INCOMPLETE BESSEL AND STRUVE FUNCTIONS

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ABSTRACT. Some properties are given of the incomplete Bessel and Struve functions defined by a Poisson-type integral. These functions are tabulated for the orders 0 and 1.

1. *Introduction*. In various problems in the diffraction theory of optical instruments, the results are found in terms of incomplete integrals, the complete forms of which are the Poisson integral defining the Bessel function of unit order and the allied integral that defines the Struve function. These integrals occur in the calculation of the illumination in a defocused image of an incoherently illuminated sinusoidal grating (Hopkins (7), Steel (9)). If the grating has a spatial frequency $2\omega\,(0 \leqslant \omega \leqslant 1)$ and ξ' is the defocusing expressed in suitable units, the image contrast is given by

$$T(\xi', \omega) = \frac{4}{\pi}\Re\left\{\exp(-4i\omega^2\xi')\int_\omega^1 \exp(4i\omega\xi' t)(1-t^2)^{\frac{1}{2}}\,dt\right\}. \tag{1}$$

The integral in (1) can be represented as the unit order of a function the general order of which is defined by

$$\mathbf{P}_\nu(x, \omega) = \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\int_\omega^1 e^{ixt}(1-t^2)^{\nu-\frac{1}{2}}\,dt, \tag{2}$$

or, in terms of the angle $\alpha = \operatorname{arc}\cos\omega$,

$$\hat{\mathbf{P}}_\nu(x, \alpha) = \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\int_0^\alpha e^{ix\cos\phi}\sin^{2\nu}\phi\,d\phi. \tag{3}$$

For α negative, $\hat{\mathbf{P}}_\nu(x, -\alpha) = -\hat{\mathbf{P}}_\nu(x, \alpha)$. (4)

Then
$$T(\xi', \omega) = \Re\{\mathbf{P}_1(4\omega\xi', \omega)\exp(-4i\omega^2\xi')\}.$$

The image of the same object in the presence of astigmatism and curvature of field can be evaluated in terms of the same function. If the wave aberration is represented in Nijboer's notation (8) as

$$\Delta = b_{120}\sigma^2r^2 + b_{022}\sigma^2r^2\cos 2(\phi-\theta),$$

ϕ and θ being azimuth angles at the pupil and in the object plane and b_{120} and b_{022} the coefficients of curvature of field and astigmatism respectively, the image contrast is given by

$$\begin{aligned} T(\Delta, \omega) &= \frac{4}{\pi x \sin\psi}\Re\left[e^{-i\omega x\cos\psi}\int_\omega^1 e^{ixt\cos\psi}\sin\{x(1-t^2)^{\frac{1}{2}}\sin\psi\}\,dt\right] \\ &= \tfrac{1}{2}\Re[e^{-i\omega x\cos\psi}\{\hat{\mathbf{P}}_1(x, \alpha-\psi) + \hat{\mathbf{P}}_1(x, \alpha+\psi)\}] - \frac{4\omega}{\pi x}\sin\psi\sin(x\sin\alpha\sin\psi), \end{aligned} \tag{5}$$

where
$$\cos\alpha = \omega,$$
$$x = 8\pi\omega\sigma^2\lambda^{-1}(b_{120}^2 + 2b_{120}b_{022}\cos 2\theta + b_{022}^2)^{\frac{1}{2}},$$
$$\tan\psi = b_{022}\sin 2\theta/(b_{120} + b_{022}\cos 2\theta).$$

A similar integral has been evaluated by Hartree (5) for certain special cases; this integral was derived by Gandy (4) for the amplitude diffraction pattern of the image of a line in an idealized lens. If the optical system has a semi-angle of aperture β, the amplitude is given as

$$F(x,y) = \Re\left\{\frac{e^{-ix}}{2\beta}\int_{-\beta}^{+\beta}\exp i(x\cos\theta - y\sin\theta)\cos\theta\,d\theta\right\}$$

for a point a distance y from the line in a plane at a distance x from the focal plane. *This integral can again be evaluated in terms of* $\mathbf{P}_1$ *in a form similar to equation* (5).

Finally, in their investigation of the effects of collimation and oblique incidence on the position of fringes in length interferometers, Bruce (2) and Thornton (10) give integrals which reduce to the zero order function. The intensity distribution in two-beam interference fringes is given as

$$I = \int_0^\theta \cos^2(K\cos\theta)\,d\theta,$$

where θ is the angular aperture and K the phase. This expression reduces to

$$I = \tfrac{1}{2}\theta + \tfrac{1}{4}\pi\Re\{\hat{\mathbf{P}}_0(2K,\theta)\},$$

while $\partial I/\partial K$ can be expressed in terms of unit-order functions.

In this paper the general properties of the functions $\mathbf{P}_\nu(x,\omega)$ are derived from the definition (2). For the convergence of the integral, $\Re(\nu)$ must be greater than $-\frac{1}{2}$, so this condition will apply to all results. The range of ω considered is $-1 \leqslant \omega \leqslant 1$; for $|\omega| > 1$ there is an analogous function, the 'incomplete Hankel function'.

