Preliminary

We consider the integral for the general Matern first as an exercise. To evaluate the integral

$$\int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}dx,$$

we use the fact that for the modified Bessel function of the second kind, $K_\nu(\cdot)$, we have

$$K_\nu(h) = \frac{\Gamma(\nu+1/2)2^\nu}{\pi^{1/2}h^\nu}\int_0^\infty \frac{\cos(hx)dx}{(1+x^2)^{\nu+1/2}}$$

(for example, see Abramowitz and Stegun 9.6.25). Then, we have

$$\begin{aligned}\int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}dx &= \int_{\mathbb{R}} \frac{\cos(hx)+i\sin(hx)}{(1+x^2)^{\nu+1/2}}dt \\ &= 2K_\nu(h)\frac{\pi^{1/2}h^\nu}{2^\nu\Gamma(\nu+1/2)} + i\int_{\mathbb{R}} \frac{\sin(hx)}{(1+x^2)^{\nu_1/2}}dt \\ &= K_\nu(h)\frac{\pi^{1/2}h^\nu}{2^{\nu-1}\Gamma(\nu+1/2)} + 0\end{aligned}$$

because the right integrand is an odd function. This gives the standard Matern class as expected.

Non-reversible

We return to the cross-covariance problem; the challenge is that the imaginary part will probably be nonzero. We have, letting $h = t - s$,

$$\begin{aligned}\mathbb{E}(Y_i(s+h)Y_j(s)) &= \int_{\mathbb{R}} e^{ihx}(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx \\ &= \int_{\mathbb{R}} (\cos(hx)+i\sin(hx))(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx\end{aligned}$$

For simplicity, let $M(h,\nu)$ be the Matern covariance at lag h and parameter ν, so that we can write

$$\mathbb{E}(Y_i(s+h)Y_j(s)) = M(h,\nu)\frac{z_{i,j}+\overline{z}_{i,j}}{2} + i\int_{\mathbb{R}} \sin(hx)(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx$$

Now, if $z_{i,j} = \overline{z}_{i,j}$, then simply $\mathbb{E}(Y_i(s+h)Y_j(s)) = M(h,\nu)z_{i,j}$, and the reversible Matern cross-covariance falls out cleanly. We turn to when cross-covariance may not be reversible.

Suppose that $z_{i,j} \neq \overline{z}_{i,j}$. If $\nu \geq 1/2$,

$$\int_{\mathbb{R}} |\sin(hx)|(1+x^2)^{-\nu-1/2} < \infty; \tag{1}$$

it is worth **checking** to see if the integral is finite if $\nu \in (0, 1/2)$ as one may expect. Supposing that (1) holds, we have

$$\begin{aligned}&\int_{\mathbb{R}} \sin(hx)(1+x^2)^{-\nu-1/2}(z_{i,j}1_{\{x>0\}} + \overline{z}_{i,j}1_{\{x<0\}})dx \\ &\quad = (z_{i,j} - \overline{z}_{i,j})\int_{(0,\infty)} \frac{\sin(hx)}{(1+x^2)^{\nu+1/2}}dx.\end{aligned}$$

This integral is similar to a Bessel function, but contains sin instead of cos. **What is the value of the integral?**

Let $C_{i,j}(h) = \mathbb{E}(Y_i(s+h)Y_j(s))$, and note that

$$C_{i,j}(-h) = C_{i,j}(h) - i2(z_{i,j} - \overline{z}_{i,j}) \int_{(0,\infty)} \frac{\sin(hx)}{(1+x^2)^{\nu+1/2}} dx$$

showing the non-reversibility of the process when $z_{i,j} \neq \overline{z}_{i,j}$.