CHAPTER 8

TECHNIQUES IN THE FRACTIONAL CALCULUS

In this chapter we discuss some of the techniques that have been developed for handling differintegration to noninteger order. Because of the special importance of the $q = \pm\frac{1}{2}$ cases, emphasis is placed on semidifferentiation and semiintegration.

Though most of the chapter deals purely with mathematics, Section 8.3 shows how the operations of the fractional calculus may be carried out using electrical circuitry. The ability to perform these operations by such analog techniques permits a hardware implementation of the fractional calculus (Meyer, 1960; Holub and Nemeč, 1966; Allegre *et al.*, 1970; Ichise *et al.*, 1971; Oldham, 1973a).

8.1 LAPLACE TRANSFORMATION

In this section we seek to Laplace transform d^qf/dx^q for all q and differintegrable f, i.e., we wish to relate

$$\mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} \equiv \int_0^\infty \exp(-sx)\,\frac{d^qf}{dx^q}\,dx$$

to the Laplace transform $\mathscr{L}\{f\}$ of the differintegrable function. Let us first recall the well-known transforms of integer-order derivatives

$$\mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = s^q\mathscr{L}\{f\} - \sum_{k=0}^{q-1} s^{q-1-k}\,\frac{d^kf}{dx^k}(0), \qquad q = 1, 2, 3, \ldots,$$

and multiple integrals

$$\mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = s^q\mathscr{L}\{f\}, \qquad q = 0, -1, -2, \ldots \tag{8.1.1}$$

and note that both formulas are embraced by

$$\text{(8.1.2)} \qquad \mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = s^q\mathscr{L}\{f\} - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k}f}{dx^{q-1-k}}(0), \qquad q = 0, \pm 1, \pm 2, \ldots .$$

Notice that the upper summation limit, written as $q - 1$ in (8.1.2), may be replaced by any integer larger than $q - 1$ and even by ∞. The only effect of the replacement is to add terms whose coefficients are $d^{-1}f(0)/dx^{-1}$, $d^{-2}f(0)/dx^{-2}$, etc. Such coefficients are necessarily zero for any function f whose Laplace transform exists. We next show that formula (8.1.2) generalizes to include noninteger q by the simple extension

$$\text{(8.1.3)} \qquad \mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = s^q\mathscr{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k}f}{dx^{q-1-k}}(0), \qquad \text{all } q,$$

where n is the integer such that $n - 1 < q \leqq n$. The sum is empty and vanishes when $q \leqq 0$.

In proving (8.1.3), we first consider $q < 0$, so that the Riemann–Liouville definition

$$\frac{d^qf}{dx^q} = \frac{1}{\Gamma(-q)}\int_0^x \frac{f(y)\,dy}{[x-y]^{q+1}}, \qquad q < 0,$$

may be adopted. Direct application of the convolution theorem (Churchill, 1948)

$$\mathscr{L}\left\{\int_0^x f_1(x-y)f_2(y)\,dy\right\} = \mathscr{L}\{f_1\}\mathscr{L}\{f_2\}$$

then gives

$$\text{(8.1.4)} \qquad \mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = \frac{1}{\Gamma(-q)}\,\mathscr{L}\{x^{-1-q}\}\mathscr{L}\{f\} = s^q\mathscr{L}\{f\}, \qquad q < 0,$$

so that equation (8.1.1) generalizes unchanged for negative q.

For noninteger positive q, we use the result (3.2.5),

$$\frac{d^qf}{dx^q} = \frac{d^n}{dx^n}\frac{d^{q-n}f}{dx^{q-n}},$$

where n is the integer such that $n - 1 < q < n$. Now, on application of formula (8.1.2), we find

$$\mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = \mathscr{L}\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n}f}{dx^{q-n}}\right]\right\} = s^n\mathscr{L}\left\{\frac{d^{q-n}f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left[\frac{d^{q-n}f}{dx^{q-n}}\right](0).$$

The difference $q - n$ being negative, the first right-hand term may be evaluated by use of (8.1.4). Since $q - n < 0$, the composition rule may be applied to the terms within the summation (see Sections 5.7 and 5.1). The result

$$\mathscr{L}\left\{\frac{d^qf}{dx^q}\right\} = s^q\mathscr{L}\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k}f}{dx^{q-1-k}}(0), \qquad 0 < q \neq 1, 2, 3, \ldots,$$

follows from these two operations and is seen to be incorporated in (8.1.3).

The transformation (8.1.3) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of f. No similar generalization exists, however, for the classical formulas

$$\mathscr{L}\left\{\frac{-f}{x}\right\} = \frac{d^{-1}\mathscr{L}\{f\}}{ds^{-1}}(s) - \frac{d^{-1}\mathscr{L}\{f\}}{ds^{-1}}(\infty),$$

$$\mathscr{L}\{-xf\} = \frac{d\mathscr{L}\{f\}}{ds},$$

$$\mathscr{L}\{[-x]^nf\} = \frac{d^n\mathscr{L}\{f\}}{ds^n}, \qquad n = 1, 2, 3, \ldots, \tag{8.1.5}$$

for the integration or differentiation of the transform. That this is so may be established by considering the example $\mathscr{L}\{f\} = s^p$, where $-1 < p < 0$. Then

$$\frac{d^q\mathscr{L}\{f\}}{[ds]^q} = \frac{\Gamma(p+1)s^{p-q}}{\Gamma(p-q+1)} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)}\mathscr{L}\left\{\frac{x^{q-p-1}}{\Gamma(q-p)}\right\}$$

$$= \frac{\Gamma(p+1)\Gamma(-p)}{\Gamma(p-q+1)\Gamma(q-p)}\mathscr{L}\{x^qf\} = \frac{\csc(\pi[-p])\mathscr{L}\{x^qf\}}{\csc(\pi[q-p])}$$

$$= [\cos(\pi q) - \cot(\pi p)\sin(\pi q)]\mathscr{L}\{x^qf\}.$$

Notice that only for integer values of q is the coefficient of $\mathscr{L}\{x^qf\}$ independent of the function f, even for such a simple example as $f = x^p$. It is futile, therefore, to seek a simple generalization of equation (8.1.5) to noninteger order.

As a final result of this section we shall establish the useful formula

$$\mathscr{L}\left\{\exp(-kx)\frac{d^q}{dx^q}[f\exp(kx)]\right\} = [s+k]^q\mathscr{L}\{f\}, \qquad q \leqq 0, \tag{8.1.6}$$

of which equation (8.1.4) may be regarded as the $k = 0$ instance. For $q = 0$, equation (8.1.6) is a trivial identity, while for $q < 0$, we may use the Riemann–Liouville definition of the differintegral to write

$$\mathscr{L}\left\{\exp(-kx)\frac{d^q}{dx^q}[\exp(kx)f(x)]\right\} = \mathscr{L}\left\{\frac{1}{\Gamma(-q)}\int_0^x \frac{\exp(-k[x-y])\mathscr{L}\{f(y)\}\,dy}{[x-y]^{q+1}}\right\}$$

$$= \frac{1}{\Gamma(-q)}\mathscr{L}\left\{\frac{\exp(-kx)}{x^{q+1}}\right\}\mathscr{L}\{f(x)\}$$

by the convolution theorem. The proof is completed by making use of

$$\mathscr{L}\left\{\frac{\exp(-kx)}{x^{q+1}}\right\} = \Gamma(-q)[s+k]^q$$

[see Churchill (1948)] for $q < 0$.

8.2 NUMERICAL DIFFERINTEGRATION

In this section, algorithms for effecting differintegration to order q will be devised and evaluated. These algorithms are designed to approximate d^qf/dx^q for arbitrary q when the value of f is known at $N+1$ evenly spaced points in the range 0 to x of the independent variable.[1] The nomenclature

$$\begin{aligned} f_N &\equiv f(0), \\ f_{N-1} &\equiv f\left(\frac{x}{N}\right), \\ &\vdots \\ f_j &\equiv f\left(x - \frac{jx}{N}\right), \\ &\vdots \\ f_0 &\equiv f(x) \end{aligned}$$

for the function values will be adopted. This possibly confusing notation is clarified by reference to Fig. 8.2.1, which depicts an $N = 5$ example.

