9 Rotational Symmetry

A television image is rectangular and uses two independent variables in its representation of a three-dimensional scene. However, the television camera has an optical system with component parts, such as the aperture containing the objective lens, which, while distinctly multidimensional, possess circular symmetry. Only one independent variable, namely radius, may be needed to specify the instrumental properties across such an aperture, because the properties may be independent of the second variable, in this case angle. Circular symmetry also occurs in objects that are under study (some astronomical objects, for example) or in objects that influence imaging (rain drops), and is a property of many artifacts. Because of the prevalence of circular symmetry, particularly in instruments, special attention to circularly symmetrical transforms is warranted. This chapter also deals briefly with objects having rotational symmetry, which are of less frequent occurrence, but related.

Bessel functions play a basic role in the following discussion, especially the zero-order Bessel function $J_0(x)$, or more aptly $J_0(r)$, since the independent variable is ordinarily radial. For reference, Fig. 9-1 shows $J_0(r)$ and $J_1(r)$. For convenience, another function introduced later is included, and, for comparison, the well-known sinc function.

R. Bracewell, *Fourier Analysis and Imaging*

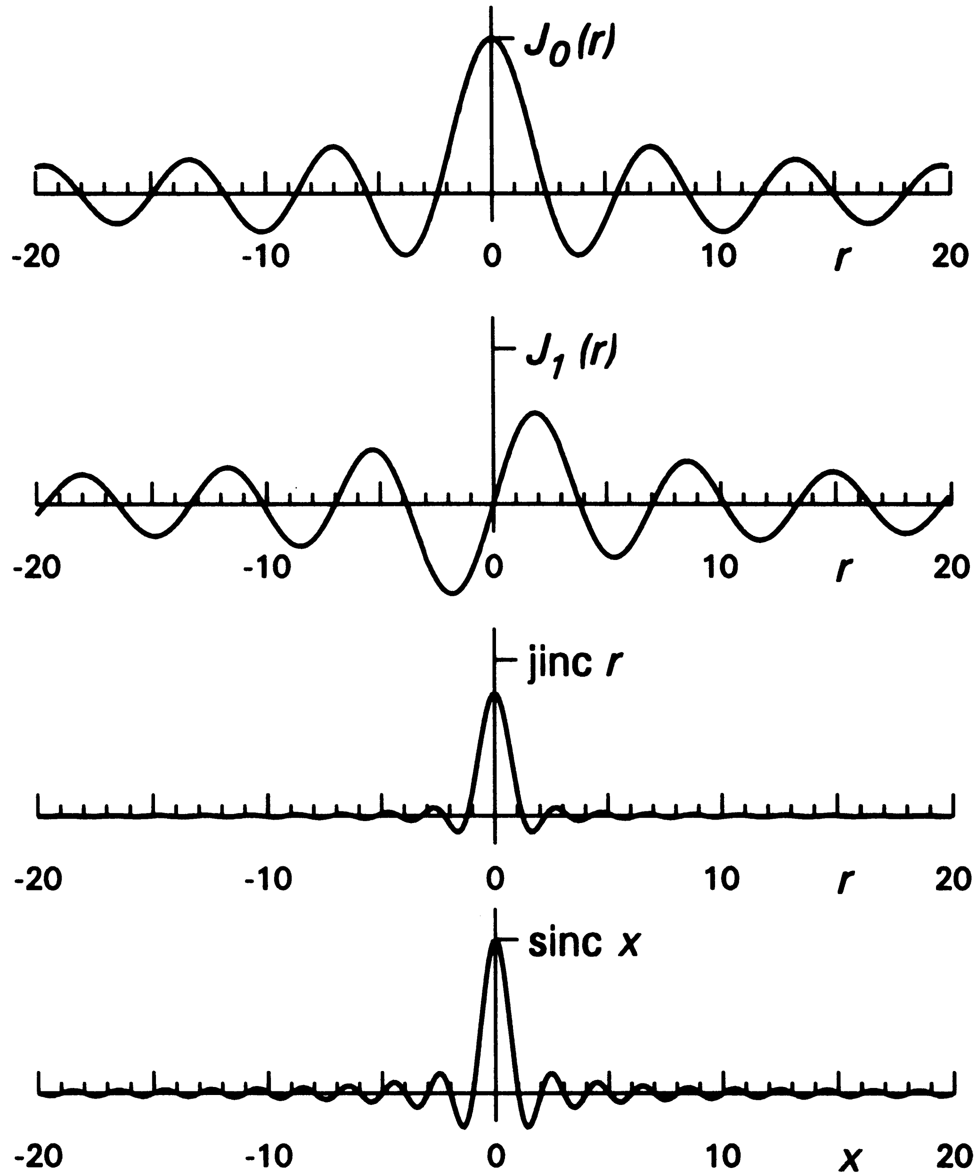

Figure 9-1 The functions $J_0(r)$, $J_1(r)$, jinc r, and sinc x.

WHAT IS A BESSEL FUNCTION?

If you allow a hanging chain to swing, it takes on the shape of a Bessel function, if you bang a drum in the center and wait for the overtones to die out, the fundamental mode of vibration follows a Bessel function of radius; and if you look at the sidebands of an FM station transmitting a tone, their amplitudes are distributed according to a Bessel function.

The chain problem is of particular interest historically, because that is where F. W. Bessel (1784–1846), a great astronomer who first measured the distance to a star and discovered the dark companion of Sirius, found his function.

We may also take a mathematical approach, purified of any physics, and define the Bessel function $J_0(x)$ as the sum of the series

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$

or as a solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0.$$

Of course we know that a second-order differential equation may possess two independent solutions, and we will automatically generate the second kind of Bessel function $Y_0(x)$ as well as $J_0(x)$ if we take the differential equation as a basis.

By taking x to be large, we find the very useful asymptotic expression

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{1}{4}\pi), \quad x \gg 1.$$

This expression, together with the series expansion, enables us to calculate with Bessel functions without recourse to tables or to special computer subprograms. Numerical computing itself leads to fascinating viewpoints, exemplified by the following polynomial approximation (A&S 1963, p. 356) for $-3 \le x \le 3$:

$$J_0(x) = 1 - 2.24999\,97(x/3)^2 + 1.26562\,08(x/3)^4 - .31638\,66(x/3)^6 + .04444\,79(x/3)^8$$
$$- .00394\,44(x/3)^{10} + .00021\,00(x/3)^{12} + \epsilon, \quad |\,\epsilon\,| < 5 \times 10^{-8}.$$

A quite different integral definition may also be encountered, namely.

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\phi)d\phi.$$

It is by no means immediately obvious how the three mathematical definitions interrelate, and one would need to exercise some care before choosing a definition as a point of departure into a particular problem.

As the chain, drum, and FM station show, Bessel functions are not at all restricted as to where they may show up, but here we will emphasize situations with uniform circular geometry.

The water in a circular tank (Fig. 9-2) may slosh so that its height is $J_0(kr)\cos\omega t$; the longitudinal electric field in a circular waveguide may follow exactly the same expression, and so may the sound pressure in a circular tunnel. There are many other cases. The light intensity on the photographic film of a circular telescope pointed at a star is described by another Bessel function (Fig. 9-3) and has the appearance of a spot surrounded by rings of decreasing brightness.

These Airy rings were first analyzed by former Astronomer Royal George B. Airy (1801–1892), who is well known for many other things including the Airy functions of

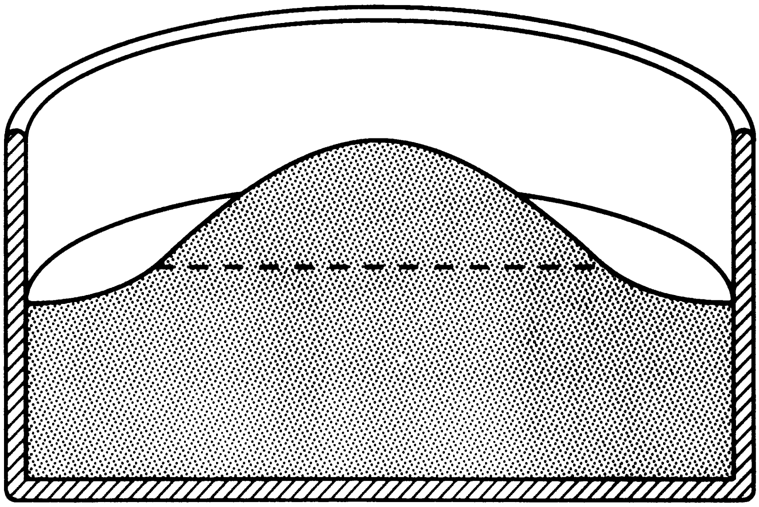

Figure 9-2 Water sloshing at low amplitude in a circular tank may take up a Bessel-function form. This tells us something about the volume under a Bessel function, when we remember that water is rather incompressible.

electromagnetic theory, which are also Bessel functions, and for having turned away J. C. Adams (1819–1892) when the latter appeared at his door at dinner time to talk about the discovery of Neptune.

As groundwork for the discussion we look at the familiar cosinusoidal corrugation as it appears in the polar coordinates which are appropriate to uniform circular circumstances. Consider the corrugation

$$f(x, y) = A\cos 2\pi ux$$

illustrated in Fig. 9-4. At a point (r, θ) the value of the function is $A\cos(2\pi ur\cos\theta)$. Hence in polar coordinates

$$f(x, y) = A\cos(2\pi ur\cos\theta).$$

As a mathematical expression, a cosine of a cosine may appear unfamiliar, but it describes a rather simple entity. Now, if we wish to describe the rotated corrugation $A\cos 2\pi ux'$, where the axis x' makes an angle ϕ with the x-axis, we may write

$$A\cos[2\pi ur\cos(\theta - \phi)].$$

Of course, the rotated corrugation may also be expressed in cartesian coordinates as

$$A\cos[2\pi u(x\cos\phi + y\sin\phi)],$$

which is just as complicated as the form taken in polar coordinates.

Imagine that the full circle is divided into a large number of equispaced directions and that for each of the directions ϕ there is a corrugation $Ad\phi\cos[2\pi ur\cos(\theta - \phi)]$, all

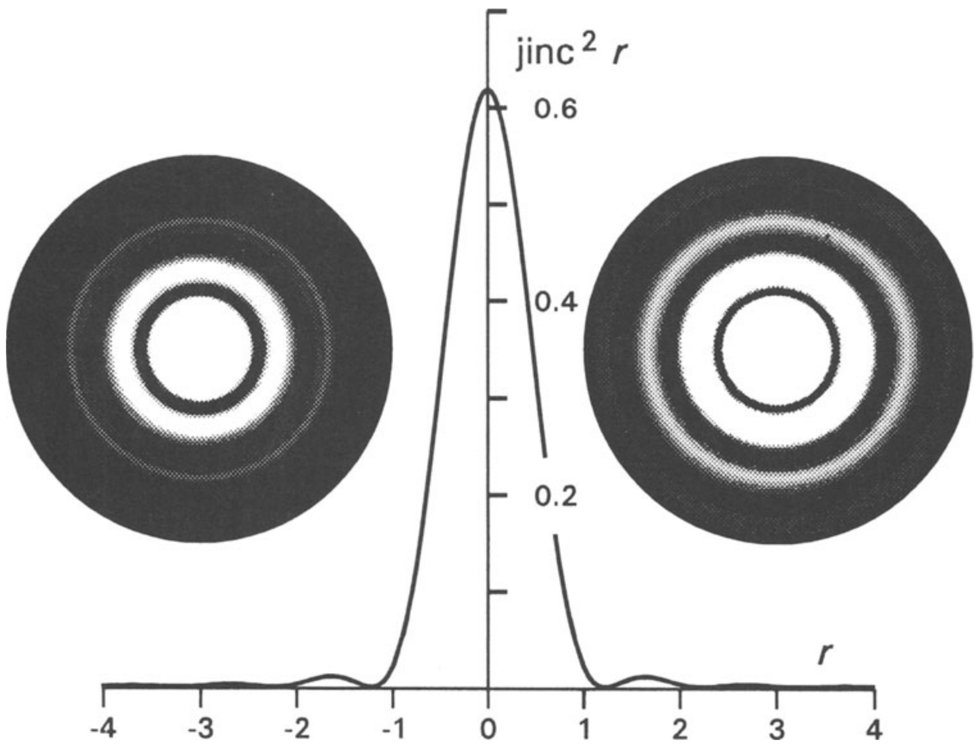

Figure 9-3 The Airy pattern jinc$^2\, r$ and two halftone renderings. The three rings, at their brightest, are respectively 18, 24, and 28 dB down relative to the central brightness maximum. The thresholds for black are set at 27 dB (left) and 31 dB (right). Comparison of the widths of the light and dark annuli with the graph will confirm that faithful halftone reproduction of the Airy pattern is impossible in a printed book.

of which corrugations are to be superimposed by addition. The sum will be

$$\int_0^{2\pi} A\cos[2\pi ur\cos(\theta-\phi)]d\phi.$$

After this integral has been evaluated, we will have a new function of r and θ; the variable of integration ϕ will have disappeared. However, if we fix attention on a particular value of r, it is clear from the circular uniformity with which the superposition was carried out that the result will not depend on θ; the integral must be independent of θ. Therefore, it makes no difference for what particular value of θ we evaluate it. Choose $\theta = 0$. Then the circular superposition integral becomes

$$\int_0^{2\pi} \cos(2\pi ur\cos\phi)d\phi.$$

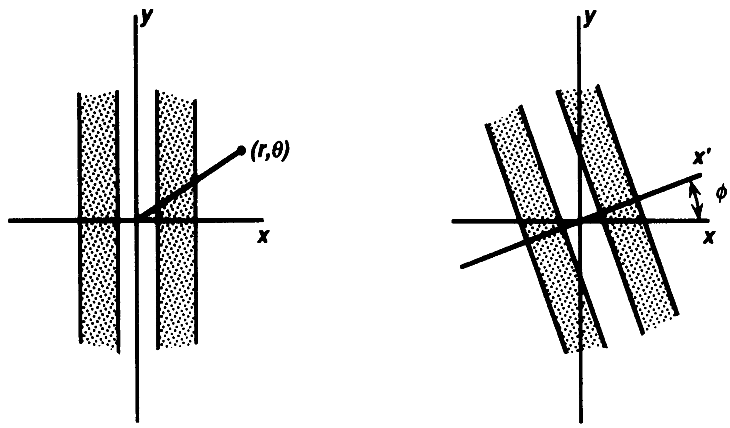

Figure 9-4 The simple corrugation $A \cos 2\pi ux$ (left) becomes $A \cos(2\pi ur \cos\theta)$ in polar coordinates, and the rotated corrugation (right) becomes $A \cos[2\pi ur \cos(\theta - \phi)]$.

