1 Functional Setup

We consider the Argo data residuals $\{\{p_{i,j}, T_{i,j}\}_{j=1}^{m_i}, t_i, s_i\}_{i=1}^{N}$ where i denotes the profile and j denotes an individual measurement. For now, we take $T_{i,j}$ to be temperature and $\mathbb{E}(T_{i,j}) = 0$ for all i and j. For now, we ignore the times t_i.

Tailen has suggested the model

$$T_{i,j} = \sum_{k=1}^{K} Y_k(s_i)\phi_k(p_{i,j}) \tag{1}$$

for some small K. The $Y_k(\cdot)$ may have different distributions across k and be dependent across both s_i and k. Here, we assume that each ϕ_k is a fixed function that has been estimated through some form of functional principal component analysis (FPCA).

Let $T^*(p)$ and $Y^*(s_0) \in \mathbb{R}^K$ be the respective predictions at a fixed location s_0, and Y to denote all $Y_k(s_i)$ in a radius of s_0 from three consecutive months of the year. Let n be the number of profiles in this radius. From here, one can see that

$$\mathbb{E}(T^*(p)|Y) = \sum_{k=1}^{K} \mathbb{E}(Y_k^*(s_0)|Y)\phi_k(p)$$

by linearity of expectation and

$$\begin{aligned} \mathrm{Var}(T^*(p)|Y) &= \mathrm{Var}\left(\sum_{k=1}^{K} Y_k^*(s_0)\phi_k(p) \middle| Y\right) \\ &= \sum_{k=1}^{K} \phi_k(p)^2 \mathrm{Var}(Y_k^*(s_0)|Y) + 2\sum_{k\neq\ell} \phi_k(p)\phi_\ell(p)\mathrm{Cov}(Y_k^*(s_0), Y_\ell^*(s_0)|Y). \end{aligned}$$

Assuming that the field of $\{Y_k(s_i)\}_{i=1,k=1}^{n,K}$ is jointly multivariate normal and the $T_{i,j}$ follow model (1), then the conditional mean and variance completely describe the predictive distribution of $T^*(p)$, since it is a linear combination of Y^*.

Suppose that one has a model for Y and let

$$\begin{aligned} \Sigma_Y &= \mathrm{Var}(Y) \in \mathbb{R}^{nK\times nK} \\ \Sigma^* &= \mathrm{Var}(Y^*(s_0)) \in \mathbb{R}^{K\times K} \\ \Sigma_{21} &= \mathrm{Cov}(Y, Y^*(s_0)) \in \mathbb{R}^{nK\times K} \end{aligned}$$

Then the conditional expectation is

$$\mathbb{E}(Y^*(s_0)|Y) = \Sigma_{21}^\top \Sigma_Y^{-1} Y$$

and the conditional variance is

$$\mathrm{Var}(Y^*(s_0)|Y) = \Sigma^* - \Sigma_{12}^\top \Sigma_Y^{-1} \Sigma_{12},$$

which then gives

$$T^*(p)|Y \sim N\left((\Sigma_{21}^\top \Sigma_Y^{-1} Y)^\top \phi(p),\ \phi(p)^\top (\Sigma^* - \Sigma_{12}^\top \Sigma_Y^{-1}\Sigma_{12})\phi(p)\right)$$

where $\phi(p) = \begin{pmatrix} \phi_1(p) & \phi_2(p) & \dots & \phi_K(p) \end{pmatrix}^\top$.

2 Multivariate Model for scores

Given a model for Y, the above gives a relevant functional model. We consider the problem here on how to model the scores Y.

Following Stilian?s formulation, when $d = 1$ and s and t are scalar, we consider

$$Y(t) = \int_{\mathbb{R}} e^{itx}((1+ix)^{-\nu-1/2}A1_{\{x>0\}} + (1+ix)^{-\nu-1/2}\overline{A}1_{\{x<0\}}\tilde{B}(dx)$$

where $\tilde{B}(dx)$ is $\mathbb{C}^k$-valued Brownian motion with

$$\tilde{B}(x) = \overline{\tilde{B}(-x)} \qquad\qquad \mathbb{E}[\tilde{B}(dx)\tilde{B}(dx)^*] = \mathbb{I}_k dx$$

and A is a complex-valued matrix. When $\nu = \nu\mathbb{I}_k$ is a scalar, this above integral gives the covariance

$$\mathbb{E}[Y(t)Y(s)^\top] = \int_{\mathbb{R}} e^{i(s-t)x}(1+x^2)^{-\nu-1/2}(AA^*1_{\{x>0\}} + \overline{AA}^*1_{\{x<0\}})dx.$$

We call this the multivariate Matern model. When AA^* is real, the model is reversible and reduces to a special case of the Gneiting et al (2010) model where each auto-covariance and cross-covariance are Matern.

