CHAPTER 1

INTRODUCTION

1.1 HISTORICAL SURVEY

The concept of differentiation and integration to noninteger order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known—Leibniz (1859) mentions it in a letter to L'Hospital in 1695.[1] The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832a), Riemann (1953), and Holmgren (1864), although Euler (1730), Lagrange (1772), and others made contributions even earlier.

It was Liouville (1832a) who expanded functions in series of exponentials and defined the qth derivative of such a series by operating term-by-term as though q were a positive integer. Riemann (1953) proposed a different definition that involved a definite integral and was applicable to power series with noninteger exponents. Evidently it was Grünwald and Krug who first unified the results of Liouville and Riemann. Grünwald (1867), disturbed by the restrictions of Liouville's approach, adopted as his starting point the definition of a derivative as the limit of a difference quotient and arrived at definite-integral formulas for the qth derivative. Krug (1890), working through Cauchy's integral formula for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit while Liouville's definition, in which no distinguishable lower limit appeared, corresponded to a lower limit $-\infty$.

[1] Authors and dates designate entries in our working list of references, to be found at the end of the book. They should not be confused with items in the chronological bibliography appearing at the end of Section 1.1. Cited dates, as in "Leibniz (1859)," refer to the year of publication of the references listed at the end of the book. As with collected works, such a date is not necessarily that of the year of original publication.

Parallel to these theoretical beginnings was a development of the applications of the fractional calculus to various problems. In a sense, the first of these was the discovery by Abel (1823, 1825) in 1823 that the solution of the integral equation for the tautochrone could be accomplished via an integral transform, which, as we shall see, benefits from being written as a semiderivative. A powerful stimulus to the use of fractional calculus to solve problems was provided by the development by Boole (1844) of symbolic methods for solving linear differential equations with constant coefficients. The essence of Boole's idea is the formal expansion of an arbitrary function $f(D)$ of the differential operator as a power series and the solution of differential equations by formal inversion of such series. Boole's methods have subsequently been made rigorous for certain classes of functions f [see Bourlet (1897) and Ritt (1917)] and extended in many directions.

The operational calculus of Heaviside (1892, 1893, 1920), developed by him to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. Heaviside (1920) introduced fractional differentiation in his investigation of transmission line theory; this concept has been extended by Gemant (1936) for use in problems of elasticity. While Heaviside seemed to scorn the "wet blankets of rigorists," at least some theorists recognized the merit of his techniques, and attempted to justify them by acceptable mathematical standards [see Carson (1926) and Wiener (1926)].

In the present century notable contributions have been made to both the theory and application of the fractional calculus. Weyl (1917), Hardy (1917), Hardy and Littlewood (1925, 1928, 1932), Kober (1940), and Kuttner (1953) examined some rather special, but natural, properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes. Erdélyi (1939, 1940, 1954) and Osler (1970a) have given definitions of differintegrals with respect to arbitrary functions, and Post (1930) used difference quotients to define generalized differentiation for operators $f(D)$, where D denotes differentiation and f is a suitably restricted function. Riesz (1949) has developed a theory of fractional integration for functions of more than one variable. Erdélyi (1964, 1965) has applied the fractional calculus to integral equations and Higgins (1967) has used fractional integral operators to solve differential equations. Other applications include those to rheology (Scott Blair *et al.*, 1947; Shermergor, 1966; Scott Blair, 1947, 1950a,b; Scott Blair and Caffyn, 1949; Graham *et al.*, 1961), to electrochemistry (Belavin *et al.*, 1964; Oldham, 1969a; Oldham and Spanier, 1970; Grenness and Oldham, 1972), to chemical physics (Somorjai and Bishop, 1970), and to general transport problems (Oldham, 1973b; Oldham and Spanier, 1972). The developments are far too numerous to give an exhaustive survey here, nor is this our purpose. The readers interested in further references to the literature may consult the chronological bibliography which ends this section. Virtually no area of classical analysis has been left untouched by the fractional calculus. Indeed, could one expect less from the natural extension of perhaps the two most basic operations of mathematics—differentiation and integration?

We close Section 1.1 with an annotated chronological bibliography on fractional calculus prepared by Professor Bertram Ross of the University of New Haven. It is reprinted here intact with his kind permission and is meant to give additional historical perspective to our subject. No attempt has been made to eliminate duplication between Professor Ross' bibliography and our own working references to be found at the end of the book.

Professor Ross has examined many of the papers and texts in the following list and has appended short comments where he deemed it appropriate. His criteria for inclusion were: first investigation of an important development, and frequency of citation.

1695 G. W. Leibniz, Letter from Hanover, Germany, September 30, 1695 to G. A. L'Hospital. *Leibnizen Mathematische Schriften*, Vol. 2, pp. 301–302. Olms Verlag., Hildesheim, Germany, 1962. First published in 1849.

Leibniz wrote prophetically, "Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt[2]{dx:x}$, an apparent paradox, from which one day useful consequences will be drawn."

1697 G. W. Leibniz, Letter from Hanover, Germany, May 28, 1697 to J. Wallis, *Leibnizen Mathematische Schriften*, Vol. 4, p. 25. Olms Verlag., Hildesheim, Germany, 1962. First published in 1859.

In this letter Leibniz discusses Wallis' infinite product for π. Leibniz mentions differential calculus and uses the notation $d^{\frac{1}{2}}y$ to denote a derivative of order $\frac{1}{2}$.

1730 L. Euler, "De Progressionibus Transcentibus, sev Quarum Termini Algebraice Dari Nequeunt." *Comment. Acad. Sci. Imperialis Petropolitanae* **5**, 38–57 (1738).

On p. 55 of "Concerning transcendental progressions whose terms can not be given algebraically," Euler writes, "When n is a positive integer, the ratio d^np, p a function of x, to dx^n can always be expressed algebraically. Now it is asked: what kind of ratio can be made if n be a fraction? If n is a positive integer, d^n can be found by continued differentiation. Such a way, however, is not evident if n is a fraction. But the matter may be expedited with the help of the interpolation of series as explained earlier in this dissertation."

1772 J. L. Lagrange, "Sur une nouvelle espèce de calcul relatif à la différentiation et à l'intégration des quantités variables." *Oeuvres de Lagrange*, Vol. 3, pp. 441–476. Gauthier-Villars, Paris, 1849. First appeared in *Nouv. Mém. Acad. Roy. Sci. Belles-Lett. Berlin* **3**, 185–206 (1772).

Lagrange's contribution in this work is the law of exponents (indices) for operators of integer order:

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y.$$

Later, when the theory of fractional calculus started, it became important to know whether this law held true if m and n were fractions.

1812 P. S. Laplace, *Théorie Analytique des Probabilités*, Courcier, Paris, 1820. First appeared in 1812.

On pp. 85 and 186 of the third edition, Laplace writes expressions for certain fractional derivatives.

1819 S. F. Lacroix, *Traité du Calcul Différentiel et du Calcul Intégral*, 2nd ed., Vol. 3 pp. 409–410. Courcier, Paris.

In this 700 page text two pages are devoted to fractional calculus. Lacroix develops a formula for fractional differentiation for the nth derivative of v^m by induction. Then, he formally replaces n with the fraction $\frac{1}{2}$, and together with the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, he obtains

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}} v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

1822 J. B. J. Fourier, "Théorie Analytique de la Chaleur." *Oeuvres de Fourier*, Vol. 1, p. 508. Didot, Paris.

Fourier makes the following generalization:

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha)\, d\alpha \int_{-\infty}^{+\infty} p^u \cos\left(px - p\alpha + \frac{u\pi}{2}\right) dp,$$

and states, "The number u will be regarded as any quantity whatever, positive or negative."

1823 N. H. Abel, "Solution de quelques problèmes à l'aide d'intégrales définites." *Oeuvres Complètes*, Vol. 1, pp. 16–18. Grondahl, Christiania, Norway, 1881. This paper first appeared in *Mag. Naturvidenkaberne* (1823).