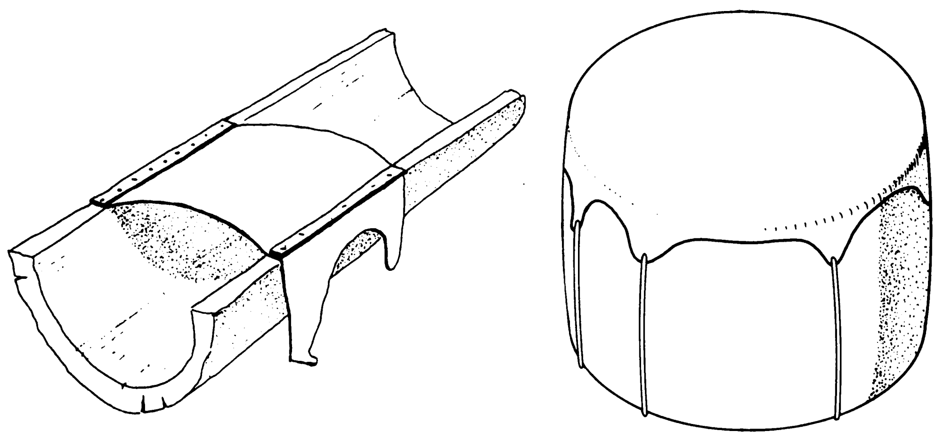

Figure 9-5 The fundamental mode of a laterally stretched membrane is cosinusoidal (left), while the corresponding mode of a circular drum is a zero-order Bessel function.

If we divide by 2π to obtain the angular average rather than the sum, we recognize this as one of the mathematical definitions of $J_0(x)$ mentioned earlier. What have we discovered?

We have found that $J_0(r)$ is to circular situations what cosine is to one-dimensional problems. A laterally stretched square membrane clamped along opposite sides can vibrate cosinusoidally like a clamped string (Fig. 9-5). Clamp the membrane on a circular rim, as with a drum, and all the possible cosinusoidal corrugations differently oriented add up to a Bessel function of radius only (Fig. 9-6).

The same can be said of water waves in square tanks and round tanks, microwaves in rectangular and round waveguides, acoustic waves in bars and rods. Just as the sinusoidal oscillation with respect to one-dimensional time derives its importance from the ubiquity

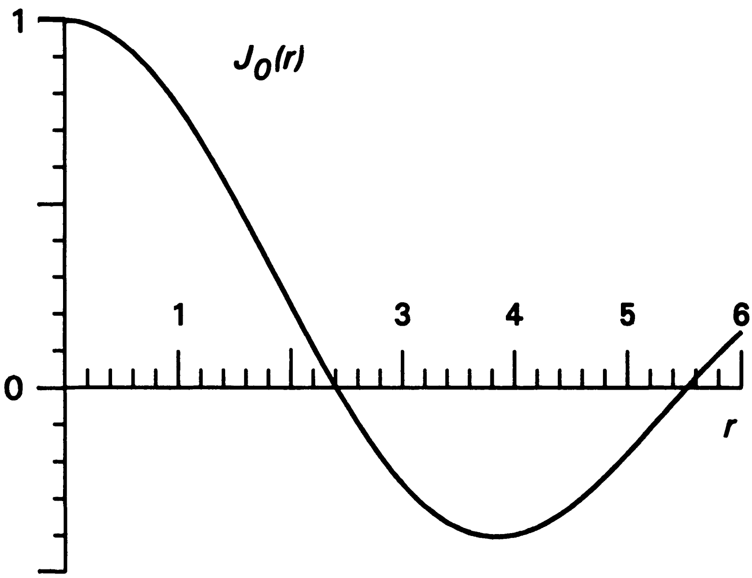

Figure 9-6 The zero-order Bessel function $J_0(r)$.

of electrical and mechanical systems that are both linear and time invariant, so the Bessel function will be found in higher dimensionalities where linearity and space invariance exist and where circular boundary conditions are imposed.

THE HANKEL TRANSFORM

To obtain the Fourier transform of a function that is constant over a central circle in the (x, y)-plane and zero elsewhere, or which, while not being constant, is a function of $r = (x^2 + y^2)^{1/2}$ only, one may of course use the standard transform definition in cartesian coordinates. Let the function $f(x, y)$ be $\mathbf{f}(r)$. Then

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}(r)e^{-i2\pi(ux+vy)}\,dx\,dy.$$

Some comments should be made about the limits of integration. They may certainly be written as above, provided it is understood that $\mathbf{f}(r)$ covers the whole plane $0 < r < \infty$. In the example mentioned, where the function is a constant K over a central circular region C of radius a and zero elsewhere, there would be a simplified alternative

$$F(u, v) = K\int\int_C e^{-i2\pi(ux+vy)}\,dx\,dy.$$

Of course, the boundary of a circle does not lend itself to the simplest kind of representation in rectangular coordinates. Nevertheless, it is possible to do so, in which case we could write

$$F(u,v) = K\int_{-a}^{a} dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi(ux+vy)}\,dx.$$

Sometimes this kind of integration over elaborate boundaries works out painlessly; let us try to proceed in this case.

$$\begin{aligned}
F(u,v) &= K\int_{-a}^{a} e^{-i2\pi vy}\,dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi ux}\,dx \\
&= K\int_{-a}^{a} e^{-i2\pi vy}\left[\frac{e^{-i2\pi ux}}{-i2\pi u}\right]_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} dy \\
&= K\int_{-a}^{a} e^{-i2\pi vy}\left[\frac{e^{-i2\pi u(a^2-y^2)^{1/2}} - e^{i2\pi u(a^2-y^2)^{1/2}}}{-i2\pi u}\right] dy \\
&= \frac{K}{\pi u}\int_{-a}^{a} e^{-i2\pi vy}\sin\left[2\pi u(a^2-y^2)^{1/2}\right] dy \\
&= 2\frac{K}{\pi u}\int_{0}^{a} \cos(2\pi vy)\sin\left[2\pi u(a^2-y^2)^{1/2}\right] dy.
\end{aligned}$$

This is the maximum simplification that we can hope to make before turning to lists of integrals for help. We find the integral on p. 399 as entry 3.711 in GR (1965). The answer is

$$F(u,v) = Ka(u^2+v^2)^{-1/2}J_1\left[2\pi a(u^2+v^2)^{1/2}\right] = Ka\frac{J_1(2\pi aq)}{q},$$

where J_1 is the first-order Bessel function of the first kind and $q = \sqrt{u^2+v^2}$.

As an alternative approach, consider making use of the circular symmetry of the exercise rather than forcing rectangular coordinates upon it. Let (r,θ) be the polar coordinates of (x,y). Then

$$F(u,v) = \int\int_C \mathsf{f}(r)e^{-i2\pi(ux+vy)}\,dx\,dy = \int_0^\infty \int_0^{2\pi} \mathsf{f}(r)e^{-i2\pi qr\cos(\theta-\phi)}r\,dr\,d\theta.$$

We are now integrating from 0 to ∞ radially and through 0 to 2π in azimuth, and, because of the independence of azimuth, the latter integral should drop out. The new symbols q and ϕ are polar coordinates in the (u,v)-plane. Thus

$$q^2 = u^2 + v^2 \quad \text{and} \quad \tan\phi = v/u.$$

Then the new kernel $\exp[-i2\pi qr\cos(\theta-\phi)]$ arises from recognizing $ux+vy$ as the scalar product of two two-dimensional vectors. Thus

$$ux + vy = \Re[(x+iy)(u-iv)] = \Re[re^{i\theta}qe^{-i\phi}] = qr\cos(\theta-\phi).$$

Removing the integration with respect to θ,

$$\begin{aligned} F(u, v) &= \int_0^\infty \mathbf{f}(r) \left[\int_0^{2\pi} e^{-i2\pi qr\cos(\theta-\phi)} d\theta \right] r\,dr \\ &= \int_0^\infty \mathbf{f}(r) \left[\int_0^{2\pi} \cos(2\pi qr\cos\theta) d\theta \right] r\,dr. \end{aligned}$$

Since u and v, and therefore q and ϕ, are fixed during the integration over the (x, y)-plane, we may drop ϕ, because it merely represents an initial angle in an integration that will run over one full rotation from 0 to 2π. Therefore, the result of the integration will be the same regardless of the value of ϕ; take it to be zero. Again, the sine component of the imaginary exponential may be dropped, because it will integrate to zero; to see this, make sketches of cos (cos θ) and sin (cos θ) for $0 < \theta < 2\pi$.

We now make use of the fundamental integral representation for the zero-order Bessel function $J_0(z)$, namely,

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \cos(z\cos\theta) d\theta.$$

This basic relation will be returned to later. Meanwhile, incorporating it into the development, we have finally

$$F(u, v) = 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi qr) r\,dr.$$

Here we have the statement of the integral transform that takes the place of the two-dimensional Fourier transform when $f(x, y)$ possesses circular symmetry and is representable by $\mathbf{f}(r)$. It follows that $F(u, v)$ also possesses circular symmetry (the rotation theorem is an expression of this) and may be written $\mathbf{F}(q)$, depending only on the radial coordinate $q = (u^2 + v^2)^{1/2}$ in the (u, v)-plane and not on azimuth ϕ. The transform

Hankel transform.

$$\mathbf{F}(q) = 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi qr) r\,dr$$

is known as the Hankel transform. It is a one-dimensional transform. The functions, **f** and **F** are functions of one variable. They are not a Fourier transform pair, but a Hankel transform pair. As functions of *one* variable they may be used to represent two-dimensional functions, which will be two-dimensional Fourier transform pairs.

Hankel Transform of a Disk

Our first example of a Hankel transform pair, obtained directly by integration, was

$$\operatorname{rect}\left(\frac{r}{2a}\right) \quad \text{has Hankel transform} \quad \frac{aJ_1(2\pi aq)}{q}.$$

This is such an important pair that we adopt the special name jinc q for the Hankel transform of rect r. Thus

$$\text{rect}\, r \quad \text{has Hankel transform} \quad \text{jinc}\, q \equiv \frac{J_1(\pi q)}{2q}.$$

From the integral transform formulation it follows that

$$\text{jinc}\, q = 2\pi \int_0^\infty \text{rect}\, r\, J_0(2\pi q r) r\, dr.$$

Likewise, from the reversibility of the two-dimensional Fourier transform, it follows that

$$\text{rect}\, r = 2\pi \int_0^\infty \text{jinc}\, q \;\; J_0(2\pi q r)\; q\, dq.$$

Frequently needed properties of the jinc function are collected below.

Hankel Transform of the Ring Impulse

As an example of the Hankel transform we used the rectangle function of radius 0.5 and found by direct integration in two dimensions that the Hankel transform was jinc q. Now we make use of the Hankel transform formula to obtain another important transform pair. Let

$$\mathbf{f}(r) = \delta(r - a)$$

which describes a unit-strength ring impulse. Its Hankel transform $\mathbf{F}(q)$ is given by

$$\begin{aligned}\mathbf{F}(q) &= 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi q r) r\, dr \\ &= 2\pi \int_0^\infty \delta(r - a) J_0(2\pi q r) r\, dr.\end{aligned}$$

Apply the sifting property to obtain immediately

$$\mathbf{F}(q) = 2\pi a J_0(2\pi a q).$$

From the reciprocal property of the Hankel transform it also follows that

$$\delta(r - a) = 2\pi \int_0^\infty 2\pi a J_0(2\pi a q) J_0(2\pi r q) q\, dq$$

a relationship that can be recognized as expressing an orthogonality relationship between zero-order Bessel functions of different "frequencies." Unless the two Bessel functions have the same "frequency," the infinite integral of their product is zero, just as with sines and cosines. Although $J_0(\omega t)$ is not a monochromatic waveform, nevertheless as t elapses, the waveform decays away rather slowly in amplitude and settles down more and more closely to a definite angular frequency ω and a fixed phase, as may be seen from the asymptotic expression

$$J_0(\omega t) \sim \sqrt{\frac{2}{\pi \omega t}} \cos(\omega t - \tfrac{1}{4}\pi).$$

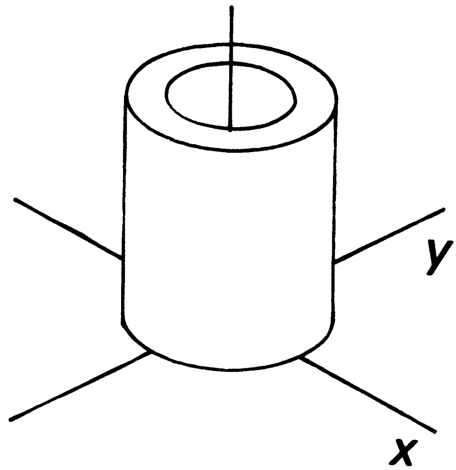

Figure 9-7 A circularly symmetrical function suitable for representing the distribution of light over a uniformly illuminated annular slit. By an extraordinary visual illusion connected with the evolution of vision, the height and outside diameter appear unequal.

Just as jinc r can be described as a circularly symmetrical two-dimensional function that contains all spatial frequencies up to a certain cutoff, in uniform amount for all frequencies and orientations, so also $J_0(r)$ can be seen as a circularly symmetrical two-dimensional function that contains only one spatial frequency but equally in all orientations. The jinc function is thus like the sinc function, and J_0 is like the cosine function.

Annular Slit

We have established the following two Hankel transform pairs:

$$\operatorname{rect} r \supset \operatorname{jinc} q$$
$$\delta(r-a) \supset 2\pi a J_0(2\pi a q),$$

and in what follows we need to recall the similarity theorem in its form applicable to circular symmetry:

$$\text{If } \mathbf{f}(r) \supset \mathbf{F}(q) \quad \text{then} \quad \mathbf{f}(r/a) \supset a^2\mathbf{F}(aq).$$

Note that the sign $\supset$ may be read "has Hankel transform" if you picture **f** and **F** as one dimensional, but may alternatively be read "has two-dimensional Fourier transform" if you prefer to think of **f** and **F** as functions of radius representing two-dimensional entities.

Both the unit circular patch represented by rect r and the ring impulse $\delta(r-a)$ are constantly needed. A third important circularly symmetrical function (Fig. 9-7) is unity over an annulus. For concreteness of description it may be referred to as an *annular slit,* but of course the function is of wider significance.

A narrow circular slit could be cut in an opaque sheet and, if uniformly illuminated from behind, would be reasonably represented by a ring impulse. If the mean radius of the annulus were a, the slit width w, and the amplitude of illumination A, then the distribution of light would be expressible as

$$A \operatorname{rect}\left(\frac{r}{2a+w}\right) - A \operatorname{rect}\left(\frac{r}{2a-w}\right).$$

The "quantity" of light would be $2\pi a w A$, and the quantity per unit arc length would be wA. Therefore, since the ring impulse $\delta(r-a)$ has unit weight per unit arc length, the appropriate impulse representation would be $wA\delta(r-a)$.

