$, $f_2^{(n)}, \ldots, f_n^{(n)}$.

2. If constants $k_1, k_2, \ldots, k_n$ exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$$

for all x in (a, b), then $W(x) = 0$ for all x in (a, b).

3. The vanishing of the Wronskian throughout (a, b) is necessary, but not sufficient, for the linear dependence of $f_1, f_2, \ldots, f_n$.

14.317 Gram-Kowalewski theorem on linear dependence

A necessary and sufficient condition for n functions $f_1, f_2, \ldots, f_n$ square integrable over $a \leq n \leq b$ to be linearly dependent in this interval is the vanishing of the Gram determinant

$$G(f_1, f_2, \ldots, f_n) = \begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x)f_2(x)\,dx & \cdots & \int_a^b f_1(x)f_n(x)\,dx \\ \int_a^b f_2(x)f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x)f_n(x)\,dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x)\,dx & \int_a^b f_n(x)f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix}. \qquad \text{SA 2 (Theorem 3)}$$

14.318 If the n functions $f_1, f_2, \ldots, f_n$ are square integrable over $a \leq n \leq b$, then the Gram determinant

$$G\left(f_1, f_2, \ldots, f_n\right) \geq 0,$$

and the equality sign holds only when the functions are linearly dependent in $a \leq n \leq b$.

SA 4 (Corollary 1)

14.319 The rank of the matrix corresponding to the Gram determinant $G(f_1, f_2, \ldots, f_n)$ gives the maximum number of linearly independent functions $f_1,\ f_2,\ \ldots\ ,\ f_n$ in $a \leq x \leq b$. If the rank is r, then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

SA 3 (Theorem 4)

15 Norms

15.1–15.9 Vector Norms

15.11 General Properties

The **vector norm** $||\mathbf{x}||$ of an $n \times 1$ column vector $\mathbf{x}$ is a nonnegative number having the property that

1. $||\mathbf{x}|| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $||\mathbf{x}|| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;
2. $||k\mathbf{x}|| = |k|||\mathbf{x}||$ for any scalar k;
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

15.21 Principal Vector Norms

15.211 The norm $||\mathbf{x}||_1$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_1 = \sum_{r=1}^{n} |x_r|. \qquad \text{VA 15}$$

15.212 The norm $||\mathbf{x}||_2$ (Euclidean or L_2 norm)

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_2 = \left(\sum_{r=1}^{n} |x_r|^2 \right)^{1/2}. \qquad \text{VA 8}$$

15.213 The norm $||\mathbf{x}||_\infty$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_\infty = \max_i |x_i|. \qquad \text{VA 15}$$

15.31 Matrix Norms

15.311 General properties

The **matrix norm** $||\mathbf{A}||$ of a square matrix $\mathbf{A}$ is a nonnegative number associated with $\mathbf{A}$ having the properties that

1. $||\mathbf{A}|| > 0$ when $\mathbf{A} \neq \mathbf{0}$ and $||\mathbf{A}|| = 0$ if, and only if, $\mathbf{A} = \mathbf{0}$;
2. $||k\mathbf{A}|| = |k|\,||\mathbf{A}||$ for any scalar k;
3. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$;
4. $||\mathbf{AB}|| \leq ||\mathbf{A}||\,||\mathbf{B}||$. VA 9

The matrix norm $||\mathbf{A}||$ associated with $\mathbf{A} = [a_{ij}]$, and the vector norm $||\mathbf{x}||$ associated with the column vector $\mathbf{x}$ for which the matrix product $\mathbf{Ax}$ is defined, are said to be **compatible** if

$$||\mathbf{Ax}|| \leq ||\mathbf{A}||\,||\mathbf{x}||.$$

15.312 Induced norms

When a vector $\mathbf{z}$ with norm $||\mathbf{z}||$ exists such that the maximum is attained in the expression

$$||\mathbf{A}|| = \max_{||\mathbf{z}||=1} ||\mathbf{Az}||,$$

then $||\mathbf{A}||$ is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm $||\mathbf{z}||$. NO 428

15.313 Natural norm of unit matrix

If $\mathbf{I}$ is the unit matrix, then for any natural norm

$$||\mathbf{I}|| = 1. \qquad \text{NO 429}$$

15.41 Principal Natural Norms

The natural matrix norms induced on matrix $\mathbf{A} = [a_{ij}]$ by the 1, 2, and ∞ vector norms are as follows:

15.411 Maximum absolute column sum norm

$$||\mathbf{A}||_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \qquad \text{NO 429}$$

15.412 Spectral norm

If $\mathbf{A}^{\mathrm{H}}$ denotes the Hermitian transpose of the square matrix $\mathbf{A} = [a_{ij}]$, so that $\mathbf{A}^{\mathrm{H}} = [\overline{a_{ji}}]$ with a bar denoting the complex conjugate operation, then

$$||\mathbf{A}||_2 = \sqrt{\text{maximum eigenvalue of } \mathbf{A}^{\mathrm{H}}\mathbf{A}},$$

or, equivalently,

$$||\mathbf{A}||_2 = \max_{||x||_2 \neq 0} \frac{||\mathbf{Ax}||_2}{||\mathbf{x}||_2}. \qquad \text{NO 429}$$

15.413 Maximum absolute row sum norm

$$||\mathbf{A}||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \qquad \text{NO 429}$$

15.51 Spectral Radius of a Square Matrix

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with elements that may be complex, and with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the **spectral radius** $\rho(\mathbf{A})$ of $\mathbf{A}$ is the number

$$\rho(\mathbf{A}) = \max_{1 \le i \le n} |\lambda_i|. \qquad \text{VA 9}$$

15.511 Inequalities concerning matrix norms and the spectral radius

1. $||\mathbf{A}||_2{}^2 \le ||\mathbf{A}||_1 ||\mathbf{A}||_\infty$. NO 431
2. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, and the $n \times n$ matrix $\mathbf{U}$ is unitary, so that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, with ^{H} denoting the Hermitian transpose of $\mathbf{A}$ (see **13.123**), then
$$||\mathbf{AU}|| = ||\mathbf{UA}|| = ||\mathbf{A}||. \qquad \text{VA 15}$$
3. If $\mathbf{A}$ is any nonsingular $n \times n$ matrix with elements that may be complex with eigenvalues λ_1, λ_2, λ_n, then
$$\frac{1}{||\mathbf{A}^{-1}||} \le |\lambda| \le ||\mathbf{A}||. \qquad \text{VA 16}$$
4. For any square matrix $\mathbf{A}$ with spectral radius $\rho(\mathbf{A})$ and any natural norm $||\mathbf{A}||$,
$$\rho(\mathbf{A}) \le ||\mathbf{A}||. \qquad \text{NO 430}$$
5. If the square matrix $\mathbf{A}$ is Hermitian, then
$$\rho(\mathbf{A}) = ||\mathbf{A}||.$$
6. If the square matrix $\mathbf{A}$ is Hermitian and $P_m(x)$ is any polynomial of degree m with real coefficients, then
$$||P_m(\mathbf{A})|| = \rho(P_m(\mathbf{A})).$$
7. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, then the sequence of matrices $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots$ converges to the null matrix as $n \to \infty$ if, and only if, $\rho(\mathbf{A}) < 1$. NO 303

15.512 Deductions from Gerschgorin's theorem (see 15.814)

1. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex; then $\rho(\mathbf{A}) \le \min\left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|, \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|\right)$. VA 17

2. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex, and $x_1, x_2, \ldots, x_n$ be any set of n positive numbers; then $\rho(\mathbf{A}) \leq \min\left(\max_{1\leq i\leq n}\left(\frac{\sum_{j=1}^{n}|a_{ij}|x_j}{x_i}\right), \max_{1\leq j\leq n}\left(x_j\sum_{i=1}^{n}\frac{|a_{ij}|}{x_i}\right)\right)$.

VA 18

15.61 Inequalities Involving Eigenvalues of Matrices

The **eigenvalues** (**characteristic values** or **latent roots**) λ of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

When expanded, the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the **characteristic polynomial**, and it has the form

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of $\mathbf{A}$. In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \qquad \text{(sum of all principal minors of } |\mathbf{A}| \text{ of order } r).$$

It then follows that

$$\begin{aligned} b_{n-1} &= (-1)^n\left(a_{11} + a_{22} + \cdots + a_{nn}\right), \\ b_{n-2} &= (-1)^n\sum_{i<j}\left(a_{ii}a_{jj} - a_{ij}a_{ji}\right), \\ b_0 &= |\mathbf{A}|. \end{aligned}$$

Since the sum of the elements of the leading diagonal of $\mathbf{A}$ is called the **trace** of $\mathbf{A}$, written $\operatorname{tr}\mathbf{A}$, it follows that $b_{n-1} = (-1)^n \operatorname{tr}\mathbf{A}$. ML 198

15.611 Cayley-Hamilton theorem

Every square matrix $\mathbf{A}$ satisfies its characteristic equation, so that

$$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + c_{n-2}\mathbf{A}^{n-2} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}.$$ ML 206

15.612 Corollaries

1. If $\mathbf{A}$ is nonsingular, then its adjoint, denoted by $\operatorname{adj}\mathbf{A}$, is

$$\operatorname{adj}\mathbf{A} = -\left[(-1)^n\mathbf{A}^{n-1} + c_{n-1}\mathbf{A}^{n-2} + c_{n-2}\mathbf{A}^{n-3} + \cdots + c_2\mathbf{A} + c_1\mathbf{I}\right].$$

2. If $\mathbf{A}$ is nonsingular, then the characteristic polynomial of $\mathbf{A}^{-1}$ is

$$(-1)^n\left(\lambda^n + \frac{c_1}{|\mathbf{A}|}\lambda^{n-1} + \frac{c_2}{|\mathbf{A}|}\lambda^{n-2} + \cdots + \frac{(-1)^n}{|\mathbf{A}|}\right).$$

15.71 Inequalities for the Characteristic Polynomial

The first group of inequalities that follow, which relate to the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ whose elements may be complex, refer directly to the coefficients of the polynomial when written in the form

$$P(\lambda) \equiv |\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n,$$

and only implicitly to the coefficients a_{ij} of $\mathbf{A}$ that give rise to the b_i.

15.711 Named and unnamed inequalities

The first group of inequalities relating to the eigenvalues λ satisfying $P(\lambda) = 0$ are unnamed and are as follows:

1. All the eigenvalues λ lie within or on the circle $||z|| \leq r$, where r is the positive root of .

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0 \qquad \text{MG 122}$$

2. All the eigenvalues λ lie within the circle

$$|z| < 1 + \max_i |b_i|. \qquad \text{MG 123}$$

3. When $b_n \neq 0$ the eigenvalue λ of smallest modulus lies in the annulus $R \leq |z| \leq \dfrac{R}{2^{1/n} - 1}$, where R is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0. \qquad \text{MG 126}$$

4. All the eigenvalues λ lie on or outside the circle

$$|z| = \min_k \left[\frac{|b_n|}{(|b_n| + |b_k|)}\right]. \qquad \text{MG 126}$$

5. If the eigenvalues λ are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > 1 \geq |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|,$$

then

$$|z_1 z_2 \ldots z_p| \leq N, \qquad |z_p| \leq N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2. \qquad \text{MG 129}$$

6. All the eigenvalues λ lie in or on the circle

$$|z| \leq \sum_{j=1}^{n} |b_j|^{1/j}. \qquad \text{MG 126}$$

7. All the eigenvalues λ lie on the disk

$$\left|z + \frac{b_1}{2}\right| \leq \left|\frac{b_1}{2}\right| + |b_2|^{1/2} + |b_3|^{1/3} + \cdots + |b_n|^{1/n}. \qquad \text{MG 145}$$

8. All the eigenvalues λ lie in the annulus $m \leq ||z|| \leq M$, where

$$m^2 = \max\left\{0, \min_{1 \leq j \leq n-1}\left[1 - |b_j|, |b_n|^2\right]\right\}$$

and

$$M^2 = \max\left\{1 + |b_j|, |b_n|^2 + 2\sum_{j=1}^{n-1} |b_j|^2\right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial $P(\lambda)$. MG 145

15.712 Parodi's theorem

The eigenvalues λ satisfying $P(\lambda) = 0$ lie in the union of the disks

$$|z| \le 1, \qquad |z + b_1| \le \sum_{j=1}^{n} |b_j|. \qquad \text{MG 143}$$

15.713 Corollary of Brauer's theorem

If

$$|b_1| > 1 + \sum_{j=2}^{n} |b_j|,$$

then one and only one eigenvalue satisfying $P(\lambda) = 0$ lies on the disk

$$|z + b_1| \le \sum_{j=2}^{n} |b_j|. \qquad \text{MG 141}$$

15.714 Ballieu's theorem

For any set $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of positive numbers, let $\mu_0 = 0$ and

$$M_\mu = \max_{0 \le k \le n-1} \left[\frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}}\right].$$

Then all the eigenvalues satisfying $P(\lambda) = 0$ lie on the disk $||z|| \le M_\mu$. MG 144

15.715 Routh-Hurwitz theorem

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0$$

determining the n eigenvalues λ of the real $n \times n$ matrix $\mathbf{A}$. Then the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \qquad \ldots, \qquad \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \ldots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \ldots & b_k \end{vmatrix}. \qquad \text{GM 230}$$

15.81–15.82 Named Theorems on Eigenvalues

In the following theorems involving eigenvalue inequalities the elements a_{ij} of matrix $\mathbf{A}$ enter directly, and not in the form of the coefficients of the characteristic polynomial.

15.811 Schur's inequalities

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with elements that may be complex, and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

1. $$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2$$
2. $$\sum_{i=1}^{n} |\operatorname{Re} \lambda_i|^2 \leq \sum_{i,j=1}^{n} \left| \frac{a_{ij} + \overline{a_{ji}}}{2} \right|^2$$
3. $$\sum_{i=1}^{n} |\operatorname{Im} \lambda_i|^2 \leq \sum_{i,j=1}^{n} \left| \frac{a_{ij} - \overline{a_{ji}}}{2} \right|^2$$ ML 309

15.812 Sturmian separation theorem

Let $\mathbf{A}_r = [a_{ij}]$ with $i,\ j = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, N$ be a sequence of N symmetric matrices of increasing order. Then if $\lambda_k(\mathbf{A}_r)$ for $k = 1, 2, \ldots, r$ denotes the kth eigenvalue of A_r, where the ordering is such that

$$\lambda_1 (A_r) \geq \lambda_2 (A_r) \geq \cdots \geq \lambda_r (A_r),$$

it follows that

$$\lambda_{k+1} (A_{i+1}) \leq \lambda_k (A_i) \leq \lambda_k (A_{i+1}).$$ BL 115

15.813 Poincare's separation theorem

Let $\{\mathbf{y}^k\}$, with $k = 1, 2, \ldots, K$, be a set of orthonormal vectors so that the inner product $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Set

$$\mathbf{x} = \sum_{k=1}^{K} u_k \mathbf{y}^k,$$

so that for any square matrix $\mathbf{A}$ for which the product $\mathbf{Ax}$ is defined, the quadratic form

$$(\mathbf{x}, \mathbf{Ax}) = \sum_{k,l=1}^{K} u_k u_l \left(\mathbf{y}^k, \mathbf{Ay}^l\right).$$

Then if

$$\mathbf{b}_K = \left(\mathbf{y}^k, \mathbf{Ay}^l\right) \text{ for } k, l = 1, 2, \ldots, K,$$

it follows that

$$\begin{aligned} \lambda_i (\mathbf{b}_K) &\leq \lambda_i(\mathbf{A}) && \text{for } i = 1, 2, \ldots, K, \\ \lambda_{K-j} (\mathbf{b}_K) &\geq \lambda_{N-j}(\mathbf{A}) && \text{for } j = 0, 1, 2, \ldots, K-1. \end{aligned}$$ BL 117

15.814 Gerschgorin's theorem

Let $\mathbf{A} = [a_{ij}]$ be any arbitrary $n \times n$ matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{j=1, i \neq j}^{n} |a_{ij}| \text{ for } i = 1, 2, \ldots, n.$$

Then all of the eigenvalues λ_i of $\mathbf{A}$ lie in the union of the n disks Γ_i, where

$$\Gamma_i : \quad |z - a_{ii}| \leq \Lambda_i \text{ for } i = 1, 2, \ldots, n.$$ VA 16

15.815 Brauer's theorem

If in Gerschgorin's theorem for a given m

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all $j \neq m$, then one and only one eigenvalue of $\mathbf{A}$ lies in the disk Γ_m. MG 141

15.816 Perron's theorem

If $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ is an arbitrary set of positive numbers, then all the eigenvalues λ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ lie on the disk $|z| \leq \mathbf{M}_\mu$, where

$$\mathbf{M}_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i} |a_{ij}|.$$ MG 141

15.817 Frobenius theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients, so that $a_{ij} > 0$ for all $i, j = 1, 2, \ldots, n$, then $\mathbf{A}$ has a positive eigenvalue λ_0, and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0.$$ MG 142

15.818 Perron–Frobenius theorem

If all elements a_{ij} of an irreducible matrix $\mathbf{A}$ are nonnegative, then $R = \min M_\lambda$ is a simple eigenvalue of $\mathbf{A}$, and all the eigenvalues of $\mathbf{A}$ lie on the disk $|z| \leq R$, where, if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a set of nonnegative numbers, not all zero,

$$M_\lambda = \inf \left\{ \mu : \mu\lambda_i > \sum_{j=1}^{n} |a_{ij}| \lambda_j, 1 \leq i \leq n \right\}$$

and $R = \min M_\lambda$.

Furthermore, if $\mathbf{A}$ has exactly p eigenvalues ($p \leq n$) on the circle $|z| = R$, then the set of all its eigenvalues is invariant under rotations $2\pi/p$ about the origin. GM 69

15.819 Wielandt's theorem

If the $n \times n$ matrix $\mathbf{A}$ satisfies the conditions of the Perron–Frobenius theorem and if in the $n \times n$ matrix $\mathbf{C} = [c_{ij}]$

$$|c_{ij}| \leq a_{ij}, \qquad i, j = 1, 2, \ldots, n,$$

then any eigenvalue λ_0 of $\mathbf{C}$ satisfies the inequality $|\lambda_0| \leq R$. The equality sign holds only when there exists an $n \times n$ matrix $\mathbf{D} = [\pm\delta_{ij}]$ such that $\delta_{ii} = 1$ for all $i, \delta_{ij} = 0$ for all $i \neq j$, and

$$\mathbf{C} = (\lambda_0/R)\,\mathbf{DAD}^{-1}. \qquad \text{GM 69}$$

15.820 Ostrowski's theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients and λ_0 is the positive eigenvalue in Frobenius' theorem, then the $n-1$ eigenvalues $\lambda_j \neq \lambda_0$ satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \qquad m = \min a_{ij} \qquad \text{for} \quad i, j = 1, 2, \ldots, n. \qquad \text{MG 145}$$

15.821 First theorem due to Lyapunov

In order that all the eigenvalues of the real $n \times n$ matrix $\mathbf{A}$ have negative real parts, it is necessary and sufficient that if V is an $n \times n$ matrix, the equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{V} + \mathbf{VA} = -\mathbf{I}$$

has as a solution the matrix of coefficients $\mathbf{V}$ of some positive-definite quadratic form $(\mathbf{x}, \mathbf{Vx})$ (see **13.21**). GM 224

15.822 Second theorem due to Lyapunov

If all the eigenvalues of the real matrix $\mathbf{A}$ have negative real parts, then to an arbitrary negative-definite quadratic form $(\mathbf{x}, \mathbf{Wx})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive-definite quadratic form $(\mathbf{x}, \mathbf{Vx})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$$

then $(\mathbf{x}, \mathbf{Vx})$ and $(\mathbf{x}, \mathbf{Wx})$ satisfy

$$\frac{d}{dt}(\mathbf{x}, \mathbf{Vx}) = (\mathbf{x}, \mathbf{Wx}).$$

Conversely, if for some negative-definite form $(\mathbf{x}, \mathbf{Wx})$ there exists a positive-definite form $(\mathbf{x}, \mathbf{Vx})$ connected to $(\mathbf{x}, \mathbf{Wx})$ by the preceding two equations, then all the eigenvalues of $\mathbf{A}$ have negative real parts (see **13.21, 13.31**). GM 222

15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov

1. The off-diagonal Hermitian matrix $\mathbf{A}$ of rank n whose elements are given by

$$a_{jk} = (1 - \delta_{jk}) \left\{ 1 + i \cot \left[\frac{(j-k)\,\pi}{n} \right] \right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \text{ for } s = 1, 2, \ldots, n,$$

and the corresponding eigenvectors $v^{(s)}$ have the components

$$v_j^{(s)} = \exp\left(-\frac{2\pi isj}{n}\right) \qquad \text{for} \quad j = 1, 2, \ldots, n.$$

2. The two off-diagonal Hermitian matrices $\mathbf{B}$ and $\mathbf{C}$ whose elements are defined by the formulas

$$b_{jk} = (1 - \delta_{jk}) \sin^{-2}\left[\frac{(j-k)\pi}{n}\right],$$
$$c_{jk} = (1 - \delta_{jk}) \sin^{-4}\left[\frac{(j-k)\pi}{n}\right],$$

are related to the matrix $\mathbf{A}$ in (1) by the equations

$$\mathbf{B} = \frac{1}{2}\left(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}\right),$$
$$\mathbf{C} = -\frac{1}{6}\left(\mathbf{B}^2 - 2\left(2 + \sigma_n^{(1)}\right)\mathbf{B} - \sigma_n^{(2)}\mathbf{I}\right),$$

where $\mathbf{I}$ is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3}\left(n^2 - 1\right), \qquad \sigma_n^{(2)} = \frac{1}{45}\left(n^2 - 1\right)\left(n^2 + 11\right).$$

The eigenvalues of $\mathbf{B}$ and $\mathbf{C}$ corresponding to the eigenvector $v_j^{(s)}$ in (1) have the form

$$\lambda_s^{(b)} = \sigma_n^{(1)} - 2s(n-s) \qquad \text{for } s = 1, 2, \ldots, n,$$
$$\lambda_s^{(c)} = \sigma_n^{(2)} - 2s(n-s)\frac{s(n-s)+2}{3} \qquad \text{for } s = 1, 2, \ldots, n.$$

3. Together, the above two results imply the following diophantine summation rules:

(a) $$\sum_{k=1}^{n-1} \cot\left(\frac{k\pi}{n}\right) \sin\left(\frac{2sk\pi}{n}\right) = n - 2s \qquad \text{for } s = 1, 2, \ldots, n-1$$

(b) $$\sum_{k=1}^{n-1} \sin^{-2}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = b_s \qquad \text{for } s = 1, 2, \ldots, n-1,$$

(c) $$\sum_{k=1}^{n-1} \sin^{-4}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = c_s \qquad \text{for } s = 1, 2, \ldots, n-1,$$

(d) $$\sum_{k=1}^{n-1} \sin^{-2p}\left(\frac{k\pi}{n}\right) = \sigma_n^{(p)},$$

with $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)}\frac{2n^4 + 23n^2 + 191}{315}, \qquad \sigma_n^{(4)} = \sigma_n^{(2)}\frac{3n^4 + 10n^2 + 227}{315}$$
$$b_s = \sigma_n^{(1)} - 2s(n-s), \qquad c_s = \sigma_n^{(2)} - \frac{2}{3}s(n-s)[s(n-s)+2].$$

15.91 Variational Principles

15.911 Rayleigh quotient

If $\mathbf{A}$ is an Hermitian matrix, the Rayleigh quotient $\rho(\mathbf{x})$ is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}. \qquad \text{NO 407}$$

15.912 Basic theorems

1. If the $n \times n$ matrix A is Hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\lambda_1 \leq \rho \leq \lambda_n,$$

where ρ is the Rayleigh quotient for any $\mathbf{x} \neq \mathbf{0}$, and

$$\lambda_1 = \min_{x \neq 0} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \qquad \text{and} \qquad \lambda_n = \max_{x \neq 0} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}. \qquad \text{NO 407}$$

2. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ corresponding to the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively, and $\mathbf{x} \neq \mathbf{0}$ is such that

$$(\mathbf{x}, \mathbf{x}_1) = (\mathbf{x}, \mathbf{x}_2) = \cdots = (\mathbf{x}, \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_x \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})},$$

and

$$\lambda_j \leq \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \leq \lambda_n. \qquad \text{NO 410}$$

3. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian, then the eigenvalue

$$\lambda_r = \max\left(\min \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}\right),$$

where first the minimum over $\mathbf{x}$ is taken subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = 1, 2, \ldots, r-1$, with the $\mathbf{b}_i$ regarded as fixed vectors, and then the maximum over all possible $\mathbf{b}_i$. Also, the eigenvalue

$$\lambda_r = \min\left(\max \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}\right),$$

where now the maximum over $\mathbf{x}$ is taken first subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = r+1, r+2, \ldots, n$ for fixed $\mathbf{b}_i$, and then the minimum over all possible $\mathbf{b}_i$. NO 414

4. The $(n-1)$ eigenvalues $\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}$ obtained from the $(n-1) \times (n-1)$ matrix derived from an Hermitian matrix $\mathbf{A}$ from which the last row and column have been omitted separate the n eigenvalues of $\mathbf{A}$, so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \cdots < \lambda'_{n-1} < \lambda_n \qquad \text{(see } \mathbf{15.812}\text{)}.$$

This page intentionally left blank

16 Ordinary Differential Equations

16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-Order Equations

16.111 Solution of a first-order equation

Consider the real function $f(t,x)$ that is defined and continuous in an open set $D \subset R^2$. Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t,x)$$

in the open interval $I \subset R$ is a real function $u(t)$ that is defined and is both continuous and differentiable in I, with the property that

(i) $(t, u(t)) \in D$ for $t \in I$,

(ii) $\dfrac{du}{dt} = f(t, u(t))$ for $t \in I$.

16.112 Cauchy problem

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t,x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where $(t_0, u(t_0)) \in D$, the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^{t} f(\tau, u(\tau))\, d\tau \qquad \text{(see } \mathbf{16.316}\text{)}.$$

16.113 Approximate solution to an equation

The real function $\phi(t)$ is said to be an **approximate solution**, to within the error ϵ, of the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if ϕ' is piecewise continuous, and for a given $\epsilon > 0$ and an open interval $I \subset R$,

$$|\phi'(t) - f(t, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative. HU 3

16.114 Lipschitz continuity of a function

The real function $f(t, x)$ defined and continuous in some open set $D \subset R^2$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points (t, x_1) and (t, x_2) belonging to D

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|. \qquad \text{HU 5}$$

16.21 Fundamental Inequalities and Related Results

16.211 Gronwall's lemma

Let the three piecewise continuous, non-negative functions u, v, and w be defined in the interval $[0, a]$ and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau)\, d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau)v(\tau) \exp\left(\int_\tau^t v(\sigma)\, d\sigma\right) d\tau. \qquad \text{BB 135}$$

16.212 Comparison of approximate solutions of a differential equation

Let f be a real function that is defined in an open set $D \subset R^2$, in which it is both continuous and Lipschitz continuous. In addition, let u_1 and u_2 be two approximate solutions of

$$\frac{dx}{dt} = f(t, x)$$

in an open set $I \subset R$ in the sense already defined, with

$$|u_1'(t) - f(t, u_1(t))| \leq \epsilon_1, \quad |u_2'(t) - f(t, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \exp\{|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)[\exp\{k|t - t_0|\} - 1]. \qquad \text{HU 6}$$

16.31 First-Order Systems

16.311 Solution of a system of equations

The **system** of n first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= f_1\left(t, x_1, x_2, \ldots, x_n\right),\\ \frac{dx_2}{dt} &= f_2\left(t, x_1, x_2, \ldots, x_n\right),\\ &\vdots\\ \frac{dx_n}{dt} &= f_n\left(t, x_1, x_2, \ldots, x_n\right),\end{aligned}$$

in which the functions $f_1, f_2, \ldots, f_n$ are real and continuous in an open set $D \subset R^{n+1}$, may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(t, \mathbf{x}\right),$$

where $\mathbf{x}$ and $\mathbf{f}$ are $n \times 1$ column vectors. Its solution in the open interval $I \subset R$ is the vector $\mathbf{u}(t)$ with elements $u_1(t), u_2(t), \ldots, u_n(t)$ with the property that

(i) $(t, \mathbf{u}(t)) \in D$ for $t \in I$,

(ii) $\dfrac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t))$ for $t \in I$. HU 24

16.312 Cauchy problem for a system

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(t, \mathbf{x}\right)$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector** condition

$$\mathbf{u}\left(t_0\right) = \mathbf{x}_0,$$

where $(t_0, \mathbf{u}(t_0) \in D$, the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{u}(\tau))\, d\tau.$$

16.313 Approximate solution to a system

The real vector $\boldsymbol{\phi}(t)$ is said to be an **approximate vector solution**, to within the order ϵ, of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(t, \mathbf{x}\right),$$

if the elements of $\boldsymbol{\phi}'$ are piecewise continuous, and for a given $\epsilon > 0$ and open interval $I \subset R$,

$$||\phi'(t) - \mathbf{f}\left(t, \phi(t)\right)|| \leq \epsilon,$$

except at points of discontinuity of the derivative, where $||\mathbf{w}||$ denotes the supremum norm

$$||\mathbf{w}|| = \sup\left(|w_1|, |w_2|, \ldots, |w_n|\right).$$ HU 25

16.314 Lipschitz continuity of a vector

The real vector $\mathbf{f}(t, x)$ defined and continuous in some open set $D \subset R^n$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points $(t, \mathbf{x}_1)$, $(t, \mathbf{x}_2)$ belonging to D,

$$||\mathbf{f}\left(t, \mathbf{x}_1\right) - \mathbf{f}\left(t, \mathbf{x}_2\right)|| \leq k||\mathbf{x}_1 - \mathbf{x}_2||.$$ HU 26

16.315 Comparison of approximate solutions of a system

Let $\mathbf{f}$ be a real vector defined in an open set $D \subset R \times R^n$ in which it is both continuous and Lipschitz continuous. In addition, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

in an open set $I \subset R$ in the sense already defined, with

$$|\mathbf{u}_1'(t) - \mathbf{f}(t, \mathbf{u}_1(t))| \leq \epsilon_1, \qquad |\mathbf{u}_2'(t) - \mathbf{f}(t, \mathbf{u}_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$||\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)|| \leq \delta,$$

it follows that

$$||\mathbf{u}_1(t) - \mathbf{u}_2(t)|| \leq \delta \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)[\exp\{k|t - t_0|\} - 1]. \qquad \text{HU 27}$$

16.316 First-order linear differential equation

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t)\, dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)}\left(\mu(t_0)\, y_0 + \int_{t_0}^{t} \mu(\xi) Q(\xi)\, d\xi\right),$$

where $y_0 = y(t_0)$.

