CHAPTER 4

DIFFERINTEGRATION OF SIMPLE FUNCTIONS

It is the purpose of this chapter to calculate the qth-order differintegral of certain simple functions. The formulas we develop will play a major role in all our later work. Because of the identities established in Chapter 3, we are at liberty to use whichever of the definitions summarized in Section 3.6 we find most convenient for a given function. We shall even find it convenient to use, for a given f, different formulas for different q ranges.

The simple functions that are considered in three of the sections of this chapter are examples of the power functions $[x-a]^p$. Thus in Sections 4.1 and 4.3 we treat the $p = 0$ and $p = 1$ instances of this function, while in Section 4.4 the general case is examined. We refer the reader back to Section 2.9 for a review of the information available from the classical calculus.

4.1 THE UNIT FUNCTION

We consider first the differintegral to order q of the function $f \equiv 1$, for which we find it convenient to reserve the special notation $[1]$. We shall refer to this function as the unit function and trust that no confusion will arise between $[1]$ and the unit *step* function: $f = 0$, $-\infty \leqq x < a$; $f = 1$, $x \geqq a$ [which we shall term the Heaviside function $H(x-a)$ and shall encounter in Section 6.9].

A straightforward application of (3.2.1) to the function $[1]$ gives

$$\frac{d^q[1]}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}.$$

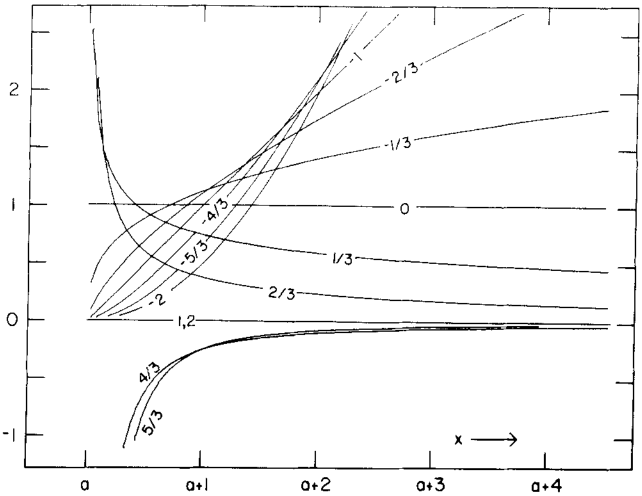

FIG. 4.1.1. Differintegrals of unity for q values in the range -2 to $+2$. Notice that $d^q[1]/[d(x-a)]^q$ is zero for all positive integer q; negative for $1 < q < 2$, $3 < q < 4$, etc.; and otherwise positive.

Application first of equation (1.3.18) and then of (1.3.14) yields

$$\text{(4.1.1)} \qquad \frac{d^q[1]}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \frac{\Gamma(N-q)}{\Gamma(1-q)\Gamma(N)}\right\} = \frac{[x-a]^{-q}}{\Gamma(1-q)}$$

as our result. Figure 4.1.1 shows some examples of these differintegrals. We note the reduction to an instance of (2.9.3) when q is an integer. Formula (4.1.1) was derived (in a different symbolism and context) in the 1890's by Heaviside (1920).

As an example of the application of the unit function and its differintegrals, consider the combination of formulas (3.6.5) and (4.1.1) into

$$\frac{d^q\phi}{[d(x-a)]^q} = \sum_{k=0}^{\infty} [-]^k \frac{\Gamma(1+k-q)}{\Gamma(-q)[k-q]k!} \frac{d^{q-k}[1]}{[d(x-a)]^{q-k}} \phi^{(k)}$$

valid for any analytic function ϕ. Application of a number of the properties of the gamma function [equations (1.3.2), (1.3.3), and (1.3.16)] then leads to the concise representation

$$\text{(4.1.2)} \qquad \frac{d^q\phi}{[d(x-a)]^q} = \sum_{k=0}^{\infty} \binom{q}{k} \frac{d^{q-k}[1]}{[d(x-a)]^{q-k}} \frac{d^k\phi}{[d(x-a)]^k},$$

reminiscent of Leibniz's theorem

$$\frac{d^n}{dx^n}[fg] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}f}{dx^{n-k}} \frac{d^k g}{dx^k}$$

for differentiation of a product, except that in (4.1.2) we have chosen $f \equiv 1$ but have allowed the order to be unrestricted. The result, as we might expect, is an infinite rather than a finite sum of terms. We have more to say on the subject of generalizations of Leibniz's theorem in Section 5.5.

