CHAPTER 5

GENERAL PROPERTIES

In this chapter we examine those properties of differintegral operators which we might expect to generalize classical formulas for derivatives and integrals. It is these properties which will provide our primary means of understanding and utilizing the fractional calculus. As we shall see, while some of these classical properties do generalize without essential change, others require modification of one sort or another. Unless a stipulation is made to the contrary, we assume throughout the rest of the book that all functions encountered are differintegrable in the sense explained in Section 3.1. In the present chapter we shall often find it useful further to restrict our attention to differintegrable series.

5.1 LINEARITY

The linearity of the differintegral operator, by which we mean

$$\frac{d^q[f_1 + f_2]}{[d(x-a)]^q} = \frac{d^q f_1}{[d(x-a)]^q} + \frac{d^q f_2}{[d(x-a)]^q}, \tag{5.1.1}$$

is an immediate consequence of any of the definitions summarized in Section 3.6.

5.2 DIFFERINTEGRATION TERM BY TERM

The linearity of differintegral operators means that they may be distributed through the terms of a finite sum; i.e.,

$$\frac{d^q}{[d(x-a)]^q} \sum_{j=0}^{n} f_j = \sum_{j=0}^{n} \frac{d^q f_j}{[d(x-a)]^q}. \tag{5.2.1}$$

We want to investigate the circumstances that permit term-by-term differintegration of an infinite series of functions. Our major goal in this section is to establish the term-by-term differintegrability of general differintegrable series (see Section 3.1). We shall make repeated use of the classical results on differentiation and integration of infinite series term by term (Section 2.8). In order to use these results we must ensure that the terms f_j of the series are either continuous or continuously differentiable. If we restrict our attention to summands f_j that are differintegrable series, the form of such a series [see equation (3.1.1)] and of its term-by-term derivative shows that the requisite continuity assumptions are valid away from the lower limit $x = a$. Henceforth we shall consider infinite sums of differintegrable series and establish results on the term-by-term differintegrability of such sums which will, in general, be valid in open intervals[1] such as $a < x < a + X$, where X is the radius of convergence of the differintegrable series. First we need to establish some facts about this radius of convergence.

Consider first an ordinary power series,

$$\phi = \sum_{j=0}^{\infty} a_j [x-a]^j, \qquad a_j = \frac{\phi^{(j)}(a)}{j!},$$

convergent for $|x - a| \leqq X$. One knows from classical results that ϕ, together with all of its term-by-term derivatives and integrals, converges uniformly in the interval $0 \leqq |x - a| < X$. What can be said about the series obtained, more generally, from term-by-term differintegration of ϕ? Making use of equation (4.4.4), the series obtained by applying $d^q/[d(x-a)]^q$ to every summand of ϕ is the series

$$\sum_{j=0}^{\infty} \frac{a_j \Gamma(j+1)}{\Gamma(j-q+1)} [x-a]^{j-q} = [x-a]^{-q} \sum_{j=0}^{\infty} \frac{\phi^{(j)}(a)}{\Gamma(j-q+1)} [x-a]^j.$$

We know that the series for ϕ converges for $|x - a| < X$, where, by the ratio test,

$$X \equiv \lim_{j\to\infty} \left| \frac{a_j}{a_{j+1}} \right| = \lim_{j\to\infty} \left| \frac{[j+1]\phi^{(j)}(a)}{\phi^{(j+1)}(a)} \right|,$$

while the differintegrated series will converge for

$$\begin{aligned} |x-a| &< \lim_{j\to\infty} \left| \frac{\phi^{(j)}(a)\Gamma(j-q+2)}{\phi^{(j+1)}(a)\Gamma(j-q+1)} \right| = \lim_{j\to\infty} \left| \frac{[j-q+1]\phi^{(j)}(a)}{\phi^{(j+1)}(a)} \right| \\ &\geqq \lim_{j\to\infty} \left| \frac{[j+1]\phi^{(j)}(a)}{\phi^{(j+1)}(a)} \right| - q \lim_{j\to\infty} \left| \frac{\phi^{(j)}(a)}{\phi^{(j+1)}(a)} \right| \\ &= X - qA, \end{aligned}$$

[1] The classical results of Section 2.8 are stated for closed intervals rather than open ones. Henceforth the phrase " convergence in an open interval " is adopted as a shorthand for " convergence in every closed subinterval of the open interval."

where

$$A \equiv \lim_{j\to\infty}\left|\frac{\phi^{(j)}(a)}{\phi^{(j+1)}(a)}\right|.$$

Examination of the possibilities for A and X reveals that A is invariably negligible in comparison with X so that the differintegrated series converges in the open interval $0 < |x-a| < X$. Furthermore, the same result is valid for the differintegrable series whose jth term is $a_j[x-a]^{j+p}$ (since this series has the same radius of convergence as its analytic part, $\sum a_j[x-a]^j$) and, therefore, for general differintegrable series. That is, if the differintegrable series f, which is a finite sum of functions each representable as

$$[x-a]^p \sum_{j_1=0}^{\infty} a_{j_1}[x-a]^{j_1} + [x-a]^{[np+1]/n} \sum_{j_2=0}^{\infty} a_{j_2}[x-a]^{j_2}$$
$$+ \cdots + [x-a]^{[np+n-1]/n} \sum_{j_n=0}^{\infty} a_{j_n}[x-a]^{j_n},$$

converges for $|x-a| < X$, then so does the series obtained by differintegrating each "unit" term by term, except possibly at the endpoint $x = a$. This fact will be important in what follows.

Let f be any differintegrable series. Since f may be decomposed as a finite sum of differintegrable series units

$$f_U = [x-a]^p \sum_{j=0}^{\infty} a_j[x-a]^j,$$

where $p > -1$ and $a_0 \neq 0$, the term-by-term differintegrability of f will follow from that of f_U. Accordingly, our objective is to establish that

$$\text{(5.2.2)} \qquad \frac{d^q}{[d(x-a)]^q}\left\{[x-a]^p \sum_{j=0}^{\infty} a_j[x-a]^j\right\} = \sum_{j=0}^{\infty} a_j \frac{d^q[x-a]^{p+j}}{[d(x-a)]^q}$$

for all q. More specifically, the equality (5.2.2) will be proven valid inside the interval of convergence of the differintegrable series $\sum a_j[x-a]^{p+j}$.

