CHAPTER 10

APPLICATIONS IN THE CLASSICAL CALCULUS

This chapter will deal briefly with a few applications of the fractional calculus to problems whose formulations and solutions are normally couched in terms of integrals or derivatives alone. The number of such problems to which techniques of the fractional calculus may be successfully applied is very large—we are able only to indicate briefly the scope of this subject. As we have often found to be so, the use of differintegral operators and their general properties greatly facilitates the problem formulation and solution.

10.1 EVALUATION OF DEFINITE INTEGRALS AND INFINITE SUMS

That differintegration provides a route for the evaluation of definite integrals is illustrated here with several examples. A single example will serve to illustrate the utility of the fractional calculus as a means of summing infinite series.

Substitution of $y = x - x\lambda$ in the Riemann–Liouville definition (3.6.2) of the differintegral of x^q, where $-1 < q < 0$, gives

$$\frac{d^q x^q}{dx^q} = \frac{1}{\Gamma(-q)} \int_0^1 \frac{[1-\lambda]^q \, d\lambda}{\lambda^{q+1}} = \Gamma(q+1).$$

The further substitution $z = -\ln(\lambda)$ then leads to

$$\int_0^\infty \frac{dz}{[\exp(z) - 1]^{-q}} = \Gamma(-q)\Gamma(q+1) = \pi \csc(q\pi),$$

a result that is quite difficult to establish otherwise.

The same $y = x - x\lambda$ substitution in the Riemann–Liouville definition of the differintegral of an arbitrary function gives

$$\int_0^{x^{-q}} f(x - x\lambda)\, d(\lambda^{-q}) = \Gamma(1-q)x^q \frac{d^q f}{dx^q}$$

for $q < 0$. Replacing λ^{-q} by z and $-1/q$ by the positive (but not necessarily integer) number p, we obtain

$$\int_0^{x^{1/p}} f(x - xz^p)\, dz = \Gamma\left(\frac{p+1}{p}\right) x^{-1/p} \frac{d^{-1/p} f}{dx^{-1/p}}$$

as a formula which has utility in definite integration, particularly for $x = 1$:

$$\int_0^1 f(1 - z^p)\, dz = \Gamma\left(\frac{p+1}{p}\right) \left.\frac{d^{-1/p} f}{dx^{-1/p}}\right|_{x=1}.$$

As one example consider

$$\begin{aligned}\int_0^1 \exp(1 - z^{\frac{2}{3}})\, dz &= \Gamma(\tfrac{5}{2}) \left.\frac{d^{-\frac{3}{2}}}{dx^{-\frac{3}{2}}} \exp(x)\right|_{x=1} \\ &= \tfrac{3}{4}\sqrt{\pi}\left[\exp(x)\operatorname{erf}(\sqrt{x}) - 2\sqrt{\frac{x}{\pi}}\right]_{x=1} = 1.5451,\end{aligned}$$

and as a second

$$\begin{aligned}\int_0^1 \sin(\sqrt{1 - z^2})\, dz &= \Gamma(\tfrac{3}{2}) \left.\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \sin(\sqrt{x})\right|_{x=1} \\ &= \tfrac{1}{2}\sqrt{\pi}\left[\sqrt{\pi x}\, J_1(\sqrt{x})\right]_{x=1} = 0.69123.\end{aligned}$$

As we have just seen, use of the Riemann–Liouville definition of a differintegral provides an avenue for the evaluation of certain definite integrals. One might well expect that use of some of the general properties of such differintegral operators will be of even greater benefit in evaluating integrals. For example, Osler (1972c) has established the integral analog of Leibniz's rule:

$$\frac{d^q[fg]}{dx^q} = \int_{-\infty}^{\infty} \binom{q}{\lambda + \gamma} \frac{d^{q-\gamma-\lambda} f}{dx^{q-\gamma-\lambda}} \frac{d^{\gamma+\lambda} g}{dx^{\gamma+\lambda}}\, d\lambda, \tag{10.1.1}$$

where γ is arbitrary. The choices $f = x^p$, $g = x^P$, and $\gamma = 0$ in (10.1.1) lead to

$$\int_{-\infty}^{\infty} \frac{\Gamma(q+1)\Gamma(p+1)\Gamma(P+1)\, d\lambda}{\Gamma(q-\lambda+1)\Gamma(\lambda+1)\Gamma(p-q+\lambda+1)\Gamma(P-\lambda+1)} = \frac{\Gamma(p+P+1)}{\Gamma(p+P-q+1)},$$

which is an integral extension of the identity (1.3.21). Specialization to $P = 0$, $p - q + 1 = 1$ leads to the interesting formula

$$\int_{-\infty}^{\infty} \frac{\sin(\pi\lambda)\, d\lambda}{\lambda\Gamma(\lambda+1)\Gamma(Q-\lambda)} = \frac{\pi}{\Gamma(Q)}$$

upon setting $q + 1 = Q$. Osler (1972c) has, in fact, made use of an equation that generalizes even equation (10.1.1) to create a short table of integrals.

