CHAPTER 6

DIFFERINTEGRATION OF MORE COMPLEX FUNCTIONS

Chapter 4 was devoted to the differintegration of certain simple functions. Having now developed, in Chapter 5, some general rules governing differintegration, we are in a position to tackle more difficult functions. The most powerful weapon in our armory is the rule

$$\text{(6.0.1)} \qquad \frac{d^q}{[d(x-a)]^q} \sum_{j=0}^{\infty} a_j [x-a]^{p+[j/n]} = \sum_{j=0}^{\infty} a_j \frac{\Gamma\left(\dfrac{pn+j+n}{n}\right)}{\Gamma\left(\dfrac{pn-qn+j+n}{n}\right)} [x-a]^{p-q+[j/n]}$$

for $p > -1$, established in Section 5.2 for the differintegral of the function $\sum a_j[x-a]^{p+[j/n]}$. As will be recalled from the discussion in Section 3.1, such a function belongs to the class we have termed "differintegrable series" provided n is a positive integer, p exceeds -1, and a_0 is nonzero.

In early sections of this chapter we shall differintegrate using an arbitrary lower limit a; but, to avoid the added complexity introduced by this generality, some later sections will adopt a lower limit of zero.

6.1 THE BINOMIAL FUNCTION $[C-cx]^p$

The binomial theorem permits, upon writing $C - cx = C - ac - c[x-a]$, the power-series expansion

$$[C-cx]^p = \sum_{j=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(j+1)\Gamma(p-j+1)} [-c]^j [C-ac]^{p-j} [x-a]^j$$

in $x - a$, provided that the quantity X, defined as $c[x - a]/[C - ca]$, lies in the range $-1 < X < +1$. Note that this expansion is valid even if p is a positive integer, but that in this event the sum is automatically finite, all terms for which j exceeds p having infinite denominators. Equation (6.0.1) may be applied straightforwardly to this sum, leading to the result

$$\frac{d^q[C - cx]^p}{[d(x - a)]^q} = \frac{[C - ca]^p[x - a]^{-q}}{\Gamma(-p)} \sum_{j=0}^{\infty} \frac{\Gamma(j - p)}{\Gamma(j - q + 1)} \left[\frac{c[x - a]}{C - ca}\right]^j$$

after use of the $\Gamma(p + 1)/\Gamma(p - j + 1) = [-]^j\Gamma(j - p)/\Gamma(-p)$ identity and removal of all j-independent factors from within the summation. The most concise representation of the summed terms is as an incomplete beta function (see Section 1.3) of argument X, yielding

$$\frac{d^q[C - cx]^p}{[d(x - a)]^q} = \frac{c^q[C - cx]^{p-q}}{\Gamma(-q)} \mathrm{B}_X(-q, q - p)$$

as the final result.

For the case in which $a = 0$, and $C = c = 1$, the simple result

$$\frac{d^q[1 - x]^p}{dx^q} = \frac{[1 - x]^{p-q}}{\Gamma(-q)} \mathrm{B}_x(-q, q - p) \tag{6.1.1}$$

emerges.

6.2 THE EXPONENTIAL FUNCTION $\exp(C - cx)$

With C and c as arbitrary constants, the power-series expansion

$$\exp(C - cx) = \exp(C - ca) \sum_{j=0}^{\infty} \frac{\{-c[x - a]\}^j}{\Gamma(j + 1)}$$

is valid for all $x - a$. Differintegration term by term with respect to $c[x - a]$ yields

$$\frac{d^q \exp(C - cx)}{[d(cx - ca)]^q} = \{c[x - a]\}^{-q} \exp(C - ca) \sum_{j=0}^{\infty} \frac{\{-c[x - a]\}^j}{\Gamma(j - q + 1)}.$$

The sum may be expressed as an incomplete gamma function [see equation (1.3.26)] of argument $-c[x - a]$ and parameter $-q$. The final result appears as

$$\frac{d^q \exp(C - cx)}{[d(x - a)]^q} = \frac{\exp(C - cx)}{[x - a]^q} \gamma^*(-q, -c[x - a]) \tag{6.2.1}$$

after use of the scale change relationship (5.4.2) to replace the variable of differintegration by $x - a$.

Since $\gamma^*(-n, y) = y^n$ for nonnegative integer n, the above result is seen to reduce to the well-known formula for multiple differentiation of an exponential function. Reduction to the simple formula

$$\frac{d^q \exp(\pm x)}{dx^q} = \frac{\exp(\pm x)}{x^q} \gamma^*(-q, \pm x) \tag{6.2.2}$$

occurs on substituting $C = a = 0$ and $c = \mp 1$ into the general result.

The incomplete gamma function has an asymptotic expansion which permits us to write

$$\frac{\gamma^*(-q, y)}{y^q} \sim 1 - \frac{\exp(-y)}{\Gamma(-q)y^{q+1}} \left[1 - \frac{q+1}{y} + O(y^{-2})\right]$$

for large y. This expansion may be invoked to evaluate the $a \to -\infty$ limit of equation (6.2.1). Thus, if we choose $C = 0$, $c = -v$,

$$\frac{d^q \exp(vx)}{[d(x+\infty)]^q} = v^q \exp(vx).$$

This simple result for differintegration with a lower limit of $-\infty$ provided the basis for much of the work of Liouville and Weyl, about which we wrote in Section 1.1. Functions other than exponentials, however, seldom yield finite differintegrals when the lower limit is minus infinity and we shall have no occasion again to make use of these "Weyl differintegrals."

The functions considered in this section and the last, $[C - cx]^p$ and $\exp(C - cx)$, are both analytic in $x - a$. The functions treated in the next section, while they are not necessarily analytic, are differintegrable and so are still subject to the term-by-term differintegration rule (6.0.1).