For $q \leqq 0$ a stronger result that directly extends the classical theorem on term-by-term integration is easy to establish: Suppose the infinite series of differintegrable functions $\sum f_j$ converges uniformly in $0 < |x-a| < X$; then

$$\text{(5.2.3)} \qquad \frac{d^q}{[d(x-a)]^q}\sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} \frac{d^q f_j}{[d(x-a)]^q}, \qquad q \leqq 0,$$

and the right-hand series also converges uniformly in $0 < |x-a| < X$. To demonstrate this result, let

$$f \equiv \sum_{j=0}^{\infty} f_j, \qquad S_N \equiv \sum_{j=0}^{N} f_j.$$

Since $q < 0$, the Riemann–Liouville representations

$$\frac{d^q f}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)\,dy}{[x-y]^{q+1}}, \qquad \frac{d^q f_j}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f_j(y)\,dy}{[x-y]^{q+1}}$$

are valid and

$$\frac{d^q f}{[d(x-a)]^q} - \frac{d^q S_N}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{[f(y) - S_N(y)]\,dy}{[x-y]^{q+1}}.$$

The assumption of uniform convergence means that, given $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that

$$|f(y) - S_n(y)| < \varepsilon$$

for $n > N$ and for all y in the interval $a \leqq y \leqq x$ with $|x - a| < X$. Then

$$\begin{aligned}
\left| \frac{d^q f}{[d(x-a)]^q} - \sum_{j=0}^{N} \frac{d^q f_j}{[d(x-a)]^q} \right| &= \frac{1}{\Gamma(-q)} \left| \int_a^x \frac{[f(y) - S_N(y)]\,dy}{[x-y]^{q+1}} \right| \\
&\leqq \frac{1}{\Gamma(-q)} \int_a^x \frac{|f(y) - S_N(y)|\,dy}{[x-y]^{q+1}} \\
&< \frac{\varepsilon}{\Gamma(-q)} \int_a^x [x-y]^{-q-1}\,dy \\
&= \frac{\varepsilon [x-a]^{-q}}{q\Gamma(-q)},
\end{aligned}$$

which can be made small independently of x in the interval $0 < |x - a| < X$. This proves that $\sum d^q f_j/[d(x-a)]^q$ converges uniformly to $d^q f/[d(x-a)]^q$ in $0 < |x - a| < X$.

The result just established shows that equation (5.2.2) is valid for $q \leqq 0$, and thus, if f is any differintegrable series, the operator $d^q/[d(x-a)]^q$ may be distributed through the several infinite series that define f as long as $q \leqq 0$. Applying result (4.4.4) to equation (5.2.2) gives

$$\text{(5.2.4)} \qquad \frac{d^q f_U}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)}{\Gamma(p-q+j+1)} [x-a]^{p+j-q}, \qquad q \leqq 0.$$

Equations (5.2.2) and (5.2.4) are also valid for $q > 0$, as we now establish. First we decompose the series for f_U into two pieces,

$$f_U = \sum_{j=0}^{\infty} a_j [x-a]^{p+j} = \sum_{j \in J_1} a_j [x-a]^{p+j} + \sum_{j \in J_2} a_j [x-a]^{p+j},$$

where J_1 is the set of nonnegative integers j for which $\Gamma(p - q + j + 1)$ is infinite and J_2 consists of all nonnegative integers not in J_1. For fixed q

the properties of the gamma function (Section 1.3) ensure that the set J_1 has only a finite number of elements. Thus,

$$\frac{d^q f_U}{[d(x-a)]^q} = \frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_1} a_j[x-a]^{p+j}\right\} + \frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\}$$

$$= \sum_{j\in J_1} \frac{d^q[x-a]^{p+j}}{[d(x-a)]^q} + \frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\},$$

making use only of the linearity of $d^q/[d(x-a)]^q$. Now we see that the proof of equation (5.2.4) for $q > 0$ depends only upon establishing that

(5.2.5)

$$\frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\} = \sum_{j\in J_2} a_j \frac{d^q[x-a]^{p+j}}{[d(x-a)]^q}$$

$$= \sum_{j\in J_2} \frac{a_j\Gamma(p+j+1)}{\Gamma(p-q+j+1)}[x-a]^{p+j-q}, \quad q > 0.$$

Assuming that the series for f_U converges uniformly in $0 < |x-a| < X$, so will the series on the right-hand side of equation (5.2.5), as we proved at the beginning of this section. Thus the operator $d^{-1}/[d(x-a)]^{-1}$ may be distributed through the terms of this series to yield

(5.2.6) $$\frac{d^{-1}}{[d(x-a)]^{-1}}\left\{\sum_{j\in J_2} \frac{a_j\Gamma(p+j+1)}{\Gamma(p-q+j+1)}[x-a]^{p+j-q}\right\}$$

$$= \sum_{j\in J_2} \frac{d^{-1}}{[d(x-a)]^{-1}}\left\{\frac{a_j\Gamma(p+j+1)}{\Gamma(p-q+j+1)}[x-a]^{p+j-q}\right\}$$

$$= \sum_{j\in J_2} \frac{a_j\Gamma(p+j+1)\Gamma(p-q+j+1)}{\Gamma(p-q+j+1)\Gamma(p-q+j+2)}[x-a]^{p+j-q+1}$$

$$= \sum_{j\in J_2} \frac{a_j\Gamma(p+j+1)}{\Gamma(p-q+j+2)}[x-a]^{p+j-q+1}$$

$$= \sum_{j\in J_2} a_j \frac{d^{q-1}}{[d(x-a)]^{q-1}}[x-a]^{p+j}$$

and the last series converges uniformly in $0 < |x-a| < X$, as does the series obtained from it by differentiating each term. The cancellation needed to obtain the penultimate expression in equation (5.2.6) may be justified since the definition of the set J_2 guarantees that $\Gamma(p-q+j+1)$ is finite. Applying the classical theorem on term-by-term differentiation (Section 2.8) to the series $\sum a_j\{d^{q-1}/[d(x-a)]^{q-1}\}[x-a]^{p+j}$ gives

$$\frac{d}{dx}\left\{\sum_{j\in J_2} a_j \frac{d^{q-1}}{[d(x-a)]^{q-1}}[x-a]^{p+j}\right\} = \sum_{j\in J_2} a_j \frac{d^q}{[d(x-a)]^q}[x-a]^{p+j}.$$

Arguing similarly we find that

$$\text{(5.2.7)} \quad \frac{d^n}{dx^n}\left\{\sum_{j\in J_2} a_j \frac{d^{q-n}}{[d(x-a)]^{q-n}}[x-a]^{p+j}\right\} = \sum_{j\in J_2} a_j \frac{d^q}{[d(x-a)]^q}[x-a]^{p+j}$$

for every positive integer n. Choosing n to make $q-n<0$ permits us to apply equation (5.2.4), with the result that

$$\sum_{j\in J_2} a_j \frac{d^{q-n}}{[d(x-a)]^{q-n}}[x-a]^{p+j} = \frac{d^{q-n}}{[d(x-a)]^{q-n}}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\}.$$

Differentiating both sides of this equation n times, we see that

$$\frac{d^n}{dx^n}\sum_{j\in J_2} a_j \frac{d^{q-n}}{[d(x-a)]^{q-n}}[x-a]^{p+j} = \frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\}.$$

Utilizing equation (5.2.7) gives, finally,

$$\frac{d^q}{[d(x-a)]^q}\left\{\sum_{j\in J_2} a_j[x-a]^{p+j}\right\} = \sum_{j\in J_2} a_j \frac{d^q}{[d(x-a)]^q}[x-a]^{p+j}, \qquad q>0,$$

as we wanted to show. Thus the representation (5.2.5) is valid for $q>0$ and, hence, for arbitrary q.