We know the Hankel transforms of both the annulus and the ring delta, and the transforms will approach equality as the slit width w approaches zero, provided at the same time the amplitude of illumination A is increased so as to maintain constant the integrated amplitude over the slit. We are saying that as the slit width w goes to zero while wA remains constant, then

$$A(2a+w)^2 \operatorname{jinc}[(2a+w)q] - A(2a-w)^2 \operatorname{jinc}[(2a-w)q] \to (wA)2\pi a J_0(2\pi a q).$$

It follows that

$$\lim_{w\to 0} w^{-1}\left[(2a+w)^2 \operatorname{jinc}[(2a+w)q] - (2a-w)^2 \operatorname{jinc}[(2a-w)q]\right] = 2\pi a J_0(2\pi a q).$$

The left-hand side is recognizable as a derivative, and therefore the conclusion implies the identity

$$\frac{\partial}{\partial a}(4a^2 \operatorname{jinc} 2qa) = 2\pi a J_0(2\pi a q),$$

a result that can be deduced independently from properties of Bessel functions.

For computing purposes we may sometimes wish to represent a ring impulse by an annulus of small but nonzero width, and we may also wish to do the reverse for purposes of theory—namely, to represent an annular slit by a ring impulse. A slit width equal to 10 percent of the mean radius may seem a rather crude example to take, but with $a = 1$ and $w = 0.1$ let us compare $10[(2.1)^2 \operatorname{jinc}(2.1q) - (1.9)^2 \operatorname{jinc}(1.9q)]$ with $2\pi J_0(2\pi q)$.

We quickly find from a few test points that the agreement is good.

q	0	0.38277	−0.5	1.0
LHS	6.28318	−0.004	−1.9061	1.365
RHS	6.28318	0	−1.9116	1.384

Thus, as far as the transform is concerned, a 10 percent slit width, which seems far from a slit of zero width, gives results within 1 percent or so.

It is worthwhile doing numerical calculations of this sort from time to time to develop a sense of how crude an approximation may be and still be useful. Over the whole range $0 < q < 1$ the discrepancy ranges between limits of 0.0192 and −0.0197 or just under 2 percent of the central value. For less crude approximations the results would, of course, be even more accurate. An approximate solution to an urgent problem is most welcome, provided you have the experience to feel confidence in the quality of the approximation. You gain this feeling for magnitudes by making a habit of comparing rough approximations with correct solutions. Reference lists of Hankel transforms can be found in FTA (1986), in Erd (1954), and others occur in GR (1965).

Table 9-1 Table of Hankel transforms.

$\mathbf{f}(r)$	$\mathbf{F}(q) = 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi qr) r dr$
$\mathbf{f}(ar)$	$a^{-2}\mathbf{f}(q/a)$
$\mathbf{f} ** \mathbf{g}$	$\mathbf{FG}$
$r^2\mathbf{f}(r)$	$-\nabla^2\mathbf{F}$
rect r	jinc q
$\delta(r-a)$	$2\pi a J_0(2\pi aq)$
$e^{-\pi r^2}$	$e^{-\pi q^2}$
$r^2 e^{-\pi r^2}$	$\pi^{-1}(\pi^{-1} - q^2)e^{-\pi q^2}$
$(1+r^2)^{-1/2}$	$q^{-1}e^{-2\pi q}$
$(1+r^2)^{-3/2}$	$2\pi e^{-2\pi q}$
$(1-4r^2)$ rect r	$J_2(\pi q)/\pi q^2$
$(1-4r^2)^\nu$ rect r	$2^{\nu-1}\nu! J_{\nu+1}(\pi q)/\pi^\nu q^{\nu+1}$
r^{-1}	q^{-1}
e^{-r}	$2\pi(4\pi^2q^2+1)^{-3/2}$
$r^{-1}e^{-r}$	$2\pi(4\pi^2q^2+1)^{-1/2}$
$^2\delta(x,y)$	1

Theorems for the Hankel Transform

Theorems for the Hankel transform are deducible from those for the two-dimensional Fourier transform, with appropriate change of notation. For example, the similarity theorem $f(ax, by) \;{}^2\!\supset\; |ab|^{-1}F(u/a, v/b)$ will apply, provided $a = b$, a condition that is necessary to preserve circular symmetry. Thus $\mathbf{f}(ar)$ has Hankel transform $a^{-2}\mathbf{F}(q/a)$. The shift theorem does not have any meaning for the Hankel transform, since shift of origin destroys circular symmetry. The convolution theorem, $f ** g \;{}^2\!\supset\; FG$, retains meaning for the Hankel transform on the understanding that $\mathbf{f}(r)$ and its Hankel transform $\mathbf{F}(q)$ are both taken as representing two-dimensional functions on the (x, y)-plane. Then $\mathbf{f} ** \mathbf{g}$ has Hankel transform $\mathbf{FG}$. Some theorems have been incorporated in Table 9-1.

Computing the Hankel Transform

It is perfectly feasible to compute the Hankel transform from the integral definition. The infinite upper limit causes no trouble in practice when the given function either cuts off or dies away rapidly. To evaluate the Bessel functions needed for all the q values one uses the series approximation given above for arguments less than 3 and an asymptotic expansion otherwise. In the following sample program the given function $\mathbf{f}(r)$ is defined to be $\exp[-\pi(r/7)^2]$, which falls to 3×10^{-6} at $r = 14$, and is integrated from 0 to 14. The Bessel function appears in the inner loop with three explicit multiplies, but at least ten more occur in the function definition for $J_0(x)$. Consequently this program is not fast.

A faster method starts by taking the Abel transform of $\mathbf{f}(r)$ (see below) followed by a standard fast Fourier transform; or, since only the real part of the complex output will be utilized, some may prefer to call a standard fast Hartley transform, which will give exactly the same result faster.

HANKEL TRANSFORM

```
DEF FNf(r)=EXP(-PI*(r/7)^2      Function definition
dr=0.1                          Step in r
FOR q=0 TO 0.25 STEP 0.05
   k=2*PI*q
   s=0
   FOR r=dr/2 TO 14 STEP dr
      s=s+FNf(r)*FNJ0(k*r)* r
   NEXT r
   PRINT q;2*PI*s*dr
NEXT q
END
```

THE JINC FUNCTION

Just as in one dimension there is a sinc function which contains all frequencies equally up to a cutoff, and no higher frequencies, so in two dimensions there is a jinc function (Figs. 9-1, 9-8 and Table 9-1) that has already been referred to. The following material, which is collected in one place for reference, mentions the Abel transform, and the Struve function of order unity, which are discussed later. A table of the jinc function is given as Table 9-4 at the end of the chapter.

Properties of the jinc Function

Definition.

$$\operatorname{jinc} x = \frac{J_1(\pi x)}{2x}.$$

Series Expansion.

$$\begin{aligned}\operatorname{jinc} x &= \frac{\pi}{4} - \frac{\pi^3}{2^5}x^2 + \frac{\pi^5}{2^8.3}x^4 - \frac{\pi^7}{2^{12}.3}x^6 + \frac{\pi^9}{2^{16}.3^2.5}x^8 - \cdots \\ &= .785398 - .968946x^2 + .398463x^4 - .245792x^6 + .010108x^8 + \cdots.\end{aligned}$$

Asymptotic Expression, x > 3.

$$\operatorname{jinc} x \sim \frac{\cos[\pi(x - 3/4)]}{\sqrt{2\pi^2 x^3}}.$$

Asymptotic Behavior. The slow decay of $J_0(r)$ with r is connected with the fact that its Hankel transform is impulsive, while the relatively rapid decay of jinc r to small values

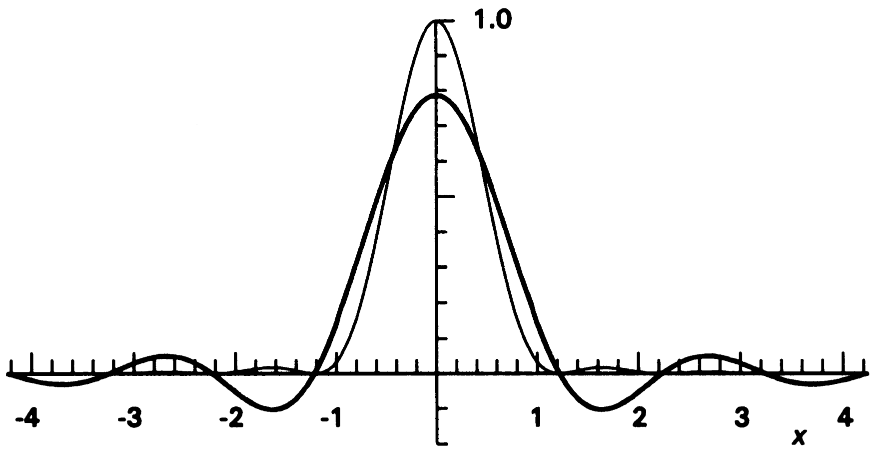

Figure 9-8 The jinc function, Hankel transform of the unit rectangle function (heavy line). The first null is at 1.22, the constant that is familiar in the expression $1.22\lambda/D$ for the angular resolution of a telescope of diameter D. See Table 9-4 for tabulated values. The jinc^2 function normalized to unity at the origin (light line) describes the intensity in the Airy disc, the diffraction pattern of a circular aperture.

occurs because its transform has only a finite discontinuity. The even greater compactness of $\text{jinc}^2(r)$ is associated with the even smoother form of its transform, the chat function. Thus $J_0(r) \sim r^{-1/2}$, $\text{jinc}\, r \sim r^{-3/2}$, $\text{jinc}^2 r \sim r^{-3}$. It appears that, if n derivatives of $\mathbf{F}(q)$ have to be taken to make the result impulsive, then $\mathbf{f}(r) \sim r^{-(n+1/2)}$. The similar theorem for the Fourier transform is that if the nth derivative of $f(x)$ is impulsive, then $F(s) \sim s^{-n}$. However, in the presence of circular symmetry a qualification is necessary, because there cannot be a finite discontinuity at the origin, and a discontinuity in slope at the origin, such as $\text{chat}\, r$ exhibits, counts for less.

Zeros. $\text{jinc}\, x_n = 0$

n	1	2	3	4	5	6	...	n
x_n	1.2197	2.2331	3.2383	4.2411	5.2428	6.2439	...	$\sim n + 1/4$

Derivative.

$$\text{jinc}' x = \frac{\pi}{2x} J_0(\pi x) - \frac{1}{x^2} J_1(\pi x) = -\frac{\pi}{2x} J_2(\pi x).$$

Maxima and Minima.

Location	1.6347	2.6793	3.6987	4.7097	5.7168	6.7217
Value	−0.1039	0.0506	−0.0314	0.0219	−0.016	0.013

Integral. The jinc function has unit area under it:

$$\int_{-\infty}^{\infty} \operatorname{jinc} x \, dx = 1.$$

Half Peak and 3 dB Point.

$$\operatorname{jinc}(0.70576) = 0.5 \operatorname{jinc} 0 = \pi/8 = 0.39270.$$

$$\operatorname{jinc} \theta_{3\mathrm{dB}} = \frac{1}{\sqrt{2}} \operatorname{jinc} 0 = 0.55536,$$

where $\theta_{3\mathrm{dB}} = 0.51456$.

Fourier Transform. The one-dimensional Fourier transform of the jinc function is semi-elliptical with unit height and unit base.

$$\int_{-\infty}^{\infty} \operatorname{jinc} x \, e^{-i2\pi sx} \, dx = \sqrt{1-(2s)^2} \, \operatorname{rect} s.$$

Hankel Transform. The Hankel transform of the jinc function is the unit rectangle function

$$\int_{0}^{\infty} \operatorname{jinc} r \, J_0(2\pi qr) \, 2\pi r \, dr = \operatorname{rect} q.$$

Abel Transform. The Abel transform (line integral) of the jinc function is the sinc function

$$2\int_{x}^{\infty} \frac{\operatorname{jinc} r \, r \, dr}{\sqrt{r^2 - x^2}} = \operatorname{sinc} x.$$

Two-dimensional Aspect. Regarded as a function of two variables x and y, jinc r (where $r^2 = x^2 + y^2$) describes a circularly symmetrical hump surrounded by null circles separating positive and negative annuli.

The Null Circles. Nulls occur at radii 1.220, 2.233, 3.239, etc. As the radii approach values of 0.25 + integer, their spacing approaches unity.

Two-dimensional Integral. The volume under jinc r is unity:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{jinc}\sqrt{x^2+y^2} \, dx \, dy = \int_{0}^{\infty} \operatorname{jinc} r 2\pi r \, dr = 1.$$

Two-dimensional Fourier Transform. A disc function of unit height and diameter:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{jinc}\sqrt{x^2+y^2} \, e^{-i2\pi(ux+vy)} dx \, dy = \operatorname{rect}(\sqrt{u^2+v^2}).$$

Two-dimensional Autocorrelation Function of the jinc function is the jinc function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{jinc}\sqrt{\xi^2+\eta^2} \;\; \operatorname{jinc}\sqrt{(\xi+x)^2+(\eta+y)^2} \, d\xi \, d\eta = \operatorname{jinc}\sqrt{x^2+y^2}.$$

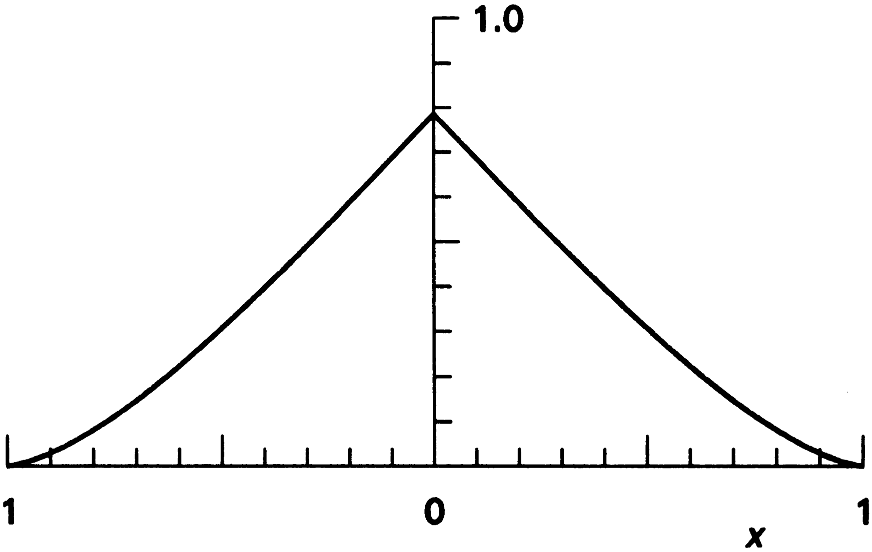

Figure 9-9 The Chinese hat function, autocorrelation function of rect r.

The Chinese Hat Function (Fig. 9-9) is the Hankel transform of jinc$^2\,r$.

$$\operatorname{chat} q \equiv \tfrac{1}{2}\left(\cos^{-1}|q| - |q|\sqrt{1-q^2}\right) \operatorname{rect} \tfrac{1}{2}q = \int_0^\infty \operatorname{jinc}^2 r \; J_0(2\pi q r)\, 2\pi r\, dr.$$

q	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
chat q	0.7854	0.6856	0.5867	0.4900	0.3963	0.3071	0.2236	0.1477	0.0817	0.0294	0.0

Autocorrelation of the Unit Disk Function is the Chinese hat function of radius 1.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{rect}(\sqrt{\alpha^2+\beta^2})\operatorname{rect}(\sqrt{(\alpha+u)^2+(\beta+v)^2})\, d\alpha\, d\beta = \operatorname{chat}\sqrt{u^2+v^2}.$$

Two-dimensional Fourier Transform of the jinc^2 function is the Chinese hat function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{jinc}^2\sqrt{x^2+y^2}\, e^{-i2\pi(ux+vy)} dx\, dy = \operatorname{chat}\sqrt{u^2+v^2}.$$

See Fig. 9-9.