2. *General properties.* 2·1 *Definitions.* The constants in equation (2) were chosen so that

$$P_\nu(0,0) = \Lambda_\nu(0) = 1,$$

where the function $\Lambda_\nu(x) = \nu!\,2^\nu x^{-\nu}J_\nu(x)$ is chosen as a model in preference to the Bessel function, since its value at $x = 0$ is unity for all ν.

The incomplete Bessel and Struve functions denoted by $\mathscr{J}_\nu(x,\omega)$ and $\mathscr{H}_\nu(x,\omega)$ are real functions defined in terms of the real and imaginary parts of $\mathbf{P}_\nu(x,\omega)$ multiplied by the factor $x^\nu/(\nu!\,2^\nu)$. Thus

$$\mathscr{J}_\nu(x,\omega) = \frac{2}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu\int_\omega^1 (1-t^2)^{\nu-\frac{1}{2}}\cos xt\,dt, \tag{6}$$

$$\mathscr{H}_\nu(x,\omega) = \frac{2}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu\int_\omega^1 (1-t^2)^{\nu-\frac{1}{2}}\sin xt\,dt. \tag{7}$$

2·2. *Special values.* (*a*) The following special values follow readily from the definition:

$$\left.\begin{aligned}\mathbf{P}_\nu(x,0) &= \nu!\,2^\nu x^{-\nu}\{J_\nu(x) + i\mathbf{H}_\nu(x)\},\\ \mathscr{J}_\nu(x,0) &= J_\nu(x),\quad \mathscr{H}_\nu(x,0) = \mathbf{H}_\nu(x),\end{aligned}\right\} \tag{8}$$

$$\mathbf{P}_\nu(x,-1) = 2\Lambda_\nu(x),\quad \mathbf{P}_\nu(x,1) = 0,$$

$$\mathbf{P}_\nu(0,\omega) = \mathbf{P}_{\nu-1}(0,\omega) - (\nu-1)!\,\omega(1-\omega^2)^{\nu-\frac{1}{2}}/\{\sqrt{\pi}\,(\nu-\tfrac{1}{2})!\}.$$

Since $\mathbf{P}_0(0,\omega) = \dfrac{2}{\pi}\arccos\omega$, it follows for integral orders that

$$\mathbf{P}_n(0,\omega) = \frac{2}{\pi}\left\{\arccos\omega - \tfrac{1}{2}\sqrt{\pi}\sum_{r=1}^{n}(r-1)!\,\omega(1-\omega^2)^{r-\frac{1}{2}}/(r-\tfrac{1}{2})!\right\}. \tag{9}$$

The form of the functions $\mathbf{P}_\nu(0,\omega)$ are shown in Fig. 1 for several values of ν.

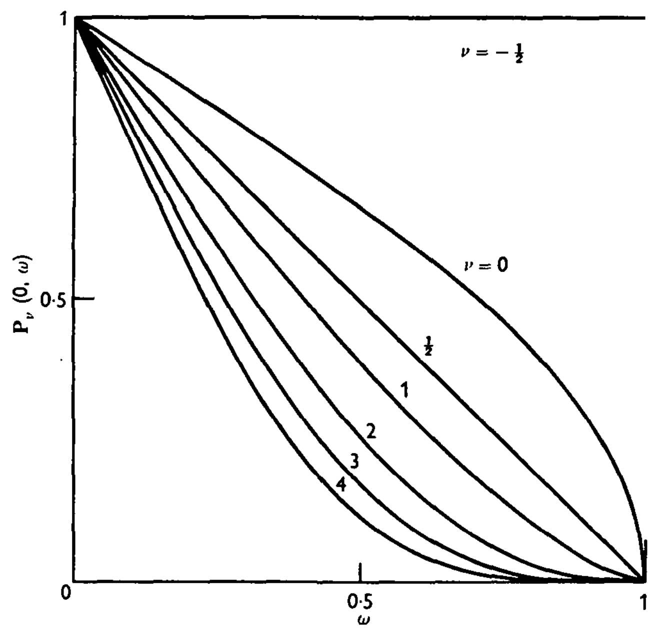

Fig. 1. $\mathbf{P}_\nu(0, \omega)$ for $\nu = -\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2, 3, 4.

For negative arguments

$$\mathbf{P}_\nu(-x,\omega) = \mathbf{P}_\nu^*(x,\omega), \tag{10}$$

$$\mathbf{P}_\nu(x,-\omega) = 2\Lambda_\nu(x) - \mathbf{P}_\nu^*(x,\omega), \tag{11}$$

where $\mathbf{P}^*$ is the complex conjugate of $\mathbf{P}$.

(*b*) When $\nu = n+\frac{1}{2}$, where n is an integer, equation (2) can be integrated in finite terms involving algebraic and trigonometrical functions of x and ω in a manner analogous to spherical Bessel functions.

Hence

$$\mathbf{P}_{\frac{1}{2}}(x,\omega) = -\frac{i}{x}(e^{ix} - e^{i\omega x})$$

$$= \frac{2}{x}\sin\{\tfrac{1}{2}(1-\omega)x\}\exp\{\tfrac{1}{2}i(1+\omega)x\}.$$

As a limit, it is found that $\mathbf{P}_{-\frac{1}{2}}(x,\omega) = e^{ix}$.

