CHAPTER 7

SEMIDERIVATIVES AND SEMIINTEGRALS

Within our experience, the most useful applications of derivatives and integrals of noninteger order involve a zero lower limit and moiety order (see Chapter 11). The present chapter, therefore, is devoted to the operators $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ and $d^{-\frac{1}{2}}/dx^{-\frac{1}{2}}$. We shall here present the effect of these operators on a large variety of functions and also some general properties of these special operators. The presentation is in the form of an extended table, no proofs being given. Generally, the formulas quoted are either straightforward specializations of those given in Chapters 3–6, or can be derived by application of the rules of Chapter 5. Though the chapter is lengthy, it by no means exhausts the possible operands to which semidifferentiation and semiintegration[1] may be applied. We know of no preexisting tabulation of semiderivatives or semiintegrals, though a table of Riemann–Liouville transforms (Erdélyi *et al.*, 1954) can, with caution, be adapted to this purpose. To a limited extent, tables of Laplace transforms (for example, Roberts and Kaufman, 1966) are useful in constructing tables of semiderivatives and semiintegrals, as a result of the transformation properties we shall discuss in Section 8.1.

Though the tables which follow have been designed to have their maximum utility when x has positive real values, most of the entries are valid more generally.

7.1 DEFINITIONS

In this section we specialize to $a = 0$, $q = \pm\frac{1}{2}$ the differintegral definitions summarized in Section 3.6. This specialization leads directly to all table

[1] The names "semidifferentiation" and "semiintegration" seem natural to denote the operations performed by the $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ and $d^{-\frac{1}{2}}/dx^{-\frac{1}{2}}$ operators. Likewise we shall term $d^{\frac{1}{2}}f/dx^{\frac{1}{2}}$ the "semiderivative" of f, and $d^{-\frac{1}{2}}f/dx^{-\frac{1}{2}}$ its "semiintegral." Less frequently we will use "sesquiderivative" of f to indicate $d^{\frac{3}{2}}f/dx^{\frac{3}{2}}$, and similarly for $d^{-\frac{3}{2}}f/dx^{-\frac{3}{2}}$, etc.

entries except the fourth. The latter is derived making use of the composition rule.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
f	$\lim\limits_{N\to\infty}\left\{\sqrt{\dfrac{N}{x}}\sum\limits_{j=0}^{N-1}\dfrac{(2j)!\,f\left(x-\dfrac{jx}{N}\right)}{[2j-1][2^j j!]^2}\right\}$	$\lim\limits_{N\to\infty}\left\{\sqrt{\dfrac{x}{N}}\sum\limits_{j=0}^{N-1}\dfrac{(2j)!\,f\left(x-\dfrac{jx}{N}\right)}{[2^j j!]^2}\right\}$
f	$\dfrac{d}{dx}\left\{\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}\right\}$	$\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^x \dfrac{f(y)\,dy}{\sqrt{x-y}}$
f	$\dfrac{f(0)}{\sqrt{\pi x}}+\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^x \dfrac{f^{(1)}(y)\,dy}{\sqrt{x-y}}$	$\dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^x \dfrac{f(y)\,dy}{\sqrt{x-y}}$
f	$\dfrac{f(x)}{\sqrt{\pi x}}+\dfrac{1}{2\sqrt{\pi}}\displaystyle\int_0^x \dfrac{[f(x)-f(y)]\,dy}{[x-y]^{\frac{3}{2}}}$	$\dfrac{F(x)-F(0)}{\sqrt{\pi x}}+\dfrac{1}{2\sqrt{\pi}}\displaystyle\int_0^x \dfrac{[F(x)-F(y)]\,dy}{[x-y]^{\frac{3}{2}}}, \quad f(y)=\dfrac{dF}{dy}$
ϕ	$\sum\limits_{k=0}^{\infty}\dfrac{[-x]^k\phi^{(k)}}{\sqrt{\pi x}\,[1-2k]k!}$	$\sum\limits_{k=0}^{\infty}\sqrt{\dfrac{x}{\pi}}\dfrac{[-x]^k\phi^{(k)}}{[k+\frac{1}{2}]k!}$
ϕ	$\sum\limits_{k=0}^{\infty}\dfrac{x^{k-\frac{1}{2}}\phi^{(k)}(0)}{\Gamma(k+\frac{1}{2})}$	$\sum\limits_{k=0}^{\infty}\dfrac{x^{k+\frac{1}{2}}\phi^{(k)}(0)}{\Gamma(k+\frac{3}{2})}$

In the preceding table, as in the following section, we use ϕ and ψ to represent real analytic functions, and f an arbitrary differintegrable function of x.

7.2 GENERAL PROPERTIES

In this section we shall specialize the findings of Chapter 5 to the cases $a = 0$, $q = \pm\frac{1}{2}$. The same sequence is followed, so that little commentary is needed to supplement the table.