What is probably the simplest algorithm is generated from the Grünwald definition (3.2.1) of differintegration

$$\frac{d^qf}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{\left[\dfrac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x - j\left[\frac{x-a}{N}\right]\right)\right\}$$

by simply omitting the $N \to \infty$ operation. Thence we find, setting $a = 0$,

$$\text{(8.2.1)} \qquad \begin{aligned} \frac{d^qf}{dx^q} \approx \left(\frac{d^qf}{dx^q}\right)_{G1} &= \frac{x^{-q}N^q}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x - \frac{jx}{N}\right) \\ &= \frac{x^{-q}N^q}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f_j. \end{aligned}$$

[1] Not all algorithms, however, utilize exactly $N+1$ values of f. The $G1$-algorithm employs only N data, f_N being unused. The $G2$- and $L2$-algorithms utilize $N+2$ points, needing an $f_{-1}[\equiv f([N+1]x/N)]$ datum in addition to the standard $N+1$.

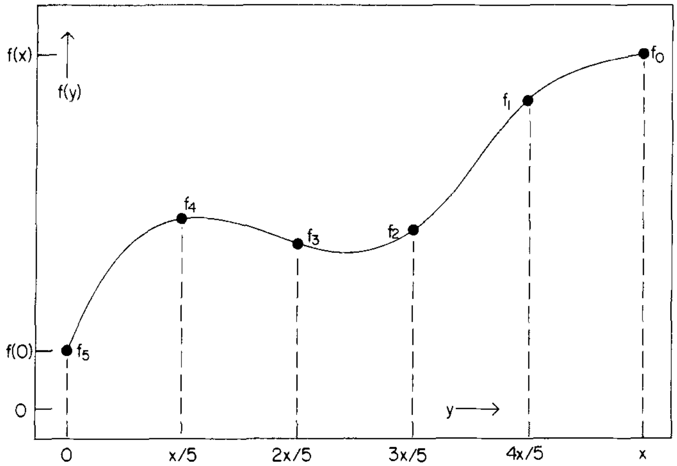

FIG. 8.2.1. An example illustrating the nomenclature used in numerical differintegration.

This is the approximation formula which we term the "$G1$-algorithm." Because of the recursion

$$\frac{\Gamma(j-q)}{\Gamma(j+1)} = \frac{j-1-q}{j}\,\frac{\Gamma(j-1-q)}{\Gamma(j)},$$

the $G1$-algorithm may be implemented by the convenient multiplication–addition–multiplication $\cdots$ multiplication–addition scheme,

$$\text{(8.2.2)}\qquad \left(\frac{d^qf}{dx^q}\right)_{G1} = \frac{N^q}{x^q}\left[\left[\left[\cdots\left[\left[f_{N-1}\left\{\frac{N-q-2}{N-1}\right\}+f_{N-2}\right]\left\{\frac{N-q-3}{N-2}\right\}\right.\right.\right.\right.$$
$$\left.\left.\left.+f_{N-3}\right]\cdots\right]\left\{\frac{1-q}{2}\right\}+f_1\right]\left\{\frac{-q}{1}\right\}+f_0\right],$$

which avoids explicit use of gamma functions and which lends itself to ease of programming.

In Section 3.4 a modified Grünwald definition was presented which was superior in its convergence properties. Rewriting equation (3.4.6) gives

$$\text{(8.2.3)}\qquad \frac{d^qf}{dx^q} = \lim_{N\to\infty}\left\{\frac{x^{-q}N^q}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x+\frac{qx}{2N}-\frac{jx}{N}\right)\right\}$$

for the $a = 0$ case. Unfortunately (unless $q = 0, \pm 2, \pm 4, \ldots$) this formula calls for the evaluation of f at points other than our known f_j values. We therefore use the approximation

$$f\left(x + \frac{qx}{2N} - \frac{jx}{N}\right) \approx \left[\frac{q}{4} + \frac{q^2}{8}\right] f\left(x + \frac{x}{N} - \frac{jx}{N}\right) + \left[1 - \frac{q^2}{4}\right] f\left(x - \frac{jx}{N}\right)$$
$$+ \left[\frac{q^2}{8} - \frac{q}{4}\right] f\left(x - \frac{x}{N} - \frac{jx}{N}\right),$$

based upon the Lagrange three-point interpolation [see Abramowitz and Stegun (1964, p. 879)] in formula (8.2.3) and use this to generate a $G2$-algorithm

$$\left(\frac{d^q f}{dx^q}\right)_{G2} = \frac{x^{-q}N^q}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} \{f_j + \tfrac{1}{4}q[f_{j-1} - f_{j+1}] + \tfrac{1}{8}q^2[f_{j-1} - 2f_j + f_{j+1}]\}$$

by relaxing the $N \to \infty$ condition. As with the $G1$-algorithm, a multiplicative–additive scheme [akin to (8.2.2)] is a convenient aid to programming this second algorithm.

The two algorithms thus far considered were based on the Grünwald definition of differintegration; the remainder originate with the Riemann–Liouville definition. The latter is

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(y)\,dy}{[x-y]^{q+1}} = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(x-y)\,dy}{y^{q+1}}$$

for negative values of q. Writing this as

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(-q)} \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \frac{f(x-y)\,dy}{y^{q+1}},$$

we can see how it is possible to produce a number of so-called R-algorithms, each arising from a different approximation for the integral. Thus, by approximating

$$\int_{jx/N}^{[jx+x]/N} \frac{f(x-y)\,dy}{y^{q+1}} \approx \frac{f\left(x - \frac{jx}{N}\right) + f\left(x - \frac{x}{N} - \frac{jx}{N}\right)}{2} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^{q+1}}$$
$$= \frac{f_j + f_{j+1}}{-2q} \left\{\left[\frac{jx+x}{N}\right]^{-q} - \left[\frac{jx}{N}\right]^{-q}\right\},$$

we arrive at the $R1$-algorithm,[2]

$$\left(\frac{d^qf}{dx^q}\right)_{R1} = \frac{x^{-q}N^q}{\Gamma(1-q)}\sum_{j=0}^{N-1}\frac{f_j+f_{j+1}}{2}\{[j+1]^{-q}-j^{-q}\}, \tag{8.2.4}$$

valid for $q < 0$. Figure 8.2.2 will explain the basis for the approximation we have just used, as well as demonstrating its inferiority to the approximation

$$\int_{jx/N}^{[jx+x]/N}\frac{f(x-y)\,dy}{y^{q+1}}$$

$$\approx \int_{jx/N}^{[jx+x]/N}\frac{\left\{\left[1+j-\dfrac{Ny}{x}\right]f\left(x-\dfrac{jx}{N}\right)+\left[\dfrac{Ny}{x}-j\right]f\left(x-\dfrac{x}{N}-\dfrac{jx}{N}\right)\right\}dy}{y^{q+1}},$$

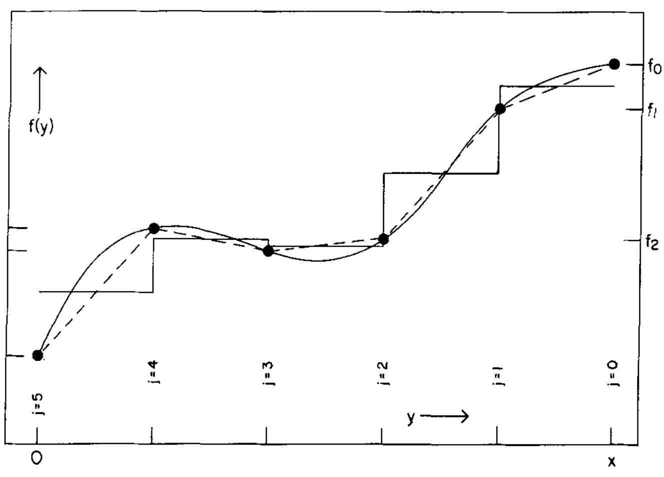

FIG. 8.2.2. The curve f is replaced by a piecewise-constant (staircase line) or a piecewise-linear (dashed line) approximation for the purpose of deriving R-algorithms.

[2] An alternative derivation of the $R1$-algorithm replaces $f(x)$ by the "staircase approximation" shown in Fig. 8.2.2 and differintegrates this approximation by treating it as a piecewise-defined function as in Section 6.9. The $R2$-algorithm likewise arises if the "ramp approximation" of Fig. 8.2.2 is piecewise differintegrated. Note that such techniques of numerical differintegration can equally be applied to data in which the abscissas are not equally spaced.

based on a linear interpolation between f_{j+1} and f_j. This latter leads directly to the $R2$-algorithm,

(8.2.5)
$$\left(\frac{d^q f}{dx^q}\right)_{R2} = \frac{x^{-q}N^q}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{[j+1]f_j - jf_{j+1}}{-q} \{[j+1]^{-q} - j^{-q}\} + \frac{f_{j+1} - f_j}{1-q} \{[j+1]^{1-q} - j^{1-q}\}$$

more complex than the $R1$, but more precise.