16.317 Linear systems of differential equations

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where $\mathbf{x}$ is an $n \times 1$ column vector and $\mathbf{A}(t)$ an $n \times n$ matrix. Then a **fundamental system** of solutions of this system is a set of n linearly independent solution vectors $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$, The square matrix $\mathbf{K}(t)$ whose columns comprise the vectors $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$ is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}(t_0)| \exp\left(\int_{t_0}^{t} \operatorname{tr} \mathbf{A}(\tau)\, d\tau\right).$$

Using the fundamental matrix $\mathbf{K}(t)$ defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, is

$$\phi(t) = \mathbf{K}(t)[\mathbf{K}(t_0)]^{-1}\mathbf{x}_0 + \mathbf{K}(t)\int_{t_0}^{t} [\mathbf{K}(\tau)]^{-1}\mathbf{b}(\tau)\, d\tau, \qquad \text{HU 43}$$

where $\mathbf{b}(t)$ is an $n \times 1$ column vector. CL 69

16.41 Some Special Types of Elementary Differential Equations

16.411 Variables separable

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y)\,dx + R(x)S(y)\,dy = 0.$$

It may then be written in the form

$$M(x)\,dx - \frac{1}{N(y)}\,dy = 0,$$

or

$$\frac{P(x)}{R(x)}\,dx + \frac{S(y)}{Q(y)}\,dy = 0,$$

provided $R(x)Q(y) \neq 0$.

16.412 Exact differential equations

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **exact** if there exists a function $h(x,y)$ such that

$$d\,[h(x,y)] = M(x,y)\,dx + N(x,y)\,dy. \qquad \text{IN 16}$$

16.413 Conditions for an exact equation

A necessary and sufficient condition that an equation of this form is exact is that the functions $M(x,y)$ and $N(x,y)$ together with their partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad \text{IN 16}$$

16.414 Homogeneous differential equations

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary k,

$$\frac{M(kx,ky)}{N(kx,ky)} = \frac{M(x,y)}{N(x,y)}.$$

Setting $y = sx$, it may then be expressed in the form

$$[M(1,s) + sN(1,s)]\,dx + xN(1)\,dx = 0,$$

in which the variables s and x are separable. IN 18

16.51 Second-Order Equations

16.511 Adjoint and self-adjoint equations

The linear second-order differential equation

$$L(u) \equiv a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2}\,[a(x)v] - \frac{d}{dx}\,[b(x)v] + c(x)v = 0.$$

The equation $L(u) = 0$ is said to be **self-adjoint** if $L(u) \equiv M(u)$.

A linear self-adjoint second-order differential equation defined on $[\alpha, \beta]$ can always be expressed in the form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

where $p(x)$ and $q(x)$ are continuous on $[\alpha, \beta]$ and $p(x) > 0$. The general equation $L(u) = 0$ can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right],$$

when

$$p(x) = \exp\int\frac{b(x)}{a(x)}\,dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right].$$

In general, if

$$L(u) = p_0\frac{d^nu}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}}\ldots + p_{n-1}\frac{du}{dx} + p_nu,$$

then its adjoint is

$$M(v) = (-1)^n\frac{d^n}{dx^n}\,[p_0v] + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\,[p_1v] + \ldots - \frac{d}{dx}\,[p_{n-1}v] + p_nv.$$ HI 391

16.512 Abel's identity

If $p(x)$ and $q(x)$ are continuous in $[\alpha, \beta]$ in which $p(x) > 0$, and $u(x)$ and $v(x)$ are suitably differentiable with

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

then the result

$$p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) \equiv \text{ const.}$$

is known as **Abel's identity**.

More generally, if we consider the linear n^{th}-order equation

$$p_0\frac{d^nu}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}} + \ldots + p_{n-1}\frac{du}{dx} + p_n = 0,$$

and Δ is the Wronskian of a (fundamental) set of linearly independent solutions $u_1, u_2, \ldots, u_n$, the Abel identity takes the form

$$\Delta = \Delta_0\exp\left(-\int_{x_0}^{x}\frac{p_1(x)}{p_0(x)}\,dx\right),$$

where Δ_0 is the value of Δ at $x = x_0$. IN 119

16.513 Lagrange identity

If the linear n^{th}-order equation $L(u) = 0$ is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx} \{P(u, v)\},$$

where $M(v)$ is the adjoint of $L(u)$, is called the **Lagrange identity**. The expression $P(u, v)$, which is linear and homogeneous in

$$u, \frac{du}{dx}, \ldots, \frac{d^{n-1} u}{dx^{n-1}} \quad \text{and} \quad v, \frac{dv}{dx}, \ldots, \frac{d^{n-1} v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**. In the case of the second-order equation

$$L(u) = a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x) u = 0,$$

with adjoint $M(v)$, the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx} \left(a(x) v \frac{du}{dx} - \frac{d}{dx}(a(x) v) u + b(x) uv \right).$$ IN 124

16.514 The Riccati equation

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x) z + b(x) z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{\left(p(x) \frac{du}{dx}\right)}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x) u = 0.$$

The further substitution $v = u \left(\exp \int_\alpha^x a(x)\, dx \right)$ in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x) v^2 + s(x) = 0,$$

with

$$r(x) = b(x) \exp \left(-\int_\alpha^x a(x)\, dx \right) \quad \text{and} \quad s(x) = c(x) \exp \left(\int_\alpha^x a(x)\, dx \right).$$ HI 273

16.515 Solutions of the Riccati equation

If in the Riccati equation

$$\frac{dv}{dx} + r(x) v^2 + s(x) = 0,$$

$r(x) \neq 0$, while $r(x)$ and $s(x)$ are continuous on the interval $[\alpha, \beta]$, then every solution $v(x)$ may be expressed in the form

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with A, B arbitrary constants, not both zero, and the prime denoting differentiation, while u and v are linearly independent solutions of

$$\frac{d}{dx}\left(\frac{1}{r(x)}\frac{dz}{dx}\right) + s(x)z = 0.$$

Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions of this last equation and A and B are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)}\frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever $Au(x) = Bv(x) \neq 0$. IN 24

16.516 Solution of a second-order linear differential equation

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} = \phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t),$$

and the solution to the inhomogeneous equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t),$$

subject to the initial conditions $x(t_0) = x_0$ and $x'(t_0) = x_1$ may be written

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^{t} \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi)\, d\xi,$$

where the constants c_1 and c_2 are chosen such that $x(t)$ satisfies the initial conditions.

The linear combination $c_1\phi_1(t) + c_2\phi_2(t)$ is known as the **complementary function** where c_1 and c_2 are arbitrary constants.

16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations

Equations whose solutions possess an infinite number of zeros in the interval $(0, \infty)$ are said to have **oscillatory** solutions. The following theorems relate to such properties:

16.611 First basic comparison theorem

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if $\psi(x) \geq \phi(x)$ and some solutions v are non-oscillatory, then so also must some solutions u be non-oscillatory. BS 119

16.622 Second basic comparison theorem

If all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0$$

are oscillatory as $x \to \infty$, and if

$$q_2(x) \geq q_1(x),$$
$$p_2(x) \geq p_1(x) > 0,$$

then all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0$$

are oscillatory. BS 120

16.623 Interlacing of zeros

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and suppose that $y_1(x)$ has at least two zeros in the interval (a, b). Then if x_1 and x_2 are two consecutive zeros of $y_1(x)$, the function $y_2(x)$ has one, and only one, zero in the interval (x_1, x_2). HI 374

16.624 Sturm separation theorem

Let $u(x)$ and $v(x)$ be two linearly independent solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) = 0,$$

in which $p(x) > 0$ and $p(x), q(x)$ are continuous on $[a, b]$. Then, between any two consecutive zeros of $u(x)$ there will be one, and only one, zero of $v(x)$. IN 224

16.625 Sturm comparison theorem

Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions in the differential equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0,$$

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0.$$

Then between any two zeros of a non-trivial solution $u(x)$ of the first equation there will be at least one zero of every non-trivial solution $v(x)$ of the second equation. IN 228

16.626 Szegö's comparison theorem

Suppose, under the conditions of the Sturm comparison theorem, that $p_1(x) \equiv p_2(x), q_1(x) \not\equiv q_2(x)$, and $u(x) > 0, v(x) > 0$ for $a < x < b$, together with

$$\lim_{x \to a} p_1(x)\left(\frac{du}{dx}v - \frac{dv}{dx}u\right) = 0.$$

Then, if $u(b) = 0$, there is a point ξ in (a, b) such that $v(\xi) = 0$. HI 379

16.627 Picone's identity

Consider the equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right)+q_1(x)u=0,$$
$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right)+q_2(x)v=0,$$

with p_1, p_2, q_1, and q_2 positive and continuous for $a<x<b$, where $q_2(x)>q_1(x)$ and $p_1(x)>p_2(x)$. Then with $a<\alpha<\beta<b$, Picone's identity is

$$\left(\frac{u}{v}\left(p_1\frac{du}{dx}v-p_2\frac{dv}{dx}u\right)\right)_\alpha^\beta=\int_\alpha^\beta (q_2-q_1)\,u^2\,ds+\int_\alpha^\beta (p_1-p_2)\left(\frac{du}{ds}\right)^2 ds+\int_\alpha^\beta \frac{p_2}{v^2}\left(v\frac{du}{ds}-u\frac{dv}{ds}\right)^2 ds.$$

IN 226

16.628 Sturm-Picone theorem

Consider the self-adjoint equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right)+q_1(x)u=0$$

and

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right)+q_2(x)v=0.$$

Let p_1, p_2, q_1, and q_2 be positive and continuous for $a<x<b$, where $q_2(x)>q_1(x)$ and $p_1(x)>p_2(x)$. Then, if x_1 and x_2 is a pair of consecutive zeros of $u(x)$ in (a,b), $v(x)$ has at least one zero in the open interval (a,b). IN 225

16.629 Oscillation on the half line

Consider the self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=0.$$

We then have the following results:

(i) Let $p(x)>0$ and p, q be continuous on $[0,\infty)$. If the two improper integrals

$$\int_1^\infty \frac{dx}{p(x)} \quad \text{and} \quad \int_1^\infty q(x)\,dx$$

diverge, then every solution $u(x)$ has infinitely many zeros on the interval $[1,\infty)$. Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)}=+\infty \quad \text{and} \quad \int_0^1 q(x)\,dx=+\infty,$$

then every solution $u(x)$ has infinitely many zeros on the interval (0, 1).

(ii) (Moore's theorem). Every non-trivial solution $u(x)$ has at most a finite number of zeros on the interval $[a,\infty)$ if the improper integral

$$\int_a^\infty \frac{dx}{p(x)}$$

converges, and if

$$\left|\int_a^x q(s)\,ds\right| < M \quad \text{for} \quad a \le x < \infty$$

with $M > 0$ a finite constant.

16.71 Two Related Comparison Theorems

16.711 Theorem 1

Consider the equations in the Sturm comparison theorem with the same assumptions on $p(x)$ and $q(x)$, and let $u(x), v(x)$ be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if $u(x)$ is increasing in $[x_1, x_2]$ and reaches a maximum at x_2, the function $v(x)$ reaches a maximum at some point x_3 such that $x_1 < x_3 < x_2$. HI 376

16.712 Theorem 2

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

in which $F(x)$ is continuous in (a, b) and such that

$$0 < m \le F(x) \le M.$$

Then, if the solution $y(x)$ has two successive zeros x_1, x_2, it follows that

$$\pi M^{-1/2} \le x_2 - x_1 \le \pi m^{-1/2}.$$

16.81–16.82 Non-Oscillatory Solutions

The real solution $y(x)$ of

$$\frac{d^2y}{dx^2} + F(x)y = 0$$

is said to be **non-oscillatory** in the wide sense in $(0, \infty)$ if there exists a finite number c such that the solution has no zeros in $[c, \infty)$. HI 376

16.811 Kneser's non-oscillation theorem

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup \left[x^2 F(x)\right] = \gamma^*,$$

$$\liminf \left[x^2 F(x)\right] = \gamma_*.$$

Then the solution $y(x)$ is non-oscillatory if $\gamma^* < \frac{1}{4}$, oscillatory if $\frac{1}{4} < \gamma_*$ and no conclusion can be drawn if either γ^* or γ_* equals $\frac{1}{4}$. HI 461

16.822 Comparison theorem for non-oscillation

Consider the differential equations

$$\frac{d^2y}{dx^2} + F(x)y = 0, \quad f(x) = x\int_x^\infty F(s)\,ds,$$

$$\frac{d^2y}{dx^2} + G(x)y = 0, \quad g(x) = x\int_x^\infty G(s)\,ds,$$

where $0 < g(x) < f(x)$. Then if the first equation is non-oscillatory in the wide sense, so also is the second. HI 460

16.823 Necessary and sufficient conditions for non-oscillation

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x\to\infty} \sup \left(x\int_x^\infty F(s)\,ds \right) = F^*,$$

$$\lim_{x\to\infty} \inf \left(x\int_x^\infty F(s)\,ds \right) = F_*,$$

it follows that:

(i) a necessary condition that the solution $y(x)$ be non-oscillatory is that $F_* \le \frac{1}{4}$ and $F^* \le 1$;

(ii) a sufficient condition that the solution $y(x)$ be non-oscillatory is that $F^* < \frac{1}{4}$.

16.91 Some Growth Estimates for Solutions of Second-Order Equations

16.911 Strictly increasing and decreasing solutions

Suppose that $G(x) > 0$ be continuous in $(-\infty, \infty)$ and such that $xG(x) \notin L(0,\infty)$. Then the equation $\dfrac{d^2y}{dx^2} - G(x)y = 0$ has one, and only one, solution $y_+(x)$ passing through the point $(0,1)$, which is positive and strictly monotonic decreasing for all x, and one and only one solution $y_-(x)$ through the point $(0,1)$, which is positive and strictly increasing for all x. The solution $y_+(x)$ has the property that

$$[G(x)]^{1/2}\, y_+(x) \in L_2(0,\infty) \text{ and } \frac{dy_+(x)}{dx} \in L_2(0,\infty).$$

If, in addition, $0 < \alpha^2 \le G(x) \le \beta^2 < \infty$, then

$$e^{-\beta x} \le y_+(x) \le e^{-\alpha x} \quad \text{for} \quad x > 0.$$

HI 359

16.912 General result on dominant and subdominant solutions

Consider the equations

$$\frac{d^2y}{dx^2} - g(x)y = 0, \quad \frac{d^2Y}{dx^2} - G(x)Y = 0,$$

where g and G are continuous on $(0,\infty)$ with $0 < g(x) < G(x)$, and $xg(x) \notin L(0,\infty)$. In addition, let y_α and Y_α be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \qquad y'_\alpha(0) = Y'_\alpha(0) = \alpha \text{ for } -\infty < \alpha < \infty.$$

Let y_ω and Y_ω be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \qquad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let y_+ and Y_+ be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while $\left[y'_+(x)\right]^2, g(x)\left[y_+(x)\right]^2, \left[Y'_+(x)\right]^2$, and $G(x)\left[Y'_+(x)\right]^2$ belong to $L(0,\infty)$. Then, if β and γ are such that $y_{-\beta} = y_+$ and $Y_{-\gamma} = Y_+$, it follows that $\beta < \gamma$ and

$$\begin{aligned} &y_\alpha(x) < Y_\alpha(x), \qquad 0 < x < \infty, \quad -\gamma \le \alpha, \\ &y_\omega(x) < Y_\omega(x), \\ &y_+(x) > Y_+(x). \end{aligned}$$

HI 440

16.913 Estimate of dominant solution

Let $G(x)$ be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G'(x) < \tfrac{5}{4}\left[G'(x)\right]^2.$$

Then there exists a **dominant solution** $y(x)$ of the fundamental solutions $Y_0(x)$ and $Y_1(x)$ of

$$\frac{d^2y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$\begin{aligned} 2Y_0(0) &= 0, \qquad Y_1(0) = 1, \\ Y'_0(0) &= 1, \qquad Y'_1(0) = 0, \end{aligned}$$

such that

$$y(x) < [G(x)]^{-1/4} \exp\left(\int_0^x [G(\xi)]^{1/2}\, d\xi\right),$$

and a positive constant C such that the normalized subdominant solution $y_+(x)$, for which $y_+(0) = 1$ and $\left[y'_+(x)\right]^2 \in L(0,\infty)$, $G(x)\left[y_+(x)\right]^2 \in L(0,\infty)$, satisfies

$$y_+(x) > CG(x)^{-1/4} \exp\left(-\int_0^x [G(\xi)]^{1/2}\, d\xi\right).$$

HI 443

16.914 A theorem due to Lyapunov

Let $y(x)$ be any solution of

$$\frac{d^2y}{dx^2} - G(x)y = 0$$

with $G(x)$ positive and continuous in $(0,\infty)$ with $xG(x) \in L(0,\infty)$. Then

$$\begin{aligned} \exp\left(-\int_0^x [G(\xi)+1]\, d\xi\right) &< [y(x)]^2 + [y'(x)]^2 \\ &< C\exp\left(\int_0^x [G(\xi)+1]\, d\xi\right), \end{aligned}$$

HI 446

where $C = [y(0)]^2 + [y'(0)]^2$.

16.92 Boundedness Theorems

16.921[6] All solutions of the equation

$$\frac{d^2u}{dx^2} + (1 + \phi(x) + \psi(x))\, u = 0$$

are bounded, provided that

(i) $\int^\infty |\phi(x)|\, dx < \infty$,

(ii) $\displaystyle\int^\infty |\psi(x)|\, dx < \infty$ and $\psi(x) \to 0$ as $x \to \infty$. BS 112

16.922 If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x) + b(x))u = 0$$

are also bounded if

$$\int^\infty |b(x)|\, dx < \infty.$$

BS 112

16.923 If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded as $x \to \infty$. BS 113

16.924 Consider the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

in which

$$\int^\infty x|a(x)|\, dx < \infty.$$

Then $\displaystyle\lim_{x\to\infty}\left(\frac{du}{dx}\right)$ exists, and the general solution is asymptotic to $d_0 + d_1x$ as $x \to \infty$, where d_0 and d_1 may be zero, but not simultaneously. BS 114

16.93[10] Growth of maxima of $|y|$

Sonin's theorem generalized by Pólya may be stated as follows: *Let y(x) satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

where $k(x) > 0, \phi(x) > 0$, and both functions $k(x), \phi(x)$ have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\phi(x)$ is decreasing or increasing. SZ 164

17 Fourier, Laplace, and Mellin Transforms

17.1–17.4 Integral Transforms

17.11 Laplace transform

The **Laplace transform** of the function $f(x)$, denoted by $F(s)$, is defined by the integral

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx, \qquad \operatorname{Re} s > 0.$$

The functions $f(x)$ and $F(s)$ are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^\infty |f(x)|e^{-c|x|}\,dx$$

is finite, then the Laplace transform exists when $s = \sigma + i\tau$ is such that $\sigma \geq c$.

Setting

$$F(s) = \mathcal{L}\left[f(x); s\right]$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}\left[F(s); x\right].$$

The inversion of the Laplace transform is accomplished for analytic functions $F(s)$ of order $O\left(s^{-k}\right)$ with $k > 1$ by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{sx}\,ds,$$

where γ is a real constant that exceeds the real part of all the singularities of $F(s)$. SN 30

17.12 Basic properties of the Laplace transform

1.[8] For a and b arbitrary constants,

$$\mathcal{L}\left[af(x) + bg(x)\right] = aF(s) + bG(s) \qquad \text{(linearity)}$$

2. If $n > 0$ is an integer and $\lim_{x\to\infty} f(x)e^{-sx} = 0$, then for $x > 0$,

$$\mathcal{L}\left[f^{(n)}(x); s\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0) \quad \text{(transform of a derivative)}$$

SN 32

3.[11] If $\lim_{x\to\infty}\left(e^{-sx}\int_0^x f(\zeta)\,d\zeta\right)=0$, then

$$\mathcal{L}\left[\int_0^x f(\xi)\,d\xi; s\right]=\frac{1}{s}F(s) \qquad \text{(transform of an integral)} \qquad \text{SN 37}$$

4. $\mathcal{L}\left[e^{-ax}f(x); s\right]=F(s+a)$ (shift theorem) SU 143

5. The **Laplace convolution** $f*g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f*g(x)=\int_0^x f(x-\xi)g(\xi)\,d\xi,$$

and it has the property that $f*g=g*f$ and $f*(g*h)=(f*g)*h$. In terms of the convolution operation

$$\mathcal{L}\left[f*g(x); s\right]=F(s)G(s) \qquad \text{(convolution (Faltung) theorem).} \qquad \text{SN 30}$$

17.13 Table of Laplace transform pairs

	$f(x)$	$F(s)$	
1	1	$1/s$	
2	$x^n, \quad n=0,1,2,\ldots$	$\dfrac{n!}{s^{n+1}}, \quad \operatorname{Re} s>0$	ET I 133(3)
3	$x^\nu, \quad \nu>-1$	$\dfrac{\Gamma(\nu+1)}{s^{\nu+1}}, \quad \operatorname{Re} s>0$	ET I 137(1)
4	$x^{n-\frac{1}{2}}$	$\dfrac{\Gamma\left(n+\frac{1}{2}\right)}{s^{n+\frac{1}{2}}}, \quad \operatorname{Re} s>0$	ET I 135(17)
5	$x^{-1/2}(x+a)^{-1}, \quad \lvert\arg a\rvert<\pi$	$\pi a^{-1/2}e^{as}\operatorname{erfc}\left(a^{1/2}s^{1/2}\right), \quad \operatorname{Re} s\geq 0$	ET I 136(25)
6	$\begin{cases} x & \text{for } 0<x<1 \\ 1 & \text{for } x>1 \end{cases}$	$\dfrac{1-e^{-s}}{s^2}, \quad \operatorname{Re} s>0$	ET I 142(14)
7	e^{-ax}	$\dfrac{1}{s+a}, \quad \operatorname{Re} s>-\operatorname{Re} a$	ET I 143(1)
8	xe^{-ax}	$\dfrac{1}{(s+a)^2}, \quad \operatorname{Re} s>-\operatorname{Re} a$	ET I 144(2)
9a	$\dfrac{e^{-ax}-e^{-bx}}{b-a}$	$(s+a)^{-1}(s+b)^{-1}, \quad \operatorname{Re} s>\{-\operatorname{Re} a,-\operatorname{Re} b\}$	AS 1022(29.3.12)

continued on next page

continued from previous page

	$f(x)$	$F(s)$
9b[11]	$\dfrac{\alpha e^{-ax}+\beta e^{-bx}+\gamma e^{-cx}}{(a-b)(b-c)(c-a)}$ a,b,c distinct , $\alpha=c-b$, $\beta=a-c$, $\gamma=b-a$	$(s+a)^{-1}(s+b)^{-1}(s+c)^{-1}$, $\operatorname{Re} s>\{-\operatorname{Re} a,-\operatorname{Re} b,-\operatorname{Re} c\}$
10[11]	$\dfrac{ae^{-ax}-be^{-bx}}{b-a}$	$s(s+a)^{-1}(s+b)^{-1}$, $\operatorname{Re} s>\{-\operatorname{Re} a,-\operatorname{Re} b\}$ AS 1022(29.3.13)
11	$\dfrac{e^{ax}-1}{a}$	$s^{-1}(s-a)^{-1}$, $\operatorname{Re} s>\operatorname{Re} a$
12	$\dfrac{e^{ax}-ax-1}{a^2}$	$s^{-2}(s-a)^{-1}$, $\operatorname{Re} s>\operatorname{Re} a$
13	$\dfrac{\left(e^{ax}-\frac{1}{2}a^2x^2-ax-1\right)}{a^3}$	$s^{-3}(s-a)^{-1}$, $\operatorname{Re} s>\operatorname{Re} a$
14	$(1+ax)e^{ax}$	$\dfrac{s}{(s-a)^2}$, $\operatorname{Re} s>\operatorname{Re} a$
15	$\dfrac{1+(ax-1)e^{ax}}{a^2}$	$s^{-1}(s-a)^{-2}$, $\operatorname{Re} s>\operatorname{Re} a$
16	$\dfrac{2+ax+(ax-2)e^{ax}}{a^3}$	$s^{-2}(s-a)^{-2}$, $\operatorname{Re} s>\operatorname{Re} a$
17	x^ne^{ax}, $n=0,1,2,\ldots$	$n!(s-a)^{-(n+1)}$, $\operatorname{Re} s>\operatorname{Re} a$
18	$\left(x+\frac{1}{2}ax^2\right)e^{ax}$	$\dfrac{s}{(s-a)^3}$, $\operatorname{Re} s>\operatorname{Re} a$
19	$\left(1+2ax+\frac{1}{2}a^2x^2\right)e^{ax}$	$\dfrac{s^2}{(s-a)^3}$, $\operatorname{Re} s>\operatorname{Re} a$
20	$\frac{1}{6}x^3e^{ax}$	$(s-a)^{-4}$, $\operatorname{Re} s>\operatorname{Re} a$
21	$\left(\frac{1}{2}x^2+\frac{1}{6}ax^3\right)e^{ax}$	$\dfrac{s}{(s-a)^4}$, $\operatorname{Re} s>\operatorname{Re} a$
22	$\left(x+ax^2+\frac{1}{6}a^2x^3\right)e^{ax}$	$s^2(s-a)^{-4}$, $\operatorname{Re} s>\operatorname{Re} a$

continued on next page

continued from previous page

	$f(x)$	$F(s)$	
23	$\left(1+3ax+\frac{3}{2}a^2x^2+\frac{1}{6}a^3x^3\right)e^{ax}$	$s^3(s-a)^{-4}$, $\operatorname{Re} s > \operatorname{Re} a$	
24	$\dfrac{ae^{ax}-be^{bx}}{a-b}$	$s(s-a)^{-1}(s-b)^{-1}$, $\operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$	
25	$\dfrac{\left(\frac{1}{a}e^{ax}-\frac{1}{b}e^{bx}+\frac{1}{b}-\frac{1}{a}\right)}{a-b}$	$s^{-1}(s-a)^{-1}(s-b)^{-1}$, $\operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$	
26	$x^{\nu-1}e^{-ax}$, $\operatorname{Re}\nu > 0$	$\Gamma(\nu)(s+a)^{-\nu}$, $\operatorname{Re} s > -\operatorname{Re} a$	ET I 144(3)
27	$xe^{-x^2/(4a)}$, $\operatorname{Re} a > 0$	$2a - 2\pi^{1/2}a^{3/2}se^{as^2}\operatorname{erfc}\left(sa^{1/2}\right)$	ET I 146(22)
28	$\exp\left(-ae^x\right)$, $\operatorname{Re} a > 0$	$a^s\,\Gamma\left(-s, a\right)$	ET I 147(37)
29[8]	$x^{1/2}e^{-a/(4x)}$, $\operatorname{Re} a \geq 0$	$\frac{1}{2}\pi^{1/2}s^{-3/2}\left(1+a^{1/2}s^{1/2}\right)\exp\left[(-as)^{1/2}\right]$, $\operatorname{Re} s > 0$	ET I 146(26)
30[8]	$x^{-1/2}e^{-a/(4x)}$, $\operatorname{Re} a \geq 0$	$\pi^{1/2}s^{-1/2}\exp\left[(-as)^{1/2}\right]$, $\operatorname{Re} s > 0$	ET I 146(27)
31[8]	$x^{-3/2}e^{-a/(4x)}$, $\operatorname{Re} a > 0$	$2\pi^{1/2}a^{-1/2}\exp\left[(-as)^{1/2}\right]$, $\operatorname{Re} s \geq 0$	ET I 146(28)
32	$\sin(ax)$	$a\left(s^2+a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$	ET I 150(1)
33	$\cos(ax)$	$s\left(s^2+a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$	ET I 154(3)
34	$\lvert\sin(ax)\rvert$, $a > 0$	$a\left(s^2+a^2\right)^{-1}\coth\left(\dfrac{\pi s}{2a}\right)$, $\operatorname{Re} s > 0$	ET I 150(2)