4.2 THE ZERO FUNCTION

When the definition (3.6.1) is applied to the function defined by $f \equiv C$, C any constant including zero, we see that

$$\frac{d^q[C]}{[d(x-a)]^q} = C\frac{d^q[1]}{[d(x-a)]^q} = C\frac{[x-a]^{-q}}{\Gamma(1-q)}. \tag{4.2.1}$$

Since $d^q[1]/[d(x-a)]^q$ is never infinite for $x > a$, we conclude by setting $C = 0$ that

$$\frac{d^q[0]}{[d(x-a)]^q} = 0 \qquad \text{for all} \quad q. \tag{4.2.2}$$

Result (4.2.2) may appear trivial or obvious. As an example of its importance, however, observe that it provides a powerful counterexample to the thesis that if

$$\frac{d^q f}{[d(x-a)]^q} = g, \qquad \text{then} \qquad \frac{d^{-q}g}{[d(x-a)]^{-q}} = f,$$

for, if f gives zero on differentiation to order q, f cannot be restored by q-order integration. Here again, as in Section 2.3, we encounter the so-called composition rule, this time for noninteger orders. This subject will be more fully explored in Section 5.7.

4.3 THE FUNCTION $x-a$

For the function $f(x) = x - a$, definition (3.6.1) gives

$$\frac{d^q[x-a]}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\left[\frac{Nx-jx+ja}{N} - a\right]\right\}$$

$$= [x-a]^{1-q}\left[\lim_{N\to\infty}\left\{N^q\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}\right.$$

$$\left. - \lim_{N\to\infty}\left\{N^{q-1}\sum_{j=0}^{N-1} j\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\right\}\right].$$

If the summation formulas (1.3.18) and (1.3.19) are now employed, followed by (1.3.14), the result

$$\frac{d^q[x-a]}{[d(x-a)]^q} = [x-a]^{1-q}\left[\frac{1}{\Gamma(1-q)} + \frac{q}{\Gamma(2-q)}\right]$$

is obtained which, on application of the recurrence formula (1.3.2), becomes

$$\frac{d^q[x-a]}{[d(x-a)]^q} = \frac{[x-a]^{1-q}}{\Gamma(2-q)}. \tag{4.3.1}$$

Alternatively, we argue from the Riemann–Liouville formula (3.6.2) that, on substituting $w \equiv x - y$,

$$\begin{aligned}\frac{d^q[x-a]}{[d(x-a)]^q} &= \frac{1}{\Gamma(-q)}\int_a^x \frac{[y-a]\,dy}{[x-y]^{q+1}} \\ &= \frac{1}{\Gamma(-q)}\int_0^{x-a} \frac{[x-a-w]\,dw}{w^{q+1}} \\ &= \frac{1}{\Gamma(-q)}\left[\int_0^{x-a} \frac{[x-a]\,dw}{w^{q+1}} - \int_0^{x-a}\frac{dw}{w^q}\right] \\ &= \frac{1}{\Gamma(-q)}\left[\frac{[x-a]^{1-q}}{-q} - \frac{[x-a]^{1-q}}{1-q}\right] \\ &= \frac{[x-a]^{1-q}}{[-q][1-q]\Gamma(-q)}, \qquad q<0,\end{aligned} \tag{4.3.2}$$

the denominator of which equals $\Gamma(2-q)$ by the recurrence formula (1.3.2). Use of equation (3.2.5),

$$\frac{d^q[x-a]}{[d(x-a)]^q} = \frac{d^n}{dx^n}\left\{\frac{d^{q-n}[x-a]}{[d(x-a)]^{q-n}}\right\},$$

is now all that is required to remove the restrictive q condition. Thus for arbitrary q one may select an integer n so large that $q-n<0$. Result (4.3.2) gives

$$\frac{d^{q-n}[x-a]}{[d(x-a)]^{q-n}} = \frac{[x-a]^{1-q+n}}{\Gamma(2-q+n)},$$

whence

$$\begin{aligned}\frac{d^q[x-a]}{[d(x-a)]^q} &= \frac{d^n}{[d(x-a)]^n}\left\{\frac{d^{q-n}[x-a]}{[d(x-a)]^{q-n}}\right\} \\ &= \frac{\Gamma(2-q+n)}{\Gamma(2-q)}\,\frac{[x-a]^{1-q}}{\Gamma(2-q+n)} = \frac{[x-a]^{1-q}}{\Gamma(2-q)}\end{aligned}$$

follows by equation (2.9.1).

