CHAPTER 2

DIFFERENTIATION AND INTEGRATION TO INTEGER ORDER

Our point of view in this work is that many results about generalized derivatives and integrals are motivated by and follow rather naturally from classical results about derivatives and integrals of integer order. Accordingly, after first establishing some notational conventions, we shall present familiar definitions and a number of results for ordinary multiple derivatives and integrals. In this way we hope to motivate the corresponding definitions and results for derivatives and integrals of arbitrary order when they are encountered in Chapter 3.

2.1 SYMBOLISM

We are accustomed to the use of the notation

$$\frac{d^n f}{dx^n}$$

for the nth derivative of a function f with respect to x when n is a nonnegative integer. Because integration and differentiation are inverse operations,[1] it is natural to associate the symbolism

$$\frac{d^{-1} f}{[dx]^{-1}}$$

with indefinite integration of f with respect to x. However, it is necessary to stipulate a lower limit of integration in order that an indefinite integral be

[1] Except when they are not! See the discussion in Section 2.3 relating to classical differentiation and integration and the more general discussion in Section 5.7.

completely specified. We choose to associate the above symbol with the lower limit zero. Hence we define

$$\frac{d^{-1}f}{[dx]^{-1}} \equiv \int_0^x f(y)\, dy.$$

Multiple integration with zero lower limit is symbolized by the natural extensions

$$\frac{d^{-2}f}{[dx]^{-2}} \equiv \int_0^x dx_1 \int_0^{x_1} f(x_0)\, dx_0,$$

$$\vdots$$

$$\frac{d^{-n}f}{[dx]^{-n}} \equiv \int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \cdots \int_0^{x_2} dx_1 \int_0^{x_1} f(x_0)\, dx_0.$$

Use is made of the identity

$$\int_a^x f(y)\, dy = \int_0^{x-a} f(y+a)\, dy$$

to extend the symbolism to lower limits other than zero. Thus we define

$$\frac{d^{-1}f}{[d(x-a)]^{-1}} \equiv \int_a^x f(y)\, dy,$$

$$\vdots$$

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} \equiv \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \cdots \int_a^{x_2} dx_1 \int_a^{x_1} f(x_0)\, dx_0.$$

Caution must be exercised against subconsciously carrying over the equivalence

$$\frac{d^n}{[d(x-a)]^n} = \frac{d^n}{dx^n}$$

characteristic of a local operator to negative orders (or, as we shall see in Section 5.8, to fractional orders of either sign), because, of course,

$$\frac{d^{-n}}{[d(x-a)]^{-n}} \neq \frac{d^{-n}}{[dx]^{-n}}$$

in general. We use these new symbols because of the clumsiness of the conventional symbolism for multiple integration and in order to achieve a unified terminology which embraces both integration and differentiation. We prefer our symbolism to

$$p^{-n} \qquad \text{and} \qquad I^n$$

used in the literature (Riesz, 1949; Duff, 1956; Courant and Hilbert, 1962) for n-fold integration because these do not specify the lower limit, nor do they lend themselves readily to symbolize integrations with respect to a variable other than the independent variable, x, of the function under discussion.

The symbol $f^{(n)}$ finds frequent use as an abbreviation for $d^nf/[dx]^n$: it has the merit of brevity. Likewise we shall occasionally use $f^{(-n)}$ to symbolize an n-fold integration of f with respect to x, the lower limits being unspecified. That is,

$$f^{(-n)} \equiv \int_{a_n}^{x} dx_{n-1} \int_{a_{n-1}}^{x_{n-1}} dx_{n-2} \cdots \int_{a_2}^{x_2} dx_1 \int_{a_1}^{x_1} f(x_0)\, dx_0 ,$$

where $a_1, a_2, \ldots, a_n$ are completely arbitrary. However, when we write a difference such as $f^{(-n)}(x) - f^{(-n)}(a)$ we intend that the same lower limits $a_1, a_2, \ldots, a_n$ attach to each integral. Finally, we notice that our habit has been to write f rather than $f(x)$ and

$$\frac{d^{-1}f}{[d(x-a)]^{-1}}$$

rather than

$$\frac{d^{-1}f}{[d(x-a)]^{-1}}(x)$$

since f and $d^{-1}f/[d(x-a)]^{-1}$ are understood to be functions of the independent variable x. When we have need to specify the value of x at which a function is to be evaluated we shall adopt symbols such as

$$f(a) \qquad \text{or} \qquad \left[\frac{d^{-1}f}{[d(x-a)]^{-1}}\right]_{x=x_0}$$

for that purpose.

2.2 CONVENTIONAL DEFINITIONS

In this section we shall review standard definitions of repeated derivatives and integrals as limits of difference quotients and sums, respectively. Our aim is to embrace both kinds of operations within a single formula and thus to pave the way for the extension we seek to differintegral[2] operators to arbitrary order.

[2] This word seems to us a natural one to describe a class of operators that includes both differential and integral operators as special cases.

We begin with the familiar definition of the first derivative in terms of a backward difference

$$\frac{d^1 f}{dx^1} \equiv \frac{d}{dx} f(x) \equiv \lim_{\delta x \to 0} \{[\delta x]^{-1}[f(x) - f(x - \delta x)]\}.$$

Similarly,

$$\frac{d^2 f}{dx^2} \equiv \lim_{\delta x \to 0} \{[\delta x]^{-2}[f(x) - 2f(x - \delta x) + f(x - 2\delta x)]\}$$

and

$$\frac{d^3 f}{dx^3} \equiv \lim_{\delta x \to 0} \{[\delta x]^{-3}[f(x) - 3f(x - \delta x) + 3f(x - 2\delta x) - f(x - 3\delta x)]\},$$

etc., where we have, of course, assumed that the indicated limits exist.

