CHAPTER 9

REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

The message of this ninth chapter is that the fractional calculus enables a plethora of transcendental functions to be expressed in terms of a basis set of only three functions: the inverse binomial, the exponential, and the zero-order Bessel. Using only simple algebraic operations (multiplication by constants and powers) in addition to differintegration, each of these reducible transcendentals may be synthesized from one or other of the three basis functions. A partial listing of reducible transcendentals includes: exponential-related functions such as exponential integrals and error functions; logarithms; inverse trigonometric functions and their hyperbolic counterparts; incomplete gamma and beta functions; circular and hyperbolic sines, cosines, Fresnel integrals, sine integrals, cosine integrals, Bessel functions, and Struve functions; generalized functions such as Kummer, Gauss and other hypergeometric functions; Legendre functions and associated Legendre functions; and elliptic integrals.

Our position is that these differintegral representations of transcendental functions constitute more appealing definitions than those customary in terms of definite integrals, differential equations, and the like. Indeed, we are tempted to go even further and assert that differintegration removes the need to recognize many transcendentals as functions in their own right. Thus, if the incomplete gamma function is nothing but the differintegral of an exponential (as Section 6.2 shows it to be), is any purpose served in regarding it as anything but the differintegral of an exponential? Need it be graced with a special name? Our viewpoint is that, in this case (as in many others), the description "exponential differintegral" and the symbol $\exp(-x)\,d^q\{\exp(x)\}/dx^q$ are more informative than the name "incomplete gamma function" and the symbol $\gamma^*(-q, x)$.

The definition of and our notation for a generalized hypergeometric function were presented in Section 2.10; the behavior of these functions under differintegration was derived in Section 6.6; and the reader will have met these useful generalized functions elsewhere in this text. In this chapter we shall encounter many instances of the product of a power of x and a generalized hypergeometric function; that is,

$$x^p\left[\pm x \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right], \qquad p > -1.$$

For brevity we shall term such products simply "hypergeometrics." They constitute examples of differintegrable series (see Section 3.1).

9.1 TRANSCENDENTAL FUNCTIONS AS HYPERGEOMETRICS

The majority of those important transcendental functions which remain finite[1] as their arguments approach zero are expressible in terms of generalized hypergeometric functions of complexity $\frac{0}{0}$, $\frac{0}{1}$, $\frac{1}{1}$, $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, or $\frac{1}{3}$. This section is devoted merely to illustrating this statement by citing copious examples.

We shall find it convenient to denote by $\pm x$ the argument of the hypergeometric function; accordingly the argument of the transcendental function is not always x (frequently being $\sqrt{x}$ or $2\sqrt{x}$). A change of independent variable is, of course, always possible. Usually, the formulas below imply $x > 0$; sometimes the restriction $0 < x < 1$ is required. Where the left-hand member of the formula incorporates two functions, such as $\genfrac{}{}{0pt}{}{\sinh(x)}{\sin(x)}$, the upper formula is associated with a positive argument $+x$ of the hypergeometric function, the lower with the negative argument $-x$.

The only instances of hypergeometric functions of $\frac{0}{0}$ complexity are

$$\text{(9.1.1)} \qquad \frac{1}{1 \mp x} = \left[\pm x \frac{}{}\right].$$

Hypergeometrics of $\frac{0}{1}$ complexity are exponential-like. The general case is

$$\exp(x)\gamma^*(c, x) = \left[x \frac{}{c}\right],$$

[1] Functions unbounded at $x = 0$ may also be expressed as hypergeometrics provided the singularity at zero is a branch point of order less than unity.

but the important instances,

$$x^n \exp(x) = \left[x \frac{}{-n}\right], \qquad n = 0, 1, 2, \ldots,$$

$$\exp(x)\,\mathrm{erf}(\sqrt{x}) = \sqrt{x}\left[x \frac{}{\frac{1}{2}}\right],$$

and Dawson's integral

$$\mathrm{daw}(\sqrt{x}) = \frac{\sqrt{\pi}}{2}\left[-x \frac{}{\frac{1}{2}}\right],$$

are included therein.

Instances of the general case

$$\frac{\mathrm{B}_x(c,\, b - c + 1)}{[1 - x]^{b-c+1}} = \frac{\Gamma(c)x^c}{\Gamma(b+1)}\left[x \frac{b}{c}\right]$$

of a $\frac{1}{1}$ hypergeometric are numerous. They include the logarithm[2] (Section 6.7) as well as binomials

$$\frac{1}{[1 \mp x]^{b+1}} = \frac{1}{\Gamma(b+1)}\left[\pm x \frac{b}{0}\right]$$

and such inverse trigonometric functions as

$$\frac{\arcsin(\sqrt{x})}{\sqrt{1-x}} = \frac{\sqrt{\pi x}}{2}\left[x \frac{0}{\frac{1}{2}}\right]$$

and hyperbolic functions as

$$\frac{\mathrm{arctanh}(\sqrt{x})}{\arctan(\sqrt{x})} = \left[\pm x \frac{-\frac{1}{2}}{\frac{1}{2}}\right].$$

Oscillatory functions of argument $2\sqrt{x}$ and their hyperbolic counterparts are mostly hypergeometrics of $\frac{0}{2}$ complexity. Thus we find the sines,

$$\frac{\sinh(2\sqrt{x})}{\sin(2\sqrt{x})} = \sqrt{\pi x}\left[\pm x \frac{}{0, \frac{1}{2}}\right],$$

the cosines,

$$\frac{\cosh(2\sqrt{x})}{\cos(2\sqrt{x})} = \sqrt{\pi}\left[\pm x \frac{}{-\frac{1}{2}, 0}\right],$$

[2] The generalized logarithm that will be discussed in Section 10.5 is also a hypergeometric of $\frac{1}{1}$ complexity.

the Bessel functions,

$$\begin{matrix} I_\nu(2\sqrt{x}) \\ J_\nu(2\sqrt{x}) \end{matrix} = x^{\frac{1}{2}\nu}\left[\pm x \frac{\quad}{0,\ \nu}\right],$$

and the Struve functions,

$$\begin{matrix} L_\nu(2\sqrt{x}) \\ H_\nu(2\sqrt{x}) \end{matrix} = x^{\frac{1}{2}+\frac{1}{2}\nu}\left[\pm x \frac{\quad}{\frac{1}{2}, \frac{1}{2}+\nu}\right], \tag{9.1.2}$$

to be examples of this general pattern.