6.3 THE FUNCTIONS $x^q/[1-x]$ AND $x^p/[1-x]$ AND $[1-x]^{q-1}$

By use of the binomial expansion of $[1-x]^{-1}$ and the technique of term-by-term differintegration, we arrive at

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right] = \sum_{j=0}^{\infty} \frac{d^q}{dx^q} x^{j+q}$$

as a formula expressing the effect of the d^q/dx^q operator with the lower limit zero on the $x^q/[1-x]$ function, subject to the proviso that x not exceed unity in magnitude. Provided also that q exceeds -1, the rules of Section 4.4 permit differintegration of the powers of x and lead to

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right] = \sum_{j=0}^{\infty} \frac{\Gamma(j+q+1)}{\Gamma(j+1)} x^j = \Gamma(q+1) \sum_{j=0}^{\infty} \binom{-q-1}{j} [-x]^j.$$

Identification of the sum as a binomial expansion produces

$$\frac{d^q}{dx^q}\left[\frac{x^q}{1-x}\right] = \frac{\Gamma(q+1)}{[1-x]^{q+1}} \tag{6.3.1}$$

as the simple final result.

The technique for differintegrating $x^p/[1-x]$ follows such a similar course that it will suffice to cite one intermediate and the final result

$$\frac{d^q}{dx^q}\left[\frac{x^p}{1-x}\right] = x^{p-q}\sum_{j=0}^{\infty}\frac{\Gamma(j+p+1)x^j}{\Gamma(j+p-q+1)} = \frac{\Gamma(p+1)\mathrm{B}_x(p-q,q+1)}{\Gamma(p-q)[1-x]^{q+1}}, \tag{6.3.2}$$

together with the restrictions, namely, $0 < x < 1$ and $p > -1$, which were assumed during the derivation.

As an illustration of the utility of the composition rule (Section 5.7) in finding differintegrals, consider the effect of the d^{-q}/dx^{-q} operator applied to each member of equation (6.3.1). The composition rule may readily be applied when $q < 0$ to yield

$$\frac{x^q}{1-x} = \Gamma(q+1)\frac{d^{-q}}{dx^{-q}}[1-x]^{-q-1}.$$

Extension to $q < 1$ follows[1] by the boundedness at zero of the function $x^q/[1-x]$. [Recall also that $q > -1$ was assumed in the derivation of (6.3.1).] Rearrangement and reversal of the sign of q then produces

$$\frac{d^q}{dx^q}[1-x]^{q-1} = \frac{x^{-q}}{\Gamma(1-q)[1-x]}$$

with $|q| < 1$. This result may be regarded as a special case of equation (6.1.1), from which it may be derived alternatively.

6.4 THE HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS $\sinh(\sqrt{x})$ AND $\sin(\sqrt{x})$

The differintegrals of the hyperbolic and circular sines of the square root of x are particularly interesting examples of the differintegration of nonanalytic functions. They provide a foretaste of the capability of differintegration to

[1] See Chapter 5, footnote 3.

interrelate important transcendental functions, a subject which is considered in detail in Chapter 9.

The now familiar technique of series expansion followed by term-by-term differintegration gives

$$\frac{d^q}{dx^q}\sinh(\sqrt{x}) = \sum_{j=0}^{\infty} \frac{d^q}{dx^q}\left[\frac{x^{j+\frac{1}{2}}}{\Gamma(2j+2)}\right] = \sum_{j=0}^{\infty} \frac{\Gamma(j+\frac{3}{2})x^{j-q+\frac{1}{2}}}{\Gamma(2j+2)\Gamma(j-q+\frac{3}{2})}.$$

The simplification

$$\frac{\Gamma(j+\frac{3}{2})}{\Gamma(2j+2)} = \frac{\sqrt{\pi}}{2^{2j+1}\Gamma(j+1)}$$

is a consequence of the duplication property of gamma functions and permits us to write

$$\text{(6.4.1)} \qquad \frac{d^q}{dx^q}\sinh(\sqrt{x}) = \frac{\sqrt{\pi}\,x^{\frac{1}{2}-q}}{2}\sum_{j=0}^{\infty}\frac{[\frac{1}{4}x]^j}{\Gamma(j-q+\frac{3}{2})\Gamma(j+1)}$$
$$= \tfrac{1}{2}\sqrt{\pi}\,[2\sqrt{x}]^{\frac{1}{2}-q} I_{\frac{1}{2}-q}(\sqrt{x}),$$

where $I_{\frac{1}{2}-q}(\sqrt{x})$ denotes the $(\frac{1}{2}-q)$th-order hyperbolic Bessel function of argument $\sqrt{x}$.

The differintegral of the circular sine is derived in a strictly analogous way, being

$$\text{(6.4.2)} \qquad \frac{d^q}{dx^q}\sin(\sqrt{x}) = \tfrac{1}{2}\sqrt{\pi}\,[2\sqrt{x}]^{\frac{1}{2}-q} J_{\frac{1}{2}-q}(\sqrt{x})$$

and involving an ordinary Bessel function of the first kind. These last two formulas constitute generalizations of Rayleigh's formulas (Abramowitz and Stegun, 1964, pp. 439, 445).

6.5 THE BESSEL FUNCTIONS

This section will evaluate the differintegrals of the functions

$$x^{\nu/2}J_\nu(2\sqrt{x}) \qquad \text{and} \qquad x^{\nu/2}I_\nu(2\sqrt{x}),$$

where $J_\nu(\ )$ and $I_\nu(\ )$ are the νth-order Bessel and modified Bessel functions. This exercise will exemplify the fact that some formulas of the classical calculus generalize unchanged into the fractional calculus, whereas others do not.

The νth-order Bessel function is defined via the series

$$J_\nu(2\sqrt{x}) \equiv x^{\nu/2} \sum_{j=0}^{\infty} \frac{[-x]^j}{\Gamma(j+1)\Gamma(j+\nu+1)}. \tag{6.5.1}$$

Therefore,

$$x^{\nu/2} J_\nu(2\sqrt{x}) = \sum_{j=0}^{\infty} \frac{[-]^j x^{j+\nu}}{\Gamma(j+1)\Gamma(j+\nu+1)}$$

and is seen to be a differintegrable function provided $\nu > -1$. Performing the differintegration with a lower limit of zero yields

$$\begin{aligned} \frac{d^q}{dx^q}\{x^{\nu/2} J_\nu(2\sqrt{x})\} &= \sum_{j=0}^{\infty} \frac{[-]^j x^{j+\nu-q}}{\Gamma(j+1)\Gamma(j+\nu-q+1)} \\ &= x^{[\nu-q]/2} J_{\nu-q}(2\sqrt{x}), \end{aligned} \tag{6.5.2}$$

a simple and appealing result.

