**CHAPTER 3**

# FRACTIONAL DERIVATIVES AND INTEGRALS: DEFINITIONS AND EQUIVALENCES

In this chapter we compare several rival definitions of the differintegral of a function $f$ to arbitrary order $q$. All definitions are, of course, required to yield multiple derivatives and integrals when the order is a positive or negative integer.

Perhaps the least ambiguous symbolism for the value at $x$ of the differintegral to order $q$ of a function $f$ defined on the interval $a \leqq y \leqq x$ would be

$$\frac{d^q f(y)}{[d(y-a)]^q}(x).$$

We shall eventually relate this differintegral to an ordinary integral in which $y$ is a "dummy" variable of integration, and $a$ and $x$ are limits of integration. In line with conventions adopted in Section 2.1, our normal abbreviations for the $q$th differintegral of a function $f$ will be

$$\frac{d^q f}{[d(x-a)]^q}, \qquad \text{and} \qquad \frac{d^q f}{[d(x-a)]^q}(x_0) \qquad \text{or} \qquad \left[\frac{d^q f}{[d(x-a)]^q}\right]_{x=x_0},$$

it being understood that $f$ and $d^q f/[d(x-a)]^q$ are functions of the independent variable $x$ when the $x$ is omitted. Since, as we shall see ultimately, differintegrals must include integral transforms as well as ordinary derivatives and integrals, for which a variety of conflicting notations abound in the literature, the selection of a single satisfactory replacement for all of these is as much a matter of taste as logic. We have attempted to combine both elements in our choice.

## 3.1 DIFFERINTEGRABLE FUNCTIONS

It is time now to delineate the class of functions to which we shall apply differintegral operators. We shall, for the most part, work within the framework of classically defined functions rather than distributions (sometimes also called symbolic functions, or generalized functions). For such classically defined functions, we take our clue from the integral calculus and require that our candidate functions be defined on the closed interval $a \leqq y \leqq x$, that they be bounded everywhere in the half-open interval $a < y \leqq x$, and be "better behaved" at the lower limit $a$ than is $[y-a]^{-1}$.

We define the class of "differintegrable series" to be all finite sums of functions, each of which may be represented

$$f(y) = [y-a]^p \sum_{j=0}^{\infty} a_j [y-a]^{j/n}, \qquad a_0 \neq 0, \quad p > -1, \tag{3.1.1}$$

as the product of a power of $[y-a]$ and an analytic function of $[y-a]^{1/n}$, $n$ a positive integer.[1] Notice that $p$ has been chosen to ensure that the leading coefficient is nonzero in equation (3.1.1). Such differintegrable series $f$ then satisfy

$$\lim_{y \to a}\{[y-a]f(y)\} = 0$$

(this is what we mean by the phrase "better behaved" at $a$ than $[y-a]^{-1}$) and, in addition to bounded examples, include such functions as $f(y) = [y-a]^{-\frac{1}{2}}$ and $f(y) = \sin(\sqrt{y-a})/[y-a]^{\frac{5}{4}}$. Indeed, most of the special functions of mathematical physics are differintegrable series according to the definition we have given. An important consequence of the representation (3.1.1) is that $f$ may be further decomposed as a finite sum

$$f(y) = [y-a]^p \sum_{j_1=0}^{\infty} a_{j_1}[y-a]^{j_1} + [y-a]^{[np+1]/n} \sum_{j_2=0}^{\infty} a_{j_2}[y-a]^{j_2}$$
$$+ \cdots + [y-a]^{[np+n-1]/n} \sum_{j_n=0}^{\infty} a_{j_n}[y-a]^{j_n}$$

of $n$ differintegrable "units" $f_U$, each of which is a product of a power (greater than $-1$) of $y-a$ and a function analytic in $y-a$. The desirability

[1] It is likely that arbitrary powers of $[y-a]$ could be treated in the infinite series factor of $f$. At a minimum, inclusion of such functions would complicate many proofs in Chapter 5 without introducing any new ideas. We therefore restrict our attention to the simpler, more tractable, series.

of this property will become more apparent in Chapter 5 (see, for example, Sections 5.2 and 5.7). Another feature motivating the selected form for differintegrable series is that this form is reproduced upon differintegration to any order (although the restriction $p > -1$ may be violated in the differintegrated series). Notice the analogy with analytic series whose form is reproduced upon differentiation or integration to integer order.

