Utility Maximization

EC 311 - Intermediate Microeconomics

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Outline

Chapter 4

- Topics
  - Optimal Consumption (4.4)
    - Slopes Logic
    - Constrained Optimization

Optimal Consumption

Putting All We've Seen Together

So far we have done 3 things:

1. We learned about measuring and analyzing benefit functions, which we called **Utility**

2. We learned about measuring and analyzing cost functions, we called those **Budgets**

3. We motivated the importance of "Constrained Optimization"

Now we will put it all together to solve problems

Gameplan

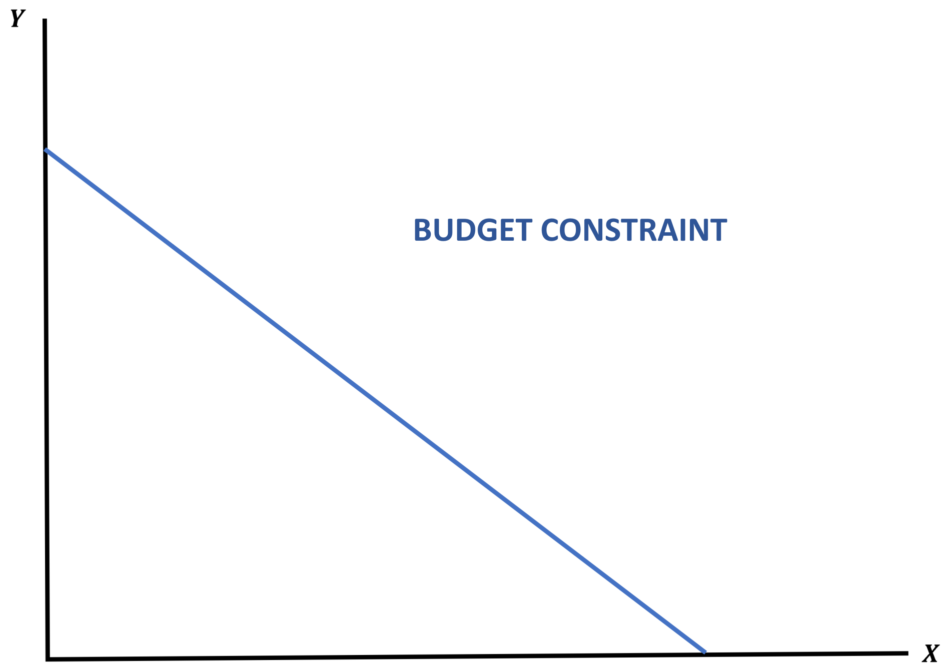
We will do it by:

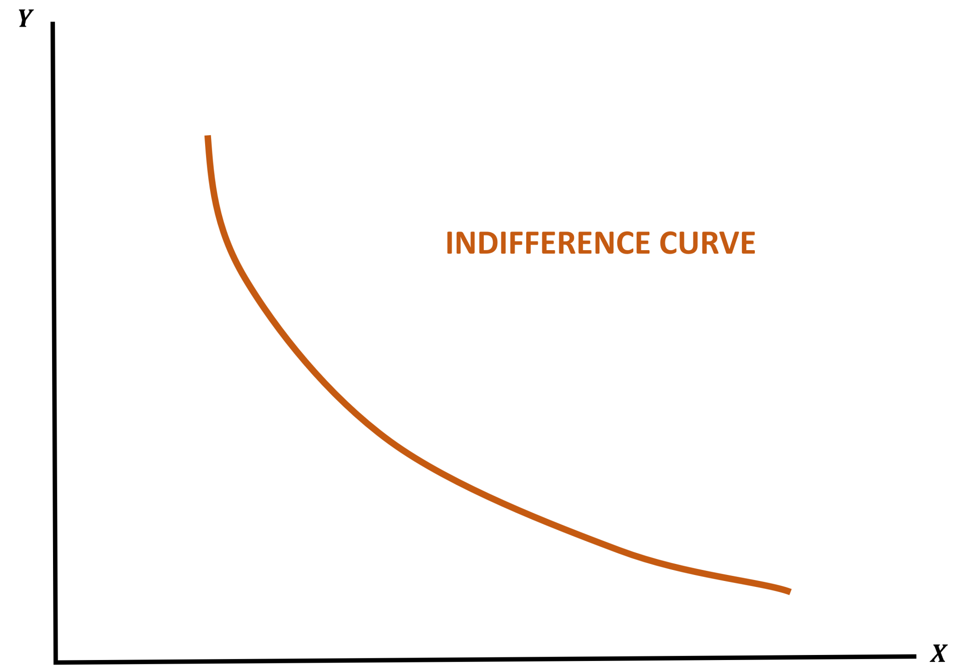
1. Solving it graphically to gain the intuition of what is happening
2. Mathematically, because we need to be able to solve these problems formally

