

# 1 Multivariate Model for scores

Given a model for  $Y$ , the above gives a relevant functional model. We consider the problem here on how to model the scores  $Y$ .

Following Stilian's formulation, when  $d = 1$  and  $s$  and  $t$  are scalar, we consider

$$Y(t) = \int_{\mathbb{R}} e^{itx} ((1+ix)^{-\nu-1/2} A 1_{\{x>0\}} + (1+ix)^{-\nu-1/2} \overline{A} 1_{\{x<0\}}) \tilde{B}(dx)$$

where  $\tilde{B}(dx)$  is  $\mathbb{C}^k$ -valued Brownian motion with

$$\tilde{B}(x) = \overline{\tilde{B}(-x)} \quad \mathbb{E}[\tilde{B}(dx) \tilde{B}(dx)^*] = \mathbb{I}_k dx$$

and  $A$  is a complex-valued matrix. When  $\nu = \nu \mathbb{I}_k$  is a scalar, this above integral gives the covariance

$$\mathbb{E}[Y(t)Y(s)^\top] = \int_{\mathbb{R}} e^{i(s-t)x} (1+x^2)^{-\nu-1/2} (AA^* 1_{\{x>0\}} + \overline{AA^*} 1_{\{x<0\}}) dx.$$

## 2 Simple Model

We consider a more simple model and aim to derive its validity for any dimension  $d$ . In particular, we let  $AA^*$  be a constant matrix. Also, specify a hyperplane that goes through the origin in  $d-1$  dimensions that is the plane of reflection of the nonreversibility. When a vector lies on the hyperplane, the model is reversible; when one lies perpendicular to the hyperplane, the model is at its most nonreversible direction. This formulation is considerably less flexible than the model described above with  $\sigma(\theta)$  and  $AA^*(\theta)$ .

### 2.1 Formulation for general $d$

Let  $\mathbf{h}, \boldsymbol{\omega}$  be vectors in  $\mathbb{R}^d$ , and let  $AA^* = R + iM$  where  $R$  is a  $k \times k$  real positive definite matrix and  $M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$  for some  $m \in \mathbb{R}$ . Let  $\mathbf{a}$  be a vector in  $\mathbb{R}^d$  that describes the plane through the origin for which the non-reversibility is reflected, defined by all  $\boldsymbol{\omega}$  such that  $\mathbf{a}^\top \boldsymbol{\omega} = 0$ .

We want to consider the covariance of

$$C(\mathbf{0}, \mathbf{h}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^\top \mathbf{h}} (1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-\frac{d}{2}} \left( AA^* 1_{\{\mathbf{a}^\top \boldsymbol{\omega} > 0\}} + \overline{AA^*} 1_{\{\mathbf{a}^\top \boldsymbol{\omega} < 0\}} \right) d\boldsymbol{\omega}$$

Plugging in  $AA^* = R + iM$  gives

$$C(\mathbf{0}, \mathbf{h}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^\top \mathbf{h}} (1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-\frac{d}{2}} \left( R + iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} > 0\}} - iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} < 0\}} \right) d\boldsymbol{\omega}$$

and by breaking up the integral we have

$$\begin{aligned} C(\mathbf{0}, \mathbf{h}) &= \int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top \mathbf{h}) (1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-\frac{d}{2}} \left( R + iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} > 0\}} - iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} < 0\}} \right) d\boldsymbol{\omega} \\ &\quad + i \int_{\mathbb{R}^d} \sin(\boldsymbol{\omega}^\top \mathbf{h}) (1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-\frac{d}{2}} \left( R + iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} > 0\}} - iM 1_{\{\mathbf{a}^\top \boldsymbol{\omega} < 0\}} \right) d\boldsymbol{\omega}. \end{aligned}$$

Using the even and odd properties of the cosine and sine functions, respectively, gives

$$C(\mathbf{0}, \mathbf{h}) = R \int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top \mathbf{h}) (1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-\frac{d}{2}} d\boldsymbol{\omega}$$

$$\begin{aligned}
& + i^2 M \int_{\mathbb{R}^d} \sin(\boldsymbol{\omega}^\top \mathbf{h})(1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu - \frac{d}{2}} \left( 1_{\{\mathbf{a}^\top \boldsymbol{\omega} > 0\}} - 1_{\{\mathbf{a}^\top \boldsymbol{\omega} < 0\}} \right) d\boldsymbol{\omega} \\
& = R \int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^\top \mathbf{h})(1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu - \frac{d}{2}} d\boldsymbol{\omega} \\
& \quad - 2M \int_{\boldsymbol{\omega} | \mathbf{a}^\top \boldsymbol{\omega} > 0} \sin(\boldsymbol{\omega}^\top \mathbf{h})(1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu - \frac{d}{2}} d\boldsymbol{\omega}.
\end{aligned}$$