Abel Transform of jinc^2 is (See Fig. 9-10.)

$$2\int_x^\infty \frac{\operatorname{jinc}^2 r\; r\, dr}{\sqrt{r^2-x^2}} = \frac{\mathbf{H}_1(2\pi x)}{4\pi x^2}.$$

Fourier Transform of Chat.

$$\int_{-\infty}^{\infty} \operatorname{chat} x e^{-i2\pi s x}\, dx = \frac{\mathbf{H}_1(2\pi s)}{4\pi s^2}.$$

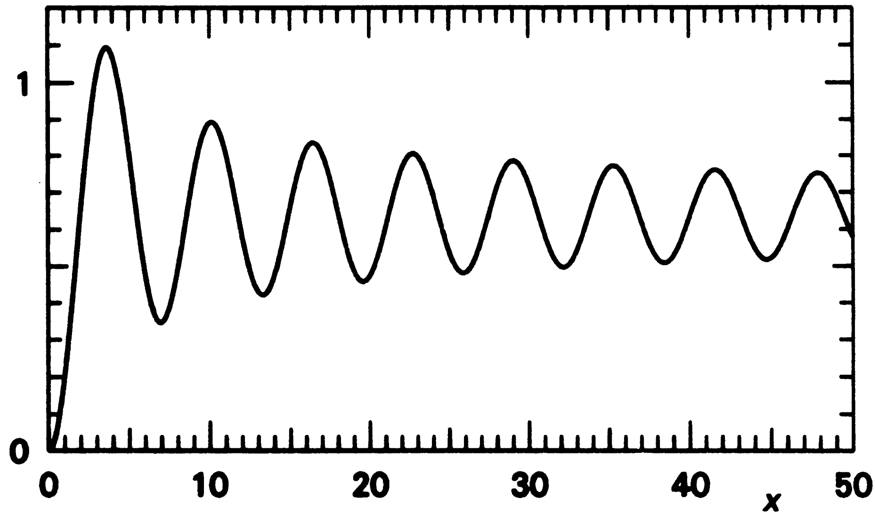

Figure 9-10 The Struve function $\mathbf{H}_1(x)$.

Abel Transform of Chat.

$$2\int_x^\infty \frac{\text{chat}\, r\; r\, dr}{\sqrt{r^2-x^2}} = \text{See Fig. 9-12.}$$

Fourier Transform of jinc^2 **is**

$$\int_{-\infty}^{\infty} \text{jinc}^2\, x\; e^{-i2\pi sx}\, dx =$$

See Fig. 9-12.

Integrals and Central Values.

$$\int_0^\infty \text{jinc}^2\, r\; 2\pi r\, dr = \text{chat}\, 0 = \pi/4,$$

$$\int_0^\infty \text{chat}\, r\; 2\pi r\, dr = \text{jinc}^2\, 0 = \pi^2/16.$$

To summarize, we arrange the various functions in groups of four to display their relationships according to the pattern shown in Fig. 9-11. Each algebraic quartet in the table can also be illustrated graphically. We have been accustomed to arranging functions on the left and transforms on the right. A certain convenience accrues from the adoption of conventions of this sort, which provide a constant framework within which different cases may be considered. In the graphical version of the new organization proposed, the

$$\begin{tikzcd} \mathbf{f}(r) \arrow[r, leftrightarrow, "\mathcal{F}"] \arrow[d, "\mathcal{A}"'] \arrow[dr, leftrightarrow, "\mathcal{H}"] & \mathcal{F}\{\mathbf{f}\} \\ \mathcal{A}\{\mathbf{f}\} \arrow[r, leftrightarrow, "\mathcal{F}"] & \mathcal{H}\{\mathbf{f}\} \arrow[u, "\mathcal{A}"'] \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{jinc} r \arrow[r, leftrightarrow, "\mathcal{F}"] \arrow[d, "\mathcal{A}"'] \arrow[dr, leftrightarrow, "\mathcal{H}"] & \sqrt{1-4s^2}\operatorname{rect} s \\ \operatorname{sinc} x \arrow[r, leftrightarrow, "\mathcal{F}"] & \operatorname{rect} q \arrow[u, "\mathcal{A}"'] \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{jinc}^2 r \arrow[r, leftrightarrow, "\mathcal{F}"] \arrow[d, "\mathcal{A}"'] \arrow[dr, leftrightarrow, "\mathcal{H}"] & \Phi(s) \\ \mathrm{H}_1(2\pi x)/4\pi x^2 \arrow[r, leftrightarrow, "\mathcal{F}"] & \operatorname{chat} q \arrow[u, "\mathcal{A}"'] \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{rect} r \arrow[r, leftrightarrow, "\mathcal{F}"] \arrow[d, "\mathcal{A}"'] \arrow[dr, leftrightarrow, "\mathcal{H}"] & \operatorname{sinc} s \\ \sqrt{1-4x^2}\operatorname{rect} x \arrow[r, leftrightarrow, "\mathcal{F}"] & \operatorname{jinc} q \arrow[u, "\mathcal{A}"'] \end{tikzcd}$$

Figure 9-11 The jinc function and its relatives arranged in quartets obeying the relationships specified in the upper left. The function $\Phi(s)$ is both the Abel transform of the chat function and the Fourier transform of the jinc^2 function.

function $f(x, y)$ (or $\mathbf{f}(r)$ in the case of circular symmetry) goes in the northwest and its two-dimensional Fourier transform $F(u, v)$ [or $\mathbf{F}(q)$] goes in the southeast, giving a diagonal arrangement on the page, because in this way the left-right juxtaposition of one-dimensional Fourier transforms can be preserved. In the example where jinc r is in the top left-hand corner and rect q in the bottom right-hand corner, the cross section of each two-dimensional function along the east-west axis can be shown rabatted into the plane. These one-dimensional functions of x and u, respectively, constitute a Hankel transform pair. Where circular symmetry happens to exist and $f(x, y) \ {}^2\!\supset F(u, v)$, then $f(x, 0)$ has Hankel transform $F(u, 0)$. The general situation of no symmetry has more to do with data than with properties of instruments, which can often be designed with cylindrical symmetry, and is taken up later in connection with the projection-slice theorem.

The cross section of $f(x, y)$ along a line $x = \text{const}$ has an area which is the ordinate of the Abel transform of $f(x, y)$, viz., sinc x. We see that sinc x and its Fourier transform rect q are arranged left-to-right as planned.

The whole story can now be repeated, since the Hankel transform is reciprocal, starting in the bottom right-hand corner. Thus the cross section of rect q has an area equal to the ordinate of $(1 - 4u^2)^{1/2}$ rect u, which in turn is the one-dimensional Fourier transform of jinc r, the function we began with.

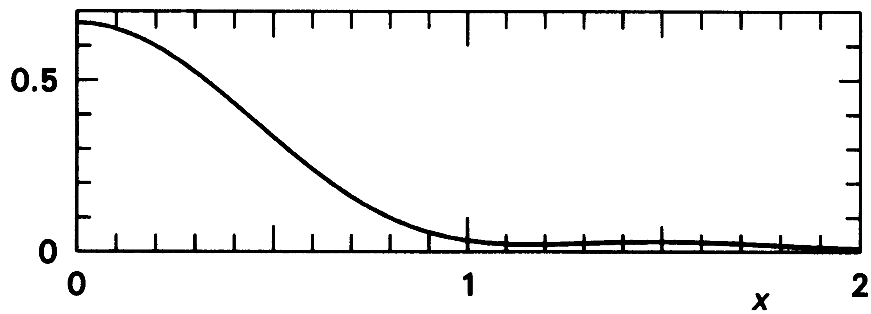

Figure 9-12 The Abel transform of $\operatorname{jinc}^2 r$. This function is also the one-dimensional Fourier transform of the chat function.

THE STRUVE FUNCTION

The Fourier transform of the chat function, which is the same as the Abel transform of the Airy diffraction pattern, or jinc^2 function, both arise naturally in optical systems and may be expressed in terms of the Struve function $\mathbf{H}_1(x)$ as $\mathbf{H}_1(2\pi x)/4\pi x^2$. For values of x up to about 5 the Struve function can be calculated from the Taylor series $(2/\pi)(1 + x^2/3 - x^4/3^2.5 + x^6/3^2 \cdot 5^2 \cdot 7 - \cdots)$. For larger values of x use the asymptotic expansion 12.1.31 given in A&S (1964). A graph is shown in Fig. 9-10; the function oscillates with a period close to 2π about a limiting value of $2/\pi$. The oscillations decay rather slowly in amplitude, inversely as the square root of x; the only null is the one at $x = 0$.

THE ABEL TRANSFORM

A two-dimensional function $\mathsf{f}(r)$ that has circular symmetry possesses a line integral, or projection, that is the same in all directions. Call this function $f_A(x)$. The subscript A refers to Abel and the variable x can be thought of as being the abscissa in the (x, y)-coordinate system to which the radial coordinate r belongs. Thus $f_A(x)$ is the projection in the y-direction, or the line integral in the y-direction. As an example, if $\mathsf{f}(r) = \operatorname{rect}(r)$, then $f_A(x) = (1 - 4x^2)^{1/2} \operatorname{rect} x$ (Fig. 9-13). This is because a disk function of unit height and unit diameter has a cross-section area $(1 - 4x^2)^{1/2}$ on the line $x = \text{const}$, provided $|x| < \frac{1}{2}$. Where $|x| \geq \frac{1}{2}$, the cross-section area is zero, a fact that the factor $\operatorname{rect} x$ reminds us of. The shape of the Abel transform in this example is semi-elliptical, which is connected with the fact that the given outline was circular. If you wanted to know what function of r has a semicircular Abel transform, the answer would be $\frac{1}{2} \operatorname{rect} r$.

In lieu of these explanatory remarks it would be sufficient simply to introduce the Abel transform $f_A(\ )$ of a function $\mathsf{f}(\ )$ by this definition:

$$f_A(x) \triangleq \int_{-\infty}^{\infty} \mathsf{f}(\sqrt{x^2 + y^2})\, dy.$$

Then if the question arose as to the Abel transform of rect( ), we would evaluate it as follows:

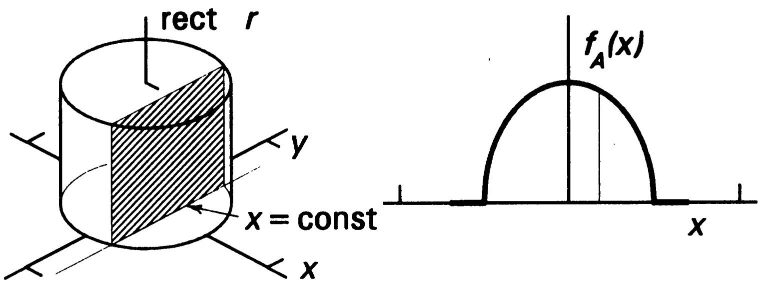

Figure 9-13 The area of the shaded cross section is the Abel transform of the function of r for the particular value of x chosen. In the case of rect r the Abel transform $f_A(x)$ is the semi-ellipse $\sqrt{1-4x^2}$, $|x| < \frac{1}{2}$.

$$\begin{aligned} f_A(x) &= \int_{-\infty}^{\infty} \operatorname{rect}(\sqrt{x^2+y^2})\, dy \\ &= \int_{-(1/4-x^2)^{1/2}}^{(1/4-x^2)^{1/2}} dy \operatorname{rect} x \\ &= 2\sqrt{\tfrac{1}{4} - x^2} \operatorname{rect} x \\ &= \sqrt{1-4x^2} \operatorname{rect} x. \end{aligned}$$

The limits of integration were arrived at by noting that points on a line parallel to the y-axis at abscissa x must lie within $y = \pm(\frac{1}{4} - x^2)^{1/2}$ in order for $\operatorname{rect}(\sqrt{x^2+y^2})$ to be unity rather than zero. The integration involved is not difficult, but it is obvious that some geometrical reasoning based on the explanatory introduction is helpful in arriving at the limits of integration.

An alternative form of the definition can be given in terms of r, which is the natural variable to think of as underlying a circularly symmetrical function $\mathbf{f}(r) = f(x, y)$. Thus

Abel transform definition.

$$f_A(x) \triangleq 2\int_x^{\infty} \frac{\mathbf{f}(r) r\, dr}{\sqrt{r^2 - x^2}}.$$

To convert from dy to dr write $dr/dy = \sin\theta = \sqrt{r^2 - x^2}/r$. Thus $dy = r\, dr/\sqrt{r^2 - x^2}$. To relate this definition to the previous one, note that the minimum value of r is the given value of x. Thus $\int_{-\infty}^{\infty} \dots dy$ may be replaced by $2\int_x^{\infty} \dots \frac{dy}{dr}\, dr$, the factor 2 arising from the equal contributions from above and below the x-axis. From $y^2 = r^2 - x^2$ we deduce that $dy/dr = r/y$ at constant x. Alternatively, from Fig. 9-14, we can see from similar triangles that $dy/dr = r/y$.

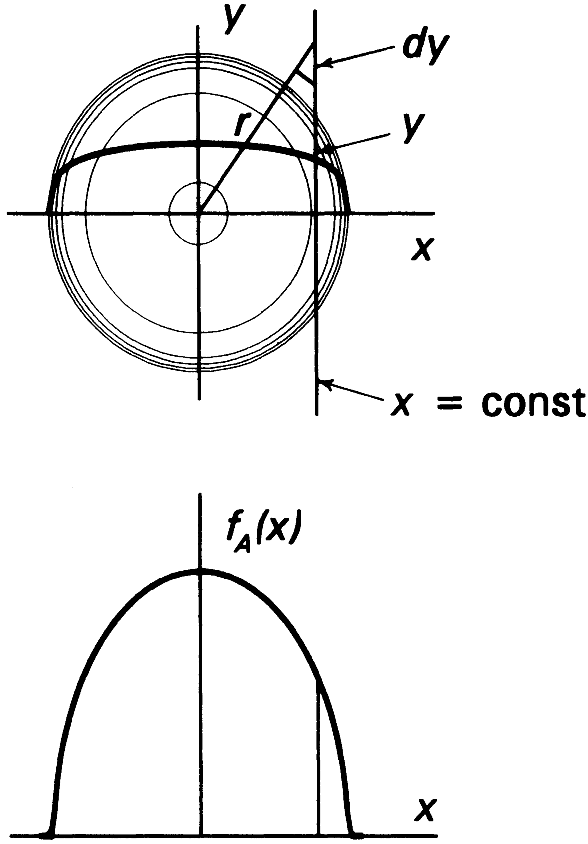

Figure 9-14 A contour map of the plane of $\mathbf{f}(r)$ (above) with a diametral cross section and a graph (below) of the line integral $\int \mathbf{f}(r)dy$ along the line x = const.