Observe that each derivative involves one more functional evaluation than the order of the derivative, and that the coefficients build up as binomial coefficients and alternate in sign. This suggests the general formula for positive integer n,

$$\frac{d^n f}{dx^n} \equiv \lim_{\delta x \to 0} \left\{ [\delta x]^{-n} \sum_{j=0}^{n} [-]^j \binom{n}{j} f(x - j\,\delta x) \right\}.$$

If the nth derivative of f exists, this last equation does indeed define $d^n f/dx^n$ as an unrestricted limit, i.e., as a limit as δx tends to zero through values that are totally unrestricted. In order to unify this formula with the one which defines an integral as a limit of a sum, it is desirable to define derivatives in terms of a restricted limit, namely, as a limit as δx tends to zero through discrete values only. To do this, choose $\delta_N x \equiv [x - a]/N$, $N = 1, 2, \ldots$, where a is a number smaller than x and plays a role akin to a lower limit. Then, since if the unrestricted limit exists so does the restricted limit and they are equal, the nth derivative may be defined as

$$\frac{d^n f}{[dx]^n} \equiv \lim_{\delta_N x \to 0} \left\{ [\delta_N x]^{-n} \sum_{j=0}^{n} [-]^j \binom{n}{j} f(x - j\,\delta_N x) \right\}.$$

Now, since $\binom{n}{j} = 0$ if $j > n$ when n is integer, the above may be rewritten

$$\text{(2.2.1)} \qquad \frac{d^n f}{[dx]^n} \equiv \lim_{\delta_N x \to 0} \left\{ [\delta_N x]^{-n} \sum_{j=0}^{N-1} [-]^j \binom{n}{j} f(x - j\,\delta_N x) \right\}$$

$$\equiv \lim_{N \to \infty} \left\{ \left[\frac{x-a}{N}\right]^{-n} \sum_{j=0}^{N-1} [-]^j \binom{n}{j} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\}.$$

Equation (2.2.1) will henceforth be adopted as defining $d^n f/[dx]^n$ with the understanding that the limit exists in the usual, unrestricted sense.

Turning our attention from derivatives to integrals, we begin with the usual definition of an integral as a limit of a Riemann sum. Using the symbolism of Section 2.1,

$$\frac{d^{-1}f}{[d(x-a)]^{-1}} \equiv \int_a^x f(y)\,dy$$
$$\equiv \lim_{\delta_N x\to 0}\{\delta_N x[f(x)+f(x-\delta_N x)+f(x-2\,\delta_N x)+\cdots$$
$$+f(a+\delta_N x)]\}$$
$$\equiv \lim_{\delta_N x\to 0}\left\{\delta_N x\sum_{j=0}^{N-1} f(x-j\,\delta_N x)\right\},$$

where $\delta_N x \equiv [x-a]/N$, as before. Application of the same definition to a double integral gives

$$\frac{d^{-2}f}{[d(x-a)]^{-2}} \equiv \int_a^x dx_1 \int_a^{x_1} f(x_0)\,dx_0$$
$$\equiv \lim_{\delta_N x\to 0}\{[\delta_N x]^2[f(x)+2f(x-\delta_N x)+3f(x-2\delta_N x)+\cdots$$
$$+Nf(a+\delta_N x)]\}$$
$$\equiv \lim_{\delta_N x\to 0}\left\{[\delta_N x]^2\sum_{j=0}^{N-1}[j+1]f(x-j\,\delta_N x)\right\}.$$

We need to do one more iteration to obtain a clearer picture of the general formula:

$$\frac{d^{-3}f}{[d(x-a)]^{-3}} \equiv \int_a^x dx_2 \int_a^{x_2} dx_1 \int_a^{x_1} f(x_0)\,dx_0$$
$$\equiv \lim_{\delta_N x\to 0}\left\{[\delta_N x]^3\sum_{j=0}^{N-1}\frac{[j+1][j+2]}{2}f(x-j\,\delta_N x)\right\}.$$

This time we notice the coefficients building up as $\binom{j+n-1}{j}$, where n is the order of the integral, and all the signs are positive. Therefore,

$$\text{(2.2.2)}\qquad \frac{d^{-n}f}{[d(x-a)]^{-n}} \equiv \lim_{\delta_N x\to 0}\left\{[\delta_N x]^n\sum_{j=0}^{N-1}\binom{j+n-1}{j}f(x-j\,\delta_N x)\right\}$$
$$\equiv \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^n\sum_{j=0}^{N}\binom{j+n-1}{j}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\}.$$

We now compare formulas (2.2.2) and (2.2.1), recalling equation (1.3.16),

$$[-]^j\binom{n}{j} = \binom{j-n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)},$$

and see that formulas (2.2.2) and (2.2.1) are embraced in the equation

$$\text{(2.2.3)} \qquad \frac{d^q f}{[d(x-a)]^q} \equiv \lim_{N\to\infty}\left\{\frac{\left[\dfrac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\},$$

where q is an integer of either sign. We shall have more to say about this formula later when we introduce it as our most basic definition of a differintegral.