Closely related are the Airy functions, of which

$$\mathrm{fai}([9x]^{\frac{1}{3}}) = \Gamma(\tfrac{2}{3})\left[x \frac{\quad}{-\frac{1}{3}, 0}\right]$$

and

$$\mathrm{gai}([9x]^{\frac{1}{3}}) = \Gamma(\tfrac{1}{3})[\tfrac{1}{3}x]^{\frac{1}{3}}\left[x \frac{\quad}{0, \frac{1}{3}}\right]$$

are examples. We here use fai( ) and gai( ) to denote those Airy functions that Abramowitz and Stegun (1964, Section 10.4) denote by $f(\ )$ and $g(\ )$.

A variety of transcendental functions comprise the hypergeometric class of $\frac{1}{2}$ complexity. Those with one denominatorial parameter equal to zero are Kummer functions (Section 2.10), of which

$$\mathrm{erf}(\sqrt{x}) = \sqrt{\frac{x}{\pi}}\left[-x \frac{-\frac{1}{2}}{0, \frac{1}{2}}\right]$$

provides one example, many others being listed by Abramowitz and Stegun (1964, p. 509). An illustration of a $\frac{1}{2}$ hypergeometric that is not a Kummer function is the function

$$\mathrm{Ei}(x) - \ln(x) = \gamma + x\left[x \frac{0}{1, 1}\right]$$

related to the exponential integral.

An example of a hypergeometric of $\frac{2}{2}$ complexity is the associated Legendre function

$$\frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2})\Gamma(-\frac{1}{2}\nu - \frac{1}{2}\mu)}{2^\mu} P_\nu{}^\mu(\sqrt{1-x}) = x^{-\frac{1}{2}\mu}\left[x \frac{\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2},\ -\frac{1}{2}\nu - \frac{1}{2}\mu - 1}{-\mu, 0}\right]. \tag{9.1.3}$$

This hypergeometric function has two disposable constants ν and μ in addition to the independent variable x. Other important $\frac{2}{2}$ hypergeometrics, such as the elliptic integral

$$K(x) = \frac{1}{2}\left[x \,\frac{-\frac{1}{2}, -\frac{1}{2}}{0, 0}\right]$$

and the inverse hyperbolic sine

$$\text{(9.1.4)} \qquad \operatorname{arcsinh}(\sqrt{x}) = \frac{1}{2}\sqrt{\frac{x}{\pi}}\left[-x \,\frac{-\frac{1}{2}, -\frac{1}{2}}{0, \frac{1}{2}}\right],$$

have no such disposable constants. If at least one of the denominatorial parameters is zero, as in the three examples cited above, the hypergeometric is an example of the Gauss function [see Section 2.10 and Abramowitz and Stegun (1964, Chap. 15)] about which a large body of theory exists.

The greatest complexity of hypergeometrics that we here consider is $\frac{1}{3}$. The sine integral and its hyperbolic analog provide examples,

$$\text{(9.1.5)} \qquad \begin{matrix}\operatorname{Shi}(2\sqrt{x}) \\ \operatorname{Si}(2\sqrt{x})\end{matrix} = \left[\pm x \,\frac{-\frac{1}{2}}{0, \frac{1}{2}, \frac{1}{2}}\right].$$

9.2 HYPERGEOMETRICS WITH $K > L$

We have chosen not to stress the convergence aspects of hypergeometric functions. From the $j \to \infty$ properties of x^j and $\Gamma(j + \text{constant})$, however, we would expect the following rule to apply:

$K < L$: convergence for all finite x,
$K = L$: convergence for $-1 \leqq x < 1$ only,
$K > L$: divergence for all x except $x = 0$.

And, indeed, this rule does seem generally applicable.

Because it represents a series which diverges for all nontrivial values of its argument, one might imagine that any hypergeometric function for which the number K of numeratorial parameters exceeds the number L of denominatorial parameters would be a useless concept. This is not quite the case because such a hypergeometric may asymptotically represent an important transcendental function. For example, we have the asymptotic relation

$$\exp\left(\frac{1}{x}\right) E_1\left(\frac{1}{x}\right) \sim x\left[-x \,\frac{0}{\quad}\right],$$

valid as $x \to 0$. Here $E_1(\,)$ is the exponential integral, defined by

$$E_1(x) = \int_x^\infty \frac{\exp(-y)\,dy}{y}.$$

Some similar relations are

$$\exp\left(\frac{1}{x}\right)\operatorname{erfc}\left(\frac{1}{\sqrt{x}}\right) \sim \frac{\sqrt{x}}{\pi}\left[-x \frac{-\frac{1}{2}}{\quad}\right], \qquad x \to 0,$$

and

$$\tfrac{1}{2}\pi \sin(x) - \mathrm{Si}(x)\sin(x) - \mathrm{Ci}(x)\cos(x) \sim \frac{2x^2}{\sqrt{\pi}}\left[-4x^{-2}\frac{0, \frac{1}{2}}{\quad}\right],$$

but we shall find no further use for such asymptotic representations.