Both R-algorithms are restricted to negative q. To construct an algorithm valid in the range $0 \leqq q < 1$ of the differintegration order, we turn to definition (3.6.4). Written for $a = 0$ and $n = 1$, this formula gives

$$\frac{d^q f}{dx^q} = \frac{x^{-q}f(0)}{\Gamma(1-q)} + \frac{1}{\Gamma(1-q)} \int_0^x \left[\frac{df}{dy}(y)\right] \frac{dy}{[x-y]^q}$$
$$= \frac{1}{\Gamma(1-q)} \left\{\frac{f(0)}{x^q} + \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \left[\frac{df}{dy}(x-y)\right] \frac{dy}{y^q}\right\}.$$

We shall put forward only one algorithm, the $L1$-algorithm, based on this formula, although more sophisticated alternatives are available. The $L1$-algorithm utilizes the approximation

$$\int_{jx/N}^{[jx+x]/N} \left[\frac{df}{dy}(x-y)\right] \frac{dy}{y^q} \approx \frac{f\left(x - \frac{jx}{N}\right) - f\left(x - \frac{x}{N} - \frac{jx}{N}\right)}{x/N} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^q}$$
$$= \frac{x^{-q}N^q}{1-q} [f_j - f_{j+1}][(j+1)^{1-q} - j^{1-q}]$$

and is

(8.2.6)
$$\left(\frac{d^q f}{dx^q}\right)_{L1} = \frac{x^{-q}N^q}{\Gamma(2-q)} \left[\frac{[1-q]f_N}{N^q} + \sum_{j=0}^{N-1} [f_j - f_{j+1}][(j+1)^{1-q} - j^{1-q}]\right],$$

being valid only for $0 \leqq q < 1$.

Turning now to the $1 \leqq q < 2$ range, we can once more start with definition (3.6.4), this time choosing $a = 0$ and $n = 2$. The resulting formula,

$$\frac{d^q f}{dx^q} = \frac{x^{-q}f(0)}{\Gamma(1-q)} + \frac{x^{1-q}f^{(1)}(0)}{\Gamma(2-q)} + \frac{1}{\Gamma(2-q)} \int_0^x \frac{f^{(2)}(y)\,dy}{[x-y]^{q-1}}$$
$$= \frac{1}{\Gamma(2-q)} \left[\frac{[1-q]f(0)}{x^q} + \frac{f^{(1)}(0)}{x^{q-1}} + \sum_{j=0}^{N-1} \int_{jx/N}^{[jx+x]/N} \frac{f^{(2)}(x-y)\,dy}{y^{q-1}}\right],$$

will serve as our base from which to construct an $L2$-algorithm. By making the approximations

$$f^{(1)}(0) \approx \frac{f\left(\dfrac{x}{N}\right) - f(0)}{x/N} = \frac{N}{x}[f_{N-1} - f_N]$$

and

$$\int_{jx/N}^{[jx+x]/N} f^{(2)}(x-y)\frac{dy}{y^{q-1}}$$

$$\approx \frac{f\left(x + \dfrac{x}{N} - \dfrac{jx}{N}\right) - 2f\left(x - \dfrac{jx}{N}\right) + f\left(x - \dfrac{x}{N} - \dfrac{jx}{N}\right)}{[x/N]^2} \int_{jx/N}^{[jx+x]/N} \frac{dy}{y^{q-1}}$$

$$= \frac{x^{-q}N^q}{2-q}[f_{j-1} - 2f_j + f_{j+1}][(j+1)^{2-q} - j^{2-q}],$$

we arrive at

$$\text{(8.2.7)} \quad \left(\frac{d^qf}{dx^q}\right)_{L2} = \frac{x^{-q}N^q}{\Gamma(3-q)}\left[\frac{[1-q][2-q]f_N}{N^q} + \frac{[2-q][f_{N-1} - f_N]}{N^{q-1}} + \sum_{j=0}^{N-1}[f_{j-1} - 2f_j + f_{j+1}][(j+1)^{2-q} - j^{2-q}]\right]$$

as our newest algorithm, valid for $1 \leqq q < 2$.

It is evident that an $L3$-algorithm covering the $2 \leqq q < 3$ range, and an $L4$-algorithm covering the $3 \leqq q < 4$ range, etc., may be devised in an analogous fashion. We shall not, however, pursue any of these algorithms further.

In general, algorithms are imperfect. A measure of the imperfection of a typical Al-algorithm is given by the relative error term

$$(\varepsilon(f))_{Al} \equiv \frac{\left(\dfrac{d^qf}{dx^q}\right)_{Al} - \left(\dfrac{d^qf}{dx^q}\right)}{\dfrac{d^qf}{dx^q}}$$

for some trial function f. This error term will generally depend on q and N,[3] and should tend to zero as N approaches infinity. For certain simple

[3] Possibly on x also, though not in the examples considered here.

Table 8.2.1. *Comparison of the relative errors of six differintegration algorithms for the functions C, x, and x^2*

Al	$(\varepsilon(C))_{Al}$	$(\varepsilon(x))_{Al}$	$(\varepsilon(x^2))_{Al}$
$G1$ (all q)	$\dfrac{q[q+1]}{2N}$	$\dfrac{q[q-1]}{2N}$	$\dfrac{q[q-2]}{2N}$
$G2$ (all q)	$\dfrac{q[q+1]}{2N}$	$\dfrac{q^2}{2N}$	$\dfrac{q[q-1][q-2]}{24N^2}$
$R1$ ($q<0$)	0	$\dfrac{1-q}{N}\left[\dfrac{\zeta(q)}{N^{-q}}-\dfrac{q}{12N}\right]$	$\dfrac{[2-q][1-q]}{N^2}\left[\dfrac{\zeta(q)}{N^{-q-1}}-\dfrac{\zeta(q-1)}{N^{-q}}+\dfrac{1}{6}\right]$
$R2$ ($q<0$)	0	0	$\dfrac{2-q}{N^2}\left[\dfrac{\zeta(1-q)}{N^{-q}}+\dfrac{1-q}{12}\right]$
$L1$ ($0 \leqq q < 1$)	0	0	$\dfrac{2-q}{N^2}\left[\dfrac{\zeta(1-q)}{N^{-q}}+\dfrac{1-q}{12}\right]$
$L2$ ($1 \leqq q < 2$)	0	0	$\dfrac{2-q}{N}$

functions f, analytical expressions for $(\varepsilon(f))_{Al}$ may be derived that are excellent asymptotic approximations for large N. Table 8.2.1 presents such expressions for six of our differintegration algorithms, and for the three functions

$$f = C \text{ (a constant)}, \qquad f = x, \qquad \text{and} \qquad f = x^2.$$

Notice that, for these simple functions, several algorithms are error free. In this tabulation, $\zeta(\ )$ denotes the Riemann zeta function (Abramowitz and Stegun, 1964, p. 807), some values of which will be found listed in Table 8.2.2. The methods by which Table 8.2.1 was constructed will be explained by consideration of three examples: $(\varepsilon(C))_{G1}$, $(\varepsilon(x))_{R1}$, and $(\varepsilon(x^2))_{L2}$.