Abel was probably the first to give an application of fractional calculus. He used derivatives of arbitrary order to solve the tautochrone (isochrone) problem. The integral he worked with

$$\int_0^x (x-t)^{-\frac{1}{2}} f(t)\, dt$$

is precisely of the same form that Riemann used to define fractional operations.

1832 [a] J. Liouville, "Mémoire sur quelques Quéstions de Géometrie et de Mécanique, et sur un nouveau genre de Calcul pour résoudre ces Quéstions." *J. Ecole Polytech.* **13**, Section 21, pp. 1–69.

The first major study of fractional calculus starts with Liouville. On p. 3, Liouville considers $(d^{\frac{1}{2}}/dx^{\frac{1}{2}})e^{2x}$. In this memoir, some problems in mechanics and geometry are solved by the use of fractional operations.

1832 [b] J. Liouville, "Mémoire sur le Calcul des différentielles à indices quelconques." *J. Ecole Polytech.* **13**, Section 21, pp. 71–162.

On p. 94, he considers the existence of a complementary function to be added to the definition of a fractional operation. Liouville was led into error in this particular area. On p. 117, he works out a method for the fractional derivative of a product of two functions.

1832 [c] J. Liouville, "Mémoire sur l'intégration de l'équation $(mx^2 + nx + p)$ $d^2y/dx^2 + (qx + r)\, dy/dx + sy = 0$ à l'aide des différentielles à indices quelconques." *J. Ecole Polytech.* **13**, Section 21, pp. 163–186.

1833 G. Peacock, "Report on the Recent Progress and Present State of Affairs of Certain Branches of Analysis." *Rep. British Assoc. Advancement Sci.* 185–352.

He enunciated the *Principle of the permanence of equivalent forms*, later echoed by Kelland in 1846. Peacock was led into error on the subject of fractional operations by assuming that the above stated principle was valid for all symbolic operations. For example, although $DD^{-1} = D^0$, $D \neq 1/D^{-1}$, where $D = d/dx$.

1834 [a] J. Liouville, "Mémoire sur une formule d'analyse." *J. Reine Angew. Math.* (*Crelle's Journal*) **12**, 273–287.

Liouville discusses the tautochrone problem.

1834 [b] J. Liouville, "Mémoire sur le théoreme des fonctions complémentaires," *J. Reine Angew. Math.* (*Crelle's Journal*) **11**, 1–19.

Liouville continues work on the complementary function (1832[b]). He argues that if the differential equation

$$\frac{d^n y}{dx^n} = 0$$

has a complementary solution, why shouldn't

$$\frac{d^u y}{dx^u} = 0$$

have a complementary solution when u is arbitrary?

1835 [a] J. Liouville, "Mémoire sur l'usage que l'on peut faire de la formule de Fourier, dans le calcul des différentielles à indices quelconques." *J. Reine Angew. Math.* (*Crelle's Journal*) **13**, 219–232.

Liouville suggests a better way of writing Fourier's (1822) formula.

1835 [b] J. Liouville, "Mémoire sur le changement de la variable dans le calcul des différentielles à indices quelconques." *J. Ecole Polytech.* **15**, Section 24, 17–54.

He gives the definition of a fractional derivative, p. 22, as an infinite series:

$$\frac{d^u y}{dx^u} = \sum A_m e^{mx} m^u,$$

where, "*u* est un nombre quelconque, entier ou fractionnaire, positif ou negativ, réel ou imaginaire."

1839 S. S. Greatheed, "On General Differentiation No. I," *Cambridge Math. J.* **1**, 11–21. In the same issue are two more papers: "On General Differentiation No. II." *Cambridge Math. J.* **1**, 109–117; "On the Expansion of a Function of a Binomial." *Cambridge Math. J.* **1**, 67–74.

In the first two papers above, Greatheed uses Liouville's definition to develop formulas for fractional differentiation. In the third paper, he supplements Taylor's theorem by use of fractional derivatives.

1839 P. Kelland, "On General Differentiation." *Trans. Roy. Soc. Edinburgh* **14**, 567–618.

1841 D. F. Gregory, *Examples of the Processes of the Differential and Integral Calculus*, 1st ed., p. 350, 2nd ed., 1846, p. 354. J. J. Deighton. Cambridge, England.

Gregory was probably the founder of what was then called the *calculus of operations*. He gives the solution of the heat equation

$$\frac{d^2 z}{dx^2} = \frac{1}{a}\frac{dz}{dy}$$

in symbolic operator form:

$$z = Ae^{y\beta^{\frac{1}{2}}} + Be^{-y\beta^{\frac{1}{2}}},$$

where $\beta = a^{-1}(d/dx)$. This form was later used by Heaviside.

1842 A. De Morgan, *The Differential and Integral Calculus Combining Differentiation, Integration, Development, Differential Equations, Differences, Summation, Calculus of Variations... with Applications to Algebra, Plane and Solid Geometry and Mechanics.* Baldwin and Cradock, London, published under the superintendence of the Society for the diffusion of useful knowledge. First published in twenty-five parts.

In this long-titled text, De Morgan devotes three pages to the subject of fractional calculus. He makes the statement that neither the system of fractional operations as given by Peacock nor that of Liouville has any claim to be considered as giving the form $D^u x^n$, though either may be a form. The controversy over different systems of fractional operations was made more explicit by Center, was cleared up in the last decades of the nineteenth century, and was raised again by Post.

1846 P. Kelland, "On General Differentiation." *Trans. Roy. Soc. Edinburgh* **16**, 241–303 (1849).

Kelland assumes that the principle of the *permanence of equivalent forms*, stated for algebra, is valid for all symbolic operations. This principle was used earlier by Peacock, and by G. Boole "On a General Method in Analysis," *Philos. Trans. Roy. Soc. London* **134**, 225–282 (1844), a paper which developed the formal theory of operators. Kelland states, "Algebraic formulae which are the results of these laws and nothing else, must be correct forms also when the algebraic symbols are replaced by such symbols of operation." The mistrust that Heaviside encountered decades later when he submitted his results obtained by the use of symbolic operators might be traced to errors of these mathematicians who misapplied the principle of the *permanance of equivalent forms*.

1847 B. Riemann, "Versuch einer Auffassung der Integration und Differentiation." *Gesammelte Werke*, 1876. ed. publ. posthumously, pp. 331–344; 1892 ed., pp. 353–366. Teubner, Leipzig. Also in *Collected Works* (H. Weber, ed.), pp. 354–360. Dover, New York, 1953.

Riemann sought a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-r}}{dx^{-r}}u(x)=\frac{1}{\Gamma(r)}\int_c^x (x-k)^{r-1}u(k)\,dk.$$

However, he saw fit to add a complementary function to the above definition. Today, this definition is in common use as a definition for fractional integration but with the complementary function taken to be identically zero, and the lower limit of integration c is usually zero.

1848 C. J. Hargreave, "On the Solution of Linear Differential Equations." *Philos. Trans. Roy. Soc. London* **138**, 31–54.

This paper is notable because it appears to be the first to generalize Leibniz's rule for the nth derivative of a product: $(uv)^{(n)}=(u+v)^{(n)}$, n an integer, generalized to $d^u(uv)/dx^u$, u arbitrary.

1848 [a] W. Center, "On the Value of $(d/dx)^\theta x^0$ When θ Is a Positive Proper Fraction." *Cambridge and Dublin Math. J.* **3**, 163–169.

Using x^0 to denote a constant, unity, Center considers the fractional derivative of x^0. He explicitly defines the controversy over two systems of fractional operations. The system by Peacock

$$\left(\frac{d}{dx}\right)^\theta x^m=\frac{\Gamma(m+1)}{\Gamma(m-\theta+1)}\,x^{m-\theta},\qquad \theta>0,$$

when $m=0$ yields a finite result. Liouville's system

$$\left(\frac{d}{dx}\right)^\theta x^{-m}=\frac{(-1)^\theta\Gamma(m+\theta)}{\Gamma(m)}\,x^{-\theta-m},\qquad \theta>0,\quad m+\theta>0,$$

when $m = 0$ equals zero. Center states on p. 166: "The whole question is therefore now plainly reduced to this, what is $(d/dx)^\theta x^0$ when θ is a positive proper fraction? For when this point is settled, we shall have determined at the same time which of the two systems we *must* adopt."