2.3 COMPOSITION RULE FOR MIXED INTEGER ORDERS

The identities

$$\text{(2.3.1)} \qquad \frac{d^n}{[dx]^n}\left\{\frac{d^N f}{[dx]^N}\right\} = \frac{d^{n+N}f}{[dx]^{n+N}} = \frac{d^N}{[dx]^N}\left\{\frac{d^n f}{[dx]^n}\right\}$$

and

(2.3.2)

$$\frac{d^{-n}}{[d(x-a)]^{-n}}\left\{\frac{d^{-N}f}{[d(x-a)]^{-N}}\right\} = \frac{d^{-n-N}f}{[d(x-a)]^{-n-N}} = \frac{d^{-N}}{[d(x-a)]^{-N}}\left\{\frac{d^{-n}f}{[d(x-a)]^{-n}}\right\}$$

are obeyed when n and N are nonnegative integers; indeed these identities are basic to the concepts of multiple differentiation and integration. This section seeks to investigate the conditions, if any, required to identify

$$\text{(2.3.3)} \qquad \frac{d^{-n}}{[d(x-a)]^{-n}}\left\{\frac{d^N f}{[d(x-a)]^N}\right\},$$

$$\text{(2.3.4)} \qquad \frac{d^{N-n}f}{[d(x-a)]^{N-n}},$$

and

$$\text{(2.3.5)} \qquad \frac{d^N}{[d(x-a)]^N}\left\{\frac{d^{-n}f}{[d(x-a)]^{-n}}\right\},$$

and to evaluate the difference when any two are unequal.

Let f be a function which is at least N-fold differentiable $[N \geqq 1]$; then by formula (2.3.1) and the fundamental theorem of the calculus

$$\frac{d^{-1}f^{(N)}}{[d(x-a)]^{-1}} = f^{(N-1)}(x) - f^{(N-1)}(a),$$

with the understanding that $f^{(0)}(x) \equiv f(x)$. Next we integrate from a to x a second time. Making use of equation (2.3.2), the left-hand side becomes $d^{-2}f^{(N)}/[d(x-a)]^{-2}$. There are, however, two cases to consider in treating the right-hand side. When $N \geqq 2$, repetition of the use of formula (2.3.1), the fundamental theorem of the calculus and the rule for integrating a constant, leads to

$$\frac{d^{-2}f^{(N)}}{[d(x-a)]^{-2}} = f^{(N-2)}(x) - f^{(N-2)}(a) - [x-a]f^{(N-1)}(a).$$

When $N = 1$, this same result is obtained utilizing the identity

$$\frac{d^{-1}f}{[d(x-a)]^{-1}} = \int_a^x f(y)\,dy = f^{(-1)}(x) - f^{(-1)}(a). \tag{2.3.6}$$

The relationship

$$\frac{d^{-n}f^{(N)}}{[d(x-a)]^{-n}} = f^{(N-n)}(x) - \sum_{k=0}^{n-1} \frac{[x-a]^k}{k!} f^{(N+k-n)}(a) \tag{2.3.7}$$

follows from n repetitions of this procedure.[3]

Notice that equation (2.3.7) becomes

$$f(x) = \sum_{k=0}^{n-1} \frac{[x-a]^k}{k!} f^{(k)}(a) + R_n$$

after setting $N = n$ and rearrangement. This is Taylor's formula in which the remainder

$$R_n = \frac{d^{-n}f^{(n)}}{[d(x-a)]^{-n}}$$

is expressed as the n-fold integral of the n-fold derivative of f.

Upon repeated integration of equation (2.3.6), we obtain

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} = f^{(-n)}(x) - \sum_{k=0}^{n-1} \frac{[x-a]^k}{k!} f^{(k-n)}(a). \tag{2.3.8}$$

Differentiation, using Leibniz's theorem (Abramowitz and Stegun, 1964, p. 11) for differentiating an integral, gives

$$\frac{d}{d(x-a)}\left\{\frac{d^{-n}f}{[d(x-a)]^{-n}}\right\} = f^{(1-n)}(x) - \sum_{k=1}^{n-1} \frac{[x-a]^{k-1}}{(k-1)!} f^{(k-n)}(a),$$

[3] By setting $N = n$ in equation (2.3.7), we see that the operators d^n and d^{-n} are not inverse to each other unless the function f as well as its first $n-1$ derivatives vanish at $x = a$. (Recall footnote 1 in this chapter.)

and after N such differentiations the equation

$$\frac{d^N}{[d(x-a)]^N}\left\{\frac{d^{-n}f}{[d(x-a)]^{-n}}\right\} = f^{(N-n)}(x) - \sum_{k=N}^{n-1} \frac{[x-a]^{k-N}}{(k-N)!} f^{(k-n)}(a) \tag{2.3.9}$$

emerges.[4]

If $N \geqq n$, we have

$$\frac{d^{N-n}f}{[d(x-a)]^{N-n}} = f^{(N-n)} \tag{2.3.10}$$

because the choice of lower limit does not affect differentiation. However, if $N < n$, by analogy with equation (2.3.8),

$$\frac{d^{N-n}f}{[d(x-a)]^{N-n}} = f^{(N-n)}(x) - \sum_{k=0}^{n-N-1} \frac{[x-a]^k}{k!} f^{(k+N-n)}(a), \tag{2.3.11}$$

a result which encompasses equation (2.3.10). After a redefinition of the summation index in expression (2.3.9), we see that the right-hand sides and, therefore, the left-hand sides of equations (2.3.9) and (2.3.11) are identical.

Summarizing, we have demonstrated that

$$\begin{aligned}\frac{d^N}{[d(x-a)]^N}\left\{\frac{d^{-n}f}{[d(x-a)]^{-n}}\right\} &= \frac{d^{N-n}f}{[d(x-a)]^{N-n}} \\ &= \frac{d^{-n}}{[d(x-a)]^{-n}}\left\{\frac{d^Nf}{[d(x-a)]^N}\right\} \\ &\quad + \sum_{k=n-N}^{n-1} \frac{[x-a]^k}{k!} f^{(k+N-n)}(a).\end{aligned} \tag{2.3.12}$$

That is, the composition "rule"

$$\frac{d^q}{[d(x-a)]^q}\left\{\frac{d^Qf}{[d(x-a)]^Q}\right\} = \frac{d^{q+Q}f}{[d(x-a)]^{q+Q}}$$

(when q and Q are integers) necessarily holds unless Q is positive and q is negative; in words, unless f is first differentiated and then integrated. The identity of expression (2.3.3) with the other two will hold only if $f(a) = 0$ and if all derivatives of f through the $(N-1)$th are also zero at $x = a$.