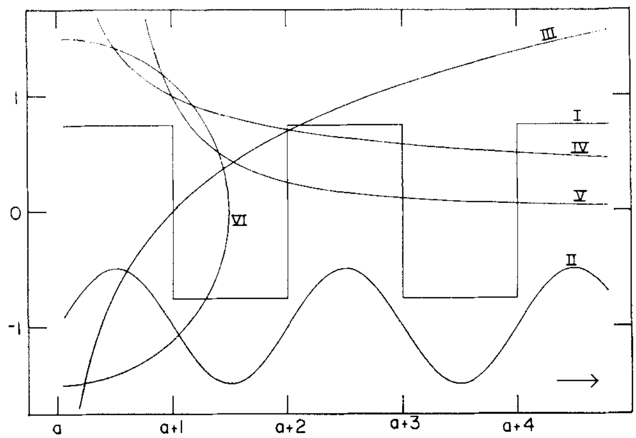


FIG. 3.1.1. Some functions that can be differintegrated: I, a square wave function; II, the sine function $\frac{1}{2}\sin(\pi[x-a])-1$; III, the logarithmic function $\ln(x-a)$; and IV, the inverse power $[x-a]^{-\frac{1}{2}}$ and one that cannot; V, the inverse power $[x-a]^{-2}$. The function VI $\sqrt{\frac{9}{4}-[x-a]^2}$, if treated as single valued, can be differintegrated, though the differintegral will be real only if $a < x \leq \frac{3}{2}+a$.

Many functions not expansible as differintegrable series are, nevertheless, differintegrable. Good examples of such functions are the logarithm and Heaviside's unit function (see Sections 6.7 and 6.8); an even simpler example is the function $f \equiv 0$ (see Section 4.2). When we use the phrase "differintegra ble function" we mean to include all of the above and, in fact, any function whose differintegrals can be determined. Figure 3.1.1 shows examples of some functions that can be differintegrated and some that cannot.

## 3.2 FUNDAMENTAL DEFINITIONS

The first definition we offer is the one we regard as the most fundamental in that it involves the fewest restrictions on the functions to which it applies and avoids explicit use of the notions of ordinary derivative and integral. This definition, which directly extends and unifies notions of difference quotients and Riemann sums, was first given by Grünwald (1867) and later extended by Post (1930).

Referring back to Section 2.2 and the discussion leading up to equation (2.2.3), we shall define the differintegral of order $q$ by the formula

$$\text{(3.2.1)}\qquad \frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\},$$

where $q$ is arbitrary.[2] Notice that definition (3.2.1) involves only evaluations of the function itself; no *explicit* use is made of derivatives or integrals of $f$.

We should like to establish that, based on the definition (3.2.1),

$$\text{(3.2.2)}\qquad \frac{d^n}{dx^n}\frac{d^q f}{[d(x-a)]^q} = \frac{d^{n+q}f}{[d(x-a)]^{n+q}}$$

for all positive integers $n$ and all $q$. One might think of this property as a limited composition law, that is, a rule for composing orders of the generalized differintegral. A complete discussion of the composition law will be presented in Section 5.7.

To establish (3.2.2), let $\delta_N x = [x-a]/N$, as in Section 2.2, and notice that

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x-j\,\delta_N x)\right\}.$$

Upon subdividing the interval $a \leqq y \leqq x - \delta_N x$ into only $N-1$ equally spaced subintervals, we see that

$$\begin{aligned}\frac{d^q f}{[d(x-a)]^q}(x-\delta_N x) &= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-2}\frac{\Gamma(j-q)}{\Gamma(j+1)}f(x-\delta_N x-j\,\delta_N x)\right\}\\ &= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)}\sum_{j=1}^{N-1}\frac{\Gamma(j-q-1)}{\Gamma(j)}f(x-\delta_N x)\right\}.\end{aligned}$$

On differentiation (making use of restricted limits as explained in Section 2.2 to define $d/dx$), one gets

$$\begin{aligned}&\frac{d}{dx}\frac{d^q f}{[d(x-a)]^q}\\ &\quad\equiv \lim_{N\to\infty}\left\{[\delta_N x]^{-1}\left[\frac{d^q f}{[d(x-a)]^q}(x) - \frac{d^q f}{[d(x-a)]^q}(x-\delta_N x)\right]\right\}\\ &\quad= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q-1}}{\Gamma(-q)}\left[\Gamma(-q)f(x) + \sum_{j=1}^{N-1}\left\{\frac{\Gamma(j-q)}{\Gamma(j+1)} - \frac{\Gamma(j-q-1)}{\Gamma(j)}\right\}\right]\right\}.\end{aligned}$$

[2] Even for $q$ a nonnegative integer [so that $\Gamma(-q)$ is infinite] the ratio $\Gamma(j-q)/\Gamma(-q)$ is finite.

But making use of the recurrence properties of gamma functions [equation (1.3.2)],

$$\frac{\Gamma(j-q)}{\Gamma(j+1)} - \frac{\Gamma(j-q-1)}{\Gamma(j)} = \frac{\Gamma(-q)\Gamma(j-q-1)}{\Gamma(-q-1)\Gamma(j+1)}$$

is obtained. Therefore,

$$\begin{aligned}\frac{d}{dx}\frac{d^q f}{[d(x-a)]^q} &= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q-1}}{\Gamma(-q-1)}\left[\sum_{j=0}^{N-1}\frac{\Gamma(j-q-1)}{\Gamma(j+1)}f(x-j\,\delta_N x)\right]\right\}\\ &= \frac{d^{q+1}f}{[d(x-a)]^{q+1}}.\end{aligned}$$

Equation (3.2.2) follows by induction.