The proof that

$$\frac{d^q}{dx^q}\{x^{\nu/2} I_\nu(2\sqrt{x})\} = x^{[\nu-q]/2} I_{\nu-q}(2\sqrt{x}) \tag{6.5.3}$$

follows an identical pattern since the definition of the modified Bessel function mirrors equation (6.5.1) except that the alternating signs within the summand are missing.

If we set $z \equiv 2\sqrt{x}$ in (6.5.2), transformation to

$$\frac{d^q}{[z\,dz]^q}\{z^\nu J_\nu(z)\} = z^{\nu-q} J_{\nu-q}(z) \tag{6.5.4}$$

occurs. This result has long been known in the classical calculus [see Abramowitz and Stegun, (1964, p. 361)], where it is restricted, of course, to integer q. It is seen to generalize unchanged into the fractional calculus.

A classical result, complementary to (6.5.4), is

$$\frac{d^q}{[z\,dz]^q}\left\{\frac{J_\nu(z)}{z^\nu}\right\} = [-]^q \frac{J_{\nu+q}(z)}{z^{\nu+q}} \tag{6.5.5}$$

for integer q. This result clearly cannot generalize unchanged into the fractional calculus, for whereas the left-hand side is real (for real z) the right-hand side would necessarily be complex for many q values. The appropriate extension of equation (6.5.5) for the special values $q = \pm\frac{1}{2}$ is given in Section 7.7.

6.6 HYPERGEOMETRIC FUNCTIONS

A definition of a generalized hypergeometric function was given in Section 2.10. Formulas were there developed showing the effect of integer order differintegration on a generalized hypergeometric function and on the product of such a function with x^p. The technique of Section 5.2 shows that these formulas [(2.10.2) and (2.10.3)] generalize unchanged if q is unrestricted. That is,

$$\text{(6.6.1)} \quad \frac{d^q}{dx^q}\left[x\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right] = x^{-q}\left[x\,\frac{0, b_1, b_2, \ldots, b_K}{-q, c_1, c_2, \ldots, c_L}\right]$$

and

$$\text{(6.6.2)} \quad \frac{d^q}{dx^q}\left\{x^p\left[x\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\}$$

$$= x^{p-q}\left[x\,\frac{p, b_1, b_2, \ldots, b_K}{p-q, c_1, c_2, \ldots, c_L}\right], \qquad p > -1$$

for all q.

It will be recognized that all hypergeometric functions of argument x (or of argument $x^{m/n}$, where m and n are integers) fall into our class of differintegrable series, as defined in Section 3.1. This class also embraces all products of a hypergeometric function with x^p (for $p > -1$) or with a second hypergeometric function.

Inasmuch as

$$[1-x]^p = \frac{1}{\Gamma(-p)}\left[x\,\frac{-1-p}{0}\right]$$

and

$$\frac{[1-x]^{p-q}}{\Gamma(-q)}\,\mathrm{B}_x(-q, q-p) = \frac{x^{-q}}{\Gamma(-p)}\left[x\,\frac{-1-p}{-q}\right],$$

result (6.1.1) is seen to be nothing but a special case of formula (6.6.1) for differintegration of a generalized hypergeometric function, coupled with the rule for canceling equal numeratorial and denominatorial parameters. Result (6.2.2) is an even simpler special case of the same formula.

Likewise, all other results we have thus far derived in this chapter [formulas (6.3.1), (6.3.2), (6.4.1), (6.4.2), (6.5.2), and (6.5.3)] may easily be shown to be special instances of the formula (6.6.2) for differintegration of a product of a power function x^p (with $p > -1$) and a generalized hypergeometric function. Consider formula (6.4.1) as an example. Using the duplication formula (1.3.9), the hyperbolic sine of $\sqrt{x}$ may be written as a $\frac{0}{2}$ hypergeometric function,

$$\sinh(\sqrt{x}) = \frac{\sqrt{\pi x}}{2}\left[\tfrac{1}{4}x \,\frac{}{0, \tfrac{1}{2}}\right],$$

so that, utilizing the scale-change theorem, we find

$$\begin{aligned}\frac{d^q}{dx^q}\sinh(\sqrt{x}) &= \frac{\sqrt{\pi}}{4^q}\,\frac{d^q}{[d(x/4)]^q}\left\{\sqrt{\frac{x}{4}}\left[\tfrac{1}{4}x \,\frac{}{0, \tfrac{1}{2}}\right]\right\}\\ &= \frac{\sqrt{\pi}}{4^q}\left[\frac{x}{4}\right]^{\frac{1}{2}-q}\left[\tfrac{1}{4}x \,\frac{\tfrac{1}{2}}{\tfrac{1}{2}-q, 0, \tfrac{1}{2}}\right]\\ &= \frac{\sqrt{\pi}}{2}\,x^{\frac{1}{2}-q}\left[\tfrac{1}{4}x \,\frac{}{\tfrac{1}{2}-q, 0}\right] = \tfrac{1}{2}\sqrt{\pi}\,[2\sqrt{x}]^{\frac{1}{2}-q} I_{\frac{1}{2}-q}(\sqrt{x})\end{aligned}$$

because of the expressibility of a modified Bessel function as the $\frac{0}{2}$ hypergeometric function,

$$I_\nu(\sqrt{x}) = \left[\frac{x}{4}\right]^{\nu/2}\left[\tfrac{1}{4}x \,\frac{}{0, \nu}\right].$$

Use was made of the scale-change property (Section 5.4) in the preceding paragraph. The procedure may be generalized,

$$\begin{aligned}&\text{(6.6.3)}\qquad \frac{d^q}{dx^q}\left\{x^p\left[\beta x \,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\}\\ &\qquad = \beta^{q-p}\,\frac{d^q}{[d(\beta x)]^q}\left\{[\beta x]^p\left[\beta x \,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\}\\ &\qquad = x^{p-q}\left[\beta x \,\frac{p, b_1, b_2, \ldots, b_K}{p-q, c_1, c_2, \ldots, c_L}\right], \qquad p > -1,\end{aligned}$$

to show that the differintegration properties of a hypergeometric of argument equal to a constant multiplied by x are not affected by the magnitude of the constant.

Thus far we have treated the generalized hypergeometric functions of argument equal to x, the variable of differintegration, or some constant multiple thereof. We now turn to situations in which the argument is a root $x^{1/n}$ or power x^n of x, n being a (typically small) positive integer.