9.3 REDUCTION OF COMPLEX HYPERGEOMETRICS

In Section 6.6 it was demonstrated that the formula

$$\frac{d^q}{dx^q}\left\{x^p\left[\beta x \frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\} = x^{p-q}\left[\beta x \frac{p, b_1, b_2, \ldots, b_K}{p-q, c_1, c_2, \ldots, c_L}\right]$$

expresses the result of differintegrating a power hypergeometric product of complexity $\frac{K}{L}$, provided p exceeds -1. Because of the cancellation property of hypergeometric parameters (Section 2.10), it follows that the sequence of operations

(i) multiplication by x^{c-p},
(ii) differintegration to order $c - b$,
(iii) multiplication by x^{P-b},

where b and c are respectively a numeratorial and a denominatorial parameter, accomplishes the transformation

$$x^{P-b}\frac{d^{c-b}}{dx^{c-b}}\left\{x^{c-p}x^p\left[\beta x \frac{b, b_2, \ldots, b_K}{c, c_2, \ldots, c_L}\right]\right\} = x^P\left[\beta x \frac{b_2, \ldots, b_K}{c_2, \ldots, c_L}\right],$$

converting one hypergeometric of complexity $\frac{K}{L}$ to another of complexity $\frac{K-1}{L-1}$. For example, setting $\beta = 1$, $K = 1$, $b = -\frac{1}{2}$, $L = 3$, $c = 0$, $c_2 = c_3 = \frac{1}{2}$, and $p = P = \frac{1}{2}$ gives

$$x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{x^{-\frac{1}{2}}\sqrt{x}\left[x \frac{-\frac{1}{2}}{0, \frac{1}{2}, \frac{1}{2}}\right]\right\} = \sqrt{x}\left[x \frac{\quad}{\frac{1}{2}, \frac{1}{2}}\right]. \tag{9.3.1}$$

The procedure may be iterated to enable a $\frac{K}{L}$ hypergeometric to be reduced in complexity to $\frac{K-n}{L-n}$. An example is

$$\text{(9.3.2)}\qquad \frac{d}{dx}\left\{x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{x^{-\frac{1}{2}}\sqrt{x}\left[-x\,\frac{-\frac{1}{2},\,-\frac{1}{2}}{0,\,\frac{1}{2}}\right]\right\}\right\}=\left[-x\,\frac{\quad}{\quad}\right].$$

The value of such expressions is that they allow an interrelation between important transcendental functions. Thus, the hypergeometrics in equation (9.3.1) will be recognized from Section 9.1 as the hyperbolic sine integral and the zero-order Struve function [see (9.1.5) and (9.1.2), respectively], so that

$$x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\{x^{-\frac{1}{2}}\operatorname{Shi}(2\sqrt{x})\}=L_0(2\sqrt{x}).$$

Likewise,

$$2\sqrt{\pi}\,\frac{d}{dx}\left\{x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\{x^{-\frac{1}{2}}\operatorname{arcsinh}(\sqrt{x})\}\right\}=\frac{1}{1+x}$$

follows upon recognition of the hypergeometrics on each side of equation (9.3.2) from expressions (9.1.4) and (9.1.1).

It is evident that K applications of the sequence (i) through (iii) will convert a $\frac{K}{L}$ hypergeometric to one of $\frac{0}{L-K}$ complexity. Such a hypergeometric has the general form

$$x^p\left[\beta x\,\frac{\quad}{c_1,c_2,\ldots,c_{L-K}}\right],$$

being without numeratorial parameters. The sequence of operations

(a) multiplication by x^{c-p},
(b) differintegration to order c,

where c is any denominatorial parameter, now serves to replace that parameter by a zero, thus:

$$\frac{d^c}{dx^c}\left\{x^{c-p}x^p\left[\beta x\,\frac{\quad}{c,c_2,\ldots,c_{L-K}}\right]\right\}=\left[\beta x\,\frac{\quad}{0,c_2,\ldots,c_{L-K}}\right].$$

Hence, if this sequence is repeated $L-K$ times, all the denominatorial parameters may be replaced by zeros.

In summary then: By a sufficient number of operations, each of which is either multiplication by a power of x or differintegration with respect to x, any hypergeometric of complexity $\frac{K}{L}$ is reducible to one of complexity $\frac{0}{L-K}$ in which all denominatorial parameters are zero.

9.4 BASIS HYPERGEOMETRICS

As we have just seen, those generalized hypergeometric functions that are without numeratorial parameters and whose denominatorial parameters are all zero play a distinguished role in that they are the end result of the reduction of all other hypergeometrics. We shall refer to them as "basis hypergeometrics." The three most important are

$$\left[x\,\frac{\quad}{\quad}\right] = \frac{1}{1-x}, \qquad -1 < x < 1,$$

$$\left[x\,\frac{\quad}{0}\right] = \exp(x), \qquad -\infty < x < \infty,$$

and

$$\left[x\,\frac{\quad}{0,0}\right] = I_0(2\sqrt{x}), \qquad 0 \leqq x < \infty.$$

Figure 9.4.1 compares these three basis functions on the interval $0 \leqq x < 1$, the only range which they share in common. Also shown on the figure are the complementary functions

$$\left[-x\,\frac{\quad}{\quad}\right] = \frac{1}{1+x},$$

$$\left[-x\,\frac{\quad}{0}\right] = \exp(-x),$$

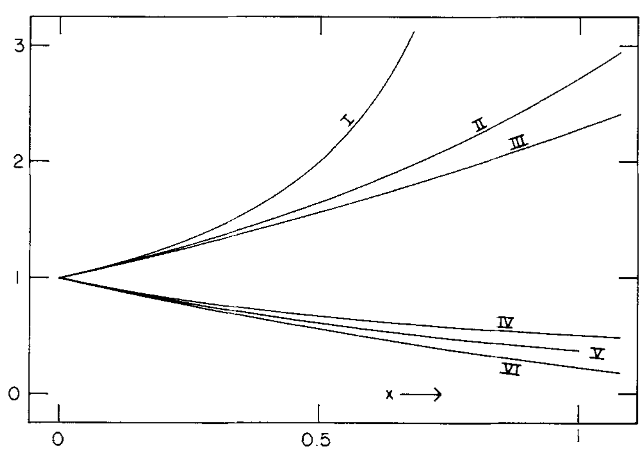

FIG. 9.4.1. The three most important basis functions: $\mathrm{I} = [1-x]^{-1}$, $\mathrm{II} = \exp(x)$, and $\mathrm{III} = I_0(2\sqrt{x})$; and their complements: $\mathrm{IV} = [1+x]^{-1}$, $\mathrm{V} = \exp(-x)$, and $\mathrm{VI} = J_0(2\sqrt{x})$.

and

$$\left[-x \frac{}{0,\, 0}\right] = I_0(2i\sqrt{x}) = J_0(2\sqrt{x}).$$

Reversing the process discussed in the preceding section, it is possible by employing only the operations of multiplication by x^p and differintegration to order q to generate from each basis hypergeometric a host[3] of transcendental functions. The next three sections will be devoted to detailing such function syntheses. In those sections we shall make use of a special kind of chart, a "function synthesis diagram," and it is appropriate to set the stage for these future sections by explaining here the significance of these diagrams.