The most frequently encountered definition of an integral of fractional order is via an integral transform called the Riemann–Liouville integral (Liouville, 1832a; Riesz, 1949; Riemann, 1953). To motivate this definition, one need only examine Cauchy's formula (2.7.2) and replace $-n$ by $q$, suggesting the generalization to noninteger $q$:

$$(3.2.3)\qquad \left[\frac{d^q f}{[d(x-a)]^q}\right]_{\mathrm{R-L}} = \frac{1}{\Gamma(-q)}\int_a^x [x-y]^{-q-1} f(y)\,dy, \qquad q<0.$$

In equation (3.2.3) we have used the symbol $[\ ]_{\mathrm{R-L}}$ to designate the Riemann–Liouville fractional integral as possibly distinct from our more basic definition (3.2.1). Presently we shall show that the two definitions yield identical results and the symbol $[\ ]_{\mathrm{R-L}}$ will be dropped at that point.

Riesz (1949) regarded $q$ as a complex variable; the integral (3.2.3) converges for $\mathrm{Re}(q) < 0$ and defines, for fixed $f$, an analytic function of $q$ in the left-half $q$-plane. Although the integral in (3.2.3) diverges when $\mathrm{Re}(q) \geqq 0$, a meaning may be attached to the operator $[\ ]_{\mathrm{R-L}}$ by a proper analytic continuation across the line $\mathrm{Re}(q) = 0$ provided the function $f$ is sufficiently differentiable in $a \leqq y \leqq x$. In fact, if $f$ is $n$ times differentiable the formula

$$(3.2.4)\qquad \left[\frac{d^q f}{[d(x-a)]^q}\right]_{\mathrm{R-L}} = \sum_{k=0}^{n-1}\frac{[x-a]^{-q+k} f^{(k)}(a)}{\Gamma(-q+k+1)} + \left[\frac{d^{q-n} f^{(n)}}{[d(x-a)]^{q-n}}\right]_{\mathrm{R-L}}$$

defines an analytic function of $q$ for $\mathrm{Re}(q) < n$ and thus provides a valid analytic continuation [see, for example, Knopp (1945)] of formula (3.2.3) (Riesz, 1949; Duff, 1956). This analytic continuation is very analogous to the way one extends the definition of the gamma function (see Section 1.3) in the complex plane. This is not surprising since the gamma function plays such a fundamental role in defining differintegrals. We shall frequently make use of this analyticity in $q$ to simplify proofs dealing with $d^q f/[d(x-a)]^q$ for general $q$.

We prefer, however, to take a different, more elementary, approach in attaching a meaning to the operator for $q \geqq 0$. Formula (3.2.3) will be retained as the $q < 0$ definition of the differintegral; it is extended to $q \geqq 0$ by insisting that equation (3.2.2) be satisfied by the Riemann–Liouville integral. That is, we shall require that

$$\left[\frac{d^q f}{[d(x-a)]^q}\right]_{\text{R-L}} \equiv \frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{[d(x-a)]^{q-n}}\right]_{\text{R-L}}, \tag{3.2.5}$$

where $d^n/dx^n$ effects ordinary $n$-fold differentiation and $n$ is an integer chosen so large that $q - n < 0$. Together with equation (3.2.5), definition (3.2.3) then defines the operator

$$\left[\frac{d^q}{[d(x-a)]^q}\right]_{\text{R-L}}$$

for all $q$.

We see that if we choose $q$ to equal the negative integer $-n$ in equation (3.2.3), we obtain

$$\left[\frac{d^{-n} f}{[d(x-a)]^{-n}}\right]_{\text{R-L}} = \frac{1}{\Gamma(n)}\int_a^x [x-y]^{n-1} f(y)\, dy.$$

Comparison with equation (2.7.2) reveals that the Riemann–Liouville definition correctly generates an $n$-fold integral of $f$. It is also evident on choosing $n = 1$ and $q = 0$ in equation (3.2.5) that

$$\left[\frac{d^0 f}{[d(x-a)]^0}\right]_{\text{R-L}} = \frac{d}{dx}\left[\frac{d^{-1} f}{[d(x-a)]^{-1}}\right]_{\text{R-L}} = f.$$

Moreover, by selecting $n = q$, we establish that

$$\left[\frac{d^n f}{[d(x-a)]^n}\right]_{\text{R-L}} = \frac{d^n f}{dx^n}$$

is the ordinary $n$th derivative when $n$ is a nonnegative integer.