We demonstrated in Section 2.10 that a $\frac{K}{L}$ hypergeometric function of argument $x^{1/n}$ could be equated to a sum of n hypergeometrics, each generally having the complexity $\frac{nK}{nL}$ and argument x. The problem of differintegrating the original function is thus solved. Formula (2.10.4) was the relevant equation and we use it in the following development of the differintegration properties of the function $[1 - x^{\frac{1}{3}}]^{-1}$:

$$\begin{aligned}
\frac{d^q}{dx^q}\left\{\frac{x^p}{1 - x^{\frac{1}{3}}}\right\} &= \frac{d^q}{dx^q}\left\{x^p\left[x^{\frac{1}{3}} \frac{}{}\right]\right\} \\
&= \frac{d^q}{dx^q}\left\{x^p\left[x \frac{}{}\right]\right\} + \frac{d^q}{dx^q}\left\{x^{p+\frac{1}{3}}\left[x \frac{}{}\right]\right\} + \frac{d^q}{dx^q}\left\{x^{p+\frac{2}{3}}\left[x \frac{}{}\right]\right\} \\
&= x^{p-q}\left[x \frac{p}{p-q}\right] + x^{p-q+\frac{1}{3}}\left[x \frac{p+\frac{1}{3}}{p-q+\frac{1}{3}}\right] \\
&\quad + x^{p-q+\frac{2}{3}}\left[x \frac{p+\frac{2}{3}}{p-q+\frac{2}{3}}\right].
\end{aligned}$$

Finally, we analyze the differintegration properties of the generalized hypergeometric function of argument x^n, using the example of a function of $\frac{1}{1}$ complexity, namely, $[x^n \frac{b}{c}]$. First, however, we shall state the result,

$$\text{(6.6.4)} \qquad \frac{\Gamma(jn + p + 1)}{\Gamma(jn + p - q + 1)} = n^q \prod_{i=1}^{n} \frac{\Gamma\left(j + \dfrac{p+i}{n}\right)}{\Gamma\left(j + \dfrac{p-q+1}{n}\right)},$$

of applying the Gauss multiplication formula (1.3.10) to the left-hand member of (6.6.1). We are now in a position to develop the following proof:

$$\begin{aligned}
&\frac{d^q}{dx^q} x^p \left[x^n \frac{b}{c}\right] \\
&\quad = \sum_{j=0}^{\infty} \frac{d^q}{dx^q} x^{jn+p} \frac{\Gamma(j + b + 1)}{\Gamma(j + c + 1)} \\
&\quad = x^{p-q} \sum_{j=0}^{\infty} x^{jn} \frac{\Gamma(jn + p + 1)\Gamma(j + b + 1)}{\Gamma(jn + p - q + 1)\Gamma(j + c + 1)} \\
&\quad = n^q x^{p-q}\left[x^n \frac{\dfrac{p-n+1}{n}, \dfrac{p-n+2}{n}, \ldots, \dfrac{p-1}{n}, \dfrac{p}{n}, b}{\dfrac{p-q-n+1}{n}, \dfrac{p-q-n+2}{n}, \ldots, \dfrac{p-q}{n}, c}\right].
\end{aligned}$$

Extension to a hypergeometric function of $\frac{K}{L}$ complexity is obvious, and is seen to yield a differintegral of $\frac{K+n}{L+n}$ complexity in the absence of parametric cancellation. For the important $n = 2$ case, the general result reads

$$\frac{d^q}{dx^q}\left\{x^p\left[x^2 \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\} \tag{6.6.5}$$

$$= 2^q x^{p-q}\left[x^2 \frac{\frac{p-1}{2}, \frac{p}{2}, b_1, b_2, b_3, \ldots, b_K}{\frac{p-q-1}{2}, \frac{p-q}{2}, c_1, \ldots, c_L}\right].$$

6.7 LOGARITHMS

Its recognition as the hypergeometric function

$$\ln(x+1) = x\left[-x \frac{0}{1}\right]$$

permits a ready differintegration of the logarithm function of argument $x + 1$. Thus

$$\frac{d^q \ln(x+1)}{dx^q} = x^{1-q}\left[-x \frac{0}{1-q}\right]$$

on application of rule (6.6.2) and cancellation of a unity parameter. Surprisingly, since it occurs ubiquitously in the generalized calculus, the simple hypergeometric function

$$\left[x \frac{0}{c}\right]$$

lacks a generic name, though instances of it are widespread (see Chapter 9).

Because of the simplicity of the chain rule for integer-order derivatives (Section 2.6) a change of function variable is readily accomplished in the classical differential calculus. As we saw in Section 5.6, however, no simple analog of the chain rule exists in the general calculus, a fact which prevents evaluation of

$$\frac{d^q \ln(x)}{dx^q} \quad \text{by way of} \quad \frac{d^q \ln(x+1)}{dx^q}.$$

Likewise, if we try to accomplish the same derivation by noting the equivalence of

$$\frac{d^q \ln(x)}{dx^q} \quad \text{to} \quad \frac{d^q \ln(X+1)}{[d(X+1)]^q},$$

(with $X = x - 1$) and attempt to relate the latter to

$$\frac{d^q \ln(X+1)}{dX^q},$$

we are impeded by the complexity of the rule (Section 5.8) for shifting the lower limit of differintegration. In fact, in the generalized calculus, differintegrals of $\ln(x)$ and $\ln(x+1)$ are astonishingly unrelated. We shall find this to be quite a common state of affairs—for example such apparently similar functions as $\sin(x)$ and $\sin(\sqrt{x})$ behave quite differently on general differintegration—and we shall learn not to be surprised thereby.