A similar idea of Osler's (1970a) was to use the discrete version of Leibniz's rule (see Section 5.5)

$$\frac{d^q[fg]}{dx^q} = \sum_{k=-\infty}^{\infty} \binom{q}{j+\gamma} \frac{d^{q-\gamma-k}f}{dx^{q-\gamma-k}} \frac{d^{\gamma+k}g}{dx^{\gamma+k}}$$

to derive summation formulas. Here we notice only that the choices $f = x^p$, $g = x^P$, and $\gamma = 0$ lead this time to the result

$$\sum_{k=0}^{\infty} \frac{\Gamma(q+1)\Gamma(p+1)\Gamma(P+1)}{\Gamma(q-k+1)\Gamma(k+1)\Gamma(p-q+k+1)\Gamma(P-k+1)} = \frac{\Gamma(p+P+1)}{\Gamma(p+P-q+1)}$$

which, upon selecting P to be a positive integer j, reproduces (1.3.21) exactly.

10.2 ABEL'S INTEGRAL EQUATION

The elegance and power of the fractional calculus is nicely illustrated by its use in formulating and solving the weakly singular integral equation discussed in this section, whose study by Abel in the early 19th century gave birth to the subject of integral equations. Abel was interested in the problem of the tautochrone; that is, determining a curve in the (x, y) plane such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve. More generally, one might specify the time required for descent as a function of initial height. Let us fix the lowest point of the curve at the origin and position the curve in the positive quadrant of the plane, denoting by (X, Y) the initial point and (x, y) any point intermediate between $(0, 0)$ and (X, Y) (see Fig. 10.2.1). Assuming no frictional losses, we may equate the gain in kinetic energy to the loss in potential energy:

$$\frac{m}{2}\left[\frac{d\sigma}{dt}\right]^2 = mg[Y - y],$$

where σ is the arc length along the curve measured from the origin, m is the mass of the particle, g the gravitational acceleration, and t the time. Thus, since $d\sigma/dt < 0$,

$$d\sigma = -\sqrt{2g[Y-y]}\, dt. \tag{10.2.1}$$

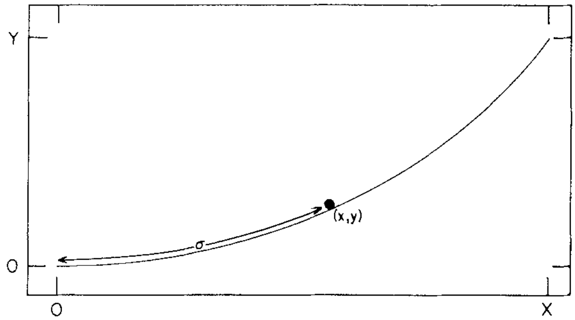

FIG. 10.2.1. The coordinate system for the tautochrone.

Now the arc length σ depends on the unknown curve. We write $\sigma = \sigma(y(t))$ to indicate the dependence of σ on the height y (which itself depends on time). Separating variables in (10.2.1) and integrating from $y = Y$ to $y = 0$ leads to

(10.2.2)
$$\sqrt{2g}\,T = \int_0^Y \frac{\sigma^{(1)}(y)\,dy}{\sqrt{Y-y}},$$

where T is the time of descent. Writing $\sqrt{2g}\,T = f(Y)$ puts (10.2.2) into the form

(10.2.3)
$$f(Y) = \int_0^Y \frac{\sigma^{(1)}(y)\,dy}{\sqrt{Y-y}}, \qquad f(0) = 0,$$

in which we may recognize it as an integral equation of convolution type for the unknown arc length σ. Abel (1823, 1825) found the solution of equation (10.2.3) to be

(10.2.4)
$$\sigma(y) = \frac{1}{\pi}\int_0^y \frac{f(Y)\,dY}{\sqrt{y-Y}},$$

essentially by Laplace transforming (10.2.3), making use of the convolution theorem, and inverting.