The second and third entries, exemplifying respectively term-by-term differintegration of analytic functions and arbitrary differintegrable series, follow directly from the results of Section 5.2. The eleventh entry, which stems from the composition rule, requires both f and d^qf/dx^q to be differintegrable series.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$f_1 \pm f_2$	$\dfrac{d^{\frac{1}{2}}f_1}{dx^{\frac{1}{2}}} \pm \dfrac{d^{\frac{1}{2}}f_2}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f_1}{dx^{-\frac{1}{2}}} \pm \dfrac{d^{-\frac{1}{2}}f_2}{dx^{-\frac{1}{2}}}$
$\sum_{j=0}^{\infty} \phi_j$	$\sum_{j=0}^{\infty} \dfrac{d^{\frac{1}{2}}\phi_j}{dx^{\frac{1}{2}}}$	$\sum_{j=0}^{\infty} \dfrac{d^{-\frac{1}{2}}\phi_j}{dx^{-\frac{1}{2}}}$
$x^p \sum_{j=0}^{\infty} a_j x^j,$ $p > -1$	$\sum_{j=0}^{\infty} a_j \dfrac{\Gamma(p+j+1)}{\Gamma(p+j+\frac{1}{2})} x^{p+j-\frac{1}{2}}$	$\sum_{j=0}^{\infty} a_j \dfrac{\Gamma(p+j+1)}{\Gamma(p+j+\frac{3}{2})} x^{p+j+\frac{1}{2}}$
Cf	$C\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$C\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$f(\beta x)$	$\sqrt{\beta}\,\dfrac{d^{\frac{1}{2}}}{[d(\beta x)]^{\frac{1}{2}}} f(\beta x)$	$\dfrac{1}{\sqrt{\beta}}\,\dfrac{d^{-\frac{1}{2}}}{[d(\beta x)]^{-\frac{1}{2}}} f(\beta x)$
$f(-x)$	$i\,\dfrac{d^{\frac{1}{2}}}{[d(-x)]^{\frac{1}{2}}} f(-x)$	$-i\,\dfrac{d^{-\frac{1}{2}}}{[d(-x)]^{-\frac{1}{2}}} f(-x)$
$\phi\psi$	$\sum_{j=0}^{\infty} \binom{\frac{1}{2}}{j} \dfrac{d^{\frac{1}{2}-j}\phi}{dx^{\frac{1}{2}-j}} [\psi^{(j)}]$	$\sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \dfrac{d^{-\frac{1}{2}-j}\phi}{dx^{-\frac{1}{2}-j}} [\psi^{(j)}]$
fp_n	$\sum_{j=0}^{n} \binom{\frac{1}{2}}{j} \dfrac{d^{\frac{1}{2}-j}f}{dx^{\frac{1}{2}-j}} [p_n^{(j)}]$	$\sum_{j=0}^{n} \binom{-\frac{1}{2}}{j} \dfrac{d^{-\frac{1}{2}-j}f}{dx^{-\frac{1}{2}-j}} [p_n^{(j)}]$
xf	$x\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} + \dfrac{1}{2}\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$	$x\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}} - \dfrac{1}{2}\dfrac{d^{-\frac{3}{2}}f}{dx^{-\frac{3}{2}}}$
$x^n f$	$\dfrac{\sqrt{\pi}}{2} \sum_{j=0}^{n} \binom{n}{j} \dfrac{x^{n-j}}{\Gamma(\frac{3}{2}-j)} \dfrac{d^{\frac{1}{2}-j}f}{dx^{\frac{1}{2}-j}}$	$\sqrt{\pi} \sum_{j=0}^{n} \binom{n}{j} \dfrac{x^{n-j}}{\Gamma(\frac{1}{2}-j)} \dfrac{d^{-\frac{1}{2}-j}f}{dx^{-\frac{1}{2}-j}}$
$\dfrac{d^q f}{dx^q}$	$\dfrac{d^{q+\frac{1}{2}}f}{dx^{q+\frac{1}{2}}},\quad \dfrac{d^{-q}}{dx^{-q}}\dfrac{d^q f}{dx^q} = f$	$\dfrac{d^{q-\frac{1}{2}}f}{dx^{q-\frac{1}{2}}},\quad \dfrac{d^{-q}}{dx^{-q}}\dfrac{d^q f}{dx^q} = f$
$\dfrac{df}{dx}$	$\dfrac{d^{\frac{3}{2}}f}{dx^{\frac{3}{2}}} - \dfrac{x^{-\frac{3}{2}}f(0)}{2\sqrt{\pi}},\quad f(0) \neq \infty$	$\dfrac{f(0)}{\sqrt{\pi}} + \dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}},\quad f(0) \neq \infty$
$\dfrac{d^n f}{dx^n},$ $n = 1, 2, \ldots$	$\dfrac{d^{n+\frac{1}{2}}f}{dx^{n+\frac{1}{2}}} + \sum_{k=0}^{n-1} \dfrac{x^{k-n-\frac{1}{2}} f^{(k)}(0)}{\Gamma(k-n+\frac{1}{2})},$ $f^{(k)}(0) \neq \infty$	$\dfrac{d^{n-\frac{1}{2}}f}{dx^{n-\frac{1}{2}}} + \sum_{k=0}^{n-1} \dfrac{x^{k-n+\frac{1}{2}} f^{(k)}(0)}{\Gamma(k-n+\frac{3}{2})},$ $f^{(k)}(0) \neq \infty$
$\phi(x+A)$	$\dfrac{d^{\frac{1}{2}}}{[d(x+A)]^{\frac{1}{2}}} \phi(x+A) - \sum_{k=1}^{\infty} \dfrac{[x+A]^{-\frac{1}{2}-k}}{\Gamma(\frac{1}{2}-k)} \dfrac{d^{-k}}{dA^{-k}} \phi(A)$	$\dfrac{d^{-\frac{1}{2}}}{[d(x+A)]^{-\frac{1}{2}}} \phi(x+A) - \sum_{k=1}^{\infty} \dfrac{[x+A]^{\frac{1}{2}-k}}{\Gamma(\frac{3}{2}-k)} \dfrac{d^{-k}}{dA^{-k}} \phi(A)$
$f,$ $x \to 0$	$\dfrac{f(0)}{\sqrt{\pi x}},\quad f(0) \neq 0$	$2f(0)\sqrt{\dfrac{x}{\pi}},\quad f(0) \neq 0$

In the table we have used p_n to represent a polynomial of degree n in x. The binomial coefficients $\binom{\frac{1}{2}}{j}$ and $\binom{-\frac{1}{2}}{j}$ occur several times in the foregoing. Values for many of these coefficients are listed in Table 7.2.1. In certain applications it is the values of finite or infinite sums of $\binom{\frac{1}{2}}{j}$ or $\binom{-\frac{1}{2}}{j}$ which are needed; such values are also to be found in the tabulation.

Table 7.2.1. *Values of* $\binom{\frac{1}{2}}{j}$, $\binom{-\frac{1}{2}}{j}$ *and their cumulative sums.*

J	$\binom{\frac{1}{2}}{J}$	$\sum_{j=0}^{J}\binom{\frac{1}{2}}{j}$	$\binom{-\frac{1}{2}}{J}$	$\sum_{j=0}^{J}\binom{-\frac{1}{2}}{j}$
0	1	1	1	1
1	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$
2	$\frac{-1}{8}$	$\frac{11}{8}$	$\frac{3}{8}$	$\frac{7}{8}$
3	$\frac{1}{16}$	$\frac{23}{16}$	$\frac{-5}{16}$	$\frac{9}{16}$
4	$\frac{-5}{128}$	$\frac{179}{128}$	$\frac{35}{128}$	$\frac{107}{128}$
5	$\frac{7}{256}$	$\frac{365}{256}$	$\frac{-63}{256}$	$\frac{151}{256}$
6	$\frac{-21}{1024}$	$\frac{1439}{1024}$	$\frac{231}{1024}$	$\frac{835}{1024}$
$\to\infty$	$\frac{-[-1]^J}{2\sqrt{\pi}J^{\frac{3}{2}}}$	$\to\sqrt{2}$	$\frac{[-1]^J}{\sqrt{\pi J}}$	$\to\frac{1}{\sqrt{2}}$