We note, as expected, that formula (4.3.1) reduces to zero when $q = 2, 3, 4, \ldots$; to unity when $q = 1$; to $x - a$ when $q = 0$; and to $[x-a]^{n+1}/(n+1)!$ when $q = -n = -1, -2, -3, \ldots$. Notice also, on comparison of formulas (4.3.1) and (4.1.1), that the qth differintegral of $x - a$ equals the $(q-1)$th differintegral of unity. For this reason, Fig. 4.1.1 can be readily adapted to illustrate the present section.

4.4 THE FUNCTION $[x - a]^p$

The final function we consider in this chapter is the important function $f = [x-a]^p$, where p is initially arbitrary. We shall see, however, that p must exceed -1 for differintegration to have the properties we demand of the operator. (Recall our discussion of differintegrable functions in Section 3.1.)

For integer q of either sign, we have the formula (2.9.1) from classical calculus. Our first encounter with noninteger q will be restricted to negative q so that we may exploit the Riemann–Liouville definition; thus

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x \frac{[y-a]^p\,dy}{[x-y]^{q+1}} = \frac{1}{\Gamma(-q)}\int_0^{x-a} \frac{v^p\,dv}{[x-a-v]^{q+1}}, \qquad q < 0,$$

where v has replaced $y - a$. By further replacement of v by $[x-a]u$, the integral may be cast into the standard beta function form,

$$\text{(4.4.1)} \qquad \frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{[x-a]^{p-q}}{\Gamma(-q)}\int_0^1 u^p[1-u]^{-q-1}\,du, \qquad q < 0.$$

The definite integral in (4.4.1) will be recognized as the beta function (see Section 1.3), $\mathrm{B}(p+1, -q)$, provided both arguments are positive. Therefore

$$\text{(4.4.2)} \qquad \frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{[x-a]^{p-q}}{\Gamma(-q)}\mathrm{B}(p+1, -q) = \frac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)},$$

$$q < 1, \quad p > -1,$$

where the beta function has been replaced by its gamma function equivalent [see equation (1.3.24)].

For comparison of technique we digress at this point to include a verification of formula (4.4.2) starting with the definition (3.4.3). We replace $x - a$ by z so that

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} \equiv \frac{d^q z^p}{dz^q} = \frac{\Gamma(q+1)}{2\pi i}\oint_C \frac{\zeta^p\,d\zeta}{[\zeta - z]^{q+1}},$$

where the contour C in the complex ζ-plane begins and ends at $\zeta = 0$ enclosing z once in the positive sense. If one sets $\zeta \equiv zs$, then

$$\frac{d^q z^p}{dz^q} = \frac{\Gamma(q+1)z^{p-q}}{2\pi i} \oint s^p[s-1]^{-q-1}\,ds,$$

where the integral is over a contour encircling the point $s = 1$ once in the positive sense and beginning and ending at $s = 0$. When such a contour is deformed into the one shown in Fig. 4.4.1, then

$$\begin{aligned}\frac{d^q z^p}{dz^q} &= \frac{\Gamma(q+1)z^{p-q}}{2\pi i}\left[1 - \exp(-2\pi i[q+1])\right]\int_0^1 s^p[s-1]^{-q-1}\,ds \\ &= \frac{\Gamma(q+1)z^{p-q}}{2\pi i}\left[1 - \exp(-2\pi i[q+1])\right][-]^{-q-1}\int_0^1 s^p[1-s]^{-q-1}\,ds \\ &= \frac{\Gamma(q+1)z^{p-q}}{2\pi i}\left[\exp(i\pi[q+1]) - \exp(-i\pi[q+1])\right]\int_0^1 s^p[1-s]^{-q-1}\,ds \\ &= \frac{\Gamma(q+1)z^{p-q}}{2\pi i}\, 2i\sin(\pi[q+1])\int_0^1 s^p[1-s]^{-q-1}\,ds \\ &= \frac{\Gamma(p+1)z^{p-q}}{\Gamma(p-q+1)}, \qquad p > -1, \quad q < 0,\end{aligned}$$

where use has been made of the reflection formula (1.3.8) and of the properties of the beta integral.

As in the previous section we may again use equation (3.2.5), together with the classical formula (2.9.1), to extend our treatment to positive q.