Of course, equation (10.2.4) still gives only an expression for the arc length $\sigma(y)$ as a function of height y along the curve in terms of the function $f(Y)$ determined by the time required for descent from an initial height Y. The relationship

$$\frac{d\sigma}{dy} = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}$$

may then be used to obtain the differential equation for the curve in terms of x and y. For the simplest case when $f(Y) = \sqrt{2g}\,T$ is independent of Y one has

$$1 + \left[\frac{dx}{dy}\right]^2 = \frac{2gT^2}{\pi^2 y},$$

and the equations for the resulting tautochrone may be obtained from this as

$$x = \tfrac{1}{2}a[\theta + \sin(\theta)], \qquad y = \tfrac{1}{2}a[1 - \cos(\theta)]$$

where $a = 2gT^2/\pi^2$ and θ is the angle depicted in Fig. 10.2.2. The tautochrone is one arch of the cycloid generated by a point P on a circle of radius $\frac{1}{2}a$ as the circle rolls along the lower side of the line $y = a$.

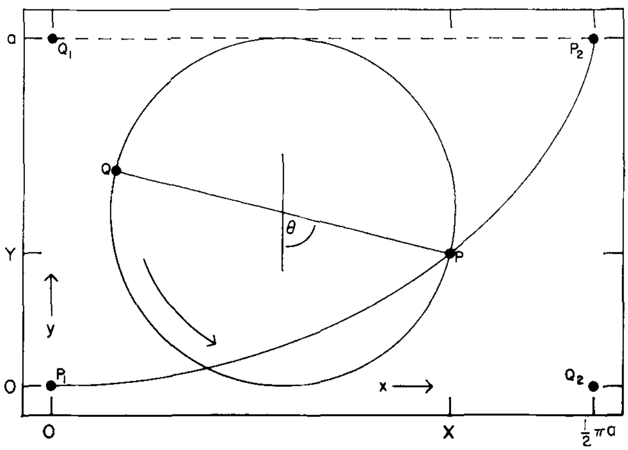

FIG. 10.2.2. The tautochrone $P_1 P_2$ is traced out by point P as the circle rolls along the line $y = a$. The point Q, initially at Q_1, ends at Q_2.

By this time the reader must surely recognize the integral in (10.2.3) as a close relative of the semiintegral of $\sigma^{(1)}$:

$$\frac{d^{-\frac{1}{2}}\sigma^{(1)}(Y)}{dY^{-\frac{1}{2}}} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^Y \frac{\sigma^{(1)}(y)\,dy}{\sqrt{Y-y}}$$

so that in the language of the fractional calculus, equation (10.2.3) reads

$$f(Y) = \Gamma(\tfrac{1}{2})\,\frac{d^{-\frac{1}{2}}\sigma^{(1)}}{dY^{-\frac{1}{2}}}(Y).$$

Use of the formula (5.7.13) with $q = -\frac{1}{2}$, $N = 1$, and Y replaced by y gives

$$f(y) = \sqrt{\pi}\left[\frac{d^{\frac{1}{2}}\sigma(y)}{dy^{\frac{1}{2}}} - \frac{y^{\frac{1}{2}}\sigma(0)}{\sqrt{\pi}}\right] = \sqrt{\pi}\,\frac{d^{\frac{1}{2}}\sigma(y)}{dy^{\frac{1}{2}}}$$

since $\sigma(0) = 0$. If we assume σ to be differintegrable, the continuity of σ at $y = 0$ may be used to rule out the presence, in σ, of a factor y^p for $p < 0$. Thus (see Section 5.7) the composition rule may be applied to give

$$\frac{d^{-\frac{1}{2}}f(y)}{dy^{-\frac{1}{2}}} = \sqrt{\pi}\,\sigma(y),$$

which is Abel's solution (10.2.4) upon writing the Riemann–Liouville form for the semiintegral. Notice how easily the inversion of the original equation (10.2.3) is performed through the use of general properties of differintegrals.

The generalized Abel equation,

$$f(Y) = \int_0^Y \frac{\sigma^{(1)}(y)\,dy}{[Y-y]^\alpha}, \qquad 0 < \alpha < 1,$$

may be inverted with equal ease through the fractional calculus.

10.3 SOLUTION OF BESSEL'S EQUATION

As an example of the way differintegration can be used to tackle classical differential equations, we here consider Bessel's equation, which arises in connection with the vibrations of a circular drumhead, as well as in other important physical applications. The modified Bessel equation, which differs only in the sign of the third term, and which arises in a number of diffusion problems, is equally amenable to the approach we here take.