2·3. *Recurrence relationships.* Recurrence relationships analogous to those for Bessel functions can be found by integrating equation (2) by parts. It is found that

$$\mathbf{P}_{\nu-1}(x,\omega)+\frac{x^2}{4\nu(\nu+1)}\mathbf{P}_{\nu+1}(x,\omega)$$
$$=\mathbf{P}_\nu(x,\omega)+\frac{(\nu-1)!}{2\sqrt{\pi}\,(\nu+\frac{1}{2})!}\,e^{i\omega x}\,(1-\omega^2)^{\nu-\frac{1}{2}}\{ix(1-\omega^2)+(2\nu+1)\,\omega\}, \tag{12}$$

$$-\frac{x}{2(\nu+1)}\mathbf{P}_{\nu+1}(x,\omega)=\mathbf{P}'_\nu(x,\omega)-\frac{i\nu!}{\sqrt{\pi}\,(\nu+\frac{1}{2})!}e^{i\omega x}\,(1-\omega^2)^{\nu+\frac{1}{2}}, \tag{13}$$

where the derivative is taken with respect to x. For the first relation $\Re(\nu)>0$.

For the incomplete Bessel and Struve functions,

$$\begin{aligned}
\mathscr{J}_{\nu-1}(x,\omega)+\mathscr{J}_{\nu+1}(x,\omega) &= 2\nu x^{-1}\mathscr{J}_\nu(x,\omega)+\omega A_\nu(x,\omega)\cos\omega x-A_{\nu+1}(x,\omega)\sin\omega x,\\
\mathscr{J}_{\nu-1}(x,\omega)-\mathscr{J}_{\nu+1}(x,\omega) &= 2\,\mathscr{J}'_\nu(x,\omega)+\omega A_\nu(x,\omega)\cos\omega x+A_{\nu+1}(x,\omega)\sin\omega x,\\
\mathscr{H}_{\nu-1}(x,\omega)+\mathscr{H}_{\nu+1}(x,\omega) &= 2\nu x^{-1}\mathscr{H}_\nu(x,\omega)+\omega A_\nu(x,\omega)\sin\omega x+A_{\nu+1}(x,\omega)\cos\omega x,\\
\mathscr{H}_{\nu-1}(x,\omega)-\mathscr{H}_{\nu+1}(x,\omega) &= 2\mathscr{H}'_\nu(x,\omega)+\omega A_\nu(x,\omega)\sin\omega x-A_{\nu+1}(x,\omega)\cos\omega x,
\end{aligned}$$

where
$$A_\nu(x,\omega)=\frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu-1}\frac{(1-\omega^2)^{\nu-\frac{1}{2}}}{(\nu-\frac{1}{2})!}.$$

2·4. *Differential equations.* The function $\mathbf{P}_\nu(x,\omega)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2}+(2\nu+1)\frac{dy}{dx}+xy=\frac{2i\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}(1-\omega^2)^{\nu+\frac{1}{2}}\,e^{i\omega x}. \tag{14}$$

If the differential operator of Bessel's equation is denoted by

$$\nabla_\nu\equiv x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+x^2-\nu^2,$$

then
$$\left.\begin{aligned}
\nabla_\nu\,\mathscr{J}_\nu(x,\omega) &= -\frac{4(1-\omega^2)^{\nu+\frac{1}{2}}}{\pi(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^{\nu+1}\sin\omega x,\\
\nabla_\nu\,\mathscr{H}_\nu(x,\omega) &= \frac{4(1-\omega^2)^{\nu+\frac{1}{2}}}{\pi(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^{\nu+1}\cos\omega x.
\end{aligned}\right\} \tag{15}$$

Since $J_\nu(x)$ and $Y_\nu(x)$ are independent solutions of Bessel's equation $\nabla_\nu y=0$, integral representations of $\mathscr{J}_\nu$ and $\mathscr{H}_\nu$ can be obtained by the method of variation of parameters. Combining the real and imaginary parts and fitting the result to the limits of $\mathbf{P}_\nu(x,\omega)$ as $x\to 0$ and $x\to\infty$, we find that

$$\mathbf{P}_\nu(x,\omega)=\mathbf{P}_\nu(0,\omega)\,\Lambda_\nu(x)-\frac{i\sqrt{\pi}\,\nu!}{(\nu-\frac{1}{2})!\,x^\nu}(1-\omega^2)^{\nu-\frac{1}{2}}$$
$$\times\left\{J_\nu(x)\int_0^x x^\nu Y_\nu(x)\,e^{i\omega x}\,dx-Y_\nu(x)\int_0^x x^\nu J_\nu(x)\,e^{i\omega x}\,dx\right\}. \tag{16}$$

2·5. *Related functions.* For an integral order n, $\mathbf{P}_n(x,\omega)$ or $\hat{\mathbf{P}}_n(x,\alpha)$ can be expressed as an incomplete Bessel integral. If equation (2) is integrated by parts n times and Jacobi's transformation (Watson (**11**), p. 27)

$$\frac{d^{n-1}(\sin^{2n-1}\theta)}{d(\cos\theta)^{n-1}}=(-1)^{n-1}\frac{(n-\frac{1}{2})!\,2^n}{n\sqrt{\pi}}\sin n\theta$$

is applied, it is found that

$$\hat{\mathbf{P}}_n(x,\alpha) = \frac{2n!}{\sqrt{\pi}\,(n-\frac{1}{2})!}\,e^{ix\cos\alpha}\sum_{r=0}^{n-1}\left(\frac{i}{x}\right)^{r+1}\frac{d^r(\sin^{2n-1}\alpha)}{d(\cos\alpha)^r}+\frac{n!\,2^{n+1}}{\pi i^n x^n}\int_0^\alpha e^{ix\cos\phi}\cos n\phi\,d\phi. \quad (17)$$