In a function synthesis diagram, each hypergeometric occupies a single[4] point in a plane. The Cartesian coordinates of the point are $(p + 2B - 2C, p)$ where p is the exponent of the power, B is the sum $[= \sum b_k]$ of the numeratorial parameters of the generalized hypergeometric function, and C is is the sum $[=\sum c_l]$ of the denominatorial parameters. The origin (0, 0) is occupied by the basis hypergeometric, but, in common with all points in the plane, the origin is also the location of an infinity of other functions. Thus, for example, the hypergeometrics

$$x^{\frac{1}{2}}\left[x \frac{\frac{1}{2}}{1}\right], \qquad x^{\frac{1}{2}}\left[x \frac{-\frac{1}{2}, \frac{1}{2}}{-1, \frac{3}{2}}\right], \qquad \text{and} \qquad x^{\frac{1}{2}}\left[x \frac{-\frac{1}{2}}{0}\right]$$

are all colocated at $(-\frac{1}{2}, \frac{1}{2})$. To depict function syntheses in such a diagram, points representing hypergeometrics are connected by diagonal arrows. An arrow thus $\nearrow$ (always at a $\pi/4$ angle) indicates multiplication by a positive power of x. Similarly, division by a positive power of x is indicated by $\swarrow$. Differintegration is represented by arrows at right angles to these: upwards $\nwarrow$ in the case of integration (to fractional or integer order) and downwards $\searrow$ for differentiation. The index (power p in x^p or order q in d^q/dx^q) may be indicated in a "box" on the arrow. The example, Fig. 9.4.2, illustrates most of the features of a function synthesis diagram.

[3] But not quite every hypergeometric! The recurrences displayed in Section 2.10 show that two hypergeometrics of the same complexity $\frac{K}{L}$ differ by a term that may vanish on differintegration. This possible coalescence of two hypergeometrics on differintegration implies that not every hypergeometric may be regenerated from the appropriate basis function.

[4] Cyclodifferential functions (Section 6.11) will occupy more than one point.

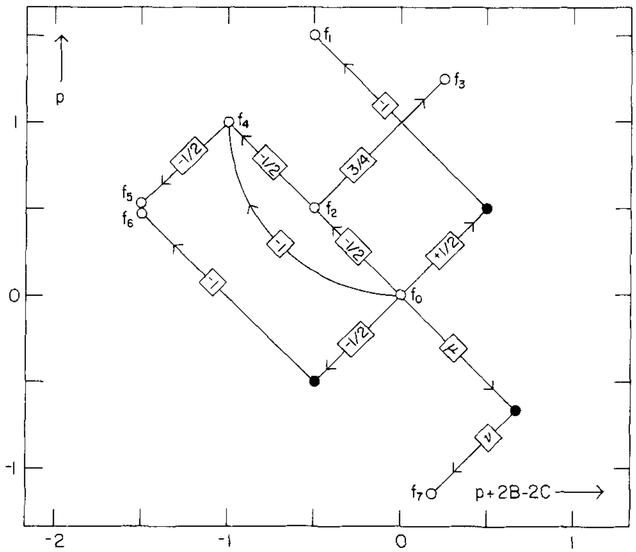

FIG. 9.4.2. Synthesis diagram for seven fictitious functions.

Figure 9.4.2 shows how seven fictitious functions may be generated from a basis hypergeometric f_0. Function f_1 is synthesized by multiplying f_0 by $x^{+\frac{1}{2}}$ and integrating the product, i.e.,

$$f_1 = \frac{d^{-1}}{dx^{-1}}\{\sqrt{x}f_0\}.$$

Function f_2 is the semiintegral of f_0, while

$$f_3 = x^{\frac{3}{4}}f_2 = x^{\frac{3}{4}}\frac{d^{-\frac{1}{2}}f_0}{dx^{-\frac{1}{2}}}$$

is the product of $x^{\frac{3}{4}}$ and the semiintegral of f_0. Notice that though the synthetic routes to f_1 and f_3 both traverse the point (0, 1), they will, in general, not be coincident there because the operations of multiplication and differintegration do not commute. Function f_4 is the semiintegral of f_2 and on account of the composition law (Section 5.7) it is also the integral of f_0. Hence two pathways —two semiintegrations, or a single integration (shown by the looped arrow) —produce f_4 from f_0; indeed, four quarter integrations would also serve. Normally only one of these alternatives would be included in a function synthesis diagram. Because of the lack of commutativity, two routes around a rectangle do not normally produce the same function. Thus, even though they occupy the same site $(-\frac{3}{2}, \frac{1}{2})$ in the plane, functions f_5 (equal to $f_4/\sqrt{x}$)

and f_6 (equal to $d^{-1}\{f_0/\sqrt{x}\}/dx^{-1}$) are distinct. Function f_7 is synthesized from the basis hypergeometric by differintegration to order μ, followed by multiplication by x^ν; that is,

$$f_7 = x^\nu \frac{d^\mu f_0}{dx^\mu},$$

where, as drawn, μ is positive and ν negative. Notice that f_7 lies below the $p = -1$ line; since it is not generally possible to differintegrate such hypergeometrics, f_7 is not a suitable starting point for further synthetic steps.

The numbers shown in boxes on Fig. 9.4.2 will be seen often to be redundant, the same information being conveyed by the lengths of the arrows and also by their projection on either axis. Accordingly, we shall often omit them from function synthesis diagrams whenever their omission leaves no danger of confusion.