1848 [b] W. Center, "On Differentiation with Fractional Indices, and on General Differentiation." *Cambridge and Dublin Math. J.* **3**, 274–285.

1849 W. Center, "On Fractional Differentiation." *Cambridge and Dublin Math. J.* **4**, 21–26.

1850 W. Center, "On Fractional Differentiation." *Cambridge and Dublin Math. J.* **5**, 206–217.

1855 J. Liouville, "Sur une formule pour les différentielles à indices quelconques à l'occasion d'un Mémoire de M. Tortolini." *J. Math. Pures Appl.* **20**, pages unnumbered (1855).

Liouville adds to his discussion of a series definition for a fractional derivative.

1859 H. R. Greer, "On Fractional Differentiation." *Quart. J. Math.* Oxford Ser. **3**, 327–330 (1858–1860).

Greer develops formulas for the semiderivatives of $\sin x$ and $\cos x$ using for his starting point Liouville's development $D^{\frac{1}{2}}e^{mx} = m^{\frac{1}{2}}e^{mx}$. He also deals with finite differences of order $\frac{1}{2}$, $\Delta^{\frac{1}{2}}$.

1861 Z. Wastchenxo, "On Fractional Differentiation." *Quart. J. Math.* **4**, 237–243.

Additional formulas to those of Greer above are developed.

1865 H. Holmgren, "Om differentialkalkylen med indices af havd natur som helst." *Kongliga Svenska Ventenkaps-Akademiens Handlingar*, Vol. 5, No. 11, 1–83 (1866). (Usually cataloged under Svenska.)

Holmgren took the same integral representation arrived at by Riemann (1847) as his starting point for a monograph on fractional differentiation.

1867 H. Holmgren, "Sur l'intégration de l'équation différentielle

$$(a_2 + b_2 x + c_2 x^2)\, d^2y/dx^2 + (a_1 + b_1 x)\, dy/dx + a_0 y = 0,"$$

Kongliga Svenska Ventenkaps-Akadamiens, Vol. 7, No. 9, 58 pages (1867–1868).

1867 A. K. Grünwald, "Ueber „begrenzte" Derivationen und deren Anwendung." *Z. Math. Phys.* **12**, 441–480.

One of three applications is inversion, p. 478. If θ is a known function of x, then by fractional operations one can determine the unknown function $f(t)$ in the integral equation

$$\theta = \int_0^x (x-t)^p f(t)\, dt.$$

1868 [a] A. V. Letnikov, "Theory of Differentiation of Fractional Order." *Mat. Sb.* **3**, 1–68.

Letnikov proves for arbitrary orders, pp. 56–58, that:

$$[D^q D^p f(x)]_{x_0}^x = [D^{q+p} f(x)]_{x_0}^x.$$

1868 [b] A. V. Letnikov, "Historical Development of the Theory of Differentiation of Fractional Order." *Mat. Sb.* **3**, 85–119.

Letnikov discusses the work of Liouville, Peacock and Kelland.

1872 [a] A. V. Letnikov, "An Explanation of Fundamental Notions of the Theory of Differentiation of Fractional Order." *Mat. Sb.* **6**, 413–445.

The main theme here is the generalization of Cauchy's integral formula.

1872 [b] A. V. Letnikov, "Studies in the Theory of Integrals of the form $\int_a^x (x-u)^{p-1} f(u)\,du$." *Mat. Sb.* 7, 5–205 (1874). Summary in French, *Bull. Sci. Math. Astron.* 7, 233–238 (1874).

In Chapter III, Letnikov applies the theory of fractional calculus to the solution of certain differential equations.

1873 J. Liouville, "Mémoire sur l'intégration des équations différentielles à indices fractionnaires." *J. Ecole Polytech.* **13**, Section 25, pp. 58–84.

1880 A. Cayley, "Note on Riemann's Paper." *Math. Ann.* **16**, 81–82.

Referring to Riemann's paper (1847) he says, "The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of a complementary function containing an infinity of arbitrary constants." The question of the existence of a complementary function caused much confusion. Liouville and Peacock were led into error, and Riemann became inextricably entangled in his concept of a complementary function.

1884 H. Laurent, "Sur le calcul des derivées à indices quelconques." *Nouv. Ann. Math.* [3], **3**, 240–252.

Laurent generalizes Cauchy's integral formula. He does work on the generalized product rule of Leibniz but leaves the result in integral form.

1888 P. A. Nekrassov, "General Differentiation." *Mat. Sb.* **14**, 45–168.

Using Liouville's starting point for pth order differentiation $d^pe^{mx}/dx^p = m^pe^{mx}$, Nekrassov, p. 152, finds the derivative of arbitrary order of $(x-a)^q$.

1890 A. Krug, "Theorie der Derivationen." *Akad. Wiss. Wien Denkenschriften, Math. Naturwiss. Kl.* **57**, 151–228.

1892 J. Hadamard, "Essai sur l'étude des fonctions données par leur développment de Taylor." *J. Math. Pures Appl.* [4], **8**, 101–186.

1892 O. Heaviside, *Electrical Papers.* The Macmillan Company, London.

1893 [a] O. Heaviside, "On Operators in Physical Mathematics." *Proc. Roy. Soc. London* **52**, 504–529 (1893); **54**, 105–143 (1894).

1893 [b] O. Heaviside, Electromagnetic theory, Vol. 1. The Electrician printing and publishing company, ltd., London. Reprinted by Benn, London 1922.

1893 G. Oltramare, *Calcul de Généralization.* Hermann, Paris, reprinted with revisions, 1899, cited by Davis (1936, pp. 94–98).

1899 O. Heaviside, Electromagnetic theory, Vol. 2. The Electrician printing and publishing company, ltd., London. Reprinted by Benn, London 1922.

1902 R. E. Moritz, "On the Generalization of the Differentiation Process." *Amer. J. Math.* **24**, 257–302.

He uses many new symbols and terms making this paper extremely difficult to read.

1902 S. Pincherle, "Sulle derivate ad indice qualunque." *Mem. Reale Accad. Inst. Sci. Bologna* [5], **9**, 745–758, cited by Davis (1936).

1912 O. Heaviside, Electromagnetic theory, Vol. 3. The Electrician printing and publishing company, ltd., London. Reprinted by Benn, London 1922.

1917 G. H. Hardy, "On Some Properties of Integrals of Fractional Order." *Messenger Math.* **47**, 145–150.

1917 H. Weyl, "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung." *Vierteljschr. Naturforsch. Gesellsch. Zurich* **62**, 296–302.

1918 E. Schuyler, Problem #360. *Amer. Math. Monthly* **25**, 173.

What interpretation must be given to $d^{\frac{1}{2}}y/dx^{\frac{1}{2}}$ so that $(d^{\frac{1}{2}}/dx^{\frac{1}{2}})(d^{\frac{1}{2}}y/dx^{\frac{1}{2}}) = dy/dx$? This problem was discussed and solved by Post (1919).

1918 L. O'Shaughnessy, Problem #433. *Amer. Math. Monthly* **25**, 172–173.

Solve the equation $d^{\frac{1}{2}}y/dx^{\frac{1}{2}} = y/x$. This problem was discussed and solved by Post (1919).