To differintegrate $\ln(x)$ we start with the Riemann–Liouville formulation and then apply the $v = [x - y]/x$ variable change:

$$\frac{d^q \ln(x)}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{\ln(y)\,dy}{[x-y]^{q+1}}, \qquad q < 0$$

$$= \frac{x^{-q}\ln(x)}{\Gamma(-q)} \int_0^1 \frac{dv}{v^{q+1}} + \frac{x^{-q}}{\Gamma(-q)} \int_0^1 \frac{\ln(1-v)\,dv}{v^{q+1}},$$

to produce two definite integrals. The first evaluates trivially to $1/[-q]$, while the second yields to the parts-integration,

$$\int_0^1 \frac{\ln(1-v)\,dv}{v^{q+1}} = \frac{1}{q} \int_0^1 \ln(1-v)\,d(1 - v^{-q})$$

$$= \left.\frac{[1 - v^{-q}]\ln(1-v)}{q}\right|_0^1 - \frac{1}{q}\int_0^1 \frac{1 - v^{-q}}{1 - v}\,dv$$

$$= 0 - \frac{\gamma + \psi(1-q)}{q},$$

where we have made use of equations (1.3.35) and (1.3.36) in deriving the term containing the psi function of $1 - q$. Putting these results together, and employing the recurrence (1.3.2),

$$\frac{d^q \ln(x)}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)} [\ln(x) - \gamma - \psi(1-q)] \tag{6.7.1}$$

is obtained. Though derived for $q < 0$, the usual appeal to analyticity establishes result (6.7.1) for all q, though the expression is indeterminate as formulated for q a positive integer.[2] Note the reduction of (6.7.1) to the classical result

$$\frac{d^{-n}\ln(x)}{dx^{-n}} = \frac{x^n}{n!}\left[\ln(x) - \sum_{j=1}^{n}\frac{1}{j}\right]$$

when q is a negative integer, in consequence of equations (1.3.34) and (1.3.35). Figure 6.7.1 shows some differintegrals of $\ln(x)$.

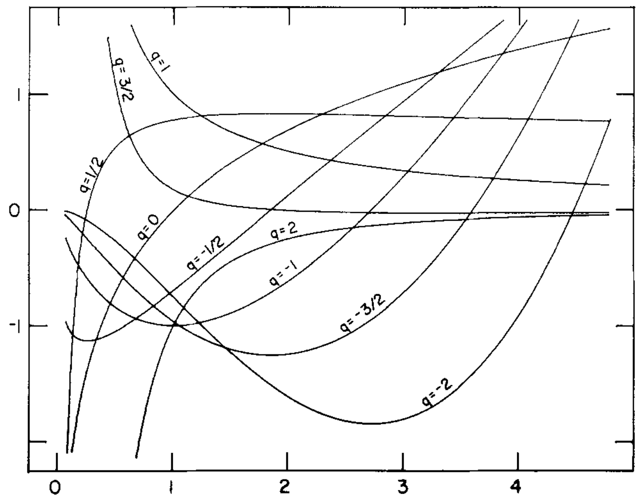

FIG. 6.7.1. Differintegrals of $\ln(x)$ for q in the range -2 to $+2$; $d^q\ln(x)/dx^q = x^{-q}[\ln(x) - \gamma - \psi(1-q)]/\Gamma(1-q)$.

Equation (6.7.1) may be regarded as a special instance of the more general result (whose proof we omit)

$$\text{(6.7.2)} \qquad \frac{d^q[x^p\ln(x)]}{dx^q} = \frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}\left[\ln(x) + \psi(p+1) - \psi(p-q+1)\right]$$

for $p > -1$ and $q < 0$.

Using the latter result it is possible to derive the formula

$$\frac{d^q}{dx^q}\frac{d^Q\ln(x)}{dx^Q} = \frac{d^{q+Q}\ln(x)}{dx^{q+Q}},$$

at least for $Q < 1$, and thereby establish the composition rule for the logarithm function for certain ranges of Q and q. It will be recalled that in Section 5.7 the composition rule was derived only for the class of functions comprised by all differintegrable series, a class from which $\ln(x)$ is excluded.

[2] From the limit of $\psi(1-q)/\Gamma(1-q)$ as q approaches the positive integer n, the rule $d^n\ln(x)/dx^n = -\Gamma(n)[-x]^{-n}$ is readily deduced.

6.8 THE HEAVISIDE AND DIRAC FUNCTIONS

The Heaviside function (or unit-step function) occurring at $x = x_0$,

$$H(x - x_0) \equiv \begin{cases} 0, & x < x_0, \\ 1, & x > x_0, \end{cases}$$

is the simplest example of a piecewise-defined function, the general class of which is considered in the next section. The Dirac function (or delta function) $\delta(x - x_0)$ is the derivative of the Heaviside function: It is everywhere zero except at $x = x_0$, where it is infinite.

The differintegration of the Heaviside function for $q < 0$ and with $a < x_0 < x$ is a trivial operation via the Riemann–Liouville definition (3.2.3). Thus

$$\text{(6.8.1)} \quad \frac{d^q}{[d(x-a)]^q} H(x - x_0) = \frac{1}{\Gamma(-q)} \int_a^x \frac{H(y - x_0)\, dy}{[x - y]^{q+1}}, \qquad q < 0$$

$$= \frac{1}{\Gamma(-q)} \int_a^{x_0} \frac{[0]\, dy}{[x - y]^{q+1}} + \frac{1}{\Gamma(-q)} \int_{x_0}^{x} \frac{[1]\, dy}{[x - y]^{q+1}}$$

$$= 0 + \frac{d^q[1]}{[d(x - x_0)]^q}, \qquad x > x_0$$

$$= \frac{[x - x_0]^{-q}}{\Gamma(1 - q)}, \qquad x > x_0,$$

where the results of Sections 4.2 and 4.1 have been employed. By invoking equation (3.2.5) it is easily demonstrated that the same equation applies for all q. Figure 6.8.1 illustrates this formula.

One of the principal uses of the Heaviside function is to delimit the range of definition of a function f. Thus, we have the product

$$f H(x - x_0) = \begin{cases} 0, & x < x_0, \\ f, & x > x_0. \end{cases}$$

We shall need to utilize differintegrals of such a product. By analogy with the derivation (6.8.1) we easily establish that

$$\text{(6.8.2)} \qquad \frac{d^q}{[d(x-a)]^q} \{f H(x - x_0)\} = H(x - x_0) \frac{d^q f}{[d(x - x_0)]^q},$$

where $a < x_0 < x$.