Before leaving the subject of basis hypergeometrics, we wish to call attention to an interesting interrelation which exists between the inverse binomial function, the exponential function, and the zero-order Bessel function of argument $2\sqrt{x}$, that is, between the most important basis hypergeometrics. It may already have been noted by the reader who is an expert in Laplace transformation that

$$\mathscr{L}\{I_0(2\sqrt{x})\} = \frac{1}{s}\exp\left(\frac{1}{s}\right)$$

and

$$\mathscr{L}\{\exp(x)\} = \frac{1}{s-1} = \frac{1}{s}\left[\frac{1}{1-[1/s]}\right].$$

Thus Laplace transformation converts a basis hypergeometric into a hypergeometric of lower complexity. These transforms are just examples of the general rule

$$\mathscr{L}\left\{\left[x \,\frac{b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]\right\} = \frac{1}{s}\left[\frac{1}{s} \,\frac{0, b_1, b_2, \ldots, b_K}{c_1, c_2, \ldots, c_L}\right]$$

for Laplace transformation of a generalized hypergeometric function. By invoking this relationship it is possible to Laplace transform any function that can be expressed as a hypergeometric,

$$\mathscr{L}\left\{x^p\left[\beta x \,\frac{b_1, \ldots, b_K}{c_1, \ldots, c_L}\right]\right\} = \frac{1}{s^{p+1}}\left[\frac{\beta}{s} \,\frac{0, 0, b_1, \ldots, b_K}{p, c_1, \ldots, c_L}\right].$$

More importantly, it is also possible to Laplace invert any transform which is expressible as a hypergeometric of s^{-1}, thus:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^p}\left[\frac{\beta}{s}\;\frac{b_1,\ldots,b_K}{c_1,\ldots,c_L}\right]\right\} = x^{p-1}\left[\beta x\;\frac{p-1,b_1,\ldots,b_K}{0,0,c_1,\ldots,c_L}\right].$$

9.5 SYNTHESIS OF $K = L$ TRANSCENDENTALS

Transcendentals that are representable by a hypergeometric having equal numbers of numeratorial and denominatorial parameters may be synthesized from the basis hypergeometric $[1-x]^{-1}$ (or its $[1+x]^{-1}$ complement). We shall give some examples in the form of function synthesis diagrams, omitting proofs. Proofs can always be established by replacing the functions by their hypergeometric equivalents (many of which will be found listed in Section 9.1) and invoking equation (6.6.3).

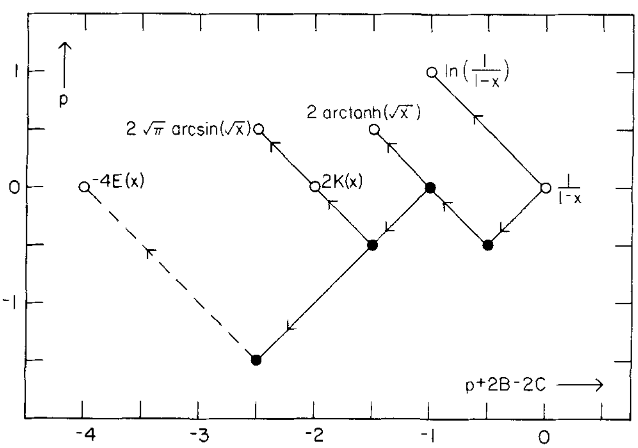

FIG. 9.5.1. Four transcendental functions synthesized from the $[1-x]^{-1}$ basis. $E(x)$ cannot be synthesized via the dashed route.

Figure 9.5.1 is a function synthesis diagram showing how four transcendentals may be generated from the basis hypergeometric $[1-x]^{-1}$. Similar syntheses starting with $[1+x]^{-1}$ would have produced $\ln(1+x)$, $2\arctan(\sqrt{x})$, $2K(-x)$,[5] and $2\sqrt{\pi}\,\operatorname{arcsinh}(\sqrt{x})$. Since

$$(9.5.1)\qquad -4E(x) = \left[x\;\frac{-\frac{3}{2},-\frac{1}{2}}{0,0}\right],$$

[5] An alternative representation of the complete elliptic integral $K(-x)$ is $[1+x]^{-\frac{1}{2}} K(x/[1+x])$.

this elliptic integral can, formally, be generated by the synthetic route indicated by the dotted arrow. However, this route involves differintegration of a term proportional to $x^{-\frac{3}{2}}$, an invalid operation. To generate $E(x)$ legitimately, we must make use of the identity

$$2\pi - 4E(x) = x\left[x \frac{-\frac{1}{2}, \frac{1}{2}}{1, 1}\right],$$

which follows from (9.5.1) and one of the recursions cited in Section 2.10. One of the two[6] synthetic routes to this hypergeometric is mapped out in Fig. 9.5.2. The route is four-legged, consisting serially of multiplication by $\sqrt{x}$, semiintegration, division by $x^{\frac{3}{2}}$, and finally sesquiintegration.

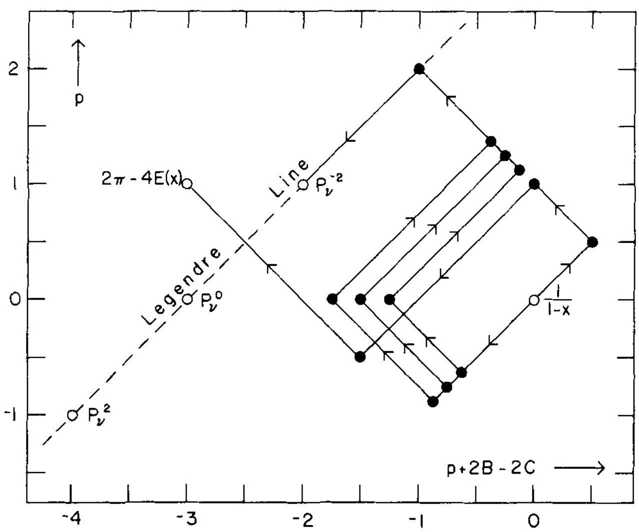

FIG. 9.5.2. All associated Legendre functions lie on the dashed line and can be synthesized from the $[1 - x]^{-1}$ basis by a five-legged route akin to the ones shown.