From equation (8.2.1) defining the $G1$-algorithm, we have

$$\left(\frac{d^qC}{dx^q}\right)_{G1} = \frac{x^{-q}N^q}{\Gamma(-q)}\, C \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)}$$

for $f = C$. Using summation (1.3.17),

$$\left(\frac{d^qC}{dx^q}\right)_{G1} = \frac{Cx^{-q}N^q}{\Gamma(1-q)}\,\frac{\Gamma(N-q)}{\Gamma(N)},$$

whereas the exact differintegral is

$$\frac{d^qC}{dx^q} = \frac{Cx^{-q}}{\Gamma(1-q)}$$

Table 8.2.2. *Values of the Riemann zeta function*

q	$\zeta(q)$	q	$\zeta(q)$	q	$\zeta(q)$
-4	0	$-\frac{2}{3}$	-0.15519690	$+1$	∞
$-3\frac{1}{2}$	$+0.00444098$	$-\frac{1}{2}$	-0.20788622	$+1\frac{1}{6}$	$+6.58921554$
-3	$+0.00833333$	$-\frac{1}{3}$	-0.27734305	$+1\frac{1}{3}$	$+3.60093775$
$-2\frac{1}{2}$	$+0.00851692$	$-\frac{1}{6}$	-0.37073766	$+1\frac{1}{2}$	$+2.61237535$
-2	0	0	-0.50000000	$+2$	$+1.64493407$
$-1\frac{1}{2}$	-0.02548520	$+\frac{1}{6}$	-0.68658158	$+2\frac{1}{2}$	$+1.34148726$
$-1\frac{1}{3}$	-0.04006133	$+\frac{1}{3}$	-0.97336025	$+3$	$+1.20205690$
$-1\frac{1}{6}$	-0.05896522	$+\frac{1}{2}$	-1.46035451	$+3\frac{1}{2}$	$+1.12673387$
-1	-0.08333333	$+\frac{2}{3}$	-2.44758074	$+4$	$+1.08232323$
$-\frac{5}{6}$	-0.11469997	$+\frac{5}{6}$	-5.43504324	∞	1

by equation (4.2.1). The error term is thus

$$\big(\varepsilon(C)\big)_{G1} = \frac{N^q\Gamma(N-q)}{\Gamma(N)} - 1 \sim \frac{q[q+1]}{2N} + O(N^{-2}),$$

where the asymptotic expansion is a consequence of (1.3.13) and is valid for large N.

The derivation of $\big(\varepsilon(x)\big)_{R1}$ proceeds as follows, starting with equation (8.2.4):

$$\begin{aligned}\left(\frac{d^qx}{dx^q}\right)_{R1} &= \frac{x^{-q}N^q}{\Gamma(1-q)}\sum_{j=0}^{N-1}\frac{[N-j]x+[N-j-1]x}{2N}\{[j+1]^{-q}-j^{-q}\}\\ &= \frac{x^{1-q}N^{q-1}}{2\Gamma(1-q)}\left[\sum_{j=1}^{N}[2N-2j+1]j^{-q}-\sum_{j=0}^{N-1}[2N-2j-1]j^{-q}\right]\\ &= \frac{x^{1-q}N^{q-1}}{2\Gamma(1-q)}\left[N^{-q}+2\sum_{j=1}^{N-1}j^{-q}\right].\end{aligned}$$

To progress further, use is made of the asymptotic expansion

$$\sum_{j=1}^{N} j^{-q} \sim \zeta(q) + \frac{N^{1-q}}{1-q} + \frac{N^{-q}}{2} - \frac{qN^{-q-1}}{12} + O(N^{-q-3})$$

[see Oldham (1970)]. Thence, recalling from Chapter 4 that $x^{1-q}/\Gamma(2-q)$ is the exact differintegral of x, we obtain

$$\big(\varepsilon(x)\big)_{R1} \sim \frac{[1-q]\zeta(q)}{N^{1-q}} - \frac{q[1-q]}{12N^2} + O(N^{-4}),$$

valid as $N \to \infty$, as the appropriate relative error term. Either of the two leading terms may dominate,[4] depending on the value of q, so both are included in the Table 8.2.1 entry.

Setting $f_j = [N - j]^2 x^2/N^2$ in the equation (8.2.7) defining the $L2$-algorithm, one easily arrives at

$$\left(\frac{d^q x^2}{dx^q}\right)_{L2} = \frac{x^{-q} N^q}{\Gamma(3-q)} \left[\frac{2-q}{N^{q-1}}\left\{\frac{x}{N}\right\}^2 + \sum_{j=0}^{N-1}\left\{\frac{2x^2}{N^2}\right\}\{[j+1]^{2-q} - j^{2-q}\}\right]$$
$$= \frac{2x^{2-q}}{\Gamma(3-q)}\left[\frac{2-q}{2N} + 1\right] = \frac{d^q x^2}{dx^q}\left[\frac{2-q}{2N} + 1\right],$$

from which the tabular entry follows exactly.

All of the differintegration algorithms we have devised are expressible as

(8.2.8) $$\left(\frac{d^q f}{dx^q}\right)_{Al} = \frac{N^q}{x^q} \sum_{j=-1}^{N} w_j(q) f_j,$$

where $w_j(q)$ is a weighting factor whose value depends on j, q, and on the algorithm in question, but not on N or f. For most algorithms, the values of the weighting factors near the two ends of the $-1 \leqq j \leqq N$ range are atypical and cannot be calculated from the $w_j(q)$ formula applicable in the middle of the j range. Table 8.2.3 (pp. 146–147) is comprehensive, listing both typical and atypical weighting factors for all six algorithms. The use of summation (8.2.8) in conjunction with Table 8.2.3 provides an alternative computational method, replacing formulas such as (8.2.1), (8.2.2), (8.2.4), (8.2.5), etc.

Notice in Table 8.2.3 that the same entries serve the $R2$-algorithm and the $L1$-algorithm. In fact, despite the apparent dissimilarity of definitions (8.2.5) and (8.2.6), these two algorithms are identical. Henceforth we shall use the term "*RL*-algorithm" to refer to this conjunction.

At this point it is appropriate to review the advantages and disadvantages of the various algorithms. The G-algorithms are valuable in that they span all q values, can be implemented by a convenient multiplication-addition scheme, and need no gamma functions. The $G1$-algorithm, moreover, is unique in not utilizing an f_N datum, permitting this algorithm to be used to differintegrate functions akin to $\ln(x)$ and $1/\sqrt{x}$, which are infinite at $x = 0$. The $R1$-algorithm was devised from a formula restricted to negative q; it is inferior to the RL-algorithm in accuracy, but is significantly simpler. The RL-algorithm is an efficient and relatively simple algorithm; it is, in fact, the one which we have adopted in routine work (Grenness and Oldham, 1972; Oldham, 1972). The $L2$-algorithm, designed for the $1 \leqq q < 2$ range,

[4] Because $\zeta(-1) = -\frac{1}{12}$ (see Table 8.2.2), the error inherent in the $R1$-algorithm is minute for q values in the vicinity of -1.

is complex to implement and (in common with the $G2$-algorithm) requires an f_{-1} datum, a requirement which cannot be met in some physical applications.

The identity of the $R2$- and $L1$-algorithms illustrates the fact that an algorithm, although designed to differintegrate for a limited q-range, may perform efficiently over an extended range. To test this possibility, and to provide a crucial comparative test for our five algorithms, Fig. 8.2.3 and 8.2.4 were constructed.

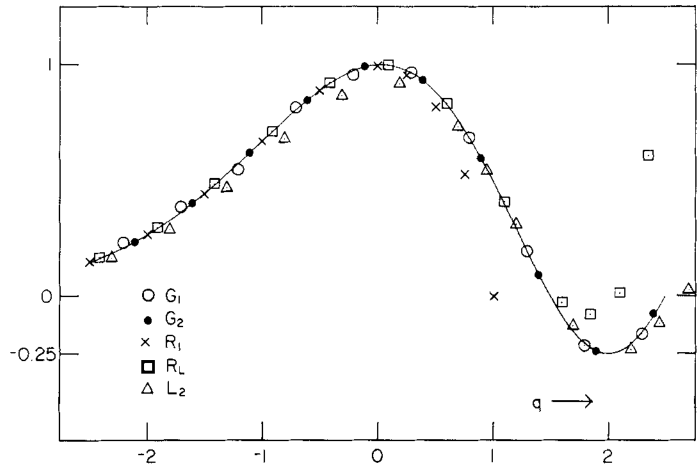

FIG. 8.2.3. The curve depicts the exact differintegral of $\sqrt{x}$, evaluated at $x = 1$. The points are approximations generated by the $G1$, $G2$, $R1$, RL, and $L2$ algorithms, utilizing 32 data points.

The curve in Fig. 8.2.3 is a plot of $\Gamma(\frac{3}{2})/\Gamma(\frac{3}{2} - q)$ versus q, for a wide range of q values. Each set of points in this diagram was produced by allowing one of the five algorithms to differintegrate $\sqrt{x}$ to order q, the differintegral then being evaluated at $x = 1$. Thus, since

$$\frac{d^q\sqrt{x}}{dx^q}(1) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - q)},$$

all points would lie on the curve if the algorithms were exact. In fact, since we chose $N = 32$ only, some points are seen to diverge significantly from the curve, particularly at large positive q values.