Computing the integral gives for each component product gives

$$\mathbb{E}[Y_i(t)Y_j(s)] = \mathrm{Re}(z_{i,j})M(s-t,\nu) + \mathrm{Im}(z_{i,j})S(s-t,\nu).$$

where $z_{i,j} = e_i^\top AA^* e_j$ where $\{e_j\}_{j=1}^K$ are the standard basis vectors,

$$M(h,\nu) = \frac{\pi^{1/2}}{2^{\nu-1}\Gamma(\nu+1/2)}|h|^\nu K_\nu(h)$$

is the standard Matern covariance, and

$$S(h,\nu) = \frac{\pi^{3/2}}{2^\nu\Gamma(\nu+1/2)\cos(\pi\nu)}\mathrm{sign}(h)|h|^\nu(I_\nu(|h|) - L_{-\nu}(|h|))$$

with I_v is the modified Bessel function of the first kind and L_ν is the modified Struve function. Note that when $\nu = 1/2$ or $\nu = 3/2$ the formula for $S(h,\nu)$ does not necessarily apply.

In the model, the $z_{i,i}$ are real and each gives the marginal variance of response i. When $\mathrm{Im}(z_{i,j}) = 0$, $z_{i,j}$ is the covariance between components i and j. When $\mathrm{Im}(z_{i,j}) = 0$, the model is reversible and each component is Matern; in this way, $\mathrm{Im}(z_{i,j})$ describes the amount of non-reversibility between components i and j.

2.1 Considering the multidimensional t case

We have mentioned the turning bands method to build the multidimensional $d > 1$ case. One challenge with this approach is that the $d = 2$ covariance that results may not easily tractable or desired.

Consider a somewhat different approach. For the Matern model, it is more common to plug in the distance $\|h\|$ into the 1-d function. We aim to take this approach, while maintaining the non-reversibility.

Let θ be a direction in $[0,\pi]$, and for a fixed direction, let $\sigma(\theta)$ (scale) and $z_{i,j}(\theta)$ (variance and covariance) be parameters that depend on θ. Then, letting $\|h\| = \|s_1 - s_2\|$ and $\theta \in [0,\pi]$ to be the angle between s_1 and s_2, one might consider a model of

$$\mathbb{E}[Y_i(s_1)Y_j(s_2)] = \mathrm{Re}(z_{i,j}(\theta))M\left(\frac{\|h\|}{\sigma(\theta)},\nu\right) + \mathrm{Im}(z_{i,j}(\theta))S\left(\mathrm{sign}(s_1,s_2)\frac{\|h\|}{\sigma(\theta)},\nu\right).$$

where $\mathrm{sign}(s_1,s_2)$ is some function such that $|\mathrm{sign}(s_1,s_2)| = 1$ and $\mathrm{sign}(s_1,s_2) = -\mathrm{sign}(s_2,s_1)$. For example, when $s \in \mathbb{R}^2$, one could take $\mathrm{sign}(s_1,s_2) = 1$ if s_1 is North of s_2 and -1 otherwise. Note that this maintains the non-reversibility since θ_k is only on the half circle; for half of the circle, $\mathrm{sign}(s_1,s_2) = 1$ and the other half, $\mathrm{sign}(s_1,s_2) = -1$. In this model, the amount of non-reversibility depends on θ, which is in some way described by $\mathrm{Im}(z_{i,j}(\theta))$.

In the setting where $Y_i(s) \in \mathbb{R}^2$, we might have

$$AA^*(\theta) = \begin{pmatrix} z_{1,1}(\theta) & z_{1,2}(\theta) \\ z_{2,1}(\theta) & z_{2,2}(\theta) \end{pmatrix} = \begin{pmatrix} a(\theta) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b(\theta) \end{pmatrix} + \begin{pmatrix} 0 & c(\theta) \\ c(\theta) & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -d(\theta) \\ d(\theta) & 0 \end{pmatrix}$$

and estimate $a(\theta)$, $b(\theta)$, $c(\theta)$, $d(\theta)$, and $\sigma(\theta)$. If there are too many parameters in this model, one could likely take $a(\theta) = a$ and $b(\theta) = b$ to be constants without too much of a loss in performance. One would want each of these to be periodic. In practice, one can take the first few Fourier basis functions, and apply necessary constraints on their coefficients.

