PREFACE

Students of mathematics early encounter the differential operators d/dx, d^2/dx^2, d^3/dx^3, etc., and some doubtless ponder whether it is necessary for the order of differentiation to be an integer. Why should there not be a $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ operator, for instance? Or d^{-1}/dx^{-1} or even $d^{\sqrt{2}}/dx^{\sqrt{2}}$? It is to these and related questions that the present work is addressed. It will come as no surprise to one versed in the calculus that the operator d^{-1}/dx^{-1} is nothing but an indefinite integral in disguise, but fractional orders of differentiation are more mysterious because they have no obvious geometric interpretation along the lines of the customary introduction to derivatives and integrals as slopes and areas. The reader who is prepared to dispense with a pictorial representation, however, will soon find that fractional order derivatives and integrals are just as tangible as those of integer order and that a new dimension in mathematics opens to him when the order q of the operator d^q/dx^q becomes an arbitrary parameter. Nor is this a sterile exercise in pure mathematics—many problems in the physical sciences can be expressed and solved succinctly by recourse to the fractional calculus.

Our interest in this subject began in 1968 with the realization that the use of half-order derivatives and integrals leads to a formulation of certain electrochemical problems which is more economical and useful than the classical approach in terms of Fick's laws of diffusion. This discovery stimulated our interest, not only in the applications of the notions of the derivative and integral to arbitrary order, but also in the basic mathematical properties of these fascinating operators. Our collaboration since 1968 has taken us far beyond the original motivation and has produced a wealth of material, some of which we believe to be original. As befits a cooperative effort between a mathematician [J. S.] and a chemist [K. B. O.], our work attempts to expose not only the theory underlying the properties of the generalized operator, but also to illustrate the wide variety of fields to which these ideas may be applied with profit. We do not presume to present an exhaustive survey of the subject, but our aim has been to introduce as many readers as possible to the beauty and utility of this material. Accordingly, we have made a deliberate attempt to keep the mathematical discussions as simple as possible. For example, we have not used techniques of modern functional analysis to deal with d^q/dx^q from an operator-theoretic point of view. This latter approach, which has been taken to some extent by Feller (1952)[1] and Hille (1939, 1948), should prove to be very fruitful but is properly the subject of a much more advanced work. Nor have we sought to incorporate the fractional calculus into the larger field of symbolic, operational mathematics (Boole, 1844; Heaviside, 1893, 1920; Mikusinski, 1959; Friedman, 1969; Bourlet, 1897; Ritt, 1917).

During our investigations of the general theory and applications of differintegrals (a term we have coined to avoid the cumbersome alternate "derivatives or integrals to arbitrary order"), we have discovered that, while this subject is old, dating back at least to Leibniz in its theory and to Heaviside in its application, it has been studied relatively little since the early papers which only hinted at its scope. In the last several years there seems to have taken place a mild revival of interest in the subject, but, in our opinion, the application of these ideas has not yet been fully exposed, primarily because of their unfamiliarity. Our studies have convinced us that differintegral operators may be applied advantageously in many diverse areas. Within mathematics, the subject makes contact with a very large segment of classical analysis and provides a unifying theme for a great many known, and some new, results. Applications outside mathematics include such otherwise unrelated topics as: transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of intergranular grooves at metal surfaces, quantum mechanical calculations, and dissemination of atmospheric pollutants.

In developing the theoretical foundations of the subject our guideline has been to view differintegrals as composing a continuum of operators which include ordinary differentiation and integration, single and multiple, as particular instances. These special cases serve as cornerstones of familiarity which we use to establish credibility for the general properties we study. Thus, after a historical introduction to the subject and a review of facts about the gamma function in Chapter 1, Chapter 2 is devoted to a discussion of the properties of derivatives and integrals to integer order as a reminder of results we shall seek to generalize later. In Chapter 3 we introduce our basic definition (due originally to Grünwald) of a generalized derivative–integral

[1] All references cited in this Preface are to works listed in the References, beginning on p. 219. They should not be confused with items in the chronological bibliography appearing at the end of Section 1.1.

and demonstrate the equivalence of this definition with the more familiar one attributed to Riemann and Liouville. It seems time in Chapter 4 to derive and display formulas exemplifying differintegration, and we have chosen simple algebraic functions for this purpose. Chapter 5 deals with the general properties of the differintegral operator which help to clinch the idea that differintegrals of noninteger order are not really so different from ordinary derivatives and integrals.