1919 E. Post, "Discussion of Problems #360 and #433." *Amer. Math. Monthly* **26**, 37–39.

When two different solutions are presented to Problem #433, Post takes the opportunity to answer Problem #360 at the same time. He explains that the two solutions are correct; however, each solution is based upon a different definition. The proposer, in his solution, used Liouville's definition of integration of fractional order which is equivalent to the definite integral

$$_cD_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\, dt$$

with lower limit of integration c being negative infinity, while Post, in his solution, used Riemann's definition, which is the above integral with c equal to zero. Although Post makes no reference to Center (1848[a]), it is clear why Center, with $f(x)$ equal to a constant, would have two different results for the arbitrary derivative.

1919 M. T. Naraniengar, "Fractional Differentiation." *J. Indian Math. Soc.* **11**, 88–95.

In the view of the present writer, Naraniengar makes an unwarranted assumption that enables him to develop the coefficient $R(n)$, namely, $\Gamma(n+1)/\Gamma(n+\frac{1}{2})$, which satisfies the relation

$$D^{\frac{1}{2}}(x^n) = R(n)x^{n-\frac{1}{2}}.$$

1919 T. J. I'a Bromwich, "Examples of Operational Methods in Mathematical Physics." *Philos. Mag.* [6], **37**, 407–419.

He states that the purpose of this paper is to encourage the use of operational methods in the solution of physical problems. In the course of attacking heat and induction problems, Bromwich is led to some general rules which confirm the accuracy of Heaviside's methods, held in doubt for twenty years.

1921 T. J. I'a Bromwich, "Symbolical Methods in the Theory of Conduction of Heat." *Proc. Cambridge Philos. Soc.* **20**, 411–427.

1922 G. H. Hardy, "Notes on Some Points in the Integral Calculus." *Messenger Math.* **51**, 186–192.

Hardy investigates the properties of integrals of fractional order, in particular, theorems of continuity and summability, seeking analogies to properties valid for integer order.

1922 W. C. Brenke, "An Application of Abel's Integral Equation." *Amer. Math. Monthly* **29**, 58–60.

The problem is to determine the shape of a weir notch when the quantity of water through the weir in a given time is a function of the height of the notch. The equation formulated from physical considerations is

$$Q(h) = c \int_0^h (h-t)^{\frac{1}{2}} f(t)\, dt.$$

To determine the unknown function $f(t)$, the process of inversion is simplified by means of fractional operations.

1923 P. Levy, "Sur le dérivation et l'intégration géneralisées." *Bull. Sci. Math.* [2], **47**, Pt. 1, 307–320; Pt. 2, 343–352.

On pp. 317–318, Levy considers the fractional derivative of e^{iz}.

1924 E. J. Berg, "Heaviside's Operators in Engineering and Physics." *J. Franklin Inst.* **198**, 647–702, cited by Davis (1936).

1924 H. T. Davis, "Fractional Operations as Applied to a Class of Volterra Integral Equations." *Amer. J. Math.* **46**, 95–109.

The lack of detailed explanation, understandable in a journal article, is made up for by a review of the theory of fractional calculus before the theory is applied to the solution of certain integral equations. This paper and Davis' 1927 article are, in the view of the present writer, distinguished not only for their contributions to the theory and applications of fractional calculus, but also as examples of how mathematics papers should be written.

1925 E. Stephens, "Bibliography on General (or Fractional) Differentiation." *Washington Univ.* (*St. Louis*) *Studies Sci. Ser.* [6], **12**, 149–152.

This bibliography, with some errors in dates and page numbers, has 32 entries without commentary.

1925 G. H. Hardy and J. E. Littlewood, "Some Properties of Fractional Integrals." *Proc. London Math. Soc.* [2], **24**, 37–41.

1927 W. O. Pennell, "A General Operational Analysis." *J. Math. and Phys.* **7**, 24–38.

Theorems on operational calculus are presented. In the course of solving a linear operational equation $p^2y + xy = 1$, $p = d/dx$, Pennell gives a detailed explanation of how p can be expanded in a series. This results in a particular integral solution of the differential equation $y'' + xy = 1$.

1927 H. T. Davis, "The Application of Fractional Operators to Functional Equations." *Amer. J. Math.* **49**, 123–142.

Davis discusses the various notations used to define fractional operations. His suggestion is the notation ${}_cD_x^{-v}f(x)$ to define $\int_c^x (x-t)^{v-1} f(t)\, dt/\Gamma(v)$. Properties of fractional operators are reviewed and then applied to the solution of certain differential equations with operators having fractional exponents, for example, ${}_cD_x^{\frac{1}{2}}u + \lambda u = f(x)$.

1928 G. H. Hardy and J. E. Littlewood, "Some Properties of Fractional Integrals, I." *Math. Z.* **27**, 565–606 (1928); "Some Properties of Fractional Integrals, II." *Math. Z.* **34**, 403–439 (1932).

In part I, their purpose is to develop properties of the Riemann–Liouville integral and derivative of arbitrary order of functions of certain standard classes, in particular the "Lebesgue class L^p." Part II is an extension of the first paper to the complex field.

1930 E. L. Post, "Generalized Differentiation." *Trans. Amer. Math. Soc.* **32**, 723–781.

1931 L. M. Blumenthal, "Note on Fractional Operators, and the Theory of Composition." *Amer. J. Math.* **53**, 483–492. Cited by Davis (1936).

1931 H. T. Davis, "Properties of the Operator $z^{-v} \log z$, where $z = d/dx$." *Bull. Amer. Math. Soc.* **37**, 468–479.

1931 Y. Watanabe, "Notes on the Generalized Derivative of Riemann–Liouville and its Application to Leibniz's Formula." *Tôhoku Math. J.* **34**, 8–41.

1933 K. S. Cole, "Electric Conductance of Biological Systems," *Proc. Cold Spring Harbor Symp. Quant. Biol.* pp. 107–116. Cold Spring Harbor, New York.

The problem is to express analytically the strength of stimulus that, when applied to the nerve bundle, will change the potential difference across the membrane of an individual fiber by a threshold amount in a given time. See also Davis (1936, pp. 288–289).

1935 A. Zygmund, *Trigonometric Series*, Vol. II, 1st. ed. Z. subwencji Funduszu kultury narodewej, Warsaw; 2nd ed. Cambridge University Press, Cambridge, 1959, pp. 132–142.

In the section entitled "Fractional Integration," Zygmund considers a definition of fractional integration introduced by Weyl more convenient for trigonometric series.

1936 H. Poritsky, "Heaviside's Operational Calculus—Its Applications and Foundations." *Amer. Math. Monthly* **43**, 331–344.

In an exceptionally well-written paper, Poritsky states that is it hoped that this paper will serve to popularize this subject in the United States and that it will induce mathematicians to include it in their research and curricula. He explains how a correct interpretation of the operator $p^{\frac{1}{2}}$, $p = d/dx$, can be obtained by expanding in powers of $p^{\frac{1}{2}}$ and *neglecting the integer powers*. This procedure, discovered by Heaviside, was never justified by him. Operational calculus, says Poritsky, effects a connection between linear functional transformations enjoying the "translational or shifting property" and analytic functions, by expressing such operations as analytic functions of p; it is thus related to the modern developments of transformations in Hilbert space.

1936 H. T. Davis, *The Theory of Linear Operators*, Principia Press, Bloomington, Indiana.

This text contains an extensive bibliography of operator theory pp. 571–616. Davis develops fractional calculus pp. 64–75 and gives applications pp. 276–292.

1936 W. Fabian, "Fractional Calculus." *J. Math. and Phys.* **15**, 83–89.

In this paper some properties of the fractional integral at infinity are studied, and Fabian deduces therefrom a method of summability of series and integrals. He extends Riemann's definition which enables him to perform fractional integrations along any simple curve in the complex plane in contrast to Hardy and Littlewood (1928) who integrated along straight lines in the complex plane.

1938 L. C. Young and E. R. Love, "On Fractional Integration by Parts." *Proc. London Math. Soc.* [2] **44**, 1–28.

1939 A. Erdélyi, "Transformation of Hypergeometric Integrals by Means of Fractional Integration by Parts." *Quart. J. Math. Oxford Ser.* **10**, 176–189.