The equation

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} + \left[x - \frac{v^2}{4}\right]w = 0 \tag{10.3.1}$$

is a form of Bessel's equation. As is the rule for second-order differential equations, its general solution is a combination of two linearly independent functions w_1 and w_2 of x, each of which depends on the parameter v. The usual method of solving (10.3.1) is via an infinite series approach, but we shall demonstrate how differintegration procedures lead to a ready solution in terms of elementary functions.

We start by making either of the substitutions

$$w = x^{\pm\frac{1}{2}v}u,$$

where v denotes the nonnegative square root of v^2, so that equation (10.3.1) is transformed to

$$x\frac{d^2u}{dx^2} + [1 \pm v]\frac{du}{dx} + u = 0. \tag{10.3.2}$$

We next assume that for every function u that satisfies (10.3.2) there exists a differintegrable function f, related to u by the equation

$$u = \frac{d^{\frac{1}{2} \pm \nu} f}{dx^{\frac{1}{2} \pm \nu}}. \tag{10.3.3}$$

Moreover, use of equation (3.2.5) permits the combination of equations (10.3.2) and (10.3.3) to give

$$x \frac{d^{\frac{5}{2} \pm \nu} f}{dx^{\frac{5}{2} \pm \nu}} + [1 \pm \nu] \frac{d^{\frac{3}{2} \pm \nu} f}{dx^{\frac{3}{2} \pm \nu}} + \frac{d^{\frac{1}{2} \pm \nu} f}{dx^{\frac{1}{2} \pm \nu}} = 0. \tag{10.3.4}$$

Application of Leibniz's rule allows the rewriting of equation (10.3.4) as

$$\frac{d^{\frac{5}{2} \pm \nu}\{xf\}}{dx^{\frac{5}{2} \pm \nu}} - \frac{3}{2} \frac{d^{\frac{3}{2} \pm \nu} f}{dx^{\frac{3}{2} \pm \nu}} + \frac{d^{\frac{1}{2} \pm \nu} f}{dx^{\frac{1}{2} \pm \nu}} = 0 \tag{10.3.5}$$

wherein the parameter ν is no longer present as a coefficient. We next plan to decompose the operators, thus

$$\frac{d^{\frac{1}{2} \pm \nu}}{dx^{\frac{1}{2} \pm \nu}} \frac{d^2\{xf\}}{dx^2} - \frac{3}{2} \frac{d^{\frac{1}{2} \pm \nu}}{dx^{\frac{1}{2} \pm \nu}} \frac{df}{dx} + \frac{d^{\frac{1}{2} \pm \nu} f}{dx^{\frac{1}{2} \pm \nu}} = 0, \tag{10.3.6}$$

an equation directly convertible to

$$\frac{d^2\{xf\}}{dx^2} - \frac{3}{2} \frac{df}{dx} + f = 0 \tag{10.3.7}$$

by the action of the $d^{-\frac{1}{2} \mp \nu}/dx^{-\frac{1}{2} \mp \nu}$ operator. Equations (10.3.6) and (10.3.5) are equivalent to each other if and only if

$$[xf]_{x=0} = 0 \qquad \text{and} \qquad \left[\frac{d\{xf\}}{dx}\right]_{x=0} = 0 \tag{10.3.8}$$

and

$$f(0) = 0, \tag{10.3.9}$$

whereas (10.3.7) and (10.3.6) are equivalent if

$$\frac{d^{-\frac{1}{2} \mp \nu}}{dx^{-\frac{1}{2} \mp \nu}} \frac{d^{\frac{1}{2} \pm \nu} g}{dx^{\frac{1}{2} \pm \nu}} = g \qquad \text{with} \quad g = f, \ \frac{df}{dx}, \ \text{and} \ \frac{d^2\{xf\}}{dx^2}. \tag{10.3.10}$$

Conversion of equation (10.3.7) to the canonical form

$$\frac{d^2 f}{[d(2\sqrt{x})]^2} + f = 0$$

is straightforward, whereby it follows that the two possible candidate functions f are

$$f_1 = \sin(2\sqrt{x}) \qquad \text{and} \qquad f_2 = \cos(2\sqrt{x}).$$

We must now inquire which, if either, of these candidate functions satisfies the requirements (10.3.8), (10.3.9), and (10.3.10), which we assumed held during our derivation. Because

$$\cos(2\sqrt{x}) = 1 - 2x + \tfrac{2}{3}x^2 - \cdots,$$

it is evident that f_2 fails to meet requirement (10.3.8) or (10.3.9) and must be rejected. However, f_1 passes these tests. The requirement