Some Theorems for the Abel Transform

Many theorems for the Fourier transform do not have a counterpart when circular symmetry is imposed, but a small number of interesting theorems for the Abel theorem can be mentioned.

Similarity Theorem. If $\mathbf{f}(r)$ is contracted by a factor a to $\mathbf{f}(ar)$, then clearly $f_A(x)$ will be contracted in the same proportion and, in addition, the values of $f_A()$ will be reduced in magnitude by the same factor. Thus

$$\mathbf{f}(ar) \quad \text{has Abel transform} \quad a^{-1}f_A(ax).$$

This result is verifiable immediately by substituting ar for r in the definition.

Linear Superposition. If $\mathbf{f}(r)$ has Abel transform $f_A(ax)$, and $\mathbf{g}(r)$ has Abel transform $g_A(ax)$, then

$$\mathbf{f}(r) + \mathbf{g}(r) \quad \text{has Abel transform} \quad f_A(x) + g_A(x),$$

for any choice of **f** and **g**.

Convolution Theorem. If $\mathbf{f}(r)$ and $\mathbf{g}(r)$ are convolved in two dimensions, then the Abel transform of the result can be obtained as follows. From each of the Abel transforms $f_A(x)$ and $g_A(x)$ construct a circularly symmetrical function with the same radial section. These two functions are correctly written $f_A(r)$ and $g_A(r)$. After f_A and g_A are convolved two-dimensionally, the radial section in any direction θ is the desired Abel transform. The theorem is:

$$\mathbf{f}(r) ** \mathbf{g}(r) \quad \text{has Abel transform} \quad \Big[f_A(r) ** g_A(r) \Big]_{\theta=\text{const}}.$$

The derivation of this theorem can be written down starting from the definition integrals, but such a simple result must have a simple explanation, and it is given in Chapter 14.

Conservation Theorem. As the Abel transform is a simple projection of a two-dimensional function, the area integral of $\mathbf{f}(r)$ equals the integral of $f_A(x)$:

$$2\pi \int_0^\infty \mathbf{f}(r) r\, dr = \int_{-\infty}^\infty f_A(x)\, dx.$$

Central Value Theorem. Putting $x = 0$ in the defining integral, we see that

$$f_A(0) = 2 \int_0^\infty \mathbf{f}(r)\, dr,$$

a relation that is useful for normalizing at the end of a computation in which unnecessary multiplications by constants are dropped.

Table 9-2 lists a variety of Abel transforms for ready reference.

Inverting the Abel Transform

Inversion of the Abel transform is performed by

$$\mathbf{f}(r) = -\frac{1}{\pi} \int_r^\infty \frac{f_A'(x)\, dx}{\sqrt{x^2 - r^2}}.$$

An important special case is where $f_A(x)$ is zero for x greater than some cutoff value r_0. Then

$$\mathbf{f}(r) = -\frac{1}{\pi} \int_r^{r_0} \frac{f_A'(x)\, dx}{\sqrt{x^2 - r^2}} + \frac{f_A(r_0-)}{\pi \sqrt{r_0^2 - r^2}}.$$

The final term, which might be overlooked, arises from the possibility of $f_A(x)$ being discontinuous at $x = r_0$. Numerical inversion of the Abel transform is important, because

Table 9-2 Table of Abel transforms. For compactness rect x is written $\Pi(x)$.

$\mathbf{f}(r)$		$f_A(x) = 2\int_x^\infty (r^2 - x^2)^{-1/2}\mathbf{f}(r) r\,dr$	
$\Pi(r/2a)$	Disk	$2(a^2 - x^2)^{1/2}\Pi(x/2a)$	Semiellipse
$(a^2 - r^2)^{-1/2}\Pi(r/2a)$		$\pi\Pi(x/2a)$	Rectangle
$(a^2 - r^2)^{1/2}\Pi(r/2a)$	Hemisphere	$\frac{1}{2}\pi(a^2 - x^2)\Pi(x/2a)$	Parabola
$(a^2 - r^2)\Pi(r/2a)$	Paraboloid	$\frac{4}{3}(a^2 - x^2)^{3/2}\Pi(x/2a)$	
$(a^2 - r^2)^{3/2}\Pi(r/2a)$		$\frac{3\pi}{8}(a^2 - x^2)^2\Pi(x/2a)$	
$(1 - \lvert r\rvert)\Pi(r/2)$	Cone	$[(a^2 - x^2)^{1/2} - (x^2/a)\cosh^{-1}(a/x)]\Pi(x/2a)$	
$\cosh^{-1}(a/r)\Pi(r/2a)$		$\pi a(1 - \lvert r/a\rvert)\Pi(r/2a)$	Triangle
$\delta(r - a)$	Ring impulse	$2a(a^2 - x^2)^{-1/2}\Pi(x/2a)$	
$\exp(-\pi r^2 W^2)$	Gaussian	$W\exp(-\pi x^2/W^2)$	Gaussian
$\exp(-r^2/2\sigma^2)$	Normal	$\sqrt{2\pi}\sigma\exp(-x^2/2\sigma^2)$	Normal
$r^2\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma(x^2 + \sigma^2)\exp(-x^2/2\sigma^2)$	
$(r^2 - \sigma^2)\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma x^2\exp(-x^2/2\sigma^2)$	
r^{-2}		π/x	
$(a^2 + r^2)^{-1}$		$\pi(a^2 + x^2)^{-1/2}$	
$J_0(2\pi ar)$	Bessel	$(\pi a)^{-1}\cos 2\pi ax$	Cosine
$2\pi[r^{-3}\int_0^r J_0(r)dr - r^{-2}J_0(r)]$		$\operatorname{sinc}^2 x$	
$\delta(r)/\pi\lvert r\rvert$		$\delta(x)$	Impulse
$2a\operatorname{sinc}(2ar)$		$J_0(2\pi ax)$	Bessel
$\frac{1}{2}r^{-1}J_1(2\pi ar)$		$\operatorname{sinc} 2ax$	
$\operatorname{jinc} r$		$\operatorname{sinc} x$	

line-integrated data can often be obtained in situations where values of $f(r)$ itself are inaccessible. In such circumstances a formula containing a derivative looks unattractive if the derivative $f'_A(x)$ must be formed by differencing, because measurement error is unfavorable. A numerical inversion procedure is described in FTA (1986) which avoids the derivative. A different method of inversion is to take advantage of the Fourier-Abel-Hankel cycle: take the one-dimensional Fourier transform and then take the Hankel transform.

One can take the Hankel transform without the need to invoke Bessel functions, by first taking the Abel transform and then taking the one-dimensional Fourier transform, as displayed in the following diagram.

$$\begin{tikzcd} f(r) \arrow[r, leftrightarrow, "\mathcal{F}"] \arrow[d, "\mathcal{A}"'] & F(\cdot) \\ f_A(x) \arrow[r, leftrightarrow, "\mathcal{F}"] & H(q) \arrow[u, "\mathcal{A}"'] \end{tikzcd}$$

It follows that $f(r)$ can be recovered from $f_A(x)$ as indicated by

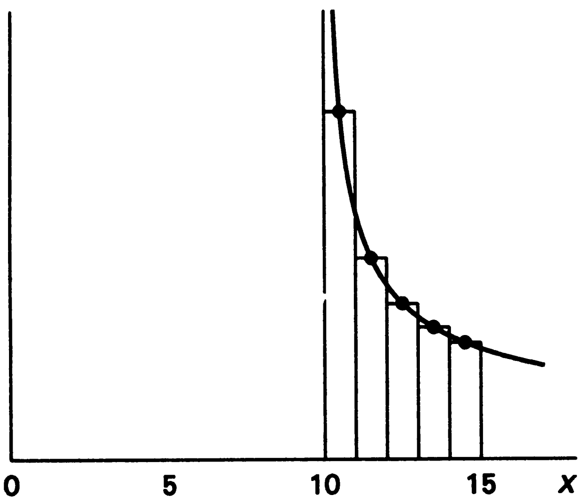

Figure 9-15 Staggering the samples of an integrand to avoid the infinite sample at the discontinuity.

$$f = \mathcal{F}A\mathcal{F}f_A.$$

Implementation of this sequence of operations numerically is straightforward.

Computing the Abel Transform

A rather interesting item of numerical analysis arises when one is called on to evaluate

$$\int_r^{r_0} \frac{dx}{\sqrt{x^2 - r^2}}$$

since the integrand is infinite at the left edge. For example, if we wanted $I = \int_{10}^{15}(x^2 - 10^2)^{-1/2}\,dx$, there would be trouble at $x = 10$, even though the full integral is finite. A way around this would be to stagger the samples (Fig. 9-15). Let $(x^2 - 10^2)^{-1/2} = \phi(x)$; then if one computes $\sum_{j=10.5}^{14.5} \phi(j)$, where $j = 10.5, 11.5, \ldots 14.5$, the result is 0.827, but the exact integral is $I = 0.962$. Obviously the approximation is crude, and fine subdivision of the interval might be needed to achieve desired accuracy. The correct approach is to note that $\int_{10}^{10+w} \phi(x)\,dx = C_1\phi(10 + 0.5w)w$ and that $\int_{10+w}^{10+2w} = C_2\phi(10 + 1.5w)w$, where C_1 and C_2 are coefficients and w is the sampling interval. As $w \to 0$, C_1 and C_2 assume definite values 1.414 and 1.015 that may be used for general-purpose integration in cases such as this, where the integrand diverges inversely as the square root of distance from the pole. Thus

$$\int_r \frac{\mathbf{f}(r)\,dx}{\sqrt{x^2 - r^2}} \approx \left[1.414f(r + 0.5w) + 1.015f(r + 1.5w) + \sum_{j=2\frac{1}{2}} f(r + jw)\right]w.$$

With $w = 1$, which is rather coarse, the approximation to the correct value 0.962 is 0.959, which is already better than 1 percent. A similar approach works with integrands that go infinite as the inverse three-halves power.

With this useful background, the reader may enjoy the following complete program for the Abel transform in which I avoid the pole at $x = r$. This application assumes that the function of radius is expressible in algebraic form and that the abscissa is scaled so that the function is zero where $r > 1$. The example applies to $f(r) = 1 - r$, whose Abel transform is known to be $\sqrt{1 - x^2} - x^2 \cosh^{-1}(1/x)$. The program can readily be modified for *data* given at equal intervals.

ABEL TRANSFORM

```
DEF FNf(r)=1-r                    Define given function as cone
d=0.1                             Step in x
dy=0.01                           Step in y
FOR x=d/2 TO 1 STEP d
   s=0.5*FNf(x)
   FOR y=dy TO SQR(1-x^2) STEP dy
      s=s+FNf(SQR(x^2+y^2))
   NEXT y
   PRINT x;2*s*dy
NEXT x
```

The results are as follows:

r	.05	.15	.25	.35	.45	.55	.65	.75	.85	.95
$f(r)$	0.98952	0.93053	0.83928	0.72718	0.60210	0.47067	0.33909	0.21404	0.10363	0.02071

Comparison with the theoretical expression shows that the largest error is one digit in the fifth decimal place. Whether the value of dy is too coarse can be checked empirically.

SPIN AVERAGING

A function $f(x)$ may be spin averaged to obtain a new function $f_S(r)$ defined by

$$f_S(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos\alpha)\, d\alpha.$$

If $f(x)$ is an even function, as in all that follows here, we may evaluate $f_S(r)$ from

$$f_S(r) = \frac{2}{\pi} \int_0^{\pi/2} f(r \cos\alpha)\, d\alpha.$$

Two ways of viewing spin averaging will now be described.

Imagine a function defined on the (x, y)-plane so that at any point (x, y) the value is $f(x)$, i.e., independent of y. If the function value represented the height of a surface above the (x, y)-plane, the surface would be a cylindrical ridge running in the y-direction. Then if we traveled on the surface of this ridge so that our track projected onto the (x, y)-plane was a circle of radius r, then our average height would be $f_S(r)$, as given by either of the

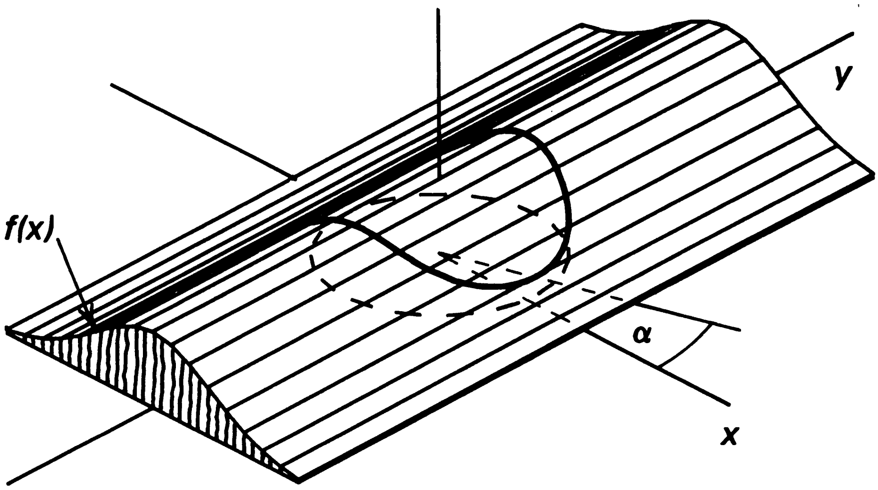

Figure 9-16 The saddle-shaped curve is a circle of radius r in plan view. Its average height at given r is the spin average of the profile $f(x)$.

above integrals (Fig. 9-16). Alternatively, if we were to sit at the point $(r, 0)$ in the (x, y)-plane and spin the distribution at uniform speed, then $f_S(r)$ would be the average of the values passing by in the course of one rotation. If a linear source of light with brightness distributed as $f(x)$ were spun about its center, then a long exposure on a photographic plate would show a radial dependence of photographic density depending on $f_S(r)$. This is the reason for adopting the term *spin-averaging.*

Example

What is the spin average of a cosine function?▹The corrugated ridge pattern $\cos x$ spin-averages into $J_0(r)$. This is an expression of the relation

$$J_0(r) = \frac{2}{\pi}\int_0^{\pi/2} \cos(r\cos\alpha)\,d\alpha,$$

which is sometimes expressed in words by saying that a disturbance $J_0(r)$ (for example, on a drum membrane) can be regarded as made up of cosinusoidal corrugations, all of the same wavelength, superimposed uniformly in all azimuths. ◃

The spin integral of the cosine function was first evaluated in 1805 by M. A. Parseval as follows:

$$1 - \frac{r^2}{2^2} + \frac{r^4}{2^2 \cdot 4^2} + \frac{r^6}{2^2 \cdot 4^2 . 6^2} + \cdots = \frac{1}{\pi}\int_0^{\pi} \cos(r\sin\alpha)\,d\alpha.$$

A traveler moving steadily around a circular track lingers longer at the larger x-values, the time spent in the vicinity of any x being proportional to the secant of the inclination of the track to the x-axis. Near $x = 0$ each element dx is passed through in about the same time. As $| x |$ approaches r, it takes longer and longer to pass through a given increment dx. Therefore, a weighting function can be imagined that tells how to average the function values $f(x)$ with extra emphasis on the larger x-values.