We have accomplished our principal goal in this section: to prove the term-by-term differintegrability of arbitrary differintegrable series. In the process we established a generalization (to the operator $d^q/[d(x-a)]^q$ for any $q \leqq 0$) of the classical theorem on term-by-term integration, valid for uniformly convergent series. One may wonder whether a similar generalization to $q>0$ of the classical theorem on term-by-term differentiation is valid. The answer is in the affirmative: If the infinite series $\sum f_j$ as well as the series $\sum d^q f_j/[d(x-a)]^q$ converge uniformly in $0<|x-a|<X$, then

$$\text{(5.2.8)} \qquad \frac{d^q}{[d(x-a)]^q}\sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} \frac{d^q f_j}{[d(x-a)]^q}, \qquad q>0,$$

for $0<|x-a|<X$. The proof of this result makes use of composition rule facts not developed fully until much later (see Sections 5.7 and 8.4) and will be omitted. We note here only that the result does provide a very natural

extension of the classical theorem on term-by-term differentiation (Section 2.8) which also required, in addition to convergence of $\sum f_j$, the uniform convergence of $\sum df_j/dx$.

Equation (5.2.4) provides a useful alternative representation for the differintegral of an analytic function

$$\frac{d^q\phi}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \frac{\phi^{(j)}(a)}{\Gamma(j-q+1)} [x-a]^{j-q}, \tag{5.2.9}$$

where

$$\phi = \sum_{j=0}^{\infty} \frac{\phi^{(j)}(a)}{\Gamma(j+1)} [x-a]^j.$$

The series in (5.2.9) converges uniformly in $0 < |x-a| < X$, where X is the radius of convergence of ϕ. Equation (5.2.9) was previously presented without proof as formula (3.6.6).

5.3 HOMOGENEITY

The proof of homogeneity,

$$\frac{d^q[Cf]}{[d(x-a)]^q} = C\frac{d^qf}{[d(x-a)]^q}, \qquad C \quad \text{any constant}, \tag{5.3.1}$$

follows directly from the definition (3.6.1) since the constant C may be brought outside the sum and limit.

5.4 SCALE CHANGE

By a scale change of the function f with respect to a lower limit a, we mean its replacement by $f(\beta x - \beta a + a)$, where β is a constant termed the scaling factor. To clarify this definition, consider $a = 0$; then the scale change converts $f(x)$ to $f(\beta x)$, in contrast to the homogeneity operation of the previous section which converted $f(x)$ to $Cf(x)$.

In this section we seek a procedure by which the effect of the generalized $d^q/[d(x-a)]^q$ operation upon $f(\beta x - \beta a + a)$ can be found, if $d^qf/[d(x-a)]^q$ is known. We shall find it convenient to use the abbreviation

$$X \equiv x + [a - a\beta]/\beta$$

and to adopt the Riemann–Liouville definition (3.6.2). Using Y as a replacement for $\beta y - \beta a + a$, we proceed as follows:

$$\frac{d^q f(\beta X)}{[d(x-a)]^q} = \frac{d^q f(\beta x - \beta a + a)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(\beta y - \beta a + a)\, dy}{[x-y]^{q+1}} \tag{5.4.1}$$

$$= \frac{1}{\Gamma(-q)} \int_a^{\beta X} \frac{f(Y)[dY/\beta]}{\{[\beta X - Y]/\beta\}^{q+1}} = \frac{\beta^q}{\Gamma(-q)} \int_a^{\beta X} \frac{f(Y)\, dY}{[\beta X - Y]^{q+1}}$$

$$= \beta^q \frac{d^q f(\beta X]}{[d(\beta X - a)]^q}.$$

The utility of formula (5.4.1) is greatest when $a = 0$, for then $X = x$ and the scale change is simply a multiplication of the independent variable by a constant, the formula being

$$\frac{d^q f(\beta x)}{[dx]^q} = \beta^q \frac{d^q f(\beta x)}{[d(\beta x)]^q}. \tag{5.4.2}$$

This result may also be found in the work of Erdélyi *et al.* (1954). When a is nonzero, the effect of replacing $f(x)$ by $f(\beta x)$ requires, in addition to a scale change, the algebraically more difficult translation process, consideration of which will be deferred until Section 5.9.

5.5 LEIBNIZ'S RULE

The rule for differentiation of a product of two functions is a familar result in elementary calculus. It states that

$$\frac{d^n[fg]}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^{n-j} f}{dx^{n-j}} \frac{d^j g}{dx^j} \tag{5.5.1}$$

and is, of course, restricted to nonnegative integers n. In Section 2.5 we have derived, based on integration by parts, the following product rule for multiple integrals:

$$\frac{d^{-n}[fg]}{[d(x-a)]^{-n}} = \sum_{j=0}^{\infty} \binom{-n}{j} \frac{d^{-n-j} f}{[d(x-a)]^{-n-j}} \frac{d^j g}{[d(x-a)]^j}.$$

When we observe that the finite sum in (5.5.1) could equally well extend to infinity $\left[\text{since } \binom{n}{j} = 0 \text{ for } j > n\right]$ we might expect the product rule to generalize to arbitrary order q as

$$\frac{d^q[fg]}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \binom{q}{j} \frac{d^{q-j} f}{[d(x-a)]^{q-j}} \frac{d^j g}{[d(x-a)]^j}. \tag{5.5.2}$$

That such a generalization is indeed valid for real analytic functions $\phi(x)$ and $\psi(x)$ will now be established.

Starting with equation (4.1.2) and substituting for ϕ the product $\phi\psi$, we obtain

$$\frac{d^q[\phi\psi]}{[d(x-a)]^q} = \sum_{k=0}^{\infty} \binom{q}{k} \frac{d^{q-k}[1]}{[d(x-a)]^{q-k}} [\phi\psi]^{(k)}$$
$$= \sum_{k=0}^{\infty} \binom{q}{k} \frac{d^{q-k}[1]}{[d(x-a)]^{q-k}} \sum_{j=0}^{k} \binom{k}{j} \phi^{(k-j)}\psi^{(j)},$$

making use of (5.5.1). Note that, since j is an integer, the repeated derivative $\psi^{(j)}$ with respect to x equals that with respect to $x - a$. The permutation (see footnote 5 in Section 2.5)

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \tag{5.5.3}$$

may be applied to give

$$\frac{d^q[\phi\psi]}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \psi^{(j)} \sum_{k=j}^{\infty} \binom{q}{k} \binom{k}{j} \frac{d^{q-k}[1]}{[d(x-a)]^{q-k}} \phi^{(k-j)}$$
$$= \sum_{j=0}^{\infty} \psi^{(j)} \sum_{l=0}^{\infty} \binom{q}{l+j} \binom{l+j}{j} \frac{d^{q-j-l}[1]}{[d(x-a)]^{q-j-l}} \phi^{(l)}$$
$$= \sum_{j=0}^{\infty} \binom{q}{j} \psi^{(j)} \sum_{l=0}^{\infty} \binom{q-j}{l} \frac{d^{q-j-l}[1]}{[d(x-a)]^{q-j-l}} \phi^{(l)}$$
$$= \sum_{j=0}^{\infty} \binom{q}{j} \psi^{(j)} \frac{d^{q-j}\phi}{[d(x-a)]^{q-j}},$$

where we have made use of the identity[2]

$$\binom{q}{l+j}\binom{l+j}{j} = \binom{q}{j}\binom{q-j}{l}$$

and a second application of (4.1.2). Since we established (3.5.3) under the assumption that ϕ is a real analytic function and used (3.5.3) to prove (4.1.2), which in turn was used to establish (5.5.2), the latter is proven only if ϕ and ψ are real analytic functions.