Figure 9.5.2. will also be used to illustrate some of the properties of the associated Legendre function $P_\nu^\mu(\sqrt{1 - x})$. Note first of all, from the equation (9.1.3) which gives the hypergeometric representation of this function, that its parameter difference $B - C$ equals $-\frac{1}{2}$ and is independent of both μ and ν. In terms of our function synthesis diagram, this means that all associated Legendre functions lie on the line annotated "Legendre line" in Fig. 9.5.2. Moreover, since p for these functions equals $-\mu/2$ and is ν-independent, it

[6] There will, in general, be four nonequivalent routes from a $\frac{0}{0}$ to a $\frac{2}{2}$ hypergeometric, but a redundancy enters into the synthesis of $E(x)$ because of the identity of the two denominatorial parameters.

follows that μ alone determines the position on the Legendre line at which a given associated Legendre function is located. For example all associated Legendre functions of orders 2, 0, and -2 are located at the points marked P_ν^{2}, P_ν^{0}, and P_ν^{-2} on the chart.[7] In synthesizing these functions from the $[1-x]^{-1}$ basis hypergeometric, it is the route alone which determines ν, the starting and ending points being independent of the degree ν. The chart shows three five-legged routes from $[1-x]^{-1}$ to the point labelled P_ν^{-2}, at which the functions $4\Gamma(\frac{3}{2}+\frac{1}{2}\nu)\Gamma(1-\frac{1}{2}\nu)P_\nu^{-2}(\sqrt{1-x})$ are located. The routes form a family that differs only in the ν value of the associated Legendre function which arises from the five-legged route. From the infinity of possible ν values, three only ($\nu=\frac{5}{4}$, $\frac{3}{2}$, and $\frac{7}{4}$) are shown on Fig. 9.5.2.

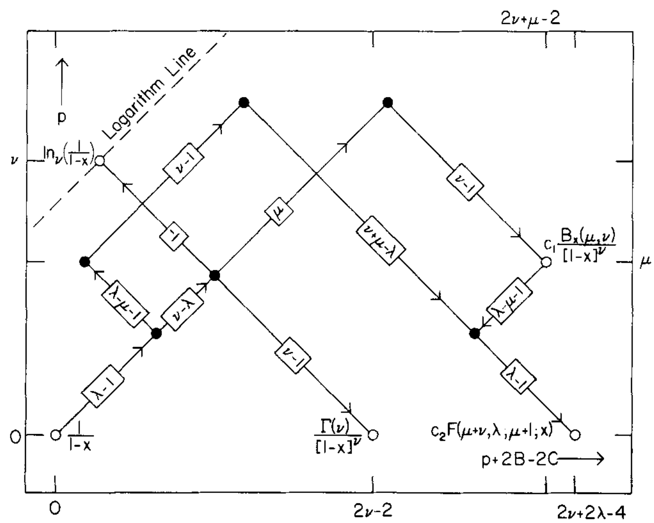

FIG. 9.5.3. Synthesis diagram for some very general $K=L$ functions. All generalized logarithms lie on the dashed line. The constants c_1 and c_2 that multiply the synthesized incomplete beta and Gauss functions are, respectively, $\Gamma(\mu+\nu)/\Gamma(\nu)$ and $\Gamma(\mu+1)/[\Gamma(\lambda)\Gamma(\mu+\nu)]$.

Three very general functions—the inverse binomial, the incomplete beta function, and the Gauss hypergeometric function—are synthesized via routes shown in Fig. 9.5.3. These functions have, respectively, one, two, and three

[7] The associated Legendre functions of zero order are usually notated P_ν rather than P_ν^{0} and are termed Legendre functions (or Legendre polynomials if the degree ν is an integer).

disposable parameters. The route to a generalized logarithm (Section 10.5) is also mapped out in this chart. These latter functions all lie on the line marked "logarithm line."

9.6 SYNTHESIS OF $K=L-1$ TRANSCENDENTALS

Figure 9.6.1 is a synthesis diagram showing how four transcendental functions are derived from the basis $K=L-1$ hypergeometric, $\exp(x)$. Starting with the complementary basis function $\exp(-x)$, similar routes lead to $\sqrt{\pi}\exp(-\frac{1}{2}x)I_1(\frac{1}{2}x)$, $E_1(x)+\ln(x)+\gamma$, and $\sqrt{\pi}\exp(-\frac{1}{2}x)I_0(\frac{1}{2}x)$.

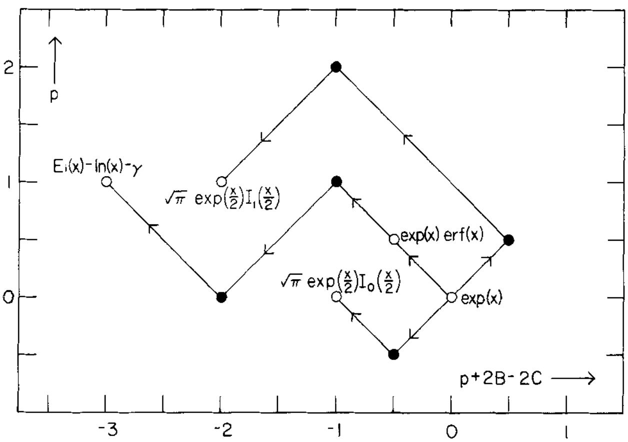

FIG. 9.6.1. Some functions synthesized from the $\frac{0}{1}$ basis hypergeometric function $\exp(x)$.

However, some additional important functions—Dawson's integral, the error function, and the error function complement integral—also arise from $\exp(-x)$, as shown in Fig. 9.6.2.

There are two important multivariate functions which are $K=L-1$ hypergeometrics. The first is the incomplete gamma function,

$$\gamma^*(c,x)=\frac{1}{\Gamma(c)}\left[-x\,\frac{c-1}{0,\,c}\right]=\frac{x^{-c}}{\Gamma(c)}\frac{d^{-1}}{dx^{-1}}\{x^{c-1}\exp(-x)\},$$

the second being the Kummer confluent function $M(\ ,\ ,\ )$. Figure 9.6.3 shows synthetic routes to these functions from $\exp(x)$.