The core question we are attempting to answer is

> **Given a specific budget constraint, what bundle should an individual choose to maximize their utility?**

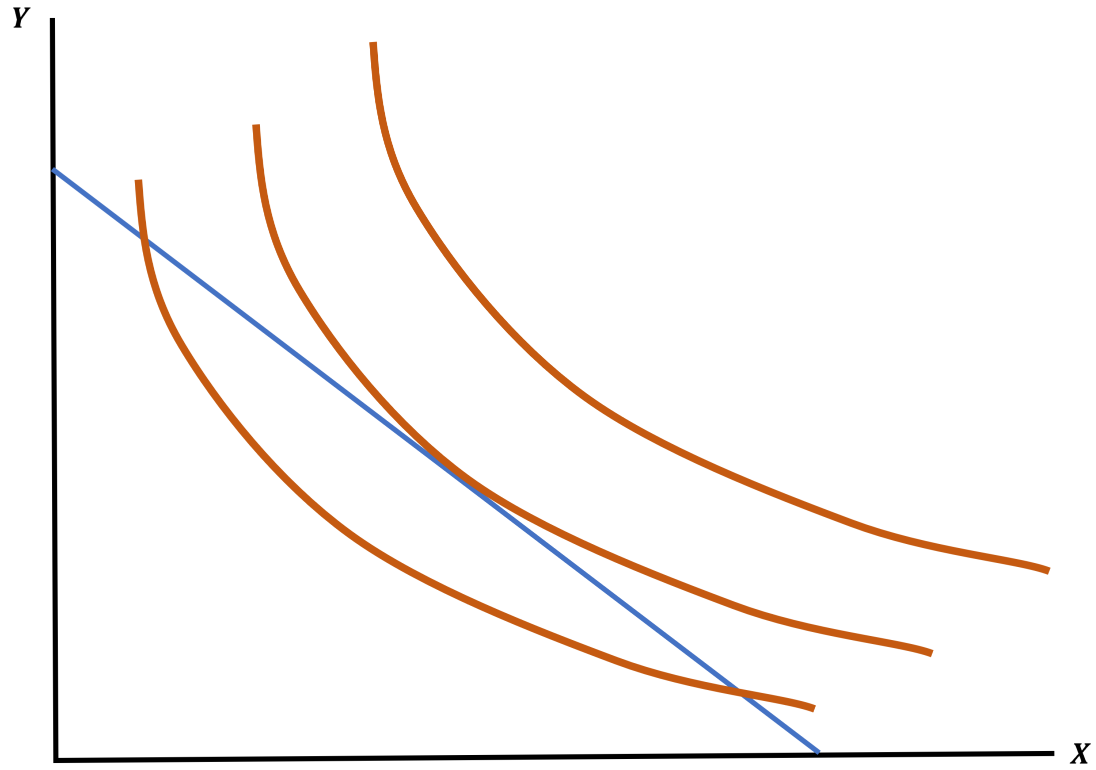
BC + IC Graph





These are our two ingredients to solving a utility maximization problem

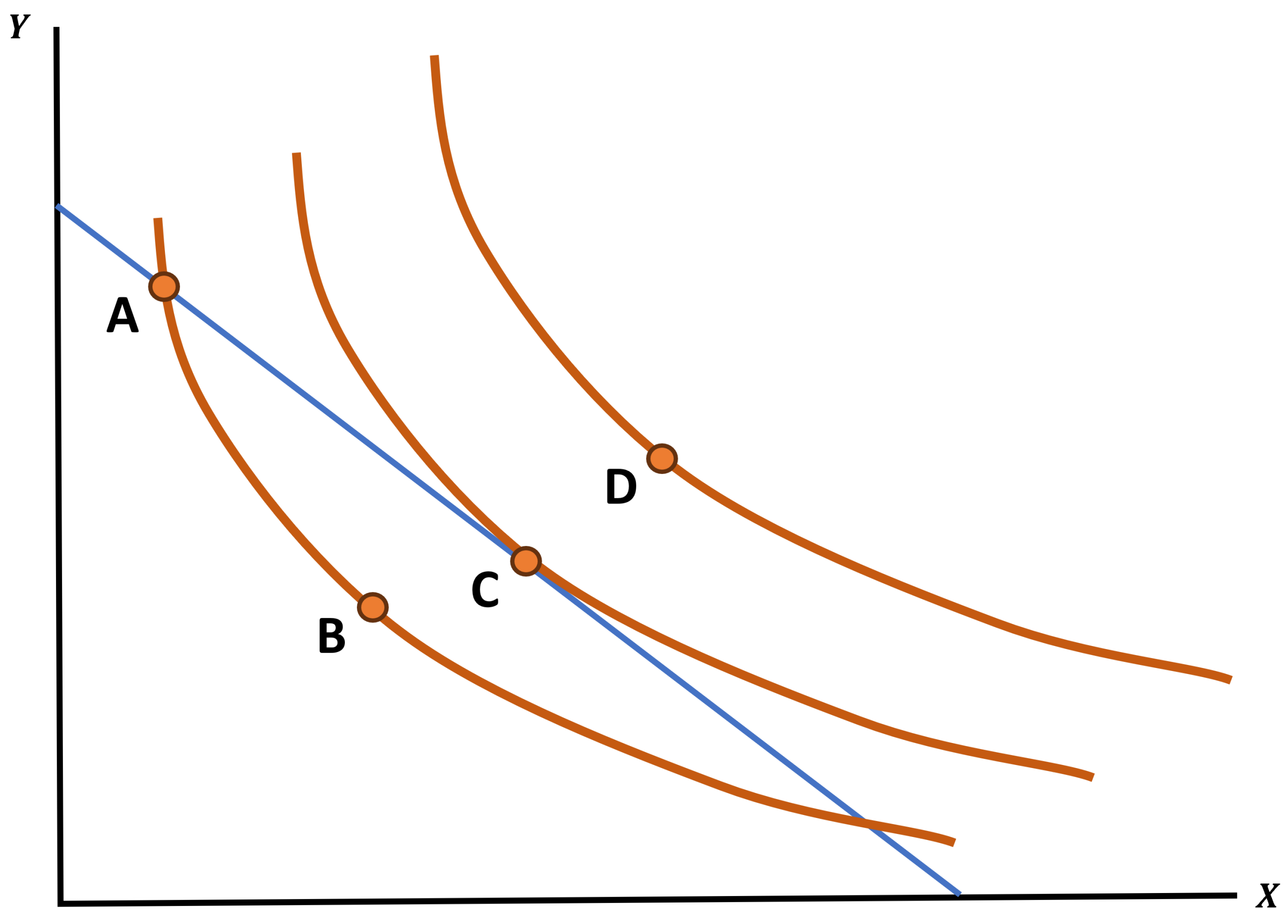
Putting It Together



The key for these graphs is to internalize the fact that there are infinite amounts of IC we can draw

> **And only one will be utility maximizing bundle**

Putting It Together



Point A

Affordable and on the Budget Line

Point B

Affordable but does not use all our budget

Point C

Affordable and on the Budget Line

Point D

Highest utility but unaffordable

Which one is the utility maximizing bundle?