Courant and Hilbert (1962) defined a semiderivative by

$$\frac{d^{\frac{1}{2}} f}{[d(x-a)]^{\frac{1}{2}}} \equiv \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_a^x \frac{f(y)\, dy}{\sqrt{x-y}} = \frac{d}{dx}\frac{d^{-\frac{1}{2}} f}{[d(x-a)]^{-\frac{1}{2}}}, \tag{3.2.6}$$

which is the same as our extended Riemann–Liouville definition, as is seen from equation (3.2.5). A careful integration by parts shows that

$$\begin{aligned} \left[\frac{d^{\frac{1}{2}} f}{[d(x-a)]^{\frac{1}{2}}}\right]_{\text{R-L}} &= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_a^x \frac{f(y)\, dy}{\sqrt{x-y}} \\ &= \frac{1}{\sqrt{\pi}}\frac{f(a)}{\sqrt{x-a}} + \frac{1}{\sqrt{\pi}}\int_a^x \frac{f^{(1)}(y)\, dy}{\sqrt{x-y}} \\ &= \frac{f(a)}{\Gamma(\frac{1}{2})\sqrt{x-a}} + \left[\frac{d^{-\frac{1}{2}} f^{(1)}}{[d(x-a)]^{-\frac{1}{2}}}\right]_{\text{R-L}}, \end{aligned} \tag{3.2.7}$$

which agrees with equation (3.2.4), as it must, for $q = +\frac{1}{2}$ and $n = 1$. We notice, however, that a semiderivative defined in this way requires explicit knowledge of the first derivative $f^{(1)}$. On the other hand, setting $q = \frac{1}{2}$ in our fundamental definition (3.2.1) avoids explicit use of the first derivative, as does equation (3.2.6).

### 3.3 IDENTITY OF DEFINITIONS

It is now pertinent to ask whether the Riemann–Liouville definition, based on equation (3.2.3) for negative $q$ and its extension to $q \geqq 0$ by means of equation (3.2.5) is equivalent to the definition (3.2.1). That is, do the operators so defined coincide for all functions $f$? We shall establish that this is, indeed, the case. First we prove the identity for a subset of $q$ values[3] and then make use of the property (3.2.2) to extend the identity to all orders $q$.

Thus we choose $f$ to be an arbitrary, but fixed, function on the interval $a \leqq y \leqq x$. As before, let $\delta_N x = [x-a]/N$. Then the difference

$$
\begin{aligned}
\Delta &\equiv \frac{d^q f}{[d(x-a)]^q}(x) - \left[\frac{d^q f}{[d(x-a)]^q}(x)\right]_{\text{R-L}} \\
&= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(x - j\,\delta_N x)\right\} - \int_0^{x-a} \frac{f(x-u)\,du}{\Gamma(-q)u^{1+q}} \\
&= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(x - j\,\delta_N x)\right\} - \lim_{N\to\infty}\left\{\sum_{j=0}^{N-1} \frac{f(x-j\,\delta_N x)\,\delta_N x}{\Gamma(-q)[j\,\delta_N x]^{1+q}}\right\} \\
&= \lim_{N\to\infty}\left\{\frac{[\delta_N x]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} f(x - j\,\delta_N x)\left[\frac{\Gamma(j-q)}{\Gamma(j+1)} - j^{-1-q}\right]\right\} \\
&= \frac{[x-a]^{-q}}{\Gamma(-q)} \lim_{N\to\infty}\left\{\sum_{j=0}^{N-1} f\left(\frac{Nx - jx + ja}{N}\right) N^q \left[\frac{\Gamma(j-q)}{\Gamma(j+1)} - j^{-1-q}\right]\right\}.
\end{aligned}
$$

The $N$ terms within the summation will be treated as two groups: $0 \leqq j \leqq J-1$ and $J \leqq j \leqq N-1$, where $J$ is independent of $N$ and large enough to validate the asymptotic expansion (1.3.13) for summands in the second group. Thus

$$
\begin{aligned}
\Delta = &\frac{[x-a]^{-q}}{\Gamma(-q)} \lim_{N\to\infty}\left\{\sum_{j=0}^{J-1} f\left(\frac{Nx - jx + ja}{N}\right) N^q \left[\frac{\Gamma(j-q)}{\Gamma(j+1)} - j^{-1-q}\right]\right\} \\
&+ \frac{[x-a]^{-q}}{\Gamma(-q)} \lim_{N\to\infty}\left\{\frac{1}{N}\sum_{j=J}^{N-1} f\left(\frac{Nx - jx + ja}{N}\right)\left[\frac{j}{N}\right]^{-2-q}\left[\frac{q[q+1]}{2N} + \frac{O(j^{-1})}{N}\right]\right\}.
\end{aligned}
$$

[3] For integer $q$ of either sign this identity is intrinsic to both the Grünwald and the Riemann–Liouville definitions.

Now, for $q < -1$, the $J$ bracketed terms within the first summation are bounded. Hence, if $f([Nx - jx + ja]/N)$ is also bounded for $j$ within the first group, the presence of the $N^q$ factor ensures that the first sum vanishes in the $N \to \infty$ limit. Examining the three factors within the second summation, we note that $[j/N]^{-2-q}$ is invariably less than unity if $q \leqq -2$ and that the third factor tends to zero when $N \to \infty$. Hence, if $f([Nx - jx + ja]/N)$ is bounded for $j$ within the second group, each term in the second summation vanishes as $1/N$ when $N \to \infty$. As there are fewer than $N$ summands in the second group, the presence of the pre-summation $1/N$ factor ensures that the second limit is zero.