Chapter 6 is viewed as transitional between the theory of differintegral operators and their application: It deals with the differintegration of certain important functions, including several which find use in later chapters. The applications of differintegration to classical mathematics itself constitute a unit which is presented as Chapters 9 and 10. In the former we demonstrate how differintegration forges powerful links among various transcendental functions, including most of those which arise naturally in mathematical physics. That differintegration can serve as a tool in unifying and extending concepts and techniques encountered in the classical calculus is the message of Chapter 10.

Chapter 11 contains what we regard as the most powerful application of this theory, namely to diffusive transport in a semiinfinite medium. Here a single equation—asserting the proportionality of a second order spatial partial derivative to a first order partial derivative with respect to time—governs a wide variety of transport phenomena: heat in solids, chemical species in homogeneous media, vorticity in fluids, electricity in resistive-capacitative lines, to mention only a few. The replacement of this equation, together with an initial and the asymptotic boundary condition, by an equation linking a first order spatial derivative to a half order temporal derivative is the essence of this application. Inasmuch as it incorporates the initial condition and one of the two boundary conditions, the latter equation represents a halfway house between the problem and its solution. In other words, it describes not one, but an entire class of boundary value problems. Besides the economy engendered by this replacement, the resulting semidifferential equation provides a simple and useful expression for the flux at the boundary, an expression which may be applied even when the boundary condition cannot be expressed as a mathematical function, and which moreover avoids the need for calculations relating to all positions in the medium. The fundamental role played in this theory by semidifferentiation and its inverse explains our preoccupation in Chapter 7 with the results of applying $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ and $d^{-\frac{1}{2}}/dx^{-\frac{1}{2}}$ to a wide variety of functions. The techniques described in Chapter 8 are likewise valuable tools which we developed primarily to handle the semidifferentiation and semiintegration operators which arise in transport applications.

In summary, then, the book is divided roughly into two parts. The first six chapters deal principally with the general properties of differintegral operators, while Chapters 7–11 are mainly oriented toward the application of these properties to mathematical and other problems.

A word about our intended readership. Our hope is that our book will be readable by, and of interest to, a broad audience. About the only prerequisite is an understanding of the classical calculus, although some familiarity with the ideas involved in solving differential equations would be useful. Among mathematicians our work should be of interest to both classical and functional analysts as well as applications-oriented mathematicians. Because of the diversity of subjects embraced by the fractional calculus, this volume could also interest physical chemists, engineers (electrical, mechanical, and petroleum engineers, among others), and many scientists who have occasion to study transport processes akin to diffusion. We have tried to maintain a readable expository style, being rigorous where it was appropriate, while always striving to tell as complete a story as possible. If we have succeeded, we feel that we have created a book which, while less than a text, is more than a monograph on an old and yet novel subject which makes contact with an amazingly large number of areas of classical and applied mathematics.

It should be stressed that, because fractional derivatives and integrals can always be expressed using ordinary derivatives and integrals (as will be apparent in Chapter 3), any result obtainable through the fractional calculus may also be derived making use only of the concepts and symbolism of classical calculus. Nevertheless, its conceptual elegance and the economy engendered through its use make the fractional calculus much more than a hollow extension of conventional theory.

Writing in 1895 Heaviside said ". . . the result is a simple fundamental one in fractional differentiation. . . . But the reader presumably cannot take in the idea of fractional differentiation yet." Our hope in presenting this treatise is to make its readers more willing than Heaviside's to feel at home with the concept of fractional differentiation and integration.