The incomplete Anger and Weber functions

$$u_p^r(\sigma) = \int_0^\sigma \cos(r\sin\sigma - p\sigma)\,d\sigma,$$

$$v_p^r(\sigma) = \int_0^\sigma \sin(r\sin\sigma - p\sigma)\,d\sigma,$$

studied by Brauer and Brauer (1), can be expressed in terms of incomplete Bessel and Struve functions when p is an integer. These authors have tabulated the functions for $p = r = 0\,(0{\cdot}05)\,0{\cdot}5$.

The incomplete Hankel function can be written as

$$\frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\int_1^\omega e^{ixt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \quad (\omega \geqslant 1).$$

The zero order can be evaluated (Erdélyi (3)) as

$$\frac{2}{\pi}\int_1^\omega e^{ixt}(t^2-1)^{-\frac{1}{2}}\,dt = \frac{2}{\pi}[\arg\cosh\omega - C\{x, x\sqrt{(\omega^2-1)}\}+iS\{x, x\sqrt{(\omega^2-1)}\}],$$

where $C(a,x)$ and $S(a,x)$ are the generalized sine and cosine integrals tabulated by Harvard University (6):

$$C(a,x) = \int_0^x (1-\cos u)\,dx/u, \quad S(a,x) = \int_0^x \sin u\,dx/u,$$

where $u = \sqrt{(x^2+a^2)}$. Hence $\mathbf{P}_0(x,\omega)$ can be expressed in terms of similar integrals where now

$$u = \sqrt{(a^2-x^2)}.$$

3. *Series representations.* 3·1. *Power series.*

$$\mathbf{P}_\nu(x,\omega) = \frac{\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\sum_{r=0}^{\infty}\frac{i^r x^r}{r!}\mathrm{B}_{1-\omega^2}\left(\frac{2\nu+1}{2},\frac{r+1}{2}\right), \quad (18)$$

where $\mathrm{B}_x(p,q)$ is the incomplete beta function $\int_0^x t^{p-1}(1-t)^{q-1}\,dt$.

3·2. *Bessel series.* $$\mathbf{P}_\nu(x,\omega) = \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\sum_{r=0}^{\infty}\epsilon_r i^r b_{\nu,r}J_r(x), \quad (19)$$

where ϵ_r is Neumann's factor,

$$\epsilon_r = 1 \quad (r=0), \qquad \epsilon_r = 2 \quad (r>0),$$

and

$$b_{\nu,r} = \int_0^\alpha \sin^{2\nu}\phi\cos r\phi\,d\phi,$$

$$\omega = \cos\alpha.$$

This series is given by Hopkins (7) for $\nu = 1$.

When ω is near unity, it is preferable to use an expansion in terms of Bessel functions of argument $y = (1-\omega)x$. It is found that

$$\mathbf{P}_\nu(x,\omega)\,e^{-ix} = \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}(1-\omega)^{2\nu}\sum_{r=0}^{\infty}\epsilon_r(-i)^r J_r(y)\int_0^1\left(\frac{2}{1-\omega}u-u^2\right)^{\nu-\frac{1}{2}}\cos(r\arccos u)\,du, \tag{20}$$

and
$$\mathbf{P}_\nu(x,\omega)\,e^{-i\omega x} = \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\sum_{r=0}^{\infty}\epsilon_r i^r J_r(y)\int_\omega^1(1-t^2)^{\nu-\frac{1}{2}}\cos r\phi\,dt, \tag{21}$$

where
$$\cos\phi = (t-\omega)/(1-\omega).$$

Expansion (21) is found to be the more convenient. Further, a good approximation for all values of ω is given by

$$\mathbf{P}_\nu(x,\omega)\,e^{-i\omega x} \simeq \mathbf{P}_\nu(y,0)\,\mathbf{P}_\nu(0,\omega) = \mathbf{P}_\nu(0,\omega)\{\Lambda_\nu(y)+i\nu!\,2^\nu x^{-\nu}\mathbf{H}_\nu(y)\}. \tag{22}$$

For functions of integral order, Jacobi's expansion for $\exp(ix\sin\theta)$ can be differentiated n times with respect to x to give an expansion for the real part of $\mathbf{P}_n(x,\omega)\,e^{-i\omega x}$ in which the leading term is $\mathbf{P}_n(0,\omega)\,\Lambda_n(y)$.