As an example of the failure of the equality for mixed signs, consider $f = \exp(2x) + 1$ with $N = 1$, $n = 3$, and $a = 0$. Then

$$\begin{aligned}\frac{d}{dx}\left\{\frac{d^{-3}}{[dx]^{-3}}[\exp(2x)+1]\right\} &= \frac{d^{-2}}{[dx]^{-2}}[\exp(2x)+1] \\ &= \tfrac{1}{4}\exp(2x) + \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{4},\end{aligned} \tag{2.3.13}$$

[4] Notice that the summation is empty if $N \geqq n$.

whereas

$$(2.3.14)\qquad \frac{d^{-3}}{[dx]^{-3}}\left\{\frac{d}{dx}[\exp(2x)+1]\right\} = \tfrac{1}{4}\exp(2x) - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{4}.$$

The difference x^2 between expressions (2.3.13) and (2.3.14) is correctly given by formula (2.3.12) as

$$\sum_{k=2}^{2} \frac{x^k}{k!} f^{(k-2)}(0).$$

2.4 DEPENDENCE OF MULTIPLE INTEGRALS ON LOWER LIMIT

A multiple indefinite integral depends in a rather complex fashion on the magnitude of its lower limit. Thus, making use of equation (2.3.8),

(2.4.1)

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} - \frac{d^{-n}f}{[d(x-b)]^{-n}} = \sum_{k=0}^{n-1} \frac{1}{k!}\left\{[x-b]^k f^{(k-n)}(b) - [x-a]^k f^{(k-n)}(a)\right\}.$$

Apart from the exceptional $n = 1$ case,

$$\frac{d^{-1}f}{[d(x-a)]^{-1}} - \frac{d^{-1}f}{[d(x-b)]^{-1}} = f^{(-1)}(b) - f^{(-1)}(a),$$

the difference in formula (2.4.1) depends not only on the lower limits, but also on the upper limit x.

If we seek the explicit dependence on $b - a$ of shifting the lower limit from b to a, we start with equation (2.4.1) and proceed as follows:

$$\begin{aligned}
\frac{d^{-n}f}{[d(x-a)]^{-n}} &- \frac{d^{-n}f}{[d(x-b)]^{-n}} - \sum_{k=0}^{n-1} \frac{[x-b]^k}{k!} f^{(k-n)}(b) \\
&= -\sum_{k=0}^{n-1} \frac{[(x-b)+(b-a)]^k}{k!} f^{(k-n)}(a) \\
&= -\sum_{k=0}^{n-1} f^{(k-n)}(a) \sum_{j=0}^{k} \frac{[x-b]^j [b-a]^{k-j}}{j!(k-j)!} \\
&= -\sum_{K=0}^{n-1} \frac{[x-b]^K}{K!} \sum_{J=K}^{n-1} \frac{[b-a]^{J-K}}{(J-K)!} f^{(J-n)}(a) \\
&= -\sum_{K=0}^{n-1} \frac{[x-b]^K}{K!} \sum_{j=0}^{n-K-1} \frac{[b-a]^j}{j!} f^{(j+K-n)}(a),
\end{aligned}$$

where there have been many changes of summation index. Identifying k with K and later replacing $n - K$ by k, we find

$$\begin{aligned}\frac{d^{-n}f}{[d(x-a)]^{-n}} - \frac{d^{-n}f}{[d(x-b)]^{-n}} &= \sum_{K=0}^{n-1} \frac{[x-b]^K}{K!}\left[f^{(K-n)}(b) - \sum_{j=0}^{n-K-1}\frac{[b-a]^j}{j!} f^{(K+j-n)}(a)\right] \\ &= \sum_{k=1}^{n} \frac{[x-b]^{n-k}}{(n-k)!}\left[f^{(-k)}(b) - \sum_{j=0}^{k-1}\frac{[b-a]^j}{j!} f^{(j-k)}(a)\right] \\ &= \sum_{k=1}^{n} \frac{[x-b]^{n-k}}{(n-k)!}\frac{d^{-k}f(b)}{[d(b-a)]^{-k}},\end{aligned}$$

where equation (2.3.8) was used once again for the final step.

2.5 PRODUCT RULE FOR MULTIPLE INTEGRALS

We are interested in establishing a rule for multiple integration of a product of two functions, similar to Leibniz's theorem for repeatedly differentiating a product. In Section 5.5 a more general result will be established that embraces both Leibniz's theorem and the result of this section.

We begin with the familiar formula for integration by parts:

$$\int_a^x g(y)\,dv(y) = g(x)v(x) - g(a)v(a) - \int_a^x v(y)\,dg(y). \tag{2.5.1}$$

Let

$$v(y) = \int_a^y f(z)\,dz$$

in (2.5.1); then

$$\int_a^x g(y)f(y)\,dy = g(x)\int_a^x f(z)\,dz - \int_a^x \left[\int_a^y f(z)\,dz\right]\frac{dg(y)}{dy}\,dy$$

or, in the symbolism of Section 2.1,

$$\frac{d^{-1}[fg]}{[d(x-a)]^{-1}} = g\frac{d^{-1}f}{[d(x-a)]^{-1}} - \frac{d^{-1}}{[d(x-a)]^{-1}}\left\{g^{(1)}\frac{d^{-1}f}{[d(x-a)]^{-1}}\right\}. \tag{2.5.2}$$