We characterize the Dirac delta function $\delta(x - x_0)$ by means of its property

$$\int_a^x \delta(y - x_0) f(y)\, dy = f(x_0), \qquad a < x_0 < x,$$

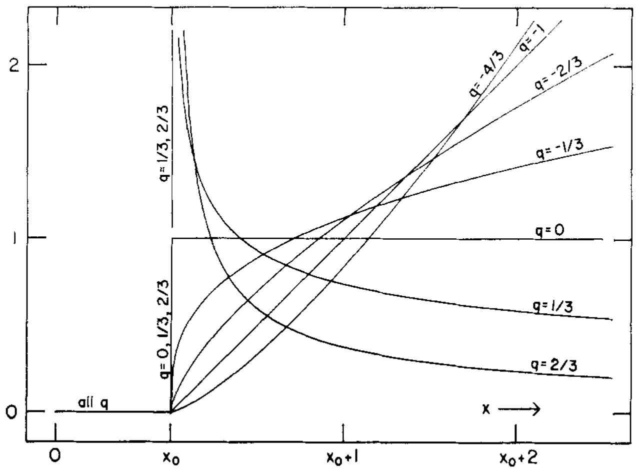

FIG. 6.8.1. Some differintegrals of the Heaviside function $H(x-x_0)$. For $q=1$ differintegration yields the ungraphable delta function and the $q>1$ differintegrals are likewise impossible to depict.

for any function f. From this we see immediately on selecting $f=[x-y]^{-q-1}$ that

$$\int_a^x \frac{\delta(y-x_0)\,dy}{[x-y]^{q+1}} = [x-x_0]^{-q-1}, \qquad q<0. \tag{6.8.3}$$

If we divide equation (6.8.3) by $\Gamma(-q)$ we recognize the left-hand side as the qth differintegral of the Dirac function; that is,

$$\frac{d^q\delta(x-x_0)}{[d(x-a)]^q} = \frac{[x-x_0]^{-q-1}}{\Gamma(-q)}. \tag{6.8.4}$$

Extension of formula (6.8.4) to negative q is once again accomplished via equation (3.2.5).

Comparing formulas (6.8.1) and (6.8.4) it will be noticed that

$$\frac{d^{q+1}H(x-x_0)}{[d(x-a)]^{q+1}} = \frac{d^q\delta(x-x_0)}{[d(x-a)]^q}.$$

The well-known relationship

$$\frac{d}{dx}H(x-x_0) = \delta(x-x_0)$$

follows upon setting $q=0$.

6.9 THE SAWTOOTH FUNCTION

In this section we will first derive a general formula for the differintegral of a piecewise-defined function, and then apply the formula to a simple example, the sawtooth function.

Consider first the function

$$f = \begin{cases} f_1, & a \leqq x < x_1, \\ f_2, & x_1 < x. \end{cases}$$

Making use of the Heaviside function, we may write f as

$$\begin{aligned} f = f_1 H(x_1 - x) + f_2 H(x - x_1) &= f_1 - f_1 H(x - x_1) + f_2 H(x - x_1) \\ &= f_1 + [f_2 - f_1] H(x - x_1) \end{aligned}$$

since

$$H(x - x_1) + H(x_1 - x) = 1, \qquad x \neq x_1.$$

Application of linearity and equation (6.8.2) is now all that is needed to establish

$$\text{(6.9.1)} \quad \frac{d^q f}{[d(x-a)]^q} = \frac{d^q f_1}{[d(x-a)]^q} + H(x - x_1) \frac{d^q [f_2 - f_1]}{[d(x - x_1)]^q}, \qquad x \neq x_1.$$

The generalization of this result to the many-sectioned piecewise-defined function

$$\begin{aligned} f &= f_k, \qquad x_{k-1} < x < x_k, \qquad k = 1, 2, 3, \ldots, N, \\ &= f_1 + [f_2 - f_1] H(x - x_1) + \cdots + [f_N - f_{N-1}] H(x - x_{N-1}), \qquad x \neq x_k, \end{aligned}$$

(where $x_0 \equiv a$) is straightforward. The general result

$$\text{(6.9.2)} \qquad \frac{d^q f}{[d(x-a)]^q} = \frac{d^q f_1}{[d(x-a)]^q} + \sum_{k=1}^{N-1} H(x - x_k) \frac{d^q [f_{k+1} - f_k]}{[d(x - x_k)]^q}$$

allows for an arbitrarily large number of sections.

The sawtooth function, defined by

$$\begin{aligned} \operatorname{saw}(x) &= [-]^k [2k - x - 2], \qquad 2k - 3 < x < 2k - 1, \quad k = 1, 2, \ldots, N \\ &= x + \sum_{k=1}^{N-1} [-]^k [2x - 4k + 2] H(x - 2k + 1) \end{aligned}$$

is shown in Fig. 6.9.1. Application of formula (6.9.2), with a lower limit of zero, yields

$$\frac{d^q \operatorname{saw}(x)}{dx^q} = \frac{d^q x}{dx^q} + \sum_{k=1}^{N-1} H(x-2k+1)\frac{d^q\{[-]^k[2x-4k+2]\}}{[d(x-2k+1)]^q}$$
$$= \frac{x^{1-q}}{\Gamma(2-q)} + 2\sum_{k=1}^{N-1} H(x-2k+1)[-]^k \frac{[x-2k+1]^{1-q}}{\Gamma(2-q)}.$$

Examples of this differintegral are incorporated into Fig. 6.9.1.

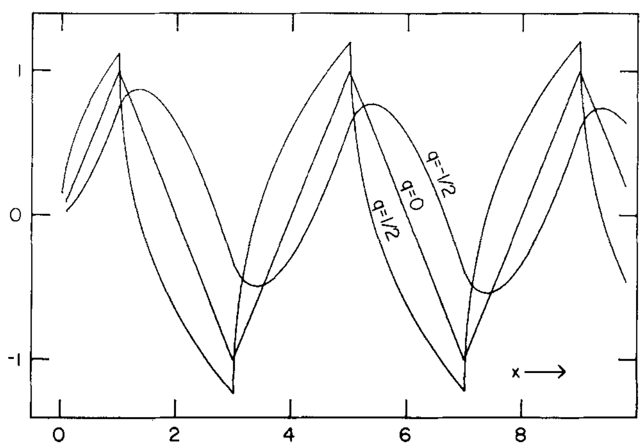

FIG. 6.9.1. The sawtooth function saw(x) and two of its differintegrals.