continued on next page

continued from previous page

	$f(x)$	$F(s)$
35[11]	$\lvert\cos(ax)\rvert, \quad a > 0$	$\left(s^2+a^2\right)^{-1}\left[s + a\,\mathrm{cosech}\left(\frac{\pi s}{2a}\right)\right],$ $\mathrm{Re}\, s > 0$ ET I 155(44)
36	$\frac{1-\cos(ax)}{a^2}$	$s^{-1}\left(s^2+a^2\right)^{-1},$ $\mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ AS 1022(29.3.19)
37	$\frac{ax-\sin(ax)}{a^3}$	$s^{-2}\left(s^2+a^2\right)^{-1},$ $\mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ AS 1022(29.3.20)
38	$\frac{\sin(ax)-ax\cos(ax)}{2a^3}$	$\left(s^2+a^2\right)^{-2},$ $\mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ AS 1022(29.3.21)
39	$\frac{x\sin(ax)}{2a}$	$s\left(s^2+a^2\right)^{-2}, \quad \mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ ET I 152(14)
40	$\frac{\sin(ax)+ax\cos(ax)}{2a}$	$s^2\left(s^2+a^2\right)^{-2},$ $\mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ AS 1023(29.3.23)
41	$x\cos(ax)$	$\left(s^2-a^2\right)\left(s^2+a^2\right)^{-2},$ $\mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$ ET I 157(57)
42	$\frac{\cos(ax)-\cos(bx)}{b^2-a^2}$	$s\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1},$ $\mathrm{Re}\, s > \{\lvert\mathrm{Im}\, a\rvert, \lvert\mathrm{Im}\, b\rvert\}$ AS 1023(29.3.25)
43	$\frac{\left[\frac{1}{2}a^2x^2-1+\cos(ax)\right]}{a^4}$	$s^{-3}\left(s^2+a^2\right)^{-1}, \quad \mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$
44	$\frac{\left[1-\cos(ax)-\frac{1}{2}ax\sin(ax)\right]}{a^4}$	$s^{-1}\left(s^2+a^2\right)^{-2}, \quad \mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$
45	$\frac{\left[\frac{1}{b}\sin(bx)-\frac{1}{a}\sin(ax)\right]}{a^2-b^2}$	$\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1},$ $\mathrm{Re}\, s > \{\lvert\mathrm{Im}\, a\rvert, \lvert\mathrm{Im}\, b\rvert\}$
46[11]	$\frac{\left[1-\cos(ax)+\frac{1}{2}ax\sin(ax)\right]}{a^2}$	$s^{-1}\left(s^2+a^2\right)^{-2}\left(2s^2+a^2\right), \quad \mathrm{Re}\, s > \lvert\mathrm{Im}\, a\rvert$

continued on next page

continued from previous page

	$f(x)$	$F(s)$
47	$\dfrac{a\sin(ax)-b\sin(bx)}{a^2-b^2}$	$s^2\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Im} a\rvert,\lvert\operatorname{Im} b\rvert\}$
48	$\sin(a+bx)$	$(s\sin a+b\cos a)\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\lvert\operatorname{Im} b\rvert$
49	$\cos(a+bx)$	$(s\cos a-b\sin a)\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\lvert\operatorname{Im} b\rvert$
50	$\dfrac{\left[\frac{1}{a}\sinh(ax)-\frac{1}{b}\sin(bx)\right]}{a^2+b^2}$	$\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Im} b\rvert\}$
51	$\dfrac{\cosh(ax)-\cos(bx)}{a^2+b^2}$	$s\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Im} b\rvert\}$
52	$\dfrac{a\sinh(ax)+b\sin(bx)}{a^2+b^2}$	$s^2\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Im} b\rvert\}$
53	$\sin(ax)\sin(bx)$	$2abs\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Im} a\rvert,\lvert\operatorname{Im} b\rvert\}$
54	$\cos(ax)\cos(bx)$	$s\left(s^2+a^2+b^2\right)\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Im} a\rvert,\lvert\operatorname{Im} b\rvert\}$
55	$\sin(ax)\cos(bx)$	$a\left(s^2+a^2-b^2\right)\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re} s>\{\lvert\operatorname{Im} a\rvert,\lvert\operatorname{Im} b\rvert\}$
56	$\sin^2(ax)$	$2a^2s^{-1}\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re} s>\lvert\operatorname{Im} a\rvert$
57	$\cos^2(ax)$	$\left(s^2+2a^2\right)s^{-1}\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re} s>\lvert\operatorname{Im} a\rvert$
58	$\sin(ax)\cos(ax)$	$a\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re} s>\lvert\operatorname{Im} a\rvert$
59	$e^{-ax}\sin(bx)$	$b\left[(s+a)^2+b^2\right]^{-1}$, $\operatorname{Re} s>\{-\operatorname{Re} a,\quad\lvert\operatorname{Im} b\rvert\}$

continued on next page

continued from previous page

	$f(x)$	$F(s)$
60	$e^{-ax}\cos(bx)$	$(s+a)\left[(s+a)^2+b^2\right]^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, \quad \lvert\operatorname{Im} b\rvert\}$
61	$x^{-1}\sin(ax)$	$\arctan(a/s)$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 152(16)
62	$x^{-1}\left[1-\cos(ax)\right]$	$\frac{1}{2}\ln\left(1+a^2/s^2\right)$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 157(59)
63	$\sinh(ax)$	$a\left(s^2-a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 162(1)
64	$\cosh(ax)$	$s\left(s^2-a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 162(2)
65	$x^{\nu-1}\sinh(ax)$, $\operatorname{Re}\nu > -1$	$\frac{1}{2}\Gamma(\nu)\left[(s-a)^{-\nu}-(s+a)^{-\nu}\right]$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 164(18)
66	$x^{\nu-1}\cosh(ax)$, $\operatorname{Re}\nu > 0$	$\frac{1}{2}\Gamma(\nu)\left[(s-a)^{-\nu}+(s+a)^{-\nu}\right]$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 164(19)
67	$x\sinh(ax)$	$2as\left(s^2-a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
68	$x\cosh(ax)$	$\left(s^2+a^2\right)\left(s^2-a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
69	$\sinh(ax)-\sin(ax)$	$2a^3\left(s^4-a^4\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Re} a\rvert, \lvert\operatorname{Im} a\rvert\}$ AS 1023(29.3.31)
70	$\cosh(ax)-\cos(ax)$	$2a^2s\left(s^4-a^4\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Re} a\rvert, \lvert\operatorname{Im} a\rvert\}$ AS 1023(29.3.32)
71	$\sinh(ax)+ax\cosh(ax)$	$2as^2\left(a^2-s^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
72	$ax\cosh(ax)-\sinh(ax)$	$2a^3\left(a^2-s^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$

continued on next page

continued from previous page

	$f(x)$	$F(s)$
73	$x\sinh(ax)-\cosh(ax)$	$s\left(a^2+2a-s^2\right)\left(a^2-s^2\right)^{-2}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} a\rvert$
74	$\dfrac{\left[\frac{1}{a}\sinh(ax)-\frac{1}{b}\sinh(bx)\right]}{a^2-b^2}$	$\left(a^2-s^2\right)^{-1}\left(b^2-s^2\right)^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
75	$\dfrac{\cosh(ax)-\cosh(bx)}{a^2-b^2}$	$s\left(s^2-a^2\right)^{-1}\left(s^2-b^2\right)^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
76	$\dfrac{a\sinh(ax)-b\sinh(bx)}{a^2-b^2}$	$s^2\left(s^2-a^2\right)^{-1}\left(s^2-b^2\right)^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
77	$\sinh(a+bx)$	$(b\cosh a+s\sinh a)\left(s^2-b^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} b\rvert$
78	$\cosh(a+bx)$	$(s\cosh a+b\sinh a)\left(s^2-b^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} b\rvert$
79	$\sinh(ax)\sinh(bx)$	$2abs\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
80[8]	$\cosh(ax)\cosh(bx)$	$s\left(s^2-a^2-b^2\right)\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
81	$\sinh(ax)\cosh(bx)$	$a\left(s^2-a^2+b^2\right)\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}, \qquad \operatorname{Re} s>\{\lvert\operatorname{Re} a\rvert,\lvert\operatorname{Re} b\rvert\}$
82	$\sinh^2(ax)$	$2a^2s^{-1}\left(s^2-4a^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} a\rvert$
83	$\cosh^2(ax)$	$\left(s^2-2a^2\right)s^{-1}\left(s^2-4a^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} a\rvert$
84	$\sinh(ax)\cosh(ax)$	$a\left(s^2-4a^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} a\rvert$
85	$\dfrac{\cosh(ax)-1}{a^2}$	$s^{-1}\left(s^2-a^2\right)^{-1}, \qquad \operatorname{Re} s>\lvert\operatorname{Re} a\rvert$

continued on next page

continued from previous page

	$f(x)$	$F(s)$		
86	$\dfrac{\sinh(ax) - ax}{a^3}$	$s^{-2}\left(s^2 - a^2\right)^{-1},$	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
87	$\dfrac{\left[\cosh(ax) - \frac{1}{2}a^2x^2 - 1\right]}{a^4}$	$s^{-3}\left(s^2 - a^2\right)^{-1},$	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
88	$\dfrac{\left[1 - \cosh(ax) + \frac{1}{2}ax\sinh(ax)\right]}{a^4}$	$s^{-1}\left(s^2 - a^2\right)^{-2},$	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
89	$x^{1/2}\sinh(ax)$	$\left(\pi^{1/2}/4\right)\left[(s-a)^{3/2} - (s+a)^{3/2}\right],$	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
90	$\ln x$	$-s^{-1}\ln(\mathbf{C}s),$	$\operatorname{Re} s > 0$	ET I 148(1)
91	$\ln(1 + ax),$ $\lvert\arg a\rvert < \pi$	$s^{-1}e^{s/a}\operatorname{Ei}(-s/a),$	$\operatorname{Re} s > 0$	ET I 148(4)
92	$x^{-1/2}\ln x$	$-(\pi/s)^{1/2}\ln(4\mathbf{C}s),$	$\operatorname{Re} s > 0$	ET I 148(9)
93	$\mathrm{H}(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases}$ (Heaviside step function)	$s^{-1}e^{-as},$	$a \geq 0$	
94	$\delta(x)$ (Dirac delta function)	1		
95	$\delta(x-a)$	$e^{-as},$	$a \geq 0$	
96	$\delta'(x-a)$	$se^{-as},$	$a \geq 0$	
97	$\operatorname{Si}(x) \equiv \int_0^x \dfrac{\sin\xi}{\xi}\,d\xi \equiv \dfrac{1}{2}\pi + \operatorname{si}(x)$	$s^{-1}\operatorname{arccot} s,$	$\operatorname{Re} s > 0$	ET I 177(17)
98	$\operatorname{Ci}(x) \equiv \operatorname{ci}(x) \equiv -\int_x^\infty \dfrac{\cos\xi}{\xi}\,d\xi$	$-\dfrac{1}{2}s^{-1}\ln\left(1 + s^2\right),$	$\operatorname{Re} s > 0$	ET I 178(19)
99[8]	$\operatorname{erf}\left(\dfrac{x}{2a}\right)$	$s^{-1}e^{a^2s^2}\operatorname{erfc}(as),$	$\operatorname{Re} s > 0, \lvert\arg a\rvert < \pi/4$	ET I 176(2)

continued on next page

continued from previous page

	$f(x)$	$F(s)$
100	$\operatorname{erf}\left(a\sqrt{x}\right)$	$as^{-1}\left(s+a^2\right)^{-1/2}$, $\operatorname{Re} s > \left\{0, -\operatorname{Re} a^2\right\}$ ET I 176(4)
101	$\operatorname{erfc}\left(a\sqrt{x}\right)$	$s^{-1}\left(s+a^2\right)^{-\frac{1}{2}}\left[\left(s+a^2\right)^{1/2}-a\right]$, $\operatorname{Re} s > 0$ ET I 177(9)
102[8]	$\operatorname{erfc}\left(\dfrac{a}{\sqrt{x}}\right)$	$s^{-1}e^{-2a\sqrt{s}}$, $\operatorname{Re} s > 0, \quad \operatorname{Re} a > 0$ ET I 177(11)
103[8]	$J_\nu(ax)$, $\operatorname{Re}\nu > -1$	$a^{-\nu}\left(\sqrt{s^2+a^2}-s\right)^{\nu}\left(s^2+a^2\right)^{-1/2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$, ET I 182(1)
104	$x\,J_\nu(ax)$, $\operatorname{Re}\nu > -2$	$a^{\nu}\left[s+\nu\left(s^2+a^2\right)^{1/2}\right]\left[s+\left(s^2+a^2\right)^{1/2}\right]^{-\nu}$ $\times\left(s^2+a^2\right)^{-3/2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$, ET I 182(2)
105	$\dfrac{J_\nu(ax)}{x}$	$a^{\nu}\nu^{-1}\left[s+\left(s^2+a^2\right)^{1/2}\right]^{-\nu}$, $\operatorname{Re} s \geq \lvert\operatorname{Im} a\rvert$ ET I 182(5)
106	$x^n\,J_n(ax)$	$1\cdot 3\cdot 5\cdots(2n-1)a^n\left(s^2+a^2\right)^{-\left(n+\frac{1}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 182(4)
107	$x^{\nu}\,J_\nu(ax)$, $\operatorname{Re}\nu > -\frac{1}{2}$	$2^{\nu}\pi^{-1/2}\,\Gamma\left(\nu+\frac{1}{2}\right)a^{\nu}\left(s^2+a^2\right)^{-\left(\nu+\frac{1}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$, ET I 182(7)
108	$x^{\nu+1}\,J_\nu(ax)$, $\operatorname{Re}\nu > -1$	$2^{\nu+1}\pi^{-1/2}\,\Gamma\left(\nu+\frac{3}{2}\right)a^{\nu}s\left(s^2+a^2\right)^{-\left(\nu+\frac{3}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 182(8)
109[8]	$I_\nu(ax)$, $\operatorname{Re}\nu > -1$	$a^{-\nu}\left[s-\sqrt{s^2-a^2}\right]^{\nu}\left(s^2-a^2\right)^{-1/2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 195(1)

continued on next page

continued from previous page		
	$f(x)$	$F(s)$
110	$x^{\nu}\, I_{\nu}(ax), \qquad \operatorname{Re}\nu > -\frac{1}{2}$	$2^{\nu}\pi^{-1/2}\,\Gamma\left(\nu+\frac{1}{2}\right) a^{\nu}\left(s^2-a^2\right)^{-\left(\nu+\frac{1}{2}\right)}, \quad \operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 195(6)
111	$x^{\nu+1}\, I_{\nu}(ax), \qquad \operatorname{Re}\nu > -1$	$2^{\nu+1}\pi^{-1/2}\,\Gamma\left(\nu+\frac{3}{2}\right) a^{\nu} s\left(s^2-a^2\right)^{-\left(\nu+\frac{3}{2}\right)}, \quad \operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 196(7)
112	$x^{-1}\, I_{\nu}(ax), \qquad \operatorname{Re}\nu > 0$	$\nu^{-1}a^{\nu}\left[s+\left(s^2-a^2\right)^{1/2}\right]^{-\nu}, \quad \operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 195(4)
113	$\sin\left(2a^{1/2}x^{1/2}\right)$	$(\pi a)^{1/2}s^{-3/2}e^{-a/s}, \qquad \operatorname{Re} s > 0$ ET I 153(32)
114	$x^{-1/2}\cos\left(2a^{1/2}x^{1/2}\right)$	$\pi^{1/2}s^{-1/2}e^{-a/s}, \qquad \operatorname{Re} s > 0$ ET I 158(67)
115	$x^{-1}e^{-ax}\, I_1(ax)$	$\left[(s+2a)^{1/2}-s^{1/2}\right]\left[(s+2a)^{1/2}+s^{1/2}\right]^{-1}, \quad \operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ AS 1024(29.3.52)
116	$\dfrac{J_k(ax)}{x}$	$k^{-1}a^{-k}\left[\left(s^2+a^2\right)^{1/2}-s\right]^{k}, \quad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert, k > -1$ AS 1025(29.3.58)
117	$\left(\dfrac{x}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(ax)$	$\Gamma(k)\pi^{-1/2}\left(s^2+a^2\right)^{k}, \quad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert, \quad k > 0$ AS 1024(29.3.57)
118	$J_0(ax) - ax\, J_1(ax)$	$s^2\left(s^2+a^2\right)^{-3/2}, \qquad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert$
119	$I_0(ax) + ax\, I_1(ax)$	$s^2\left(s^2-a^2\right)^{-3/2}, \qquad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert$

17.21 Fourier transform

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function $f(x)$, denoted by $F(\xi)$, is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,dx.$$

The functions $f(x)$ and $F(\xi)$ are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered. Setting $F(\xi) = \mathcal{F}\left[f(x);\xi\right]$, to emphasize the nature of the transform, we have

the symbolic inverse result $f(x) = \mathcal{F}^{-1}[F(\xi); x]$. The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x}\, d\xi.$$

17.22 Basic properties of the Fourier transform

1. For a and b arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \qquad \text{(linearity)}$$

2. If $n > 0$ is an integer, and $\lim_{|x|\to\infty} f^{(r)}(x) = 0$ for $r = 0, 1, \ldots, n-1$ with $f^{(0)}(x) \equiv f(x)$, then

$$\mathcal{F}\left[f^{(n)}(x); \xi\right] = (-i\xi)^n F(\xi) \qquad \text{(transform of a derivative)}$$ SN 27

3. The **Fourier convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi,$$

and it has the property $f * g = g * f$, and $f * (g * h) = (f * g) * h$. In terms of the convolution operation,

$$\mathcal{F}[f * g(x); \xi] = F(\xi) G(\xi) \qquad \text{(convolution [Faltung] theorem).}$$ SN 24

17.23 Table of Fourier transform pairs

	$f(x)$	$F(\xi)$	
1	1	$(2\pi)^{1/2}\,\delta(\xi)$	SU 496
2[7]	$\dfrac{1}{x}$	$(\pi/2)^{1/2}\, i \operatorname{sign} \xi$	SU 50
3	$\delta(x)$	$(2\pi)^{-1/2}$	SU 496
4[8]	$\delta(ax+b), \qquad a, b \in \mathbb{R}, \quad a \neq 0$	$(2\pi)^{-1/2} e^{ib\xi/a}$	SU 517
5	$\begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}, \qquad a > 0$	$(2/\pi)^{1/2} \xi^{-1} \sin(a\xi)$	
6[8]	$\mathrm{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$	$-\dfrac{1}{i\xi\sqrt{2\pi}} + \sqrt{\dfrac{\pi}{2}}\,\delta(\xi)$	SN 523

continued on next page

continued from previous page

	$f(x)$	$F(\xi)$	
7	$\dfrac{1}{\lvert x\rvert^{a}}, \quad 0 < \operatorname{Re} a < 1$	$\dfrac{(2/\pi)^{1/2}\,\Gamma(1-a)\sin\left(\frac{1}{2}a\pi\right)}{\lvert\xi\rvert^{1-a}}$	SN 523
8	$e^{iax}, \quad a \in \mathbb{R}$	$(2\pi)^{1/2}\,\delta(\xi + a)$	SU 50
9	$e^{-a\lvert x\rvert}, \quad a > 0$	$\dfrac{a(2/\pi)^{1/2}}{a^2+\xi^2}$	SU 50
10[7]	$xe^{-a\lvert x\rvert}, \quad a > 0$	$\dfrac{2ai\xi(2/\pi)^{1/2}}{\left(a^2+\xi^2\right)^2}, \quad \xi > 0$	SU 50
11	$\lvert x\rvert e^{-a\lvert x\rvert}, \quad a > 0$	$\dfrac{(2/\pi)^{1/2}\left(a^2-\xi^2\right)}{\left(a^2+\xi^2\right)^2}$	SU 50
12	$\dfrac{e^{-a\lvert x\rvert}}{\lvert x\rvert^{1/2}}, \quad a > 0$	$\dfrac{\left[a+\left(a^2+\xi^2\right)^{1/2}\right]^{1/2}}{x\left(a^2+\xi^2\right)^{1/2}}$	SN 523
13	$e^{-a^2x^2}, \quad a > 0$	$\left(a\sqrt{2}\right)^{-1} e^{-\xi^2/4a^2}$	SU 51
14	$\dfrac{1}{a^2+x^2}, \quad \operatorname{Re} a > 0$	$\dfrac{(\pi/2)^{1/2}\,e^{-a\lvert\xi\rvert}}{a}$	SU 51
15[7]	$\dfrac{x}{a^2+x^2}, \quad \operatorname{Re} a > 0$	$i\operatorname{sign}\xi\,(\pi/2)^{1/2}\,e^{-a\lvert\xi\rvert}$	
16[9]	$\sin\left(ax^2\right)$	$\dfrac{1}{(2a)^{1/2}}\cos\left(\dfrac{\xi^2}{4a}+\dfrac{\pi}{4}\right)$	SN 523
17	$\cos\left(ax^2\right)$	$\dfrac{1}{(2a)^{1/2}}\cos\left(\dfrac{\xi^2}{4a}-\dfrac{\pi}{4}\right)$	SN 523
18	$e^{-a\lvert x\rvert}\cos(bx), \quad a > 0, \quad b > 0$	$a(2\pi)^{-1/2}\left[\dfrac{1}{a^2+(b+\xi)^2}+\dfrac{1}{a^2+(b-\xi)^2}\right]$	
19	$e^{-\frac{1}{2}ax^2}\sin(bx), \quad a > 0, \quad b > 0$	$\dfrac{1}{2}ia^{-1/2}\left\{\exp\left[-\dfrac{1}{2}\dfrac{(\xi-b)^2}{a}\right] - \exp\left[-\dfrac{1}{2}\dfrac{(\xi+b)^2}{a}\right]\right\}$	
20[9]	$\dfrac{\sinh(ax)}{\sinh(bx)}, \quad \lvert a\rvert < \lvert b\rvert$	$\dfrac{(\pi/2)^{1/2}\sin(\pi a/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$	SU 123
21[9]	$\dfrac{\cosh(ax)}{\sinh(bx)}, \quad \lvert a\rvert < \lvert b\rvert$	$\dfrac{i\,(\pi/2)^{1/2}\sinh(\pi\xi/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$	SU 123

continued on next page

continued from previous page		
	$f(x)$	$F(\xi)$
22	$\dfrac{\sin(ax)}{x}$	$\begin{cases}(\pi/2)^{1/2} & \lvert\xi\rvert < a,\\ 0 & \lvert\xi\rvert > a\end{cases}$ SN 523
23[11]	$\dfrac{x}{\sinh x}$	$\dfrac{\left(2\pi^3\right)^{1/2} e^{\pi\xi}}{\left(1+e^{\pi\xi}\right)^2}$ SU 123
24[7]	$x^n \operatorname{sign} x,\qquad n = 1, 2, \ldots$	$(2/\pi)^{1/2}(-i\xi)^{-(1+n)} n!$ SU 506
25[7]	$\lvert x\rvert^{\nu},$ $-1 < \nu < 0$, but not integral	$(2/\pi)^{1/2}\,\Gamma(\nu+1)\lvert\xi\rvert^{-\nu-1}\cos\left[\pi(\nu+1)/2\right]$ SU506
26[7]	$\lvert x\rvert^{\nu}\operatorname{sign} x,$ $-1 < \nu < 0$, but not integral	$\dfrac{i\operatorname{sign}\xi(2/\pi)^{1/2}\sin\left[(\pi/2)\,(\nu+1)\right]\Gamma(\nu+1)}{\lvert\xi\rvert^{\nu+1}}$ SU 506
27	$e^{-ax}\ln\left\lvert 1-e^{-x}\right\rvert,$ $-1 < \operatorname{Re} a < 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\cot\left(\pi a - i\xi\pi\right)}{a-i\xi}$ ET I 121(26)
28	$e^{-ax}\ln\left(1+e^{-x}\right),$ $-1 < \operatorname{Re} a < 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\csc\left(\pi a - i\xi\pi\right)}{a-i\xi}$ ET I 121 (27)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $1/(2\pi)^{1/2}$ employed in our definition of F has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

17.24 Table of Fourier transform pairs for spherically symmetric functions

	$f(\lvert\lvert\mathbf{r}\rvert\rvert) = \dfrac{1}{(2\pi)^{3/2}}\iiint E(\lvert\lvert\mathbf{k}\rvert\rvert)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{k}$	$E(\lvert\lvert\mathbf{k}\rvert\rvert) = \dfrac{1}{(2\pi)^{3/2}}\iiint f(\lvert\lvert\mathbf{r}\rvert\rvert)e^{-i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}$
1	$f(r) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{r}\displaystyle\int_0^\infty E(k)\sin(kr)k\,dk$	$E(k) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{k}\displaystyle\int_0^\infty f(r)\sin(kr)r\,dr$
2	e^{-ar}	$\sqrt{\dfrac{2}{\pi}}\dfrac{2a}{\left(a^2+k^2\right)^2}$
3[11]	$\dfrac{e^{-ar}}{r}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{1}{\left(a^2+k^2\right)^2}$
4[11]	1	$(2\pi)^{3/2}\,\delta(\mathbf{k})$

17.31 Fourier sine and cosine transforms

The **Fourier sine** and **cosine transforms** of the function $f(x)$, denoted by $F_s(\xi)$ and $F_c(\xi)$, respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\xi x)\,dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\xi x)\,dx.$$

The functions $f(x)$ and $F_s(\xi)$ are called a **Fourier sine transform pair**, and the functions $f(x)$ and $F_c(\xi)$ a **Fourier cosine transform pair**, and knowledge of either $F_s(\xi)$ or $F_c(\xi)$ enables $f(x)$ to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s\left[f(x);\xi\right] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c\left[f(x);\xi\right],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s{}^{-1}\left[F_s(\xi);x\right] \quad \text{and} \quad f(x) = \mathcal{F}_c{}^{-1}\left[F_c(\xi);x\right].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\xi)\sin(\xi x)\,d\xi \qquad [x \geq 0]$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\xi)\cos(\xi x)\,d\xi \qquad [x \geq 0]. \qquad \text{SN 17}$$

17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s\left[af(x) + bg(x)\right] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c\left[af(x) + bg(x)\right] = aF_c(\xi) + bG_c(\xi) \qquad \text{(linearity)}$$

2. If $\lim\limits_{x\to\infty} f^{(r-1)}(x) = 0$ and $\lim\limits_{x\to\infty}\sqrt{\frac{2}{\pi}}f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by $F_s{}^{(r)}$ and $F_c{}^{(r)}$, respectively,

(i) $F_c{}^{(r)}(\xi) = -a_{r-1} + \xi F_s{}^{(r-1)}$.

(ii) $F_s{}^{(r)}(\xi) = -\xi F_c{}^{(r-1)}(\xi)$,

(iii) $F_c{}^{(2r)}(\xi) = -\sum\limits_{n=0}^{r-1}(-1)^n a_{2r-2n-1}\xi^{2n} + (-1)^r\xi^{2n}F_c(\xi)$,

(iv) $F_c{}^{(2r+1)}(\xi) = -\sum\limits_{n=0}^{r-1}{}'(-1)^n a_{2r-2n}\xi^{2n} + (-1)^r\xi^{2r+1}F_s(\xi)$,

(v) $F_s{}^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s{}^{(r-2)}(\xi)$,

(vi)[6] $F_s{}^{(2r)}(\xi) = -\sum\limits_{n=1}^{r}(-1)^n\xi^{2n-1}a_{2r-2n} + (-1)^r\xi^{2r}F_s(\xi)$,

(vii) $F_s{}^{(2r+1)}(\xi) = -\sum\limits_{n=1}^{r}{}'(-1)^n\xi^{2n-1}a_{2r-2n+1} + (-1)^{r+1}\,\xi^{2r+1}F_c(\xi)$. SN 28

3. (i) $$\int_0^\infty F_s(\xi)G_s(\xi)\cos(\xi x)\,d\xi = \frac{1}{2}\int_0^\infty g(s)\left[f(s+x)+f(s-x)\right]\,ds,$$

(ii) $$\int_0^\infty F_c(\xi)G_c(\xi)\cos(\xi x)\,d\xi = \frac{1}{2}\int_0^\infty g(s)\left[f(s+x)+f\left(|x-s|\right)\right]\,ds$$

(convolution (Faltung) theorem) SN 24

4. (i) If $F_s(\xi)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $F_s(x)$ is $f(\xi)$.

(ii) If $F_c(\xi)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $F_c(x)$ is $f(\xi)$.

(iii) If $f(x)$ is an odd function in $(-\infty,\infty)$, then the Fourier sine transform of $f(x)$ in $(0,\infty)$ is $-iF(\xi)$.

(iv) If $f(x)$ is an even function in $(-\infty,\infty)$, then the Fourier cosine transform of $f(x)$ in $(0,\infty)$ is $F(\xi)$.

(v) The Fourier sine transform of $f(x/a)$ is $aF_s(a\xi)$.

(vi) The Fourier cosine transform of $f(x/a)$ is $aF_c(a\xi)$.