7.3 CONSTANTS AND POWERS

These provide the simplest subjects for semidifferentiation and semi-integration.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
0	0	0
C, any constant	$\dfrac{C}{\sqrt{\pi x}}$	$2C\sqrt{\dfrac{x}{\pi}}$
$x^{-\alpha}, \quad \alpha = 0.79195\ldots$	$-x^{-\alpha-\frac{1}{2}}$	$\dfrac{x^{\frac{1}{2}-\alpha}}{\alpha - \frac{1}{2}}$
$\dfrac{1}{\sqrt{x}}$	0	$\sqrt{\pi}$
$x^0 \equiv 1$	$\dfrac{1}{\sqrt{\pi x}}$	$2\sqrt{\dfrac{x}{\pi}}$
$x^\beta, \quad \beta = 0.22119\ldots$	$[\beta + \frac{1}{2}]x^{\beta-\frac{1}{2}}$	$x^{\beta+\frac{1}{2}}$
$\sqrt{x}$	$\frac{1}{2}\sqrt{\pi}$	$\frac{1}{2}\sqrt{\pi}\,x$
$x^{\beta+\frac{1}{2}}$	x^β	$\dfrac{x^{\beta+1}}{\beta+1}$
x	$2\sqrt{\dfrac{x}{\pi}}$	$\dfrac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}$
$x^{\frac{3}{2}}$	$\frac{3}{4}\sqrt{\pi}\,x$	$\frac{3}{8}\sqrt{\pi}\,x^2$
x^2	$\dfrac{8x^{\frac{3}{2}}}{3\sqrt{\pi}}$	$\dfrac{16x^{\frac{5}{2}}}{15\sqrt{\pi}}$
$x^n, \quad n = 0, 1, 2, \ldots$	$\dfrac{[n!]^2[4x]^n}{(2n)!\sqrt{\pi x}}$	$\dfrac{[n!]^2[4x]^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}}$
$x^{n+\frac{1}{2}}$	$\dfrac{(2n+1)!\sqrt{\pi}}{2[n!]^2}\left[\dfrac{x}{4}\right]^n$	$\dfrac{(2n+2)!\sqrt{\pi}}{[(n+1)!]^2}\left[\dfrac{x}{4}\right]^{n+1}$
$x^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2})}x^{p-\frac{1}{2}}$	$\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}x^{p+\frac{1}{2}}$

The exponents α and β appear in the solution of the equations

$$\sqrt{x}\,\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}} \pm f = 0,$$

which are two of the simplest semidifferential equations (see Section 8.5 for a definition of a semidifferential equation and for more complex examples).

7.4 BINOMIALS

By this title we mean powers of $1 \pm x$; we also include instances of powers of $1 \pm x$ multiplied by powers of x. The validity of the semidifferentiation and semiintegration formulas in this section are generally restricted to $0 < x < 1$, though $x = 1$ is often permitted.

Notice in the table below that the same entry, $\sqrt{\pi}/\{2[1-x]^{\frac{3}{2}}\}$, occurs in the semiderivative column opposite two distinct operands, $1/\{\sqrt{x}\,[1-x]\}$ and $\sqrt{x}/[1-x]$. This result is correct, though it may appear questionable. The final paragraphs of Section 5.7, which discuss the occasional failure of the $d^{-q}\{d^q f/dx^q\}/dx^{-q} \equiv f$ "rule," provide the key to the apparent anomaly. The paradox is paralleled in the classical calculus by the observation that two distinct functions, say x^2 and $x^2 - 1$, may have the same derivative.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\sqrt{1+x}$	$\dfrac{1}{\sqrt{\pi x}} + \dfrac{\arctan(\sqrt{x})}{\sqrt{\pi}}$	$\sqrt{\dfrac{x}{\pi}} + \dfrac{[1+x]\arctan(\sqrt{x})}{\sqrt{\pi}}$
$\sqrt{1-x}$	$\dfrac{1}{\sqrt{\pi x}} - \dfrac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{\pi}}$	$\sqrt{\dfrac{x}{\pi}} + \dfrac{[1-x]\operatorname{arctanh}(\sqrt{x})}{\sqrt{\pi}}$
$\dfrac{1}{\sqrt{1+x}}$	$\dfrac{1}{\sqrt{\pi x}\,[1+x]}$	$\dfrac{2}{\sqrt{\pi}}\arctan(\sqrt{x})$
$\dfrac{1}{\sqrt{1-x}}$	$\dfrac{1}{\sqrt{\pi x}\,[1-x]}$	$\dfrac{2}{\sqrt{\pi}}\operatorname{arctanh}(\sqrt{x})$
$\dfrac{1}{1+x}$	$\dfrac{\sqrt{1+x} - \sqrt{x}\operatorname{arcsinh}(\sqrt{x})}{\sqrt{\pi x}\,[1+x]^{\frac{3}{2}}}$	$\dfrac{2\operatorname{arcsinh}(\sqrt{x})}{\sqrt{\pi}\,[1+x]}$
$\dfrac{1}{1-x}$	$\dfrac{\sqrt{1-x} + \sqrt{x}\arcsin(\sqrt{x})}{\sqrt{\pi x}\,[1-x]^{\frac{3}{2}}}$	$\dfrac{2\arcsin(\sqrt{x})}{\sqrt{\pi}\,[1-x]}$
$\dfrac{1}{[1\pm x]^{\frac{3}{2}}}$	$\dfrac{1\mp x}{\sqrt{\pi x}\,[1\pm x]^2}$	$\dfrac{2}{1\pm x}\sqrt{\dfrac{x}{\pi}}$