$$\frac{d^{-\frac{1}{2}+\nu}}{dx^{-\frac{1}{2}+\nu}}\frac{d^{\frac{1}{2}-\nu}g}{dx^{\frac{1}{2}-\nu}} = g \qquad \text{with} \quad g = f, \quad \frac{df}{dx}, \quad \text{and} \quad \frac{d^2\{xf\}}{dx^2},$$

one part of (10.3.10), is met by the function

$$\sin(2\sqrt{x}) = 2x^{\frac{1}{2}} - \tfrac{4}{3}x^{\frac{3}{2}} + \tfrac{4}{15}x^{\frac{5}{2}} - \cdots$$

for all values of ν (recall that we restricted ν to nonnegative values), while the other part,

$$\frac{d^{-\frac{1}{2}-\nu}}{dx^{-\frac{1}{2}-\nu}}\frac{d^{\frac{1}{2}+\nu}g}{dx^{\frac{1}{2}+\nu}} = g \qquad \text{with} \quad g = f, \quad \frac{df}{dx}, \quad \text{and} \quad \frac{d^2\{xf\}}{dx^2},$$

is met by f_1 for all ν values except the nonnegative integers.

Returning to equation (10.3.3) then, we conclude that the function

$$u_1 = \frac{d^{\frac{1}{2}-\nu}}{dx^{\frac{1}{2}-\nu}}\sin(2\sqrt{x})$$

is a solution to equation (10.3.2) for all ν values, and that

$$u_2 = \frac{d^{\frac{1}{2}+\nu}}{dx^{\frac{1}{2}+\nu}}\sin(2\sqrt{x})$$

is another solution when ν is not an integer. Our sought solutions to the original Bessel equation are thus

$$w_1(\nu, x) = x^{-\frac{1}{2}\nu}u_1 = x^{-\frac{1}{2}\nu}\frac{d^{\frac{1}{2}-\nu}\sin(2\sqrt{x})}{dx^{\frac{1}{2}-\nu}}, \qquad \text{all} \quad \nu \geqq 0,$$

and

$$w_2(\nu, x) = x^{\frac{1}{2}\nu}u_2 = x^{\frac{1}{2}\nu}\frac{d^{\frac{1}{2}+\nu}\sin(2\sqrt{x})}{dx^{\frac{1}{2}+\nu}}, \qquad 0 \leqq \nu \neq 1, 2, \ldots .$$

The problem is now completely solved, except that a second solution is needed for integer ν values. Our technique cannot reveal this second solution.

The relationship of w_1 and w_2 to the conventional notation for Bessel functions is simply

$$w_1(\nu, x) = \sqrt{\pi}\, J_{-\nu}(2\sqrt{x}) \qquad \text{and} \qquad w_2(\nu, x) = \sqrt{\pi}\, J_\nu(2\sqrt{x}).$$

10.4 CANDIDATE SOLUTIONS FOR DIFFERENTIAL EQUATIONS

As is the case in several facets of the fractional calculus, the composition rule lies at the crux of the derivation of the last section. Much care and labor needs to be expended in examining, at every step in an argument such as the one involved in solving Bessel's equation, the demands of this exacting rule. Such examination is especially difficult and tedious when one is dealing with unknown functions. In the present section we shall bypass all these difficulties by blithely assuming that the composition rule applies universally! We excuse this cavalier treatment of a vital theorem of the fractional calculus on the basis of the uses that will be made of the techniques of the present section.

The major task in solving a difficult ordinary differential equation is the search for candidate solutions. Having found a candidate, it is a comparatively simple matter to test whether it does or does not satisfy the original differential equation. Hence, if lack of attention to the details of the composition rule lets through a few illicit solutions, no great harm is done: These offenders will be found wanting when they are processed in an attempt to reproduce the original problem equation. Thus, in the last section, had we not so painstakingly checked the requirements of the composition rule at each stage of our derivations, the two functions

$$x^{\pm\frac{1}{2}\nu} \frac{d^{\frac{1}{2}\pm\nu} \cos(2\sqrt{x})}{dx^{\frac{1}{2}\pm\nu}}$$

would have emerged as possible solutions to Bessel's equation. It would, however, have been an easy step to show that these particular candidate solutions fail to regenerate Bessel's equation. On the other hand, inattention to the requirements of the composition rule may, as by excluding certain additive power terms which ought properly to have been present, allow one to miss certain correct solutions. Even then, however, the exercise may not have been a complete waste of time, since a careful examination of why a candidate solution fails to reproduce the original ordinary differential equation can give valuable clues as to how this potential solution needs to be "patched up" so as to fill the needs of the original equation.