Figure 8.2.4 is likewise a graph versus q of the function

$$\frac{d^q[1 - x^{\frac{3}{2}}]}{dx^q}(1) = \frac{\Gamma(1)}{\Gamma(1 - q)} - \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2} - q)},$$

Table 8.2.3. *Weighting factors for six numerical differintegration algorithms*

Al	Typical $w_j(q)$	Range of typicality	Values of j for which $w_j(q) = 0$	Atypical values
*G*1	$\dfrac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}$	$0 \leqq j \leqq N-1$	$-1, N$	none
*G*2	$\dfrac{\Gamma(j-1-q)}{\Gamma(-q)\Gamma(j+2)}\left\{j^2 - \dfrac{jq}{2}[q+3] - [q+1]\left[\dfrac{q^3}{8} + \dfrac{q^2}{2} - 1\right]\right\}$	$0 \leqq j \leqq N-1$	none	$w_{-1}(q) = \dfrac{q^2+2q}{8\Gamma(-q)}$
				$w_N(q) = \dfrac{\Gamma(N-1-q)}{\Gamma(-q)\Gamma(N)}\left[\dfrac{q^2}{8} - \dfrac{q}{4}\right]$
*R*1	$\dfrac{[j+1]^{-q} - [j-1]^{-q}}{\Gamma(1-q)}$	$1 \leqq j \leqq N-1$	-1	$w_0(q) = \dfrac{1}{2\Gamma(1-q)}$
				$w_N(q) = \dfrac{N^{-q} - [N-1]^{-q}}{2\Gamma(1-q)}$

$R2$	$\dfrac{[j+1]^{1-q} - 2j^{1-q} + [j-1]^{1-q}}{\Gamma(2-q)}$	$1 \leqq j \leqq N-1$	-1	$w_0(q) = 1/\Gamma(2-q)$ $w_N(q) = \dfrac{[N-1]^{1-q} - N^{1-q} + [1-q]N^{-q}}{\Gamma(2-q)}$
$L1$	$\dfrac{[j+1]^{1-q} - 2j^{1-q} + [j-1]^{1-q}}{\Gamma(2-q)}$	$1 \leqq j \leqq N-1$	-1	$w_0(q) = 1/\Gamma(2-q)$ $w_N(q) = \dfrac{[N-1]^{1-q} - N^{1-q} + [1-q]N^{-q}}{\Gamma(2-q)}$
$L2$	$\dfrac{[j+2]^{2-q} - 3[j+1]^{2-q} + 3j^{2-q} - [j-1]^{2-q}}{\Gamma(3-q)}$	$1 \leqq j \leqq N-2$	none	$w_{-1}(q) = 1/\Gamma(3-q)$ $w_0(q) = \dfrac{2^{2-q} - 3}{\Gamma(3-q)}$ $w_{N-1}(q) = [\Gamma(3-q)]^{-1}\{[2-q]N^{1-q} - 2N^{2-q} + 3[N-1]^{2-q} - [N-2]^{2-q}\}$ $w_N(q) = [\Gamma(3-q)]^{-1}\{[1-q][2-q]N^{-q} - [2-q]N^{1-q} + N^{2-q} - [N-1]^{2-q}\}$

constructed to display the extent to which each algorithm reproduces the true differintegral, using only 32 data points. The functions $\sqrt{x}$ and $1 - x^{\frac{3}{2}}$ were chosen for these comparisons because, in the range $0 < x \leqq 1$, they place different emphases on the two ends of the range, the former stressing the points close to $x = 1$, the latter emphasizing data close to the $x = 0$ origin.

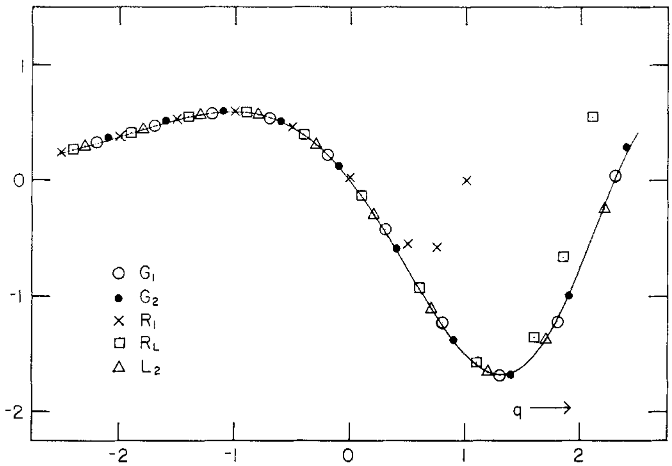

FIG. 8.2.4. The exact differintegral $d^q[1 - x^{\frac{3}{2}}]/dx^q$ at $x = 1$ and approximations to it given by the five algorithms, calculated using $N = 32$.

Inspection of the two diagrams discloses that all algorithms except the $L2$ are efficient for negative q, and extend into the positive q range with varying success. Of course, increasing N beyond 32 will increase the range of efficiency of all the algorithms. The $L2$-algorithm, as befits its mode of construction, is most successful for modestly positive q values.

8.3 ANALOG DIFFERINTEGRATION

Electrical circuits have long been used to perform the operations of differentiation and integration. Probably the simplest example is that shown in Fig. 8.3.1 in which a capacitor is used to provide a voltage output proportional to the integral of the applied current

$$e(t) = \frac{1}{C} \frac{d^{-1}}{dt^{-1}} i(t),$$

provided the initial voltage $e(0)$ is zero. Using the example of semiintegration, this section will demonstrate how circuits can be designed that perform differintegration with respect to time, to noninteger orders.

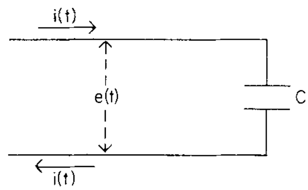

FIG. 8.3.1. An integrating circuit. If $e(0)$ is zero, the potential $e(t)$ across C is proportional to the integral of the input current $i(t)$.

The semiintegrating circuits (for which the name "semiintegrators" seems appropriate) that we shall discuss are imperfect in two respects. First, they are imprecise in that the response is only approximately proportional to the semiintegral of the input signal. Nevertheless, the output may be made to approach the true semiintegral to any specified degree of accuracy. Second, the semiintegration is accomplished only over a time interval which has a finite upper limit and a nonzero lower limit.[5] Nevertheless, these time limits may be chosen to embrace as long and as short a time as desired. Increasing perfection in both accuracy and time span is bought at the price of an increase in the number of components of which the circuit is constructed. Thus, it is a relatively simple matter to design a semiintegrator that performs within an accuracy[6] of $\pm 10\%$ over a time span from 0.1 to 10 sec, whereas a $\pm 1\%$ semiintegrator spanning the time period 10^{-2} to 10^{2} sec requires very many more (and closer tolerance) components.

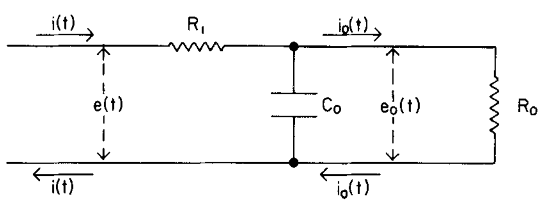

FIG. 8.3.2. A simple three-component circuit.

First consider the simple circuit depicted in Fig. 8.3.2, in which the current signal $i(t)$ must flow through the resistor R_1, following which it experiences a parallel combination of the capacitor C_0 and the resistor R_0. Imagine that

[5] We may replace this sentence in the somewhat more exact parlance of communications theory by saying that the semiintegrator has a frequency bandpass.

[6] What exactly is meant by "within an accuracy" will be apparent later. The accuracy criterion is actually prescribed in Laplace transform space, rather than in a real time domain.

prior to $t = 0$ the circuit is at rest, the applied current and the potential output both being zero:

(8.3.1) $$i(t \leqq 0) = 0 = e(t \leqq 0).$$

A nonzero signal is applied after $t = 0$.

The meanings of the symbols $i_0(t)$ and $e_0(t)$ will be clarified on inspection of Fig. 8.3.2. The current $i_0(t)$ flows through resistor R_0 and therefore is, by Ohm's law, proportional to the potential across the resistor, the constant of proportionality being the reciprocal of the resistance; that is,

$$i_0(t) = \frac{e_0(t)}{R_0}.$$

On the other hand, the current through the capacitor is proportional to the time derivative of the potential across it, the proportionality constant being the capacitance of the capacitor; that is,

$$i(t) - i_0(t) = C_0 \frac{de_0}{dt}(t).$$

Ohm's law also permits us to write

$$i(t) = \frac{e(t) - e_0(t)}{R_1}$$

for the leftmost resistor. The variables $e_0(t)$ and $i_0(t)$ can be eliminated from the above three expressions to yield the differential equation

(8.3.2) $$[R_0 + R_1]i(t) + R_0 R_1 C_0 \frac{di}{dt}(t) = e(t) + R_0 C_0 \frac{de}{dt}(t),$$

interrelating the current $i(t)$ and the potential $e(t)$.