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\dfrac{1}{[1-x]^p}$	$\dfrac{-\mathrm{B}_x(-\frac{1}{2}, p+\frac{1}{2})}{2\sqrt{\pi}[1-x]^{p+\frac{1}{2}}}$	$\dfrac{\mathrm{B}_x(\frac{1}{2}, p-\frac{1}{2})}{\sqrt{\pi}[1-x]^{p-\frac{1}{2}}}$
$\dfrac{1}{\sqrt{x[1-x]}}$	$\dfrac{E(x)-[1-x]K(x)}{x[1-x]\sqrt{\pi}}$	$\dfrac{2}{\sqrt{\pi}}K(x)$
$\dfrac{1}{\sqrt{x[1+x]}}$	$\dfrac{E\left(\dfrac{x}{1+x}\right)-K\left(\dfrac{x}{1+x}\right)}{x\sqrt{\pi[1+x]}}$	$\dfrac{2K\left(\dfrac{x}{1+x}\right)}{\sqrt{\pi[1+x]}}$
$\sqrt{\dfrac{x}{1-x}}$	$\dfrac{E(x)}{[1-x]\sqrt{\pi}}$	$\dfrac{2}{\sqrt{\pi}}[K(x)-E(x)]$
$\{x[1+x]\}^p$, $\frac{1}{2}-2\lambda=p>-1$	$\dfrac{\Gamma(p+1)}{\{x[1+x]\}^{\lambda}}P_p^{2\lambda}(2x-1)$	$\dfrac{\Gamma(p+1)}{\{x[1+x]\}^{\lambda-\frac{1}{2}}}P_p^{2\lambda-1}(2x-1)$
$\dfrac{1}{\sqrt{x}[1\pm x]}$	$\dfrac{\mp\sqrt{\pi}}{2[1\pm x]^{\frac{3}{2}}}$	$\sqrt{\dfrac{\pi}{1\pm x}}$
$\dfrac{\sqrt{x}}{1\pm x}$	$\dfrac{\sqrt{\pi}}{2[1\pm x]^{\frac{3}{2}}}$	$\mp\sqrt{\dfrac{\pi}{1\pm x}}\pm\sqrt{\pi}$
$\dfrac{\sqrt{x}}{[1-x]^2}$	$\dfrac{\sqrt{\pi}[2+x]}{4[1-x]^{\frac{5}{2}}}$	$\dfrac{\sqrt{\pi x}}{2[1-x]^{\frac{3}{2}}}$
$\dfrac{x^p}{[1-x]^{p+\frac{3}{2}}}$	$\dfrac{[p+\frac{1}{2}+\frac{1}{2}x]\Gamma(p+1)x^{p-\frac{1}{2}}}{\Gamma(p+\frac{3}{2})[1-x]^{2+p}}$	$\dfrac{\Gamma(p+1)x^{p+\frac{1}{2}}}{\Gamma(p+\frac{3}{2})[1-x]^{p+1}}$
$\sqrt{\dfrac{1-x}{x}}$	$\dfrac{E(x)-K(x)}{x\sqrt{\pi}}$	$\dfrac{2}{\sqrt{\pi}}E(x)$
$\sqrt{\dfrac{1+x}{x}}$	$\dfrac{[1+x]E\left(\dfrac{x}{1+x}\right)-K\left(\dfrac{x}{1+x}\right)}{x\sqrt{\pi[1+x]}}$	$2\sqrt{\dfrac{1+x}{\pi}}E\left(\dfrac{x}{1+x}\right)$
$\dfrac{x^p}{[1\pm x]^r}$,	$\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2})}x^{p-\frac{1}{2}}F(r,p+1;p+\frac{1}{2};\mp x)$	$\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}x^{p+\frac{1}{2}}F(r,p+1;p+\frac{3}{2};\mp x)$
$p>-1$	$\equiv\dfrac{x^{p-\frac{1}{2}}}{\Gamma(-r)}\left[\mp x\,\dfrac{r-1,p}{0,p-\frac{1}{2}}\right]$	$\equiv\dfrac{x^{p+\frac{1}{2}}}{\Gamma(-r)}\left[\mp x\,\dfrac{r-1,p}{0,p+\frac{1}{2}}\right]$

In the preceding table $K(x)$ and $E(x)$ denote the (complete) elliptic integrals of the first

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x\sin^2(\theta)}}$$

and second

$$E(x) = \int_0^{\pi/2} \sqrt{1 - x\sin^2(\theta)}\, d\theta$$

kinds. The Legendre function of the first kind of degree ν, order μ, and argument $y[-1 < y < 1]$ is symbolized here and elsewhere by $P_\nu^{\,\mu}(y)$: Its definition and properties are discussed by Abramowitz and Stegun (1964, Chap. 8). The symbols $\mathrm{B}_x(\ ,\ )$ and $F(\ ,\ ;\ ;\ x)$ for the incomplete beta and Gauss hypergeometric functions were introduced in Sections 1.3 and 2.10.

7.5 EXPONENTIAL AND RELATED FUNCTIONS

The exponential function $\exp(x)$ is unique in the classical calculus in being preserved on repeated differentiation with respect to x. On repeated integration with zero lower limit, $\exp(x)$ is again preserved but a constant or power of x is subtracted at each integration. This is a suitable place to enquire if a function of x exists that is preserved on semidifferentiation with zero lower limit, and thus plays a similar role to the exponential in classical calculus. Inspection of the table below shows that

$$\frac{1}{\sqrt{\pi x}} + \exp(x)\operatorname{erfc}(-\sqrt{x})$$

is such a function.[2] Moreover, on repeated semiintegration, this function is preserved but a constant or a power of x is subtracted at each step; thus, the parallel to $\exp(x)$ is complete. Figure 7.5.1 shows this important function and some of its successive semiintegrals.

[2] In other words, $[1/\sqrt{\pi x}] + \exp(x)\operatorname{erfc}(-\sqrt{x})$ is an eigenfunction of the $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ operator. More generally, an eigenfunction of the d^q/dx^q operator $(q > 0)$ is the differintegrable series $x^{q-1}\sum C^j x^{jq}/\Gamma([j+1]q)$, where the summation is from $j = 0$ to $j = \infty$ and C is the eigenvalue.