6.10 PERIODIC FUNCTIONS

Any periodic function is expressible as

$$\text{(6.10.1)} \qquad \operatorname{per}(x) = \sum_{k=1}^{\infty}\left[C_k \exp\left(\frac{2\pi ikx}{X}\right) + \bar{C}_k \exp\left(\frac{-2\pi ikx}{X}\right)\right],$$

where C_k and $\bar{C}_k$ are conjugate complex constants and X is the period of the function. Differintegration of a periodic function thus devolves into determining

$$\frac{d^q}{dx^q}\exp\left(\frac{\pm 2\pi ikx}{X}\right),$$

an operation to which we now turn our attention.

Since our proofs of the scale-change theorem (Section 5.4) and of the differintegration of exponential functions (Section 6.2) in no way precluded imaginary constants or variables, we can write

$$
\begin{aligned}
(6.10.2)\quad \frac{d^q}{dx^q}\exp\left(\frac{\pm 2\pi ikx}{X}\right) &= \left[\frac{\pm 2\pi ik}{X}\right]^q \frac{d^q}{[d(\pm 2\pi ikx/X)]^q}\exp\left(\frac{\pm 2\pi ikx}{X}\right) \\
&= x^{-q}\exp\left(\frac{\pm 2\pi ikx}{X}\right)\gamma^*\left(-q, \frac{\pm 2\pi ikx}{X}\right).
\end{aligned}
$$

Introduced into (6.10.1), this formula completely describes the differintegration of any periodic function for any value of x. However, it is of interest to find the limiting forms of the differintegral corresponding to small and to large x values.

For small x values a power series of the incomplete gamma function permits the rewriting of equation (6.10.2) as

$$
\frac{d^q}{dx^q}\exp\left(\frac{\pm 2\pi ikx}{X}\right) = x^{-q}\sum_{j=0}^{\infty}\frac{[\pm 2\pi ikx/X]^j}{\Gamma(j-q+1)}
$$

so that, in the limit of small x, the differintegral of a periodic function is

$$
\frac{d^q}{dx^q}\operatorname{per}(x\to 0) = \sum_{k=1}^{\infty}\left[\frac{[C_k+\bar{C}_k]x^{-q}}{\Gamma(1-q)} + \frac{i[C_k-\bar{C}_k]kx^{1-q}}{X\Gamma(2-q)} + \cdots\right].
$$

Notice that the coefficients $[C_k + \bar{C}_k]$ and $i[C_k - \bar{C}_k]$ are real and that the leading term in this expansion is simply the differintegral of the initial value per(0) of the periodic function, treated as a constant.

Of more interest is the limiting form for large x. An asymptotic expansion (Abramowitz and Stegun, 1964, p. 263) of the incomplete gamma function leads to

$$
\frac{d^q}{dx^q}\exp\left(\frac{\pm 2\pi ikx}{X}\right) \sim \left[\frac{\pm 2\pi ik}{X}\right]^q \exp\left(\frac{\pm 2\pi ikx}{X}\right) - \sum_{j=0}^{\infty}\frac{[\pm 2\pi ikx/X]^{-1-j}x^q}{\Gamma(-q-j)}
$$

as equivalent to (6.10.2) in the limit as x tends to infinity. Because $(\pm i)^q$ are the complex numbers $\exp(\pm\pi iq/2)$, this expansion yields

$$
\begin{aligned}
&\frac{d^q}{dx^q}\operatorname{per}(x\to\infty) \\
&\quad = \sum_{k=1}^{\infty}\left[\frac{2\pi k}{X}\right]^q\left\{C_k\exp\left(2\pi i\left[\frac{kx}{X}+\frac{q}{4}\right]\right) + \bar{C}_k\exp\left(-2\pi i\left[\frac{kx}{X}+\frac{q}{4}\right]\right)\right\} \\
&\quad\quad + \sum_{k=1}^{\infty}\left\{\frac{i[C_k-\bar{C}_k]x^{-1-q}X}{2\pi k\Gamma(-q)} + \frac{[C_k+\bar{C}_k]x^{-2-q}X^2}{4\pi^2k^2\Gamma(-q-1)} + \cdots\right\}
\end{aligned}
$$

upon incorporation into the general expression for the differintegral of a periodic function. The terms grouped within the first summation are periodic: They show that the effect of differintegration to order q has been to change the amplitude of each component of the original function by a factor of $[2\pi k/X]^q$ and to change its phase by an angle $\pi q/2$ radians. Within the second summation the terms are aperiodic: Provided q exceeds -1, they represent transients which eventually (i.e., at large x) become insignificant.

The simplest periodic functions are sin(x) and cos(x). In the x approaching infinity limit, the differintegrals of these functions are

$$\frac{d^q}{dx^q}\sin(x) = \sin\left(x + \frac{\pi q}{2}\right) + \frac{x^{-1-q}}{\Gamma(-q)} - \frac{x^{-3-q}}{\Gamma(-q-2)} + \cdots \tag{6.10.3}$$

and

$$\frac{d^q}{dx^q}\cos(x) = \cos\left(x + \frac{\pi q}{2}\right) + \frac{x^{-2-q}}{\Gamma(-q-1)} - \frac{x^{-4-q}}{\Gamma(-q-3)} + \cdots. \tag{6 10.4}$$

Notice how these equations reduce to well-known formulas in the classical cases of positive or negative integer q.

6.11 CYCLODIFFERENTIAL FUNCTIONS

By this term we mean functions such as $\exp(\pm x)$, cosh(x), and sin(x), which are regenerated after sufficient differentiations. In Section 7.6 we shall seek functions which are preserved under repeated semidifferentiation, but for now we restrict consideration to regeneration after a small number of integer order differentiations.