Point C is the highest level of indifference curve that is on the budget line only once!

BC + IC Graph Takeaways

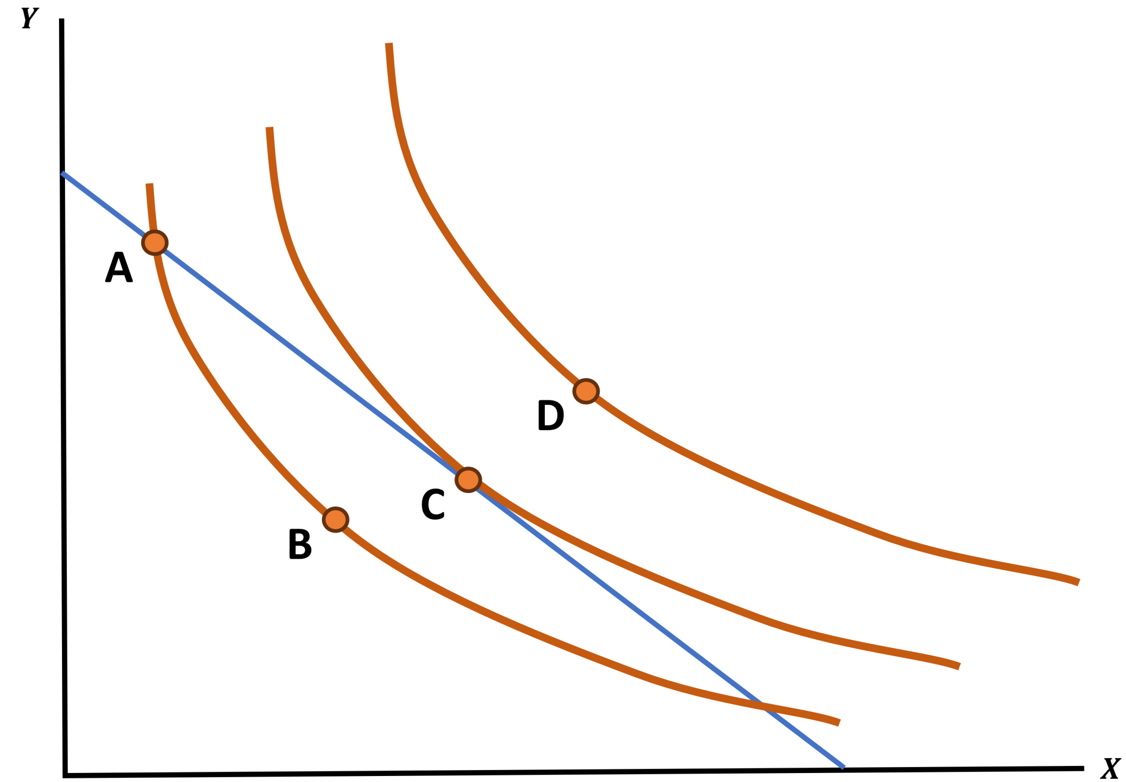
ICs **above** the BC:

- If the IC **never touches the budget constraint**, then none of the bundles on that curve (that utility level) are affordable
- It is impossible to achieve that level of utility with the given amount of income

ICs **below** the BC:

- If the IC is **within the bounds of the BC**, it is possible to attain that level of utility with the given amount of income
- It is also possible to attain more utility than the level given by this IC and still use less than the given income