Nevertheless, a somewhat different argument may be used to establish equation (5.5.2) when one of the functions is a polynomial. The argument

[2] This identity is immediately apparent when the binomial coefficients are replaced by gamma function equivalents.

begins with consideration of the product $xf(x)$ and $q < 0$. Making use of the Riemann-Liouville definition,

$$\frac{d^q[xf]}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x \frac{yf(y)\,dy}{[x-y]^{q+1}} + \frac{x}{\Gamma(-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q+1}} - \frac{x}{\Gamma(-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q+1}} \tag{5.5.4}$$

$$= \frac{x}{\Gamma(-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q+1}} - \frac{1}{\Gamma(-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q}}$$

$$= x\frac{d^qf}{[d(x-a)]^q} + q\frac{d^{q-1}f}{[d(x-a)]^{q-1}},$$

where, in the last step, use was made of the recurrence formula (1.3.2) for the gamma function. Extension of this result to $q \geqq 0$ is now quite easy, for if $n - 1 \leqq q < n$, $n = 1, 2, 3, \ldots$, then

$$\frac{d^q[xf]}{[d(x-a)]^q} = \frac{d^n}{dx^n}\left\{\frac{d^{q-n}[xf]}{[d(x-a)]^{q-n}}\right\}$$

$$= \frac{d^n}{dx^n}\left\{x\frac{d^{q-n}f}{[d(x-a)]^{q-n}} + [q-n]\frac{d^{q-n-1}f}{[d(x-a)]^{q-n-1}}\right\}$$

$$= x\frac{d^qf}{[d(x-a)]^q} + n\frac{d^{q-1}f}{[d(x-a)]^{q-1}} + [q-n]\frac{d^{q-1}f}{[d(x-a)]^{q-1}}$$

$$= x\frac{d^qf}{[d(x-a)]^q} + q\frac{d^{q-1}f}{[d(x-a)]^{q-1}}$$

as before. Equation (3.2.5) has been used repeatedly in the preceding derivation. An inductive argument establishes equation (5.5.2) when $g = x^k$ for any nonnegative integer k and any f, thus, for g any polynomial and f arbitrary.

When g is a polynomial in $x - a$ in equation (5.5.2) the sum on the right-hand side is, of course, finite. Convergence difficulties are encountered, however, when one tries to extend the previous argument to the case f arbitrary and g analytic. An example that illustrates this is obtained by considering $f = 1$, $g = [1-x]^{-1} = 1 + x + x^2 + \cdots$ for $|x| < 1$. Then

$$\frac{d^q[fg]}{dx^q} = \frac{d^q\{[1-x]^{-1}\}}{dx^q} = \frac{d^q}{dx^q}\sum_{k=0}^{\infty} x^k, \qquad |x| < 1.$$

Term-by-term differintegration is justified by the results of Section 5.2 since

$\sum x^k$ is certainly a differintegrable (in fact, an analytic) series. Therefore,

$$\frac{d^q}{dx^q}\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \frac{d^q x^k}{dx^q} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-q+1)} x^{k-q},$$

which converges for $0 < |x| < 1$. On the other hand, use of Leibniz's rule would give

$$\begin{aligned}\frac{d^q}{dx^q}\sum_{k=0}^{\infty} x^k &= \sum_{j=0}^{\infty} \binom{q}{j} \frac{d^{q-j}[1]}{dx^{q-j}} \frac{d^j\{[1-x]^{-1}\}}{dx^j} \\ &= \sum_{j=0}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)} \frac{x^{j-q}}{\Gamma(j-q+1)} \Gamma(j+1)[1-x]^{-j-1} \\ &= \frac{\Gamma(q+1)x^{-q}}{1-x} \sum_{j=0}^{\infty} \frac{[-]^j}{[j-q]\pi\csc(\pi q)} \frac{x^j}{[1-x]^j},\end{aligned}$$

which converges only for $x \leqq \frac{1}{2}$.

Leibniz's rule has been thoroughly studied recently by Osler (1970a, 1971, 1972b, 1972c). He was led to wonder whether equation (5.5.2) is a special case of a still more general result in which the interchangeability of f and g is more apparent. The more general result proved by Osler is

$$(5.5.5) \qquad \frac{d^q[fg]}{dx^q} = \sum_{j=-\infty}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-\gamma-j+1)\Gamma(\gamma+j+1)} \frac{d^{q-\gamma-j}f}{dx^{q-\gamma-j}} \frac{d^{\gamma+j}g}{dx^{\gamma+j}},$$

where γ is arbitrary, which reduces to (5.5.2) with $a = 0$ when $\gamma = 0$. Watanabe derived equation (5.5.5) in 1931, but his method does not yield the precise region of convergence in the complex plane. Osler pointed this out in his paper (1970a) and used the Cauchy integral formula representation [epuation (3.4.3)] for fractional derivatives to delineate the appropriate region of convergence (a star-shaped region in the complex plane) for the formula (5.5.5). He also used (5.5.5) to generate certain infinite series expansions interrelating special functions of mathematical physics. The interested reader is referred to the work of Osler (1970a, 1972b) for further details.

A further generalization of Leibniz's rule due to Osler (1972c) is the integral form

$$(5.5.6) \qquad \frac{d^q[fg]}{dx^q} = \int_{-\infty}^{\infty} \frac{\Gamma(q+1)}{\Gamma(q-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)} \frac{d^{q-\gamma-\lambda}f}{dx^{q-\gamma-\lambda}} \frac{d^{\gamma+\lambda}g}{dx^{\gamma+\lambda}}\, d\lambda$$

in which a discrete sum is replaced by an integral. Formula (5.5.6) has been used by Osler to obtain a generalization of Parseval's integral formula [see Titchmarsh (1948)] of Fourier analysis. As one might expect, equation (5.5.6) is also useful in deriving formulas for definite integrals. A table of these is presented in Osler's work (1972c).

5.6 CHAIN RULE

The chain rule for differentiation,

$$\frac{d}{dx} g(f(x)) = \frac{d}{df(x)} g(f(x)) \frac{d}{dx} f(x),$$

lacks a simple counterpart in the integral calculus. Indeed, if there were such a counterpart, the process of integration would pose no greater difficulty than does differentiation. Since any general formula for $d^q g(f(x))/[d(x-a)]^q$ must encompass integration as a special case, little hope can be held out for a useful chain rule for arbitrary q. Nevertheless, as we shall see, a formal chain rule may be derived quite simply.