For $n = 1$

$$\Re\{\mathbf{P}_1(x,\omega)\,e^{-i\omega x}\} = \frac{8}{\pi y}\sum_{r=0}^{\infty}(2r+1)\,J_{2r+1}(y)\int_\omega^1\frac{\cos(2r+1)\theta}{\cos\theta}\sqrt{(1-t^2)}\,dt, \tag{23}$$

where
$$\sin\theta = (t-\omega)/(1-\omega).$$

When ω is small ($\leqslant 0{\cdot}4$), a rapidly convergent series can be found in terms of spherical Bessel functions. On integrating by parts and applying Brauer's formula (Watson (11), p. 368) it is found that

$$\mathbf{P}_\nu(x,\omega) = \frac{\nu!\,2^\nu}{x^\nu}\{J_\nu(x)+i\mathbf{H}_\nu(x)\} - \frac{2i\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!\,x}\{1-e^{i\omega x}(1-\omega^2)^{\nu-\frac{1}{2}}\}$$
$$+\frac{2\nu!}{(\nu-\frac{3}{2})!\,x}\sqrt{\left(\frac{2}{\omega x}\right)}\sum_{r=0}^{\infty}i^{r+1}(2r+1)\,J_{r+\frac{1}{2}}(\omega x)\int_0^\omega t(1-t^2)^{\nu-\frac{3}{2}}P_r\left(\frac{t}{\omega}\right)dt, \tag{24}$$

where $P_r(t/\omega)$ is a Legendre polynomial.

3·3. *Asymptotic expansion*. If the expression

$$\mathbf{P}_\nu(x,\omega) = \nu!\,2^\nu x^{-\nu}\{J_\nu(x)+i\mathbf{H}_\nu(x)\} - \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}\int_0^\omega e^{ixt}(1-t^2)^{\nu-\frac{1}{2}}\,dt$$

is integrated by parts p times and the asymptotic expansion of $\mathbf{H}_\nu(x)$ inserted in the result, it is found that

$$\mathbf{P}_\nu(x,\omega) \sim \frac{\nu!\,2^\nu}{x^\nu}\{J_\nu(x)+iY_\nu(x)\} + \frac{2\nu!}{\sqrt{\pi}\,(\nu-\frac{1}{2})!}e^{i\omega x}\sum_{r=0}^{p-1}\left(\frac{i}{x}\right)^{r+1}\frac{d^r}{d\omega^r}(1-\omega^2)^{\nu-\frac{1}{2}} + o(x^{-p-1}). \tag{25}$$

For $p > \nu-\frac{1}{2}$ the expansion contains negative powers of $x(1-\omega^2)$, and hence the use of this expansion is restricted to very large values of x when ω is near unity.

4. *Tabulation.* The series given above have been used to calculate the functions of order 0 and 1. It is found that the incomplete Bessel and Struve functions are too irregular for simple tabulation for they show both the oscillation of Bessel functions and trigonometrical functions. The real and imaginary parts of $\mathbf{P}_n(x,\omega)\exp(-i\omega x)$ are much more regular for interpolation in x while a change of argument to $x/(1-\omega)$ gives a smooth function in ω. The table can be further simplified by applying the approximation (22) and subtracting the real and imaginary parts of $\mathbf{P}_n(x,0)\,\mathbf{P}_n(0,\omega)$. The result is compact tables permitting interpolation to 4 decimals.

Tables 1 and 2 give, for $x = 0(0{\cdot}5)12{\cdot}5$, $\omega = -0{\cdot}1(0{\cdot}1)1$; $4D$, the values of the real and imaginary parts of

$$\mathfrak{R}_n(x) + i\mathfrak{I}_n(x) = \mathbf{P}_n\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right) - \mathbf{P}_n(x,0)\,\mathbf{P}_n(0,\omega) \tag{26}$$

for $n = 0$ and 1. The real and imaginary parts of $\mathbf{P}_n(x,0)$ are given as an indication of the magnitude of the function; they are however extensively tabulated elsewhere. $\mathbf{P}_n(0,\omega)$ is given in Table 3 with second differences. It may also be computed from equation (9). The tables are rounded off from computations to 6 decimals and are suitable for interpolation using second differences to within one or two units in the fourth decimal, except for $\mathbf{P}_0(x,\omega)$ when ω is near 1, when the error may be larger.

The approximation

$$\mathbf{P}_n\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right) \simeq \mathbf{P}_n(x,0)\,\mathbf{P}_n(0,\omega)$$

is seen to be good to $0{\cdot}028$ for $\mathbf{P}_0$ and to $0{\cdot}017$ for $\mathbf{P}_1$. By inspection of Tables 1 and 2 it is seen that the remainder can again be expressed approximately as a product of functions of x and ω only. These functions have been chosen empirically to fit the tabulated results and they are given in Table 3 with second differences.

From this table $\mathbf{P}_n(x,\omega)$ is computed using

$$\mathbf{P}_n\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right) \simeq \mathbf{P}_n(x,0)\,\mathbf{P}_n(0,\omega) + g_n(\omega)\{a_n(x) + ib_n(x)\}\times 10^{-4}, \tag{27}$$

where the error is now less than $0{\cdot}0007$ for $\mathbf{P}_0$ and less than $0{\cdot}00015$ for $\mathbf{P}_1$. Hence for this latter order the approximation is as accurate as interpolation in Table 2, while being more convenient.