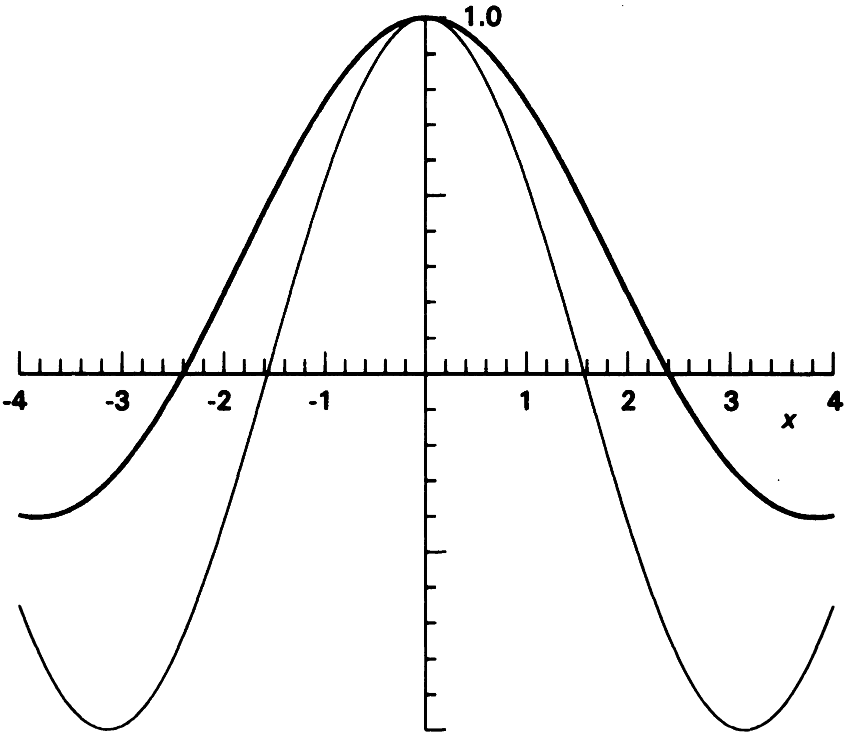

Figure 9-17 Showing how $J_0(2.405) = 0$ from the spin-average viewpoint.

Since the weighting function has to depend on r as well as x, write it $w(x, r)$. It has to be proportional to sec (inclination) = sec $|\arcsin(x/r)| = r(r^2 - x^2)^{-1/2}$. We also want the weighting function to have unit area, as is usual with taking weighted averages, because when r is very small and $f(x)$ is therefore essentially constant around the small circular track, the spin average must approach $f(0)$ in value. Noting that the integral of $r(r^2 - x^2)^{-1/2}$ from 0 to r is $\pi r/2$, (or one-quarter of the perimeter of the circle of radius r), we find

$$w(x, r) = \frac{2}{\pi} \frac{1}{\sqrt{r^2 - x^2}}.$$

Thus the alternative expression for the spin average is

$$f_S(r) = \frac{2}{\pi} \int_0^r \frac{f(x)\, dx}{\sqrt{r^2 - x^2}}.$$

Theorems for Spin Averaging

Denoting spin average by $\mathcal{S}$, the Fourier transform by $\mathcal{F}$, and the Hankel transform by $\mathcal{H}$, we can enunciate the following theorems.

$$f_S(0) = f(0)$$
$$\mathcal{S}(f+g) = \mathcal{S}(f) + \mathcal{S}(g)$$
$$\text{If } \mathcal{S}[f(x)] = f_S(r) \text{ then } \mathcal{S}[bf(ax)] = bf_S(ar)$$
$$\pi \mid q \mid \mathcal{H}[f_S(r)] = \mathcal{F}[f(x)].$$
$$\lim_{r\to\infty} f_S(r) = \frac{\int_{-\infty}^{\infty} f(x)\,dx}{\pi r}$$

Example

Use the fourth theorem to calculate the spin integral of $f(x) = \cos x$.▷First rewrite the theorem as

$$\mathcal{H}\left\{\frac{1}{\pi \mid q \mid}\mathcal{F}[f(x)]\right\} = f_S(r).$$

Then

$$\mathcal{F}[\cos x] = \tfrac{1}{2}\delta(q + 1/2\pi) + \tfrac{1}{2}\delta(q - 1/2\pi)$$

and

$$\frac{1}{\pi \mid q \mid}\mathcal{F}[\cos x] = \delta(q + 1/2\pi) + \delta(q - 1/2\pi),$$

because $\pi \mid q \mid \delta(q \pm 1/2\pi)$ is a half-strength impulse. Finally,

$$f_S(r) = 2\pi \int_0^\infty \frac{1}{\pi \mid q \mid}\mathcal{F}[\cos x] J_0(2\pi r q) q\,dq = J_0(r).◁$$

Table 9-3 presents a list of spin average functions. The functions of x are shown as even functions of x. However, any entry $f(x)$ may be replaced by $f(x)\mathbf{H}(x)$, which is zero for $x < 0$, and $f_S(r)$ will remain the same because the averaging is done over only one quadrant.

Computing the Spin Average

The spin average is convenient to compute, as shown in the following program.

SPIN AVERAGE

```
DEF FNf(x)=x^2              Define given function as parabola
d=0.1                       Step in radius
N=20                        Number of sample points per quadrant
FOR r=0 TO 1 STEP d
   da=PI/2/N                Step in angle
   s=0
   FOR a=da/2 TO PI/2 STEP da
      s=s+FNf(r*COS(a))
   NEXT a
   spinav=s/N               Take the average
   PRINT r;FNf(r);spinav
NEXT r
END
```

Table 9-3 Spin-average functions

$f(x)$	$f_S(r) = \frac{2}{\pi}\int_0^{\pi/2} f(r\cos\alpha)d\alpha$
$\cos x$	$J_0(r)$
$\operatorname{sinc} x - \frac{1}{2}\operatorname{sinc}^2(x/2)$	$\operatorname{jinc} r$
$\operatorname{sinc} x - \frac{1}{2}\operatorname{sinc}^2(x/2)x\mathrm{III}(x)$	$\sum_1^\infty 2\pi n J_0(2\pi n r)$
$\delta(x+a)+\delta(x-a)$	$(2r/\pi)(r^2-a^2)^{-1/2}H(r-a)$
$\delta(x)$	$1/\pi r$
$\operatorname{rect}(x/2a)$	$1-(2/\pi)\cos^{-1}(a/r)H(r-a)$
$1-\operatorname{rect}(x/2b)$	$(2/\pi)\cos^{-1}(b/r)H(r-b)$
$\operatorname{rect}(x/2a)-\operatorname{rect}(x/2b),\ b<a$	$(2/\pi)[\cos^{-1}(b/r)H(r-b)-\cos^{-1}(a/r)H(r-a)]$
$J_1(\|x\|)$	$2\pi r(1-\cos r)$
$J_0(x)$	$[J_0(\frac{1}{2}r)]^2$
$\cos^2 x$	$\frac{1}{2}+\frac{1}{2}J_0(2(r)$
$(1-x^2)^{-1/2}$	$(2/\pi)E(r^2)=(2/\pi)\int_0^{\pi/2}(1-r^2\cos^2\alpha)^{1/2}d\alpha$
$\Lambda(x/2w)$	$(2/\pi)(\pi/2-\alpha_1)-(2/\pi)(2r/w)(1-\sin\alpha_1)$ where $\cos\alpha_1 = H(w/2r)+(w/2r)H(r-w/2)$
1	1
$\|x\|$	$2r/\pi$
x^2	$r^2/2$
$\|x\|^3$	$4r^3/3\pi$
x^4	$3r^4/8$
$\|x\|^5$	$16r^5/15\pi$
x^6	$15r^6/16$
$\|x\|^7$	$52r^7/35\pi$
$\sin x$	$(2/\pi)\Sigma_0^\infty(-1)^k r^{2k+1}/[(2k+1)!!]^2\dagger$
$x\sin x$	$rJ_1(r)$
$\cosh x$	$I_0(r)$
$J_2(x)$	$J_1^2(r/2)$
$N_0(x)$	$J_0(r/2)N_0(r/2)$
$\|x\|\,J_0(x)$	$(2/\pi)\sin r$
$J_{2\nu(x)}$	$J_\nu^2(r/2),\ \operatorname{Re}2\nu>1$

$\dagger\,(2k+1)!! = 1\cdot 3\cdot\cdots\cdot(2k+1)$.

ANGULAR VARIATION AND CHEBYSHEV POLYNOMIALS

When $n = 0$, a function of the form $f(r)\cos n\theta$ is circularly symmetrical. In a typical application it might describe a wave disturbance inside a circular pipe when there is no dependence on azimuth. When $n = 1$, the disturbance would be zero where $\theta = \pm\pi/2$. It could describe a higher-order mode with a null line on one diameter and antiphased disturbances in the two halves. For $n = 2,\ 3,\ 4,\ldots$, we say that the function has n-

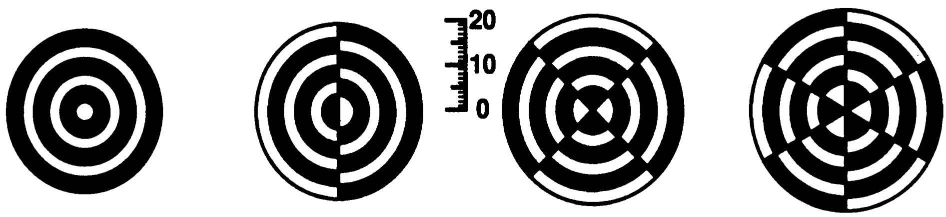

Figure 9-18 Modulated ring impulses $\delta(r-0.5)\cos n\theta$ have transforms $J_n(\pi q)\cos n\phi$ represented by nodal lines delineating antiphase regions (shaded) for $n = 0, 1, 2, 3$.

fold rotational symmetry. Many functions $f(r)\cos n\theta$ arising from wave propagation in circular structures and from diffraction have transforms that are simply generalizations of cases discussed under circular symmetry, and some will be mentioned here.

The simplest example is a ring impulse of radius a modulated by a cosine function of azimuth, namely

$$\delta(r-a)\cos n\theta.$$

The two-dimensional Fourier transform is

$$(-i)^n 2\pi a J_n(2\pi aq)\cos n\phi.$$

We verify that this reduces to $2\pi a J_0(2\pi aq)$ when $n = 0$. Figure 9-18 shows, for the first four modulated ring impulses $\delta(r-1)\cos n\theta$, the configuration of the nodal lines of the transforms $J_n(\pi q)\cos n\phi$.

Because $\delta(r-a)\cos n\theta$ is not circularly symmetrical, it does not have an Abel transform, but it has a projection $f_0(x)$ onto the x-axis given by

$$\int_{-\infty}^{\infty} \delta(r-a)\cos n\theta\, dy$$

$$= \int_{-\infty}^{\infty} \delta(r-a)\frac{\cos n\theta}{\sin\theta}dr = 2\frac{\cos n\theta}{\sin\theta}.$$

The integral is taken in the y-direction along a line $x = \text{const}$ which intersects the ring impulse where $\theta = \pm\cos^{-1}(x/a)$ and at an angle θ. Consequently the integral is the product of the local strength $\cos n\theta$ with the factor $1/\sin\theta$, all doubled because there are two intersections.

Since $\cos\theta = x/a$ and $\sin\theta = \sqrt{1-x^2/a^2}$,

$$f_0(x) = \frac{2\cos(n\cos^{-1}x/a)}{\sqrt{1-x^2/a^2}}$$

$$= \frac{2T_n(x/a)}{\sqrt{1-x^2/a^2}},$$

where $T_n(x)$ is a Chebyshev polynomial ($T_1(x) = x$, $T_2(x) = 2x^2-1$, $T_3(x) = 4x^3-3x$, $T_4(x) = 8x^4-8x^2+1, \ldots$). This result allows us to understand the Chebyshev polynomials in terms of the simple projections of $\delta(r-1)\cos n\theta$ onto the x-axis. It follows that the one-dimensional Fourier transform of $2/\sqrt{(1-x^2/a^2}T_n(x/a)$ is $(-i)^n 2\pi a J_n(2\pi aq)$.

Any distribution of strength around a circular ring impulse can now be handled readily by one-dimensional Fourier analysis on the circle. Because such a distribution is of necessity periodic, discrete coefficients will result, and the 2D Fourier transform of the general nonuniform ring impulse will take the form

$$\sum J_n(2\pi aq)\big(a_n \cos n\phi + b_n \sin n\phi\big).$$

We shall also be able to get the projection of the nonuniform ring impulse. Clearly, this is of general value, since any two-dimensional function can be regarded as built up of concentric nonuniform annuli of infinitesimal width, and in addition to the analytic aspect there is the very practical feature that decomposition into fine annuli may be appropriate in numerical computation with data acquired within a broad annular zone.

As an analytic application consider the 2D Fourier transform of rect $r\cos n\theta$. Regard this as a superposition of ring impulses $\delta(r-a)$, where a ranges from 0 to 0.5. Then the desired transform should be

$$\int_0^{1/2} (-i)^n 2\pi a J_n(2\pi qa)\cos n\phi \, da = (-i)^n 2\pi a \cos n\phi \int_0^{1/2} J_n(2\pi qa)\, da.$$

Integrals of Bessel functions are computable. In the particular case of $n=1$ we find that the 2D Fourier transform of rect $r\cos\theta$ is

$$\begin{aligned}-i2\pi a\cos\phi \int_0^{1/2} J_1(2\pi qa)\, da &= i2\pi a\cos\phi(2\pi q)^{-1} J_0(2\pi qa)\Big]_0^{1/2} \\ &= i(a/q)\cos\phi[J_0(\pi q)-1].\end{aligned}$$

The projections of $\exp(-\pi r^2)\cos n\theta$ will play a role in the discussion of the Radon transform.

Modulated Ring Delta

Just as a function of the single variable x can be interpreted as a two-dimensional surface $z=f(x)$ above the (x,y)-plane, so also can functions of the single variables r and θ. For example, $z=r^2/4a$, which is a function of r but not of θ, describes a paraboloidal dish with focal length a that is concave upward. The function $\cos n\theta$ which arises in the discussion of rotational symmetry, can be thought of correspondingly. The surface is like a magic carpet with azimuthal undulations. The contour lines are all level, and at the origin, where the contours converge, there is a nasty singularity. The illustration (Fig. 9-19) terminates on a cylinder of unit radius. The curve of intersection with the cylinder of unit radius describes the strength of the modulated ring delta $\delta(r-1)\cos n\theta$, whose character is conveyed in Fig. 9-20 for $n=0$, 1, and 2, left to right.