When  $M$  is the 0 matrix, each component is Matern, and the above is a simplified version of Gneiting et al (2010) with scale parameter 1, which is a valid covariance iff  $R$  is positive definite and  $\nu > 0$ . The derivation of the Matern covariance in the univariate case in arbitrary dimension is given in Stein 1999 *Interpolation of Spatial Data*.

Therefore, in the following, we focus on evaluating the second integral:

$$\int_{\boldsymbol{\omega} | \mathbf{a}^\top \boldsymbol{\omega} > 0} \sin(\boldsymbol{\omega}^\top \mathbf{h})(1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu - \frac{d}{2}} d\boldsymbol{\omega} \tag{1}$$

## 2.2 $d = 1$

Consider the case  $d = 1$  where  $\omega$  and  $h$  are scalars and we have

$$\begin{aligned}
& \int_{\omega > 0} \sin(\omega h)(1 + \omega^2)^{-\nu - \frac{1}{2}} d\omega \\
& = \text{sign}(h) |h|^\nu 2^{-\nu-1} \sqrt{\pi} \Gamma(-\nu + 1/2) (I_\nu(|h|) - \mathbf{L}_{-\nu}(|h|))
\end{aligned}$$

given on page 332 of Watson: *Theory of Bessel Functions* and  $I_\nu$  is the modified Bessel function and  $\mathbf{L}_\nu$  is the modified Struve function. Note that the above is undefined for  $\nu = 1/2, 3/2, \dots$  due to the gamma function, where we have extended the gamma function to the negative nonintegers.

What happens when  $\nu = 1/2, 3/2, \dots$ ?

## 2.3 $d = 2$

For the 2-dimensional case, we switch to polar coordinates. Note that  $\sin(\boldsymbol{\omega}^\top \mathbf{h}) = \sin(r \|\mathbf{h}\| \cos(\theta))$  where  $\theta$  is the angle between  $\boldsymbol{\omega}$  and  $\mathbf{h}$  and  $r = \|\boldsymbol{\omega}\|$ . Assume these polar coordinates, and let  $c$  be the angle that  $\mathbf{h}$  makes with the  $x$  axis. Finally, assume without loss of generality that  $\mathbf{a} = (0, 1)^\top$  so that we integrate over the upper half of  $\mathbb{R}^2$ . See the figure below for a visual representation. Then the integral (1) is

$$\begin{aligned}
& \int_{\boldsymbol{\omega} | \mathbf{a}^\top \boldsymbol{\omega} > 0} \sin(\boldsymbol{\omega}^\top \mathbf{h})(1 + \boldsymbol{\omega}^\top \boldsymbol{\omega})^{-\nu-1} d\boldsymbol{\omega} \\
& = \int_0^\infty \int_{-\pi+c}^c \sin(r \|\mathbf{h}\| \cos(\theta))(1 + r^2)^{-\nu-1} r d\theta dr.
\end{aligned}$$

Now, by 12.1.7 of Abramowitz and Stegun *Handbook of Mathematical Functions*,

$$\int_0^{\pi/2} \sin(r \|\mathbf{h}\| \cos(\theta)) d\theta = \frac{\pi}{2} \mathbf{H}_0(r \|\mathbf{h}\|)$$

where  $\mathbf{H}_\nu$  is the Struve function. One can further see that

$$\int_{\pi/2}^\pi \sin(r \|\mathbf{h}\| \cos(\theta)) d\theta = -\frac{\pi}{2} \mathbf{H}_0(r \|\mathbf{h}\|) \quad \int_{-\pi/2}^0 \sin(r \|\mathbf{h}\| \cos(\theta)) d\theta = \frac{\pi}{2} \mathbf{H}_0(r \|\mathbf{h}\|)$$

based on properties of  $\sin$  and  $\cos$ . We can now evaluate for two values of  $c$ :