(vii) $\mathcal{F}_s\left[f(x);\xi\right] = F_s\left(|\xi|\right)\operatorname{sign}\xi$ SU 45

17.33 Table of Fourier sine transforms

	$f(x)$	$F_s(\xi)$ $(\xi>0)$		
1	x^{-1}	$(\pi/2)^{1/2}$,	$\xi>0$	ET I 64(3)
2	$x^{-\nu}$, $0<\operatorname{Re}\nu<2$	$(2/\pi)^{1/2}\xi^{\nu-1}\,\Gamma(1-\nu)\cos(\nu\pi/2)$,	$\xi>0$	ET I 68(1)
3	$x^{-1/2}$	$\xi^{-1/2}$,	$\xi>0$	ET I 64(6)
4	$x^{-3/2}$	$2\xi^{1/2}$,	$\xi>0$	ET I 64(9)
5	$\begin{cases}1 & 0<x<a\\ 0 & x>a\end{cases}$	$(2/\pi)^{1/2}\xi^{-1}\left[1-\cos(a\xi)\right]$,	$\xi>0$	ET I 63(1)
6	$\begin{cases}x^{-1} & 0<x<a\\ 0 & x>a\end{cases}$	$(2/\pi)^{1/2}\operatorname{Si}(a\xi)$,	$\xi>0$	ET I 64(4)
7	$\dfrac{1}{a-x}$, $a>0$	$(2/\pi)^{1/2}\left\{\sin(a\xi)\operatorname{Ci}(a\xi)-\cos(a\xi)\left[\frac{1}{2}\pi+\operatorname{Si}(a\xi)\right]\right\}$,	$\xi>0$	ET I 64(11)

continued on next page

continued from previous page

	$f(x)$	$F_s(\xi)$ $(\xi > 0)$	
8[7]	$\dfrac{1}{x^2+a^2}$, $a>0$	$(2\pi)^{-1/2}a^{-1}\left[e^{-a\xi}\operatorname{Ei}(a\xi) - e^{a\xi}\operatorname{Ei}(-a\xi)\right]$, $\xi>0$	ET I 65(14)
9	$x\left(x^2+a^2\right)^{-3/2}$, $\operatorname{Re} a>0$	$(2/\pi)^{1/2}\xi\, K_0(a\xi)$, $\xi>0$	ET I 66(27)
10	$x^{-1/2}\left(x^2+a^2\right)^{-1/2}$, $\operatorname{Re} a>0$	$\xi^{1/2}\, I_{\frac{1}{4}}\left(\frac{1}{2}a\xi\right) K_{\frac{1}{4}}\left(\frac{1}{2}a\xi\right)$, $\xi>0$	ET I 66(28)
11[7]	$x\left(x^2+a^2\right)^{-\nu-\frac{3}{2}}$, $\operatorname{Re}\nu>-1$, $\operatorname{Re} a>0$	$\dfrac{\xi^{\nu+1}}{\sqrt{2}(2a)^{\nu}\,\Gamma\left(\nu+\frac{3}{2}\right)} K_{\nu}(a\xi)$,	
12	$\dfrac{x}{a^2+x^2}$, $\operatorname{Re} a>0$	$\left(\dfrac{\pi}{2}\right)^{1/2} e^{-a\xi}$, $\xi>0$	ET I 65(15)
13	$\dfrac{x}{\left(a^2+x^2\right)^2}$	$\sqrt{\pi/8}\,a^{-1}\xi e^{-a\xi}$, $\xi>0$	ET I 67(35)
14	$x^{-1}\left(x^2+a^2\right)^{-1}$, $\operatorname{Re} a>0$	$\dfrac{\sqrt{\pi/2}}{a^2}\left(1-e^{-a\xi}\right)$, $\xi>0$	ET I 65(20)
15	$x^{-1}e^{-ax}$, $\operatorname{Re} a>0$	$(2/\pi)^{1/2}\tan^{-1}\left(\dfrac{\xi}{a}\right)$, $\xi>0$	ET I 72(2)
16	$x^{\nu-1}e^{-ax}$, $\operatorname{Re}\nu>-1$, $\operatorname{Re} a>0$	$(2/\pi)^{1/2}\,\Gamma(\nu)\left(a^2+\xi^2\right)^{-\nu/2}\sin\left[\nu\tan^{-1}\left(\dfrac{\xi}{a}\right)\right]$, $\xi>0$	ET I 72(7)
17	e^{-ax}, $\operatorname{Re} a>0$	$\dfrac{\sqrt{2/\pi}\,\xi}{a^2+\xi^2}$, $\xi>0$	ET I 72(1)
18	xe^{-ax}, $\operatorname{Re} a>0$	$\dfrac{(2/\pi)^{1/2}2a\xi}{\left(a^2+\xi^2\right)^2}$, $\xi>0$	ET I 72(3)
19	xe^{-ax^2}, $\lvert\arg a\rvert<\pi/2$	$(2a)^{-3/2}\xi\exp\left(\dfrac{-\xi^2}{4a}\right)$, $\xi>0$	ET I 73(19)
20	$\dfrac{\sin ax}{x}$, $a>0$	$\dfrac{1}{(2\pi)^{1/2}}\ln\left\lvert\dfrac{\xi+a}{\xi-a}\right\rvert$, $\xi>0$	ET I 78(1)
21	$\dfrac{\sin ax}{x^2}$, $a>0$	$\begin{cases}\xi\left(\frac{\pi}{2}\right)^{1/2} & 0<\xi<a\\ a\left(\frac{\pi}{2}\right)^{1/2} & a<\xi<\infty\end{cases}$, $\xi>0$	ET I 78(2)

continued on next page

continued from previous page

	$f(x)$	$F_s(\xi) \qquad (\xi > 0)$
22	$\sin\left(\frac{a^2}{x}\right), \qquad a > 0$	$a\left(\frac{\pi}{2}\right)^{1/2} \xi^{-1/2} J_1\left(2a\xi^{\frac{1}{2}}\right),$ $\xi > 0$ ET I 83(6)
23	$x^{-1}\sin\left(\frac{a^2}{x}\right), \qquad a > 0$	$\left(\frac{\pi}{2}\right)^{1/2} Y_0\left(2a\xi^{1/2}\right) + \left(\frac{2}{\pi}\right)^{1/2} K_0\left(2a\xi^{1/2}\right)$ ET I 83(7)
24	$x^{-2}\sin\left(\frac{a^2}{x}\right), \qquad a > 0$	$\left(\frac{\pi}{2}\right)^{1/2} a^{-1}\xi^{1/2} J_1\left(2a\xi^{1/2}\right),$ $\xi > 0$ ET I 83(8)
25[10]	$\operatorname{cosech}(ax), \qquad \operatorname{Re} a > 0$	$(\pi/2)^{1/2} a^{-1} \tanh\left(\frac{1}{2}\pi a^{-1}\xi\right),$ $\xi > 0$ ET I 88(2)
26	$\coth\left(\frac{1}{2}ax\right) - 1, \qquad \operatorname{Re} a > 0$	$(2\pi)^{1/2} a^{-1} \coth\left(\pi a^{-1}\xi\right) - \xi,$ $\xi > 0$ ET I 88(3)
27	$\left(1 - x^2\right)^{-1} \sin(\pi x)$	$\begin{cases} (2/\pi)^{1/2}\sin\xi & 0 \le \xi \le \pi \\ 0 & \pi < \xi \end{cases}$ ET I 78(4)
28	$e^{-ax^2}\sin(bx), \qquad \operatorname{Re} a > 0$	$(2a)^{-1/2} \exp\left[-\left(\xi^2 + b^2\right)/(4a)\right] \sinh\left(b\xi/2a\right),$ $\xi > 0$ ET I 78(7)
29	$\frac{\sin^2(ax)}{x}, \qquad a > 0$	$\begin{cases} \pi^{1/2}2^{-3/2} & 0 < \xi < 2a \\ \pi^{1/2}2^{-5/2} & \xi = 2a \\ 0 & 2a < \xi \end{cases}$ ET I 78(8)
30	$\sin\left(ax^2\right), \qquad a > 0$	$a^{-1/2}\left\{\cos\left(\xi^2/4a\right) C\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $+ \sin\left(\xi^2/4a\right) S\left[(2\pi a)^{-1/2}\xi\right],$ $\xi > 0$ ET I 82(1)
31	$\cos\left(ax^2\right), \qquad a > 0$	$a^{-1/2}\left\{\sin\left(\xi^2/4a\right) C\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $- \cos\left(\xi^2/4a\right) S\left[(2\pi a)^{-1/2}\xi\right],$ $\xi > 0$

continued on next page

continued from previous page

	$f(x)$	$F_s(\xi)$ $(\xi > 0)$	
32	$\arctan\left(\frac{x}{a}\right), \quad a > 0$	$(\pi/2)^{1/2}\,\xi^{-1}e^{-a\xi}, \quad \xi > 0$	ET I 87(3)
33[7]	$\arctan\left(\frac{2a}{x}\right), \quad \operatorname{Re} a > 0$	$(2\pi)^{-1/2}e^{-a\xi}\sinh(a\xi), \quad \xi > 0$	ET I 87(8)
34	$\frac{\ln x}{x}$	$-\left(\pi/2\right)^{1/2}\left(\boldsymbol{C} + \ln\xi\right), \quad \xi > 0$	ET I 76(2)
35	$\ln\left\lvert\frac{x+a}{x-a}\right\rvert, \quad a > 0$	$(2\pi)^{1/2}\xi^{-1}\sin(a\xi), \quad \xi > 0$	ET I 77(11)
36[7]	$\frac{\ln\left(1+a^2x^2\right)}{x}, \quad a > 0$	$-(2\pi)^{1/2}\operatorname{Ei}\left(-\xi/a\right), \quad \xi > 0$	ET I 77(14)
37	$J_0(ax), \quad a > 0$	$\begin{cases} 0 & 0 < \xi < a \\ (2/\pi)^{1/2}\left(\xi^2 - a^2\right)^{-1/2} & a < \xi < \infty \end{cases}$	ET I 99(1)
38	$J_\nu(ax), \quad \operatorname{Re}\nu > -2, \quad a > 0$	$(2/\pi)^{1/2}\left(a^2 - \xi^2\right)^{-1/2}\sin\left[\nu\sin^{-1}\left(\frac{\xi}{a}\right)\right]$ for $0 < \xi < a$; $\frac{a^\nu\cos\left(\frac{1}{2}\nu\pi\right)}{\left(\xi^2 - a^2\right)^{1/2}\left[\xi + \left(\xi^2 - a^2\right)^{1/2}\right]^\nu}$ for $a < \xi < \infty$	ET I 99(3)
39	$\frac{J_0(ax)}{x}, \quad a > 0$	$\begin{cases} (2/\pi)^{1/2}\sin^{-1}\left(\frac{\xi}{a}\right) & 0 < \xi < a \\ (\pi/2)^{1/2} & a < \xi < \infty \end{cases}$	ET I 99(4)
40[7]	$\left(x^2 + b^2\right)^{-1}J_0(ax), \quad a > 0, \quad \operatorname{Re} b > 0$	$(2/\pi)^{1/2}\sinh(b\xi)\,K_0(ab)/b, \quad 0 < \xi < a$	ET I 100(12)
41	$x\left(x^2 + b^2\right)^{-1}J_0(ax), \quad a > 0, \quad \operatorname{Re} b > 0$	$(\pi/2)^{1/2}\,e^{-b\xi}\,I_0(ab), \quad a < \xi < \infty$	ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_s has not been used in those tables.

17.34 Table of Fourier cosine transforms

	$f(x)$	$F_c(\xi)$	
1	$x^{-\nu}, \quad 0 < \operatorname{Re}\nu < 1$	$(\pi/2)^{1/2}\,[\Gamma(\nu)]^{-1}\sec\left(\frac{1}{2}\nu\pi\right)\xi^{\nu-1}, \quad \xi > 0$	ET I 10(1)
2	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$	$(2/\pi)^{1/2}\dfrac{\sin(a\xi)}{\xi}, \quad \xi > 0$	ET I 7(1)
3	$\begin{cases} 0 & 0 < x < a \\ 1/x & x > a \end{cases}$	$-(2/\pi)^{1/2}\operatorname{Ci}(a\xi), \quad \xi > 0$	ET I 8(3)
4	$\begin{cases} x^{-1/2} & 0 < x < a \\ 0 & x > a \end{cases}$	$2\xi^{-1/2}\,C(a\xi), \quad \xi > 0$	ET I 8(5)
5	$\begin{cases} 0 & 0 < x < a \\ x^{-1/2} & x > a \end{cases}$	$2\xi^{-1/2}\left[\frac{1}{2} - C(a\xi)\right], \quad \xi > 0$	ET I 8(6)
6[9]	$x^{\nu-1}, \quad 0 < \nu < 1$	$(2/\pi)^{1/2}\,\Gamma(\nu)\xi^{-\nu}\cos\left(\frac{1}{2}\nu\pi\right), \quad 0 < \nu < 1$	ET I 10(1)
7	$\dfrac{1}{x^2+a^2}, \quad \operatorname{Re} a > 0$	$\dfrac{(\pi/2)^{1/2}\,e^{-a\xi}}{a}, \quad \xi > 0$	ET I 11(7)
8[11]	$\dfrac{1}{(x^2+a^2)^2}, \quad \operatorname{Re} a > 0$	$\dfrac{(\pi/2)^{\frac{1}{2}}\,(1+a\xi)e^{-a\xi}}{2a^3}, \quad \xi > 0$	ET I 11(7)
9	$\left(x^2+a^2\right)^{-\nu-\frac{1}{2}}, \quad \operatorname{Re} a > 0, \operatorname{Re}\nu > -\dfrac{1}{2}$	$\sqrt{2}\left(\dfrac{\xi}{2a}\right)^{\nu}\dfrac{K_\nu(a\xi)}{\Gamma\left(\nu+\frac{1}{2}\right)}, \quad \xi > 0$	ET I 11(7)
10	$\begin{cases} \left(a^2-x^2\right)^{\nu} & 0 < x < a \\ 0 & x > a \end{cases}, \quad \operatorname{Re}\nu > -1$	$2^{\nu}\,\Gamma(\nu+1)(a/\xi)^{\nu+\frac{1}{2}}\,J_{\nu+\frac{1}{2}}(a\xi), \quad \xi > 0$	ET I 11(8)
11	$\begin{cases} 0 & 0 < x < a \\ \left(x^2-a^2\right)^{-\nu-\frac{1}{2}} & x > a \end{cases}, \quad -\dfrac{1}{2} < \operatorname{Re}\nu < \dfrac{1}{2}$	$-2^{-\left(\nu+\frac{1}{2}\right)}\,\Gamma\left(\frac{1}{2}-\nu\right)(\xi/a)^{\nu}\,Y_\nu(a\xi), \quad \xi > 0$	ET I 11(9)
12	$e^{-ax}, \quad \operatorname{Re} a > 0$	$(2/\pi)^{1/2}a\left(a^2+\xi^2\right)^{-1}, \quad \xi > 0$	ET I 14(1)

continued on next page

continued from previous page

	$f(x)$	$F_c(\xi)$
13	xe^{-ax}, $\operatorname{Re} a > 0$	$(2/\pi)^{1/2}\left(a^2-\xi^2\right)\left(a^2+\xi^2\right)^{-2}$, $\xi > 0$ ET I 15(7)
14[7]	$x^{\nu-1}e^{-ax}$, $\operatorname{Re} a > 0$, $\operatorname{Re}\nu > a$	$(2/\pi)^{1/2}\,\Gamma(\nu)\left(a^2+\xi^2\right)^{-\nu/2}\cos\left[\nu\tan^{-1}\left(\frac{\xi}{a}\right)\right]$, $\xi > 0$ ET I 15(7)
15	$x^{-1/2}e^{-ax}$, $\operatorname{Re} a > 0$	$\left(a^2+\xi^2\right)^{-1/2}\left[\left(a^2+\xi^2\right)^{1/2}+a\right]^{1/2}$, $\xi > 0$ ET I 14(4)
16[7]	$e^{-a^2x^2}$, $\operatorname{Re} a > 0$	$2^{-1/2}\lvert a\rvert^{-1}e^{-\xi^2/4a^2}$, $\xi > 0$ ET I 15(11)
17	$x^{-1}e^{-x}\sin x$	$(2\pi)^{-1/2}\tan^{-1}\left(\frac{2}{\xi^2}\right)$, $\xi > 0$ ET I 19(7)
18	$\sin\left(ax^2\right)$, $a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right)-\sin\left(\frac{\xi^2}{4a}\right)\right]$, $\xi > 0$ ET I 23(1)
19	$\cos\left(ax^2\right)$, $a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right)+\sin\left(\frac{\xi^2}{4a}\right)\right]$, $\xi > 0$ ET I 24(7)
20	$\frac{\sin(ax)}{x}$, $a > 0$	$\begin{cases}(\pi/2)^{1/2} & \xi < a\\ \frac{1}{2}(\pi/2)^{1/2} & \xi = a\\ 0 & \xi > a\end{cases}$ ET I 18(1)
21[7]	$\frac{\sin^2(ax)}{x^2}$, $a > 0$	$\begin{cases}(\pi/2)^{1/2}\left(a-\frac{1}{2}\xi\right) & \xi < 2a\\ 0 & 2a < \xi\end{cases}$ ET I 19(8)
22[7]	$e^{-bx}\sin(ax)$, $a > 0$, $\operatorname{Re} b > 0$	$(2\pi)^{-1/2}\left[\frac{a+\xi}{b^2+(a+\xi)^2}+\frac{a-\xi}{b^2+(a-\xi)^2}\right]$, $\xi > 0$ ET I 19(6)
23	$\frac{\sin\left[b\left(x^2+a^2\right)^{1/2}\right]}{\left(x^2+a^2\right)^2}$ $a > 0$	$(b/a)\,(\pi/2)^{1/2}\,e^{-a\xi}$, $\xi > 0$ ET I 26(29)
24	$\left(x^2+a^2\right)^{-1/2}\sin\left[b\left(x^2+a^2\right)^{1/2}\right]$, $a > 0$	$\begin{cases}(\pi/2)^{1/2}\,J_0\left[a\left(b^2-\xi^2\right)^{1/2}\right] & 0 < \xi < b\\ 0 & b < \xi\end{cases}$ ET I 26(30)

continued on next page

continued from previous page

	$f(x)$	$F_c(\xi)$
25	$\dfrac{1-\cos(ax)}{x^2}, \quad a>0$	$\begin{cases}(\pi/2)^{1/2}(a-\xi) & \xi<a \\ 0 & a<\xi\end{cases}$ ET I 20(16)
26	$e^{-ax^2}\sin\left(bx^2\right), \quad \operatorname{Re} a>\lvert\operatorname{Im} b\rvert$	$2^{-1/2}\left(a^2+b^2\right)^{-1/4}\exp\left\{-a\xi^2/\left[4\left(a^2+b^2\right)\right]\right\}$ $\times\sin\left[\frac{1}{2}\arctan(b/a)-\frac{1}{4}b\xi^2\left(a^2+b^2\right)^{-1}\right],$ $\xi>0$ ET I 23(5)
27	$e^{-ax^2}\cos\left(bx^2\right), \quad \operatorname{Re} a>\lvert\operatorname{Im} b\rvert$	$2^{-1/2}\left(a^2+b^2\right)^{-1/4}\exp\left\{-a\xi^2/\left[4\left(a^2+b^2\right)\right]\right\}$ $\times\cos\left[\frac{1}{4}b\xi^2\left(a^2+b^2\right)^{-1}-\frac{1}{2}\arctan(b/a)\right],$ $\xi>0$ ET I 24(6)
28	$\dfrac{\sinh(ax)}{\sinh(bx)} \quad \lvert\operatorname{Re} a\rvert<\operatorname{Re} b$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\sin(\pi a/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]},$ $\xi>0$ ET I 31(14)
29	$\dfrac{\cosh(ax)}{\cosh(bx)}, \quad \lvert\operatorname{Re} a\rvert<\operatorname{Re} b$	$\dfrac{(2\pi)^{1/2}\cos(\pi a/2b)\cosh(\pi\xi/2b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]},$ $\xi>0$ ET I 31(12)
30	$\operatorname{sech}(ax), \quad \operatorname{Re} a>0$	$a^{-1}(\pi/2)^{1/2}\operatorname{sech}(\pi\xi/2a),$ $\xi>0$ ET I 30(1)
31	$\left(x^2+a^2\right)\operatorname{sech}\left(\dfrac{\pi x}{2a}\right), \quad \operatorname{Re} a>0$	$2(2/\pi)^{1/2}a^3\operatorname{sech}^3(a\xi), \quad \xi>0$ ET I 32(19)
32	$\ln\left(1+\dfrac{a^2}{x^2}\right), \quad \operatorname{Re} a>0$	$(2\pi)^{1/2}\xi^{-1}\left(1-e^{-a\xi}\right), \quad \xi>0$ ET I 18(10)
33[7]	$\ln\left(\dfrac{a^2+x^2}{b^2+x^2}\right),$ $\operatorname{Re} a>0, \operatorname{Re} b>0$	$(2\pi)^{1/2}\left(e^{-b\xi}-e^{-a\xi}\right), \quad \xi>0$ ET I 18(12)
34	$\left(x^2+b^2\right)^{-1}J_0(ax),$ $a>0, \quad \operatorname{Re} b>0$	$(\pi/2)^{1/2}b^{-1}e^{-b\xi}I_0(ab),$ $a<\xi<\infty$ ET I 45(14)

continued on next page

continued from previous page		
	$f(x)$	$F_c(\xi)$
35	$x\left(x^2+b^2\right)^{-1} J_0(ax),$ $\quad a>0, \quad \operatorname{Re} b>0$	$(2/\pi)^{1/2}\cosh(b\xi)\, K_0(ab),$ $\quad 0<\xi<a$ ET I 45(15)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_c has not been used in those tables.

17.35 Relationships between transforms

The following relationships exist between transforms, and they may be used to derive further transform pairs from among the results given in Sections 17.13–17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

17.351

Fourier cosine transform and Laplace transform relationship

$$\mathcal{F}_c\left[f(x);\xi\right]=\frac{1}{\sqrt{2\pi}}\mathcal{L}\left[f(x);i\xi\right]+\frac{1}{\sqrt{2\pi}}\mathcal{L}\left[f(x);-i\xi\right].$$

17.352

Fourier sine transform and Laplace transform relationship

$$\mathcal{F}_s\left[f(x);\xi\right]=\frac{i}{\sqrt{2\pi}}\mathcal{L}\left[f(x);i\xi\right]-\frac{i}{\sqrt{2\pi}}\mathcal{L}\left[f(x);-i\xi\right].$$

17.353

Exponential Fourier transform and Laplace transform relationship

$$\mathcal{F}\left[f(x);\xi\right]=\sqrt{2\pi}\mathcal{L}\left[f(x);-i\xi\right]+\sqrt{2\pi}\mathcal{L}\left[f(-x);i\xi\right].$$

17.41[10] Mellin transform

The **Mellin transform** of the function $f(x)$, denoted by $f^*(s)$, is defined by the integral

$$f^*(s)=\int_0^\infty f(x)x^{s-1}\,dx.$$

The functions $f(x)$ and $f^*(s)$ are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

The transform exists, provided the integral

$$\int_0^\infty |f(x)|x^{k-1}\,dx$$

is bounded for some $k>0$, and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}\,ds,$$

where $c > k$.

Setting

$$f^*(s) = \mathcal{M}\left[f(x); s\right]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = \mathcal{M}^{-1}\left[f^*(s); x\right]. \qquad \text{MS 397(6)}$$

17.42 Basic properties of the Mellin transform

1. For a and b arbitrary constants,

$$\mathcal{M}\left[af(x) + bg(x)\right] = af^*(s) + bg^*(s) \qquad \text{(linearity)}$$

2. If $\lim\limits_{x\to 0} x^{s-r-1}f^{(r)}(x) = 0, \quad r = 0, 1, \ldots, n-1,$

(i) $$\mathcal{M}\left[f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n)$$

(transform of a derivative) SU 267 (4.2.3)

(ii) $$\mathcal{M}\left[x^n f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s)$$

(transform of a derivative) SU 267 (4.2.5)

3. Denoting the nth repeated integral of $f(x)$ by $I_n\left[f(x)\right]$, where

$$I_n\left[f(x)\right] = \int_0^x I_{n-1}\left[f(u)\right]\,du,$$

(i) $$\mathcal{M}\left[I_n\left[f(x)\right]; s\right] = (-1)^n \frac{\Gamma(s)}{\Gamma(n+s)} f^*(s+n)$$

(transform of an integral) SU 269 (4.2.15)

(ii) $$\mathcal{M}\left[I_n^\infty\left[f(x)\right]; s\right] = \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n),$$

where

$$I_n^\infty\left[f(x)\right] = \int_x^\infty I_{n-1}^\infty\left[f(u)\right]\,du \qquad \text{(transform of an integral)} \qquad \text{SU 269 (4.2.18)}$$

4. $$\mathcal{M}\left[f(x)g(x); s\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(u)g^*(s-u)\,du$$

(Mellin convolution theorem) SU 275(4.4.1)

17.43 Table of Mellin transforms

	$f(x)$	$f^*(s)$		
1	e^{-x}	$\Gamma(s)$,	$\operatorname{Re} s > 0$	SU 521(M13)
2	e^{-x^2}	$\frac{1}{2}\,\Gamma\left(\frac{1}{2}s\right)$,	$\operatorname{Re} s > 0$	SU 521(M14)
3	$\cos x$	$\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)$,	$0 < \operatorname{Re} s < 1$	SU 521(M15)
4	$\sin x$	$\Gamma(s)\sin\left(\frac{1}{2}\pi s\right)$,	$0 < \operatorname{Re} s < 1$	SU 521(M16)
5	$\dfrac{1}{1-x}$	$\pi\cot(\pi s)$,	$0 < \operatorname{Re} s < 1$	SU 521(M1)
6	$\dfrac{1}{1+x}$	$\pi\operatorname{cosec}(\pi s)$,	$0 < \operatorname{Re} s < 1$	SU 521(M2)
7	$(1+x^a)^{-b}$	$\dfrac{\Gamma(s/a)\,\Gamma(b-s/a)}{a\,\Gamma(b)}$,	$0 < \operatorname{Re} s < ab$	SU 521(M3)
8	$\dfrac{T_n(x)\,\mathrm{H}(1-x)}{\sqrt{(1-x^2)}}$	$\dfrac{2^{-s}\pi\,\Gamma(s)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)}$,	$\operatorname{Re} s > 0$	SU 521(M4)
9	$\dfrac{T_n\left(x^{-1}\right)\mathrm{H}(1-x)}{\sqrt{(1-x^2)}}$	$\dfrac{2^{s-2}\,\Gamma\left(\frac{1}{2}n+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}n\right)}{\Gamma(s)}$,	$\operatorname{Re} s > n$	SU 521(M5)
10	$P_n(x)\,\mathrm{H}(1-x)$	$\dfrac{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\right)}{2\,\Gamma\left(\frac{1}{2}s-\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}n+1\right)}$,	$\operatorname{Re} s > 0$	SU 521(M6)
11	$P_n\left(x^{-1}\right)\mathrm{H}(1-x)$	$\dfrac{2^{s-1}\,\Gamma\left(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}n\right)}{\sqrt{\pi}\,\Gamma(s+1)}$,	$\operatorname{Re} s > n$	SU 521(M7)
12	$\dfrac{1+x\cos\phi}{1-2x\cos\phi+x^2}$	$\dfrac{\pi\cos(s\phi)}{\sin(s\pi)}$,	$0 < \operatorname{Re} s < 1$	SU 521(M11)
13	$\dfrac{x\sin\phi}{1-2x\cos\phi+x^2}$, $-\pi < \phi < \pi$	$\dfrac{\pi\sin(s\phi)}{\sin(s\pi)}$,	$0 < \operatorname{Re} s < 1$	SU 521(M12)

continued on next page

continued from previous page

	$f(x)$	$f^*(s)$	
14	$e^{-x\cos\phi}\cos(x\sin\phi)$, $\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s)\cos(s\phi)$, $\operatorname{Re} s > 0$	SU 522(M17)
15	$e^{-x\sin\phi}\sin(x\sin p\phi)$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s)\sin(s\phi)$, $\operatorname{Re} s > -1$	SU 522(M18)
16	$x^{-\nu}J_\nu(x)$, $\nu > -\frac{1}{2}$	$\dfrac{2^{s-\nu-1}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\nu-\frac{1}{2}s+1\right)}$, $0 < \operatorname{Re} s < 1$	SU 522(M19)
17	$Y_\nu(x)$, $\nu \in \mathbb{R}$	$-2^{s-1}\pi^{-1}\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)$ $\times\cos\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\pi$, $\lvert\nu\rvert < \operatorname{Re} s < \frac{3}{2}$	SU 522(M20)
18	$K_\nu(x)$, $\nu \in \mathbb{R}$	$2^{s-2}\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)$, $\operatorname{Re} s > \nu > 0$	SU 522(M21)
19	$\mathbf{H}_\nu(x)$, $\nu \in \mathbb{R}$	$\dfrac{2^{s-1}\tan\left(\frac{1}{2}\pi s+\frac{1}{2}\pi\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}$, $-1-\nu < \operatorname{Re} s < \min\left(\frac{3}{2}, 1-\nu\right)$	SU 522(M22)
20	$\dfrac{1}{a+x^n}$, $\lvert\arg a\rvert < \pi$, $n = 1, 2, 3, \ldots$,	$\pi n^{-1}\operatorname{cosec}\left(\dfrac{\pi s}{n}\right)a^{(s/n)-1}$, $0 < \operatorname{Re} s < n$	MS 453
21	$\left(1+ax^h\right)^{-\nu}$, $h > 0$, $\lvert\arg a\rvert < \pi$	$h^{-1}a^{-s/h}\,\mathrm{B}\left(s/h, \nu-(s/h)\right)$ $0 < \operatorname{Re} s < h\operatorname{Re}\nu$	MS 454
22	$\begin{cases}\left(1-x^h\right)^{\nu-1} & \text{for } 0 < x < 1\\ 0 & \text{for } x > 1\end{cases}$, $h > 0$, $\operatorname{Re}\nu > 0$	$h^{-1}\,\mathrm{B}\left(\nu, s/h\right)$	MS 454
23	$\ln(1+ax)$, $\lvert\arg a\rvert < \pi$	$\pi s^{-1}a^{-s}\operatorname{cosec}(\pi s)$, $-1 < \operatorname{Re} s < 0$	MS 454
24	$\arctan x$	$-\frac{1}{2}\pi s^{-1}\sec(\pi s/2)$, $-1 < \operatorname{Re} s < 0$	MS 454

continued on next page

continued from previous page

	$f(x)$	$f^*(s)$
25	$\operatorname{arccot} x$	$\frac{1}{2}\pi s^{-1}\sec(\pi s/2)$, $0 < \operatorname{Re} s < 1$ MS 454
26	$\operatorname{cosech}(ax)$ $\operatorname{Re} a > 0$	$a^{-s}2\left(1-2^{-s}\right)\Gamma(s)\,\zeta(s)$, $\operatorname{Re} s > 1$ MS 454
27	$\operatorname{sech}^2(ax)$, $\operatorname{Re} a > 0$	$4a^{-s}(1-2^{2-s})\,\Gamma(s)2^{-s}\,\zeta(s-1)$, $\operatorname{Re} s > 2$ MS 454
28	$\operatorname{cosech}^2(ax)$, $\operatorname{Re} a > 0$	$4a^{-s}\,\Gamma(s)2^{-s}\,\zeta(s-1)$, $\operatorname{Re} s > 2$ MS 454
29[11]	$\left(x^2+b^2\right)^{-\frac{1}{2}\nu} J_\nu\left[a\left(x^2+b^2\right)^{1/2}\right]$	$2^{\frac{1}{2}s-1}a^{-\frac{1}{2}s}b^{\frac{1}{2}s-\nu}\,\Gamma\left(\frac{1}{2}s\right)J_{\nu-s/2}(ab)$, $0 < \operatorname{Re} s < \frac{3}{2} + \operatorname{Re}\nu$ ET I 328
30	$\begin{cases}\left(a^2-x^2\right)^{\frac{1}{2}\nu} J_\nu\left[a\left(b^2-x^2\right)^{1/2}\right] & \text{for } 0 < x < a\\ 0 & \text{for } x > a\end{cases}$ $\operatorname{Re}\nu > -1$	$2^{\frac{1}{2}s-1}\,\Gamma\left(\frac{1}{2}s\right)b^{-\frac{1}{2}s}a^{\nu+\frac{1}{2}s}\,J_{\nu+\frac{1}{2}s}(ab)$, $\operatorname{Re} s > 0$ MS 455
31	$\begin{cases}\left(a^2-x^2\right)^{-\frac{1}{2}\nu} J_\nu\left[b\left(a^2-x^2\right)^{1/2}\right] & \text{for } 0 < x < a\\ 0 & \text{for } x > a\end{cases}$	$2^{1-\nu}\left[\Gamma(\nu)\right]^{-1}a^{\frac{1}{2}s-\nu}b^{-\frac{1}{2}\nu}\,s_{\nu-1+\frac{1}{2}s,\frac{1}{2}s-\nu}(ab)$, $\operatorname{Re} s > 0$ MS 455
32	$K_\nu(\alpha x)$	$\alpha^{-s}2^{s-2}\,\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)$, $\operatorname{Re} s > \lvert\operatorname{Re}\nu\rvert$ MS 455
33	$\left(\beta a^2+x^2\right)^{-\frac{1}{2}\nu}$ $\times K_\nu\left[\alpha\left(\beta a^2+x^2\right)^{1/2}\right]$ $\operatorname{Re}(\alpha,\beta) > 0$	$\alpha^{-\frac{1}{2}s}2^{\frac{1}{2}s-1}\beta^{\frac{1}{2}s-\nu}\,\Gamma,\left(\frac{1}{2}s\right)K_{\nu-\frac{1}{2}s}(\alpha\beta)$, $\operatorname{Re} s > 0$ MS 455

This page intentionally left blank

18 The z-Transform

18.1–18.3 Definition, Bilateral, and Unilateral z-Transforms

18.1 Definitions

The **z-transform** converts a numerical sequence $x[n]$ into a function of the complex variable z, and it takes two different forms. The **bilateral** or **two-sided z-transform**, denoted here by $Z_b\{x[n]\}$, is used mainly in signal and image processing, while the **unilateral** or **one-sided z-transform**, denoted here by $Z_u\{x[n]\}$, is used mainly in the analysis of discrete time systems and the solution of linear difference equations.