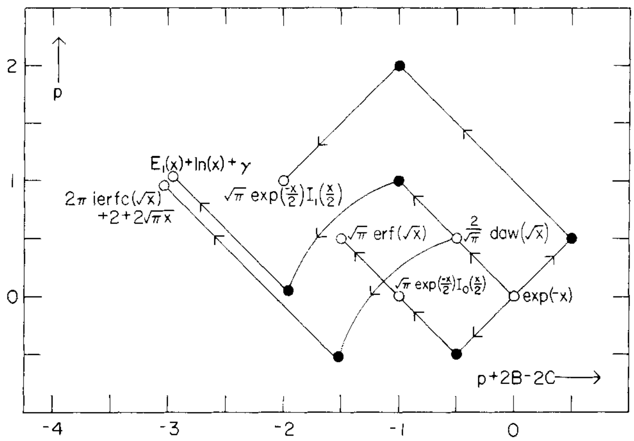

FIG. 9.6.2. Some functions synthesized from the complementary $\frac{0}{1}$ basis function $\exp(-x)$.

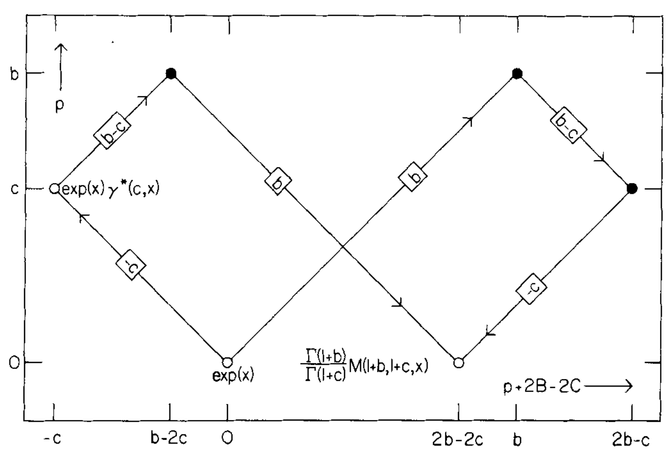

FIG. 9.6.3. Additional functions that may be synthesized from $\exp(x)$.

9.7 SYNTHESIS OF $K=L-2$ TRANSCENDENTALS

Most of the important transcendental functions of mathematical physics belong to the $K = L - 2$ class of hypergeometrics and may be synthesized from the basis function $I_0(2\sqrt{x})$ or its complementary $J_0(2\sqrt{x})$.

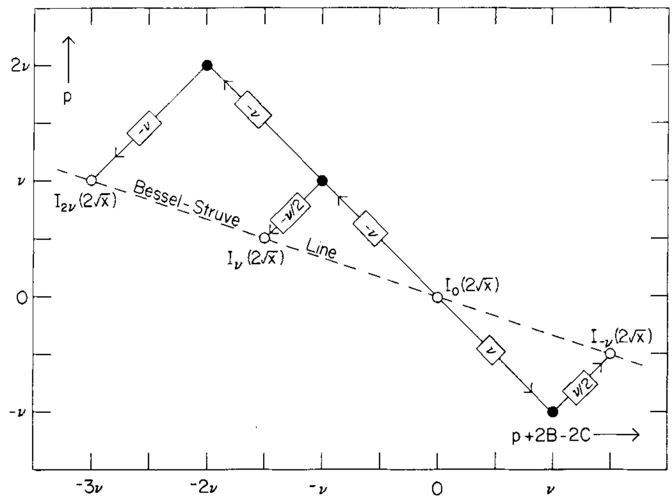

FIG. 9.7.1. Synthesis of Bessel functions. All Bessel and Struve functions of argument $2\sqrt{x}$ lie on the dashed line.

First consider the synthesis of a Bessel function $I_\nu(2\sqrt{x})$ of nonzero order from the basis hypergeometric. As depicted in Fig. 9.7.1, this synthesis is accomplished by what a chess player might term a "knight's move," the first leg of the route being twice as long as the second. From this it follows that all Bessel functions are located in our function synthesis plane on a line through the origin and inclined at an angle $\arctan(-\frac{1}{3})$. Next, consider the hypergeometric representation

$$I_{\mu+\frac{1}{2}}(2\sqrt{x}) = x^{\frac{1}{2}\mu+\frac{1}{4}}\left[x \frac{}{0,\ \mu+\frac{1}{2}}\right]$$

of a Bessel function of order $\mu + \frac{1}{2}$, compared with that of a Struve function

$$L_{\mu-\frac{1}{2}}(2\sqrt{x}) = x^{\frac{1}{2}\mu+\frac{1}{4}}\left[x \frac{}{\frac{1}{2},\ \mu}\right]$$

of order $\mu - \frac{1}{2}$. Their p and $B - C$ values being identical, it is evident that $I_{\mu+\frac{1}{2}}(2\sqrt{x})$ and $L_{\mu-\frac{1}{2}}(2\sqrt{x})$ must be colocated on what we shall henceforth term the Bessel–Struve line. It is equally evident, however, that $I_{\mu+\frac{1}{2}}(2\sqrt{x})$ and $L_{\mu-\frac{1}{2}}(2\sqrt{x})$ are distinct functions for all μ except $\mu = 0$. For this unique value, the Bessel and Struve functions coalesce:

$$I_{\frac{1}{2}}(2\sqrt{x}) = L_{-\frac{1}{2}}(2\sqrt{x}) = \frac{\sinh(2\sqrt{x})}{\sqrt{\pi}\, x^{\frac{1}{4}}}.$$

The Bessel–Struve line also appears on Fig. 9.7.2, with the synthetic routes to as many as nine important transcendentals, starting from the basis hypergeometric $I_0(2\sqrt{x})$. Of course, similar synthetic routes from the complementary $J_0(2\sqrt{x})$ basis will yield the corresponding circular functions, $\sin(2\sqrt{x})/\sqrt{\pi}$, $\cos(2\sqrt{x})/\sqrt{\pi}$, and $\mathrm{Si}(2\sqrt{x})$, as well as J Bessel functions and H Struve functions.