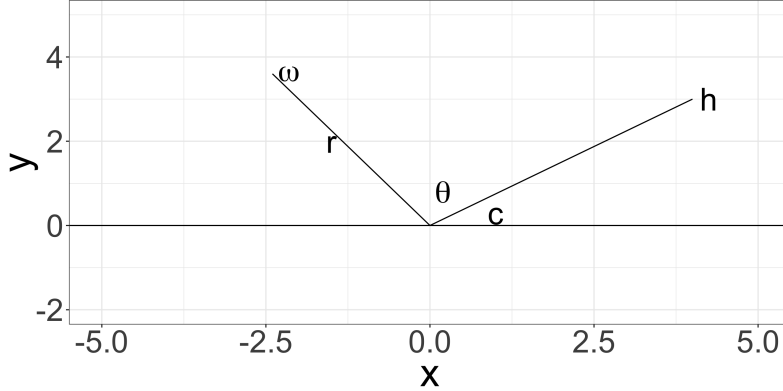
1.  $c = \pi$ : the above integral is 0, which makes sense since this is the reversible direction as defined. This non-reversible part of the covariance is 0.
2.  $c = \pi/2$ : We have

$$\begin{aligned} \int_0^\infty \int_{-\pi/2}^{\pi/2} \sin(r \|\mathbf{h}\| \cos(\theta)) (1+r^2)^{-\nu-1} r d\theta dr &= \int_0^\infty \pi \mathbf{H}_0(r \|\mathbf{h}\|) (1+r^2)^{-\nu-1} r dr \\ &= \pi \frac{2^{-\nu-1} \pi \|\mathbf{h}\|^\nu}{\Gamma(\nu+1) \cos(\nu\pi)} (I_\nu(\|\mathbf{h}\|) - \mathbf{L}_{-\nu}(\|\mathbf{h}\|)) \end{aligned}$$

by 6.814 of I.S. Gradshteyn and I.M. Ryzhik *Tables of Integrals, Series, and Products*. This gives the non-reversible part perpendicular to the reversible direction. It looks similar to the 1-d case.

However, this does not work in general because  $c \notin \{\pi, \pi/2\}$  necessarily.

An example of what is going on below. For a fixed  $\mathbf{h}$ , one integrates  $\omega$  against the upper half plane. When  $\mathbf{h}$  is in line with the  $x$  axis, the integral is 0 and the model is reversible across this axis; when  $\mathbf{h}$  points directly up or down,  $c = \pi/2$  and the model is at its most non-reversible direction.



By (1.8) on page 24 of *Theory of incomplete cylindrical functions and their applications* by M. M. Agrest M. S. Maksimov, we have

$$\int_0^c \sin(r \|\mathbf{h}\| \cos(\theta)) d\theta = \frac{\pi}{2} \mathbf{H}_0(c, r \|\mathbf{h}\|)$$

where  $\mathbf{H}_\nu(c, z)$  is the incomplete Struve function. Thus, we are left to evaluate

$$\int_0^\infty \frac{\pi}{2} (\mathbf{H}_0(c, r \|\mathbf{h}\|) + \mathbf{H}_0(\pi - c, r \|\mathbf{h}\|)) (1+r^2)^{-\nu-1} r dr$$

What is the value of the integral  $\int_0^\infty \mathbf{H}_0(c, r \|\mathbf{h}\|) (1+r^2)^{-\nu-1} r dr$ ?

By (5.10) on page 183 of *Theory of incomplete cylindrical functions and their applications* and setting  $\mu = 2$  and  $\nu = 0$ , we have

$$\int_0^\infty \frac{-\mathbf{H}_0(c, ax)}{(x^2 + k^2)^{m+1}} x dx = \frac{\pi}{m!} (-1)^{m+1} \left( \frac{d}{dk^2} \right)^m \left\{ \frac{I_0(c, ak) - \mathbf{L}_0(c, ak)}{2} \right\}$$

where  $I_0(\cdot, \cdot)$  is the incomplete modified Bessel function of the first kind and  $\mathbf{L}_0(\cdot, \cdot)$  is the incomplete modified Struve function. When  $c = \pi/2$  these reduce to their “complete” versions. This looks close but not exactly right ( $m$  is an integer,  $\nu$  should be flexible, derivative).