3 Requirements for validity

We give Cramer?s theorem, directly taken from Chiles and Delfiner page 325.

Consider the set of continuous functions $C_{ij}(h)$ of marginal and cross covariances that has the spectral representation

$$C_{ij}(h) = \int e^{2\pi i\langle u,h\rangle}F_{ij}(du)$$

where $u, h \in \mathbb{R}^d$.

Theorem 1 (Cramer?s Theorem)**.** *The continuous functions $C_{ij}(h)$ are the elements of the covariance matrix of a multidimensional stationary RF of order 2 if and only if the cross-spectral matrix $M(B) = [F_{ij}(B)]$ is positive definite for any (Borel) set B of $\mathbb{R}^n$, namely,*

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\bar{\lambda}_jF_{ij}(b) \geq 0$$

for any set of complex coefficients $\lambda_1, \ldots, \lambda_p$.

3.1 d = 2

For the two-dimensional model, we have something that looks like, where $(\theta_h, r = \mathrm{sign}(h,0)\,\|h\|)$ are the polar coordinates of h

$$C_{ij}(h) = \mathrm{Re}(z_{i,j}(\theta_h))M\left(\frac{r}{\sigma(\theta_h)},\nu\right) + \mathrm{Im}(z_{i,j}(\theta_h))S\left(\frac{r}{\sigma(\theta_h)},\nu\right).$$

Note that, by page 85-86 of Chiles-Delfiner,

$$\mathrm{Re}(z_{i,j})M\left(\frac{r}{\sigma},\nu\right) = \mathrm{Re}(z_{i,j})\int_{\mathbb{R}^2}\left(1+\frac{r^2}{\sigma^2}\right)^{-\nu-1}$$

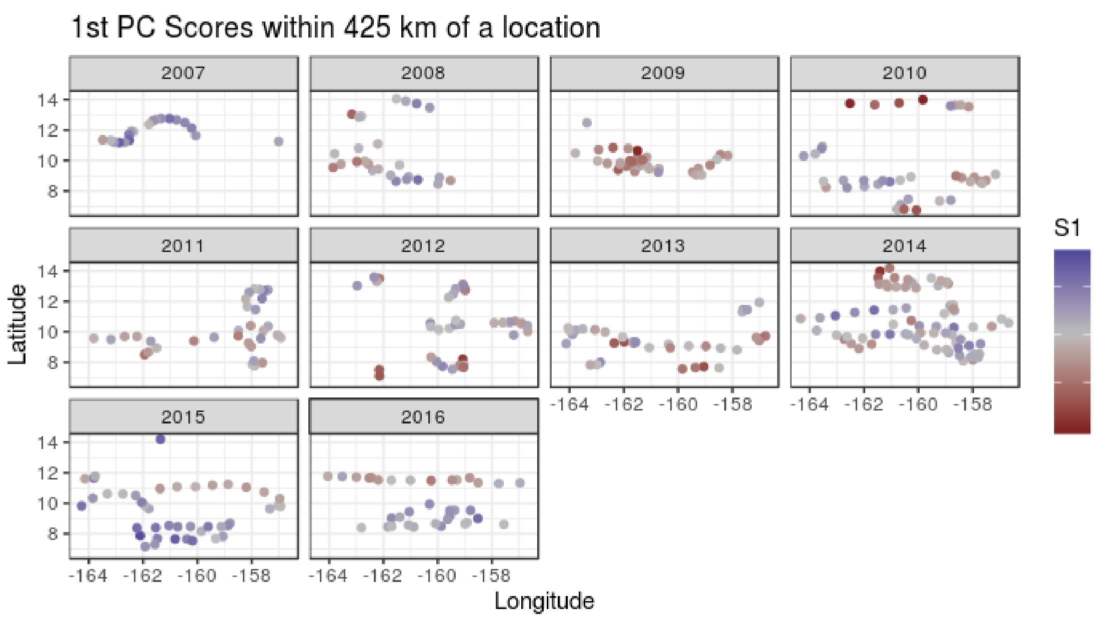
It is questionable how that can be extended for $\sigma(\theta)$. Also,

$$\mathrm{Re}(z_{i,j})M\left(\frac{r}{\sigma},\nu\right) = \mathrm{Im}(z_{i,j})\int_{\mathbb{R}^2}\left(1+\frac{r^2}{\sigma^2}\right)^{-\nu-1}$$

Theorem 2 (Stability Property). *If $C(h;t)$ is a covariance in $\mathbb{R}^n$ for all values $t \in A \subset \mathbb{R}$ of the parameter t, and if $\mu(dt)$ is a positive measure on A, then $\int C(h;t)\mu(dt)$ is a covariance in $\mathbb{R}^n$ provided that the integral exists for all h.*

4 Other thoughts for implementation

- For each of the 12 months, I aim to use data from that month and two neighboring months. I consider data from different years independent, but combine the samples to estimate the covariance parameters over all years.