With the philosophy of the foregoing in mind, let us consider the following third-order differential equation,

$$p_3 \frac{d^3 w}{dx^3} + p_2 \frac{d^2 w}{dx^2} + p_1 \frac{dw}{dx} + p_0 w = 0, \tag{10.4.1}$$

where p_n is a polynomial in x of degree not exceeding n; that is, p_3 may be a cubic in x, p_2 a quadratic, p_1 is linear, and p_0 is necessarily a constant. In our search for candidate functions of x which hopefully will satisfy equation (10.4.1), we assume the existence of a function f of x that satisfies

$$w = \frac{d^{-1-q}f}{dx^{-1-q}}, \tag{10.4.2}$$

q being presently unspecified. Assuming the composition rule, the equation

(10.4.3)

$$\frac{d^q}{dx^q}\left\{p_3 \frac{d^{2-q}f}{dx^{2-q}}\right\} + \frac{d^q}{dx^q}\left\{p_2 \frac{d^{1-q}f}{dx^{1-q}}\right\} + \frac{d^q}{dx^q}\left\{p_1 \frac{d^{-q}f}{dx^{-q}}\right\} + \frac{d^q}{dx^q}\left\{p_0 \frac{d^{-q-1}f}{dx^{-q-1}}\right\} = 0$$

arises from combining equations (10.4.1) and (10.4.2), followed by application of the d^q/dx^q operator to each term. Consider the first term in equation (10.4.3), recalling that p_3 is, in general, a cubic, say $c_3 x^3 + c_2 x^2 + c_1 x + c_0$. Application of the product rule permits the evaluation of the first term as

$$[c_3 x^3 + c_2 x^2 + c_1 x + c_0] \frac{d^2 f}{dx^2} + q[3c_3 x^2 + 2c_2 x + c_1] \frac{df}{dx}$$
$$+ q[q-1][3c_3 x + 2c_2] f + q[q-1][q-2] c_3 \frac{d^{-1}f}{dx^{-1}},$$

the composition rule having been assumed valid yet again. Similar expressions arise from each of the four terms in (10.4.3), so that the result

$$\pi_3 \frac{d^2 f}{dx^2} + \pi_2 \frac{df}{dx} + \pi_1 f + \pi_0 \frac{d^{-1}f}{dx^{-1}} = 0 \tag{10.4.4}$$

finally emerges, where π_3 is a cubic, π_2 is a quadratic, π_1 is linear, and π_0 is a constant. The constant π_0 is a composite of q and of the coefficients in the original polynomials p_3, p_2, p_1, and p_0. Since q is hitherto unrestricted we are free to assign any value to it, and it is likely that several q values (three, in general) will cause π_0 to vanish. The selection of any one such value converts equation (10.4.4) to a second-order differential equation.

Obviously, the same technique can convert a differential equation of second order to one of first order, as Liouville (1832b) demonstrated. We shall use an example of this conversion to illustrate this section. The equation

$$[x^2 - 1]\frac{d^2w}{dx^2} + 2x\frac{dw}{dx} - v[v + 1]w = 0 \tag{10.4.5}$$

is Legendre's equation. To solve it by the technique just advocated, we write

$$\frac{d^q}{dx^q}\left\{[x^2 - 1]\frac{d^{1-q}f}{dx^{1-q}}\right\} + \frac{d^q}{dx^q}\left\{2x\frac{d^{-q}f}{dx^{-q}}\right\} - \frac{d^q}{dx^q}\left\{v[v + 1]\frac{d^{-q-1}f}{dx^{-q-1}}\right\} = 0$$

and then apply Leibniz's rule. Subject to our assumption that the composition rule always holds, we get

$$[x^2 - 1]\frac{df}{dx} + 2[q + 1]xf + [q^2 + q - v^2 - v]\frac{d^{-1}f}{dx^{-1}} = 0.$$

By choosing either

$$q = v \qquad \text{or} \qquad q = -v - 1, \tag{10.4.6}$$

the final right-hand term vanishes leaving a first-order differential equation. The variables are separable, so that

$$\int \frac{df}{f} = 2[q + 1]\int \frac{x\,dx}{1 - x^2}.$$

After integration and exponentiation, the result

$$f = C[x^2 - 1]^{-1-q}$$

emerges, C being an arbitrary constant. Recalling (10.4.2) and (10.4.6), we find that

$$w_1(v, x) = \frac{d^{-1-v}[x^2 - 1]^{-1-v}}{dx^{-1-v}} \qquad \text{and} \qquad w_2(v, x) = \frac{d^v[x^2 - 1]^v}{dx^v}$$

appear as our two candidate solutions.