We start with formula (4.1.2) valid for an analytic function ϕ:

$$\frac{d^q \phi}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \binom{q}{j} \frac{d^{q-j}[1]}{[d(x-a)]^{q-j}} \frac{d^j \phi}{dx^j}.$$

The formula (4.1.1) permits the evaluation of the effect of the differintegral operator upon unity, allowing us to write

$$\frac{d^q \phi}{[d(x-a)]^q} = \frac{[x-a]^{-q}}{\Gamma(1-q)} \phi + \sum_{j=1}^{\infty} \binom{q}{j} \frac{[x-a]^{j-q}}{\Gamma(j-q+1)} \frac{d^j \phi}{dx^j},$$

where the $j = 0$ term has been separated from the others. Up to this point we have regarded ϕ as a function of x. Now we consider $\phi = \phi(f(x))$ and evaluate $d^j \phi(f(x))/dx^j$ by Faà de Bruno's formula, as developed in Section 2.6:

$$\frac{d^q}{[d(x-a)]^q} \phi(f(x)) = \frac{[x-a]^{-q}}{\Gamma(1-q)} \phi(f(x))$$
$$+ \sum_{j=1}^{\infty} \binom{q}{j} \frac{[x-a]^{j-q}}{\Gamma(j-q+1)} j! \sum_{m=1}^{j} \phi^{(m)} \sum \prod_{k=1}^{j} \frac{1}{P_k!} \left[\frac{f^{(k)}}{k!}\right]^{P_k},$$

where the third summation and the P_k's have the significances explained in Section 2.6.

The complexity of this result will inhibit its general utility. We see on inserting $q = -1$ that even for the case of a single integration,

$$\int_a^x \phi(f(y))\, dy = [x-a]\phi(f(x))$$
$$+ \sum_{j=1}^{\infty} [-]^j \frac{[x-a]^{j+1}}{j+1} \sum_{m=1}^{j} \phi^{(m)} \sum \prod_{k=1}^{j} \frac{1}{P_k!} \left[\frac{f^{(k)}}{k!}\right]^{P_k},$$

the chain rule gives an infinite series that offers little hope of being expressible in closed form, except for trivially simple instances of the functions f and ϕ.

A case in which the generalized chain rule could be of limited utility is provided by $f(x) = \exp(x)$. Then

$$\frac{d^q}{[d(x-a)]^q}\phi(\exp(x)) = \frac{[x-a]^{-q}}{\Gamma(1-q)}\phi(\exp(x))$$
$$+ \sum_{j=1}^{\infty}\binom{q}{j}\frac{[x-a]^{j-q}}{\Gamma(j-q+1)}\exp(jx)\sum_{m=1}^{j}S_j^{[m]}\phi^{(m)},$$

where $S_j^{[m]}$ is a Stirling number of the second kind (see the discussion in Section 1.3).

Extension of the chain rule in a somewhat different direction has recently been accomplished by Osler (1970b). For composite functions $f(x) = F(h(x))$ his aim was to generalize to fractional derivatives the classical formula

$$\frac{d^N f(x)}{dx^N} = \sum_{n=0}^{N}\frac{U_n(x)}{n!}\frac{d^n f(x)}{[dh(x)]^n},$$

where

$$U_n(x) = \sum_{r=0}^{n}\binom{n}{r}[-h(x)]^r\frac{d^N}{dx^N}h(x)^{n-r}.$$

He obtained such a generalization and, with the help of this and the generalized Leibniz rule (5.5.5), he derived some known and some apparently new relationships involving special functions of mathematical physics.

The absence of a simple chain rule impedes the process of differintegration with respect to a variable other than the argument of the differintegrand. In other words, considering here only those cases where the lower limit $a = 0$, it is difficult to relate

$$\frac{d^q}{dX^q}f(x) \qquad \text{to} \qquad \frac{d^q}{dx^q}f(x)$$

even where X is a very simple function of x. Of course if $X = \beta x$, where β is a constant, the scale change theorem of Section 5.4 does provide such a relationship.

Another useful case in which a relationship may be derived is the case $X = x^n$, n being a positive integer, and f being one of the many functions expressible as a generalized hypergeometric function (Section 2.10 and Chapter 9). In that event, we have

$$\frac{d^q}{dX^q}f(x) = \frac{d^q}{dX^q}\left[X^{1/n}\,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]$$

so that the differintegration may be carried out via the mediation of equation (2.10.4).

5.7 COMPOSITION RULE

In seeking a general composition rule for the operator $d^q/[d(x-a)]^q$ we search for the relationship between

$$\frac{d^q}{[d(x-a)]^q}\frac{d^Q f}{[d(x-a)]^Q} \quad \text{and} \quad \frac{d^{q+Q}f}{[d(x-a)]^{q+Q}},$$

which we temporarily abbreviate to $d^q d^Q f$ and $d^{q+Q}f$. Of course, if these symbols are to be generally meaningful we need to assume not only that f is differintegrable but that $d^Q f$ is differintegrable as well. In the present section we restrict attention to differintegrable series as defined in Section 3.1.

We saw in Section 3.1 that the most general nonzero differintegrable series is a finite sum of differintegrable "units," each having the form

$$f_U = [x-a]^p \sum_{j=0}^{\infty} a_j [x-a]^j, \qquad p > -1, \quad a_0 \neq 0. \tag{5.7.1}$$

We shall see that the composition rule may be valid for some units of f but possibly not for others. It follows from the linearity of differintegral operators that

$$d^q d^Q f = d^{q+Q} f \tag{5.7.2}$$

if

$$d^q d^Q f_U = d^{q+Q} f_U \tag{5.7.3}$$

for every unit f_U of f. Accordingly we shall first assess the validity of the composition rule (5.7.3) for a differintegrable series unit function f_U.

Obviously, if $f_U = 0$, then $d^Q f_U = 0$ for every Q by equation (4.2.2), and so

$$d^q d^Q[0] = d^{q+Q}[0] = 0.$$

While the composition rule is trivially satisfied for the differintegrable function $f_U = 0$, we shall see that the possibility

$$f_U \neq 0, \qquad \text{but} \qquad d^Q f_U = 0,$$

is exactly the condition that prevents the composition rule (5.7.3), and therefore (5.7.2), from being satisfied generally.

Having dealt with the case $f_U = 0$ we now assume $f_U \neq 0$ and use equation (5.2.4) to evaluate $d^Q f_U$:

$$d^Q f_U = \sum_{j=0}^{\infty} a_j d^Q [x-a]^{p+j} = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)[x-a]^{p+j-Q}}{\Gamma(p+j-Q+1)}. \tag{5.7.4}$$

Furthermore, we note that since $p > -1$, it follows that $p + j > -1$ so that $\Gamma(p + j + 1)$ is always finite but nonzero. Individual terms in $d^Q f_U$ will vanish, therefore, only when the coefficient a_j is zero or when the denominatorial gamma function $\Gamma(p + j - Q + 1)$ is infinite. We see, then, that a necessary and sufficient condition for $d^Q f_U \neq 0$ is

(5.7.5) $\Gamma(p + j + 1 - Q)$ is finite for each j for which $a_j \neq 0$.