For $x > 12{\cdot}5$ the asymptotic expansion may be used for all ω. Writing equation (25) as

$$\mathbf{P}_n\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right) \sim n!\,2^n x^{-n}\exp\left(-\frac{i\omega x}{1-\omega}\right)\left\{J_n\left(\frac{x}{1-\omega}\right) + iY_n\left(\frac{x}{1-\omega}\right)\right\} + p_n(x_1\omega),$$

to four decimals, for $x > 12{\cdot}5$,

$$p_0(x,\omega) = \frac{2}{\pi}(1-\omega^2)^{\frac{1}{2}}\{iz - \omega z^2 - i(1+2\omega)z^3 + \dots\},$$

$$p_1(x,\omega) = \frac{4}{\pi}(1-\omega^2)^{\frac{3}{2}}\{iz - \omega z^2 + iz^3 - \dots\},$$

where $$z = \{(1+\omega)x\}^{-1}.$$

Table 1

$$\mathfrak{R}_0(x,\omega)+i\mathfrak{J}_0(x,\omega) = \mathbf{P}_0\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right)-\mathbf{P}_0(x,0)\,\mathbf{P}_0(0,\omega)$$

$\mathfrak{R}_0(x,\omega)\times 10^4$

x	ω												$J_0(x)$
	−0·1	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	
0·0	0	0	0	0	0	0	0	0	0	0	0	0	+1·0000
0·5	+ 6	0	− 5	− 9	− 12	− 14	− 16	− 16	− 16	− 14	− 11	0	0·9385
1·0	23	0	19	34	45	54	59	61	59	53	41	0	7652
1·5	47	0	38	69	92	109	119	123	120	108	83	0	5118
2·0	73	0	59	106	142	168	184	190	185	166	128	0	+ 2239
2·5	+95	0	−77	−137	−184	−217	−237	−244	−238	−214	−164	0	−0·0484
3·0	107	0	87	155	207	244	267	274	266	239	184	0	2601
3·5	108	0	86	155	206	242	264	271	263	236	181	0	3801
4·0	95	0	76	135	180	211	229	235	227	203	155	0	3971
4·5	73	0	57	101	133	155	167	170	164	146	111	0	3205
5·0	+44	0	−33	− 57	− 74	− 85	− 90	− 90	− 86	− 75	− 56	0	−0·1776
5·5	+14	0	− 8	− 13	− 14	− 14	− 12	− 10	− 7	− 4	− 1	0	− 0068
6·0	−11	0	+12	+ 24	+ 35	+ 45	+ 52	+ 57	+ 58	+ 55	+ 44	0	+ 1506
6·5	28	0	25	48	66	81	92	98	98	90	71	0	2601
7·0	33	0	29	54	74	90	102	107	107	98	76	0	3001
7·5	−26	0	+24	+ 44	+ 60	+ 73	+ 82	+ 86	+ 86	+ 78	+ 61	0	+0·2663
8·0	−12	0	+11	+ 21	+ 29	+ 35	+ 40	+ 42	+ 42	+ 39	+ 30	0	1717
8·5	+ 7	0	− 5	− 9	− 11	− 12	− 13	− 13	− 12	− 11	− 8	0	+ 0419
9·0	26	0	21	37	49	58	63	65	64	58	44	0	− 0903
9·5	39	0	32	57	77	91	100	103	101	91	71	0	1939
10·0	+45	0	−36	− 66	− 88	−104	−114	−118	−116	−105	− 81	0	−0·2459
10·5	41	0	33	60	80	95	104	108	105	95	73	0	2366
11·0	30	0	24	42	57	67	73	75	73	65	50	0	1712
11·5	+13	0	−10	− 17	− 23	− 26	− 28	− 28	− 27	− 24	− 18	0	− 0677
12·0	− 5	0	+ 5	+ 9	+ 13	+ 17	+ 19	+ 21	+ 21	+ 20	+ 16	0	+ 0477
12·5	−19	0	+17	+ 31	+ 43	+ 52	+ 58	+ 61	+ 61	+ 56	+ 43	0	+0·1469

$\mathfrak{J}_0(x,\omega)\times 10^4$

x	ω												$\mathbf{H}_0(x)$
	−0·1	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	
0·0	0	0	0	0	0	0	0	0	0	0	0	0	0·0000
0·5	−22	0	+18	+ 32	+ 43	+ 51	+ 55	+ 57	+ 55	+ 50	+ 38	0	+ 3096
1·0	39	0	31	56	75	88	96	99	96	86	66	0	5687
1·5	46	0	36	65	87	102	111	114	110	99	76	0	7367
2·0	40	0	32	56	74	87	94	96	93	83	63	0	7909
2·5	−22	0	+17	+ 29	+ 38	+ 44	+ 47	+ 48	+ 45	+ 40	+ 30	0	+0·7300
3·0	+ 4	0	− 5	− 10	− 14	− 18	− 22	− 24	− 24	− 23	− 18	0	5743
3·5	35	0	30	54	74	89	100	104	103	94	73	0	3608
4·0	63	0	53	96	130	155	171	178	175	159	123	0	+ 1350
4·5	84	0	70	126	171	203	224	232	227	205	158	0	− 0585
5·0	+95	0	−78	−140	−189	−224	−246	−255	−249	−224	−173	0	−0·1852
5·5	92	0	75	136	182	215	236	244	237	214	164	0	2268
6·0	78	0	64	114	152	179	196	202	196	176	135	0	1846
6·5	57	0	45	81	107	125	136	139	134	120	91	0	− 0773
7·0	32	0	25	43	56	65	69	69	65	58	− 43	0	+ 0634
7·5	+ 9	0	− 6	− 9	− 10	− 10	− 9	− 7	− 4	− 2	0	0	+0·2009
8·0	− 7	0	+ 7	+ 15	+ 22	+ 28	+ 33	+ 36	+ 37	+ 35	+ 29	0	3020
8·5	13	0	12	24	34	42	48	52	53	49	39	0	3442
9·0	− 9	0	+ 9	+ 18	+ 25	32	37	40	40	38	30	0	3199
9·5	+ 3	0	− 1	− 1	0	+ 1	+ 3	+ 4	+ 5	+ 6	+ 5	0	2375
10·0	+20	0	−15	− 26	− 35	− 40	− 43	− 44	− 42	− 37	− 28	0	+0·1187
10·5	37	0	29	53	70	83	90	93	90	81	62	0	− 0074
11·0	50	0	41	73	97	115	126	130	127	114	88	0	1114
11·5	57	0	46	82	110	130	143	147	143	129	99	0	1703
12·0	54	0	44	79	105	124	136	141	137	123	95	0	1725
12·5	+45	0	−36	− 64	− 85	−100	−109	−112	−109	− 98	− 75	0	−0·1206