Multiplication of a function $f(x,y)$ by $\delta(r-a)\,{}^{\cos}_{\sin}\,n\theta$ has the effect of singling out the infinitesimal annulus of radius a and taking the first step toward finding the Fourier series coefficients for the annular section. We now show how the modulated ring delta $\delta(r-a)\cos\theta$ can arise from differentiation of the familiar function $f(x,y)=\mathrm{rect}(r/2a)$

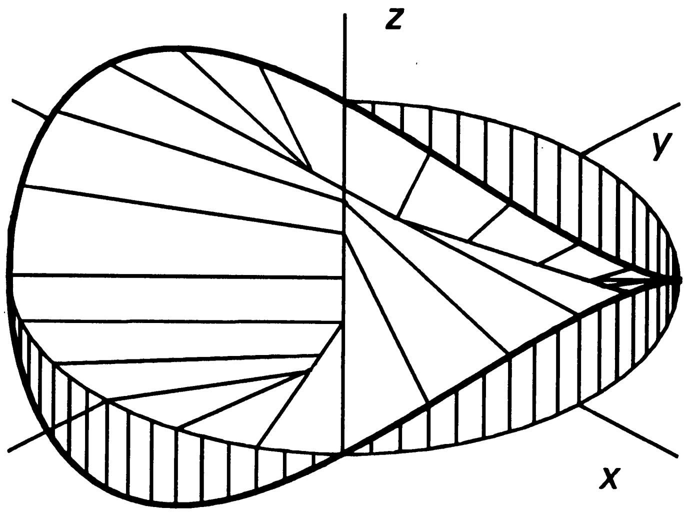

Figure 9-19 The function $\cos n\theta$ exhibited by contours of the surface $z = \cos n\theta$ above the (x, y)-plane for the case $n = 2$.

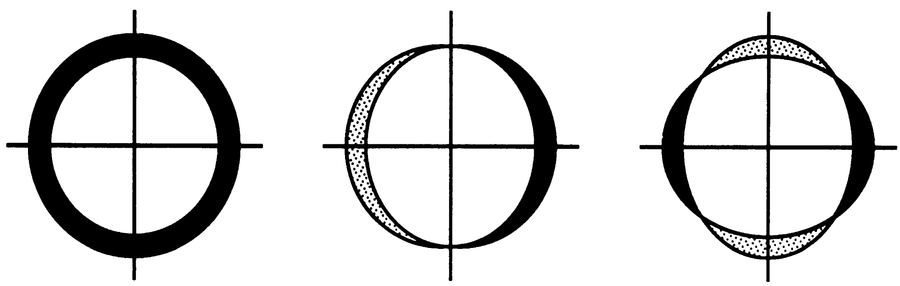

Figure 9-20 Modulated ring deltas $\delta(r-1)$, $\delta(r-1)\cos\theta$, and $\delta(r-1)\cos 2\theta$. Strength is represented by radial line width, and by shading where negative.

and thereby gain a useful algebraic way of relating rotational symmetry to circular symmetry. The proposition is that

$$\frac{\partial}{\partial x}\operatorname{rect}\left(\frac{r}{2a}\right) = -\delta(r-a)\cos\theta.$$

We know that differentiation of $f(x, y)$ with respect to x multiplies the Fourier transform $F(u, v)$ by $i2\pi u$. This example, where $F(u, v)$ is known to be the jinc function $4a^2 \operatorname{jinc} 2aq$, implies that $-\delta(r-a)\cos\theta$ has a Fourier transform $i8\pi a^2 u \operatorname{jinc} 2aq$, which simplifies to $i2\pi a J_1(2\pi aq)\cos\phi$. This relation has in fact already been derived by manipulation of integrals. Knowing the simple proposition stated above provides another way of thinking with a fundamental entity such as the modulated ring delta.

Since $\operatorname{rect}(r/2a)$ is discontinuous on the circle $r = a$, differentiation is not in ordinary circumstances defined on the very locus where the ring delta is situated. Thus, to set

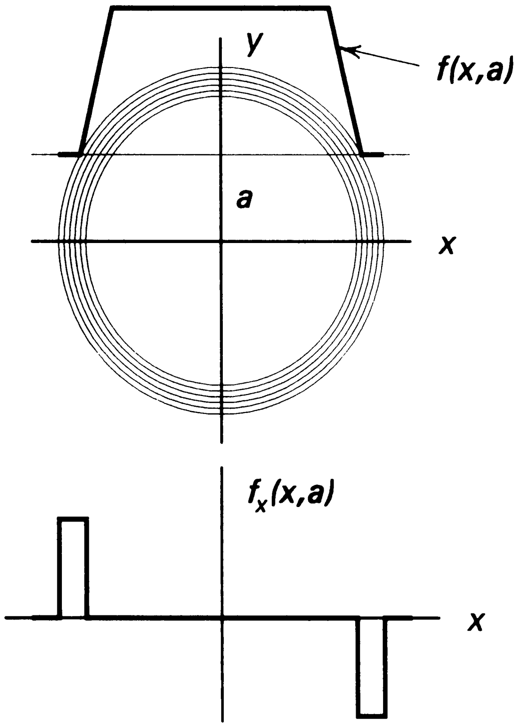

Figure 9-21 Contour representation of an upside-down bucket of top and bottom radii $a - \tau$ and a, a cross section at constant y, and the derivative of this cross section with respect to x (below).

up a consistent relation we think of a sequence of differentiable functions that approach $\text{rect}(r/2a)$ in the limit. The simplest such function is equal to zero outside the circle $r = a$, equal to unity inside the circle of radius $a - \tau$, and equal to the height of the cone forming a continuous surface between the two circular rims of radius a and $a - \tau$. We now differentiate with respect to x and consider the sequence of outcomes as $\tau \to 0$. Along each line $y = \text{const}$ the derivative produces two unit-area rectangle functions of width τ and heights $1/\tau$ and $-1/\tau$, respectively, sequences representing delta functions of strength 1 and -1, situated on the locus $r = a$. How then do we arrive at a strength modulated in accordance with $\cos\theta$? The answer is that the strength of a curvilinear impulse is measured by the cross section area in the direction normal to the curve. What we have done is to measure in the x-direction, thus amplifying the normal cross section by $\cos\theta$. Allowing for this, it is confirmed that $(\partial/\partial x)\,\text{rect}(r/2a) = -\delta(r - a)\cos\theta$. This rather elegant technique is original and is applicable to other situations where partial differentiation in two or three dimensions is probably the last thing that would occur to you.

SUMMARY

Although circular symmetry is a special case, a rich set of useful tools reveals itself when we restrict attention to radial dependence only. Some important transforms emerge, especially the Hankel transform, many of whose particular instances are in everyday use for computation and for thinking about symmetrical situations. Outstanding examples are

Table 9-4 The jinc function.

	.00	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	0.7854	0.7853	0.7850	0.7845	0.7838	0.7830	0.7819	0.7807	0.7792	0.7776
0.1	0.7757	0.7737	0.7715	0.7691	0.7666	0.7638	0.7609	0.7577	0.7544	0.7509
0.2	0.7473	0.7434	0.7394	0.7352	0.7309	0.7264	0.7217	0.7168	0.7118	0.7067
0.3	0.7014	0.6959	0.6903	0.6845	0.6786	0.6725	0.6663	0.6600	0.6536	0.6470
0.4	0.6402	0.6334	0.6264	0.6194	0.6122	0.6049	0.5975	0.5900	0.5824	0.5747
0.5	0.5669	0.5590	0.5510	0.5430	0.5348	0.5267	0.5184	0.5101	0.5017	0.4932
0.6	0.4847	0.4762	0.4676	0.459	0.4503	0.4416	0.4329	0.4241	0.4153	0.4066
0.7	0.3978	0.3890	0.3802	0.3714	0.3626	0.3538	0.3450	0.3363	0.3276	0.3189
0.8	0.3102	0.3016	0.2930	0.2845	0.2760	0.2676	0.2593	0.2510	0.2428	0.2346
0.9	0.2266	0.2186	0.2107	0.2030	0.1953	0.1877	0.1735	0.1656	0.1577	0.1500
1.0	0.1423	0.1347	0.1272	0.1198	0.1125	0.1053	0.0982	0.0912	0.0843	0.0776
1.1	0.0709	0.0643	0.0578	0.0515	0.0453	0.0392	0.0332	0.0273	0.0216	0.0159
1.2	0.0104	0.0051	−0.0002	−0.0053	−0.0103	−0.0151	−0.0199	−0.0245	−0.0289	−0.0333
1.3	−0.0375	−0.0416	−0.0455	−0.0493	−0.0530	−0.0565	−0.0599	−0.0632	−0.0663	−0.0694
1.4	−0.0722	−0.0750	−0.0776	−0.0801	−0.0824	−0.0847	−0.0868	−0.0887	−0.0906	−0.0923
1.5	−0.0939	−0.0954	−0.0967	−0.0979	−0.0990	−0.1000	−0.1009	−0.1017	−0.1023	−0.1028
1.6	−0.1033	−0.1036	−0.1038	−0.1039	−0.1039	−0.1038	−0.1036	−0.1033	−0.1029	−0.1024
1.7	−0.1018	−0.1011	−0.1004	−0.0995	−0.0986	−0.0976	−0.0965	−0.0953	−0.0941	−0.0928
1.8	−0.0914	−0.0900	−0.0885	−0.0869	−0.0853	−0.0836	−0.0818	−0.0801	−0.0782	−0.0763
1.9	−0.0744	−0.0724	−0.0704	−0.0683	−0.0662	−0.0641	−0.0620	−0.0598	−0.0576	−0.0553
2.0	−0.0531	−0.0508	−0.0485	−0.0462	−0.0439	−0.0416	−0.0393	−0.0369	−0.0346	−0.0323
2.1	−0.0299	−0.0276	−0.0253	−0.0230	−0.0207	−0.0184	−0.0161	−0.0138	−0.0116	−0.0093
2.2	−0.0071	−0.0050	−0.0028	−0.0007	0.0014	0.0035	0.0056	0.0076	0.0096	0.0115
2.3	0.0134	0.0153	0.0171	0.0189	0.0206	0.0224	0.0240	0.0256	0.0272	0.0287
2.4	0.0302	0.0316	0.0330	0.0344	0.0356	0.0369	0.0380	0.0392	0.0403	0.0413
2.5	0.0423	0.0432	0.0440	0.0448	0.0456	0.0463	0.0470	0.0476	0.0481	0.0486
2.6	0.0490	0.0494	0.0497	0.0500	0.0503	0.0504	0.0506	0.0506	0.0506	0.0506
2.7	0.0505	0.0504	0.0502	0.0500	0.0497	0.0494	0.0491	0.0487	0.0482	0.0477
2.8	0.0472	0.0466	0.0460	0.0454	0.0447	0.0440	0.0432	0.0424	0.0416	0.0407
2.9	0.0398	0.0389	0.0379	0.0370	0.0360	0.0349	0.0339	0.0328	0.0317	0.0306
3.0	0.0295	0.0283	0.0271	0.0259	0.0247	0.0235	0.0223	0.0211	0.0198	0.0186
3.1	0.0173	0.0161	0.0148	0.0135	0.0123	0.0110	0.0097	0.0085	0.0072	0.0060
3.2	0.0047	0.0035	0.0022	0.0010	−0.0002	−0.0014	−0.0026	−0.0038	−0.0049	−0.0061
3.3	−0.0072	−0.0083	−0.0094	−0.0105	−0.0116	−0.0126	−0.0136	−0.0146	−0.0156	−0.0165
3.4	−0.0174	−0.0183	−0.0192	−0.0200	−0.0209	−0.0216	−0.0224	−0.0231	−0.0238	−0.0245
3.5	−0.0252	−0.0258	−0.0264	−0.0269	−0.0274	−0.0279	−0.0284	−0.0288	−0.0292	−0.0296
3.6	−0.0299	−0.0302	−0.0304	−0.0307	−0.0309	−0.0311	−0.0312	−0.0313	−0.0314	−0.0314
3.7	−0.0314	−0.0314	−0.0314	−0.0313	−0.0312	−0.0310	−0.0309	−0.0307	−0.0304	−0.0302
3.8	−0.0299	−0.0296	−0.0292	−0.0289	−0.0285	−0.0281	−0.0276	−0.0272	−0.0267	−0.0268
3.9	−0.0257	−0.0251	−0.0245	−0.0239	−0.0233	−0.0227	−0.0221	−0.0214	−0.0207	−0.0200
4.0	−0.0193	−0.0186	−0.0179	−0.0171	−0.0164	−0.0156	−0.0148	−0.0140	−0.0132	−0.0124
4.1	−0.0116	−0.0108	−0.0100	−0.0092	−0.0083	−0.0075	−0.0067	−0.0058	−0.0050	−0.0042
4.2	−0.0034	−0.0025	−0.0017	−0.0009	−0.0001	0.0007	0.0015	0.0023	0.0031	0.0039
4.3	0.0046	0.0054	0.0061	0.0069	0.0076	0.0083	0.0090	0.0097	0.0104	0.0110
4.4	0.0117	0.0123	0.0129	0.0135	0.0141	0.0146	0.0152	0.0157	0.0162	0.0167
4.5	0.0171	0.0176	0.0180	0.0184	0.0188	0.0191	0.0195	0.0198	0.0201	0.0204
4.6	0.0206	0.0208	0.0210	0.0212	0.0214	0.0215	0.0217	0.0218	0.0218	0.0219
4.7	0.0219	0.0219	0.0219	0.0219	0.0218	0.0218	0.0217	0.0215	0.0214	0.0212
4.8	0.0211	0.0209	0.0207	0.0204	0.0202	0.0199	0.0196	0.0193	0.0190	0.0186
4.9	0.0183	0.0179	0.0175	0.0171	0.0167	0.0163	0.0158	0.0153	0.0149	0.0144
5.0	0.0139	0.0134	0.0129	0.0124	0.0118	0.0113	0.0107	0.0102	0.0096	0.0090

the relation of the annular slit to the J_0 Bessel function and the corresponding relation between the circular disc and the jinc function. In the world of symmetry which our machines manufacture, these two relationships are as common as the familiar relationships of the cosine function and the rectangle function to their one-dimensional Fourier transforms. Of course, the Hankel transform is one-dimensional, but we use it for handling two-dimensional things. Special instances of the Abel transform are of similar importance—the annular slit and circular disk are again examples. The spin average transform is closely related, but less heard of, not being associated with a great name. It can be recommended for attention. Circular symmetry is a special case of rotational symmetry. Superposition of rotationally symmetrical components permits synthesis of general two-dimensional functions and is important in other ways. It is appropriate to mention the topic in this chapter and to introduce the Chebyshev polynomial, which will be found indispensable if rotational symmetry is pursued.

Professor Pafnutii L'vovich Chebyshev (1821–1894) was a distinguished mathematician of St. Petersburg who worked on the kinematics of machines, the distribution of prime numbers, and many other subjects. He is known as Tchebicheff in French, Tschebischew in German, and Cebiscev in Italian.

Niels Henrik Abel (1802–1829) was a brilliant Norwegian mathematician who solved the spin integral equation in 1823 (*J. für Math.*, vol. 1, p. 153, 1826). Herman Hankel (1839–1873) was Professor of Mathematics at Leipzig. Hermann Ottovich Struve (1854–1920), working at the Pulkovo Observatory near St. Petersburg, introduced his special functions in connection with studies of the diffraction pattern in a telescope.