The **bilateral z-transform**, $X_b(z)$ of the sequence $x[n] = \{x_n\}_{n=-\infty}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} = X_b(z),$$

and the **unilateral z-transform** $X_u(z)$ of the sequence $x[n] = \{x_n\}_{n=0}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=0}^{\infty} x_n z^{-n} = X_u(z),$$

where each has its own domain of convergence (DOC). The series $X_b(z)$ is a Laurent series, and $X_u(z)$ is the principal part of the Laurent series for $X_b(z)$. When $x_n = 0$ for $n < 0$, the two z-transforms $X_b(z)$ and $X_u(z)$ are identical. In each case the sequence $x[n]$ and its associated z-transform is a called a **z-transform pair**.

The inverse z-transformation $x[n] = Z^{-1}\{X(z)\}$ is given by

$$x[n] = \frac{1}{2\pi i}\int_{\Gamma} X(z) z^{n-1}\, dz,$$

where $X(z)$ is either $X_b(z)$ or $X_u(z)$, and Γ is a simple closed contour containing the origin and lying entirely within the domain of convergence of $X(z)$. In many practical situations, the z-transform is either found by using a series expansion of $X(z)$ in the inversion integral or, if $X(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are polynomials in z, by means of partial fractions and the use of an appropriate table of z-transform pairs. In order for the inverse z-transform to be unique, it is necessary to specify the domain of convergence, as can be seen by comparison of entries 3 and 4 of Table 18.2. Table 18.1 lists general properties of the bilateral z-transform, and Table 18.2 lists some bilateral z-transform pairs. In what follows, use is made of the **unit integer function** $h(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$, that is, a generalization of the Heaviside step function, and the **unit integer pulse function** $\Delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$, that is, a generalization of the delta function.

18.2 Bilateral z-transform

Table 18.1 General properties of the bilateral z-transform $X_b(n) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$.

	Term in sequence	z-Transform $X_b(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_b(z) + \beta Y_b(z)$	Intersection of DOC's of $X_b(z)$ and $Y_b(z)$ with α, β constants
2	x_{n-N}	$z^{-n} X_b(z)$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin or the point at infinity
3	nx_n	$-z\dfrac{dX_b(z)}{dz}$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_b\left(\dfrac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $\lvert z_0\rvert$
5	$nz_0^n x_n$	$-z\dfrac{dX_b(z/z_0)}{dz}$	DOC of $X_b(z)$ scaled by $\lvert z_0\rvert$ to which it may be necessary to add or delete the origin and the point at infinity
6	x_{-n}	$X_b(1/z)$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
7	nx_{-n}	$-z\dfrac{dX_b(1/z)}{dz}$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
8	$\bar{x}_n$	$\overline{X_b}(\overline{z})$	The same DOC as x_n
9	$\operatorname{Re} x_n$	$\frac{1}{2}\left[X_b(z) + \overline{X_b}(\overline{z})\right]$	DOC contains the DOC of x_n
10	$\operatorname{Im} x_n$	$\frac{1}{2i}\left[X_b(z) - \overline{X_b}(\overline{z})\right]$	DOC contains the DOC of x_n
11	$\sum_{k=-\infty}^{\infty} x_k y_{n-k}$	$X_b(z)Y_b(z)$	DOC contains the intersection of the DOCs of $X_b(z)$ and $Y_b(z)$ (convolution theorem)
12	$x_n y_n$	$\dfrac{1}{2\pi i}\displaystyle\int_\Gamma X_b(\xi) Y_b\left(\dfrac{z}{\xi}\right)\xi^{-1}\,d\xi$	DOC contains the DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin (convolution theorem)
13	Parseval formula	$\sum_{n=-\infty}^{\infty} x_n \bar{y}_n = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma X_b(\xi)\overline{Y_b}\left(\dfrac{z}{\overline{\xi}}\right)\xi^{-1}\,d\xi$	DOC contains the intersection of DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin
14	Initial value theorem for $x_n h(n)$	$x_0 = \lim_{z\to\infty} X_b(z)$	

Table 18.2 Basic bilateral z-transforms

	Term in sequence	z-Transform $X_b(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n-N)$	z^{-n}	When $N>0$ convergence is for all z except at the origin. When $N<0$ convergence is for all z except at ∞
3	$a^n h(n)$	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
4	$a^n h(-n-1)$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
5	$na^n h(n)$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > a > 0$
6	$na^n h(-n-1)$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert < a, \quad a > 0$
7	$n^2 a^n h(n)$	$\dfrac{az(z+a)}{(z-a)^3}$	$\lvert z\rvert > a > 0$
8	$\left(\dfrac{1}{a^n}+\dfrac{1}{b^n}\right)h(n)$	$\dfrac{az}{az-1}+\dfrac{bz}{bz-1}$	$\lvert z\rvert > \max\left(\frac{1}{\lvert a\rvert}, \frac{1}{\lvert b\rvert}\right)$
9	$a^n h(n-N)$	$\dfrac{z\left(1-(a/z)^N\right)}{z-a}$	$\lvert z\rvert > 0$
10	$a^n h(n)\sin\Omega n$	$\dfrac{az\sin\Omega}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a > 0$
11	$a^n h(n)\cos\Omega n$	$\dfrac{z(z-a\cos\Omega)}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a > 0$
12	$e^{an}h(n)$	$\dfrac{z}{z-e^a}$	$\lvert z\rvert > e^{-a}$
13	$e^{-an}h(n)\sin\Omega n$	$\dfrac{ze^a\sin\Omega}{z^2e^{2a}-2ze^a\cos\Omega+1}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}h(n)\cos\Omega n$	$\dfrac{ze^a(ze^a-\cos\Omega)}{z^2e^{2a}-2ze^a\cos\Omega+1}$	$\lvert z\rvert > e^{-a}$

18.3 Unilateral z-transform

The relationship between the Laplace transform of a continuous function $x(t)$ sampled at $t = 0$, T, $2T$, $\ldots$ and the unilateral z-transform of the function $\hat{x}(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT)$ follows from the result

$$\begin{aligned}\mathcal{L}\{\hat{x}(t)\} &= \int_0^{\infty} \left[\sum_{k=0}^{\infty} x(kT)\delta(t - kT)\right] e^{-st}\, dt \\ &= \sum_{k=0}^{\infty} x(kT)e^{-ksT}.\end{aligned}$$

Setting $z = e^{sT}$, this becomes:

$$\mathcal{L}\{\hat{x}(t)\} = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z),$$

showing that the unilateral z-transform $X_u(z)$ can be considered to be the Laplace transform of a continuous function $x(t)$ for $t \geq 0$ sampled at $t = 0$, T, $2T$, $\ldots$.

Table 18.3 lists some general properties of the unilateral z-transform, and Table 18.4 lists some unilateral z-transform pairs.

Table 18.3 General properties of the unilateral z-transform

	Term in sequence	z-Transform $X_u(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_u(z) + \beta Y_u(z)$	Intersection of DOC's of $X_u(z)$ and $Y_u(z)$ with α, β constants
2	x_{n+k}	$z^k X_u(z) - z^k x_0 - z^{k-1} x_1$ $-z^{k-2} x_2 - \cdots - z x_{k-1}$	
3	$n x_n$	$-z \dfrac{dX_u(z)}{dz}$	DOC of $X_u(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_u\left(\dfrac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $\lvert z_0 \rvert$, to which it may be necessary to add or delete the origin and the point at infinity
5	$n z_0^n x_n$	$-z \dfrac{dX_u(z/z_0)}{dz}$	DOC of $X_u(z)$ scaled by $\lvert z_0 \rvert$, to which it may be necessary to add or delete the origin and the point at infinity
6	$\bar{x}_n$	$\overline{X_u}(\overline{z})$	The same DOC as x_n
7	$\operatorname{Re} x_n$	$\frac{1}{2}\left[X_u(z) + \overline{X_u}(\overline{z})\right]$	DOC contains the DOC of x_n
8	$\dfrac{\partial}{\partial \alpha} x_n(\alpha)$	$\dfrac{\partial}{\partial \alpha} X_u(z, \alpha)$	Same DOC as $x_n(\alpha)$
9	Initial value theorem	$x_0 = \lim\limits_{z \to \infty} X_u(z)$	
10	Final value theorem	$\lim\limits_{n \to \infty} x_n = \lim\limits_{z \to 1} \left[\left(\dfrac{z-1}{z}\right) X_u(z)\right]$	When $X_u(z) = N(z)/D(z)$ with $N(z)$, $D(z)$ polynomials in z and the zeros of $D(z)$ inside the unit circle $\lvert z \rvert = 1$ or at $z = 1$

Table 18.4 Basic unilateral z-transforms

	Term in sequence	z-Transform $X_u(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n-k)$	z^{-k}	Convergence for all $z \neq 0$
3	$a^n h(n)$	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n h(n)$	$\dfrac{az}{(z-az)^2}$	$\lvert z\rvert > a > 0$
5	$n^2 a^n h(n)$	$\dfrac{az(z+a)}{(z-a)^3}$	$\lvert z\rvert > a > 0$
6	$na^{n-1} h(n)$	$\dfrac{z}{(z-a)^2}$	$\lvert z\rvert > a > 0$
7	$(n-1)a^n h(n)$	$\dfrac{z(2a-z)}{(z-a)^2}$	$\lvert z\rvert > a > 0$
8	$e^{-an} h(n)$	$\dfrac{ze^a}{ze^a-1}$	$\lvert z\rvert > e^{-a}$
9	$ne^{-an} h(n)$	$\dfrac{ze^a}{(ze^a-1)^2}$	$\lvert z\rvert > e^{-a}$
10	$n^2 e^{-an} h(n)$	$\dfrac{ze^a(1+ze^a)}{(ze^a-1)^3}$	$\lvert z\rvert > e^{-a}$
11	$e^{-an} h(n) \sin \Omega n$	$\dfrac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$\lvert z\rvert > e^{-a}$
12	$e^{-an} h(n) \cos \Omega n$	$\dfrac{ze^a(ze^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$\lvert z\rvert > e^{-a}$
13	$h(n) \sinh an$	$\dfrac{z \sinh a}{z^2 - 2z \cosh a + 1}$	$\lvert z\rvert > e^{-a}$
14	$h(n) \cosh an$	$\dfrac{z(z - \cosh a)}{z^2 - 2z \cosh a + 1}$	$\lvert z\rvert > e^{-a}$
15	$h(n) a^{n-1} e^{-an} \sin \Omega n$	$\dfrac{ze^a \sin \Omega}{z^2 e^{2a} - 2zae^a \cos \Omega + a^2}$	$\lvert z\rvert > e^{-a}$
16	$h(n) a^n e^{-an} \cos \Omega n$	$\dfrac{ze^a\,(ze^a - a \cos \Omega)}{z^2 - 2zae^a \cos \Omega + a^2}$	$\lvert z\rvert > e^{-a}$

Bibliographic References Used in Preparation of Text

(See the introduction for an explanation of the letters preceding each bibliographic reference.)

AS Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions*, Dover Publications, New York, 1972.

AD Adams, E. P. and Hippisley, R. L., *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Smithsonian Institute, Washington, D.C., 1922.

AK Appell, P. and Kampé de Fériet, *Fonctions hypergéometriques et hypersphériques, polynomes d'Hermite*, Gauthier Villars, Paris, 1926.

BB Beckenbach, E. F. and Bellman, R., *Inequalities*, 3rd printing. Springer–Verlag, Berlin, 1971.

BE Bertrand, J., *Traite de calcul différentiel et de calcul intégral*, vol. 2, *Calcul intégral, intégrales définies et indéfinies*, Gauthier-Villars, Paris, 1870.

BI Bierens de Haan, D., *Nouvelles tables d'intégrales définies*, Amsterdam, 1867. (Reprint) G. E. Stechert & Co., New York, 1939.

BL Bellman, R., *Introduction to Matrix Analysis*, McGraw Hill, New York, 1960.

BR Bromwich, T. I'A., *An Introduction to the Theory of Infinite Series*, Macmillan, London, 1908, 2nd edition, 1926.*

BS Bellman, R., *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

BU Buchholz, H., *Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen*, Springer–Verlag, Berlin, 1953. Also an English edition: *The confluent Hypergeometric Function*, Springer–Verlag, Berlin, 1969.

BY Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer–Verlag, Berlin, 1954.

CA Carslaw, H. S., *Introduction to the Theory of Fourier's Series and Integrals*, Macmillan, London, 1930.

CE Cesàro, Z., *Elementary Class Book of Algebraic Analysis and the Calculation of Infinite Limits*, 1st ed. ONTI, Moscow and Leningrad, 1936.

CL Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw Hill, New York, 1955.

CO Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Wiley (Interscience), New York, 1953.

DW Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1934.

The Bibliographic Reference BR refers to the 1908 edition of Bromwich T. I.'A., *An Introduction to the Theory of Infinite Series*; BR refers to the 1926 edition.

DW61 Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, 1961.

EF Efros, A. M. and Danilevskiy, A. M., *Operatsionnoye ischisleniye i konturnyye integraly* (Operational calculus and contour integrals). GNTIU, Khar'kov, 1937.

EH Erdélyi, A., et al., *Higher Transcendental Functions*, vols. I, II, and III. McGraw Hill, New York, 1953–1955.

ET Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.

EU Euler, L., *Introductio in Analysin Infinitorum*, Bousquet, Lausanne, 1748.

FI Fikhtengol'ts, G. M., *Kurs differentsial'nogo i integral'nogo ischisleniya* (Course in differential and integral calculus), vols. I, II, and III. Gostekhizdat, Moscow and Leningrad, 1947–1949. Also a German edition: *Differential-und Integralrechnung I–III*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986–1987.

GA Gauss, K. F., *Werke*, Bd. III. Göttingen, 1876.

GE Gel'fond, A. O., *Ischisleniye konechnykh raznostey* (Calculus of finite differences), part I. ONTI, Moscow and Leningrad, 1936.

GH2 Gröbner, W. and Hofreiter, N., *Integraltafel*, vol. 2, *Bestimmte Integrale*, Springer, Wien, 1961.

GI Giunter, N. M. and Kuz'min, R. O. (eds.), *Sbornik zadach po vysshey matematike* (Collection of problems in higher mathematics), vols. I, II, and III. Gostekhizdat, Moscow and Leningrad, 1947.

GM Gantmacher, F. R., *Applications of the Theory of Matrices*, translation by J. L. Brenner. Wiley (Interscience), New York, 1959.

GO Goursat, E. J. B., *Cours d'Analyse*, vol. I, Gauthier–Villars, Paris, 1923.

GU Gröbner, W. et al., *Integraltafel*, Teil I, *Unbestimmte Integrale*, Akad. Verlag, Braunschweig, 1944.

GW Gröbner, W. and Hofreiter, N., *Integraltafel*, Teil II, *Bestimmte Integrale*, Springer–Verlag, Wien and Innsbruck, 1958.

HI Hille, E., *Lectures on Ordinary Differential Equations*, Addison- Wesley, Reading, Massachusetts, 1969.

HL Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge University Press, London, 2nd ed., 1952.

HO Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, London, 1931.

HU Hurewicz, W., *Lectures on Ordinary Differential Equations*, MIT Press, Cambridge, Massachusetts, 1958.

IN Ince, E. L., *Ordinary Differential Equations*, Dover, New York, 1944.

JA Jahnke, E. and Emde, F., *Tables of Functions with Formulas and Curves*, Dover, New York, 1943.

JAC Jackson, J. D., *Classical Electrodynamics*, Wiley, New York, 1975.

JE James, H. M. et al. (eds.), *Theory of Servomechanisms*, McGraw Hill, New York, 1947.

JO Jolley, L., *Summation of Series*, Chapman and Hall, London, 1925.

KE Kellogg, O. D., *Foundations of Potential Theory*, Dover, New York, 1958.

KR Krechmar, V. A., *Zadachnik po algebre* (Problem book in algebra), 2nd ed. Gostekhizdat, Moscow and Leningrad, 1950.

KU Kuzmin, R. O., *Besselevy funktsii* (Bessel functions). ONTI, Moscow and Leningrad, 1935.

LA Laska, W., *Sammlung von Formeln der reinen und angewandten Mathematik*, Friedrich Viewig und Sohn, Braunschweig, 1888–1894.

LE Legendre, A. M., *Exercises calcul intégral*, Paris, 1811.

LI Lindeman, C. E., *Examen des nouvelles tables d'intégrales définies de M. Bierens de Haan*, Amsterdam, 1867, Norstedt, Stockholm, 1891.

LO Lobachevskiy, N. I., *Poloye sobraniye sochineniy* (Complete works), vols. I, III, and V. Gostekhizdat, Moscow and Leningrad, 1946–1951.

LUKE Luke, Y. L., *Mathematical Functions and their Approximations*, Academic Press, New York, 1975.

LW Lawden, D. F., *Elliptic Functions and Applications*, Springer–Verlag, Berlin, 1989.

MA McLachlan, N. W., *Theory and Application of Mathieu Functions*, Oxford University Press, London, 1947.

MC Computation by Mathematica.

ME McLachlan, N. W. and Humbert, P., *Formulaire pour le calcul symbolique*, L'Acad. des Sciences de Paris, Fasc. 100, 1950.

MF Morse, M. P. and Feshbach, H., *Methods of Theoretical Physics*, vol. I, McGraw Hill, New York, 1953.

MG Marden, M., *Geometry of Polynomials*, American Mathematical Society, Mathematical Survey 3, Providence, Rhode Island, 1966.

MI McLachlan, N. W. et al., *Supplément au formulaire pour le calcul symbolique*, L'Acad. des Sciences de Paris et al., Fasc. 113, 1950.

ML Mirsky L., *An Introduction to Linear Algebra*, Oxford University Press, London, 1963.

MM MacMillan, W. D., *The Theory of the Potential*, Dover, New York, 1958.

MO Magnus, W. and Oberhettinger, F., *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer–Verlag, Berlin, 1948.

MS Magnus, W., Oberhettinger, F. and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed. Springer–Verlag, Berlin, 1966.

MS2 Mathai, A. M. and Saxens, R. K. , *Generalized Hypergeometrics Functions With Applications in Statistics and Physical Science*, Springer–Verlag, Berlin, 1973.

MT Mitrinović, D. S., *Analytic Inequalities*, Springer–Verlag, Berlin, 1970.

MV Milne, E. A., *Vectorial Mechanics*, Methuen, London, 1948.

MZ Meyer Zur Capellen, W., *Integraltafeln, Sammlung unbestimmer Integrale elementarer Funktionen*, Springer–Verlag, Berlin, 1950.

NA Natanson, I. P., *Konstruktivnaya teoriya funktsiy* (Constructive theory of functions). Gostekhizdat, Moscow and Leningrad, 1949.

NH Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Teubner, Leipzig, 1906.

NO Noble, B., *Applied Linear Algebra*, Prentice Hall, Englewood Cliffs, New Jersey, 1969.

NT Nielsen, N., *Theorie des Integrallogarithmus und verwandter Transcendenten*, Teubner, Leipzig, 1906.

NV Novoselov, S. I., *Obratnyye trigonometricheskiye funktsii, posobive dlya uchiteley* (Inverse trigonometric functions, textbook for students), 3rd ed. Uchpedgiz, Moscow and Leningrad, 1950.

OB Oberhettinger, F., *Tables of Bessel Transforms*, Springer–Verlag, New York: 1972.

PBM Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series*, Gordan and Breach, New York, vols. I (1986), II (1986), III (1990).

PE Peirce, B. O., *A Short Table of Integrals*, 3rd ed. Ginn, Boston, 1929.

SA Sansone, G., *Orthogonal Functions* (Revised English Edition), Interscience, New York, 1959.

SI Sikorskiy, Yu. S., *Elementy teorii ellipticheskikh funktsiy s prilozheniyama k mekhanike* (Elements of theory of elliptic functions with applications to mechanics). ONTI, Moscow and Leningrad, 1936.

SN Sneddon, I. N., *Fourier Transforms*, 1st ed. McGraw Hill, New York, 1951.

SM Smirnov, V. I., *Kurs vysshey matematiki* (A course of higher mathematics), vol. III, Part 2, 4th ed. Gostekhizdat, Moscow and Leningrad, 1949.

ST Strutt, M. J. O., *Lamésche, Mathieusche und verwandte Funktionen in Physik and Technik*, Springer–Verlag, Berlin, 1932.

STR Stratton, J. C., *Phys. Rev A*, **43**(3), pages 1381–1388, 1991.

SU Sneddon, I. N., *The Use of Integral Transforms*, McGraw Hill, New York, 1972.

SZ Szegö, G., *Orthogonal Polynomials*, Revised Edition, Colloquium Publications XXIII, American Mathematical Society, New York, 1959.

TF Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, 2nd ed. Oxford University Press, London, 1948.

TI Timofeyev, A. F. *Integrirovaniye funktsiy* (Integration of functions), part I. GTTI, Moscow and Leningrad, 1933.

VA Varga, R. S., *Matrix Iterative Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.

VL Vladimirov, V. S., *Equations of Mathematical Physics*, Dekker, New York, 1971.

WA Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge University Press, London, 1966.

WH Whittaker, E. T. and Watson, G. N., *Modern Analysis*, 4th ed. Cambridge University Press, London, 1927, part II, 1934.

ZH Zhuravskiy, A. M., *Spravochnik po ellipticheskim funktsiyam* (Reference book on elliptic functions). Izd. Akad. Nauk. U.S.S.R., Moscow and Leningrad, 1941.

ZY Zygmund, A., *Trigonometrical Series*, 2nd ed. Chelsea, New York, 1952.

Classified Supplementary References

(Prepared by Alan Jeffrey for the English language edition.)

General reference books

1. Bromwich, T. I'A., *An Introduction to the Theory of Infinite Series*, 2nd ed., Macmillan, London, 1926 (Reprinted 1942).
2. Carlitz, L., "Generating Functions", 1969, *Fibonacci Quarterly*, 7 (4): 359–393.
3. Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, London, 1935.
4. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Interscience Publishers, New York, 1953.
5. Davis, H. T., *Summation of Series*, Trinity University Press, San Antonio, Texas, 1962.
6. Erdélyi, A. et al. *Higher Transcendental Functions*, vols. I to III, McGraw Hill, New York 1953–1955.
7. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.
8. Fletcher, A., Miller, J. C. P., and Rosenhead, L., *An Index of Mathematical Tables*, 2nd ed., Scientific Computing Service, London, 1962.
9. Gröbner, W. and Hofreiter, N., *Integraltafel*, I, II. Springer–Verlag, Wien and Innsbruck, 1949.
10. Hardy, G. H., Littlewood, J. E., and Pólya, G., *Inequalities*, 2nd ed., Cambridge University Press, London, 1952.
11. Hartley, H. O. and Greenwood, J. A., *Guide to Tables in Mathematical Statistics*, Princeton University Press, Princeton, New Jersey, 1962.
12. Jeffreys, H. and Jeffreys, B. S., *Methods of Mathematical Physics*, Cambridge University Press, London, 1956.
13. Jolley, L. B. W., *Summation of Series*, Dover Publications, New York, 1962.
14. Knopp, K., *Theory and Application of Infinite Series*, Blackie, London, 1946, Hafner, New York, 1948.
15. Lebedev, N. N., *Special Functions and their Applications*, Prentice Hall, Englewood Cliffs, New Jersey, 1965.
16. Magnus, W. and Oberhettinger, F., *Formulas and Theorems for the Special Functions of Mathematical Physics*, Chelsea, New York, 1949.
17. McBride, E. B., *Obtaining Generating Functions*, Springer–Verlag, Berlin, 1971.
18. National Bureau of Standards, *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington, D.C., 1964.
19. Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series*, Vols. 1–5, Gordon and Breach, New York, 1986–1992.
20. Truesdell, C. *A Unified Theory of Special Functions*, Princeton University Press, Princeton, New Jersey, 1948.

21. Vein, R. and Dale, P., *Determinants and Their Applications in Mathematical Physics*, Springer–Verlag, New York, 1999.
22. Whittaker, E. T. and Watson, G. N., *A Course of Modern Analysis*, 4th ed., Cambridge University Press, London, 1940.

Asymptotic expansions

1. De Bruijn, N. G., *Asymptotic Methods in Analysis*, North-Holland Publishing Co., Amsterdam, 1958.
2. Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 3rd ed., Springer, New York, 1971.
3. Copson, E. T., *Asymptotic Expansions*, Cambridge University Press, London, 1965.
4. Erdélyi, A., *Asymptotic Expansions*, Dover Publications, New York, 1956.
5. Ford, W. B., *Studies on Divergent Series and Summability*, Macmillan, New York, 1916.
6. Hardy, G. H., *Divergent Series*, Clarendon Press, Oxford, 1949.
7. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

Bessel functions

1. Bickley, W. G., *Bessel Functions and Formulae*, Cambridge University Press, London, 1953.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vols. I and II. McGraw Hill, New York, 1954.
3. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II. McGraw Hill, New York, 1954.
4. Gray, A., Mathews, G. B. and MacRobert, T. M., *A Treatise on Bessel Functions and Their Applications to Physics*, 2nd ed., Macmillan, 1922.
5. McLachlan, N. W., *Bessel Functions for Engineers*, 2nd ed., Oxford University Press, London, 1955.
6. Luke, Y. L., *Integrals of Bessel Functions*, McGraw Hill, New York, 1962.
7. Petiau, G., *La théorie des fonctions de Bessel*, Centre National de la Recherche Scientifique, Paris, 1955.
8. Relton, F. E., *Applied Bessel Functions*, Blackie, London, 1946.
9. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.
10. Wheelon, A. D., *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, San Francisco, 1968.

Complex analysis

1. Ahlfors, L. V., *Complex Analysis*, 3rd ed., McGraw Hill, New York, 1979.
2. Ahlfors, L. V. and Sario, L., *Riemann Surfaces*, Princeton University Press, Princeton, New Jersey, 1971.
3. Bieberbach, L., *Conformal Mapping*, Chelsea, New York, 1964.
4. Henrici, P., *Applied and Computational Complex Analysis*, 3 vols, Wiley, New York, 1988, 1991, 1977.
5. Hille, E., *Analytic Function Theory*, 2 vols. 2nd ed., Chelsea, New York, 1990, 1987.
6. Kober, H., *Dictionary of Conformal Representations*, Dover Publications, New York, 1952.
7. Titchmarsh, E. C., *The Theory of Functions*, 2nd ed., Oxford University Press, London, 1939. (Reprinted 1975).