Formula (3.6.5) provides a means of differintegrating cyclodifferential functions which is unusually powerful in that a nonzero lower limit may be employed even with relatively complex functions. We illustrate this by setting $\phi(x) = \sin(x)$ in formula (3.6.5), whereby

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\left\{\sin(x)\sum_{k=0,4,\ldots} - \cos(x)\sum_{k=1,5,\ldots} - \sin(x)\sum_{k=2,6,\ldots} + \cos(x)\sum_{k=3,7,\ldots}\right\}\frac{[x-a]^{k-q}}{k![k-q]}$$

is obtained. This formulation is intended to convey that the same summand, $[x-a]^{k-q}/\{k!\,[k-q]\}$, is summed using four different k-sequences. On regrouping terms,

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{\sin(x)}{\Gamma(-q)} \sum_{j=0,1,\ldots}^{\infty} \frac{[-]^j [x-a]^{2j-q}}{(2j)!\,[2j-q]} - \frac{\cos(x)}{\Gamma(-q)} \sum_{j=0,1,\ldots}^{\infty} \frac{[-]^j [x-a]^{2j-q+1}}{(2j+1)!\,[2j-q+1]}$$

results. At this stage we could proceed to express the summations as hypergeometric functions. However, we shall instead identify them as the indefinite integrals,

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{\sin(x)}{\Gamma(-q)} \int_0^{x-a} \frac{\cos(u)}{u^{q+1}}\,du - \frac{\cos(x)}{\Gamma(-q)} \int_0^{x-a} \frac{\sin(u)}{u^{q+1}}\,du, \tag{6.11.1}$$

provided $q < 0$. To cover the $q < 2$ range, we can withdraw the $j = 0$ terms from the summation, leading to

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{\sin(x)}{\Gamma(-q)} \left[\frac{[x-a]^{-q}}{-q} + \int_0^{x-a} \frac{\cos(u)-1}{u^{q+1}}\,du \right] - \frac{\cos(x)}{\Gamma(-q)} \left[\frac{[x-a]^{1-q}}{1-q} + \int_0^{x-a} \frac{\sin(u)-u}{u^{q+1}}\,du \right].$$

Notice that by defining $u = x - y$, equation (6.11.1) could have been generated directly from the Riemann–Liouville definition

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{\sin(y)\,dy}{[x-y]^{q+1}}$$

of the differintegral of $\sin(x)$. It is unusual to be able to remove the upper differintegration limit x from within the Riemann–Liouville integral in this way. It is possible in the case $f(x) = \sin(x)$ because $\sin(x-u)$ is expressible as a sum of products $F(x)G(u)$ with separated variables. Our discussion here suggests that the ability of a function $f(x-u)$ to be factored in this way to $\sum F(x)G(u)$ is an inherent property of a cyclodifferential function. A search for corresponding algebraic properties in functions (see Section 7.6) that play an analogous role to cyclodifferentials in the semicalculus, however, proved unsuccessful.

The integrals in equation (6.11.1), and in the corresponding equation for the differintegration of the cosine function, are not evaluable in terms of established functions (except for special q values), but they are expressible as generalized $\frac{1}{3}$ hypergeometric functions. The formulas

$$\frac{d^q \sin(x)}{[d(x-a)]^q} = \frac{\sqrt{\pi}\sin(x)}{2\Gamma(-q)[x-a]^q}\left[-\tfrac{1}{4}[x-a]^2 \frac{-\frac{1}{2}q - 1}{-\frac{1}{2}q, -\frac{1}{2}, 0}\right]$$
$$- \frac{\sqrt{\pi}\cos(x)}{2\Gamma(-q)[x-a]^q}\left[-\tfrac{1}{4}[x-a]^2 \frac{-\frac{1}{2}q - \frac{1}{2}}{-\frac{1}{2}q + \frac{1}{2}, 0, \frac{1}{2}}\right]$$

and

$$\frac{d^q \cos(x)}{[d(x-a)]^q} = \frac{\sqrt{\pi}\cos(x)}{2\Gamma(-q)[x-a]^q}\left[-\tfrac{1}{4}[x-a]^2 \frac{-\frac{1}{2}q - 1}{-\frac{1}{2}q, -\frac{1}{2}, 0}\right]$$
$$+ \frac{\sqrt{\pi}\sin(x)}{2\Gamma(-q)[x-a]^q}\left[-\tfrac{1}{4}[x-a]^2 \frac{-\frac{1}{2}q - \frac{1}{2}}{-\frac{1}{2}q + \frac{1}{2}, 0, \frac{1}{2}}\right]$$

thereby emerge for $q < 0$.

6.12 THE FUNCTION $x^{q-1}\exp[-1/x]$

In the previous section we noted that the exponential function, in common with only a few other special functions f, possesses the property

$$f(x \pm u) = \sum_i F_i(x)G_i(u),$$

by which an argument that is a linear combination of two variables can be decomposed to a finite sum of products. This property also permits the function $x^{q-1}\exp(-1/x)$ to be differintegrated to order q with a zero lower limit, as we now demonstrate.

Using the Riemann–Liouville definition followed by the substitution $y = x/[xz + 1]$, we find

$$\frac{d^q}{dx^q}\left\{\frac{\exp(-1/x)}{x^{1-q}}\right\} = \frac{1}{\Gamma(-q)}\int_0^x \frac{y^{q-1}\exp(-1/y)\,dy}{[x-y]^{q+1}}$$
$$= \frac{\exp(-1/x)}{\Gamma(-q)x^{q+1}}\int_0^\infty \frac{\exp(-z)dz}{z^{q+1}}.$$

From definition (1.3.1), the integral is evaluated simply as $\Gamma(-q)$ so that the final result

$$\text{(6.12.1)} \qquad \frac{d^q}{dx^q}\left\{\frac{\exp(-1/x)}{x^{1-q}}\right\} = \frac{\exp(-1/x)}{x^{q+1}}$$

emerges. We shall omit the proof of the more general result

$$\frac{d^q}{dx^q}\left\{x^{q-n}\exp\left(\frac{-1}{x}\right)\right\} = \exp\left(\frac{-1}{x}\right)\sum_{j=0}^{n-1}\frac{\Gamma(j-q)}{\Gamma(-q)}\binom{n-1}{j}x^{j-n-q}, \quad n = 1, 2, 3, \ldots$$

of which equation (6.12.1) is the simplest instance.

The corresponding trigonometric differintegrals are

$$\frac{d^q}{dx^q}\left\{x^{q-1}\sin\left(\frac{1}{x}\right)\right\} = x^{-q-1}\sin\left(\frac{1}{x} - \frac{q\pi}{2}\right)$$

and

$$\frac{d^q}{dx^q}\left\{x^{q-1}\cos\left(\frac{1}{x}\right)\right\} = x^{-q-1}\cos\left(\frac{1}{x} - \frac{q\pi}{2}\right).$$

The proof of these results, for $0 < q < 1$, is established by an argument very similar to that of the previous paragraph. It is of interest to note that differintegration of these trigonometric functions has introduced a "phase shift" of $\pi q/2$, as was found in equations (6.10.3) and (6.10.4), but that the shift is in the opposite sense.