Stilian proposes that in $d = 1$, the multivariate matern process is

$$Y(t) = \int_{\mathbb{R}} e^{itx} \left((1+ix)^{-\nu-1/2} A 1_{x>0} + (1+ix)^{-\nu-1/2} \overline{A} 1_{x<0} \right) \tilde{B}(x)$$

where A is a $k \times k$ complex valued matrix, ν is a $k \times k$ real matrix, and $\tilde{B}(x)$ is a $\mathbb{C}^k$-valued Brownian motion such that

$$\tilde{B}(x) = \overline{\tilde{B}(-x)} \qquad \mathbb{E}(\tilde{B}(x)\tilde{B}(x)^*) = \mathbb{I}_k dx.$$

Below, we developed two versions of this integral that give extensions of the multivariate matern model of Gneiting et al (2010).

- (1) The first gives an asymmetric cross covariances when the two processes have the same smoothness parameter ν, with the asymmetric portion coming from a complex valued A.
- (2) The second takes a real-valued A, and allows the smoothness of the two processes to be different. In contrast to the OFBM case, a real-valued AA^* does not imply time-reversibility, and this second approach also allows asymmetries to be modelled!

Presumably, there exists a complete version that has the two above special cases, but I haven't found it yet. The first seems to deal more with the shape of the cross-covariance, while the second seems to deal more with the lag of the cross-covariance. We outline results for the general integral below. I've specified where we do or do not have a closed-form of the covariance:

- $d = 1$
  - ν is a constant
    - Covariance: Matern covariance
    - Cross-Covariance:
      - AA^* is real: Matern covariance
      - AA^* is complex (1): Matern covariance + (combo of Bessel + Struve functions)
  - ν is a diagonal matrix
    - Covariances: Matern covariance
    - Cross covariances:
      - When the two smoothness parameters are the same, you get the same thing as when ν is a constant above.
      - AA^* is real and different smoothness (2): Whittaker function
      - AA^* is complex and different smoothness: Not solved
  - ν is diagonalizable: Not solved, but probably could follow naturally from diagonal case
  - ν even more general?: Not solved
- $d > 1$: Not solved, but some work has been done for it.

The surprising thing, to me, is that even when you consider only when A is real, the Multivariate Matern of Gneiting et al (2010) is not the complete class of cross covariances!! There are more of them when the processes have different smoothnesses. In fact, when two of the processes have different smoothness, the Multivariate Matern of Gneiting et al (2010) does not follow from the spectral density.

1 The case $d = 1$

1.1 $\nu = \nu \mathbb{I}_k$

Stilian proposed first looking at the subset of processes where ν is a constant diagonal matrix. In the following, we write ν to be a real constant. In this case

$$\mathbb{E}\left(Y(t)Y(s)^*\right) = \int_{\mathbb{R}} e^{i(t-s)x}(1+x^2)^{-\nu-1/2}(AA^* 1_{x>0} + \overline{AA^*} 1_{x<0})dx$$

Now, since $e^{i(t-s)x} = \cos((t-s)x) + i\sin((t-s)x)$, we can write

$$\begin{aligned}\mathbb{E}\left(Y(t)Y(s)^*\right) = &\int_{\mathbb{R}} \cos((t-s)x)(1+x^2)^{-\nu-1/2}(AA^* 1_{x>0} + \overline{AA^*} 1_{x<0})dx \\ &+ i\int_{\mathbb{R}} \sin((t-s)x)(1+x^2)^{-\nu-1/2}(AA^* 1_{x>0} + \overline{AA^*} 1_{x<0})dx\end{aligned}$$

provided that the integrals exist. Then, using Gradshteyn and Ryzhik 3.771 1 and 2, we have

$$\begin{aligned}\mathbb{E}\left(Y(t)Y(s)^*\right) = &\operatorname{Re}(AA^*)\left(c_1\left(|t-s|\right)^{\nu} K_\nu(|t-s|)\right) \\ &- \operatorname{Im}(AA^*)\left(\operatorname{sign}(t-s)c_2\left(|t-s|\right)^{\nu}\left(I_\nu(|t-s|) - L_{-\nu}(|t-s|)\right)\right)\end{aligned}$$

when $\nu \neq 1/2, 3/2, \ldots$ and

$$\begin{aligned}c_1 &= \frac{2}{\sqrt{\pi}}\frac{1}{2^\nu}\cos(-\pi\nu)\Gamma(-\nu+1/2) \\ c_2 &= \frac{\sqrt{\pi}}{2^\nu}\Gamma(-\nu+1/2) \\ K_\nu(t) &: \text{ modified Bessel function of the second kind} \\ I_\nu(t) &: \text{ modified Bessel function of the first kind} \\ L_\nu(t) &: \text{ modified Struve function}\end{aligned}$$

The first part is the standard Matern covariance. See `https://argo.stat.lsa.umich.edu/shiny/ShinyApps/multivariate_matern/` to see various plots of the covariances and cross covariances.

The second function has been mentioned, for example in the Gnieting turning bands paper.