- We will want to consider time in the covariance in the future as well.

- One can do salinity in the same way with delayed mode data.

- Later future question. Consider

$$T_{i,j} = \sum_{k=1}^{K} Y_{i,k}\phi_k(p_{i,j}) + \sum_{k=1}^{K} S_{i,k}\phi_k(p_{i,j})$$

  where $S_{i,k}$ is the salinity field to jointly model the temperature and salinity. This could then be extended to oxygen data and other geochemical data for the co-kriging problem.

The standard Matern model in d dimensions is, letting $r = \|h\|$,

$$C(r) = \frac{1}{2^{\nu-1}\Gamma(\nu)} \left(\frac{r}{a}\right)^{\nu} K_\nu\left(\frac{r}{a}\right) = \int_{\mathbb{R}^d} e^{iry}\left(1+\frac{y^2}{a^2}\right)^{-\nu-d/2} dy$$

Now, let?s consider the integral

$$\begin{aligned}&\int_{\mathbb{R}} e^{ixh}\left(1+x^2\right)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}}+\overline{z_{i,j}}1_{\{x<0\}})dx\\ &= \int_{\mathbb{R}}(\cos(hx)+i\sin(hx))\left(1+x^2\right)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}}+\overline{z_{i,j}}1_{\{x<0\}})dx.\end{aligned}$$

Focusing on the cosine term, we have

$$\begin{aligned}&\int_{\mathbb{R}} \cos(hx)\left(1+x^2\right)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}}+\overline{z_{i,j}}1_{\{x<0\}})dx\\ &= (z_{i,j}+\overline{z}_{i,j})\int_0^\infty \cos(hx)\left(1+x^2\right)^{-\nu-1/2}dx\\ &= \mathrm{Re}(z_{i,j})\frac{\sqrt{\pi}|h|^\nu}{\Gamma(\nu+1/2)2^{\nu-1}}K_\nu(|h|)\end{aligned}$$

which gives some variation of the standard Matern class. See Stein 1999 pg 48-50 for alternative and more common parameterizations. Now, considering the sine term, we have

$$\begin{aligned}&i\int_{\mathbb{R}} \sin(hx)\left(1+x^2\right)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}}+\overline{z_{i,j}}1_{\{x<0\}})dx\\ &= \frac{\Gamma(-\nu+1/2)\Gamma(1/2)}{2(\frac{1}{2}h)^{-\nu}}\left(I_\nu(|h|)-L_{-\nu}(|h|)\right)\\ &= \mathrm{sign}(h)|h|^\nu 2^{-\nu-1}\sqrt{\pi}\Gamma(-\nu+1/2)\left(I_\nu(|h|)-L_{-\nu}(|h|)\right)\end{aligned}$$

given on page 332 of Watson: *Theory of Bessel Functions* and I_ν is the modified Bessel function and L_ν is the modified Struve function. Note that the above is undefined for $\nu = 1/2, 3/2, \ldots$ due to the gamma function, where we have extended the gamma function to the negative nonintegers. $\boxed{\text{What happens when } \nu = 1/2, 3/2, \ldots?}$

4.1 d=2

Give function that gives level of reversibility. Let ϕ be a $d-1$-dimensional function.

Now, we want

$$\begin{aligned}&\int_{\mathbb{R}^2} \cos(h\sqrt{x^2+y^2})(1+x^2+y^2)^{-\nu-1/2}dxdy\\ &= \int_0^\infty\int_{-\pi}^{\pi}\cos(hr)(1+r^2)^{-\nu-1/2}d\theta dr\\ &=\end{aligned}$$

Now, let?s consider the integral

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{iry}\left(1+\frac{r^2}{a^2}\right)^{-\nu+d/2}(c_1 1_{\{r>0\}}+c_2 1_{\{r<0\}})dr\\ &= \int_{\mathbb{R}^d} e^{iry}\left(1+\frac{r^2}{a^2}\right)^{-\nu+d/2}(c_1 1_{\{r>0\}}+c_2 1_{\{r<0\}})dr\end{aligned}$$

4.2 d arbitrary, varying reversibility

Let $\boldsymbol{h}$ be an arbitrary vector in $\mathbb{R}^d$, and let $AA^*(\omega) = R + i\begin{pmatrix} 0 & -b(\omega) \\ b(\omega) & 0 \end{pmatrix}$ where R is a $k \times k$ real positive definite matrix. Here, call $F(\omega) = \begin{pmatrix} 0 & -b(\omega) \\ b(\omega) & 0 \end{pmatrix}$.