When (2.5.2) is applied to the product in braces, and the composition rule (2.3.2) evoked, one has

$$\frac{d^{-1}[fg]}{[d(x-a)]^{-1}} = g\frac{d^{-1}f}{[d(x-a)]^{-1}} - g^{(1)}\frac{d^{-2}f}{[d(x-a)]^{-2}} + \frac{d^{-1}}{[d(x-a)]^{-1}}\left\{g^{(2)}\frac{d^{-2}f}{[d(x-a)]^{-2}}\right\}$$

and indefinite repetition of this process gives [utilizing equation (1.3.16)]

$$\frac{d^{-1}[fg]}{[d(x-a)]^{-1}} = \sum_{j=0}^{\infty}[-]^{j}g^{(j)}\frac{d^{-1-j}f}{[d(x-a)]^{-1-j}} = \sum_{j=0}^{\infty}\binom{-1}{j}g^{(j)}\frac{d^{-1-j}f}{[d(x-a)]^{-1-j}}. \tag{2.5.3}$$

When (2.5.3) is integrated, using this same formula inside the summation, and the composition rules for integrals and derivatives (2.3.2) and (2.3.1) are applied, we obtain

$$\begin{aligned}\frac{d^{-2}[fg]}{[d(x-a)]^{-2}} &= \sum_{j=0}^{\infty}\binom{-1}{j}\sum_{k=0}^{\infty}\binom{-1}{k}g^{(j+k)}\frac{d^{-j-k-2}f}{[d(x-a)]^{-j-k-2}} \\ &= \sum_{j=0}^{\infty}\sum_{l=j}^{\infty}\binom{-1}{j}\binom{-1}{l-j}g^{(l)}\frac{d^{-2-l}f}{[d(x-a)]^{-2-l}} \\ &= \sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{-1}{j}\binom{-1}{l-j}g^{(l)}\frac{d^{-2-l}f}{[d(x-a)]^{-2-l}} \\ &= \sum_{l=0}^{\infty}\binom{-2}{l}g^{(l)}\frac{d^{-2-l}f}{[d(x-a)]^{-2-l}}.\end{aligned} \tag{2.5.4}$$

In the penultimate and final steps of (2.5.4), we have made use of the permutation[5]

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}$$

and of the equation (1.3.20).

Iteration of the procedure that produced (2.5.4) from (2.5.3) leads to the desired formula:

$$\frac{d^{-n}[fg]}{[d(x-a)]^{-n}} = \sum_{j=0}^{\infty}\binom{-n}{j}g^{(j)}\frac{d^{-n-j}f}{[d(x-a)]^{-n-j}}, \qquad n = 1, 2, 3, \ldots .$$

[5] This identity is easily established, much as is the analogous permutation of variables in the double integral $\int_0^\infty dy\int_0^y dx = \int_0^\infty dx\int_x^\infty dy$.

2.6 THE CHAIN RULE FOR MULTIPLE DERIVATIVES

The chain rule for differentiation,

$$\frac{d}{dx} g(f(x)) = \frac{d}{du} g(u) \frac{d}{dx} f(x) = g^{(1)} f^{(1)}, \tag{2.6.1}$$

which enables $g(u)$ to be differentiated with respect to x if the derivatives of $g(u)$ with respect to u and of u with respect to x are known, is one of the most useful in the differential calculus. In this section we report the extension of this rule to higher orders of differentiation. For brevity, we shall use $g^{(m)}$ and $f^{(n)}$ to denote

$$\frac{d^m}{du^m} g(u) \qquad \text{and} \qquad \frac{d^n}{dx^n} f(x),$$

respectively.

By application of (2.6.1) and Leibniz's rule for differentiating a product, we find

$$\begin{aligned}
\frac{d^2}{dx^2} g(f(x)) &= \frac{d}{dx}\left[\frac{d}{dx} g(f(x))\right] \\
&= \frac{d}{dx}\left[\frac{d}{du} g(u) \frac{d}{dx} f(x)\right] \\
&= \left[\frac{d}{dx}\frac{d}{du} g(u)\right]\left[\frac{d}{dx} f(x)\right] + \left[\frac{d}{du} g(u)\right]\left[\frac{d^2}{dx^2} f(x)\right] \\
&= \left[\frac{du}{dx}\frac{d^2}{du^2} g(u)\right]\left[\frac{d}{dx} f(x)\right] + \left[\frac{d}{du} g(u)\right]\left[\frac{d^2}{dx^2} f(x)\right] \\
&= g^{(1)} f^{(2)} + g^{(2)}[f^{(1)}]^2.
\end{aligned}$$

The repetition of similar procedures yields successively

$$\frac{d^3}{dx^3} g(f(x)) = g^{(1)} f^{(3)} + 3g^{(2)} f^{(1)} f^{(2)} + g^{(3)}[f^{(1)}]^3,$$

$$\begin{aligned}
\frac{d^4}{dx^4} g(f(x)) &= g^{(1)} f^{(4)} + 4g^{(2)} f^{(1)} f^{(3)} + 6g^{(2)}[f^{(2)}]^2 \\
&\quad + 6g^{(3)}[f^{(1)}]^2 f^{(2)} + g^{(4)}[f^{(1)}]^4,
\end{aligned}$$

$$\begin{aligned}
\frac{d^5}{dx^5} g(f(x)) &= g^{(1)} f^{(5)} + 5g^{(2)} f^{(1)} f^{(4)} + 10g^{(3)}[f^{(1)}]^2 f^{(3)} \\
&\quad + 30g^{(3)} f^{(1)}[f^{(2)}]^2 + 10g^{(4)}[f^{(1)}]^3 f^{(2)} + g^{(5)}[f^{(1)}]^5,
\end{aligned}$$

etc. The generalization to order n produces Faà di Bruno's formula (Abramowitz and Stegun, 1964, p. 823)