TABLE OF THE JINC FUNCTION

Table 9-4 (on page 380) is provided to facilitate use of the jinc function. In the event that values are needed for machine use, the series expansion and asymptotic expression given earlier are available. For greater precision, refer to A&S (1964) for computation of $J_1(x)$.

PROBLEMS

9–1. *First null of J_0.* Evaluate the two expressions (i) $\cos[r\cos(15°)] + \cos[r\cos(45°)] + \cos[r\cos(75°)]$, (ii) $0.5\cos r + \cos[r\cos(30°)] + \cos[r\cos(60°)] + 0.5\cos[r\cos(90°)]$ for $r = 2.4048255577$ and comment on the results. ☞

9–2. *Approximation to Bessel function.* A student claims that you can evaluate $J_0(r)$ with an error less than 0.001 from $f_1(r) = \frac{1}{2}(\cos 0.9239r + \cos 0.3827r)$, where $r < 3$. Obviously this claim is easy to check if you have a table of Bessel functions, and you might wish to do so.

(a) Explain how to test the claim without needing a table.

(b) Devise a comparable formula that is good to 0.00002 where $r < 3$. [*Hint.* If you do not recognize the source of the formula, identify the coefficients by looking at their arccosines.]

9–3. *Hankel transform theorem.* Observing that $\operatorname{rect} r$ and $\operatorname{jinc} q$, which form a Hankel transform pair, both have unit area, a student conceives the following conjectural theorem.

"If $\mathbf{f}(r)$ has Hankel transform $\mathbf{F}(q)$, then $\int_0^\infty \mathbf{f}(r)\,dr = \int_0^\infty \mathbf{F}(q)\,dq$."

A skeptical friend says that one swallow does not make a summer and that the thoughtful normalization build into the definitions of the two functions explains the student's observation. Write a short report on the above material. ☞

9–4. *Annular slit approximation.* We expect that the Hankel transform of $\text{rect}[r/(1+\epsilon)] - \text{rect}[r/(1-\epsilon)]$ is closely approximated by $\mathbf{F}(q) = \pi\epsilon J_0(\pi q)$ when ϵ is small. For example, when $\epsilon = 0.05$, the error is less than 0.02 over the range $0 < q < 1$. Explain why the approximation would not be expected to work for larger values of q.

9–5. *Smooth function.* The circularly symmetrical function $(1-4r^2)^2 \text{rect}\, r$ falls to zero at $r = \frac{1}{2}$ continuously, and in addition the slope at the cutoff boundary is continuous. What is the Hankel transform and how does it differ from that of $(1-4r^2)\,\text{rect}\, r$, which also falls to zero continuously?

9–6. *Orthogonality.*
(a) Deduce the orthogonality relation $\int_{-\infty}^{\infty} J_0(2\pi ar)J_0(2\pi br)r\,dr = 0$ when $b \neq a$ directly from the Hankel transform pair $\delta(r-a) \supset 2\pi a J_0(2\pi aq)$.
(b) What can you say about $\int_{-\infty}^{\infty} J_0(2\pi ar)J_0(2\pi br)\,dr$?
(c) When $a = b$, what useful quantitative statement can you make?

9–7. *Asymptotic variation.* The Hankel transform of a ring impulse consists of a central hump surrounded by fringes whose amplitude decays as $q^{-3/2}$, where q is the radial coordinate in the transform plane, but is the same true of a circular slit? Consider an equivalent narrow annulus of width ϵ centered on a ring impulse, equivalent in the sense that the integral over the (x, y)-plane is the same. The annulus can be represented as the difference between almost equal circular apertures each having a transform that dies out as $q^{-3/2}$.
(a) Explain why the fringes will be of less amplitude than before.
(b) Well away from the origin, how will the fringe amplitude vary with q? ☞

9–8. *Hankel transform of annular dipole.* Show that the Hankel transform of $\delta'(r-a)$ is $2\pi[2\pi aq J_1(2\pi aq) - J_0(2\pi aq)]$. ☞

9–9. *Hankel transform notation.* The Hankel transform of $f(x)$ is defined in the extensive "Tables of Integral Transforms" [Erd (1954), vol. 2] as

$$g(y) = \int_0^\infty f(x)J_0(xy)\sqrt{xy}\,dx, \quad y > 0.$$

Show that if $f(x)$ and $g(y)$ appear as a transform pair in these tables, then $f(x)/\sqrt{x}$ and $2\pi g(y)/\sqrt{y}$ will be a Hankel transform pair under the definition of this book when x is replaced by r and y is replaced by $2\pi q$.

9–10. *Bessel function identity.* Show that

$$\frac{d}{dx}(x^2 \,\text{jinc}\, x) = \tfrac{1}{2}\pi x J_0(\pi x). \quad ☞$$

9–11. *Property of jinc function.* Show that $\text{jinc}\, x = \frac{1}{4}\pi[J_0(\pi x) + J_2(\pi x)]$.

9–12. *Identify diffraction pattern.* State whether the function illustrated in Fig. 9-22 represents a J_0 □ sinc □ jinc □ $[J_0]^2$ □ sinc^2 □ jinc^2 □ or other □ function of radius r, giving the reason for your opinion.

9–13. *Power spectrum of zone plate.* A glass plate is alternately opaque and transparent on annuli

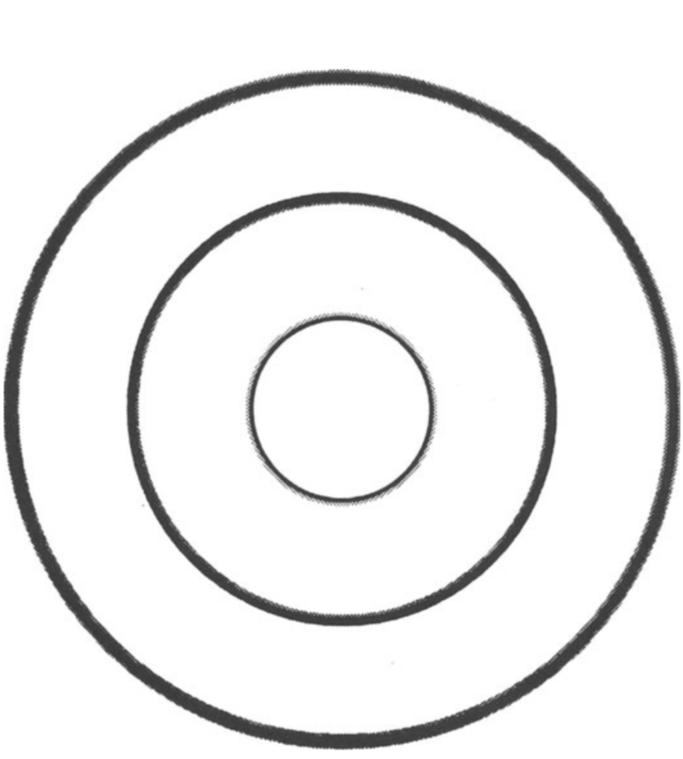

Figure 9-22 A function to be identified.

bounded by radii 1, $\sqrt{2}$, $\sqrt{3}$, ..., $\sqrt{8}$, $\sqrt{9}$ (the unit-radius disc at the center being opaque). Present the power spectrum graphically, giving numerical values for spatial frequency.

9–14. *Central value of Abel transform.* Prove that $f_A(0) = 2\int_0^\infty f(r)\,dr$.

9–15. *Integral of Abel transform.* Prove that $2\int_0^\infty f_A(x)\,dx = 2\pi \int_0^\infty f(r)r\,dr$, a useful result for numerical checking.

9–16. *Repeated Abel transformation.* Show that $f_{AA}(x)$, the Abel transform of the Abel transform, is equal to π times the volume under that part of $f(r)$ where $r > x$.

9–17. *Annulus seen as a convolution.* Is there a function $\phi(r)$ with the following property that

$$\phi(r) ** \delta(r - R) = \begin{cases} 1, & |r - R| < \frac{1}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

In other words, if $\phi(r)$ were convolved with a ring impulse of radius R, the result would have to be unity over an annulus of unit width straddling the circle of radius R.

9–18. *Abel transform by convolution.* A function $f(r)$ whose Abel transform is desired can be expressed as $\hat{f}(\rho)$, where $\rho = r^2$. Consider a kernel $K(\rho)$ which is zero where $\rho \geq 0$ but for negative ρ is given by $1/\sqrt{-\rho}$. Show that by convolving $K(\rho)$ with $\hat{f}(\rho)$ we obtain a function $\hat{F}_A(x^2)$ which, for any x, is the desired Abel transform $f_A(x)$.

9–19. *Accuracy of Abel transform.* Take $f(r) = 64(1 - r/64)\,\text{rect}(r/128)$, a function with a known Abel transform, and form a set of samples at $r = 0, 1, 2, \ldots, 63$. Compute the

Abel transform from the samples and compare with the known transform. Start again with a set of samples at $r = 0.5, 1.5, 2.5, \ldots, 63.5$ and find out whether better accuracy results.

9–20. *Computed Abel transform.* A function of radius is measured as follows.

r	.05	.15	.25	.35	.45	.55	.65	.75	.85	.95
$f(r)$	1.98	1.39	1.04	0.91	0.78	0.64	0.48	0.32	0.16	0.03

Compute and plot the Abel transform.

9–21. *Abel transform of conical crater.* Plot the Abel transform of $f(r) = |r| \operatorname{rect}(r/2)$ and compute values for $x = 0[0.2]1$ to an accuracy of ± 0.01. ☞

9–22. *Abel transform to be guessed.*
(a) Compute the Abel transform of $f(r) = (1 - r^2)^2 \operatorname{rect}(r/2)$ for $x = 0[0.1]1$.
(b) The analytic form is likely to be fairly simple in form; try to guess the formula, and verify.

9–23. *Spin average.* What is the spin average of the parabolic function $f(x) = 1 - x^2/a^2$?

9–24. *Sinusoidal spin integral.* Show that $\frac{1}{2}\pi |x| J_0(x)$ is the function whose spin integral is $\sin r$. ☞

9–25. *Spin averaging over the full circle.*
(a) Show that $f_1(x) = \delta(x + a) + \delta(x - a)$ has the same spin average as $f_2(x) = \delta(x - a)$ under the standard definition.
(b) What would the situation be if the integration limits were changed to $(1/2\pi) \int_0^{2\pi} \ldots$?
(c) How would the case of $\delta(x)$ be affected? ☞

9–26. *Spin averaging theorems for initial derivatives.* Show that if $f_s(r)$ is the spin average of $f(x)$, then $f_s(0) = f(0)$, $f_s'(0) = (2/pi)f$, $f_s''(0) = (1/2)f''(0) = 4/3\pi)f'''(0)$, $f_s^{(4)}(0) = (3/8)f^{(4)}$, $f_s^{(5)}(0) = (16/15\pi)f^{(5)}(0)$, $f_s^{(6)}(0) = (15/16)f^{(6)}(0)$, $f_s^{(7)}(0) = 52/35\pi)f^{(7)}(0), \ldots$. ☞

9–27. *Chebyshev polynomials.* A 2D function is a nonuniform ring impulse of unit radius and strength $\cos n\theta$ as represented by $f(x, y) = \delta(r - 1)\cos n\theta$. To get the projection $g_n(x)$ on the x-axis we take the strength at the point $P(x, \sqrt{1 - x^2})$, double it because there are two points on the circle with the same abscissa, and multiply by $(1 - x^2)^{-1/2} = 1/\sin\theta$ to allow for the angle between the line of projection and the tangent at P. The result is $2\cos n\theta / \sin\theta$, or, as a function of x,

$$g_n(x) = 2\cos(n \arccos x)(1 - x^2)^{-1/2} = T_n(x) 2(1 - x^2)^{-1/2}.$$

The factor $2(1 - x^2)^{-1/2}$, which is just the projection of $\delta(r - 1)$, represents the envelope under which $g_n(x)$ oscillates. Show that for $n = 1$ to 4, the oscillatory expressions $\cos(n \arccos x)$, are

$$T_1(x) = x, \qquad T_2(x) = 2x^2 - 1, \qquad T_3(x) = 4x^3 - 3x, \qquad T_4(x) = 8x^4 - 8x^2 + 1,$$

and verify that these functions are the Chebyshev polynomials defined by

$$T_n(x) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n - m - 1)!}{m!(n - 2m)!} (2x)^{n-2m}.$$

Compute and plot $T_{40}(x)$ for $0 < x < 1$ using the expression $\cos(40 \arccos x)$. ☞

9–28. *A function with rotational symmetry.* Show that rect $r \cos\theta$ has the 2D Fourier transform $i(a/q)\cos\phi[J_0(\pi q) - 1]$.

9–29. *Analysis into rings.*

(a) Confirm the following transform pair for a ring impulse whose strength varies cosinusoidally in azimuth:

$$\delta(r-1)\cos n\theta \;{}^2\!\supset\; (-1)^n 2\pi J_n(2\pi q)\cos n\phi.$$

(b) Show that any nonuniform ring impulse $\delta(r-1)\Theta(\theta)$ can be transformed in two dimensions by analyzing $\Theta(\theta)$ into a Fourier series $\Theta(\theta) = c_n \exp in\theta$ to find the coefficients c_n and then superposing patterns of the form $J_n(2\pi q)\exp in\theta$ to get

$$\delta(r-1)\Theta(\theta) \;{}^2\!\supset\; \sum(-1)^n c_n 2\pi J_n(2\pi q)e^{in\theta}.$$

(c) Show further that any function $f(r,\theta)$ can be decomposed into concentric annuli of infinitesimal radial width, each of which can be Fourier analyzed azimuthally, and that the transform $F(q,\phi)$ can therefore be expressed as

$$F(q,\phi) = \int_0^\infty \sum(-1)^n c_{n,a} 2\pi a J_n(2\pi a q)e^{in\phi}\,da,$$

where the coefficients $c_{n,a}$ relate to the azimuthal cut $\Theta(\theta) = f(a,\theta)$ whose Fourier series is $\sum(-1)^n c_{n,a}\exp in\theta$.

(d) Explain the special circumstances under which it might be advantageous to express a two-dimensional Fourier transform as a relatively complicated sum of Bessel functions. ☞

9–30. *Modulated ring impulse.* Establish the following two transform pairs:

$$\delta(r-a)\cos\theta \;{}^2\!\supset\; -i2\pi a J_1(2\pi a q)\cos\phi,$$

$$\delta(r-a)\cos 2\theta \;{}^2\!\supset\; 2\pi a J_2(2\pi a q)\cos 2\phi. \quad ☞$$

9–31. *Chebyshev polynomial.* By considering the Abel transform of the cosinusoidally modulated ring impulse, show that the Fourier transform of a Bessel function is related to the Chebyshev polynomial T_n as follows (where n is an even integer):

$$J_n(2\pi x) \supset (-1)^{n/2}(1-s^2)^{-1/2}T_{n/2}(s)\,\mathrm{rect}(s/2). \quad ☞$$