Now notice in equation (10.4.5) that replacement of v by $-v - 1$ leaves the equation unchanged. It follows that solutions w_1 and w_2, which interchange on replacing v by $-v - 1$, are actually identical. Our technique has, therefore, in this case, yielded a single solution. It is, however, a valid solution, being related to the conventional notation for Legendre functions by

$$w_2(v, x) = \frac{d^v[x^2 - 1]^v}{dx^v} = 2^v\Gamma(v + 1)P_v(x). \tag{10.4.7}$$

We may note in passing that the differintegral representation[1] for $P_\nu(x)$ embodied in (10.4.7) constitutes a generalization of the well-known Rodrigues formula (Abramowitz and Stegun, 1964, p. 334)

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n[x^2-1]^n}{dx^n},$$

used, for integer n of course, to generate Legendre polynomials.

10.5 FUNCTION FAMILIES

Many functions are customarily defined by an integral, that is, by a $q = -1$ differintegral. By considering the behavior of the definition when q takes values other than -1, one may suitably embed the function in a family parametrized by q. We will illustrate this concept by evolving a generalized logarithm function.

The logarithm is unique in function theory in being the only transcendental function that can be generated by applying the operations of integration or differentiation to a power of $C - cx$, where C and c are constant. A collection of functions that includes a logarithm is generated by nth-order integration of $[C - cx]^{-n}$, that is, by

$$\frac{d^{-n}[C-cx]^{-n}}{dx^{-n}}, \qquad n = 1, 2, 3, \ldots,$$

where, for convenience, a lower limit of zero has been selected. The only other functions generated by

$$\frac{d^q[C-cx]^p}{dx^q}, \qquad q = 0, \pm 1, \pm 2, \ldots,$$

are algebraic and finite:

$$\frac{d^q[C-cx]^p}{dx^q} = \begin{cases} \dfrac{\Gamma(p+1)[C-cx]^{p-q}}{\Gamma(p-q+1)}, & q = 0, 1, 2, \ldots, \\ \dfrac{\Gamma(p+1)}{\Gamma(p-q+1)}\left[[C-cx]^{p-q} - \displaystyle\sum_{j=0}^{n-1}\binom{p+n}{j}C^{p+n-j}[-cx]^j\right], & \\ & p \neq q = -1, -2, \ldots, \end{cases}$$

for all p and all nonzero C and c.

[1] Essentially the same formula was obtained in a different context by Osler (1970a).

The coalescence of the differintegration order with the power of the binomial is evidently the key to ensuring that the generated function family includes a logarithm. Thus, choosing $C = c = 1$, we have

$$\frac{d^{-1}[1-x]^{-1}}{dx^{-1}} = \ln\left(\frac{1}{1-x}\right),$$

$$\frac{d^{-2}[1-x]^{-2}}{dx^{-2}} = \ln\left(\frac{1}{1-x}\right) - x,$$

$$\frac{d^{-n}[1-x]^{-n}}{dx^{-n}} = \frac{1}{(n-1)!}\left[\ln\left(\frac{1}{1-x}\right) - x - \frac{x^2}{2} - \cdots - \frac{x^{n-1}}{n-1}\right].$$

Let us now define a generalized logarithm of order n, thus

$$\ln_n\left(\frac{1}{1-x}\right) = (n-1)!\,\frac{d^{-n}[1-x]^{-n}}{dx^{-n}}$$

for $n = 1, 2, \ldots$. Then we see that

$$\ln_1\left(\frac{1}{1-x}\right) = \ln\left(\frac{1}{1-x}\right)$$

and generally

$$\text{(10.5.1)}\qquad \ln_n\left(\frac{1}{1-x}\right) = \ln\left(\frac{1}{1-x}\right) - \sum_{j=1}^{n-1}\frac{x^j}{j} = \sum_{j=1}^{\infty}\frac{x^j}{j} - \sum_{j=1}^{n-1}\frac{x^j}{j} = \sum_{j=n}^{\infty}\frac{x^j}{j},$$

where a Taylor expansion of $\ln(1/[1-x])$ was used. Thus the $\ln_n(\ )$ function is a "beheaded" logarithm, the first $n-1$ terms of the regular logarithmic series being absent.