The above demonstrates that if $f$ is bounded on $a < y \leqq x$ and if $q \leqq -2$, then

$$\frac{d^q f}{[d(x-a)]^q}(x) - \left[\frac{d^q f}{[d(x-a)]^q}(x)\right]_{\mathrm{R-L}} \equiv \Delta = 0, \tag{3.3.1}$$

so that the two definitions when applied to functions so bounded[4] are indeed identical for $q \leqq -2$. This fact, coupled with the property (3.2.2) and the requirement (3.2.5), shows that the two definitions are identical for any $q$. Indeed, for arbitrary $q$ we know that for any positive integer $n$

$$\frac{d^q f}{[d(x-a)]^q} = \frac{d^n}{dx^n}\left\{\frac{d^{q-n} f}{[d(x-a)]^{q-n}}\right\}$$

and

$$\left[\frac{d^q f}{[d(x-a)]^q}\right]_{\mathrm{R-L}} = \frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{[d(x-a)]^{q-n}}\right]_{\mathrm{R-L}}.$$

One need only choose $n$ sufficiently large that $q - n \leqq -2$ and make use of (3.3.1) to complete the proof.

## 3.4 OTHER GENERAL DEFINITIONS

We have presented in Section 3.2 the two definitions of generalized differintegration which we favor. Other starting points for generalizing ordinary derivatives and integrals have been used in the literature, however. In the present section some of these are discussed briefly. Our treatment is

[4] The astute reader may have noticed that, while our argument was made only for functions bounded on $a < y \leqq x$, the proof may be sharpened to admit all differintegrable series (see Section 3.1). We have omitted the rather tedious proof.

cursory rather than exhaustive since we shall have little occasion throughout the rest of this book to refer to any definitions other than those of Section 3.2.

Riemann (1953) considered power series with noninteger exponents to be extensions of Taylor's series and built up a generalized derivative for such functions by use of the formula

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q}, \tag{3.4.1}$$

this being an obvious generalization of the formula

$$\frac{d^n x^p}{dx^n} = p[p-1][p-2]\cdots[p-n+1]x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}$$

for $n$ a nonnegative integer. This is similar to the approach taken by Scott Blair (1947), Heaviside (1920), and others. Liouville (1832a) defined a generalized derivative for functions expansible as a series of exponentials, $f = \sum c_j \exp(b_j x)$, by

$$\frac{d^q f}{dx^q} \equiv \sum_{j=0}^{\infty} c_j b_j^q \exp(b_j x),$$

a definition that leads to a different operator than those of Section 3.2. Krug (1890), in fact, showed that this definition corresponds to a lower limit $a = -\infty$ in the Riemann–Liouville integral (3.2.3). Civin (1941) used a similar idea for integrals rather than sums to define

$$\frac{d^q f}{dx^q} \equiv \int_{-\pi}^{\pi} i^q t^q \exp(ixt)\, ds(t),$$

where $f$ is given by

$$f \equiv \int_{-\pi}^{\pi} \exp(ixt)\, ds(t).$$

Weyl (1917), working with periodic functions, defined the $q$th differintegral of $f$ via

$$\frac{d^q f}{dx^q} \equiv \int_{-\infty}^{x} \frac{f(y)\, dy}{[x-y]^{1+q}},$$

this integral being often called the Weyl integral. More recently, Gaer (1968) and Gaer and Rubel (1971) have given a definition closely related to the Weyl integral. Based on their definition, Gaer and Rubel developed a formula for the fractional derivative of a product which does not reduce to Leibniz's formula in the classical case. We shall have more to say about extensions of Leibniz's formula in Section 5.5.

A different avenue for motivating the definition of a differintegral stems from consideration of Cauchy's integral formula:

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{[\zeta - z]^{n+1}}, \tag{3.4.2}$$

where $C$ describes a closed contour surrounding the point $z$ and enclosing a region of analyticity of $f$. When the positive integer $n$ is replaced by a non-integer $q$, then $[\zeta - z]^{-q-1}$ no longer has a pole at $\zeta = z$ but a branch point. One is no longer free to deform the contour $C$ surrounding $z$ at will, since the integral will depend on the location of the point at which $C$ crosses the branch line for $[\zeta - z]^{-q-1}$. This point is chosen to be 0 and the branch line to be the straight line joining 0 and $z$ and continuing indefinitely in the quadrant $\mathrm{Re}(\zeta) \leqq 0$, $\mathrm{Im}(\zeta) \leqq 0$. Then one simply defines, for $q$ not a negative integer,

$$\frac{d^q f}{dz^q} = \frac{\Gamma(q+1)}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{[\zeta - z]^{q+1}}, \tag{3.4.3}$$

where the contour $C$ begins and ends at $\zeta = 0$ enclosing $z$ once in the positive sense. To uniquely specify the denominator of the integrand, one defines

$$[\zeta - z]^{q+1} = \exp\bigl([q+1]\ln(\zeta - z)\bigr),$$

where $\ln(\zeta - z)$ is real when $\zeta - z > 0$. We can relate the definition (3.4.3) to that of Riemann–Liouville by first deforming the contour $C$ into a contour $C'$ lying on both sides of the branch line (see Fig. 3.4.1). One finds that[5]

$$\begin{aligned}\frac{\Gamma(q+1)}{2\pi i} \oint_{C'} \frac{f(\zeta)\, d\zeta}{[\zeta - z]^{q+1}} &= \frac{\Gamma(q+1)}{2\pi i}[1 - \exp(-2\pi i[q+1])] \int_0^z \frac{f(\zeta)\, d\zeta}{[\zeta - z]^{q+1}} \\ &= \frac{1}{\Gamma(-q)} \int_0^z \frac{f(\zeta)\, d\zeta}{[z - \zeta]^{q+1}},\end{aligned}$$

which is the Riemann–Liouville definition (3.2.3) with $a = 0$. The definition (3.4.3) is attributed by Osler (1970a) to Nekrassov (1888).