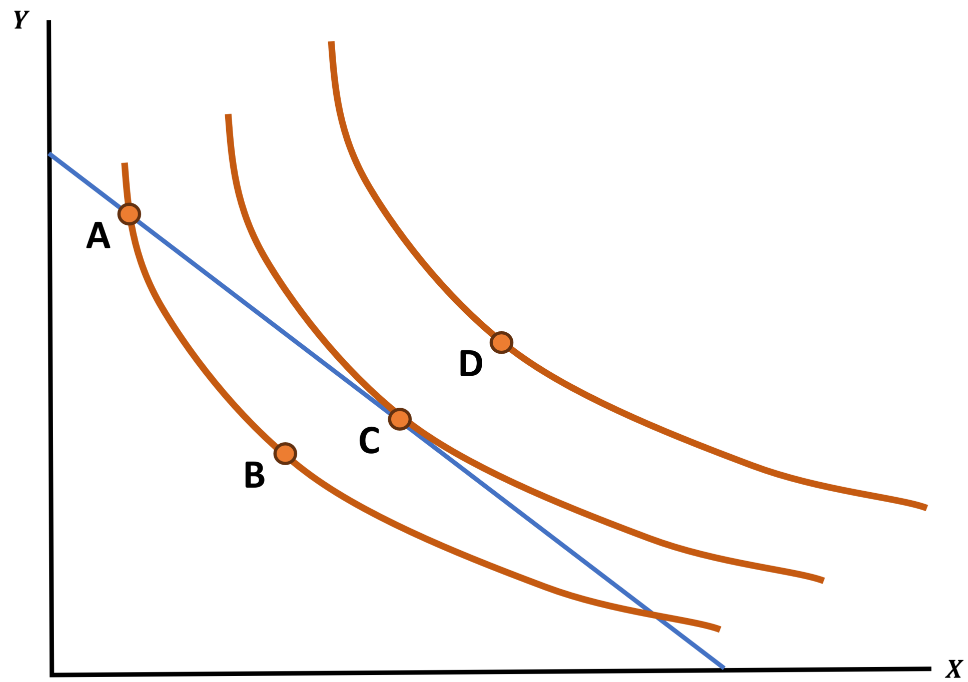
IC With The Highest Utility



Point C is a magical point

- It is where the IC touches the BC **exactly once**
- Every other point on the **green IC** is unaffordable

IC With The Highest Utility



- We call **Point C** the **Utility Maximizing Bundle**
- Moving along the BC, to other affordable bundles like **Point A** only makes you worse off

But what is so special about **Point C**? Math does!

Slope Logic (Graphical Approach)

Math Makes Point C special

This is the point where the BC line and the IC curve have the exact same slope, this is also known as **the point where they are tangent**

This helps us put the intuition we gained from the graph together with the math

> **The key to finding the utility maximizing bundle (the best mix of x and y) you have to find the bundle where the BC and IC have the same slope**

Why Does This Work?

Let's recall what the **Slope of the IC** means:

- It is the **Negative MRS**
- It tells you how much y you would be **willing to** give up to get another unit of x?

Now recall what the **Slope of the BC** means:

- It is the **Negative Price Ratio**
- It tells you how much y you would **have to** give up to get another unit of x?

Why Does This Work?

Imagine what it would it mean if these two slopes were not equal

You are consuming Bundle A where:

The **Negative MRS** is -4

- What does this mean?
  - **You are WILLING to** give up 4 units of y for 1 unit of x

Let's also say that the **Negative Price Ratio** is -3

- What does this mean?
  - **You HAVE to** give up 3 units of y for 1 unit of x

Because it is "cheaper" to trade y for x than you are willing to, you will make the trade

> **Is this a utility maximizing bundle? Why or Why not?**

Why Does This Work?