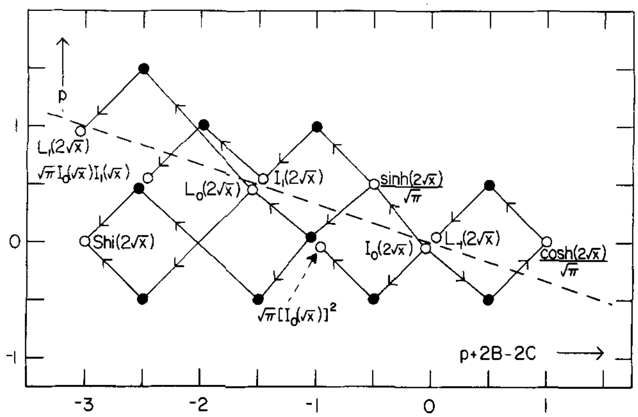

FIG. 9.7.2. Synthetic routes from the $\frac{0}{2}$ basis hypergeometric function $I_0(2\sqrt{x})$.

Notice the interesting synthetic relationship between the basis hypergeometric $I_0(2\sqrt{x})$ and the square of the closely related $I_0(\sqrt{x})$ function. This may be derived starting from the well-known binomial coefficient summation

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

[a special case of equation (1.3.20)] that may be rewritten in gamma function notation as

$$\sum_{k=0}^{n} \frac{1}{[\Gamma(k+1)]^2[\Gamma(n-k+1)]^2} = \frac{\Gamma(2n+1)}{[\Gamma(n+1)]^4}.$$

If, now, the duplication formula (1.3.9) is employed to expand the $\Gamma(2n+1)$ term, and each side of the equation is multiplied by $[\frac{1}{4}x]^n$,

$$\sum_{k=0}^{n} \frac{[\frac{1}{4}x]^k}{[\Gamma(k+1)]^2} \frac{[\frac{1}{4}x]^{n-k}}{[\Gamma(n-k+1)]^2} = \frac{x^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,[\Gamma(n+1)]^3}$$

results. The natural number n may now be regarded as an index, and the summation from $n=0$ to $n=\infty$ carried out. With use of the

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \equiv \sum_{k=0}^{\infty}\sum_{n-k=0}^{\infty}$$

permutation, this becomes

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{4}x]^k}{[\Gamma(k+1)]^2} \sum_{n-k=0}^{\infty} \frac{[\frac{1}{4}x]^{n-k}}{[\Gamma(n-k+1)]^2} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^n\Gamma(n+\frac{1}{2})}{[\Gamma(n+1)]^3},$$

which is easily expressed as

$$\left[\tfrac{1}{4}x \frac{}{0,\,0}\right]\left[\tfrac{1}{4}x \frac{}{0,\,0}\right] = \frac{1}{\sqrt{\pi}}\left[x \frac{-\frac{1}{2}}{0,\,0,\,0}\right]$$

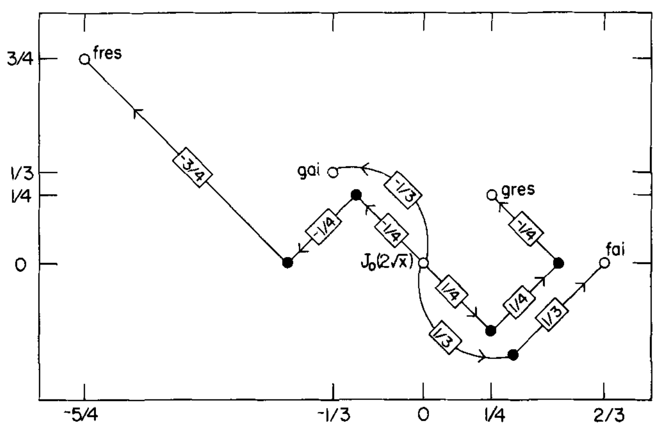

FIG. 9.7.3. A synthesis diagram involving small steps. The abbreviations

$$\text{gai} = 3^{\frac{1}{3}}\,\text{gai}(-[9x]^{\frac{1}{3}})/\Gamma(\tfrac{1}{3}),$$

$$\text{fai} = \text{fai}(-[9x]^{\frac{1}{3}})/\Gamma(\tfrac{2}{3}),$$

$$\text{fres} = \frac{1}{2\sqrt{\pi}}\,\text{fres}\left(\frac{2x^{\frac{1}{4}}}{\sqrt{\pi}}\right) - \frac{1}{\sqrt{\pi}}\cos(2\sqrt{x}+\tfrac{1}{4}\pi),$$

and

$$\text{gres} = \frac{1}{\sqrt{\pi}}\sin(2\sqrt{x}+\tfrac{1}{4}\pi) - \sqrt{\frac{2}{\pi}}\,\text{gres}\left(\frac{2x^{\frac{1}{4}}}{\sqrt{\pi}}\right)$$

are employed.

in terms of hypergeometric functions. From here it is only a short step to

$$\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\left\{\frac{I_0(2\sqrt{x})}{\sqrt{x}}\right\} = [I_0(\sqrt{x})]^2, \tag{9.7.1}$$

as portrayed in Fig. 9.7.2. Reminiscent of the classical result

$$\frac{d^{-1}}{dx^{-1}}\{\sin(2x)\} = [\sin(x)]^2,$$

equation (9.7.1) is one of many similar relationships that can be derived from the properties of binomial coefficients [see Gradshteyn and Ryzhik (1965, Chap. 0)].

Whereas Fig. 9.7.2 is largely a grid with spacings of one-half, many $K = L - 2$ transcendentals arise from smaller units. Thus Fresnel integrals arise when the $J_0(2\sqrt{x})$ basis hypergeometric is operated on by $d^{\pm\frac{1}{4}}/dx^{\pm\frac{1}{4}}$, while Airy functions are produced by the $d^{\pm\frac{1}{3}}/dx^{\pm\frac{1}{3}}$ operators. Figure 9.7.3 shows how the fres( ) and gres( ) functions (see Section 7.6) and the fai( ) and gai( ) functions (see Section 9.1) arise in this way.