$$\frac{d^n}{dx^n} g(f(x)) = n! \sum_{m=1}^{n} g^{(m)} \sum \prod_{k=1}^{n} \frac{1}{P_k!} \left[\frac{f^{(k)}}{k!}\right]^{P_k},$$

where $\sum$ extends over all combinations of nonnegative integer values of $P_1, P_2, \ldots, P_n$ such that

$$\sum_{k=1}^{n} kP_k = n \qquad \text{and} \qquad \sum_{k=1}^{n} P_k = m.$$

Faà di Bruno's formula is sufficiently complicated to be of little general utility for large n. However, certain specific instances of f and g reveal its power. Thus when $f = \exp(x)$,

$$\frac{d^n}{dx^n} g(\exp(x)) = \exp(nx) \sum_{m=1}^{n} S_n^{[m]} g^{(m)},$$

where $S_n^{[m]}$ is a Stirling number of the second kind (see Section 1.3). Again, if $g = u^2$, we find

$$\frac{d^n}{dx^n} [f]^2 = 2f^{(n)}f + \sum_{k=1}^{n-1} \binom{n}{k} f^{(k)} f^{(n-k)},$$

a result that may also be obtained from Leibniz's theorem for the multiple differentiation of a product.

2.7 ITERATED INTEGRALS

Consider the formula

$$\text{(2.7.1)} \qquad \frac{d^{-1}f}{[d(x-a)]^{-1}} \equiv \int_a^x f(y)\,dy = \frac{1}{n!}\frac{d^n}{dx^n} \int_a^x [x-y]^n f(y)\,dy,$$

$$n = 0, 1, 2, \ldots .$$

For $n = 0$, (2.7.1) is an identity, while for $n = 1$ it follows easily from Leibniz's theorem for differentiating an integral (Abramowitz and Stegun, 1964, p. 11). For general integer n one need only notice that the evaluation of the integrand on the right-hand side at the upper limit x gives 0, while differentiation n times inside the integral produces $n!f(y)$.

A single integration of (2.7.1) for $n = 1$ produces

$$\frac{d^{-2}f}{[d(x-a)]^{-2}} \equiv \int_a^x dx_1 \int_a^{x_1} f(x_0)\,dx_0 = \frac{1}{1!}\int_a^x [x-y] f(y)\,dy,$$

and an $[n-1]$-fold integration produces Cauchy's formula for repeated integration:

(2.7.2)

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} \equiv \int_a^x dx_{n-1} \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(x_0)\, dx_0 = \frac{1}{(n-1)!} \int_a^x [x-y]^{n-1} f(y)\, dy.$$

Thus an iterated integral may be expressed as a weighted single integral with a very simple weight function, a fact that provides an important clue for generalizations involving noninteger orders.

2.8 DIFFERENTIATION AND INTEGRATION OF SERIES

A great many functions are traditionally described by infinite series expansions. It is of utmost importance, therefore, to understand the conditions that permit term-by-term differentiation or integration of such infinite series. We recall here the two classical results that apply; Section 5.2 gives extensions of these classical results to differintegrals of arbitrary order.

Suppose $f_0, f_1, \ldots$ are functions defined and continuous on the closed interval $a \leqq x \leqq b$. Then

$$\text{(2.8.1)} \qquad \frac{d^{-1}}{[d(x-a)]^{-1}} \left\{ \sum_{j=0}^{\infty} f_j \right\} = \sum_{j=0}^{\infty} \frac{d^{-1} f_j}{[d(x-a)]^{-1}}, \qquad a \leqq x \leqq b,$$

provided the series $\sum f_j$ converges uniformly in the interval $a \leqq x \leqq b$.

The hypotheses required to distribute a derivative through the terms of an infinite series are somewhat different. For this we need each f_j to have continuous derivatives on $a \leqq x \leqq b$. Then

$$\text{(2.8.2)} \qquad \frac{d}{dx} \left\{ \sum_{j=0}^{\infty} f_j \right\} = \sum_{j=0}^{\infty} \frac{df_j}{dx}, \qquad a \leqq x \leqq b,$$

provided $\sum f_j$ converges pointwise and $\sum df_j/dx$ converges uniformly on the interval $a \leqq x \leqq b$.

Thus we see that a uniformly convergent series of continuous functions (which itself defines a continuous function) may be integrated term by term, and that a continuous series of continuously differentiable functions may be differentiated term by term provided that the derived series is uniformly convergent. The reader is referred to the work of Widder (1947) for proofs and further discussion of these theorems.

2.9 DIFFERENTIATION AND INTEGRATION OF POWERS

The previous section dealt with differentiation and integration of infinite series of functions. Throughout much of the rest of the book we shall often encounter series whose general term is the simple power $[x - a]^p$. We collect here the elementary formulas that express $d^q[x - a]^p/[d(x - a)]^q$ for positive and negative integer values of q. We have

$$\text{(2.9.1)} \qquad \frac{d^n[x-a]^p}{dx^n} = p[p-1]\cdots[p-n+1][x-a]^{p-n}, \qquad n = 0, 1, \ldots$$

and

$$\text{(2.9.2)} \qquad \frac{d^{-n}[x-a]^p}{[d(x-a)]^{-n}} \equiv \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \cdots \int_a^{x_1} [x_0 - a]^p \, dx_0$$

$$= \begin{cases} \dfrac{[x-a]^{p+n}}{[p+1][p+2]\cdots[p+n]}, & p > -1, \\ \infty, & p \leqq -1, \end{cases}$$

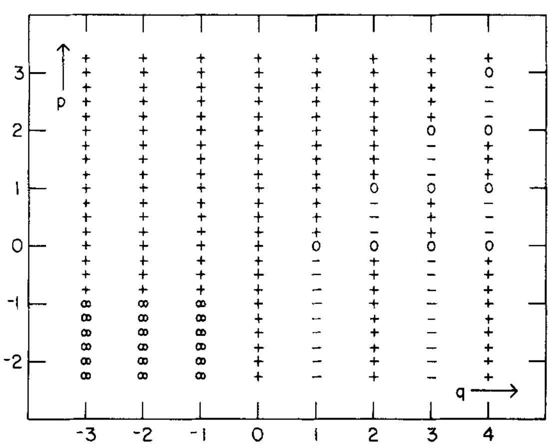

FIG. 2.9.1 The sign of $d^q[x - a]^p/[d(x - a)]^q$ for positive and negative integer values of q.