Table 2

$$\mathfrak{R}_1(x,\omega)+i\mathfrak{J}_1(x,\omega) = \mathbf{P}_1\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right) - \mathbf{P}_1(x,0)\,\mathbf{P}_1(0,\omega)$$

$\mathfrak{R}_1(x,\omega)\times 10^4$

x	ω												$\frac{2}{x}J_1(x)$
	−0·1	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1	
0·0	0	0	0	0	0	0	0	0	0	0	0	0	+1·0000
0·5	− 4	0	+ 3	+ 5	+ 6	+ 6	+ 6	+ 5	+ 4	+ 2	+ 1	0	0·9691
1·0	17	0	12	20	24	25	23	19	15	9	4	0	8801
1·5	36	0	25	42	50	52	49	41	31	19	7	0	7439
2·0	58	0	41	67	81	84	79	66	49	30	12	0	5767
2·5	−79	0	+56	+ 92	+111	+115	+108	+ 91	+68	+42	+16	0	+0·3977
3·0	98	0	69	113	136	142	132	112	83	51	20	0	2260
3·5	109	0	77	127	153	158	148	125	93	57	22	0	+ 0785
4·0	113	0	80	131	158	164	153	129	96	59	23	0	− 0330
4·5	109	0	77	126	151	157	146	123	92	56	22	0	1027
5·0	−98	0	+69	+112	+135	+139	+130	+109	+81	+50	+19	0	−0·1310
5·5	81	0	57	92	111	114	106	89	66	40	16	0	1242
6·0	61	0	43	69	83	85	79	66	49	30	12	0	0922
6·5	42	0	29	46	55	57	52	44	32	19	8	0	0473
7·0	25	0	17	27	31	32	29	24	18	11	4	0	− 0013
7·5	−12	0	+ 8	+ 12	+ 14	+ 14	+ 12	+ 10	+ 7	+ 4	+ 2	0	+0·0361
8·0	5	0	3	4	4	3	2	2	+ 1	0	0	0	0587
8·5	3	0	1	1	1	1	0	0	− 1	0	0	0	0643
9·0	5	0	3	4	4	4	3	2	+ 2	1	0	0	0545
9·5	9	0	6	10	11	11	10	8	6	4	1	0	0340
10·0	−15	0	+10	+ 16	+ 19	+ 20	+ 18	+ 15	+11	+ 7	+ 3	0	+0·0087
10·5	19	0	13	22	26	27	25	21	16	9	4	0	− 0150
11·0	22	0	15	25	30	31	29	24	18	11	4	0	0321
11·5	22	0	16	26	31	32	29	25	18	11	4	0	0397
12·0	20	0	14	23	27	28	26	22	16	10	4	0	0372
12·5	−16	0	+11	+ 18	+ 21	+ 22	+ 20	+ 17	+13	+ 8	+ 3	0	−0·0265