Circuit analysis is frequently more tractable in transform space, and we now therefore consider the Laplace transform,

$$\bar{i}(s)\,[R_0 + R_1 + R_0 R_1 C_0 s] - R_0 R_1 C_0\, i(0) = \bar{e}(s)\,[1 + R_0 C_0 s] - R_0 C_0 e(0),$$

of equation (8.3.2). Here $\bar{i}(s)$ and $\bar{e}(s)$ are, respectively, the transforms of $i(t)$ and $e(t)$, s being the dummy variable of Laplace transformation with respect to time [see Churchill (1948) for a thorough treatment of Laplace transformation theory and techniques]. This transform equation can be recast as

(8.3.3) $$\frac{\bar{e}(s)}{R_1 \bar{i}(s)} = 1 + \frac{\dfrac{1}{R_1 C_0}}{s + \dfrac{1}{R_0 C_0}}$$

after invoking the initial conditions (8.3.1).

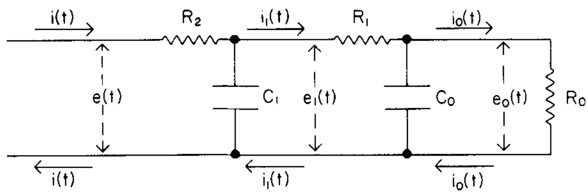

FIG. 8.3.3. Two additional components, a series resistor R_2 and a shunt capacitor C_1, have been added to the circuit in Fig. 8.3.2.

Next, transfer consideration to circuit shown in Fig. 8.3.3, which contains two components, C_1 and R_2, additional to those shown in Fig. 8.3.2. The equations

$$i(t) = \frac{e(t) - e_1(t)}{R_2} \qquad \text{and} \qquad i(t) - i_1(t) = C_1 \frac{de_1}{dt}(t)$$

interrelate the currents and voltages in the left-hand side of this circuit, and yield

$$\text{(8.3.4)} \qquad \frac{\bar{e}(s)}{R_2 \bar{i}(s)} = 1 + \frac{\dfrac{1}{R_2 C_1}}{s + \dfrac{\bar{i}_1(s)}{C_1 \bar{e}_1(s)}}$$

on Laplace transformation, combination, and rearrangement. The right-hand side of the circuit, moreover, obeys the transform equation

$$\text{(8.3.5)} \qquad \frac{\bar{e}_1(s)}{R_1 \bar{i}_1(s)} = 1 + \frac{\dfrac{1}{R_1 C_0}}{s + \dfrac{1}{R_0 C_0}}$$

by analogy with equation (8.3.3).

Progress is aided if we define the frequencies

$$\omega_0 \equiv \frac{1}{R_0 C_0}, \qquad \omega_1 \equiv \frac{1}{R_1 C_0}, \qquad \omega_2 \equiv \frac{1}{R_1 C_1}, \qquad \omega_3 \equiv \frac{1}{R_2 C_1},$$

characteristic of pairs of adjacent components. Using these abbreviations, equations (8.3.4) and (8.3.5) can be combined to give

$$\frac{\bar{e}(s)}{R_2 \bar{i}(s)} = 1 + \cfrac{\omega_3}{s + \cfrac{\omega_2}{1 + \cfrac{\omega_1}{s + \omega_0}}},$$

an expression that may be rewritten

$$\frac{\bar{e}(s)}{R_2\,\bar{i}(s)} = 1 + \frac{\omega_3}{s+}\,\frac{\omega_2}{1+}\,\frac{\omega_1}{s+}\,\frac{\omega_0}{1} \tag{8.3.6}$$

in the notation of continued fractions.

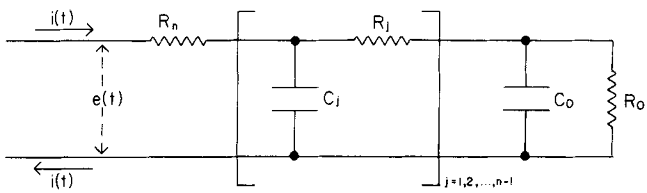

FIG. 8.3.4. A circuit that can perform the operation of semiintegration.

The generalization of equation (8.3.6) to cover the circuit shown in Fig. 8.3.4 is straightforward and yields

$$\frac{\bar{e}(s)}{R_n\,\bar{i}(s)} = 1 + \frac{\omega_{2n-1}}{s+}\,\frac{\omega_{2n-2}}{1+}\,\frac{\omega_{2n-3}}{s+}\cdots\frac{\omega_2}{1+}\,\frac{\omega_1}{s+}\,\frac{\omega_0}{1}, \tag{8.3.7}$$

where

$$\omega_{2j} = \frac{1}{R_j C_j} \qquad \text{and} \qquad \omega_{2j+1} = \frac{1}{R_{j+1} C_j}.$$

A simplification ensues if each ω_j is undimensionalized through division by s. Thus if

$$\frac{\omega_j}{s} \equiv v_j,$$

equation (8.3.7) may be expressed as the continued fraction expression

$$\frac{\bar{e}(s)}{R_n\,\bar{i}(s)} = 1 + \frac{v_{2n-1}}{1+}\,\frac{v_{2n-2}}{1+}\,\frac{v_{2n-3}}{1+}\cdots\frac{v_2}{1+}\,\frac{v_1}{1+}\,\frac{v_0}{1}. \tag{8.3.8}$$

Only for certain simple relationships between the v values is the continued fraction in equation (8.3.8) expressible in a more convenient form. We shall discuss only one such simplifying circumstance. This arises when all capacitors have a common capacitance value $C_0 = C_1 = C_2 = \cdots = C_{n-1} = C$ and all resistors *except one* have a common resistance. The exceptional resistor is R_n and this is accorded a resistance value equal to one-half that of the others; thus if

$$R_0 = R_1 = R_2 = \cdots = R_{n-1} = R,$$

then

$$R_n = \tfrac{1}{2}R.$$

If we suitably define v, thus,

$$v_0 = v_1 = v_2 = \cdots = v_{2n-3} = v_{2n-2} = \frac{1}{RCs} \equiv v,$$

then

$$v_{2n-1} = \frac{2}{RCs} = 2v$$

and consequently, from equation (8.3.8),

$$\frac{2\bar{e}(s)}{R\bar{\imath}(s)} = 1 + 2\,\frac{v}{1+}\,\frac{v}{1+}\cdots\frac{v}{1+}\,\frac{v}{1}, \tag{8.3.9}$$

where the continued fraction has $2n$ numeratorial v's. Continued fractions are discussed by Wall (1948) who shows that

$$\frac{v}{1+}\,\frac{v}{1+}\cdots\frac{v}{1+}\,\frac{v}{1} = \frac{\sqrt{4v+1}}{1+\left[\dfrac{\sqrt{4v+1}-1}{\sqrt{4v+1}+1}\right]^{2n+1}} - \frac{\sqrt{4v+1}}{2} - \frac{1}{2}, \tag{8.3.10}$$

as can also be established inductively. Combining equations (8.3.9) and (8.3.10) and dividing by $2\sqrt{v}$, we obtain

$$\frac{\bar{e}(s)}{\bar{\imath}(s)}\sqrt{\frac{Cs}{R}} = \sqrt{\frac{4v+1}{4v}}\left[\frac{[\sqrt{4v+1}+1]^{2n+1} - [\sqrt{4v+1}-1]^{2n+1}}{[\sqrt{4v+1}+1]^{2n+1} + [\sqrt{4v+1}-1]^{2n+1}}\right] \tag{8.3.11}$$

as our final result interrelating the transforms of $e(t)$ and $i(t)$.