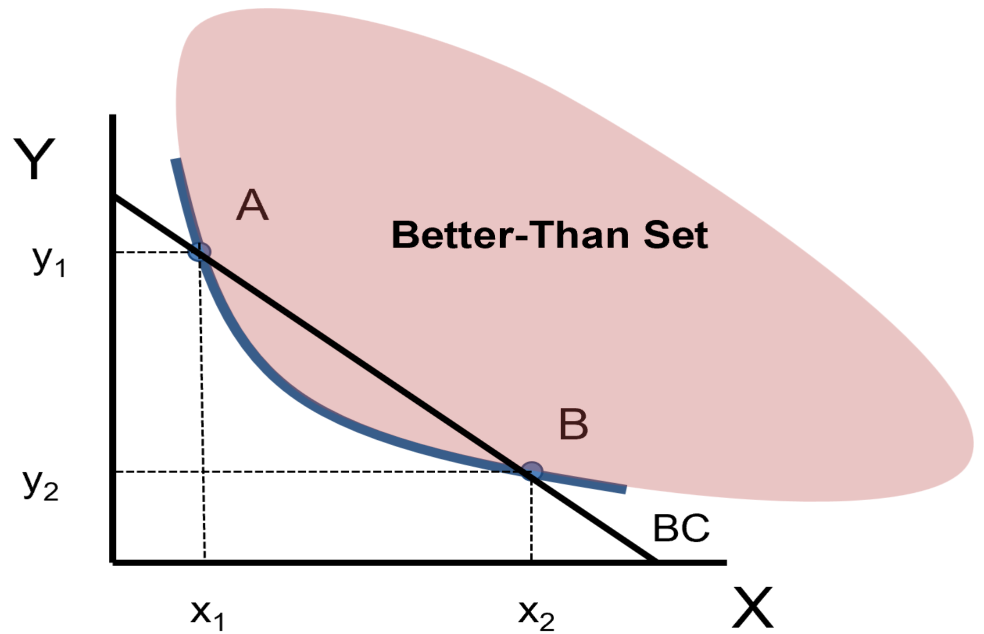
If $\text{Negative MRS} \neq \text{Price Ratio}$ then you trade that bundle for another, which means that you were **not** at a utility maximizing bundle

The logic here is:

- **If you willingly move away from a chosen bundle, then that bundle, by definition, cannot be utility maximizing**

Let's draw it!

Graph Takeaway



If $\text{Negative MRS} < \text{Price Ratio}$ **(Point A)**

- Graphically, this means that the **IC** is **steeper than** the **BC**
- Trading away some y for some x moves you down and to the right in the better-than set

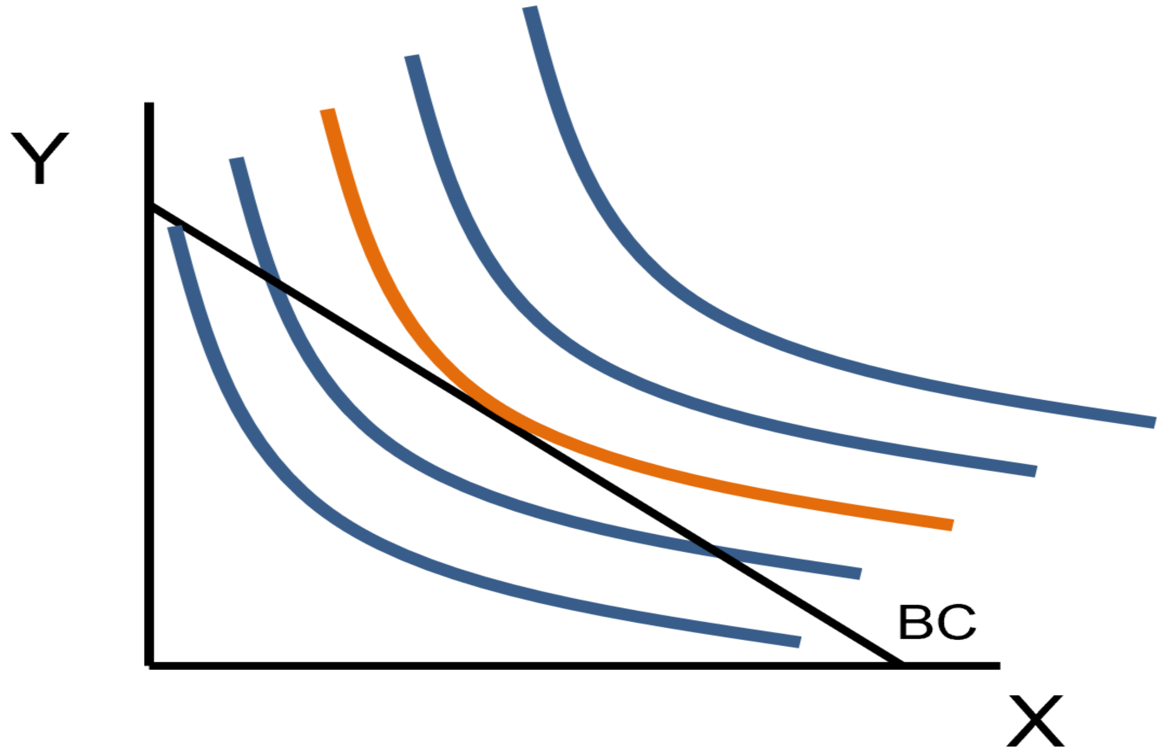
If $\text{Negative MRS} > \text{Price Ratio}$ **(Point B)**