We want to consider the covariance of

$$\begin{aligned}
C(0, \boldsymbol{h}) &= \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(AA^*(\omega) 1_{\{\omega > 0\}} + \overline{AA^*}(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&= \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left((R + iF(\omega)) 1_{\{\omega > 0\}} + (R - iF(\omega)) 1_{\{\omega < 0\}} \right) d\omega \\
&= \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&= \int_{\mathbb{R}^d} (\cos(\omega^\top \boldsymbol{h}) + i \sin(\omega^\top \boldsymbol{h}))(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&= \int_{\mathbb{R}^d} \cos(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&\quad + i \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&= R \int_{\mathbb{R}^d} \cos(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} d\omega \\
&\quad + i \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega
\end{aligned}$$

by using the even and odd properties of the cosine and sine functions, respectively.

It is clear that the first part of the above is the version of Gneiting 2010 with scale parameter 1, which is a valid covariance iff R is positive definite and $\nu > 0$.

Focusing on the second part, and letting $r = \|\omega\|$ we have

$$\begin{aligned}
& i \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(iF(\omega) 1_{\{\omega > 0\}} - iF(\omega) 1_{\{\omega < 0\}} \right) d\omega \\
&= -2 \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h})(1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} F(\omega) 1_{\{\omega > 0\}} d\omega \\
&= -2 \int_0^\infty \int_{S^{d-1}} \sin(r\theta^\top \boldsymbol{h})(1 + r^2)^{-\nu - \frac{d}{2}} F(\theta) d\theta d\omega
\end{aligned}$$

Considering $r = \pm \|\omega\|$ and $\theta = \omega / r$, we have

$$\begin{aligned}
&= \int_0^\infty \int_{S^{d-1}} e^{ir\theta^\top \boldsymbol{h}} (1 + r^2)^{-\nu - \frac{d}{2}} \left(F(\theta) 1_{\{r > 0\}} - F(\theta) 1_{\{r < 0\}} \right) d\theta dr \\
&= \int_0^\infty \int_{S^{d-1}} (\cos(r\theta^\top \boldsymbol{h}) + i \sin(r\theta^\top \boldsymbol{h}))(1 + r^2)^{-\nu - \frac{d}{2}} \left(F(\theta) 1_{\{r > 0\}} - F(\theta) 1_{\{r < 0\}} \right) d\theta dr
\end{aligned}$$

4.3 d arbitrary, varying reversibility

Let $\boldsymbol{h}$ be an arbitrary vector in $\mathbb{R}^d$, and let $AA^* = R + iM$ where R is a $k \times k$ real positive definite matrix and $M = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$ for some $m \in \mathbb{R}$. Let a be a vector in $\mathbb{R}^d$ that describes the plane through the origin satisfying $a^\top \omega = 0$ for which the non-reversibility is reflected.

We want to consider the covariance of

$$\begin{aligned} C(0, \boldsymbol{h}) &= \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(AA^* 1_{\{a^\top \omega > 0\}} + \overline{AA^*} 1_{\{a^\top \omega < 0\}} \right) d\omega \\ &= \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left((R + iM) 1_{\{a^\top \omega > 0\}} + (R - iM) 1_{\{a^\top \omega < 0\}} \right) d\omega \end{aligned}$$

Plugging in $AA^* = R + iM$ gives

$$C(0, \boldsymbol{h}) = \int_{\mathbb{R}^d} e^{i\omega^\top \boldsymbol{h}} (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iM 1_{\{a^\top \omega > 0\}} - iM 1_{\{a^\top \omega < 0\}} \right) d\omega$$

and by breaking up the integral we have

$$\begin{aligned} C(0, \boldsymbol{h}) = &\int_{\mathbb{R}^d} \cos(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iM 1_{\{a^\top \omega > 0\}} - iM 1_{\{a^\top \omega < 0\}} \right) d\omega \\ &+ i \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(R + iM 1_{\{a^\top \omega > 0\}} - iM 1_{\{a^\top \omega < 0\}} \right) d\omega. \end{aligned}$$