1940 H. Kober, "On Fractional Integrals and Derivatives." *Quart. J. Math. Oxford Ser.* **11**, 193–211.

In the first part of the paper, Kober extends some results of Hardy and Littlewood (1925) over a wider range. In the second part of the paper he deals with Mellin transforms, and also a uniqueness theorem for a solution to the equation

$$g(x) = \int_a^x (x - t)^{\alpha - 1} f(t)\, dt.$$

1941 D. V. Widder, *The Laplace Transform*, pp. 70–75. Princeton Univ. Press, Princeton, New Jersey.

Widder discusses the connection of the Laplace transform with fractional integrals.

1945 A. Zygmund, "Theorem on Fractional Derivatives." *Duke Math. J.* **12**, 455–464.

1949 M. Riesz, "L'intégrale de Riemann-Liouville et le Probléme de Cauchy." *Acta Math.* **81**, 1–223.

This work, in collaboration with Hagstrom, was started in 1933. The fundamental aspects of fractional calculus are given on pp. 10–16. The remainder of the text deals with various aspects of the fractional integral

$$I^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x - t)^{v - 1} f(t)\, dt$$

in the theory of potentials, Lorentz space, relativistic theory, and wave equation in Riemann space.

1950 N. Stuloff, "Die Differentiation beliebiger reelen Ordnung." *Math. Ann.* **122**, 400–410.

Differences of fractional order are discussed:

$$\Delta^{\alpha} x_n = \sum_{v=0}^{\infty} (-1)^v \binom{\alpha}{v} x_{n+v}.$$

1953 B. Kuttner, "Some Theorems on Fractional Derivatives." *Proc. London Math. Soc.* [3] **3**, 480–497.

Kuttner considers the relation between the integrals:

$$\frac{d^n}{dx^n} \frac{1}{\Gamma(n-k)} \int_0^x (x-t)^{n-k-1} f(t)\, dt$$

and

$$(-1)^n \frac{d^n}{dx^n} \frac{1}{\Gamma(n-k)} \int_x^1 (t-x)^{n-k-1} f(t)\, dt.$$

1953 I. I. Hirschmann, "Fractional Integration." *Amer. J. Math.* **75**, 531–546.

1954 A. Erdélyi and staff of the Bateman Manuscript Project, *Tables of Integral Transforms*, Vol. 2, pp. 181–214. McGraw-Hill, New York.

A bibliography of twenty entries on p. 184, without commentary, deals mainly with fractional integrals.

1959 J. L. Lions, "Sur l'existence de solutions des équations de Navier–Stokes," *C. R. Acad. Sci.* **248**, 2847–2849.

A weak solution to the Navier–Stokes equations is a function $u(t)$ from the negative real numbers to L^2, saitsfying a certain functional equation. Lions was the first to pose the question as to whether a weak solution possesses a fractional derivative with respect to t. Then he proceeds to show that if the number of dimensions does not exceed four, then, corresponding to any initial data, there is a weak solution with a fractional derivative of any order less than $\frac{1}{4}$. See also Shinbrot (1971).

1960 A. Erdélyi and I. N. Sneddon, "Fractional Integration and Dual Integral Equations." *Canad. J. Math.* **14**, 685–693.

1961 M. A. Bassam, "Some Properties of the Holmgren–Riesz Transform." *Ann. Scoula Norm. Sup. Pisa* [3] **15**, 1–24.

Bassam shows the equivalence between two definite integrals of arbitrary order given by Holmgren and Riesz, and thus establishes one combined definition. Bassam also wrote a dissertation, "Holmgren–Riesz Transforms." Ph. D. Thesis, Univ. of Texas, Austin, June 1951.

1961 R. Courant, *Differential and Integral Calculus* (translated by E. J. McShane), Vol. 2, pp. 339–341. Wiley (Interscience), New York.

A brief exposition of integrals and derivatives of arbitrary order is given. It is a curious fact worthy of mention that these generalized operators have secured only passing references in standard works in calculus.

1961 A. S. Peters, "Certain Dual Integral Equations and Sonine's Integrals." *Tech. Rep.* No. 225, IMM-NYU. Courant Inst. Math. Sci. New York University. Cited by Buschman (1964).

1962 A. Erdélyi, *Operational Calculus and Generalized Functions*, pp. 2–4. Holt, New York.

A brief discussion of the Heaviside operator $(d/dt)^{\frac{1}{2}}$ is given.

1964 A. Erdélyi, "An Integral Equation Involving Legendre Functions." *SIAM J. Appl. Math.* **12**, 15–30.

1964 T. P. G. Liverman, *Generalized Functions and Direct Operational Methods*, Vol. I, pp. 28–32. Prentice-Hall, Englewood Cliffs, New Jersey.

1964 I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. 1, pp. 115–122. Academic Press, New York.

Many special functions can be written as derivatives of arbitrary order of elementary functions. Gel'fand and Shilov give two examples, the hypergeometric function and the Bessel function.

1964 R. G. Buschman, "Fractional Integration," *Math. Japon.* **9**, 99–106.

In the analysis of mixed boundary value problems, dual integral equations are often encountered, as for example $\int_0^\infty y^p J_\mu(xy) f(y)\, dy = g(x)$ and $\int_0^\infty y^s J_\nu(xy) f(y)\, dy = h(x)$, where J is the usual Bessel function, $g(x)$ and $h(x)$ are given, and $f(x)$ is to be determined. By showing the connection between fractional integral operators and the algebra of functions that have the Mellin convolution as product, Buschman shows that certain identities can be obtained which have previously appeared as special cases. By using fractional operators, he shows how the dual equations above can be reduced to a single integral equation of the type $\int_0^\infty y^t J_\lambda(xy) f(y)\, dy = F(x)$.

1964 V. A. Belavin, R. Sh. Nigmatullin, A. I. Miroshnikov, and N. K. Lutskaya, "Fractional differentiation of oscillographic polarograms by means of an electrochemical two-terminal network." *Tr. Kazan. Aviacion. Inst.* **5**, 144–145.

1964 T. P. Higgins, "A Hypergeometric Function Transform." *SIAM J. Appl. Math.* **12**, 601–612.

1965 T. P. Higgins, "The Rodrigues Operator Transform, Preliminary Report." Document DI-82-0492, Boeing Sci. Res. Lab., Seattle, Washington.

1965 T. P. Higgins, "The Rodrigues Operator Transform, Table of Generalized Rodrigues Formulas." Document DI-42-0493, Boeing Sci. Res. Lab., Seattle, Washington.

1965 L. von Wolfersdorf, "Über eine Beziehung zwischen Integralen nichtganzer Ordnung." *Math. Z.* **90**, 24–28.

1965 A. Erdélyi, "Axially Symmetric Potentials and Fractional Integration." *SIAM J. Appl. Math.* **13**, 216–228.

1966 I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, pp. 46–52. Wiley, New York.

1967 R. N. Kesarwani, "Fractional Integration and Certain Dual Integral Equations." *Math. Z.* **98**, 83–88.

Kesarwani extends the earlier work of Buschman (1964).

1967 T. P. Higgins, "The Use of Fractional Integral Operators for Solving Nonhomogeneous Differential Equations." Document DI-82-0677, Boeing Sci. Res. Lab., Seattle, Washington.

Although results using fractional integral operators, states Higgins, can always be obtained by other methods, the succinct simplicity of the formulation may often suggest approaches not evident in a classical approach. In this paper, some applications to nonhomogeneous differential equations are given.

1967 G. K. Kalisch, "On Fractional Integrals of Pure Imaginary Order in L_p." *Proc. Amer. Math. Soc.* **18**, 136–139.

1968 S. G. Samko, "A Generalized Abel Equation and Fractional Integral Operators." *Differencial'nye Uravnenija* **4**, 298–314.