- Graphically, this means that the **IC** is **flatter than** the **BC**
- Trading away some x for some y moves you up and to the left in the better-than set

Graph Takeaway - Part II

The graph implicitly says:

- There are an infinite number of ICs that all correspond to different levels of utility
- There is only one BC
- Utility maximization happens **only** when you find the point on the BC that touches the highest possible IC (once)



Slope Logic Summary

The logic takaways here are:

- When you have a curved IC, you have to set the slope of the IC equal to the slope of the BC to find the utility-maximizing bundle
- If they are not equal, you would be willing to make some trade to a different bundle
- The magical point of maximization only occurs if the ICs are **Non-Crossing**, **Monotonic**, and **Convex**
  - The **weakly monotonic** and **weakly convex** arguments we saw satisfy this
- Most importantly, **there is only one magical point**

Constrained Optimization

How to Tackle This Mathematically

We know the steps:

1. Calculate the MRS of a given utility function
2. Calculate the Price Ratio using the Budget Constraint
3. Set them equal to each other

> **It really is this simple, but the math makes it look more complicated**

Defining the Problem

The problem will take the form of

$$\max \; U(x, y) \quad \text{subject to} \quad P_x \cdot x + P_y \cdot y = M$$

This is what we call a **Constrained Optimization** problem

- You are trying to maximize your utility subject to a budget constraint

Different Utility Functional Forms

We had four different forms that utility can take:

- Cobb-Douglas
- Quasi-linear
- Perfect Substitutes
- Perfect Complements

Each have their quirks and procedure to follow

> **Solving these type of problems is the crux of the course. I know it can be intimidating at first glance, but my personal advice is to not over think the process, that's what the steps are for**

> **The theory is what will guide you and allow you to interpret results, and it may seem wonky at first but that's just how economists think**

Cobb-Douglas Utility Constrained Maximization

There is a 4-step approach that we will follow to solve these problems when the utility function is **Cobb-Douglas**

1. Calculate the MRS and set it equal to the Price Ratio
2. Re-arrange this equality to isolate one good (x or y) as a function of the other (It does not matter which good you choose here).
   - The resulting equation is called an "optimality condition"
3. Write down your budget constraint and plug the optimality condition into it.
   - This allows you to solve for the **demand** of one of the two goods
4. Use your demand for one good and plug it into either the optimality condition or the BC to find the demand for the other good