We now study the implications of extending our definition to fractional v instances. From the general definition we find

$$\ln_v\left(\frac{1}{1-x}\right) = \Gamma(v)\,\frac{d^{-v}[1-x]^{-v}}{dx^{-v}} = \frac{d^{-v}}{dx^{-v}}\left[x\,\frac{v-1}{0}\right]$$

$$= x^v\left[x\,\frac{v-1}{v}\right] = \sum_{j=0}^{\infty}\frac{x^{j+v}}{j+v}$$

for all v. When we notice that equation (10.5.1) can equally well be written

$$\ln_n\left(\frac{1}{1-x}\right) = \sum_{j=0}^{\infty}\frac{x^{j+n}}{j+n},$$

the generalization is seen as perfect from the series expansion standpoint.

The three representations

$$\text{(10.5.2)}\qquad \Gamma(v)\,\frac{d^{-v}[1-x]^{-v}}{dx^{-v}}, \qquad x^v\left[x\,\frac{v-1}{v}\right], \qquad \text{and} \qquad \sum_{j=0}^{\infty}\frac{x^{j+v}}{j+v}$$

are equivalent, except for $v = 0, -1, -2, \ldots$ and any one may be used to represent $\ln_v(1/[1 - x])$. Because of the recursion

$$\ln_{v+1}\left(\frac{1}{1-x}\right) = \ln_v\left(\frac{1}{1-x}\right) - \frac{x^v}{v},$$

if the properties of $\ln_v(\ )$ are known on the interval $0 < v \leq 1$, they are known everywhere. For the important $v = \frac{1}{2}$ case, we have

$$\ln_{\frac{1}{2}}\left(\frac{1}{1-x}\right) = 2\operatorname{arctanh}(\sqrt{x}) = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$$

so that a half-order logarithm is merely an ordinary logarithm with changed argument. Generalized logarithms of order $\frac{1}{4}$ and $\frac{3}{4}$ are also expressible; thus

$$\ln_{\frac{1}{4}}\left(\frac{1}{1-x}\right) = 2\operatorname{arctanh}(x^{\frac{1}{4}}) - 2\arctan(x^{\frac{1}{4}}),$$

$$\ln_{\frac{3}{4}}\left(\frac{1}{1-x}\right) = 2\operatorname{arctanh}(x^{\frac{1}{4}}) + 2\arctan(x^{\frac{1}{4}}),$$

in terms of more familiar functions.

For $v > 0$ the function $\ln_v(1/[1 - x])$ is (for x real and greater than zero) single-valued, real, and finite. The corresponding function $\ln_v(1/[1 + x])$, however, enters the complex plane, even for real x, unless v is an integer. This is clear from consideration of a change in the sign of x in any one of the three representations (10.5.2). In each, there is a new factor $[-1]^v$ introduced:

$$\ln_v\left(\frac{1}{1+x}\right) = \Gamma(v)\frac{d^{-v}[1+x]^{-v}}{[d(-x)]^{-v}} = (-x)^v\left[-x\,\frac{v-1}{v}\right] = \sum_{j=0}^{\infty}\frac{[-x]^{j+v}}{j+v}.$$

Interpreting this factor as $\cos(v\pi) + i\sin(v\pi)$, we see that for $v = \frac{1}{2}$ or $n + \frac{1}{2}$, the modified logarithm of $[1 + x]^{-1}$ is purely imaginary. For example,

$$\ln_{\frac{1}{2}}\left(\frac{1}{1+x}\right) = i\sqrt{x}\left[-x\,\frac{-\frac{1}{2}}{\frac{1}{2}}\right] = 2i\arctan(\sqrt{x}).$$

Such behavior need not surprise us, for as simple a function as x^v behaves similarly.

The exercise we have just concluded generalized a single function into a continuum of functions. It is also possible to use the concept of differintegration to generalize a set of functions that are parametrized by a set of integers. Thus, the repeated integrals of the error function complement,

$$\text{(10.5.3)}\qquad \frac{\sqrt{\pi}}{2}\,i^n\operatorname{erfc}(x) = \int_x^{\infty}\frac{[y-x]^n}{n!}\exp(-y^2)\,dy,$$

parametrized by the integer n, are generalized by the differintegral definition

$$\frac{\sqrt{\pi}}{2}\,\mathrm{i}^{\nu}\mathrm{erfc}\left(\frac{1}{x}\right) = x^{-\nu}\,\frac{d^{-\nu-1}}{dx^{-\nu-1}}\left\{\frac{\exp(-x^{-2})}{x^{\nu+2}}\right\} \tag{10.5.4}$$

into a continuum of functions. By a suitable change of variable, it may readily be demonstrated that definition (10.5.4) reduces to (10.5.3) when ν equals the integer n.