$n = 1, 2, \ldots$. Invoking the properties of the gamma function (see Section 1.3) and recalling that $d^n/dx^n = d^n/[d(x-a)]^n$, both of these formulas may be unified as

$$\text{(2.9.3)}\qquad \frac{d^q[x-a]^p}{[d(x-a)]^q} = \begin{cases} \dfrac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)} & \begin{cases} q = 0, 1, \ldots; & \text{all } p, \\ q = -1, -2, \ldots; & p > -1, \end{cases} \\ \infty, & \quad q = -1, -2, \ldots; \quad p \leqq -1. \end{cases}$$

The coefficient $\Gamma(p+1)/\Gamma(p-q+1)$ may be positive, negative, or zero, according to the values of p and q. Figure 2.9.1 displays these instances graphically. Chapter 4 will extend formula (2.9.3) to noninteger q values, at which time the utility of the information conveyed by Fig. 2.9.1 should become more apparent.

2.10 DIFFERENTIATION AND INTEGRATION OF HYPERGEOMETRICS

In this section we discuss a generalized hypergeometric function and develop some of its properties, including its behavior under multiple differentiation and integration. This function will prove invaluable in Chapter 9 and is also considered in Section 6.6.

We adopt the abbreviated symbolism

$$\text{(2.10.1)}\qquad \left[x \,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right] = \sum_{j=0}^{\infty} x^j \frac{\prod_{k=1}^{K} \Gamma(j+1+b_k)}{\prod_{l=1}^{L} \Gamma(j+1+c_l)}$$

for what we shall term a $\frac{K}{L}$ hypergeometric. Note that leading terms within this summation will be zero if one of the denominatorial parameters is a negative integer: if $c_l = -n$, terms in $x^0, x^1, x^2, \ldots, x^{n-1}$ are absent. Conversely, infinite leading terms will be present if one of the numeratorial parameters, say b_1, is a negative integer; we consequently encounter this possibility only as the quotient

$$\frac{1}{\Gamma(1-n)}\left[x \,\frac{-n, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]$$

[see the discussion in Section 1.3 surrounding equation (1.3.4)] which represents a polynomial of degree $n-1$. We assume that parameter values and the value of the argument are such as to ensure convergence of the series; this will often imply the restriction $|x|<1$. Representation (2.10.1) proves to be more convenient than the usual symbol (Gradshteyn and Ryzhik, 1965, p. 1045) for generalized hypergeometric functions; the equation

$$\left[x\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]$$
$$= {}_{K+1}F_L(1, 1+b_1, 1+b_2, \ldots, 1+b_K; 1+c_1, \ldots, 1+c_L; x)\,\frac{\prod_{k=1}^{K}\Gamma(1+b_k)}{\prod_{l=1}^{L}\Gamma(1+c_l)}$$

relates the two. Some familiar functions which are instances of such hypergeometrics include

$$\left[x\,\frac{\quad}{\quad}\right] = \frac{1}{1-x},$$

$$\left[x\,\frac{\quad}{0}\right] = \exp(x),$$

$$\left[x\,\frac{\quad}{c}\right] = \exp(x)\gamma^*(c, x),$$

$$\left[x\,\frac{b}{0}\right] = \frac{\Gamma(1+b)}{[1-x]^{1+b}},$$

$$\left[x\,\frac{b}{c}\right] = \frac{\Gamma(1+b)}{\Gamma(c)}\,\frac{\mathrm{B}_x(c, 1+b-c)}{x^c[1-x]^{1+b-c}},$$

$$\left[x\,\frac{b}{0, c}\right] = \frac{\Gamma(1+b)}{\Gamma(1+c)}\,M(1+b, 1+c, x),$$

and

$$\left[x\,\frac{b_1, b_2}{0, c}\right] = \frac{\Gamma(1+b_1)\Gamma(1+b_2)}{\Gamma(1+c)}\,F(1+b_1, 1+b_2; 1+c; x),$$

where $\gamma^*(\,, x)$, $\mathrm{B}_x(\,,\,)$, $M(\,,\,, x)$, and $F(\,,\,;\,; x)$ denote the incomplete gamma, incomplete beta, Kummer, and Gauss functions of x, as defined by Abramowitz and Stegun (1964, Chapters 6, 13, 15).