Error function and Fresnel integrals

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1953.
2. Erdélyi, A. et al., *Tables of Integral Transforms*, vol. I, McGraw Hill, New York, 1954.
3. Slater, L. J., *Confluent Hypergeometric Functions*, Cambridge University Press, London, 1960.
4. Tricomi, F. G., *Funzioni ipergeometriche confluenti*, Edizioni Cremonese, Turan, Italy, 1954.
5. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

Exponential integrals, gamma function and related functions

1. Artin, E., *The Gamma Function*, Holt, Rinehart, and Winston, New York, 1964.
2. Busbridge, I. W., *The Mathematics of Radiative Transfer*, Cambridge University Press, London, 1960.
3. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1953.
4. Erdélyi, A. et al., *Tables of Integral Transforms*, vols. I and II, McGraw Hill, New York, 1954.
5. Hastings, Jr., C., *Approximations for Digital Computers*, Princeton University Press, Princeton, New Jersey, 1955.
6. Kourganoff, V., *Basic Methods in Transfer Problems*, Oxford University Press, London, 1952.
7. Lösch, F. and Schoblik, F., *Die Fakultät (Gammafunktion) und verwandte Funktionen*, Teubner, Leipzig, 1951.
8. Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Teubner, Leipzig, 1906.
9. Oberhettinger, F., *Tabellen zur Fourier Transformation*, Springer–Verlag, Berlin, 1957.

Hypergeometric and confluent hypergeometric functions

1. Appell, P., *Sur les Fonctions Hypergéometriques de Plusieures Variables*, Gauthier-Villars, Paris, 1926.
2. Bailey, W. N., *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1935.
3. Buchholz, H., *Die konfluente hypergeometrische Funktion*, Springer–Verlag, Berlin, 1953.
4. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. I, McGraw Hill, New York, 1953.
5. Jeffreys, H. and Jeffreys, B. S., *Methods of Mathematical Physics*, Cambridge University Press, London, 1956.
6. Klein, F., *Vorlesungen über die hypergeometrische Funktion*, Springer–Verlag, Berlin, 1933.
7. Nörlund, N. E., *Sur les Fonctions Hypergéometriques d'Ordre Superior*, North–Holland, Copenhagen, 1956.
8. Slater, L. J., *Confluent Hypergeometric Functions*, Cambridge University Press, London, 1960.
9. Slater, L. J. *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.
10. Snow, C., *The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, 2nd ed., National Bureau of Standards, Washington, D.C., 1952.
11. Swanson, C. A. and Erdélyi, A., *Asymptotic Forms of Confluent Hypergeometric Functions*, Memoir 25, American Mathematical Society, Providence, Rhode Island, 1957.
12. Tricomi, F. G., *Lezioni sulla funzioni ipergeometriche confluenti*, Gheroni, Torino, 1952.

Integral transforms

1. Bochner, S., *Vorlesungen über Fouriersche Integrale*, Akad. Verlag, Leipzig, 1932. Reprint Chelsea, New York, 1948.

2. Bochner, S. and Chandrasekharan, K., *Fourier Transforms*, Princeton University Press, Princeton, New Jersey, 1949.
3. Campbell, G. and Foster, R., *Fourier Integrals for Practical Applications*, Van Nostrand, New York, 1948.
4. Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, London, 1948.
5. Doetsch, G., *Theorie und Anwendung der Laplace-Transformation*, Springer–Verlag, Berlin, 1937. (Reprinted by Dover Publications, New York, 1943)
6. Doetsch, G., *Theory and Application of the Laplace-Transform*, Chelsea, New York, 1965.
7. Doetsch, G., *Handbuch der Physik, Mathematische Methoden II*, 1st ed., Springer–Verlag, Berlin, 1955.
8. Doetsch, G., *Guide to the Applications of the Laplace and Z-Transforms*, 2nd ed., Van Nostrand-Reinhold, London, 1971.
9. Doetsch, G., *Handbuch der Laplace-Transformation*, Vols. I–IV, Birkhäuser Verlag, Basel, 1950–56.
10. Doetsch, G., Kniess, H., and Voelker, D., Tabellen zur Laplace- Transformation, Springer–Verlag, Berlin, 1947.
11. Erdélyi, A., *Operational Calculus and Generalized Functions*, Holt, Rinehart and Winston, New York, 1962.
12. Exton, H., *Multiple Hypergeometric Functions and Applications,* Horwood, Chichester, 1976.
13. Exton, H., *Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs*, Horwood, Chichester, 1978.
14. Hirschmann, J. J. and Widder, D. V., *The Convolution Transformation*, Princeton University Press, Princeton, New Jersey, 1955.
15. Marichev, O. I., *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood Ltd., Chichester (1982).
16. Oberhettinger, F., *Tabellen zur Fourier Transformation,* Springer–Verlag, Berlin (1957).
17. Oberhettinger, F., *Tables of Bessel Transforms*, Springer–Verlag, New York (1972).
18. Oberhettinger, F., *Fourier Expansions: A Collection of Formulas,* Academic Press, New York, 1973.
19. Oberhettinger, F., *Fourier Transforms of Distributions and Their Inverses*, Academic Press, New York, 1973.
20. Oberhettinger, F., *Tables of Mellin Transforms*, Springer–Verlag, Berlin, 1974.
21. Oberhettinger, F. and Badii, L., *Tables of Laplace Transforms,* Springer–Verlag, Berlin, 1973.
22. Oberhettinger, F. and Higgins, T. P., *Tables of Lebedev, Mehler and Generalized Mehler Transforms*, Math. Note No. 246, Boeing Scientific Research Laboratories, Seattle, Wash., 1961.
23. Roberts, G. E. and Kaufman, H., *Table of Laplace Transforms,* McAinsh, Toronto, 1966.
24. Sneddon, I. N., *Fourier Transforms*, McGraw Hill, New York, 1951.
25. Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London, 1937.
26. Van der Pol, B. and Bremmer, H., *Operational Calculus Based on the Two Sided Laplace Transformation*, Cambridge University Press, London, 1950.
27. Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton, New Jersey, 1941.
28. Wiener, N., *The Fourier Integral and Certain of its Applications*, Dover Publications, New York, 1951.

Jacobian and Weierstrass elliptic functions and related functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1953.
2. Byrd, P. F. and Friedman, M. D., *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer–Verlag, Berlin, 1954.
3. Graeser, E., *Einführung in die Theorie der Elliptischen Funktionen und deren Anwendungen*, Oldenbourg, Munich, 1950.
4. Hancock, H., *Lectures on the Theory of Elliptic Functions*, vol. I, Dover Publications, New York, 1958.

5. Neville, E. H., *Jacobian Elliptic Functions*, Oxford University Press, London, 1944 (2nd ed. 1951).
6. Oberhettinger, F. and Magnus, W., *Anwendungen der Elliptischen Funktionen in Physik und Technik*, Springer–Verlag, Berlin, 1949.
7. Roberts, W. R. W., *Elliptic and Hyperelliptic Integrals and Allied Theory*, Cambridge University Press, London, 1938.
8. Tannery, J. and Molk, J., *Eléments de la Théorie des Fonctions Elliptiques*, 4 volumes. Gauthier-Villars, Paris, 1893–1902.
9. Tricomi, F. G., *Elliptische Funktionen*, Akad. Verlag, Leipzig, 1948.

Legendre and related functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. I, McGraw Hill, New York, 1953.
2. Helfenstein, H., *Ueber eine Spezielle Lamésche Differentialgleichung*, Brunner and Bodmer, Zurich, 1950 (Bibliography).
3. Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, London, 1931. Reprinted by Chelsea, New York, 1955.
4. Lense, J., *Kugelfunktionen*, Geest and Portig, Leipzig, 1950.
5. MacRobert, T. M., *Spherical Harmonics: An Elementary Treatise on Harmonic Functions with Applications*, Methuen, England, 1927. (Revised ed. 1947; reprinted Dover Publications, New York, 1948).
6. Snow, C., *The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, 2nd ed., National Bureau of Standards, Washington, D.C., 1952.
7. Stratton, J. A., Morse, P. M., Chu, L. J. and Hunter, R. A., *Elliptic Cylinder and Spheroidal Wave Functions Including Tables of Separation Constants and Coefficients*, Wiley, New York, 1941.

Mathieu functions

1. Erdélyi, A., *Higher Transcendental Functions*, vol. III, McGraw Hill, New York, 1955.
2. McLachlan, N. W., *Theory and Application of Mathieu Functions*, Oxford University Press, London, 1947.
3. Meixner, J. and Schäfke, F. W., *Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf Physikalische und Technische Probleme*, Springer–Verlag, Heidelberg, 1954.
4. Strutt, M. J. O., *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Ergeb. Math. Grenzgeb. *1*, 199–323 (1932). Reprint Edwards Bros., Ann Arbor, Michigan, 1944.

Orthogonal polynomials and functions

1. *Bibliography on Orthogonal Polynomials*, Bulletin of National Research Council No. 103, Washington, D.C., 1940.
2. Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.
3. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.
4. Kaczmarz, St. and Steinhaus, H., *Theorie der Orthogonalreihen*, Chelsea, New York, 1951.
5. Lorentz, G. G., *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.
6. Sansone, G., *Orthogonal Functions*, Interscience, New York, 1959.
7. Shohat, J. A. and Tamarkin, J. D., *The Problem of Moments*, American Mathematical Society, Providence, Rhode Island, 1943.

8. Szegö, G., *Orthogonal Polynomials*, American Mathematical Society Colloquim Pub. No. 23, Providence, Rhode Island, 1959.
9. Titchmarsh, E. C., *Eigenfunction Expansions Associated with Second Order Differential Equations*, Oxford University Press, London, part I (1946), part II (1958).
10. Tricomi, F. G., *Vorlesungen über Orthogonalreihen*, Springer–Verlag, Berlin, 1955.

Parabolic cylinder functions

1. Buchholz, H., *Die konfluente hypergeometrische Funktion*, Springer–Verlag, Berlin, 1953.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.

Probability function

1. Cramer, H., *Mathematical Methods of Statistics*, Princeton University Press, Princeton, New Jersey, 1951.
2. Erdélyi, A. et al., *Higher Transcendental Functions*, vols. I, II, and III. McGraw Hill, New York, 1953–1955.
3. Kendall, M. G. and Stuart, A., *The Advanced Theory of Statistics,* vol. I: *Distribution Theory*, Griffin, London, 1958.

Riemann zeta function

1. Titchmarsh, E. C., *The Zeta Function of Riemann*, Cambridge University Press, London, 1930.
2. Titchmarsh, E. C., *The Theory of the Riemann Zeta Function*, Oxford University Press, London, 1951.

Struve functions

1. Erdélyi, A. et al., *Higher Transcendental Functions*, vol. II, McGraw Hill, New York, 1954.
2. Gray, A., Mathews, G. B. and MacRobert, T. M., *A Treatise on Bessel Functions and Their Applications to Physics*, 2nd ed., Macmillan, London, 1922.
3. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London, 1958.

Index of Functions and Constants

This index shows the occurrence of functions and constants used in the expressions within the text. The numbers refer to pages on which the function or constant appears.

Symbols

! and !! . . . . . *see* factorials
$\mathfrak{S}_n^{(m)}$ . . . . . *see* Stirling numbers, second kind
$\mathfrak{Z}_n^{(m)}$ . . . . . *see* Bessel functions, $\mathfrak{Z}$
∇ . . . . . xliv, 767, 1050–1053, 1055–1057
β . . . . . *see* beta function
$\delta(x)$ . . . . . *see* delta function
δ_{ij} . . . . . *see* Kronecker delta
γ and Γ . . . . . *see* gamma functions
λ function . . . . . xxxix, 1043
μ function . . . . . xxxix, 1043
ν function . . . . . xxxix, 1043
Φ . . . . . *see* Lerch function and hypergeometric functions, confluent
Ψ . . . . . *see* Euler function and hypergeometric functions, confluent
Θ function . . . . . *see* Jacobi theta function
$\wp(x)$ . . . . . *see* Weierstrass function
ξ function . . . . . xxxix, 1040
$||\cdot||$ 1081–1083, 1085, 1086, 1095, 1096, 1120
$||\cdot||_1$ 1081–1083
$||\cdot||_2$ 1081–1083
$||\cdot||_\infty$ 1081, 1083

A

Airy function (Ai) xxxviii
am function . . . . . xxxix, 625, 866, 867
Anger function (**J**) . . . . . xli, 339, 352, 371, 384, 421, 423, 444–446, 670, 671, 946, 948, 949, 992
arccos function . . xxxi, xxxii, 23, 56–60, 64, 99, 135, 139, 173, 179–183, 210, 211, 241–244, 263, 272, 279, 293–296, 307, 312, 313, 318, 393, 454, 511, 558, 562, 589, 600, 601, 607, 610, 624, 695, 730, 742, 767, 768, 861, 890, 936, 993, 994
arccosec function 242, 244, 728
arccosh function . . . . . xxxi, xxxii, 56, 60–62, 64, 97, 126, 133, 135, 137, 138, 241, 382, 386, 511, 532, 621, 624, 729, 768
arccot function . . . . . xxxi, xxxii, 51, 56–58, 64, 242, 244, 245, 263, 274, 279, 325, 326, 499, 556, 561, 599, 601–607, 625, 767, 892, 1115, 1133
arccoth function . . . . . xxxi, xxxii, 56, 60, 62, 75, 131–134, 172, 177, 178, 241, 644, 647
arcosech function 62
arcsec function 61, 66, 99, 242, 244
arcsech function 62
arcsin function . . . . . xxxi, xxxii, 27, 56–61, 64, 66, 94, 97, 99, 116, 124–126, 133–138, 173, 179–182, 186–193, 195, 196, 198, 202–213, 225, 241–245, 254, 263, 265, 272, 275, 279, 297, 307, 382, 558, 563, 566–568, 583, 588, 589, 591, 600, 601, 604, 605, 607, 621, 622, 624, 625, 631, 632, 637, 638, 662, 668, 700–702, 713, 717, 718, 727, 728, 743, 744, 748, 755, 767, 768, 793, 815, 860, 914, 989, 1007
arcsinh function . . xxxi, xxxii, 54, 56, 60–62, 64, 94, 97, 126, 133, 135, 139, 240, 241, 371, 382, 386, 448, 588, 624, 637, 638, 1007
arctan function . . . . . xxxi, xxxii, 27, 30, 49, 51, 52, 55–61, 63–67, 71–77, 79, 83–85, 87, 90, 97, 103, 104, 106, 114, 116, 117, 119, 126, 128–133, 147, 148, 171–175, 177, 178, 190, 205, 239, 240, 242–245, 254, 263, 272, 274, 279, 294, 295, 307, 317, 324, 329, 346, 363, 372, 373, 381, 393, 409, 453, 493–501, 507, 509, 516–521, 524, 556, 557, 560, 563–565, 593, 599–607, 612, 622–624, 631, 632, 637, 639, 640, 643, 644, 646–649, 748, 763, 860, 884, 885, 890–893, 898, 923, 1007, 1036, 1113, 1125, 1128, 1132
arctanh function . . . . . xxxi, xxxii, 56, 60–62, 64, 75, 79, 97, 125–128, 131, 132, 134, 172, 177, 178, 241, 621–623, 977

associated Legendre functions
first kind (P) . . . . xli, 326, 327, 333, 336, 374, 406, 407, 486, 660, 661, 665, 666, 686, 699, 703, 705, 706, 727, 755, 760, 761, 767–789, 792, 793, 797, 806–808, 810, 823, 831, 839, 840, 848, 958–972, 974, 975, 980–983, 992
second kind (Q) . . . . . xli, 333, 336, 374, 383, 407, 511, 661, 662, 666, 685, 686, 700, 702, 703, 705, 727, 769–781, 783–785, 791, 795, 823, 831, 839, 840, 958–973, 981

B

$B_n(x)$ . . . . . *see* Bernoulli polynomials
$B(x)$ . . . . . *see* Beta function
Bateman function (k) . . . . . xli, 349, 1023
bei(z) . . . . . *see* Thomson functions
ber(z) . . . . . *see* Thomson functions
Bernouli number (B_n) . . xxxii, xxxiii, xxxix, 1–3, 8, 26, 42, 43, 46, 55, 145, 146, 148, 221–224, 353, 356, 376, 379–382, 387, 472, 550–552, 554, 560, 567, 574, 580, 581, 587, 589, 591, 764–766, 899, 906, 936, 1038–1045
Bernouli number (B_n^*) . . . . . xxxiii, 62
Bernoulli polynomial ($B_n(x)$) . . . xxxii, xxxiii, xxxix, 46, 1037, 1041, 1042, 1045
Bessel functions
$I_n(x)$ . . . . . xxxviii, xli, 13, 320, 339, 340, 345, 347, 350, 351, 368, 382, 385, 419, 435, 441, 444, 445, 470, 477–480, 491, 494, 496, 507, 513–515, 524, 595, 605, 616, 617, 660, 661, 663–681, 684–687, 689, 691, 692, 695–699, 702–716, 719, 720, 722–725, 727, 729, 730, 735, 736, 738, 741, 743, 745–747, 751–760, 762, 779–781, 783–787, 789, 794, 797, 800, 820, 832, 833, 838, 846, 847, 901, 911, 916, 917, 919, 920, 925–933, 943, 954, 1002, 1027, 1028, 1116, 1117, 1123, 1125, 1128
$J_n(x)$ . . . . . xxxvii, xxxviii, xli, 13, 339, 350, 352, 371, 384, 385, 417–421, 423, 435, 440–443, 445, 446, 477–480, 482, 483, 491, 492, 507, 514, 515, 522, 524, 525, 578, 629, 642, 653, 659–694, 696–753, 756–759, 761–763, 767, 768, 777, 779, 780, 782–787, 792–794, 797–799, 802, 803, 808, 811, 812, 818–820, 830–838, 841, 845–848, 854, 855, 900, 910–914, 916, 918–931, 933–950, 954–957, 963, 964, 972, 992, 1000, 1002–1004, 1017, 1023–1025, 1028, 1034, 1116, 1117, 1124–1129, 1132, 1133
Bessel functions (*continued*)
$K_n(x)$ . . . . . xxxviii, xli, 2, 337, 339, 345–348, 350–353, 364–368, 370, 371, 384, 385, 417, 419, 435, 442, 444, 445, 477–482, 490, 491, 504, 505, 507, 514, 515, 529, 573, 575, 576, 578, 595, 638, 645, 648, 653, 654, 657, 660–682, 684–696, 698–700, 702–716, 718–724, 726, 727, 729–732, 735, 736, 738, 740, 742, 745–753, 756–759, 761, 768, 776–787, 789, 794, 800, 803, 811, 814, 817–820, 828, 832–834, 837, 838, 841, 845, 846, 848, 854, 855, 900, 911, 917–920, 923, 925–933, 939, 942, 945, 955, 957, 1027, 1028, 1035, 1123–1126, 1129, 1132, 1133
$N_n(x)$ . . . . . xxxvii, 910
$Y_n(x)$ . . . . . xxxvii, xlii, 338, 339, 345, 346, 351–353, 371, 384, 385, 419, 435, 440, 442, 443, 445, 446, 477–480, 482, 483, 492, 507, 514, 515, 573, 578, 647, 654, 659–664, 666–682, 684–693, 695–700, 705–709, 711, 714–720, 722–724, 726, 728, 729, 732–736, 738, 740–742, 745, 747–752, 756–759, 761, 767, 768, 777, 779, 782, 783, 785, 787, 793, 794, 799, 817–819, 832, 833, 835–837, 847, 848, 854, 855, 910, 911, 914, 918–920, 922, 923, 925–931, 933, 937–939, 941–943, 945, 946, 949, 954–957, 975, 1025, 1034, 1124, 1126, 1132
$Z_n(x)$ . . . . . xlii, 483, 629, 630, 767, 910, 911, 926, 931–933, 937, 940, 941, 975
$\mathfrak{Z}_n(x)$ . . . . . xlii, 629, 630
Hankel . . . . . *see* Hankel function
beta function (β) 319, 322, 324, 325, 334, 371, 375, 383, 395, 396, 403, 432, 471, 553, 558, 562–564, 573, 586, 602, 904, 906, 907
Beta function (B) . . . . . xxxix, 6, 129, 175, 315–318, 320, 322–330, 332, 333, 335, 338, 347–349, 351, 359, 360, 364, 368, 370, 372, 374, 375, 382, 383, 395–397, 399–402, 407, 408, 411–413, 440, 442, 444, 460, 469, 472, 485, 486, 490, 512, 539–543, 548, 553, 559, 585, 705, 749, 754, 760, 801, 810, 813, 814, 816, 821, 894, 895, 908–910, 991, 1005, 1023, 1025
Bi function xxxviii
bilateral z transform 1135–1137, 1139, 1159
binomial coefficients . . . . . xxxiii, xliii, 1–6, 11, 12, 15, 22, 23, 25, 31, 33, 46, 84, 86, 88, 89, 100–103, 106, 110, 111, 114, 115, 119, 120, 140, 143, 148, 153, 157, 161, 173, 215, 220, 221, 223, 228, 232–238, 241, 316, 326, 329, 354, 357, 361, 362, 386, 393, 394, 397–402, 416, 431, 437, 459–461, 466, 469, 470, 478, 488, 498, 499, 504, 505, 545, 546, 548, 549, 552, 612, 808, 910, 934, 993, 994, 996, 998, 1003, 1023, 1030, 1040, 1041, 1044, 1046, 1047

C

$C_n(x)$ . . . . . *see* Gegenbauer polynomials
$C(x)$ . . *see* Fresnel sine integral and Young function
Catalan constant ($\boldsymbol{G}$) . . . . . xxxii, xl, 9, 375, 380, 433, 434, 448, 449, 452, 453, 470–472, 530–534, 536–538, 556, 558, 560, 563, 564, 580, 600–603, 632, 633, 1046
cd function xxxiii
Ce function . . . . . xxxviii, xl, 763–766, 953, 954
ce function . . . . . xxxviii, xl, 763–767, 951–957
Chebyshev polynomials
  first kind ($T_n(x)$) . . . . . xli, 448, 667, 718, 790, 800–803, 983, 988, 993–996, 999, 1131
  second kind ($U_n(x)$) . . . . . xxxvii, xli, 800–802, 994–996
chi function . . . . . xxxvi, xl, 142–144, 644, 645, 886
ci function . . . . xxxv, xl, 219–221, 340–344, 423–426, 436, 437, 447, 495, 505, 506, 528, 529, 571, 572, 578, 581, 594, 595, 599, 605, 628, 629, 638–645, 647, 656, 658, 748, 762, 886, 887, 1115
Cin function xxxvi
cn function . . . . . xxxiii, xxxiv, xl, 623–626, 714, 866–873, 875, 879, 880
complex conjugate . . . . . xliii, 293, 341, 342, 421, 422, 424–426, 511, 528, 529, 927, 931, 933, 1060, 1061, 1070, 1071, 1082, 1087, 1136, 1139
confluent hypergeometric functions . . . . . *see* hypergeometric functions, confluent
constants
  Catalan . . . . . *see* Catalan constant
  Euler . . . . . *see* Euler constant
cos function . . . . . xxix, xxxvi, xxxviii, 4, 13, 19, 20, 26–39, 41–52, 54–56, 64, 74–76, 78, 79, 126, 147, 151–237, 249, 250, 253, 254, 317, 318, 320, 322, 323, 326–329, 331–333, 338–345, 353, 358, 372, 373, 377–383, 385, 388–534, 537, 540–545, 550, 551, 561, 563, 565, 567–573, 576, 578, 579, 581–601, 604–608, 610–612, 616, 621–623, 628, 629, 631, 632, 634–644, 647–650, 652, 655, 656, 658, 659, 662–664, 667, 669, 671, 673–675, 677–681, 686, 688, 689, 691, 692, 695, 703–707, 709, 711, 713, 715, 717–748, 750–752, 754, 756–770, 779–782, 784, 785, 787–789, 792–794, 797–800, 802, 806, 808, 811, 812, 815, 817, 818, 823, 825, 829–831, 833, 836, 837, 844, 845, 847–849, 854, 862–869, 877–880, 882–886, 888–894, 896, 898–900, 904, 906–910, 912–918, 920, 922, 924, 925, 928, 930, 933–943, 945–951, 953, 954, 958–964, 966–981, 984–993, 997, 998, 1000, 1003, 1005–1007, 1023, 1025, 1026, 1029, 1030, 1037, 1038, 1041, 1042, 1044, 1053, 1057, 1060, 1061, 1066, 1067, 1090, 1110–1113, 1115, 1117, 1119–1122, 1124–1128, 1131, 1132, 1137, 1140
cosec function . . xxvii, xxix, 36–39, 43, 44, 49, 50, 64, 113–115, 126, 155, 156, 160, 225, 254, 315–323, 325, 327–332, 334, 335, 339, 349, 352, 354, 355, 358, 373, 381–383, 387, 388, 400, 401, 403–408, 410–414, 421, 422, 434, 437–439, 446, 453, 454, 471, 479, 480, 484, 496, 508, 509, 529, 540, 541, 547, 551, 565, 586, 588, 593, 600, 602, 604, 658, 659, 663, 664, 669, 670, 677–682, 689–692, 695, 697, 700, 713, 715, 717, 718, 723, 724, 755, 760, 761, 780, 787, 788, 844, 853, 869, 900, 921, 927, 929, 930, 932, 964, 967, 1131, 1132
cosech function 27, 43, 113–115, 126, 387, 501, 513, 634, 715, 751, 1111, 1124, 1133
cosh function . . . . . xxviii, xxxvi, 27–36, 38, 42, 43, 45, 47, 48, 50–52, 64, 110–151, 231–237, 251, 323, 338, 339, 371–390, 407, 419, 425, 432, 433, 439, 448, 451, 452, 454, 455, 468, 482, 484, 485, 502, 504, 509–527, 568, 570, 573, 578–581, 595, 596, 605, 610, 611, 621, 622, 634, 643–645, 648, 686, 702, 705, 710, 713–717, 722, 723, 729, 735, 747, 750–752, 755, 760, 763–767, 778, 781, 787–789, 791, 792, 802, 806, 843, 844, 886, 888, 896, 906, 908, 909, 912–917, 921, 922, 952–958, 960–963, 967–969, 973, 977, 980–982, 998, 1006, 1043, 1112–1115, 1119, 1128, 1129, 1140
cosine integral (Ci) . . . . xxxv, xxxvi, 886, 930, 1115, 1122, 1126
cot function . . xxvii, xxviii, 28, 36, 37, 39, 42, 44, 46, 49, 52, 56, 64, 147, 157–161, 168, 174, 176–178, 185, 188, 189, 194, 195, 204–207, 213, 222–225, 229, 230, 274, 318–323, 330, 331, 334, 354, 355, 358, 372, 379, 381–384, 388, 395, 396, 400, 401, 403–406, 411–414, 422, 434, 448, 454, 455, 472, 484, 485, 492, 493, 496, 506, 509, 511, 532–534, 540, 542, 543, 546, 558, 559, 565, 567, 568, 587, 588, 590, 594, 595, 604, 606, 611, 623, 631, 632, 637, 663, 664, 669, 670, 676, 678–681, 690–692, 697, 700, 717, 723, 753, 754, 849, 867, 868, 876, 882, 903–905, 912, 914, 915, 922, 927, 929, 930, 932, 954, 967–969, 971, 979, 989, 1013, 1089, 1090, 1120
coth function 28, 39, 40, 42, 44, 64, 110, 116, 118–120, 124, 130–134, 138, 145–148, 381, 384, 386–388, 485, 489, 502, 509–513, 520, 523, 580, 582, 595, 622, 634, 715, 716, 876, 921, 922, 956, 957, 967, 981, 1029, 1110, 1124
cs function xxxiii
curl 1050–1053, 1057, 1058
cylinder function . . . . *see* parabolic cylinder function

D

dc function xxxiii
degrees 263–265
delta function ($\delta(x)$) 661, 1115, 1118–1120
determinant 1070, 1075–1077, 1084–1086, 1096, 1100
dilogarithm (L_2) 642
div 1050, 1051, 1053, 1055, 1057, 1058
dn function . . . . . xxxiii, xxxiv, xl, 623–626, 714, 866–873, 875, 879, 880
double factorials . . . . . *see* factorial, double
ds function xxxiii