1968 M. C. Gaer, *Fractional Derivatives and Entire Functions*. Ph.D. Thesis, Univ. of Illinois, Urbana, Illinois.

1968 G. V. Welland, "Fractional Differentiation of Functions." *Proc. Amer. Math. Soc.* **19**, 135–141.

This paper, a portion of Welland's doctoral dissertation at Purdue University, gives some results of a special nature for functions which have lacunary Fourier series. Using Liouville's definition for fractional integration $[1/\Gamma(\beta)]\int_{-\infty}^{x} (x-t)^{\beta-1} f(t)\,dt$, this paper extends some of the work of Zygmund (1935).

1970 M. C. Gaer and L. A. Rubel, "The Fractional Derivative via Entire Functions." *J. Math. Anal. Appl.* **34**, 289–301.

1970 H. Kober, "New Properties of the Weyl Extended Integral." *Proc. London Math. Soc.* [3] **21**, 557–575.

1970 D. M. Bishop and R. L. Somorjai, "Integral-Transformation Trial Functions of the Fractional-Integral Class." *Phys. Rev. A* [3] **1**, 1013–1018.

1970 K. B. Oldham and J. Spanier, "The Replacement of Fick's Laws by a Formulation Involving Semidifferentiation." *J. Electroanal. Chem.* **26**, 331–341.

The operation of order $\frac{1}{2}$, $d^{\frac{1}{2}}f(t)/dt^{\frac{1}{2}}$, which these authors have called semidifferentiation, enables the concentration of an electroactive species at the surface of the electrode to be related straightforwardly to the faradaic current density. Through fractional operations, Oldham and Spanier claim to have uncovered a novel method of elucidating electrochemical kinetics.

1970 [a] T. J. Osler, "Leibniz Rule for Fractional Derivatives Generalized and an Application to Infinite Series." *SIAM J. Appl. Math.* **16**, 658–674.

Certain generalizations of the Leibniz rule for the derivative of the product of two functions are examined and used to generate several infinite series expansions relating special functions.

1970 [b] T. J. Osler, "The Fractional Derivative of a Composite Function." *SIAM J. Math. Anal.* **1**, 288–293.

Osler derives a generalized chain rule, and examines a few special cases of this rule.

1971 [a] T. J. Osler, "Taylor's Series Generalized for Fractional Derivatives and Applications." *SIAM J. Math. Anal.* **2**, 37–47.

1971 [b] T. J. Osler, "Fractional Derivatives and Leibniz Rule." *Amer. Math. Monthly* **78**, 645–649.

The fractional derivative is defined by generalizing Cauchy's integral formula. This definition is used to generalize the Leibniz rule for the derivative of a product by means of which the value of the hypergeometric function of unit argument is evaluated in terms of the gamma function.

1971 E. R. Love, "Fractional Derivatives of Imaginary Order." *J. London Math. Soc.* [2] **3**, 241–259.

In the usual definitions of fractional differentiation, the real part of the order of differentiation is restricted to values greater than zero. Love defines fractional differentiation when the order is purely imaginary, $\mathrm{Re}(\alpha) = 0$, in such a way that the properties of the usual definition are extended to the case of pure imaginary order.

1971 M. Shinbrot, "Fractional Derivatives of Solutions of Navier–Stokes Equations." *Arch. Rational Mech. Anal.* **40**, 139–154.

Shinbrot answers the question posed by Lions (1959) and proves that the order of fractional differentiation can be extended to $\frac{1}{2}$.

1971 P. Butzer and R. Nessel, *Fourier Analysis with Approximation*, pp. 400–403. Academic Press, New York.

1972 K. B. Oldham, "A Signal-Independent Electroanalytical Method." *Anal. Chem.* **44**, 196–198.

1972 M. Grenness and K. B. Oldham, "Semiintegral Electroanalysis: Theory and Verification." *Anal. Chem.* **44**, 1121–1129.

1972 K. B. Oldham and J. Spanier, "A General Solution of the Diffusion Equation for Semiinfinite Geometries." *J. Math. Anal. Appl.* **39**, 655–669.

1972 [a] T. J. Osler, "A Further Extension of the Leibniz Rule to Fractional Derivatives and Its Relation to Parseval's Formula." *SIAM J. Math. Anal.* **3**, 1–15.

1972 [b] T. J. Osler, "An Integral Analogue of Taylor's Series and Its Use in Computing Fourier Transforms." *Math. Comp.* **26**, 449–460.

1972 [c] T. J. Osler, "Integral Analog of the Leibniz Rule," *Math. Comp.* **26**, 903–915.

1972 R. K. Juberg, "Finite Hilbert Transforms in L_p." *Bull. Amer. Math. Soc.* **78**, 3, 435–438.

1972 T. R. Prabhakar, "Hypergeometric Integral Equations of a General Kind and Fractional Integration." *SIAM J. Math. Anal.* **3**, 422–425.

Some integral equations containing hypergeometric functions in two variables are studied with the use of fractional integration.

1974 To appear in late 1974 or early 1975 B. Ross, "*A Profile of Fractional Calculus*," Chapter V. Ph.D. Thesis, New York Univ., New York.

1975 The Proceedings of the University of New Haven Colloquium on Fractional Calculus and Its Applications to the Mathematical Sciences to be held June 1974 is expected to be published late in 1975.

1.2 NOTATION

Wherever possible we have followed the nomenclature and symbolism of Abramowitz and Stegun's excellent reference text (1964). To avoid unusual fonts, however, we adopt the following symbols:

$S_n^{[m]}$ for Stirling numbers of the second kind (Abramowitz and Stegun, 1964, p. 824).

$H_\nu(\ )$ and $L_\nu(\ )$ for Struve functions and modified Struve functions of order ν (Abramowitz and Stegun, Chapter 12).

We trust that our use of $H_\nu(\ )$ and $L_\nu(\ )$ will not cause confusion with Abramowitz and Stegun's choice of these same symbols for Hermite and Laguerre polynomials (for which we find no use in the present work). The symbolism

$$\left[x \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]$$

and the terminology "generalized hypergeometric function of complexity $\frac{K}{L}$" are introduced and explained in Section 2.10. Its relationship to the usual symbol for the generalized hypergeometric function is also discussed there.

Equation (x.y.z) denotes the zth numbered equation of section y of chapter x. Figures and tables are treated similarly, with the numbering system Figure x.y.z, Table x.y.z used consecutively in each section. Authors' names with publication dates are used to make reference to our bibliography (to be found at the end of the book).

The term "differintegral," by which we mean derivative or integral to arbitrary order, was mentioned in our Preface. We freely make use of the blood relatives of this term: "differintegration," "differintegrand," "semi-integral," etc. Our special symbolism for such operators is introduced and fully explained in the preamble to Chapter 3. Symbols such as f, g denote general differintegrable functions while the symbols ϕ, ψ are reserved for analytic functions.

Few other nonstandard notations are employed. Those that are will be fully defined at their introduction.

1.3 PROPERTIES OF THE GAMMA FUNCTION

The complete gamma function $\Gamma(x)$ plays an important role in the theory of differintegration. Accordingly, it is convenient to collect here certain formulas relating to this function. A comprehensive definition of $\Gamma(x)$ is that provided by the Euler limit

$$\Gamma(x) \equiv \lim_{N\to\infty}\left[\frac{N!\,N^x}{x[x+1][x+2]\cdots[x+N]}\right],$$

but the integral transform definition

$$\Gamma(x) \equiv \int_0^\infty y^{x-1}\exp(-y)\,dy, \qquad x>0, \tag{1.3.1}$$

is often more useful, although it is restricted to positive x values.