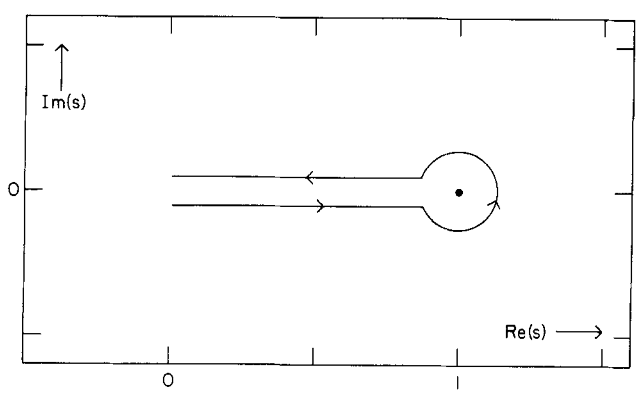

FIG. 4.4.1. Path of the contour for integration around the point $s = 1$.

Following this technique

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{d^n}{dx^n}\left[\frac{d^{q-n}[x-a]^p}{[d(x-a)]^{q-n}}\right] \tag{4.4.3}$$

$$= \frac{d^n}{dx^n}\left[\frac{[x-a]^{p-q+n}}{\Gamma(n-q)}\int_0^1 u^p[1-u]^{n-q-1}\,du\right]$$

$$= \frac{d^n}{dx^n}\left[\frac{\Gamma(p+1)[x-a]^{p-q+n}}{\Gamma(p-q+n+1)}\right], \qquad p > -1,$$

where, since we chose $n > q \geqq 0$, we were able to use (4.4.1) to evaluate the $[q-n]$th differintegral of $[x-a]^p$. The classical formula (2.9.1) then leads to

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)}, \qquad q \geqq 0, \quad p > -1,$$

straightforwardly. Unification of this result with (4.4.2) yields the formula

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \frac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)}, \qquad p > -1, \tag{4.4.4}$$

valid for all q. As required for an acceptable formula in our generalized calculus, equation (4.4.4) incorporates the classical formula (2.9.3).

Historically, the formula

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)x^{p-q}}{\Gamma(p-q+1)}$$

was important in being the basis of the concept of fractional differentiation as developed by Gemant (1936). This formulation was used by him, and later more extensively by Scott Blair *et al.* (1947) in rheology.

Thus far this section has been concerned only with the $p > -1$ instances of $[x-a]^p$. We now briefly deal with $p \leqq -1$. The generalized derivatives (4.4.1) and (4.4.3) break down for $p \leqq -1$ because the beta integrals then diverge. An infinite result

$$\frac{d^q[x-a]^p}{[d(x-a)]^q} = \infty, \qquad p \leqq -1, \quad \text{all } q,$$

would, however, be unacceptable because it would fail to incorporate the classical result (2.9.1) for positive integer q. Likewise, the formula (4.4.4) cannot be extended to $p \leqq -1$ because, though this does incorporate (2.9.1) it does not reproduce (2.9.2) for negative integer q. Moreover, we know of no generalization of formula (4.4.4) that incorporates both of the requirements (2.9.1) and (2.9.2) for $p \leqq -1$. The breakdown of (4.4.4) for $p \leqq -1$

is associated with the pole of order unity or greater which occurs at $x = a$ for the functions $[x - a]^p$, $p \leqq -1$. Functions for which such a pole occurs anywhere on the open interval from a to x lead to similar difficulties and for reasons such as this we have purposely excluded these functions from the class of differintegrable series, as explained in Section 3.1.

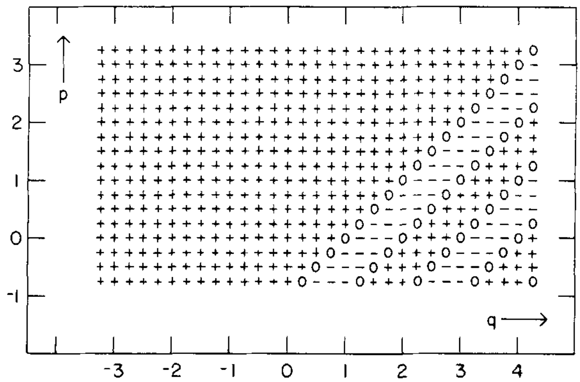

FIG. 4.4.2. The sign of the coefficient $\Gamma(p+1)/\Gamma(p-q+1)$ in the differintegrals $d^q[x-a]^p/[d(x-a)]^q$ for ranges of p and q values. Notice that entries stop short of the $p = -1$ line because functions $[x-a]^p$ are not differintegrable if $p \leqq -1$.

Figure 4.4.2 shows the sign of the gamma function ratio that is the coefficient in the differintegral $d^q[x-a]^p/[d(x-a)]^q$ for ranges of values of p and q. Note that entries stop short of the $p = -1$ line and that there exist no conflicts between this diagram and Fig. 2.9.1.