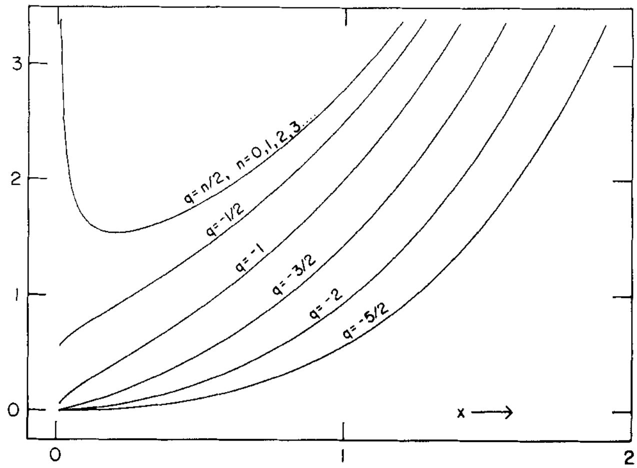

FIG. 7.5.1. The function $[1/\sqrt{\pi x}] + \exp(x)\ \mathrm{erfc}(-\sqrt{x})$, an eigenfunction of the semidifferentiation operator, and some of its successive semiintegrals. This function is preserved by the operator d^q/dx^q for $q = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, etc., but repeated semiintegration yields the function illustrated.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\exp(x)$	$\dfrac{1}{\sqrt{\pi x}} + \exp(x)\ \mathrm{erf}(\sqrt{x})$	$\exp(x)\ \mathrm{erf}(\sqrt{x})$
$\exp(-x)$	$\dfrac{1}{\sqrt{\pi x}} - \dfrac{2}{\sqrt{\pi}}\ \mathrm{daw}(\sqrt{x})$	$\dfrac{2}{\sqrt{\pi}}\ \mathrm{daw}(\sqrt{x})$
$\exp(x)\ \mathrm{erf}(\sqrt{x})$	$\exp(x)$	$\exp(x) - 1$
$\mathrm{daw}(\sqrt{x})$	$\frac{1}{2}\sqrt{\pi}\exp(-x)$	$\frac{1}{2}\sqrt{\pi}[1 - \exp(-x)]$
$\exp(x)\ \mathrm{erfc}(\sqrt{x})$	$\dfrac{1}{\sqrt{\pi x}} - \exp(x)\ \mathrm{erfc}(\sqrt{x})$	$1 - \exp(x)\ \mathrm{erfc}(\sqrt{x})$
$\exp(x)\ \mathrm{erfc}(-\sqrt{x})$	$\dfrac{1}{\sqrt{\pi x}} + \exp(x)\ \mathrm{erfc}(-\sqrt{x})$	$\exp(x)\ \mathrm{erfc}(-\sqrt{x}) - 1$
$\mathrm{erf}(\sqrt{x})$	$\exp(-\frac{1}{2}x)I_0(\frac{1}{2}x)$	$x\exp(-\frac{1}{2}x)[I_1(\frac{1}{2}x) + I_0(\frac{1}{2}x)]$
$\exp(\pm x)/\sqrt{x}$	$\frac{1}{2}\sqrt{\pi}\exp(\pm\frac{1}{2}x)[I_1(\frac{1}{2}x) \pm I_0(\frac{1}{2}x)]$	$\sqrt{\pi}\exp(\pm\frac{1}{2}x)I_0(\frac{1}{2}x)$

table continues

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\dfrac{1}{\sqrt{x}}\exp\left(\dfrac{-1}{x}\right)$	$\dfrac{1}{x^{\frac{3}{2}}}\exp\left(\dfrac{-1}{x}\right)$	$\sqrt{\pi}\,\mathrm{erfc}\left(\dfrac{1}{\sqrt{x}}\right)$
$\mathrm{erfc}\left(\dfrac{1}{\sqrt{x}}\right)$	$\dfrac{1}{\sqrt{\pi x}}\exp\left(\dfrac{-1}{x}\right)$	$2\sqrt{x}\,\mathrm{ierfc}\left(\dfrac{1}{\sqrt{x}}\right)$
$\sqrt{x}\exp(x)\gamma^*(c, x)$	$\dfrac{\sqrt{\pi}\,M(\frac{3}{2}, c+1, x)}{2\Gamma(c+1)}$	$\dfrac{\sqrt{\pi}}{\Gamma(c)}[M(\frac{1}{2}, c, x) - 1]$

7.6 TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

As was demonstrated in Section 6.4, the circular and hyperbolic sines of $\sqrt{x}$ give Bessel functions on differintegration. The corresponding cosines similarly yield Struve functions. With argument x, the trigonometric and hyperbolic functions are less easily differintegrated, but a few such results are tabulated below. Some inverse functions are also included.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\sin(\sqrt{x})$	$\frac{1}{2}\sqrt{\pi}\,J_0(\sqrt{x})$	$\sqrt{\pi x}\,J_1(\sqrt{x})$
$\cos(\sqrt{x})$	$\dfrac{1}{\sqrt{\pi x}} - \dfrac{\sqrt{\pi}\,H_0(\sqrt{x})}{2}$	$\sqrt{\pi x}\,H_{-1}(\sqrt{x})$
$\sinh(\sqrt{x})$	$\frac{1}{2}\sqrt{\pi}\,I_0(\sqrt{x})$	$\sqrt{\pi x}\,I_1(\sqrt{x})$
$\cosh(\sqrt{x})$	$\dfrac{1}{\sqrt{\pi x}} + \dfrac{\sqrt{\pi}\,L_0(\sqrt{x})}{2}$	$\sqrt{\pi x}\,L_{-1}(\sqrt{x})$
$\dfrac{\sin(\sqrt{x})}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{x}}\dfrac{H_{-1}(\sqrt{x})}{2}$	$\sqrt{\pi}\,H_0(\sqrt{x})$
$\dfrac{\cos(\sqrt{x})}{\sqrt{x}}$	$-\sqrt{\dfrac{\pi}{x}}\dfrac{J_1(\sqrt{x})}{2}$	$\sqrt{\pi}\,J_0(\sqrt{x})$
$\dfrac{\sinh(\sqrt{x})}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{x}}\dfrac{L_{-1}(\sqrt{x})}{2}$	$\sqrt{\pi}\,L_0(\sqrt{x})$