An integration by parts applied to the definition (1.3.1) leads to the recurrence relationship

$$\Gamma(x+1) = x\Gamma(x), \tag{1.3.2}$$

which is the most important property of the gamma function. The same result is a simple consequence of the Euler limit definition. Since

$$\Gamma(1) = 1,$$

this recurrence shows that for a positive integer n

(1.3.3)

$$\Gamma(n+1) = n\Gamma(n) = n[n-1]\Gamma(n-1) = \cdots = n[n-1]\cdots 2\cdot 1\cdot \Gamma(1) = n!.$$

Rewritten as

$$\Gamma(x-1) = \Gamma(x)/[x-1],$$

the recurrence formula also serves as an analytic continuation, extending the definition of the gamma function to the negative arguments to which definition (1.3.1) is inapplicable. This extension shows $\Gamma(0)$ to be infinite, as is $\Gamma(-1)$ and the value of the gamma function at all negative integers. Ratios of gamma functions of negative integers are, however, finite; thus if N and n are positive integers,

$$(1.3.4)\quad \frac{\Gamma(-n)}{\Gamma(-N)} = [-N][-N+1]\cdots[-n-2][-n-1] = [-]^{N-n}\frac{N!}{n!}.$$

The reciprocal $1/\Gamma(x)$ of the gamma function is single-valued and finite for all x. Figure 1.3.1 shows a graph of this function. Note the continuous alternation of sign for negative argument and the asymptotic approach to zero for large positive x, an approach described by

$$(1.3.5)\qquad \frac{1}{\Gamma(x)} \sim \frac{x^{\frac{1}{2}-x}}{\sqrt{2\pi}}\exp(x), \qquad x \to \infty.$$

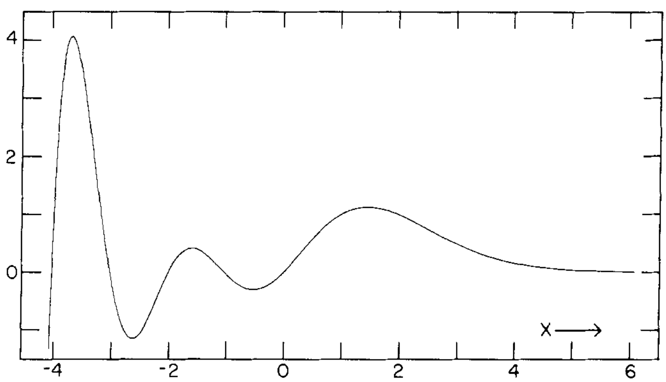

FIG. 1.3.1 The reciprocal, $1/\Gamma(x)$, of the gamma function for $-4 \leqq x \leqq 6$.

As we have seen, the gamma function of a positive integer n is itself a positive integer, while the gamma function $\Gamma(-n)$ of a negative integer is invariably infinite. The gamma functions $\Gamma(\frac{1}{2}+n)$ and $\Gamma(\frac{1}{2}-n)$ turn out to be multiples of $\sqrt{\pi}$; thus,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

$$(1.3.6)\qquad \Gamma(\tfrac{1}{2}+n) = \frac{(2n)!\sqrt{\pi}}{4^n n!},$$

and

$$\Gamma(\tfrac{1}{2} - n) = \frac{[-4]^n n! \sqrt{\pi}}{(2n)!}. \tag{1.3.7}$$

Some frequently encountered examples are included in Table 1.3.1.

Table 1.3.1 *Some values of the gamma function $\Gamma(x)$ for integer and half-integer x*

$\Gamma(-\tfrac{3}{2}) = \tfrac{4}{3}\sqrt{\pi}$	$\Gamma(1) = 1$
$\Gamma(-1) = \pm\infty$	$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}$
$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$	$\Gamma(2) = 1$
$\Gamma(0) = \pm\infty$	$\Gamma(\tfrac{5}{2}) = \tfrac{3}{4}\sqrt{\pi}$
$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$	$\Gamma(3) = 2$

Two useful properties of the gamma function are its reflection

$$\Gamma(-x) = \frac{-\pi \csc(\pi x)}{\Gamma(x+1)} \tag{1.3.8}$$

and its duplication

$$\Gamma(2x) = \frac{4^x \Gamma(x)\Gamma(x+\tfrac{1}{2})}{2\sqrt{\pi}}, \tag{1.3.9}$$

the latter being an instance of the Gauss multiplication formula

$$\Gamma(nx) = \sqrt{\frac{2\pi}{n}} \left[\frac{n^x}{\sqrt{2\pi}}\right]^n \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right). \tag{1.3.10}$$

Table 1.3.2 *Some examples of Stirling numbers of the first kind, $S_j^{(m)}$*

j	m = 0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	−1	1	0	0	0
3	0	2	−3	1	0	0
4	0	−6	11	−6	1	0
5	0	24	−50	35	−10	1

In Chapter 3 the gamma function expression

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \tag{1.3.11}$$

will be encountered, where j is a nonnegative integer and q may take any value. For small numerical values of j, expression (1.3.11) is readily simplified to a polynomial in q by application of rules (1.3.2) and (1.3.3). This procedure generalizes to give

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{[-]^j}{j!}\sum_{m=0}^{j} S_j^{(m)} q^m, \tag{1.3.12}$$

where $S_j^{(m)}$ is a Stirling number of the first kind, examples of which are to be found in Table 1.3.2. These numbers are defined by the recurrence

$$S_{j+1}^{(m)} \equiv S_j^{(m-1)} - jS_j^{(m)}; \qquad S_0^{(m)} = S_j^{(0)} = 0, \qquad \text{except} \quad S_0^{(0)} = 1.$$

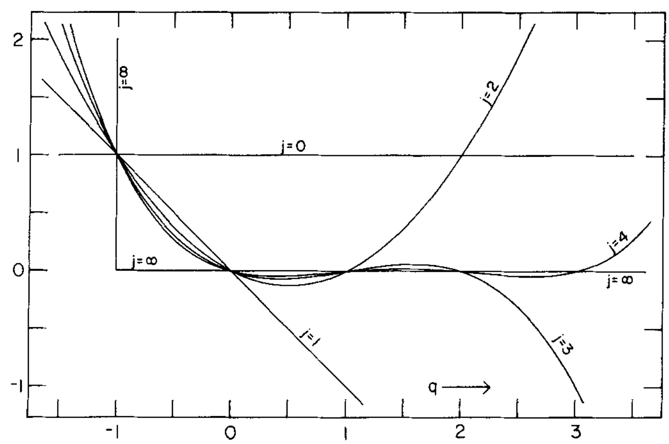

FIG. 1.3.2 The polynomial $\Gamma(j-q)/[\Gamma(-q)\Gamma(j+1)]$ for $j = 0, 1, 2, 3, 4$, and ∞.

Its expressibility in (1.3.12) as a polynomial in q establishes that expression (1.3.11) is finite and single-valued for all finite values of q and j. Figure 1.3.2 displays the numerical value of the polynomial for real values of q in the range $-2 \leqq q \leqq 5$ and for $j = 0, 1, 2, 3, 4$, and ∞.

Equation (1.3.12) provides an expression for the gamma function quotient $\Gamma(j-q)/\Gamma(-q)$. The other quotient $\Gamma(j-q)/\Gamma(j+1)$ appearing in expression (1.3.11) is also of interest. This quotient has the asymptotic expansion

$$\frac{\Gamma(j-q)}{\Gamma(j+1)} \sim j^{-1-q}\left[1 + \frac{q[q+1]}{2j} + O(j^{-2})\right], \qquad j \to \infty, \tag{1.3.13}$$

a representation which establishes that the $j \to \infty$ limit of $j^{q+1}\Gamma(j-q)/\Gamma(j+1)$ is unity. Thence one may show that

$$\lim_{j\to\infty}\left[j^{c+q+1}\frac{\Gamma(j-q)}{\Gamma(j+1)}\right] = \lim_{j\to\infty}\left[j^{c+q}\frac{\Gamma(j-q)}{\Gamma(j)}\right] = \begin{cases} +\infty, & c>0,\\ 1, & c=0,\\ 0, & c<0,\end{cases} \tag{1.3.14}$$

a result that may be generalized to

$$\lim_{j\to\infty}\left[j^{c+q+1}\frac{\Gamma(j+k-q)}{\Gamma(j+k+1)}\right] = \begin{cases} +\infty, & c>0,\\ 1, & c=0,\\ 0, & c<0,\end{cases} \tag{1.3.15}$$

for any finite integer k.