Erdélyi (1964) defined a $q$th-order differintegral of a function $f(z)$ with respect to the function $z^n$ by

$$\frac{d^q f}{[d(z^n - a^n)]^q} \equiv \frac{1}{\Gamma(-q)} \int_a^z \frac{f(\zeta) n \zeta^{n-1}\, d\zeta}{[z^n - \zeta^n]^{1+q}}.$$

[5] In establishing the second equality we have used $[-1]^{-q-1} = \exp(i\pi[q+1])$ and the reflection formula $\Gamma(x)\Gamma(1-x) = \pi \csc(\pi x) = 2\pi i/[\exp(\pi i x) - \exp(-\pi i x)]$ for the gamma function.

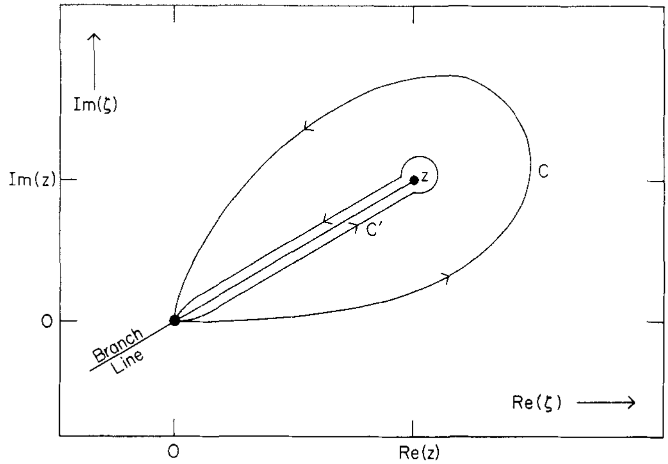


FIG. 3.4.1 The Cauchy contour $C$ is deformed into $C'$ for the purpose of implementing definition (3.4.3).

Osler (1970a) has extended Erdélyi's work by defining a differintegral of a function $f(z)$ with respect to an arbitrary function $g(z)$ by considering the Riemann–Liouville integral

$$\frac{d^q f}{[d(g(z) - g(a))]^q} \equiv \frac{1}{\Gamma(-q)} \int_a^z \frac{f(\zeta) g^{(1)}(\zeta)\, d\zeta}{[g(z) - g(\zeta)]^{q+1}},$$

where $a$ is chosen to give $g(a) = 0$, that is $a = g^{-1}(0)$. Upon setting $g(z) = z - a$, one obtains the Riemann–Liouville integral once again. Making the change $u \equiv g(\zeta)/g(z)$ gives

$$\frac{d^q f}{[d(g(z) - g(a))]^q} = \frac{g(z)^{-q}}{\Gamma(-q)} \int_0^1 \frac{f(g^{-1}(g(z)u))}{[1 - u]^{q+1}}\, du.$$

Certain choices of $g$ have been shown by Erdélyi and by Osler to lead to a number of formulas of interest in classical analysis.

We close this section with a short discussion of the pros and cons of the various definitions of generalized differintegrals. The fundamental definition (3.2.1) based on difference quotients and Riemann sums has the great merit of general applicability but is awkward to use as a working definition except for fairly simple functions. We favor it as a basic definition because the functions to which it may be applied are not limited in any intrinsic way, nor does it explicitly make use of the classical notions of derivative or integral. It may also have merit as a tool for generating numerical approximations to the differintegral of an arbitrary function.

The Riemann–Liouville definition (3.2.3) is convenient to implement but requires $q < 0$ for the integral to converge. Application of (3.2.5) to circumvent this restriction may also lead to difficulties with implementation since one must compute an $n$-fold derivative of an integral. Nevertheless, because of its convenient formulation in terms of a single integral it enjoys great popularity as a working definition of a differintegral.

The definition (3.4.3) based on Cauchy's integral formula is, as we have seen, closely related to the Riemann–Liouville definition. However, it applies only to locally analytic functions and is inapplicable when $q$ is a negative integer. A desirable feature of the definition (3.4.3) is the flexibility provided through the choice of the contour $C$. Making use of this flexibility, Osler (1972a) has extended several notions valid for real-valued functions "into the complex plane." Thus he has used (3.4.3) to develop an integral analog of Taylor's series and also to develop an integral analog of Lagrange's expansion, which he shows extends some features of Fourier analysis into the complex plane. Indeed, an appropriate choice of contour $C$ reveals that, apart from a constant factor, $z^q d^q f/dz^q$ is essentially the Fourier transform in $q$-space of the function $f$. Many useful results of Fourier analysis may be developed from this starting point by making use of the general properties of differintegral operators.