The relationships

$$\left[x\,\frac{b_1+1, b_2+1, \ldots, b_K+1}{c_1+1, c_2+1, \ldots, c_L+1}\right] = \frac{1}{x}\left[x\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right] - \frac{\Gamma(b_1+1)\Gamma(b_2+1)\cdots\Gamma(b_K+1)}{x\Gamma(c_1+1)\Gamma(c_2+1)\cdots\Gamma(c_L+1)}$$

and

$$\left[x\,\frac{b_1-1, b_2-1, \ldots, b_K-1}{c_1-1, c_2-1, \ldots, c_L-1}\right] = x\left[x\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right] + \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_K)}{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_L)}$$

are immediate consequences of definition (2.10.1). These formulas permit *all* the parameters to be increased or decreased by unity, but recurrences such as

$$\left[x\,\frac{b+1, b_2, \ldots, b_K}{c+1, c_2, \ldots, c_L}\right] = \left[x\,\frac{b, b_2, \ldots, b_K}{c, c_2, \ldots, c_L}\right] + [b-c]\left[x\,\frac{b, b_2, \ldots, b_K}{c+1, c_2, \ldots, c_L}\right]$$

and

$$\left[x\,\frac{b-1, b_2, \ldots, b_K}{c-1, c_2, \ldots, c_L}\right] = \left[x\,\frac{b, b_2, \ldots, b_K}{c, c_2, \ldots, c_L}\right] + [c-b]\left[x\,\frac{b, b_2, \ldots, b_K}{c-1, c_2, \ldots, c_L}\right]$$

enable single parameters to be selectively incremented or decremented. If one of the numeratorial parameters equals one of the denominatorial parameters, they may be "cancelled" as in the example

$$\left[x\,\frac{b}{b, c}\right] = \left[x\,\frac{}{c}\right],$$

thereby reducing the complexity of the hypergeometric function from $\frac{K}{L}$ to $\frac{K-1}{L-1}$.

The rule for multiple differentiation or integration of a generalized hypergeometric function is readily derived from the defining formula (2.10.1) because the differentiation or integration operator may be distributed through the infinite sum of terms. The rule is

$$(2.10.2)\quad \frac{d^q}{dx^q}\left[x \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right] = x^{-q}\left[x \frac{0, b_1, b_2, \ldots, b_K}{-q,\ c_1, c_2, \ldots, c_L}\right],$$

$$q = 0, \pm 1, \pm 2, \ldots .$$

Of perhaps greater interest is the similarly derived formula

$$(2.10.3)\quad \frac{d^q}{dx^q}\left\{x^p\left[x \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\}$$

$$= x^{p-q}\left[x \frac{p, b_1, b_2, \ldots, b_K}{p-q, c_1, c_2, \ldots, c_L}\right], q = 0, \pm 1, \ldots,$$

expressing the operation of differintegration of the product of the power x^p with a generalized hypergeometric function, provided p exceeds -1. These operations will normally convert a $\frac{K}{L}$ hypergeometric function to $\frac{K+1}{L+1}$, increasing its complexity. However, if p and q are suitably chosen, cancellation with existing parameters will be possible, leaving the complexity of the hypergeometric function unchanged, or even reducing it to $\frac{K-1}{L-1}$. This device, generalized to embrace differintegration to arbitrary order, will prove invaluable in Chapter 9.

We conclude this section by demonstrating that a generalized hypergeometric function of argument $x^{1/n}$ (n being a positive integer) and complexity $\frac{K}{L}$ may be equated to the sum of n hypergeometrics of argument $n^{n[K-L]}x$ and complexity $\frac{nK}{nL}$. The lengthy proof, which makes use of the Gauss multiplication formula [equation (1.3.10)] and of the abbreviations $B = \sum b_k$, $C = \sum c_l$, and $2D = K - L$ will be found on p. 44. This rather forbidding formula condenses remarkably for small values of n, K, and L. Thus we find, for example,

$$\left[\sqrt{x} \frac{}{0}\right] = \sqrt{\pi}\left[\tfrac{1}{4}x \frac{}{0, -\frac{1}{2}}\right] + \frac{\sqrt{\pi x}}{2}\left[\tfrac{1}{4}x \frac{}{\frac{1}{2}, 0}\right].$$

$$
\text{(2.10.4)}\quad \left[x^{1/n}\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]
$$

$$
= \sum_{m=0}^{n-1}\sum_{j=0}^{\infty} x^{j+[m/n]}\,\frac{\prod_{k=1}^{K}\Gamma(nj+1+m+b_k)}{\prod_{l=1}^{L}\Gamma(nj+1+m+c_l)}
$$

$$
= \sum_{m=0}^{n-1} n^{m/n}\sum_{j=0}^{\infty} x^{j}\,\frac{\prod_{k=1}^{K}[\sqrt{2\pi}]^{1-n}[n]^{nj+\frac{1}{2}+m+b_k}\prod_{i=0}^{n-1}\Gamma\left(j+1+\dfrac{m+b_k-i}{n}\right)}{\prod_{l=1}^{L}[\sqrt{2\pi}]^{1-n}[n]^{nj+\frac{1}{2}+m+c_l}\prod_{i=0}^{n-1}\Gamma\left(j+1+\dfrac{m+c_l-i}{n}\right)}
$$

$$
= [2\pi]^{D-nD}n^{B-C+D}\sum_{m=0}^{n-1} n^{2Dm}x^{m/n}\sum_{j=0}^{\infty} n^{2nDj}x^{j}\,\frac{\prod_{k=1}^{K}\prod_{i=0}^{n-1}\Gamma\left(j+1+\dfrac{m+b_k-i}{n}\right)}{\prod_{l=1}^{L}\prod_{i=0}^{n-1}\Gamma\left(j+1+\dfrac{m+c_l-i}{n}\right)}
$$

$$
= [2\pi]^{D-nD}n^{B-C+D}\sum_{m=0}^{n-1} n^{2Dm}x^{m/n}
$$

$$
\times\left[n^{2nD}x\,\frac{\dfrac{m+b_1}{n}, \dfrac{m+b_1-1}{n}, \ldots, \dfrac{m+b_1-n+1}{n}, \dfrac{m+b_2}{n}, \ldots, \dfrac{m+b_K-n+1}{n}}{\dfrac{m+c_1}{n}, \dfrac{m+c_1-1}{n}, \ldots, \dfrac{m+c_1-n+1}{n}, \dfrac{m+c_2}{n}, \ldots, \dfrac{m+c_L-n+1}{n}}\right].
$$