Figure 8.3.5 is a plot of the right-hand side of equation (8.3.11) versus v for various values of n. Notice that, for the larger n values, the function

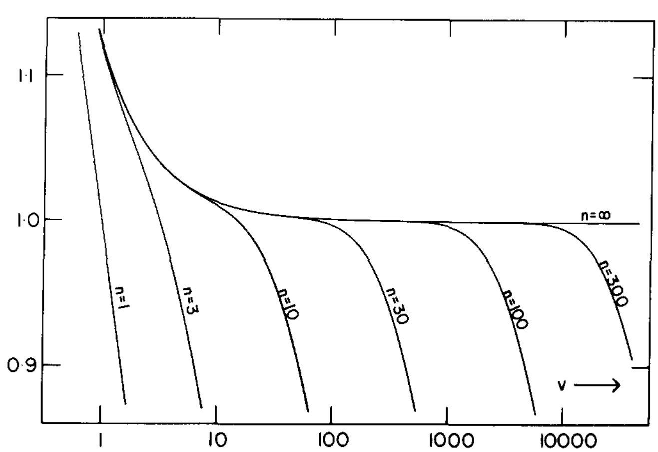

FIG. 8.3.5. A semilogarithmic plot illustrating equation (8.3.11).

is very close to unity over a wide range of v values. In fact, for n in excess of about 10, the function lies within 2% of unity,

$$\frac{\bar{e}(s)}{\bar{i}(s)}\sqrt{\frac{Cs}{R}} \approx 1,$$

provided that

$$6 \leqq v \leqq \tfrac{1}{6}n^2.$$

Recalling the definition of v, this means that

$$\text{(8.3.12)} \qquad \bar{e}(s) \approx \sqrt{\frac{R}{Cs}}\,\bar{i}(s) \qquad \text{for} \qquad 6RC \leqq \frac{1}{s} \leqq \tfrac{1}{6}n^2 RC.$$

The Laplace inversion of this relationship poses problems if it is to be performed rigorously; however, (8.3.12) implies

$$\text{(8.3.13)} \qquad e(t) \approx \sqrt{\frac{R}{C}}\,\frac{d^{-\frac{1}{2}}i}{dt^{-\frac{1}{2}}}(t) \qquad \text{for} \qquad 6RC \leqq t \leqq \tfrac{1}{6}n^2 RC$$

at least approximately. Thus, over a certain time range, the circuit of Fig. 8.3.4 performs as an efficient semiintegrator in that it develops a potential accurately proportional to the semiintegral of the input current. The lower time limit for accurate semiintegration is about six times the RC time constant of the standard components used, and can be made arbitrarily small by selecting small-valued resistors and capacitors. The upper time limit depends on the number of components and can be made arbitrarily large by employing enough resistors and capacitors.

Although it is mathematically straightforward, the circuitry of Fig. 8.3.4 is not the most efficient semiintegrator. By this we mean that a given number $2n + 1$ of components can be used to construct a semiintegrator that accurately semiintegrates over a wider time span ratio than the $n^2/36$ given by (8.3.13). For details of such circuits the interested reader is referred to Biorci and Ridella (1970), Ichise *et al.* (1971), and Oldham (1973a).

8.4 EXTRAORDINARY DIFFERENTIAL EQUATIONS

A relationship involving one or more derivatives of an unknown function f with respect to its independent variable x is known as an ordinary differential equation. Similar relationships involving at least one differintegral of noninteger order may be termed extraordinary differential equations. Such an equation is solved when an explicit expression for f is exhibited. As with ordinary differential equations, the solutions of extraordinary differential equations often involve integrals and contain arbitrary constants.

We begin by considering perhaps the simplest of all extraordinary differential equations,

$$\frac{d^Q f}{dx^Q} = F, \tag{8.4.1}$$

where Q is arbitrary, F is a known function, and f is the unknown function (we have chosen the lower limit $a = 0$ in the differintegral for simplicity). It is tempting to apply the operator d^{-Q}/dx^{-Q} to both sides of equation (8.4.1) and perform the "inversion"

$$f = \frac{d^{-Q}F}{dx^{-Q}},$$

but this is not the most general solution. In fact, referring to our discussion of the composition law in Section 5.7, we recall that it is precisely the condition

$$f - \frac{d^{-Q}}{dx^{-Q}} \frac{d^Q f}{dx^Q} = 0,$$

which guarantees obedience to the composition rule for general differintegrable series f. The difference $f - d^{-Q}/dx^{-Q}\{d^Q f/dx^Q\}$ will not, in general, vanish but will consist of those portions of the differintegrable series units f_U in f that are sent to zero under the action of d^Q/dx^Q. We decompose f into differintegrable units $f_{U,i}$, where

$$f_{U,i} \equiv x^{p_i} \sum_{j=0}^{\infty} a_{ij} x^j, \qquad p_i > -1, \quad a_{i0} \neq 0, \quad i = 1, \ldots, n, \tag{8.4.2}$$

and investigate the conditions on $f_{U,i}$ required to give

$$f - \frac{d^{-Q}}{dx^{-Q}} \frac{d^Q f}{dx^Q} \neq 0. \tag{8.4.3}$$

Inspection of condition (5.7.5) or of (5.7.6), to which it is equivalent, tells us that condition (8.4.3) obtains if and only if, for some i in the range $1 \leqq i \leqq n$,

$$\Gamma(p_i - Q + 1) \tag{8.4.4}$$

is infinite. This condition can occur, however, only when $p_i - Q + 1 = 0, -1, -2, \ldots$, that is, when $p_i = Q - 1, Q - 2, \ldots$. Putting these facts together shows that, in the most general case,

$$f - \frac{d^{-Q}}{dx^{-Q}} \frac{d^Q f}{dx^Q} = C_1 x^{Q-1} + C_2 x^{Q-2} + \cdots + C_m x^{Q-m},$$

where $C_1, \ldots, C_m$ are arbitrary constants and $0 < Q \leqq m < Q + 1$; for $Q \leqq 0$ the right-hand member of the equation is zero. Thus

$$f - C_1 x^{Q-1} - C_2 x^{Q-2} - \cdots - C_m x^{Q-m} = \frac{d^{-Q}}{dx^{-Q}} \frac{d^Q f}{dx^Q} = \frac{d^{-Q} F}{dx^{-Q}}$$

and the most general solution of equation (8.4.1) is

$$f = \frac{d^{-Q} F}{dx^{-Q}} + C_1 x^{Q-1} + C_2 x^{Q-2} + \cdots + C_m x^{Q-m}. \tag{8.4.5}$$

Notice the presence in this general solution of m arbitrary constants, where $0 < Q \leqq m < Q + 1$ or $m = 0$ for $Q \leqq 0$.

As an example of the foregoing theory, consider the extraordinary differential equation

$$\frac{d^{\frac{3}{2}} f}{dx^{\frac{3}{2}}} = x^5,$$

whose general solution is

$$f = \frac{d^{-\frac{3}{2}} x^5}{dx^{-\frac{3}{2}}} + C_1 x^{\frac{1}{2}} + C_2 x^{-\frac{1}{2}} = \frac{\Gamma(6)}{\Gamma(\frac{15}{2})} x^{13/2} + C_1 x^{\frac{1}{2}} + C_2 x^{-\frac{1}{2}},$$

and contains two arbitrary constants.

Next consider the equation

$$\frac{d^Q f}{dx^Q} + A \frac{d^{Q-1} f}{dx^{Q-1}} = F(x), \tag{8.4.6}$$

where Q is again arbitrary, A is a known constant, and F is known. Application of the operator d^{1-Q}/dx^{1-Q} to equation (8.4.6) yields, by techniques like those discussed in connection with the inversion of equation (8.4.1),

$$\frac{df}{dx} + Af = \frac{d^{1-Q} F}{dx^{1-Q}} + C_1 x^{Q-2} + C_2 x^{Q-3} + \cdots + C_m x^{Q-m-1},$$

a first-order ordinary differential equation for f whose solution may be accomplished by standard methods (Murphy, 1960).

The two extraordinary differential equations just treated were quite special; the solution of the even slightly more general equation

$$\frac{d^q f}{dx^q} + \frac{d^Q f}{dx^Q} = F$$

encounters very great difficulties except when the difference $q - Q$ is integer or half-integer. We have illustrated in the present section how to deal with the case $q - Q = n$, an integer (which leads to an nth-order ordinary differential equation). The next section will treat the important case $q - Q =$

$[2n+1]/2$, n integer. As we shall see in Chapter 11, these so-called semidifferential equations arise very naturally in solving a wide variety of transport problems and, in our experience, constitute by far the most important class of extraordinary differential equations.