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\dfrac{\cosh(\sqrt{x})}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{x}}\dfrac{I_1(\sqrt{x})}{2}$	$\sqrt{\pi}I_0(\sqrt{x})$
$\dfrac{1-\cos(\sqrt{x})}{x}$	$\dfrac{\sqrt{\pi}}{2}H_0(\sqrt{x})$	$\sqrt{\pi x}H_1(\sqrt{x})$
$\dfrac{1-\cosh(\sqrt{x})}{x}$	$\dfrac{\sqrt{\pi}}{2}L_0(\sqrt{x})$	$\sqrt{\pi x}L_1(\sqrt{x})$
$\sin(x)$	$\sin(x+\frac{1}{4}\pi)$ $-\sqrt{2}\,\mathrm{gres}\left(\sqrt{\dfrac{2x}{\pi}}\right)$	$\sin(x-\frac{1}{4}\pi)$ $+\sqrt{2}\,\mathrm{fres}\left(\sqrt{\dfrac{2x}{\pi}}\right)$
$\cos(x)$	$\dfrac{1}{\sqrt{\pi x}}+\cos(x+\frac{1}{4}\pi)$ $-\sqrt{2}\,\mathrm{fres}\left(\sqrt{\dfrac{2x}{\pi}}\right)$	$\cos(x-\frac{1}{4}\pi)$ $-\sqrt{2}\,\mathrm{gres}\left(\sqrt{\dfrac{2x}{\pi}}\right)$
$\sinh(x)$	$\dfrac{\mathrm{daw}(\sqrt{x})}{\sqrt{\pi}}-\dfrac{\exp(x)\,\mathrm{erf}(\sqrt{x})}{2}$	$\dfrac{\exp(x)\,\mathrm{erf}(\sqrt{x})}{2}-\dfrac{\mathrm{daw}(\sqrt{x})}{\sqrt{\pi}}$
$\cosh(x)$	$\dfrac{1}{\sqrt{\pi x}}+\dfrac{\exp(x)\,\mathrm{erf}(\sqrt{x})}{2}-\dfrac{\mathrm{daw}(\sqrt{x})}{\sqrt{\pi}}$	$\dfrac{\exp(x)\,\mathrm{erf}(\sqrt{x})}{2}+\dfrac{\mathrm{daw}(\sqrt{x})}{\sqrt{\pi}}$
$\dfrac{\arcsin(\sqrt{x})}{\sqrt{1-x}}$	$\dfrac{\sqrt{\pi}}{2[1-x]}$	$-\dfrac{\sqrt{\pi}}{2}\ln(1-x)$
$\arctan(\sqrt{x})$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{1+x}}$	$\sqrt{\pi}[\sqrt{1+x}-1]$
$\dfrac{\mathrm{arcsinh}(\sqrt{x})}{\sqrt{1+x}}$	$\dfrac{\sqrt{\pi}}{2[1+x]}$	$\dfrac{\sqrt{\pi}}{2}\ln(1+x)$
$\mathrm{arctanh}(\sqrt{x})$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{1-x}}$	$\sqrt{\pi}[1-\sqrt{1-x}]$

In the preceding table, $J_\nu(\ )$ and $I_\nu(\ )$ denote respectively the Bessel and modified Bessel functions of order ν; $H_\nu(\ )$ and $L_\nu(\ )$ similarly denote the Struve and modified Struve functions of order ν. The functions we denote fres( ) and gres( ) are those auxiliary Fresnel integrals that Abramowitz and Stegun (1964, p. 300) symbolize as $f(\ )$ and $g(\ )$. These functions occur in the

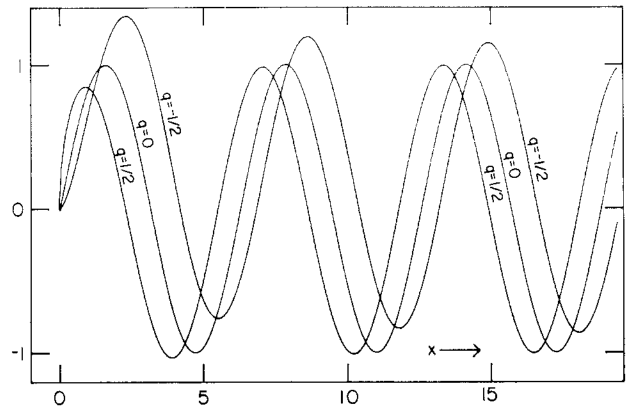

FIG. 7.6.1. The semiderivative ($q = +\frac{1}{2}$) and semiintegral ($q = -\frac{1}{2}$) of $\sin(x)$.

semiderivative and semiintegral of the sine (see Fig. 7.6.1) and cosine (see Fig. 7.6.2) functions.

Note that in this section (as, indeed, throughout the book) we have chosen real arguments for all functions. Because of the relationships

$$\sinh(x) = -i\sin(ix),$$
$$\cosh(x) = \cos(ix),$$
$$\sin(x) = \tfrac{1}{2}\exp(-ix) - \tfrac{1}{2}\exp(ix),$$

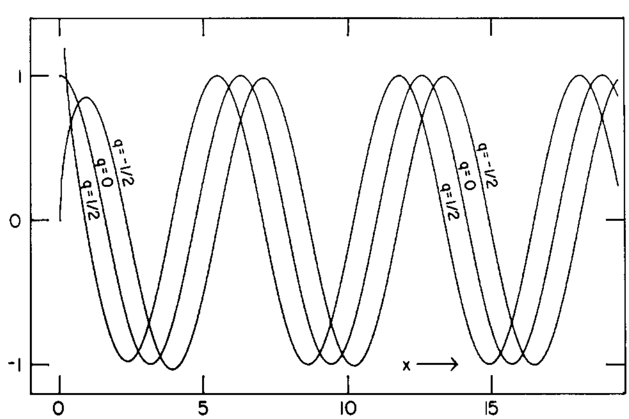

FIG. 7.6.2. The semiderivative ($q = +\frac{1}{2}$) and semiintegral ($q = -\frac{1}{2}$) of $\cos(x)$.

etc., it is easy to replace the tabular entries by functions with imaginary arguments, leading to such formulas as

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \frac{\sinh(\sqrt{x})}{\sqrt{x}} = -i\sqrt{\frac{\pi}{x}} \frac{H_{-1}(\sqrt{ix})}{2}.$$