1.2 ν is diagonal

The more general case is where ν is a matrix. We then get

$$\begin{aligned}\mathbb{E}(Y(t)Y(s)^*) = \int_{\mathbb{R}} e^{i(t-s)x}\Big(&(1+ix)^{-\nu-1/2\mathbb{I}_k}AA^*\overline{(1+ix)^{-\nu-1/2\mathbb{I}_k}}1_{x>0} \\ &+ (1+ix)^{-\nu-1/2\mathbb{I}_k}\overline{AA^*}\,\overline{(1+ix)^{-\nu-1/2\mathbb{I}_k}}1_{x<0}\Big)dx\end{aligned}$$

Throughout, we use the fact that $\overline{(1+ix)^{-\nu-1/2\mathbb{I}_k}} = \overline{(1+ix)}^{-\nu-1/2\mathbb{I}_k} = (1-ix)^{-\nu-1/2\mathbb{I}_k}$, since ν is a real-valued matrix.

We now consider the more general case where ν does not necessarily have equal diagonal entries. Let $\nu = \operatorname{diag}(\nu_1, \ldots, \nu_k)$. Then the j_1, j_2 element of the covariance is

$$\mathbb{E}(Y_{j_1}(t)Y_{j_2}(s)) = \int_{\mathbb{R}} e^{i(t-s)x}(1+ix)^{-\nu_{j_1}-\frac{1}{2}}(1-ix)^{-\nu_{j_2}-\frac{1}{2}}(B_{j_1,j_2}1_{x>0} + \overline{B_{j_1,j_2}}1_{x<0})dx$$

where $B = AA^*$. It's not clear how to evaluate this in generality, so we focus now on a few special cases.

For $j_1 = j_2$, $(1+ix)^a(1-ix)^a = (1+x^2)^a$ as Stilian noted, and $B_{j_1,j_1} = \overline{B_{j_1,j_1}}$ so the integral reduces to

$$\int_{\mathbb{R}} e^{i(t-s)x}(1+x^2)^{-\nu_{j_1}-\frac{1}{2}} B_{j_1,j_1} dx$$

giving the familiar Matern covariance. Thus, the marginal covariances of each process are Matern.

Consider another special case when $B_{j_1,j_2} = \overline{B_{j_1,j_2}}$ (i.e. $(AA^*)_{j_1,j_2}$ is real). We should end up with something like the Multivariate Matern of Gneiting et al (2010). The formula 3.384 9 of Gradshteyn and Ryzhik can be adjusted to show

$$\int_{\mathbb{R}} e^{ipx}(1+ix)^{-2\mu}(1-ix)^{-2\nu}dx = \begin{cases} \pi 2^{-\nu-\mu+1}\frac{|p|^{\nu+\mu-1}}{\Gamma(2\nu)} W_{\nu-\mu,1/2-\nu-\mu}(2|p|) & \text{if } p>0 \\ \pi 2^{-\nu-\mu+1}\frac{|p|^{\nu+\mu-1}}{\Gamma(2\mu)} W_{\mu-\nu,1/2-\nu-\mu}(2|p|) & \text{if } p<0 \end{cases}$$

where $W_{\mu,\nu}(z)$ is the Whittaker function. Thus, in the above equation, we set $\mu = \nu_{j_1}/2 + \frac{1}{4}$ and $\nu = \nu_{j_2}/2 + \frac{1}{4}$, so that

$$\mathbb{E}(Y_{j_1}(t)Y_{j_2}(s)) = B_{j_1,j_2}\pi 2^{-\nu_{j_1,j_2}+1/2}|t-s|^{\nu_{j_1,j_2}-1/2} \begin{cases} \frac{1}{\Gamma(\nu_{j_2}+1/2)} W_{\frac{-\nu_{j_1}+\nu_{j_2}}{2},-\nu_{j_1,j_2}}(2|t-s|) & \text{if } t-s>0 \\ \frac{1}{\Gamma(\nu_{j_1}+1/2)} W_{\frac{\nu_{j_1}-\nu_{j_2}}{2},-\nu_{j_1,j_2}}(2|t-s|) & \text{if } t-s<0 \end{cases}$$

where $\nu_{j_1,j_2} = \frac{1}{2}(\nu_{j_1}+\nu_{j_2})$. That's a bit messy. One can visualize for different smoothness values at the same shiny app `https://argo.stat.lsa.umich.edu/shiny/ShinyApps/multivariate_matern/`.

Now, consider the case where $\nu_{j_1} = \nu_{j_2}$, the processes have the same smoothness. Using the fact that $W_{0,\nu}(2z) = \sqrt{2z/\pi}K_\nu(z)$ and $K_\nu(z) = K_{-\nu}(z)$, we have

$$\mathbb{E}(Y_{j_1}(t)Y_{j_2}(s)) = B_{j_1,j_2}\frac{\sqrt{\pi}2^{-\nu_{j_1}+1}|t-s|^{\nu_{j_1}}}{\Gamma(\nu_{j_1}+1/2)}K_{\nu_{j_1}}(|t-s|)$$

back to the Matern that we would expect! Thus, the Whittaker functions provide a natural cross-covariance to the Matern when the two processes have different smoothness.

Now, in order to evaluate the version where AA^* contains non-zero imaginary part, we must compute

$$\int_0^\infty e^{i(t-s)x}(1+ix)^{\nu_{j_1}-1/2}(1-ix)^{\nu_{j_2}-1/2}dx$$

which I have not found in any formula book yet.

1.3 ν is diagonalizable

Suppose that we can write $\nu = V\Lambda V^{-1}$.