Before leaving this section, let us return briefly to the Grünwald definition. Recall that the motivation for definition (3.2.1),

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{\left[\dfrac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\},$$

is that it correctly reproduces standard classical definitions when $q$ is an integer of either sign. Thus when $q = -1$ or $+1$, to take the simplest examples, the Grünwald definition reduces respectively to a Riemann sum limit,

$$\text{(3.4.4)} \qquad \frac{d^{-1} f}{[d(x-a)]^{-1}} = \lim_{N\to\infty}\left\{\frac{x-a}{N}\sum_{j=0}^{N-1} f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\},$$

or a backward difference quotient limit,

$$\text{(3.4.5)} \qquad \frac{d^1 f}{[d(x-a)]^1} = \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^{-1}\sum_{j=0}^{1}[-]^j f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\}.$$

Notice, however, that the Riemann sum would converge more rapidly to the integral as $N\to\infty$ were formula (3.4.4) replaced by

$$\frac{d^{-1} f}{[d(x-a)]^{-1}} = \lim_{N\to\infty}\left\{\frac{x-a}{N}\sum_{j=0}^{N-1} f\left(x-[j+\tfrac{1}{2}]\left[\frac{x-a}{N}\right]\right)\right\}.$$

Similarly, the difference quotient would converge more rapidly to the true derivative were (3.4.5) replaced by the central difference formulation

$$\frac{d^1 f}{[d(x-a)]^1} = \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^{-1}\sum_{j=0}^{1}[-]^j f\left(x-[j-\tfrac{1}{2}]\left[\frac{x-a}{N}\right]\right)\right\}.$$

Similar modifications improve the Grünwald definition for other integer $q$ values and suggest a generalization to

$$\text{(3.4.6)} \qquad \frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x-[j-\tfrac{1}{2}q]\left[\frac{x-a}{N}\right]\right)\right\}.$$

The improved convergence of this modified Grünwald formula permits the design of a more efficient differintegration algorithm, as we shall see in Section 8.2.

## 3.5 OTHER FORMULAS APPLICABLE TO ANALYTIC FUNCTIONS

The purpose of this section is to discuss alternate representations for $d^q/[d(x-a)]^q$ for real analytic functions, i.e., functions $\phi$ that have convergent power series expansions in the interval $a \leqq y \leqq x$ of interest. Such representations offer computational variety when it comes to the evaluation of $q$th order differintegrals for specific choices of $\phi$.

We restrict attention initially to $q < 0$, since this permits us to work with the Riemann–Liouville definition (3.2.3). Thus

$$\text{(3.5.1)} \qquad \frac{d^q\phi}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)}\int_a^x \frac{\phi(y)\,dy}{[x-y]^{q+1}} = \frac{1}{\Gamma(-q)}\int_0^{x-a}\frac{\phi(x-v)\,dv}{v^{q+1}}$$

with $v \equiv x - y$. Upon Taylor expansion of $\phi(x-v)$ about $x$ one has

$$\text{(3.5.2)} \qquad \phi(x-v) = \phi - v\phi^{(1)} + \frac{v^2}{2!}\phi^{(2)} - \cdots = \sum_{k=0}^{\infty}\frac{[-]^k v^k \phi^{(k)}}{k!}.$$

The representation (3.5.2) involves no remainder since we have assumed $\phi$ to have a convergent power series expansion and since such an expansion is unique. When this expansion is inserted into (3.5.1) and term-by-term integration performed, the result is

$$\text{(3.5.3)} \qquad \frac{d^q\phi}{[d(x-a)]^q} = \sum_{k=0}^{\infty}\frac{[-]^k[x-a]^{k-q}\phi^{(k)}}{\Gamma(-q)[k-q]k!}.$$

An analyticity (in $q$) argument[6] may be used to establish the formula (3.5.3) for all $q$ even though it was derived on the basis of the assumption $q < 0$.

Another proof of equation (3.5.3) that starts with definition (3.2.1) proceeds as follows, making use of the equation (1.3.32):

$$\frac{d^q\phi}{[d(x-a)]^q}$$

$$= [x-a]^{-q} \lim_{N\to\infty}\left\{N^q \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\, \phi\left(x + j\left[\frac{a-x}{N}\right]\right)\right\}$$

$$= [x-a]^{-q} \lim_{N\to\infty}\left\{N^q \sum_{j=0}^{N-1} \binom{j-q-1}{j} \sum_{m=0}^{\infty} \binom{j}{m} G_m\left(\phi, x, \frac{a-x}{N}\right)\right\}$$

$$= [x-a]^{-q} \lim_{N\to\infty}\left\{N^q \sum_{m=0}^{\infty} G_m\left(\phi, x, \frac{a-x}{N}\right) \sum_{j=0}^{N-1} \binom{j-q-1}{j}\binom{j}{m}\right\}$$