7.7 BESSEL AND STRUVE FUNCTIONS

Here we restrict consideration almost solely to the argument $\sqrt{x}$.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$J_0(\sqrt{x})$	$\dfrac{\cos(\sqrt{x})}{\sqrt{\pi x}}$	$\dfrac{2\sin(\sqrt{x})}{\sqrt{\pi}}$
$\dfrac{J_1(\sqrt{x})}{\sqrt{x}}$	$\dfrac{\cos(\sqrt{x}) + \sqrt{x}\sin(\sqrt{x}) - 1}{\sqrt{\pi}\,x^{\frac{3}{2}}}$	$\dfrac{2[1-\cos(\sqrt{x})]}{\sqrt{\pi x}}$
$\dfrac{J_\nu(\sqrt{x})}{x^{\nu/2}}$	$\dfrac{1}{2^\nu\Gamma(\nu+1)\sqrt{\pi x}} - \dfrac{H_{\nu+\frac{1}{2}}(\sqrt{x})}{\sqrt{2}x^{[\nu/2]+\frac{1}{4}}}$	$\dfrac{\sqrt{2}\,H_{\nu-\frac{1}{2}}(\sqrt{x})}{x^{[\nu/2]-\frac{1}{4}}}$
$\sqrt{x}\,J_1(\sqrt{x})$	$\dfrac{\sin(\sqrt{x})}{\sqrt{\pi}}$	$\dfrac{2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x})}{\sqrt{\pi}}$
$x^{\nu/2}J\,(\sqrt{x})$	$\dfrac{x^{[\nu/2]-\frac{1}{4}}J_{\nu-\frac{1}{2}}(\sqrt{x})}{\sqrt{2}}$	$\sqrt{2}\,x^{[\nu/2]+\frac{1}{4}}J_{\nu+\frac{1}{2}}(\sqrt{x})$
$I_0(\sqrt{x})$	$\dfrac{\cosh(\sqrt{x})}{\sqrt{\pi x}}$	$\dfrac{2\sinh(\sqrt{x})}{\sqrt{\pi}}$
$\dfrac{I_1(\sqrt{x})}{\sqrt{x}}$	$\dfrac{\cosh(\sqrt{x}) - \sqrt{x}\sinh(\sqrt{x}) - 1}{\sqrt{\pi}\,x^{\frac{3}{2}}}$	$\dfrac{2[1-\cosh(\sqrt{x})]}{\sqrt{\pi x}}$
$\dfrac{I_\nu(\sqrt{x})}{x^{\nu/2}}$	$\dfrac{1}{2^\nu\Gamma(\nu+1)\sqrt{\pi x}} + \dfrac{L_{\nu+\frac{1}{2}}(\sqrt{x})}{\sqrt{2}\,x^{[\nu/2]+\frac{1}{4}}}$	$\dfrac{\sqrt{2}\,L_{\nu-\frac{1}{2}}(\sqrt{x})}{x^{[\nu/2]-\frac{1}{4}}}$
$\sqrt{x}\,I_1(\sqrt{x})$	$\dfrac{\sinh(\sqrt{x})}{\sqrt{\pi}}$	$\dfrac{2\sqrt{x}\cosh(\sqrt{x}) - 2\sinh(\sqrt{x})}{\sqrt{\pi}}$

table continues

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$x^{\nu/2} I_\nu(\sqrt{x})$	$\dfrac{x^{[\nu/2]-\frac{1}{4}} I_{\nu-\frac{1}{2}}(\sqrt{x})}{\sqrt{2}}$	$\sqrt{2}\, x^{[\nu/2]+\frac{1}{4}} I_{\nu+\frac{1}{2}}(\sqrt{x})$
$\exp(-x) I_0(x)$	$\dfrac{\exp(-2x)}{\sqrt{\pi x}}$	$\dfrac{\text{erf}(\sqrt{2x})}{\sqrt{2}}$
$H_0(\sqrt{x})$	$\dfrac{\sin(\sqrt{x})}{\sqrt{\pi x}}$	$\dfrac{2[1-\cos(\sqrt{x})]}{\sqrt{\pi}}$
$\dfrac{H_0(\sqrt{x})}{x}$	$\dfrac{\sin(\sqrt{x}) - \text{Si}(\sqrt{x})}{\sqrt{\pi}\, x^{\frac{3}{2}}}$	$\dfrac{2\,\text{Si}(\sqrt{x})}{\sqrt{\pi x}}$
$\sqrt{x}\, H_1(\sqrt{x})$	$\dfrac{1-\cos(\sqrt{x})}{\sqrt{\pi}}$	$\dfrac{x+2-2\sqrt{x}\sin(\sqrt{x})-2\cos(\sqrt{x})}{\sqrt{\pi}}$
$x^{\nu/2} H_\nu(\sqrt{x})$	$\dfrac{x^{[\nu/2]-\frac{1}{4}} H_{\nu-\frac{1}{2}}(\sqrt{x})}{\sqrt{2}}$	$\sqrt{2}\, x^{[\nu/2]+\frac{1}{4}} H_{\nu+\frac{1}{2}}(\sqrt{x})$
$L_0(\sqrt{x})$	$\dfrac{\sinh(\sqrt{x})}{\sqrt{\pi x}}$	$\dfrac{2[\cosh(\sqrt{x})-1]}{\sqrt{\pi}}$
$\dfrac{L_0(\sqrt{x})}{x}$	$\dfrac{\sinh(\sqrt{x}) - \text{Shi}(\sqrt{x})}{\sqrt{\pi}\, x^{\frac{3}{2}}}$	$\dfrac{2\,\text{Shi}(\sqrt{x})}{\sqrt{\pi x}}$
$\sqrt{x}\, L_1(\sqrt{x})$	$\dfrac{\cosh(\sqrt{x})-1}{\sqrt{\pi}}$	$\dfrac{2-x+2\sqrt{x}\sinh(\sqrt{x})-2\cosh(\sqrt{x})}{\sqrt{\pi}}$
$x^{\nu/2} L_\nu(\sqrt{x})$	$\dfrac{x^{[\nu/2]-\frac{1}{4}} L_{\nu-\frac{1}{2}}(\sqrt{x})}{\sqrt{2}}$	$\sqrt{2}\, x^{[\nu/2]+\frac{1}{4}} L_{\nu+\frac{1}{2}}(\sqrt{x})$

The functions Si( ) and Shi( ) appearing in this table are the sine integral

$$\text{Si}(x) \equiv \int_0^x \frac{\sin(y)\, dy}{y}$$

and its hyperbolic counterpart [see Abramowitz and Stegun (1964, p. 231)]

$$\text{Shi}(x) \equiv \int_0^x \frac{\sinh(y)\, dy}{y}.$$

7.8 GENERALIZED HYPERGEOMETRIC FUNCTIONS

The results reported here are specializations of the general rules deduced in Section 6.5.