This awkward condition (5.7.5) may be shown to be equivalent to

$$f_U - d^{-Q} d^Q f_U = 0; \tag{5.7.6}$$

that is, to the condition that the differintegrable unit f_U be regenerated upon the application, first of d^Q, then d^{-Q}. Assuming (5.7.6) temporarily, we find that d^q may then be applied to equation (5.7.4) to give

$$d^q d^Q f_U = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p + j + 1)\Gamma(p + j - Q + 1)[x - a]^{p+j-Q-q}}{\Gamma(p + j - Q + 1)\Gamma(p + j - Q - q + 1)}, \tag{5.7.7}$$

by another application of equation (5.2.4) valid since $d^Q f_U$ was assumed to be a differintegrable series. With the condition (5.7.6) [or its equivalent (5.7.5)] in effect, we may safely cancel the $\Gamma(p + j - Q + 1)$ factors in (5.7.7), arriving at

$$d^q d^Q f_U = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p + j + 1)[x - a]^{p+j-Q-q}}{\Gamma(p + j - Q - q + 1)}. \tag{5.7.8}$$

On the other hand, the same technique shows that

$$d^{q+Q} f_U = \sum_{j=0}^{\infty} a_j d^{q+Q} f_U = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p + j + 1)[x - a]^{p+j-Q-q}}{\Gamma(p + j - Q - q + 1)} = d^q d^Q f_U.$$

Thus, the composition rule (5.7.3) is obeyed for the unit f_U as long as condition (5.7.6) is satisfied.[3] However, when (5.7.6) is violated, $d^Q f_U = 0$ so that $d^q d^Q f_U = 0$. On the other hand, it is not necessarily the case that $d^{q+Q} f_U = 0$. For example, we may choose $f_U = x^{-\frac{1}{2}}$, $a = 0$, $Q = \frac{1}{2}$, and $q = -\frac{1}{2}$. Then

$$f_U - d^{-Q} d^Q f_U = x^{-\frac{1}{2}} - d^{-\frac{1}{2}} d^{\frac{1}{2}} x^{-\frac{1}{2}} = x^{-\frac{1}{2}} - d^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(0)} x^{-1} = x^{-\frac{1}{2}} \neq 0$$

so that condition (5.7.6) is certainly violated. Therefore $d^Q f_U = 0$ and $d^q d^Q f_U = 0$ while $d^{q+Q} f_U = d^0 x^{-\frac{1}{2}} = x^{-\frac{1}{2}} \neq 0$. Generalizing, we easily see the

[3] Examination of condition (5.7.5) shows that it, and therefore (5.7.6), are invariably satisfied as long as $Q < 0$ (in fact, as long as $Q < p + 1$) since, in that case, $\Gamma(p + j + 1 - Q)$ is necessarily finite for all j. We see, then, that $d^q d^Q f_U = d^{q+Q} f_U$ at least whenever $Q < 0$, and even when $Q < 1$ if f_U is bounded at the lower limit a.

relationship between $d^q d^Q f_U$ and $d^{q+Q} f_U$ in the case $f_U - d^{-Q} d^Q f_U \neq 0$ to be

$$0 = d^q d^Q f_U = d^{q+Q} f_U - d^{q+Q}\{f_U - d^{-Q}\, d^Q f_U\}. \tag{5.7.9}$$

The preceding discussion for differintegrable units f_U is summarized in Table 5.7.1.

Table 5.7.1. *Summary of the composition rule*[a] *for differintegrable units* f_U

	$f_U = 0$	$f_U \neq 0$
$d^Q f_U = 0$	$f_U - d^{-Q} d^Q f_U = 0$ $d^q d^Q f_U = d^{q+Q} f_U = 0$	$f_U - d^{-Q} d^Q f_U \neq 0$ $0 = d^q d^Q f_U = d^{q+Q} f_U$ $- d^{q+Q}[f_U - d^{-Q} d^Q f_U]$
$d^Q f_U \neq 0$	Not attainable	$f_U - d^{-Q} d^Q f_U = 0$ $d^q d^Q f_U = d^{q+Q} f_U.$

[a] The requirements for obedience to this rule are that f_U and $d^Q f_U$ both be differintegrable.

While equation (5.7.9) is a trivial identity for differintegrable units, we shall see that it is less trivial and, therefore, more useful for general differintegrable series. Because equation (5.7.2) is valid for general differintegrable series f if and only if equation (5.7.3) is valid for every differintegrable unit f_U of f, it is straightforward to apply the theory just developed for units f_U to obtain the composition rule for general f. The only difference is that while the conditions

$$f_U \neq 0 \qquad \text{and} \qquad f_U - d^{-Q}\, d^Q f_U = 0 \tag{5.7.10}$$

for units f_U guaranteed that $d^Q f_U \neq 0$, this is no longer the case for arbitrary f. The reason, of course, is that some units of f may satisfy (5.7.10) while others do not. This will make it possible to violate the composition rule (5.7.2) even though

$$f \neq 0 \qquad \text{and} \qquad d^Q f \neq 0.$$

The condition

$$f - d^{-Q}\, d^Q f = 0 \tag{5.7.11}$$

for general differintegrable series f is, however, still necessary and sufficient to guarantee (5.7.2). We mention in passing that for general differintegrable f, as was the case for differintegrable units f_U,

$$d^q\, d^Q f = d^{q+Q} f,$$

at least when $Q < 0$ (see footnote 3 earlier in this section) and even when $Q < 1$ for functions f bounded at $x = a$. The facts for general differintegrable series f are summarized in Table 5.7.2.

Table 5.7.2. *Summary of the composition rule[a] for arbitrary differintegrable functions, f*

	$f = 0$	$f \neq 0$
$d^Q f = 0$	$f - d^{-Q} d^Q f = 0$ $d^q d^Q f = d^{q+Q} f = 0$	$f - d^{-Q} d^Q f \neq 0$ $0 = d^q d^Q f = d^{q+Q} f$ $- d^{q+Q}[f - d^{-Q} d^Q f]$
$d^Q f \neq 0$	Not attainable	If $f - d^{-Q} d^Q f = 0$, then $d^q d^Q f = d^{q+Q} f$ If $f - d^{-Q} d^Q f \neq 0$, then $d^q d^Q f = d^{q+Q} f$ $- d^{q+Q}[f - d^{-Q} d^Q f]$

[a] The requirements for obedience to this rule are that f and $d^Q f$ both be differintegrable.

We have noticed previously that, in cases where the composition rule is violated, the equation

(5.7.12) $$d^q\, d^Q f = d^{q+Q} f - d^{q+Q}\{f - d^{-Q}\, d^Q f\}$$

relates $d^q\, d^Q f$ to $d^{q+Q} f$. The utility of equation (5.7.12) as a means of calculating its left-hand side is marginal in the general case. However, there is one important instance in which equation (5.7.12) is very useful: the case when $Q = N$, a positive integer. Indeed, we may then use equations (2.3.12) and (4.4.4) to see that

(5.7.13)
$$d^q\, d^N f = d^{q+N} f - d^{q+N}\{f - d^{-N}\, d^N f\} = d^{q+N} f - \sum_{k=0}^{N-1} \frac{[x - a]^{k-q-N} f^{(k)}(a)}{\Gamma(k - q - N + 1)}.$$

Furthermore, equation (5.7.13) can be established even under the relaxed assumptions[4] that f be N-fold differentiable and that $f^{(k)}(a)$ be finite, $k = 0, 1, \ldots, N - 1$. We omit the proof of this assertion, which makes use of the representation (3.6.4). Table 5.7.3 summarizes the composition rule facts for functions f that are N-fold differentiable and whose Nth derivatives are differintegrable.

[4] Since functions such as x^p, $p \leqq -1$ are differentiable (with derivative px^{p-1}) but not differintegrable.