$$= [x-a]^{-q} \lim_{N\to\infty}\left\{N^q \sum_{m=0}^{\infty} \binom{m-q-1}{m}\binom{N-q-1}{N-m-1} G_m\left(\phi, x, \frac{a-x}{N}\right)\right\},$$

where the final step is a consequence of formula (1.3.22). Now we use the expression for $G_m(\phi, x, y)$ given in (1.3.33) to get

$$\frac{d^q\phi}{[d(x-a)]^q}$$

$$= [x-a]^{-q} \lim_{N\to\infty}\left\{N^q \sum_{m=0}^{\infty} m! \binom{m-q-1}{m}\binom{N-q-1}{N-m-1}\right.$$

$$\left.\times \sum_{k=m}^{\infty} \left[\frac{a-x}{N}\right]^k S_k^{[m]} \frac{\phi^{(k)}(x)}{k!}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m-q)}{\Gamma(-q)} \sum_{k=m}^{\infty} [-]^k \frac{[x-a]^{k-q}}{k!}\, S_k^{[m]} \phi^{(k)}(x) \lim_{N\to\infty}\left\{N^{q-k}\binom{N-q-1}{N-m-1}\right\}.$$

The summation on $k$ involves only one nonzero term by virtue of

$$\lim_{N\to\infty}\left\{N^{q-k}\binom{N-q-1}{N-m-1}\right\} = \begin{cases} \dfrac{1}{\Gamma(m-q+1)}, & k = m, \\ 0, & k > m, \end{cases}$$

which follows from expression (1.3.15). This results in

$$\text{(3.5.4)} \qquad \frac{d^q\phi}{[d(x-a)]^q} = \sum_{m=0}^{\infty} \frac{\Gamma(m-q)}{\Gamma(-q)} \frac{[-]^m[x-a]^{m-q}}{m!}\, S_m^{[m]} \frac{\phi^{(m)}(x)}{\Gamma(m-q+1)},$$

[6] See p. 49.

which, by virtue of equation (1.3.29) and the recurrence property (1.3.2) of the gamma function, is seen to identify with equation (3.5.3).

We have presented this alternative proof of formula (3.5.3) to illustrate how it is possible to handle definition (3.2.1). In general, however, as with this example, the algebraic manipulation required to utilize definition (3.2.1) exceeds that required for an approach that employs the Riemann–Liouville definition followed by an analyticity argument. Accordingly, the latter method will constitute our usual approach to the development of formulas for differintegrals.

Formula (3.5.3) is useful inasmuch as it involves only integer-order derivatives of $\phi(x)$. This formula may be written somewhat more concisely, as will be demonstrated in Section 4.1.

## 3.6 SUMMARY OF DEFINITIONS

In summary, this chapter has shown that

$$\lim_{N\to\infty}\left\{\left[\frac{N}{x-a}\right]^q \sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} f\left(\frac{Nx-jx+ja}{N}\right)\right\} \tag{3.6.1}$$

is a valid expression for a $q$th order derivative or a $(-q)$th order integral of $f$ whether or not $q$ is a real integer, and that

$$\frac{1}{\Gamma(-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q+1}} \tag{3.6.2}$$

is identical with (3.6.1) for $q < 0$. To replace (3.6.2) for $q \geqq 0$ one must utilize either

$$\frac{d^n}{dx^n}\left[\frac{1}{\Gamma(n-q)}\int_a^x \frac{f(y)\,dy}{[x-y]^{q-n+1}}\right], \qquad n > q, \tag{3.6.3}$$

or

$$\sum_{k=0}^{n-1}\frac{[x-a]^{k-q} f^{(k)}(a)}{\Gamma(k-q+1)} + \frac{1}{\Gamma(n-q)}\int_a^x \frac{f^{(n)}(y)\,dy}{[x-y]^{q-n+1}}, \qquad n > q, \tag{3.6.4}$$

both these forms yielding results identical with those given by (3.6.1).

If $\phi$ is analytic, the expression

$$\sum_{k=0}^{\infty}\frac{[-]^k[x-a]^{k-q}\phi^{(k)}}{\Gamma(-q)[k-q]k!}, \tag{3.6.5}$$

developed in Section 3.5, is identical with those above. We shall prove in Section 5.2 that the expression

$$\sum_{k=0}^{\infty} \frac{[x-a]^{k-q}\phi^{(k)}(a)}{\Gamma(k-q+1)} \tag{3.6.6}$$

is also a valid representation of $d^q\phi/[d(x-a)]^q$. We notice that (3.6.6) is obtained from (3.6.4) by letting $n$ tend to infinity and dropping the integral remainder term.

It is appropriate to restate that the first four formulas define

$$\frac{d^q f}{[d(x-a)]^q}$$

for all $q$ (real or complex, integer or fractional) and for all differintegrable[7] $f$. The last two formulas have been established only for analytic functions $\phi$. The operation of differintegration is thereby defined between the limits $a$ and $x$, where $x > a$. Evaluation at the point $x = a$ is specifically excluded.

[7] See Section 3.1 for a discussion concerning the notion of a differintegrable function.