The full range of possible numeratorial and denominatorial parameters has been spelled out in the first six tabular entries; thereafter only ellipses are used to represent their possible presence.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\left[\beta x \,\dfrac{b_1,\ldots,b_K}{c_1,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{0,b_1,\ldots,b_K}{-\frac{1}{2},c_1,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{0,b_1,\ldots,b_K}{\frac{1}{2},c_1,\ldots,c_L}\right]$
$\left[\beta x \,\dfrac{-\frac{1}{2},b_2,\ldots,b_K}{c_1,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{0,b_2,\ldots,b_K}{c_1,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{0,-\frac{1}{2},b_2,\ldots,b_K}{\frac{1}{2},c_1,\ldots,c_L}\right]$
$\left[\beta x \,\dfrac{\frac{1}{2},b_2,\ldots,b_K}{c_1,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{0,\frac{1}{2},b_2,\ldots,b_K}{-\frac{1}{2},c_1,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{0,b_2,\ldots,b_K}{c_1,\ldots,c_L}\right]$
$\left[\beta x \,\dfrac{b_1,\ldots,b_K}{0,c_2,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{b_1,\ldots,b_K}{-\frac{1}{2},c_2,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{b_1,\ldots,b_K}{\frac{1}{2},c_2,\ldots,c_L}\right]$
$\left[\beta x \,\dfrac{-\frac{1}{2},b_2,\ldots,b_K}{0,c_2,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{b_2,\ldots,b_K}{c_2,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{-\frac{1}{2},b_2,\ldots,b_K}{\frac{1}{2},c_2,\ldots,c_L}\right]$
$\left[\beta x \,\dfrac{\frac{1}{2},b_2,\ldots,b_K}{0,c_2,\ldots,c_L}\right]$	$\dfrac{1}{\sqrt{x}}\left[\beta x \,\dfrac{\frac{1}{2},b_2,\ldots,b_K}{-\frac{1}{2},c_2,\ldots,c_L}\right]$	$\sqrt{x}\left[\beta x \,\dfrac{b_2,\ldots,b_K}{c_2,\ldots,c_L}\right]$
$x^p\left[\beta x \,\dfrac{\ldots}{\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \,\dfrac{p,\ldots}{p-\frac{1}{2},\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \,\dfrac{p,\ldots}{p+\frac{1}{2},\ldots}\right]$
$x^p\left[\beta x \,\dfrac{p-\frac{1}{2},\ldots}{\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \,\dfrac{p,\ldots}{\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \,\dfrac{p-\frac{1}{2},p,\ldots}{p+\frac{1}{2},\ldots}\right]$
$x^p\left[\beta x \,\dfrac{p+\frac{1}{2},\ldots}{\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \,\dfrac{p,p+\frac{1}{2},\ldots}{p-\frac{1}{2},\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \,\dfrac{p,\ldots}{\ldots}\right]$
$x^p\left[\beta x \,\dfrac{\ldots}{p,\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \,\dfrac{\ldots}{p-\frac{1}{2},\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \,\dfrac{\ldots}{p+\frac{1}{2},\ldots}\right]$
$x^p\left[\beta x \,\dfrac{p-\frac{1}{2},\ldots}{p,\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \,\dfrac{\ldots}{\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \,\dfrac{p-\frac{1}{2},\ldots}{p+\frac{1}{2},\ldots}\right]$

table continues

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$x^p\left[\beta x \dfrac{p+\frac{1}{2},\ldots}{p,\ldots}\right]$	$x^{p-\frac{1}{2}}\left[\beta x \dfrac{p+\frac{1}{2},\ldots}{p-\frac{1}{2},\ldots}\right]$	$x^{p+\frac{1}{2}}\left[\beta x \dfrac{\ldots}{\ldots}\right]$
$\left[\beta x^2 \dfrac{\ldots}{\ldots}\right]$	$\sqrt{\dfrac{2}{x}}\left[\beta x^2 \dfrac{-\frac{1}{2},0,\ldots}{-\frac{3}{4},-\frac{1}{4},\ldots}\right]$	$\sqrt{\dfrac{x}{2}}\left[\beta x^2 \dfrac{-\frac{1}{2},0,\ldots}{-\frac{1}{4},\frac{1}{4},\ldots}\right]$
$x^p\left[\beta x^2 \dfrac{\ldots}{\ldots}\right]$	$\sqrt{2}\,x^{p-\frac{1}{2}}\left[\beta x^2 \dfrac{\frac{1}{2}p-\frac{1}{2},\frac{1}{2}p,\ldots}{\frac{1}{2}p-\frac{3}{4},\frac{1}{2}p-\frac{1}{4},\ldots}\right]$	$\dfrac{x^{p+\frac{1}{2}}}{\sqrt{2}}\left[\beta x^2 \dfrac{\frac{1}{2}p-\frac{1}{2},\frac{1}{2}p,\ldots}{\frac{1}{2}p-\frac{1}{4},\frac{1}{2}p+\frac{1}{4},\ldots}\right]$

Notice that semidifferentiation or semiintegration of a $\frac{K}{L}$ hypergeometric of argument x leads generally to a $\frac{K+1}{L+1}$ hypergeometric, but that parametric cancellation may reduce this complexity to $\frac{K}{L}$ or $\frac{K-1}{L-1}$. Likewise semidifferintegration of a $\frac{K}{L}$ hypergeometric of argument x^2 leads in general to a complexity of $\frac{K+2}{L+2}$. Cancellation of parameters in the x^2 case, however, may alter the complexity to $\frac{K+1}{L+1}$, $\frac{K}{L}$, $\frac{K-1}{L-1}$, or even to $\frac{K-2}{L-2}$.

7.9 MISCELLANEOUS FUNCTIONS

The functions collected here are the logarithm, the two complete elliptic integrals, and the Heaviside function.

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$\ln(x)$	$\dfrac{\ln(4x)}{\sqrt{\pi x}}$	$2\sqrt{\dfrac{x}{\pi}}\,[\ln(4x)-2]$
$\sqrt{x}\,\ln(x)$	$\dfrac{\sqrt{\pi}}{2}\left[\ln\left(\dfrac{x}{4}\right)+2\right]$	$\dfrac{x\sqrt{\pi}}{2}\left[\ln\left(\dfrac{x}{4}\right)+1\right]$
$\dfrac{\ln(x)}{\sqrt{x}}$	$\dfrac{\sqrt{\pi}}{x}$	$\sqrt{\pi}\,\ln\left(\dfrac{x}{4}\right)$

f	$\dfrac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$	$\dfrac{d^{-\frac{1}{2}}f}{dx^{-\frac{1}{2}}}$
$K(x)$	$\dfrac{\sqrt{\pi}}{2\sqrt{x[1-x]}}$	$\sqrt{\pi}\arcsin(\sqrt{x})$
$E(x)$	$\dfrac{1}{2}\sqrt{\dfrac{\pi[1-x]}{x}}$	$\dfrac{\sqrt{\pi x[1-x]}}{2}+\dfrac{\sqrt{\pi}\arcsin(\sqrt{x})}{2}$
$H(x-x_0)$	$\dfrac{H(x-x_0)}{\sqrt{\pi[x-x_0]}}$	$\dfrac{2\sqrt{x-x_0}\,H(x-x_0)}{\sqrt{\pi}}$