Using the even and odd properties of the cosine and sine functions, respectively, gives

$$\begin{aligned} C(0, \boldsymbol{h}) &= R \int_{\mathbb{R}^d} \cos(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} d\omega \\ &\quad + i^2 M \int_{\mathbb{R}^d} \sin(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} \left(1_{\{a^\top \omega > 0\}} - 1_{\{a^\top \omega < 0\}} \right) d\omega \\ &= R \int_{\mathbb{R}^d} \cos(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} d\omega \\ &\quad - 2M \int_{\omega | a^\top \omega > 0} \sin(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - \frac{d}{2}} d\omega. \end{aligned}$$

When M is the 0 matrix, the above is a simplified version of Gneiting 2010 with scale parameter 1, which is a valid covariance iff R is positive definite and $\nu > 0$. See Appendix.

Consider the case $d = 1$ and we have

$$\begin{aligned} &- 2M \int_{\omega > 0} \sin(\omega h) (1 + \omega^2)^{-\nu - \frac{1}{2}} d\omega \\ &= -2M \mathrm{sign}(h) |h|^\nu 2^{-\nu - 1} \sqrt{\pi} \Gamma(-\nu + 1/2) \left(I_\nu(|h|) - L_{-\nu}(|h|) \right) \end{aligned}$$

given on page 332 of Watson: *Theory of Bessel Functions* and I_ν is the modified Bessel function and L_ν is the modified Struve function. Note that the above is undefined for $\nu = 1/2, 3/2, \ldots$ due to the gamma function, where we have extended the gamma function to the negative nonintegers.

$\boxed{\text{What happens when } \nu = 1/2, 3/2, \ldots ?}$

For the 2-dimensional case, we switch to polar coordinates with respect to the direction of $\boldsymbol{h}$ and assume WLOG that $a = (0, 1)^\top$ so that we integrate over the upper half of $\mathbb{R}^2$:

$$\int_{\omega | a^\top \omega > 0} \sin(\omega^\top \boldsymbol{h}) (1 + \omega^\top \omega)^{-\nu - 1} d\omega$$

$$= \int_0^\infty \int_{-\pi+c}^{c} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))(1+r^2)^{-\nu-1} r\,d\theta\,dr$$

where θ is the angle between $\boldsymbol{h}$ and ω. Here c is the angle that $\boldsymbol{h}$ makes with the x-axis.

Now note that

$$\int_0^{\pi/2} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))d\theta = \frac{\pi}{2}\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|)$$
$$\int_{\pi/2}^{\pi} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))d\theta = -\frac{\pi}{2}\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|)$$

by 12.1.7 of abramowitz and stegun where $\boldsymbol{H}_\nu$ is the Struve function. Therefore, when $c = \pi$, the above integral is 0, which makes sense since this is the reversible direction as defined. Furthermore, when $c = \pi/2$, we have

$$\begin{aligned}\int_0^\infty \int_{-\pi/2}^{\pi/2} \sin(r\,\|\boldsymbol{h}\|\cos(\theta))(1+r^2)^{-\nu-1} r\,d\theta\,dr &= \int_0^\infty \pi\boldsymbol{H}_0(r\,\|\boldsymbol{h}\|)(1+r^2)^{-\nu-1} r\,dr \\ &= \pi\frac{2^{-\nu-1}\pi\,\|\boldsymbol{h}\|^\nu}{\Gamma(\nu+1)\cos(\nu\pi)}\left(\boldsymbol{I}_\nu(\|\boldsymbol{h}\|) - \boldsymbol{L}_{-\nu}(\|\boldsymbol{h}\|)\right)\end{aligned}$$

by 6.814 of I.S. Gradshteyn and I.M. Ryzhik. (compare with Stein for Matern). This gives the nonreversible part in the most nonreversible direction.

However, this does not work in general because $c \notin \{\pi, \pi/2\}$ necessarily.

$\boxed{\text{Find } \int_{-\pi+c}^{c} \sin(z\cos(\theta))d\theta \text{ for } z > 0 \text{ to evaluate the covariance for arbitrary direction in } \mathbb{R}^2.}$ (an equivalent problem is solving $\int_0^\pi \sin(z\cos(\theta - a))d\theta$ for each $a \in (0, \pi)$). Mathematica and Wolfram Alpha do not help.