Table 5.7.3. *Summary of the composition rule*[a] *for differentiable (but not necessarily differintegrable) functions* f

	$f=0$	$f\neq 0$
$d^N f=0$	$f-d^{-N}d^N f=0$ $d^q d^N f=d^{q+N}f=0$	$f-d^{-N}d^N f\neq 0$ $0=d^q d^N f=d^{q+N}f-\sum_{k=0}^{N-1}\frac{[x-a]^{k-q-N}f^{(k)}(a)}{\Gamma(k-q-N+1)}$
$d^N f\neq 0$	Not attainable	If $f-d^{-N}d^N f=0$, then $d^q d^N f=d^{q+N}f$ If $f-d^{-N}d^N f\neq 0$, then $d^q d^N f=d^{q+N}f-\sum_{k=0}^{N-1}\frac{[x-a]^{1-q-N}f^{(k)}(a)}{\Gamma(k-q-N+1)}$

[a] The requirements for obedience to this rule are that f be N-fold differentiable and that $d^N f$ be differintegrable.

The general facts just presented about composing d^Q with d^q make it clear that, while the operators d^Q and d^{-Q} are *usually* inverse to each other, this is not always the case. In fact, as we have pointed out repeatedly, one need not look beyond integer orders to find illustrations of this. Indeed, if we choose $f_U = x$, then

$$\frac{d^2 f_U}{dx^2}=0 \qquad \text{and} \qquad \frac{d^{-2}}{dx^{-2}}\frac{d^2 f_U}{dx^2}=0$$

so the the operators d^{-2} and d^2 are certainly not inverse to each other. The difficulty, of course, is that condition (5.7.6) is violated for the differintegrable unit $f_U = x$. Nor is this problem restricted to integer q, Q. In fact, if we choose $Q=\frac{3}{2}$, $q=-\frac{3}{2}$, $a=0$, and $f_U=x^{\frac{1}{2}}$, then from equation (4.4.4) we see that

$$d^Q f_U=\frac{d^{\frac{3}{2}}x^{\frac{1}{2}}}{dx^{\frac{3}{2}}}=0 \qquad \text{so that} \qquad d^q\, d^Q f_U=\frac{d^{-\frac{3}{2}}}{dx^{-\frac{3}{2}}}\frac{d^{\frac{3}{2}}x^{\frac{1}{2}}}{dx^{\frac{3}{2}}}=0 \qquad \text{once again,}$$

while, as we well know,

$$d^{q+Q}f=\frac{d^0 f_U}{dx^0}=f_U=x^{\frac{1}{2}}.$$

This time, too, we have chosen $f_U\neq 0$ but $d^Q f_U=0$ which guarantees the violation of condition (5.7.6). Finally, to exemplify the possibility of the failure of the composition law even when $f\neq 0$, $d^Q f\neq 0$, we choose $f=\sqrt{x}+1$,

$N = 1$, $a = 0$, and arbitrary q. This time f is the sum of two differintegrable units and $df = \frac{1}{2}x^{-\frac{1}{2}}$, yet

$$d^q\, df = \frac{\Gamma(\frac{1}{2})}{2\Gamma(-q-\frac{1}{2})}\, x^{-q-\frac{1}{2}}$$

and

$$\begin{aligned} d^{q+1}f &= \frac{\Gamma(\frac{3}{2})}{\Gamma(-q-\frac{1}{2})}\, x^{-q-\frac{1}{2}} + \frac{1}{\Gamma(-q)}\, x^{-q-1} \\ &= \frac{\Gamma(\frac{1}{2})}{2\Gamma(-q-\frac{1}{2})}\, x^{-q-\frac{1}{2}} + \frac{1}{\Gamma(-q)}\, x^{-q-1}. \end{aligned}$$

As must be so, the condition (5.7.11) is violated ($f - d^{-1}\,df = 1$) and equation (5.7.13) correctly relates $d^q\,df$ and $d^{q+1}f$.

The discussion just completed also reveals the dangers that lurk when one inquires about the commutativity of differintegral operators ($d^q\, d^Q = d^Q\, d^q$) or the invertibility of differintegration ($d^Q f = g$ implies $f = d^{-Q}g$). Of course the latter comes into play in attempting to solve differintegral equations of arbitrary order; we shall have more to say about this subject in Chapter 9. The results of the present section may be used to derive conditions under which commutativity holds, provided, as always, that the operators are applied to a suitably restricted class of functions.

5.8 DEPENDENCE ON LOWER LIMIT

We have postponed until now any discussion of the manner in which $d^qf/[d(x-a)]^q$ depends on the lower limit a. In the present section we derive a formula that exhibits this dependence in a rather concise way, at least for analytic functions ϕ.

Assume $a < b < x$ and that $\phi(y)$ is any function analytic in the interval $a \leqq y \leqq x$. Writing

$$\Delta = \frac{d^q\phi}{[d(x-a)]^q} - \frac{d^q\phi}{[d(x-b)]^q}, \tag{5.8.1}$$

one might expect that the difference Δ is expressible in terms of integrals of $\phi(y)$ confined to the interval $a \leqq y \leqq b$. The correct formula is

$$\Delta = \sum_{l=1}^{\infty} \frac{d^{q+l}[1]}{[d(x-b)]^{q+l}}\, \frac{d^{-l}\phi(b)}{[d(b-a)]^{-l}}. \tag{5.8.2}$$

To prove this we first establish equation (5.8.2) for $q < 0$ making use of (3.6.2), and then utilize the identity theorem for analytic functions to argue that the result is valid in general.

Thus, assuming $q < 0$, we write

$$\begin{aligned}
\Delta &= \frac{1}{\Gamma(-q)}\int_a^x \frac{\phi(y)\,dy}{[x-y]^{q+1}} - \frac{1}{\Gamma(-q)}\int_b^x \frac{\phi(y)\,dy}{[x-y]^{q+1}} \\
&= \frac{1}{\Gamma(-q)}\int_a^b \frac{\phi(y)\,dy}{[x-y]^{q+1}} \\
&= \frac{1}{\Gamma(-q)}\int_a^b \frac{\phi(y)\,dy}{[x-b+b-y]^{q+1}} \\
&= \frac{1}{\Gamma(-q)}\int_a^b \left[\sum_{l=0}^{\infty}\binom{-1-q}{l}[x-b]^{-1-q-l}[b-y]^l\right]\phi(y)\,dy
\end{aligned} \tag{5.8.3}$$

upon binomial expansion of $[x-b+b-y]^{-1-q}$. When j is replaced by l and q by $q+l$, equation (1.3.16) becomes

$$\binom{-1-q}{l} = \frac{\Gamma(-q)}{\Gamma(-q-l)\Gamma(l+1)}. \tag{5.8.4}$$

Putting (5.8.4) into (5.8.3) produces

$$\begin{aligned}
\Delta &= \int_a^b \sum_{l=0}^{\infty} \frac{[x-b]^{-1-q-l}}{\Gamma(-q-l)}\,\frac{[b-y]^l}{\Gamma(l+1)}\,\phi(y)\,dy \\
&= \sum_{l=0}^{\infty} \frac{d^{q+l+1}[1]}{[d(x-b)]^{q+l+1}} \int_a^b \frac{\phi(y)\,dy}{\Gamma(l+1)[b-y]^{-l}} \\
&= \sum_{l=0}^{\infty} \frac{d^{q+l+1}[1]}{[d(x-b)]^{q+l+1}}\,\frac{d^{-l-1}\phi(b)}{[d(b-a)]^{-l-1}} = \sum_{l=0}^{\infty} \frac{d^{q+l}[1]}{[d(x-b)]^{q+l}}\,\frac{d^{-l}\phi(b)}{[d(b-a)]^{-l}},
\end{aligned}$$

establishing (5.8.2) for $q < 0$. We now argue that, since both sides of equation (5.8.2) are analytic in q, the equation is valid for all q by the identity theorem for analytic functions (see the discussion in Section 3.2).