Expression (1.3.11) may be regarded as a binomial coefficient,

$$\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \binom{j-q-1}{j} = [-]^j\binom{q}{j}, \tag{1.3.16}$$

the above equalities being readily established from the definition of a binomial coefficient and the reflection formula (1.3.8). Many relationships for gamma functions are derivable from the corresponding binomial coefficient identities [see Abramowitz and Stegun (1964, pp. 10, 822), as well as Gradshteyn and Ryzhik (1965, pp. 3, 4) for many of these]. Thus, since

$$\sum_{j=0}^{n}\binom{j-q-1}{j} = \binom{n-q}{n}, \tag{1.3.17}$$

the relationship

$$\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} = \frac{\Gamma(N-q)}{\Gamma(1-q)\Gamma(N)} \tag{1.3.18}$$

follows on setting $N \equiv n+1$ and expressing the binomial coefficients as their equivalent gamma function combination. Similarly, after multiplication by $-q$, it is readily shown that

$$\sum_{j=1}^{n}\binom{j-q-1}{j-1} = \binom{n-q}{n-1}$$

[which follows straightforwardly from (1.3.17) on redefinition of q, j, and n] leads to the identity

$$\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j)} = \frac{-q\Gamma(N-q)}{\Gamma(2-q)\Gamma(N-1)}. \tag{1.3.19}$$

Likewise, the well-known summation formula

$$\sum_{k=0}^{j}\binom{q}{k}\binom{Q}{j-k} = \binom{q+Q}{j} \tag{1.3.20}$$

becomes the identity

(1.3.21)

$$\sum_{k=0}^{j} \frac{\Gamma(q+1)\Gamma(p+1)\Gamma(j+1)}{\Gamma(q-k+1)\Gamma(k+1)\Gamma(p-q+k+1)\Gamma(j-k+1)} = \frac{\Gamma(p+j+1)}{\Gamma(p-q+j+1)}$$

on setting Q equal to $[p-q+j]$ and multiplication by $\Gamma(j+1)$. Yet again, the binomial relationship

$$\sum_{j=0}^{n} \binom{j-q-1}{j-m} = \binom{n-q}{n-m},$$

which is obtainable from (1.3.17), leads easily to the gamma function equivalent of

(1.3.22) $$\sum_{j=0}^{n} \binom{j-q-1}{j}\binom{j}{m} = \binom{m-q-1}{m}\binom{n-q}{n-m},$$

a relationship which we shall need in Section 3.5.

A function that is closely related to the gamma function is the complete beta function $\mathrm{B}(p, q)$. For positive values of the two parameters, p and q, the function is defined by the beta integral,

(1.3.23) $$\mathrm{B}(p, q) = \int_0^1 y^{p-1}[1-y]^{q-1}\,dy, \qquad p > 0 < q,$$

also known as Euler's integral of the second kind. If either p or q is nonpositive, the integral diverges and the beta function is then defined by the relationship

(1.3.24) $$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

valid for all p and q.

Both the beta function and the gamma function have "incomplete" analogs. The incomplete beta function of argument x is defined by the integral

(1.3.25) $$\mathrm{B}_x(p, q) = \int_0^x y^{p-1}[1-y]^{q-1}\,dy.$$

There are many alternative formulations of the incomplete gamma function. The one which we shall adopt exclusively is that defined by

(1.3.26) $$\gamma^*(c, x) = \frac{c^{-x}}{\Gamma(x)}\int_0^c y^{x-1}\exp(-y)\,dy = \exp(-x)\sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j+c+1)};$$

$\gamma^*(c, x)$ is a finite single-valued analytic function of x and c. Properties of the incomplete gamma function of which we shall have need in Chapter 8 are its recursion

(1.3.27) $$\gamma^*(c-1, x) = x\gamma^*(c, x) + \frac{\exp(-x)}{\Gamma(c)},$$

and its value

(1.3.28) $$\gamma^*(\tfrac{1}{2}, x) = \frac{\operatorname{erf}(\sqrt{x})}{\sqrt{x}}$$

for a parameter of moiety.

Powers of numbers may be expressed in terms of complete gamma functions, thus

$$q^j = \sum_{m=0}^{j} [-]^m S_j^{[m]} \frac{\Gamma(m-q)}{\Gamma(-q)}.$$

Table 1.3.3 *Some examples of Stirling numbers of the second kind,* $S_j^{[m]}$

j	m					
	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

The coefficients $S_j^{[m]}$ appearing in this expansion are Stirling numbers of the second kind. Examples of these numbers, which obey the recurrence

$$S_j^{[m]} = S_j^{[m-1]} + mS_j^{[m]}; \qquad S_0^{[m]} = S_j^{[0]} = 0, \qquad \text{except} \quad S_0^{[0]} = 1,$$

are listed in Table 1.3.3. Notice that

(1.3.29) $$S_j^{[j]} = 1$$

for all j, and that

(1.3.30) $$S_j^{[m]} = 0, \qquad j = 0, 1, \ldots, m-1.$$

The formula

(1.3.31) $$\sum_{l=0}^{m} [-]^{l+m} \binom{m}{l} l^k = m!\, S_k^{[m]}$$

may be regarded as defining Stirling numbers of the second kind.

In Section 3.5 we shall need to expand the analytic function ϕ of argument $(x + jy)$ in a rather special way. We proceed to sketch the proof of this expansion, which involves Stirling numbers of the second kind. First, we relate $\phi(x + jy)$ to the values $\phi(x)$, $\phi(x + y)$, $\phi(x + 2y), \ldots, \phi(x + jy)$ by the formula

$$\phi(x + jy) = \sum_{m=0}^{j} \binom{j}{m} \sum_{l=0}^{m} [-]^{l+m} \binom{m}{l} \phi(x + ly).$$

The inner summation is now symbolized $G_m(\phi, x, y)$ and Taylor expanded to give

$$\begin{aligned} G_m(\phi, x, y) &= \sum_{l=0}^{m} [-]^{l+m} \binom{m}{l} \sum_{k=0}^{\infty} \frac{[ly]^k}{k!} \phi^{(k)}(x) \\ &= \sum_{k=0}^{\infty} \frac{y^k}{k!} \phi^{(k)}(x) \sum_{l=0}^{m} [-]^{l+m} \binom{m}{l} l^k. \end{aligned}$$

It now only requires the application of summation (1.3.31) to yield the final result:

(1.3.32) $$\phi(x + jy) = \sum_{m=0}^{j} G_m(\phi, x, y) \binom{j}{m},$$

where

(1.3.33) $$G_m(\phi, x, y) = \sum_{k=0}^{\infty} \frac{m!}{k!} S_k^{[m]} y^k \phi^{(k)}(x).$$

Because $\binom{j}{m}$ vanishes when m exceeds the integer j, the upper summation limit in (1.3.32) may be replaced by ∞. Similarly, but this time as a consequence of equation (1.3.30), the $k = 0$ lower summation limit in expansion (1.3.33) may be replaced by $k = m$.

Yet another relative of the gamma function is the psi function that we shall come across in Section 6.7. Defined by

$$\psi(x) \equiv \frac{1}{\Gamma(x)} \frac{d\Gamma(x)}{dx},$$

the psi function obeys the recursion

$$\psi(x + 1) = \psi(x) + x^{-1},$$

whence it follows that

(1.3.34) $$\psi(n + 1) = \psi(1) + \sum_{j=1}^{n} \frac{1}{j},$$

where

(1.3.35) $$-\psi(1) = \gamma = 0.5772157 \cdots$$

is Euler's constant. Our encounter with $\psi(x)$ in Section 6.7 is via the definite integral

(1.3.36) $$\int_0^1 \frac{[v^x - v^y]\,dv}{1 - v} = \psi(y + 1) - \psi(x + 1).$$