E

$E_n(x)$ . . . . . *see* Euler polynomials
$E(x)$ . . . . . *see* MacRobert function
elliptic functions 859
  D . . . . . xl, 860
    complete . . . . . xl, 860, 861
  E . . . . . xxxi, xl, 60, 135–139, 179–183, 185–195, 197, 198, 200, 202–214, 225, 255–262, 264–274, 280–283, 285–296, 300–315, 410, 604, 606, 621–623, 625, 632, 672, 777, 814, 852, 853, 855, 857, 860, 862–864, 880
    complete . . . . . xxxiv, xl, 313, 394, 408–410, 472–475, 562, 592, 593, 596–600, 619–622, 632, 633, 671, 696, 704, 714, 729, 860–865, 869, 880, 990
  F . . . . . xxxi, 60, 61, 134–139, 179–183, 186–195, 197–200, 202–214, 225, 250, 251, 254–315, 407, 408, 410, 411, 470, 486, 563, 568, 602, 604, 606, 621–623, 631, 632, 654, 660, 661, 683, 684, 687, 699, 700, 704, 707, 730, 731, 733, 734, 758, 804, 809–820, 822, 823, 843, 849, 856, 860–864, 889, 918, 959, 963, 964, 968, 970, 971, 974–976, 979, 981, 982, 984, 986, 991, 994
  K 672
    complete . . . . . xxxiv, xli, 274, 275, 313, 394, 408–410, 472–475, 538, 539, 562, 567, 585, 588, 592, 593, 596–600, 611, 612, 619, 620, 622, 631–633, 696, 704, 713, 714, 719, 720, 729, 756, 788, 860–870, 875, 879–881, 990
  Π . . . . . xxxi, xxxiv, xxxix, 51, 52, 135, 137, 138, 180, 181, 183, 197, 198, 200–202, 204, 205, 210, 212–214, 255, 262, 263, 265, 276–279, 284, 285, 293, 295–300, 605, 620, 623, 632, 860, 880
erf . . . . . xxxvi, xl, 107–109, 336, 365, 635, 646, 887–889, 1115, 1116
erfc . . . . . xxxvi, xl, 887, 888, 890, 891, 1108, 1110, 1115, 1116
error functions . . . . . *see* erf and erfc
Euler constant ($\boldsymbol{C}$) . . . . xxxii, xxxv, xxxvi, xl, 3, 15, 321, 323, 330–332, 334, 335, 359, 361, 362, 364, 367, 369–371, 411, 412, 422, 447, 476, 478, 483, 484, 501, 534, 535, 538, 541, 553–555, 558, 559, 570–574, 578–581, 585, 586, 588, 593, 594, 599, 605, 628, 639, 644, 656, 658, 747, 748, 883, 884, 886, 894–896, 898, 903–906, 911, 919, 937–939, 944, 1037, 1038, 1046, 1125
Euler function (ψ) . . . . . xxxix, 318, 321, 323, 330–332, 334, 335, 356, 359, 360, 364, 369, 382, 387, 388, 400, 403, 405, 411–414, 466, 486, 490, 501, 509, 510, 523, 535–538, 540–543, 553–555, 558, 559, 562, 570–574, 576–579, 585, 586, 588, 594, 595, 607, 617, 658, 659, 747, 769, 770, 820, 842, 902–907, 911, 919, 929, 969, 1011–1013
Euler number (E_n) . . . . . xxxii, xxxiii, xl, 8, 43, 145, 146, 221, 223, 376, 379, 380, 533, 550, 580, 1043–1045
Euler polynomial ($E_n(x)$) . . xxxii, xxxiii, 1044, 1045
exponential function (exp) 27, 108, 109, 143, 144, 215, 335–340, 345–349, 352, 364–371, 383–385, 390, 426, 428–430, 439, 440, 480, 482, 484–486, 488, 489, 492–498, 501–509, 513, 516, 521, 523, 524, 526, 574–576, 581, 617, 639, 645–649, 651, 652, 655–658, 693, 697–699, 706–710, 712, 713, 722, 723, 748–753, 759, 760, 768, 778, 781, 785, 791, 805, 810, 811, 815, 826, 828, 829, 831, 834, 837, 841–844, 848–850, 855, 876, 877, 879, 880, 886–888, 890–893, 895, 896, 913, 915–917, 920–923, 928, 933, 935, 967, 997, 1002, 1024–1026, 1029, 1030, 1066, 1090, 1094, 1096, 1098, 1099, 1105, 1110, 1119, 1123, 1124, 1128
exponential integral ($E_n(x)$) xxxv
exponential integral ($\mathrm{Ei}(x)$) . . . . . xxxv, xl, 107, 109, 143, 144, 150, 151, 338, 340–344, 361, 370, 375, 386, 421, 422, 424–426, 432, 461, 468, 483, 484, 492, 495, 527–530, 535, 553, 555, 571–573, 577, 578, 594, 595, 605–607, 627, 628, 638–647, 649, 656, 658, 748, 883–887, 900, 902, 931, 1115, 1123, 1125

F

$\mathcal{F}$ . . . . . *see* Fourier transform
$F(x)$ . . . . . *see* hypergeometric function

factorial
! . . . . . xxxii–xxxiv, xxxvii, xliii, 2–5, 8, 12, 13, 18, 19, 22, 23, 25–27, 33, 34, 42–44, 46, 49–51, 54, 55, 60–62, 66, 68, 77, 79, 85, 90, 92, 94, 106–109, 114, 115, 127, 128, 140–143, 145, 146, 148, 152, 155, 156, 163–167, 174, 215–219, 221–224, 228, 230, 241, 315, 316, 321, 323, 325, 326, 328–330, 332, 336, 340–342, 344, 346, 353, 354, 359, 361, 364, 365, 367, 379, 381, 382, 386, 389, 393, 396–398, 400–402, 405, 408, 417, 419, 430, 431, 436, 437, 441, 444, 466, 469, 470, 472, 486–488, 495–499, 502–505, 508, 510, 512, 517, 522, 528, 530, 531, 533, 535, 550–552, 555, 559, 560, 567, 572–578, 580, 585–587, 591, 593, 601, 607, 612, 613, 616, 619, 627, 635, 636, 660, 672, 677, 687, 688, 698, 704, 705, 707, 709, 725, 769–771, 789–793, 795–801, 803–812, 840–842, 844, 860–862, 866, 867, 869, 884–886, 889, 892, 893, 895–897, 899–901, 904, 907, 909–911, 913, 918–921, 923–925, 929, 930, 934–936, 940, 941, 944, 949, 950, 961, 962, 968, 973–975, 977, 979, 982–984, 986, 988–993, 995, 997–1002, 1005, 1010–1013, 1018, 1022, 1023, 1025–1027, 1031, 1034, 1038–1044, 1046, 1047, 1074, 1108, 1109, 1120
double (!!) . . . . . xliii, 23, 77, 79, 94, 110, 111, 113, 114, 127, 128, 146, 152, 155, 156, 174, 222, 226, 245, 250, 316, 319, 324–326, 336, 345, 346, 363, 364, 367, 369, 395–398, 401, 402, 405, 408–410, 420, 430, 435, 459, 460, 466, 467, 469, 470, 472, 478, 488, 531, 538, 539, 543, 551, 573–576, 585, 586, 601, 607, 616, 793, 805, 860–862, 889, 897, 909, 923, 934, 940, 974, 977, 982, 984, 986, 988–990, 992, 994, 997
Fe function . . . . . xl, 953–955
fe function . . . . . xl, 953–955
Fek function . . . . . xl, 955, 957
Fey function . . . . . xl, 765, 767, 955, 956
Fourier transform . . . . . xliv, 1117, 1118, 1121, 1122, 1129
cosine . . . . . xliv, 1121, 1122, 1126–1129
sine . . . . . xliv, 1121–1125, 1129
Fresnel integral
cosine (C) . . . . . xxxvi, xl, 171, 225, 226, 415, 434, 475, 476, 492, 629, 641, 649, 650, 659, 887–890, 935, 1126
sine (S) . . xxxvi, xli, 170, 171, 225, 226, 415, 434, 475, 476, 492, 629, 641, 649, 650, 659, 887–890, 935, 1057, 1124

G

$G^{pq}_{nm}(x \mid {}^{a_1,\dots}_{b_1,\dots})$ . . . . . *see* Meijer G function
gamma function
$\Gamma(x)$ . . . . . xxxiv, xxxvii–xxxix, xliii, 6, 9, 68, 107–109, 121–123, 163–167, 264, 296, 317, 318, 321, 322, 324, 326–333, 336–338, 346–355, 358–361, 365–368, 370, 374, 376, 377, 379–384, 386–390, 395, 396, 398–401, 406, 407, 411, 414, 419, 421, 423, 436–445, 459, 460, 462, 466, 472, 479, 486, 491, 492, 497–499, 503, 506, 509, 511, 512, 515, 521–523, 529, 535, 538, 539, 545–548, 550–553, 555, 560, 566–568, 570–574, 576–580, 585, 588, 594, 595, 602, 604, 605, 613–617, 632–640, 645, 646, 648–663, 665–668, 670, 672–688, 690–694, 696–700, 702–712, 715–717, 724–727, 730–734, 736–738, 741, 744–749, 752–761, 769–789, 791–801, 803–853, 856, 864, 889, 892–902, 904, 909, 910, 912–921, 923, 929, 940–949, 959–964, 966–975, 978, 979, 981–983, 991–993, 995, 999, 1002, 1003, 1005, 1008, 1009, 1013, 1019–1030, 1032, 1033, 1035–1040, 1043, 1046, 1048, 1056, 1108, 1110, 1113, 1116, 1117, 1119, 1120, 1122, 1123, 1126, 1127, 1130–1133
$\gamma(x)$ . . 215, 335, 338, 340, 346, 347, 370, 440, 492, 496, 639, 657, 677, 706, 899, 902, 1027
incomplete ($\Gamma(x,y)$) . . . . . xxxix, 215, 338, 340, 346–348, 352, 366, 368, 436, 438, 498, 576, 657, 658, 710, 749, 787, 899–902, 1002, 1027, 1110
incomplete ($\gamma(x,y)$) . . . . . xxxix, 439, 899–902
gd(x) . . . . . *see* Gudermannian function
Ge function . . . . . xl, 953–955
ge function . . . . . xl, 953, 955
Gegenbauer polynomial ($C_n(x)$) . . . . . xl, 327, 406, 795–800, 927, 940, 941, 969, 983, 990–993, 995, 997, 999, 1017
Gek function . . . . . xl, 955, 957
Gey function . . . . . xl, 765, 955–957
grad 1050, 1051, 1053, 1055, 1056
Gudermannian (gd) . . . . . xl, 52, 53, 116

H

H function . . . . . xli, 879, 880
$H_n(x)$ . . . . . *see* Hermite polynomials
$\mathbf{H}(x)$ . . . . . *see* Struve function
$H(x)$ . . . . . *see* Hankel function
H(x) . . . . . *see* Heaviside function
Hankel function ($H_n(x)$) . . . . . xxxvii, xli, 339, 350, 351, 368, 370, 385, 492, 653, 663, 688, 691, 693–695, 698, 702, 709, 723, 750, 752, 753, 768, 778, 789, 850, 910, 911, 914–916, 920, 922, 923, 925–928, 931, 940, 944
$He_n(x)$ . . . . . *see* Hermite polynomials

Heaviside Function (H(x)) . . . . . xliv, 642, 750, 1115, 1118, 1131
hei$_\nu(z)$ . . . . . *see* Thomson functions
her$_\nu(z)$ . . . . . *see* Thomson functions
Hermite polynomials
  $H_n(x)$ . . . . . xxxvi, xxxvii, xli, 365, 503, 803–806, 810–812, 983, 992, 996–998, 1001, 1030
  $He_n(x)$ . . . . . xxxvi, xxxvii
Hermitian . . . . . xliv, 1070, 1071, 1082, 1083
hyperbolic
  cosine integral . . . . . *see* chi function
  sine integral . . . . . *see* shi function
hypergeometric functions
  F . . . . . xxxix, xl, 315–318, 320, 327, 329, 330, 335, 347–349, 351, 368, 370, 374, 375, 394, 398, 436, 438–440, 442, 444, 488, 490, 503, 512, 517, 639, 646, 648, 654, 657, 663, 670, 671, 673, 677–681, 683, 685, 688, 690, 699, 703, 704, 706, 707, 711, 712, 736, 737, 745, 749, 754, 755, 759, 760, 771–776, 779, 780, 784, 791, 792, 794–797, 801, 803, 805, 807–810, 813–818, 821–824, 826–835, 838, 841, 844, 846, 848, 849, 889, 910, 946, 982, 999, 1005–1013, 1015–1023, 1025, 1033, 1035, 1037, 1039
  confluent (Φ) . . . . . xxxix, 1022–1024, 1027, 1028, 1030–1032
  confluent (Ψ) . . 816, 1023, 1027, 1028, 1038, 1039

I

incomplete beta function
  I . . . . . xli, 910
  B . . . . . xxxix, 910, 1132
incomplete Gamma function . . *see* gamma function, incomplete
inverse functions 1118, 1121

J

Jacobi elliptic functions . . . *see* cd, cn, cs, dc, dn, ds, nc, nd, ns, sc, sd, sn
Jacobi polynomial ($p_n(x)$) . . . . . xli, 998–1000, 1003
Jacobi theta function (Θ) . . . . xxxiv, xxxix, 879, 880
Jacobi zeta function (zn) xxxiv

K

kei(z) . . . . . *see* Thomson functions
ker(z) . . . . . *see* Thomson functions
K' 867
k' . . . . . xliv, 134, 135, 184–200, 204, 206, 225, 263, 410, 472–475, 562, 567, 568, 585, 588, 592, 593, 596–602, 604–606, 619–626, 631–633, 859–868, 870–873, 875, 879, 881, 1006
Kronecker delta . . . . . xliv, 1046, 1047, 1076, 1088

L

$\mathcal{L}$ . . . . . *see* Laplace transform
$L_2(x)$ . . . . . *see* dilogarithm function
$\mathrm{L}_n(x)$ or $\mathrm{L}_n^\alpha(x)$ . . . . . *see* Laguerre polynomials
$L(x)$ . . . . . *see* Lobachevskiy function
$\mathbf{L}(x)$ . . . . . *see* Struve function
Laguerre
  function ($L_n^\alpha(x)$) 348, 441, 707, 709, 803–806, 808–812, 840, 901, 983, 1000–1004, 1028
  polynomial ($L_n(x)$) . . xli, 344, 806, 808, 809, 811, 812, 844
Laplace transform . . . . . xliv, 1107, 1108, 1129, 1138
Legendre functions
  first kind ($P_n(x)$) . . . . . xli, 93, 106, 327, 390, 405, 406, 409, 513, 612, 698, 707, 719, 769–772, 774, 776–782, 785, 786, 788–794, 801, 809, 815, 829, 933, 936, 940, 941, 959–961, 963–969, 972–990, 992, 999, 1017, 1131
  second kind ($Q_n(x)$) . . . . . xli, 324, 373, 383, 696, 719, 769–771, 773, 777, 780, 785, 788, 790, 791, 959, 960, 965, 966, 968, 972, 973, 975–981, 986
Lerch function (Φ) . . . . . xxxix, 642, 1039
li function . . . xxxv, xli, 238, 340, 527, 553, 636, 637, 883, 884, 887, 902, 1027
limit . . . . . xxxii, 6–8, 14, 21, 26, 53, 250–252, 511, 610, 611, 617, 635, 883, 887, 890, 894, 895, 904, 905, 931, 951, 963, 992, 1000, 1003–1006, 1023, 1038–1040, 1067, 1101, 1104, 1106–1108, 1118, 1121, 1130, 1136, 1139
ln function . . . . xxvii–xxix, xxxi, xxxii, xxxiv–xxxvii, 3, 9–11, 23, 26, 27, 43, 44, 46, 47, 49, 51–56, 61–67, 69–85, 87, 90, 94, 97, 99, 103, 104, 106, 113–120, 123–130, 133, 143, 145–148, 150, 155–161, 167–172, 174–176, 178, 186–197, 199, 200, 204–207, 220–225, 237–245, 250, 316, 321, 324, 326, 330–332, 334, 338–340, 353–364, 369–373, 375, 376, 378–381, 383, 386–390, 395, 402, 410, 431, 433, 434, 438, 447–449, 451–457, 462–466, 470–473, 483–485, 495, 497, 499–502, 517–521, 527–607, 622–628, 631–633, 636–645, 647–649, 656, 658, 659, 661, 668, 671, 672, 695, 702, 718, 719, 728, 747, 748, 755, 763, 861, 862, 868, 880, 882–887, 891–893, 895, 898–900, 902–907, 909, 911, 914, 919, 929, 937–939, 944, 963, 969, 972, 977–979, 981, 982, 990, 1006, 1011–1013, 1026, 1027, 1037–1040, 1046, 1048, 1056, 1113, 1115, 1120, 1123, 1125, 1128, 1132, *see* log function
Lobachevskiy function (L) . . . xli, 147, 225, 375, 380, 381, 530–534, 588, 589, 593, 891
log function . . . . . 27, 642, *see* ln function

Lommel function (S) ..... xxxvi, xli, 339, 346, 352, 371, 384, 386, 417, 670, 674, 676–678, 680, 681, 756, 758, 760, 761, 779, 782, 783, 785, 787, 788, 794, 815, 816, 819, 828, 945–947, 950, 1035
Lommel function (s) .... xli, 419–421, 439, 443, 670, 692, 725, 760, 761, 945, 946, 1133
Lommel function (U) ..... xlii, 947
Lommel function (V) ..... xlii, 947, 948

M

$\mathcal{M}$ ..... *see* Mellin transform
$M_{\lambda,\mu}(z)$ ..... *see* Whittaker functions
MacRobert function (E) 1035, 1036
Mathieu functions
Se ..... xxxviii, xli, 764–766, 953, 954
se ..... xxxviii, xli, 763–766, 951–957
max 851, 854, 856, 987, 1066, 1081–1086, 1088, 1091
Meijer function (G) ..... xl, 351, 444, 654, 690, 691, 704, 711, 758, 776, 778, 817–819, 825–832, 835, 838, 844, 845, 847, 850–856, 1032–1035
Mellin transform ..... xliv, 1130
min 851, 854, 856, 1085, 1091

N

$N_\nu(z)$ ..... *see* Bessel function, Y
nc function xxxiii
nd function xxxiii
Neumann function ..... *see* Bessel function, Y
Neumann polynomial ($O_n(x)$) ..... xxxvii, xli, 346, 384, 386, 946, 949, 950
norm ..... *see* $||\cdot||$ and $||\cdot||_p$
ns function xxxiii

O

$O_n(x)$ ..... *see* Neumann polynomials
orthogonal function 798

P

$\mathrm{P}_n(x)$ ..... *see* Jacobi polynomials and Legendre polynomials
$P_n(x)$ ..... *see* Legendre functions (first kind)
$P_n^m(x)$ ..... *see* Legendre functions (associated, first kind)
parabolic cylinder function (D) ..... xxxviii, xl, 348, 349, 352, 365, 384, 390, 503, 504, 506, 653, 657, 658, 697, 708, 712, 740, 746, 802, 805, 811, 841–850, 1028–1031
Phi function (Φ) ..... xxxvi, xxxix, xl, 239, 336–338, 344, 345, 353, 354, 358, 364, 367, 371, 376, 379, 381, 384, 390, 489, 503, 504, 526, 574, 604, 629, 640, 645–649, 748, 749, 755, 781, 802, 835, 838, 887–891, 899, 902, 997, 998, 1001, 1050, 1051, 1056, 1057
Pochhammer symbol ..... xliii, 321, 330, 635, 672, 705, 900, 918, 947, 1010–1012, 1018, 1022, 1031, 1048
polynomials ..... *see* specific name
principal value (PV) ... xliii, 322, 329, 335, 337, 433, 454, 528, 534, 563, 572, 883

Q

$Q_n(x)$ ..... *see* Legendre functions (second kind)
$Q_n^m(x)$ ... *see* Legendre function (associated, second kind)

R

root 15, 84, 104, 331, 539, 542, 553, 576
3 .... 72, 86–88, 264, 330, 331, 363, 364, 539, 570, 918
4 ..... 73, 78, 83, 105, 135, 136, 139, 210, 211, 263–265, 272, 295, 296, 312–315, 483, 493–495, 507, 524, 525, 868, 875, 878–881, 997
8 105
2^k 894
rot 1050

S

$\mathrm{S}_n(x)$ ..... *see* Schlafli polynomials
$S_n^{(m)}$ ..... *see* Stirling numbers, first kind
s(x) ..... *see* Lommel function
$S(x)$ ..... *see* Lommel function and Fresnel cosine integral
sc function xxxiii
Schlafli polynomial ($S_n(x)$) ..... xli, 949, 950
sd function xxxiii
Se(x) ..... *see* Mathieu functions
se(x) ..... *see* Mathieu functions
sec function 36, 39, 43, 44, 50, 52, 64, 113, 114, 155, 156, 315, 323, 328, 329, 371, 372, 377–379, 389, 395, 396, 400, 401, 403–405, 410–414, 421, 422, 436, 438, 439, 446, 471, 472, 479, 541, 551, 586, 646, 653, 661, 663, 664, 669, 670, 674, 682, 689, 691, 699, 706–708, 710, 713, 716, 718, 719, 726, 728, 734, 740, 749, 757, 761, 783, 806, 820, 825, 845, 846, 864, 921, 922, 932, 1007, 1126, 1132, 1133
sech function 27, 43, 62, 113–115, 323, 387, 509, 634, 715, 750, 751, 787, 800, 802, 841, 883, 1128, 1133

shi function . . xxxvi, xli, 142–144, 495, 644, 645, 886
Si function . . . . . xxxv, 643, 886, 930, 1115, 1122
si function . . . . . xxxv, xli, 219–221, 340–344, 421, 423–426, 447, 495, 505, 506, 528, 529, 571, 572, 578, 581, 594, 595, 599, 605, 628, 629, 638–644, 647, 649, 650, 656, 658, 748, 762, 886, 887, 992, 1115
sigma function . . . . . xxxix, 876, 877
sign function . . xlv, 46, 177, 241, 243, 251, 322, 350, 351, 365, 370, 423, 437, 438, 447, 465, 485, 594, 596, 603, 604, 610, 611, 640, 642, 652, 750, 768, 885, 1118–1120, 1122
sin function . . . . . xxvii, xxix, xxxi, xxxii, xxxiv–xxxviii, 4, 13, 19, 20, 23, 26–52, 55, 56, 64, 74–76, 79, 147, 151–237, 249, 250, 253, 254, 263–265, 317, 318, 321–323, 325–329, 331–333, 339–345, 348, 354, 355, 358, 359, 371–373, 375, 377–383, 385, 387–534, 537, 539–547, 550, 551, 554, 558, 559, 561–563, 565, 567–573, 576, 578, 579, 581–601, 604–606, 608, 610–612, 616, 621–623, 628, 629, 631–633, 635–644, 647–651, 655, 656, 658, 659, 662–665, 667, 669, 671–675, 677, 678, 680, 684, 686, 688, 689, 691, 692, 695, 698, 703, 705, 706, 708, 709, 711, 713–715, 717–748, 751, 752, 754–756, 758–760, 762–770, 773, 774, 776, 777, 779–782, 784, 789, 793, 794, 797–800, 802, 806, 808, 810–812, 817, 820, 825, 829, 830, 833, 836, 837, 839, 842, 844–846, 848–850, 853, 859–869, 876–880, 882–893, 896, 898, 900, 904, 906–910, 912–918, 920, 922, 924, 925, 927, 928, 930, 933–940, 942, 943, 945–951, 954, 958, 959, 961–964, 966–981, 984–992, 994, 997, 998, 1000, 1005–1007, 1009, 1013, 1025, 1026, 1029, 1036–1038, 1042, 1044, 1053, 1057, 1060, 1061, 1066, 1067, 1090, 1110–1113, 1115, 1117–1128, 1131, 1132, 1137, 1140
sinh function . . . . . xxxvi, 27–36, 38, 40, 42, 43, 45, 47, 48, 50–52, 64, 110–151, 231–237, 251, 338, 339, 358, 371–390, 407, 419, 425, 432, 433, 438, 439, 448, 451, 452, 454, 455, 461, 466–468, 477, 484, 485, 489, 491, 496, 502, 504, 508–527, 570, 573, 578–582, 595, 596, 606, 610, 611, 621, 622, 634, 643–645, 664, 686, 698, 702, 704, 705, 710, 711, 713–716, 723, 729, 735, 747, 751–753, 755, 760, 763–766, 778, 787–789, 806, 843, 844, 886, 888, 896, 898, 913–915, 917, 943, 953, 955–957, 960–963, 967–969, 980, 981, 997, 1006, 1025, 1029, 1040, 1112–1115, 1119, 1120, 1124, 1125, 1128, 1140
sn function . . . . . xxxiii, xxxiv, xli, 623–626, 714, 866–873, 875, 879, 880
special functions 859
square root . . . . . xxxii, xxxiv, xxxvi, xxxviii, xliv, 2, 9–11, 14, 15, 23, 25, 26, 30, 37, 43, 44, 54–61, 63–67, 71–79, 83–99, 103–109, 125–139, 158, 170–175, 177–184, 197, 199, 200, 202–214, 225, 226, 230, 239–245, 249, 251, 254–315, 317–319, 321, 324–328, 330, 333, 336, 337, 339, 344–353, 355, 359, 363–376, 380, 382–385, 390, 391, 393–396, 400–402, 404–411, 414–419, 421, 425, 426, 428–430, 434–436, 440–446, 448, 451–454, 456, 457, 460, 472–479, 481–483, 485, 486, 488–497, 499, 501–507, 511, 513–515, 517, 518, 522–527, 529, 531, 532, 534, 535, 537–539, 542, 543, 545, 549–551, 553, 554, 556–558, 560, 562, 563, 565–568, 570–576, 578–581, 584, 585, 588, 590–593, 595–606, 608–613, 615–617, 619, 621–623, 629, 631–635, 637–642, 644–651, 653, 657, 659, 661–668, 670, 672–675, 677, 678, 680–683, 685–763, 766–768, 771, 773, 777, 778, 780–782, 785–789, 791–794, 800–808, 810–812, 814, 815, 819, 828, 829, 837, 841, 843–845, 848, 853, 854, 856, 859–866, 868, 870–876, 879–881, 887–891, 893–902, 905, 908, 909, 913–915, 917, 918, 920–926, 928, 931–946, 950, 951, 958, 960–974, 976–983, 985, 987, 988, 990–998, 1000, 1002, 1003, 1007, 1009, 1018, 1019, 1023, 1026–1030, 1035, 1038, 1052, 1054, 1055, 1060, 1082, 1116–1121, 1123, 1125–1127, 1129, 1131
step function 798
Stirling number
  first kind (S_n^m) . . . . . xlv, 1046–1048
  second kind ($\mathfrak{S}_n^m$) . . . . . xlv, 1046–1048
Struve function
  $\mathbf{H}(x)$ . . xli, 345, 351, 421, 435, 442, 443, 573, 647, 659, 660, 663, 664, 669, 675, 677, 679, 680, 692, 694, 708, 722, 725, 735, 753–759, 787, 838, 848, 856, 942, 943, 946, 1035, 1132
  modified ($\mathbf{L}(x)$) . . . . . xli, 345, 350, 351, 435, 441, 515, 595, 605, 663, 664, 669, 671, 675, 676, 678, 679, 692, 722, 736, 753–759, 787, 794, 942, 943

T

$T_n(x)$ . . . . . *see* Chebyshev polynomials
tan function . . . . . xxvii–xxix, 27–30, 35, 36, 39, 40, 42, 44, 46, 50, 52, 53, 55, 56, 60, 64, 126, 151, 155–161, 167–178, 180, 181, 183–185, 188, 189, 194, 195, 199, 200, 202–207, 209, 212, 214, 222–225, 229, 230, 274, 322, 332, 339, 340, 355, 371, 375–379, 381, 382, 384, 388, 389, 393, 395, 396, 400, 403–405, 409–414, 421–423, 433, 434, 451–455, 457–459, 471–475, 483–486, 492, 493, 496, 498, 505, 506, 509, 515, 518, 532–535, 537, 541, 545, 546, 567–570, 579, 582, 586–589, 591–594, 597–599, 604, 606, 622, 631, 632, 659, 664, 669–671, 682, 686, 699, 716, 717, 725, 728, 744, 745, 754, 766, 849, 862–865, 867–869, 883, 886, 890, 894, 905, 909, 922, 932, 954, 969, 971, 979, 986, 990, 1006, 1007, 1023, 1123, 1127, 1132
tanh function 12, 27–29, 31, 39, 40, 42, 44, 51, 52, 62, 64, 110, 113–120, 123–126, 128–132, 134–137, 139, 145–148, 338, 380–383, 387, 390, 472, 484, 485, 489, 502, 509, 512, 513, 516, 518, 520, 569, 606, 621, 622, 716, 717, 751, 753, 766, 788, 789, 841, 921, 922, 956, 957, 1025, 1124
theta function (θ) . . . . xxxiv, 521, 633, 634, 877–883
Thomson functions
  bei(x) . . . . . xxxix, 761–763, 944, 945
  ber(x) . . . . . xxxix, 761–763, 944, 945
  hei(x) . . . . . xli, 944
  her(x) . . . . . xli, 944
  kei(x) . . . . . xli, 641, 663, 672, 674, 748, 762, 763, 944, 945
  ker(x) . . . . . xli, 641, 663, 671, 674, 747, 762, 763, 944, 945
toroidal function 981
tr . . . . . *see* trace
trace 1084
transpose . . . . . xlv, 1069–1073, 1075, 1089

U

$U_n(x)$ . . . . . *see* Chebyshev polynomials
unilateral z transform 1135, 1138–1140, 1159

W

$W_{\lambda,\mu}(z)$ . . . . . *see* Whittaker functions
Weber function (**E**) 338, 339, 346, 353, 371, 384, 421, 423, 670, 671, 751, 943, 946, 948, 949
Weierstrass function ($\wp$) . . . . xxxix, xl, 626, 873–877, 880
Whittaker functions
  M . . . . . xli, 338, 348, 445, 654, 682, 697, 703, 705, 706, 709, 710, 715–717, 736, 748, 749, 784, 785, 787, 819–841, 1024–1027
  W . . . . . xlii, 338, 346–349, 367, 368, 384, 423, 445, 635, 652, 654, 682, 697, 698, 704, 706, 707, 709, 710, 712, 715, 716, 726, 727, 736, 745, 749, 756, 759, 761, 776–778, 781, 782, 784–788, 803, 814–817, 819–841, 843, 844, 846, 847, 857, 979, 1024–1028, 1035

X

X_b . . . . . *see* bilateral z transform
X_u . . . . . *see* unilateral z transform

Y

Y . . . . . *see* Bessel function, Y
Young function (C) . . . . . xxxvi, xl, 417, 439, 440

Z

zeta function (ζ) . . xxxix, 8, 338, 353, 354, 358, 359, 376, 377, 379–381, 386–389, 433, 434, 449, 471, 509, 540, 542, 543, 550, 552, 560, 567, 569, 576, 577, 580, 587, 591, 593, 607, 626, 633, 634, 658, 659, 802, 876, 877, 880, 894, 898, 903–905, 907, 909, 1036–1041, 1133
zn(x) . . . . . *see* Jacobi zeta function

This page intentionally left blank

Index of Concepts

This index refers to concepts appearing in the text.