$\mathfrak{J}_1(x,\omega)\times 10^4$

x	ω												$\frac{2}{x}\mathbf{H}_1(x)$
	−0·1	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1	
0·0	0	0	0	0	0	0	0	0	0	0	0	0	+0·0000
0·5	+20	0	−14	− 23	− 28	− 29	− 27	− 23	−17	−10	− 4	0	2087
1·0	37	0	26	43	51	53	50	42	31	19	7	0	3969
1·5	47	0	33	55	66	68	64	54	40	24	10	0	5471
2·0	49	0	35	57	68	71	66	56	41	25	10	0	6468
2·5	+43	0	−30	− 49	− 59	− 61	− 57	− 48	−35	−22	− 8	0	+0·6905
3·0	28	0	20	32	38	39	37	31	23	14	5	0	6801
3·5	+ 8	0	− 5	− 8	− 9	− 9	− 8	− 7	− 5	− 3	− 1	0	6238
4·0	−15	0	+11	+ 19	+ 24	+ 25	+ 24	+ 20	+15	+ 9	+ 4	0	5349
4·5	39	0	28	46	56	59	55	47	35	22	9	0	4293
5·0	−58	0	+42	+ 69	+ 84	+ 87	+ 82	+ 69	+52	+32	+13	0	+0·3231
5·5	73	0	52	86	104	108	101	86	64	39	15	0	2300
6·0	80	0	57	94	114	118	111	94	70	43	17	0	1594
6·5	81	0	58	95	114	119	111	94	70	43	17	0	1159
7·0	76	0	54	88	106	110	103	87	65	40	16	0	0989
7·5	−66	0	+47	+ 77	+ 92	+ 95	+ 89	+ 75	+56	+34	+13	0	+0·1036
8·0	55	0	38	63	75	78	72	61	45	28	11	0	1220
8·5	43	0	30	49	59	61	56	47	35	21	8	0	1456
9·0	34	0	23	38	45	47	43	36	27	16	6	0	1663
9·5	28	0	19	31	37	38	35	29	22	13	5	0	1782
10·0	−25	0	+17	+ 28	+ 33	+ 34	+ 32	+ 26	+19	+12	+ 5	0	+0·1784
10·5	26	0	18	29	35	36	33	28	20	12	5	0	1668
11·0	29	0	20	33	39	40	37	31	23	14	6	0	1464
11·5	33	0	23	38	45	47	43	36	27	16	6	0	1216
12·0	37	0	26	42	51	52	49	41	31	19	7	0	0973
12·5	−39	0	+28	+ 45	+ 54	+ 56	+ 52	+ 44	+33	+20	+ 8	0	+0·0779

Table 3. *Functions for the approximate computation of* $\mathbf{P}_n(x,\omega)$ *using*

$$\mathbf{P}_n\left(\frac{x}{1-\omega},\omega\right)\exp\left(-\frac{i\omega x}{1-\omega}\right)\simeq \mathbf{P}_n(x,0)\,\mathbf{P}_n(0,\omega)+g_n(\omega)\{a_n(x)+ib_n(x)\}\times 10^{-4}$$

x	$a_0(x)$	δ^2	$b_0(x)$	δ^2	$a_1(x)$	δ^2	$b_1(x)$	δ^2	ω	$\mathbf{P}_0(0,\omega)$	δ^2	$g_0(\omega)$	δ^2	$\mathbf{P}_1(0,\omega)$	δ^2	$g_1(\omega)$	δ^2
0·0	0	−32	0	0	0	+13	0	0	0	+1·0000	0	0·000	−18	+1·0000	0	0·000	−53
0·5	−16	28	+57	−15	+6	12	−29	+5	0·05	0·9682	−1	+0·165	17	0·9364	+2	+0·270	49
1·0	60	18	98	27	25	9	53	9	0·1	9362	2	313	16	8729	3	492	45
1·5	122	−4	113	32	52	+5	68	12	0·15	9042	2	445	15	8097	5	670	42
2·0	188	+12	96	31	84	0	71	13	0·2	8718	3	562	14	7471	6	805	38
2·5	−243	+25	+48	−23	+115	−5	−61	+12	0·25	+0·8391	−4	+0·665	−14	+0·6850	+8	+0·903	−35
3·0	272	32	−23	−9	142	9	39	8	0·3	8060	5	753	13	6238	10	966	31
3·5	270	33	103	+6	158	12	−9	+4	0·35	7724	7	827	13	5636	12	0·998	28
4·0	234	27	176	20	164	12	+25	0	0·4	7380	8	891	13	5046	14	1·000	25
4·5	170	16	229	30	157	11	59	−5	0·45	7028	10	940	13	4470	16	0·977	22
5·0	−91	+1	−252	+34	+139	−8	+88	−8	0·5	+0·6667	−12	+0·977	−13	+0·3910	+18	+0·932	−19
5·5	−11	−14	241	30	114	−4	108	10	0·55	6293	15	1·000	14	3368	21	866	16
6·0	+55	25	200	21	85	0	118	10	0·6	5903	19	1·009	15	2848	24	784	13
6·5	96	31	138	+7	57	+4	119	9	0·65	5495	23*	1·051	16	2351	27	689	10
7·0	105	30	69	−7	32	7	110	6	0·7	5064	30*	0·985	17	1881	31	583	6
7·5	+85	−23	−8	−19	+14	+8	+95	−3	0·75	+0·4601	−40*	+0·947	−21	+0·1443	+36	+0·471	−3
8·0	+42	−11	+35	27	3	8	78	0	0·8	4097	57*	888	26	1041	42*	356	+1
8·5	−13	+2	50	28	1	6	61	+3	0·85	3532	86*	804	36	0681	51*	244	7
9·0	65	15	38	23	4	4	47	5	0·9	2871	—	684	62	0374	62*	139	18
9·5	102	23	+4	13	11	+1	38	5	0·95	2021	—	503	−322	0133	—	052	+35
10·0	−117	+25	−44	−6	+20	−1	+34	+5	1·0	0·0000	—	0·000	—	0·0000	—	0·000	—
10·5	107	22	92	+11	27	3	36	3									
11·0	74	14	129	20	31	4	40	+2									
11·5	−28	+2	146	23	32	4	47	−1									
12·0	+20	−9	139	21	28	3	52	2									
12·5	+60	−18	−111	+14	+32	−1	+56	−3									

* Modified second difference $\delta^2-0{\cdot}184\,\delta^4$.

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