Some special cases of formula (5.8.2) require consideration. When q is zero or a positive integer, all the derivatives of unity in the formula vanish, so that

$$\Delta = 0, \qquad q = 0, 1, 2, \ldots.$$

When $q = -1$, all the derivatives vanish except for the first, whence

$$\Delta = \frac{d^{-1}\phi(b)}{[d(b-a)]^{-1}}\int_a^b \phi(y)\,dy, \qquad q = -1.$$

For all other values of q, Δ is nonzero and its value depends not only upon a and b, but also on x. For example, when q is a negative integer $-n$, we find

$$\Delta = \sum_{l=1}^{n} \frac{d^{l-n}[1]}{[d(x-b)]^{l-n}} \frac{d^{-l}\phi(b)}{[d(b-z)]^{-l}}$$

$$= \sum_{l=1}^{n} \frac{[x-b]^{n-l}}{\Gamma(1-l+n)} \frac{d^{-l}\phi(b)}{[d(b-a)]^{-l}}, \qquad q-n=-1,\,-2,\,-3,\ldots.$$

This result is identical with that derived in Section 2.4 by a classical argument.

5.9 TRANSLATION

By the translation of a function f we mean the replacement of $f(x)$ by the function $f(A+x)$, where A is a constant which we take to be positive. In this section we seek a rule for evaluating the effect of a differintegral operator on a translated function, that is, we seek to relate

$$\frac{d^q f(A+x)}{[d(x-a)]^q} \quad \text{to} \quad \frac{d^q f(x)}{[d(x-a)]^q},$$

assuming, of course, that f is defined wherever needed to have these differintegrals make sense, i.e., between $\min(a, a+A)$ and $\max(x, x+A)$.

As usual, the Riemann-Liouville definition is the most tractable and from it we see immediately that

$$\frac{d^q f(x+A)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y+A)\,dy}{[x-y]^{q+1}} = \frac{1}{\Gamma(-q)} \int_{a+A}^{x+A} \frac{f(Y)\,dY}{[x+A-y]^{q+1}},$$

where $Y = y + A$. It is evident that translation by a distance A is equivalent to a shift in the upper limit from x to $x+A$ and a shift in the lower limit from a to $a+A$. Representing the effect of the latter shift by Δ, we therefore find

(5.9.1)

$$\frac{d^q f(x+A)}{[d(x-a)]^q} = \frac{d^q f(x+A)}{[d(x+A-a)]^q} - \Delta$$

$$= \frac{d^q f(x+A)}{[d(x+A-a)]^q} - \sum_{l=1}^{\infty} \frac{d^{q+l}[1]}{[d(x+A-a)]^{q+l}} \frac{d^{-l}f(a+A)}{[d(a+A-a)]^{-l}},$$

where the results of Section 5.8 have been used to evaluate Δ. Though the use of the Riemann–Liouville definition requires $q < 0$, the usual analyticity argument based on the identity theorem serves to remove this restriction.

Formula (5.9.1) involves an infinite sum that is not, in general, amenable to expression in closed form. Translation, then, represents a process which is difficult to handle in our generalized calculus. Fortunately, however, the properties of many of the important functions considered in Chapter 6 enable the difficulties inherent in (5.9.1) to be short-circuited.

5.10 BEHAVIOR NEAR LOWER LIMIT

When q is a positive integer or zero, the operator $d^q/[d(x-a)]^q$ is, as we have seen, local, i.e.,

$$\frac{d^q f}{[d(x-a)]^q} = \frac{d^q f}{dx^q}, \qquad q = n = 0, 1, 2, \ldots,$$

and the behavior of $d^q f/[d(x-a)]^q$ near $x = a$ is unexceptional. For all other values of q, however, it will now be demonstrated that $d^q f/[d(x-a)]^q$ usually approaches either zero or infinity as x approaches a, for all differintegrable series f. If f is a differintegrable series it can be decomposed as a finite sum of differintegrable series units f_U,

$$f_U = [x-a]^p \sum_{j=0}^{\infty} a_j [x-a]^j, \qquad p > -1, \quad a_0 \neq 0.$$

We shall demonstrate that $d^q f_U/[d(x-a)]^q$ normally approaches either zero or infinity as x approaches a for all such units f_U. Thus, the same conclusion will be valid for f by virtue of the linearity of $d^q/[d(x-a)]^q$.

If f_U is a differintegrable series unit, we have by equation (5.2.4),

$$\text{(5.10.1)} \qquad \frac{d^q f_U}{[d(x-a)]^q} = \sum_{j=0}^{\infty} \frac{a_j \Gamma(p+j+1)}{\Gamma(p+j-q+1)} [x-a]^{p+j-q}.$$

We know that $1/\Gamma(p+j-q+1)$ is always finite, while $\Gamma(p+j+1)$ is finite since $p > -1$. Thus, the right-hand side of (5.10.1) is dominated by its first term for small $x - a$. That is

(5.10.2)

$$\lim_{x \to a} \left\{ \frac{d^q f_U}{[d(x-a)]^q} \right\} = \lim_{x \to a} \left\{ \frac{[x-a]^{p-q} a_0 \Gamma(p+1)}{\Gamma(p-q+1)} \right\} = \begin{cases} 0, & p-q > 0, \\ a_0 \Gamma(p+1), & p-q = 0, \\ \infty, & p-q < 0, \end{cases}$$

since $a_0 \neq 0$.

5.11 BEHAVIOR FAR FROM LOWER LIMIT

In the present section we restrict attention to analytic functions ϕ. If $x \gg a$, we may write

$$
\begin{aligned}
[x-a]^{k-q} &= x^{k-q}\left[1-\frac{a}{x}\right]^{k-q} \\
&= x^{k-q}\left[1-\frac{[k-q]a}{x}+O\left(\frac{a^2}{x^2}\right)\right] \\
&\sim x^{k-q}+\frac{[q-k]ax^k}{x^{q+1}}
\end{aligned}
$$

and substitution into equation (3.5.3) yields [after use of (1.3.8)]

(5.11.1)

$$
\begin{aligned}
\frac{d^q\phi}{[d(x-a)]^q} &\approx \frac{\Gamma(q+1)\sin(\pi q)}{\pi}\left[\sum_{k=0}^{\infty}\frac{[-]^k x^{k-q}\phi^{(k)}}{[q-k]k!}+\frac{a}{x^{q+1}}\sum_{k=0}^{\infty}\frac{[-]^k x^k\phi^{(k)}}{k!}\right] \\
&= \frac{d^q\phi}{dx^q}+\frac{a\Gamma(q+1)\sin(\pi q)\phi(0)}{\pi x^{q+1}}.
\end{aligned}
$$

We observe that if q is a positive integer, the final term in equation (5.11.1) vanishes, reminding us once again of the local character of integer order derivatives.