A

Abel's identity . . . 1098
absolute convergence . . . 6
absolute values . . . 63
addition theorems . . . 973, 975
adjoint . . . 1099
  equations . . . 1098
algebraic
  inequalities . . . 1059, 1060
algebraic functions . . . 82, 253
  and arccosine . . . 242
  and arccotangent . . . 244
  and arcsine . . . 242
  and arctangent . . . 244
  and associated Legendre functions . . . 789
  and Bessel functions . . . 674, 715
  and exponentials . . . 344, 363
  and hyperbolic functions . . . 132, 375, 715
  and logarithmic functions . . . 538
  and logarithms . . . 238
  and powers . . . 363
  and rational functions . . . 789
  and trigonometric functions . . . 434
alternating series . . . 7
amplitudes . . . 866
analytic continuation . . . 970, 1012
Anger functions . . . 948
angle of parallelism . . . 51
anticommutative . . . 1049
approximate solution . . . 1093–1096
approximation by tangents . . . 921
arccosecant . . . 242
  and powers . . . 244
arccosine . . . 241
  and algebraic functions . . . 242
arccotangent . . . 242
  and algebraic functions . . . 244
arcsecant . . . 242
  and powers . . . 244
arcsine . . . 241
  and algebraic functions . . . 242
arctangent . . . 242
  and algebraic functions . . . 244
  and Bessel functions . . . 747
argument . . . 866
  of a complex number . . . xliv
arithmetic mean theorem . . . 1056
arithmetic progression . . . 1
arithmetic-geometric inequality . . . 1060
arithmetic-geometric progression . . . 1
associated Legendre functions . . . 769, 788, 958, 972, 974
  and algebraic functions . . . 789
  and Bessel functions . . . 782, 787
  and exponentials . . . 776
  and hyperbolic functions . . . 778
  and powers . . . 770, 776, 779
  and probability integral . . . 781
  and rational functions . . . 789
  and trigonometric functions . . . 779
associated Mathieu functions . . . 952
asymptotic expansions . . . 1146
asymptotic result . . . 21, 356, 895, 1026, 1029
asymptotic series . . . 21

B

Ballieu theorem . . . 1086
basic theorems . . . 1091
Bateman's function . . . 1023
Bernoulli
  numbers . . . 1040, 1045
  polynomials . . . 1040, 1041
Bessel functions . . . 629, 659, 748, 749, 753, 910, 912, 914, 916–920, 924, 925, 928, 931, 933–937, 940, 941, 954, 1146, *see* Constant/Function index
  and algebraic functions . . . 674, 715
  and arctangent . . . 747
  and associated Legendre functions . . . 782, 787

Bessel functions (*continued*)
and Chebyshev polynomials . . . 803
and exponentials . . . 694, 699, 708, 711, 713, 715, 742, 834
and Gegenbauer functions . . . 798
and hyperbolic functions . . . 713, 715, 747
and hypergeometric functions . . . 817
confluent . . . 830, 831, 834
and Legendre polynomials . . . 794
and logarithms . . . 747
and MacRobert functions . . . 854
and Mathieu functions . . . 767
and Meijer functions . . . 854
and parabolic cylinder functions . . . 845
and powers . . . 664, 675, 689, 699, 708, 711, 727, 742, 831, 834
and rational functions . . . 670
and Struve functions . . . 756
and trigonometric functions . . . 717, 727, 742, 747
generating functions . . . 933
imaginary arguments . . . 911
Bessel inequality . . . 1068
bilateral z-transform . . . 1135, 1136
bilinear concomitant . . . 1099
binomial coefficients . . . 3
symbol . . . xliii
binomials . . . 25
and powers . . . 315, 322
Bonnet–Heine formula . . . 988
bounded variation . . . 20
boundedness theorems . . . 1106
branch points . . . 866, 1024
Brauer theorem . . . 1086, 1088
Buniakowsky inequality . . . 1059, 1061, 1064

C

Calogero . . . 1089
Carleman inequality . . . 1060, 1066
Catalan constant . . . xxxii, 1046, *see* Constant/Function index
Cauchy principal value . . . 528
Cauchy problem . . . 1093, 1095
Cauchy–Schwarz–Buniakowsky inequality . . . 1059, 1061, 1064
Cayley–Hamilton theorem . . . 1084
change of variables . . . 248, 607, 608
characteristic equation . . . 1071
characteristic polynomial . . . 1084
characteristic values . . . 1084
Chebyshev inequality . . . 1059, 1065
Chebyshev polynomials . . . 988, 993
and Bessel functions . . . 803
and elementary functions . . . 802
and powers . . . 800
Christoffel formula . . . 983
Christoffel summation formula . . . 986
circle of convergence . . . 16
circulants . . . 1078
classification system . . . xxxi
classified references . . . 1145
column norm . . . 1082
comparison of approximate solutions . . . 1094, 1096
comparison theorem . . . 1100, 1101, 1103, 1104
complementary error function . . . *see* error functions, complementary
complementary modulus . . . 859
complete elliptic integrals . . . 619, 632, 859
complex analysis . . . 1146
complex conjugate . . . xliii
conditional convergence . . . 6
conditions, Dirichlet . . . 19
confluent hypergeometric function . . . *see* hypergeometric function, confluent
conical functions . . . 980
constant of integration . . . 63
constants . . . *see* Constant/Function index
Catalan . . . *see* Catalan constant
Euler . . . *see* Euler constant
continued fraction . . . 902
continuity, Lipschitz . . . 1094, 1095
converge
absolutely . . . 6
conditionally . . . 6
uniformly . . . 15
convergence
circle . . . 16
radius . . . 16
tests . . . 6, 19
convexity . . . 1066
convolution . . . 1118
theorem . . . 1108, 1118, 1122, 1130, 1136
coordinates, curvilinear . . . 1052
cosine
and rational functions . . . 171, 390
and square roots . . . 472
integral . . . 628, 639, 886
hyperbolic . . . 644, 886
multiple angles . . . 161
cosine-amplitude . . . 866
Cramer's rule . . . 1077
cube roots . . . 86
curl . . . 1050

curvilinear coordinates 1052
cycles 1046
cylinder function . . . . *see* parabolic cylinder function

D

Darboux–Christoffel formula 983
de Moivre's theorem 1060
decreasing solutions 1104
definite integrals . . . . . 247, *see* integrals, definite
delta amplitude 866
derivative of a composite function 22
determinants 1075, 1076, 1078
  Gram 1080
  Hessian 1079
  Jacobian 1078
  Vandermonde 1078
  Wronskian 1079
differential equations 873, 874, 910, 931, 944, 947–950, 952, 958, 974, 975, 980, 981, 983, 993, 995, 998, 1000, 1003, 1011, 1013, 1015, 1024, 1031, 1034, 1093
  adjoint 1098
  exact 1097
  homogeneous 1097
  hypergeometric 1010
  partial 1018, 1031
  Riccati 1099
  Riemann 1014, 1022
  second-order 1017, 1098, 1100, 1104
  self-adjoint 1098
  special types 1097
  variables separable 1097
differentiation
  of integrals 21, 1064
  of matrices 1073
  of vectors 1050
dilogarithm 642
diophantine relations 1089
directional derivative 1051
Dirichlet conditions 19
Dirichlet lemma 1067
div 1050
divergence theorem 1055
DOC (domain of convergence) 1135
domain of convergence (DOC) 1135
dominant solutions 1104, 1105
double factorial symbol xliii
double integrals 610, 1021
doubling formula 896
doubly-periodic function 866, 874

E

eigenvalues 951, 1071, 1084, 1087
eigenvectors 1071
elementary functions 25, 247, 1006
  and Chebyshev polynomials 802
  and Gegenbauer polynomials 797
  and Legendre polynomials 792
  and MacRobert functions 850
  and Meijer functions 850
  indefinite integrals 63
elliptic functions 619, 631, 865, 1148
  Jacobian 866, 870
  order 865
  Weierstrass . . . . . 626, *see* Weierstrass elliptic functions
elliptic integrals . . . 104, 184, 619, 621, 631, 632, 859
  complete 394, 472–474, 632, 859
  derivatives 394, 863, 865
  functional relations 863
  generalized 635
  Jacobian 623
  kinds 859
equations
  differential . . . . . *see* differential equations
  first-order 1093, 1096
  linear 1096
  special types 1097
  system 1094, 1095
error functions 887, 1147
  complementary 887
essential singularity 1024
Euclidean norm 1081
Euler
  constant . . . . . xxxii, *see* Constant/Function index
  dilogarithm 642
  integrals 892, 908
  numbers 1040, 1043, 1045
  polynomials 1044
  substitutions 92
exact differential equations 1097
expansion of determinants 1075, 1076
expansions, asymptotic 1146
expansions, Weierstrass 869
exponential integrals . . 627, 636, 638, 883, 885, 1147
  and exponentials 628
  and powers 627
exponentials 26, 106, 334
  and algebraic functions 344, 363
  and associated Legendre functions 776
  and Bessel functions 694, 699, 708, 711, 713, 715, 742, 834
  and complicated arguments 336

exponentials (*continued*)
and exponential integrals 628
and gamma functions 652
and hyperbolic functions 148, 338, 382, 386, 522, 525, 713, 715
and hypergeometric functions 814
confluent 822, 834
and inverse trigonometric functions 605
and logarithmic functions 339, 571, 573, 599
and parabolic cylinder functions 842
and powers 148, 346, 353, 363, 364, 386, 497, 525, 573, 699, 708, 711, 742, 754, 776, 834, 842
and rational functions 106, 340, 353
and Struve functions 754
and trigonometric functions . . . 227, 339, 485, 493, 495, 497, 522, 525, 599, 742
matrix 1074
of exponentials 338
series 27

F

factorial symbol xliii
field theory 1049
figures 608–610, 892, 913, 915, 916, 1036
final value theorem 1139
finite sums 1
first mean value theorem 1063
first-order equations 1093, 1096
first-order systems 1094
footnotes . . . . . xxix, xxxi, 82, 132, 247, 248, 274, 397, 410, 547, 656, 859, 867, 908, 920, 931, 981, 991, 1039, 1141
Fourier series 19, 46, 1066, 1067
generalized 1067, 1068
Fourier transform 1107, 1117
basic properties 1118
cosine 1121, 1129
properties 1121
table 1126
exponential 1129
sine 1121, 1129
properties 1121
table 1122
tables 1118, 1120
fourth roots 313
fractional transformation 1014
Fresnel integrals 629, 649, 887, 1147
Frobenius theorem 1088
functional series 15
functions . . . . . *see* Constant/Function index
inner xxviii
ordering xxviii
orthogonal 798
outer xxvii
fundamental inequalities 1094
fundamental system 1100

G

gamma functions 650, 892, 894, 895, 1147
and exponentials 652
and logarithms 656
and powers 652
and trigonometric functions 655
incomplete 657, 899
Gauss divergence theorem 1055
Gegenbauer functions and Bessel functions 798
Gegenbauer polynomials 990
and elementary functions 797
and powers 795
general formulas 65, 249
generalized elliptic integrals 635
generalized Fourier series 1067, 1068
generalized Legendre polynomials 990
generating functions
Bernoulli numbers 1040
Bernoulli polynomials . . . . . xxxii, 1041
Bessel functions 933
Chebyshev polynomials 995
Euler numbers 1043
Euler polynomials . . . . . xxxii, 1044
Hermite polynomials 997
Jacobi polynomials 1000
Legendre polynomials 988
Neumann polynomials . . . . . xxxvii, 950
Stirling numbers 1046, 1047
geometric progression 1
Gerschgorin theorem 1083, 1088
grad 1050
gradient 1050
Gram determinant 1080
Gram inequality 1065
Gram–Kowalewski theorem 1080
Green theorem 1055, 1056
Gronwall's lemma 1094
growth estimates 1104
growth of maxima 1106
Gudermannian (gd) 52

H

Hadamard's inequality 1077
Hadamard's theorem 1077
Hankel functions 910, 925
Heaviside step function xliv
Heine formula 988
Helmholtz equation 767, 1052
Hermite method 67
Hermite polynomials 803, 996, 997
Hermitian matrix 1077, 1089
Hessian determinant 1079
Hölder inequality 1059, 1061, 1064
homogeneity 875
homogeneous differential equations 1097
hyperbolic
  amplitude 52
  cosine integral 644, 886
  sine integral 644, 886
hyperbolic functions 28, 110, 371
  and algebraic functions 132, 375, 715
  and associated Legendre functions 778
  and Bessel functions 713, 715, 747
  and exponentials 148, 338, 382, 386, 522, 525, 713, 715
  and inverse trigonometric functions 605
  and linear functions 120
  and logarithmic functions 578
  and Mathieu functions 763
  and parabolic cylinder functions 843
  and powers 139, 148, 386, 516, 525
  and rational functions 125
  and trigonometric functions . . . 231, 509, 516, 522, 525, 747, 763
  inverse 56, 240
    and logarithms 237
  powers 110, 120
hypergeometric
  differential equation 1010
  series 1005, 1008
    confluent 1031
    generalized 1010
hypergeometric functions 812, 841, 946, 1005, 1006, 1039, 1147
  and Bessel functions 817
  and exponentials 814
  and powers 812
  and trigonometric functions 817
hypergeometric functions (*continued*)
  confluent 820, 841, 1022, 1023, 1147
    and Bessel functions 830, 831, 834
    and exponentials 822, 834
    and Legendre functions 839
    and parabolic cylinder functions 849
    and polynomials 840
    and powers 820, 831, 834
    and special functions 839
    and Struve functions 838
    and trigonometric functions 829
  several variables 1022
  two variables 1018

I

identities
  Abel 1098
  Lagrange 1099
  Picone 1102
improper integrals 251, 252
incomplete beta functions 910
incomplete gamma function 657
increasing solutions 1104
indefinite integrals
  elementary functions 63
  special functions 619
induced norm 1082
inequalities 950, 963, 979, 987, 997, 1041, 1061, 1083–1085, 1094
  algebraic 1059, 1060
  Carleman 1060, 1066
  for sets 1061
  Hadamard 1077
  integral 1063–1066
  Schur 1087
  triangle 1061
inertia 1072
infinite products 6, 14, 862
initial value theorem 1136, 1139
inner function xxxi
integer function 1135
integer pulse function 1135
integral
  differentiation 21, 1064
  formula 985
  inequalities 1063–1066
  inversion 1107, 1118, 1121, 1129
  part (symbol) xliii
  representations 887, 888, 892, 898, 900, 902, 906, 908, 912, 914, 916, 942, 946, 950, 960, 974, 976, 980, 981, 985, 991, 996, 1005, 1021, 1023, 1025, 1028, 1035, 1036, 1039, 1040, 1067

integral (*continued*)
theorems . . . 1049
transforms . . . 1107, 1147
relationships . . . 1129
integrals
definite . . . 247
special functions . . . 631
double . . . 610, 1021
elliptic . . . 104, 859
Fresnel . . . 1147
improper . . . 251, 252
indefinite . . . *see* indefinite integrals
Mellin–Barnes . . . 1021
multiple . . . 607, 612
pseudo-elliptic . . . 105
triple . . . 610
integration
constant . . . 63
techniques . . . 92
termwise . . . 16
interlacing of zeros . . . 1101
invariants . . . 874
inverse z-transformation . . . 1135
inverse hyperbolic functions . . . *see* hyperbolic functions
inverse trigonometric functions . . . 599, *see* trigonometric functions
and exponentials . . . 605
and hyperbolic functions . . . 605
and logarithms . . . 607
and powers . . . 600, 601, 607
and trigonometric functions . . . 605, 607
inversion integral . . . 1107, 1118, 1121, 1129

J

Jacobi polynomials . . . 806, 998
Jacobi theorem . . . 1076
Jacobian determinant . . . 1078
Jacobian elliptic functions . . . 866, 870, 879, 1148
Jacobian elliptic integrals . . . 623
Jensen inequality . . . 1066

K

Kneser's non-oscillation theorem . . . 1103
Kowalewski theorem . . . 1080

L

L_2 norm . . . 1081
Lagrange identity . . . 1059, 1099
Laguerre polynomials . . . 808, 1000
Laplace formula . . . 987
Laplace integral formula . . . 985
Laplace transform . . . 1107, 1129
basic properties . . . 1107
table . . . 1108
Laplacian . . . 767, 1051
latent roots values . . . 1084
Laurent series . . . 1135
least common factor . . . 798
least common multiple . . . 798
Lebesgue lemma . . . 1067
Legendre functions . . . 975, 1149
and hypergeometric functions
confluent . . . 839
associated . . . *see* associated Legendre functions
special values . . . 969
Legendre normal form . . . 859
Legendre polynomials . . . 983, 988
and Bessel functions . . . 794
and elementary functions . . . 792
and powers . . . 791
lemmas
Dirichlet . . . 1067
Gronwall . . . 1094
Riemann–Lebesgue . . . 1067
letters, conventions . . . 63
linear dependence . . . 1080
linear equations . . . 1096
L_∞ norm . . . 1081
Lipschitz continuity . . . 1094, 1095
Lobachevskiy's "angle of parallelism" . . . 51
Lobachevskiy's function . . . 891
logarithm integrals . . . 636, 887
logarithms . . . 53, 237, 527, 529
and algebraic functions . . . 238, 538
and Bessel functions . . . 747
and exponentials . . . 339, 571, 573, 599
and gamma functions . . . 656
and hyperbolic functions . . . 578
inverse . . . 237
and inverse trigonometric functions . . . 607
and powers . . . 540, 542, 553, 555, 573, 594
and rational functions . . . 535, 553
and trigonometric functions . . . 339, 581, 594, 599
gamma functions . . . 898
Lommel functions . . . 760, 945
two variables . . . 947
Lyapunov theorem . . . 1089, 1105

M

MacRobert functions . . . 850, 1035
and Bessel functions . . . 854
and elementary functions . . . 850
and special functions . . . 856

Mathieu functions 763, 950, 951, 953, 954, 1149
and Bessel functions 767
and hyperbolic functions 763
and trigonometric functions 763
imaginary argument 952
matrix
adjoint 1070
cofactors 1075
determinants . . . . . *see* determinants
diagonal 1069
diagonally dominant 1071
differentiation 1073
equivalent 1069
exponential 1074
Hermitian 1070, 1077, 1089
idempotent 1071
identity 1069
inverse 1070
irreducible 1069
minors 1075
principal 1076
nilpotent 1071
non-negative definite 1071
norm 1082, 1083
null 1069
orthogonal 1070
positive definite 1071
reducible 1069
skew-symmetric 1070
special 1069
symmetric 1070
trace 1070
transpose 1069, 1070
triangular 1070
unitary 1071
maxima 1106
mean value theorems 247, 1063, 1064
Meijer functions 850, 1032
and Bessel functions 854
and elementary functions 850
and special functions 856
Mellin transform 1107, 1129
basic properties 1130
table 1131
Mellin–Barnes integrals 1021
metric coefficients 1052
metrical coefficients 1054
Minkowski inequality 1059, 1061, 1065
modulus 632, 859, 860
multiple angle expansion 31
multiple integrals 607, 612

N

named theorems 1087
natural norm 1082
natural numbers xliv
necessary conditions 1104
Neumann functions 910
Neumann polynomials 949
nome 877
non-oscillation 1100, 1103, 1104
normal form 859
norms 1081
column 1082
compatible 1082
Euclidean 1081
induced 1082
matrix 1082, 1083
natural 1082
row 1083
spectral 1082
vector 1081
notation xliii

O

one-sided z-transform 1135
order of presentation xxvii
ordinary differential equations 1093
orthogonal curvilinear coordinates 1052
orthogonal polynomials 795, 982, 1149
oscillation 1100, 1102
Ostrogradskiy–Hermite method 67
Ostrowski inequality 1066
Ostrowski theorem 1089
outer function xxxi

P

parabolic cylinder functions 841, 849, 1028, 1150
and Bessel functions 845
and exponentials 842
and hyperbolic functions 843
and hypergeometric functions 849
and powers 842
and Struve functions 848
and trigonometric functions 844
parameter 877
parameter of the integral 859
Parodi theorem 1086
Parseval formula 1136
Parseval theorem 1067, 1068
partial fractions 66
partial sums 1067
Perelomov 1089

periodic 19
Mathieu functions 951
periods 865, 870
permutations 1046
Perron theorem 1088
Perron–Frobenius theorem 1088
Picone identity 1102
Picone theorem 1102
Pochhammer symbol xliii
Poincare's separation theorem 1087
points, singular 958
Poisson integral 1056, 1057
poles 865, 870, 874, 892
polynomials 254, 313, 322
and hypergeometric functions confluent 840
characteristic 1084
Chebyshev . . . . . *see* Chebyshev polynomials
degree 3 or 4 859
Gegenbauer 990
Hermite . . . . . *see* Hermite polynomials
Jacobi . . . . . *see* Jacobi polynomials
Laguerre . . . . . *see* Laguerre polynomials
Legendre . . . . . *see* Legendre polynomials
orthogonal 795, 982, 1149
positive definite 1071, 1072
positive semidefinite 1072
power series 16–18, 25
expansion 42
powers 253
and algebraic functions 363
and arccosecant 244
and arcsecant 244
and associated Legendre functions . . 770, 776, 779
and Bessel functions 664, 675, 689, 699, 708, 711, 727, 742, 831, 834
and binomials 315, 322
and Chebyshev polynomials 800
and exponential integrals 627
and exponentials 148, 346, 353, 363, 364, 386, 497, 525, 573, 699, 708, 711, 742, 754, 776, 834, 842
and gamma functions 652
and Gegenbauer polynomials 795
and hyperbolic functions . . 139, 148, 386, 516, 525
and hypergeometric functions 812
confluent 820, 831, 834
and inverse trigonometric functions 600, 601, 607
and Legendre polynomials 791
and logarithmic functions 540, 542, 553, 555, 573, 594
and parabolic cylinder functions 842
powers (*continued*)
and rational functions 353, 401, 553
and square roots 472
and Struve functions 754
and trigonometric functions . . . 214, 397, 401, 405, 411, 436, 459, 475, 497, 516, 525, 594, 607, 727, 742, 779
binomials 25
hyperbolic functions 110, 120
trigonometric functions 151, 395
principal
function xxxi
natural norms 1082
values 56, 252, 528
vector norms 1081
probability function 1150
probability integrals 629, 645, 887
and associated Legendre functions 781
problem, Cauchy 1093, 1095
product
finite 41
infinite 6, 14, 45
of vectors 1049
theorem 896
progressions 1, 8
pseudo-elliptic integrals 105, 184
pulse function 1135

Q

q-series 880
quadratic forms 1071
quasiperiodicity 878

R

radius of convergence 16
rank 1072
rate of change theorems 1057
rational functions 66, 253, 254
and algebraic functions 789
and associated Legendre functions 789
and Bessel functions 670
and cosine 171, 390
and exponentials 106, 340, 353
and hyperbolic functions 125
and logarithmic functions 535, 553
and powers 353, 401, 553
and sine 171, 390
and trigonometric functions 401, 423, 447
Rayleigh quotient 1091
real numbers xlv
reciprocal theorem 1056
reciprocals 3, 12

references 1141
  supplementary 1145
remainder 18
representation theorem 1056
residues 870
Riccati equation 1099
Riemann differential equation 1014
Riemann hypothesis 1038
Riemann zeta functions 1036, 1150
Riemann–Lebesgue lemma 1067
Rodrigues' formula 993, 995, 998, 1000
roots . . . . . *see* square roots and Constant/Function index
  fourth 313
Routh–Hurwitz theorem 1086
row norm 1083

S

saltus 19
scalar product 1049
Schläfli integral formula 985
Schläfli polynomials 949
Schur's inequalities 1087
Schwarz inequality 1059, 1061, 1064
second mean value theorem 1063, 1064
second-order equations 1017, 1098, 1100, 1104
self-adjoint equations 1098
semiconvergent series 21
separation theorem 1087, 1101
series . . . . . 860, *see* specific type
  alternating 7
  asymptotic 21
  convergence 6
  diverge 21
  Fourier 19, 46, 1066–1068
    generalized 1067, 1068
  functional 15
  hyperbolic functions 51
  hypergeometric 1005, 1008
    generalized 1010
  of exponentials 27
  of logarithms 55
  power 16–18
  rational fractions 26
  remainder 18
  semiconvergent 21
  Taylor 18
  trigonometric 46, 862
sign function xlv
signature 1072
signum function xlv
sine
  and rational functions 171, 390
  and square roots 472
  integral 628, 639, 886
    hyperbolic 644, 886
  multiple angles 161
sine-amplitude 866
singular points 958, 1038
solenoidal fields 1052
Sonin theorem 1106
special functions xxxix
  and hypergeometric functions
    confluent 839
  and MacRobert functions 856
  and Meijer functions 856
  indefinite integrals 619
spectral norm 1082
spectral radius 1083
spherical functions 974
square roots . . . 84, 88, 92, 94, 99, 103, 179, 184, 254
  and cosine 472
  and powers 472
  and sin 472
  trigonometric functions 408
Steffensen inequality 1065
step function xliv
Stieltjes' theorems 987
Stirling numbers 1046, 1048
  table 1047, 1048
Stokes phenomenon 920
Stokes theorem 1057
Struve functions 753, 942, 1150
  and Bessel functions 756
  and exponentials 754
  and hypergeometric functions confluent 838
  and parabolic cylinder functions 848
  and powers 754
  and trigonometric functions 755
Sturm comparison theorem 1101
Sturm separation theorem 1101
Sturm–Picone theorem 1102
Sturmian separation theorem 1087
subdominant solutions 1104
subordinate norm 1082
substitutions, Euler 92
sufficient conditions 1104
summation formula 986
summation theorems 940, 986, 992, 998, 1002, 1030, 1042

sums
binomial coefficients . . . 3
partial . . . 7
powers . . . 1
powers of trigonometric functions . . . 37
products . . . 3
products of trigonometric functions . . . 38
reciprocals . . . 3, 12
tangents of multiple angles . . . 39
trigonometric and hyperbolic functions . . . 36
supplementary references . . . 1145
Sylvester's law of inertia . . . 1072
symbol
binomial coefficient . . . xliii
factorial . . . xliii
double . . . xliii
integral part . . . xliii
Pochhammer . . . xliii
synonyms . . . xxvii
system of equations . . . 1094, 1095
linear . . . 1096
Szegö comparison theorem . . . 1101

T

table usage . . . xxxi
tangent approximation . . . 921
Taylor series . . . 18
termwise integration . . . 16
tests, convergence . . . 6, 19
theorems
addition . . . 973, 975
arithmetic mean . . . 1056
Ballieu . . . 1086
basic . . . 1091
boundedness . . . 1106
Brauer . . . 1086, 1088
Cayley–Hamilton . . . 1084
comparison . . . 1100, 1101, 1103, 1104
convolution . . . 1108, 1118, 1122, 1130, 1136
de Moivre . . . 1060
divergence . . . 1055
final value . . . 1139
Frobenius . . . 1088
Gauss . . . 1055
general nature . . . 247
Gerschgorin . . . 1083, 1088
Gram–Kowalewski . . . 1080
Green . . . 1055, 1056
Hadamard . . . 1077
initial value . . . 1136, 1139
integral . . . 1049
Jacobi . . . 1076
theorems (*continued*)
Kneser . . . 1103
Lyapunov . . . 1089, 1105
mean value . . . 247, 1063, 1064
named . . . 1087
non-oscillation . . . 1100, 1103, 1104
oscillation . . . 1100
Ostrowski . . . 1089
Parodi . . . 1086
Parseval . . . 1067, 1068
Perron . . . 1088
Perron–Frobenius . . . 1088
Poincare's . . . 1087
product . . . 896
quadratic forms . . . 1072
rate of change . . . 1057
reciprocal . . . 1056
representation . . . 1056
Routh–Hurwitz . . . 1086
second-order equations . . . 1100
separation . . . 1087
Sonin . . . 1106
Stieltjes' . . . 987
Stokes . . . 1057
Sturm comparison . . . 1101
Sturm separation . . . 1101
Sturm–Picone . . . 1102
Sturmian . . . 1087
summation . . 940, 986, 992, 998, 1002, 1030, 1042
Szegö comparison . . . 1101
vector integral . . . 1055
Wielandt . . . 1088
theta functions . . . 633, 877
Thomson functions . . . 761, 944
total variation . . . 20
trace . . . 1084
transformation formulas . . . 1008
transforms
Fourier . . . *see* Fourier transform
fractional . . . 1014
Hankel . . . *see* Hankel transform
integral . . . 1147
Laplace . . . *see* Laplace transform
Mellin . . . *see* Mellin transform
of a derivative . . . 1118
triangle inequality . . . 1061
trigonometric functions . . . 28, 151, 390, 415
and algebraic functions . . . 434
and associated Legendre functions . . . 779
and Bessel functions . . . 717, 727, 742, 747

trigonometric functions (*continued*)
and exponentials 227, 339, 485, 493, 495, 497, 522, 525, 599, 742
and gamma functions 655
and hyperbolic functions 231, 509, 516, 522, 525, 747, 763
and hypergeometric functions 817
confluent 829
and inverse trigonometric functions 605, 607
and logarithmic functions 339, 581, 594, 599
and Mathieu functions 763
and parabolic cylinder functions 844
and powers 214, 395, 397, 401, 405, 411, 436, 459, 475, 497, 516, 525, 594, 607, 727, 742, 779
and rational functions 401, 423, 447
and square roots 408
and Struve functions 755
inverse 56, 241
powers 151, 459
trigonometric series 46, 862
triple integrals 610
triple vector product 1049
two-sided z-transform 1135

U

uniform convergence 15
unilateral z-transform 1135, 1138
unit integer function 1135
unit integer pulse function 1135
use of the tables xxxi

V

Vandermonde determinant 1078
variables separable 1097
variational principles 1091
vector
differentiation 1050
field theory 1049
integral theorems 1055
norms 1081
operators 1049
product 1049

W

Weber functions 948
Weierstrass elliptic functions 626, 873, 880, 1148
Weierstrass expansions 869
weight function 982
Whittaker functions 1024
Wielandt theorem 1088
Wronskian determinant 1079

Y

Young inequality 1065

Z

zeros 865, 870, 879, 972, 1038
interlacing 1101
simple 1000
zeta function 1150
z-transforms 1135