8.5 SEMIDIFFERENTIAL EQUATIONS

By a semidifferential equation we shall understand a relationship involving differintegrals of an unknown function, each differintegral order occurring as some multiple of $\frac{1}{2}$, at least one of which must be an odd multiple of $\frac{1}{2}$. For example, the equation

$$\frac{d^6 f}{dx^6} + \sin(x)\frac{d^{\frac{7}{2}} f}{dx^{\frac{7}{2}}} = \exp(x)$$

is a semidifferential equation, as is

$$\frac{d^{\frac{3}{2}} f}{dx^{\frac{3}{2}}} - \frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} + 2f = 0,$$

while

$$\frac{d^2 f}{dx^2} + \frac{df}{dx} = \frac{d^{\frac{1}{2}} F}{dx^{\frac{1}{2}}}$$

is not if F is regarded as a known function. We shall discover by examples that two principal techniques are available for solving semidifferential equations: (1) Transformation to an ordinary differential equation and (2) Laplace transformation. As is often the case when dealing with the fractional calculus we are not able to discuss solutions of very general semidifferential equations but are forced to content ourselves with examples intended to reveal solution techniques.

Consider the semidifferential equation

$$\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} + f = 0. \tag{8.5.1}$$

Applying $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ and utilizing composition rule arguments as in Section 8.4 yields

$$\frac{df}{dx} - C_1 x^{-\frac{3}{2}} + \frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = 0.$$

This result may now be combined with the original semidifferential equation (8.5.1) to give

$$\frac{df}{dx} - f = C_1 x^{-\frac{3}{2}},$$

a first-order ordinary differential equation for the unknown f. Standard methods [see Murphy (1960)] enable us to solve for f, with the result

$$f = C\exp(x) + \exp(x)\int_0^x \exp(-y)C_1 y^{-\frac{3}{2}}\,dy. \tag{8.5.2}$$

The solution (8.5.2) is puzzling on two counts. First, it involves two constants, C_1 and C, rather than the single arbitrary constant we might have expected based on analogy with the results of Section 8.4. Second, it involves a divergent integral, namely,

$$I \equiv \int_0^x \exp(-y)y^{-\frac{3}{2}}\,dy,$$

which we recognize from equation (1.3.26) as the incomplete gamma function $-2\sqrt{\pi/x}\,\gamma^*(-\frac{1}{2}, x)$. A finite value may be assigned to I,

$$I = -2\sqrt{\pi}\,\operatorname{erf}(\sqrt{x}) - 2\frac{\exp(-x)}{\sqrt{x}},$$

by means of formulas (1.3.27) and (1.3.28). We see then that

$$\begin{aligned} f &= C\exp(x) - 2C_1\exp(x)\left[\sqrt{\pi}\,\operatorname{erf}(\sqrt{x}) + \frac{\exp(-x)}{\sqrt{x}}\right] \\ &= C\exp(x) - 2\sqrt{\pi}\,C_1\exp(x)\operatorname{erf}(\sqrt{x}) - \frac{2C_1}{\sqrt{x}}. \end{aligned} \tag{8.5.3}$$

Now upon semidifferentiation of equation (8.5.3), making use of Sections 7.5 and 7.3, we see that

$$\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} = \frac{C}{\sqrt{\pi x}} + C\exp(x)\operatorname{erf}(\sqrt{x}) - 2\sqrt{\pi}\,C_1\exp(x).$$

The original semidifferential equation now demands that

$$\begin{aligned} &\frac{C}{\sqrt{\pi x}} + C\exp(x)\operatorname{erf}(\sqrt{x}) - 2\sqrt{\pi}\,C_1\exp(x) \\ &\qquad = \frac{2C_1}{\sqrt{x}} + 2\sqrt{\pi}\,C_1\exp(x)\operatorname{erf}(\sqrt{x}) - C\exp(x). \end{aligned}$$

This condition is satisfied only if

$$\frac{C}{\sqrt{\pi}} = 2C_1$$

and shows that, indeed, there is but a single arbitrary constant in the solution of (8.5.1), which takes the final form

$$f = C\exp(x)\operatorname{erfc}(\sqrt{x}) - [C/\sqrt{\pi x}\,]. \tag{8.5.4}$$

Our second example in this section is the semidifferential equation

$$\frac{df}{dx} + \frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} - 2f = 0. \tag{8.5.5}$$

This time it is useful to Laplace transform, obtaining

$$s\bar{f}(s) - f(0) + \sqrt{s}\,\bar{f}(s) - \frac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}(0) + \bar{f}(s) = 0$$

after making use of the results of Section 8.1. Hence

$$\bar{f}(s) = \frac{f(0) + \dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}(0)}{s + \sqrt{s} - 2} = \frac{C}{[\sqrt{s} - 1][\sqrt{s} + 2]},$$

where C is a constant. A partial fraction decomposition gives

$$\bar{f}(s) = \frac{C}{3[\sqrt{s} - 1]} - \frac{C}{3[\sqrt{s} + 2]},$$

which upon Laplace inversion produces

$$\begin{aligned} f &= \frac{C}{3}\left[\frac{1}{\sqrt{\pi x}} + \exp(x)\operatorname{erfc}(-\sqrt{x}\,)\right] - \frac{C}{3}\left[\frac{1}{\sqrt{\pi x}} - 2\exp(4x)\operatorname{erfc}(2\sqrt{x}\,)\right] \\ &= \frac{C}{3}\left[2\exp(4x)\operatorname{erfc}(2\sqrt{x}\,) + \exp(x)\operatorname{erfc}(-\sqrt{x}\,)\right] \end{aligned}$$

as the final solution.

The reader has, no doubt, developed a healthy respect for the complexity hidden in solving even the simplest semidifferential equations. In Chapter 11 we reduce the solution of certain diffusion problems "*simply* to that of a semidifferential equation." It is indeed fortunate that such equations usually have the form (8.4.1) with $Q = \pm\frac{1}{2}$, a form which is amenable to easy inversion as demonstrated in Section 8.4.

8.6 SERIES SOLUTIONS

In the same way that resort must often be made to series methods in solving the more difficult ordinary differential equations, so such methods can be used for difficult semidifferential equations, and more generally for extraordinary differential equations. The techniques closely parallel those used for classical differential equations [see Murphy (1960)] and it will suffice here to present a single example.

The semidifferential equation

$$\sqrt{x}\,\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} + x^{w}f = 1 \tag{8.6.1}$$

occurs in the theory of voltammetry at expanding electrodes [see Oldham (1969b)]. Here w is a number in the range $0 < w \leqq \frac{1}{2}$, the value $w = \frac{3}{14}$ having special practical importance. As encountered practically, w will always be rational, and we shall equate it to the ratio μ/ν.

To solve equation (8.6.1), we assume that the differintegrable series

$$f = x^p \sum_{j=0}^{\infty} a_j x^{j/n} \tag{8.6.2}$$

is a solution for some value of p exceeding -1, for some integer n, and for some assignment of coefficients $a_0, a_1, a_2, \ldots$ such that $a_0 \neq 0$. If so, then we have

$$x^p \sum_{j=0}^{\infty} a_j x^{j/n} \left[\frac{\Gamma\left(p + \dfrac{j}{n} + 1\right)}{\Gamma\left(p + \dfrac{j}{n} + \frac{1}{2}\right)} + x^{\mu/\nu} \right] = 1$$

on combining equations (8.6.1) and (8.6.2) and making use of the results of Sections 5.2 and 7.3. To satisfy this equation, we select $p = 0$, $n = \nu$ and the following coefficients:

$$a_0 = \sqrt{\pi}, \qquad a_1 = a_2 = \cdots = a_{\mu-1} = 0,$$

$$a_j = -a_{j-\mu} \frac{\Gamma\left(\dfrac{j}{\nu} + \frac{1}{2}\right)}{\Gamma\left(\dfrac{j}{\nu} + 1\right)}, \qquad j = \mu, \mu + 1, \mu + 2, \ldots .$$

The semidifferential equation is now effectively solved.

This solution, most concisely written as

$$f_w(x) = \sum_{k=0}^{\infty} [-x]^{kw} \prod_{l=0}^{k} \frac{\Gamma(\frac{1}{2} + lw)}{\Gamma(1 + lw)},$$

was derived independently by Koutecky (1953) and by Matsuda and Ayabe (1955), although not in the context of the fractional calculus and only in the $w = \frac{3}{14}$ instance. Convergence is rapid for small argument, but the asymptotic alternative

$$f_w(x) \sim x^{-w} \sum_{k=0}^{\infty} [-x]^{-kw} \prod_{l=1}^{k} \frac{\Gamma(1 - lw)}{\Gamma(\frac{1}{2} - lw)}$$

is more useful for large x. The so-called Koutecky function of polarography (Delahay, 1954) is simply

$$F(y) = \tfrac{1}{2} y f_{3/14}([y/2]^{14/3}).$$

This function describes the current passed by an expanding spherical electrode under control by both diffusion and kinetics.