1 - Calculate MRS and Set Equal to Price Ratio

$$\text{Let } U(x,y) = xy,\ P_x = 1,\ P_y = 2,\ M = 12$$

Find MRS

$$MRS = \frac{MU_x}{MU_y} = \frac{y}{x}$$

Find Price Ratio

$$\text{Price Ratio} = \frac{P_x}{P_y} = \frac{1}{2}$$

Set them equal to each other

$$\frac{y}{x} = \frac{1}{2}$$

2 - Re-arrange the Equality to Isolate for One Good

$$\frac{y}{x} = \frac{1}{2}$$

You can isolate either x or y

Isolating y

$$y = \frac{x}{2}$$

Isolating x

$$x = 2y$$

These are optimality conditions for x and y

They tell us the demand of a good conditional on other model parameters

3 - Write Down BC and Plug An Optimality Condition

$$y = \frac{x}{2} \;\; \& \;\; BC: x + 2y = 12$$

Here I will plug in my known value of y anywhere I find a y in the BC

$$\begin{aligned} x + 2y &= 12 \\ x + 2(\frac{x}{2}) &= 12 \\ x + x &= 12 \\ 2x &= 12 \\ x^* &= 6 \end{aligned}$$

> **Note: I will denote the optimal amount for any good as x^* or y^* or whatever variable the problem uses**

4 - Plug Demand for One Good into Optimality Condition or BC

$$\text{We found that } x^* = 6$$

I'll do it both ways just to show you that it works but you only have to do one (I recommend finding which one is simpler for you)

Plugging into Optimality Condition

$$\begin{aligned} y &= \frac{x^*}{2} \\ y &= \frac{6}{2} \\ y^* &= 3 \end{aligned}$$

Plugging into Budget Constraint

$$\begin{aligned} x^* + 2y &= 12 \\ 6 + 2y^* &= 12 \\ 2y^* &= 6 \\ y^* &= 3 \end{aligned}$$

> **Lastly, in exams I will expect the answers to be identified at the end of your work so I know you actually answered the question that was asked**

Cobb-Douglas 2: Example For You

$$U(x,y) = x^{1/4}y^{3/4},\ P_x = 2,\ P_y = 3,\ M = 32$$

What are the optimal amounts of x and y to maximize this individual's utility subject to their budget constraint?

$$x^* = 4 \ \&\ y^* = 8$$

What utility level U^* is achieved?

$$U^* = (x^*)^{1/4}(y^*)^{3/4} = (4)^{1/4}(8)^{3/4} = \dots$$

Cobb-Douglas 2: Solution

$$MRS = \frac{MU_x}{MU_y} = \frac{1/4\, x^{-3/4} y^{3/4}}{3/4\, x^{1/4} y^{-1/4}} = \frac{1/4}{3/4} \cdot \frac{y^{3/4} y^{1/4}}{x^{1/4} x^{3/4}} = \frac{1}{3} \cdot \frac{y}{x}$$

Also recall:

$$MRS = \frac{a}{b} \cdot \frac{y}{x} = \frac{1/4}{3/4} \cdot \frac{y}{x} = \frac{1}{3} \cdot \frac{y}{x}$$

$$\text{Price Ratio} = \frac{P_x}{P_y} = \frac{2}{3}$$

$$\frac{y}{3x} = \frac{2}{3} \;\rightarrow\; y = 2x$$

Quasi-linear Utility Constrained Optimization

To solve problems with this utility functional form we repeat the same steps as C-D but we skip step 3

1. Find the MRS and set it equal to the Price Ratio
2. Solve the equality for the non-linear good (usually inside the $ln()$)
3. Use the BC to find the demand of the other good

1 - Find the MRS and set it equal to the Price Ratio

$$\text{Let } U(x,y) = 10 \cdot ln(x) + \frac{y}{2},\ P_x = 5,\ P_y = 2,\ M = 70$$

Find MRS

$$MRS = \frac{MU_x}{MU_y} = \frac{10/x}{1/2} = \frac{20}{x}$$

Find Price Ratio

$$\text{Price Ratio} = \frac{P_x}{P_y} = \frac{5}{2}$$

Set them equal to each other

$$\frac{20}{x} = \frac{5}{2}$$

2 - Solve the Equality for the Non-linear good

$$\frac{20}{x} = \frac{5}{2}$$

We solve for x (notice there is no y)

$$40 = 5x$$
$$x^* = 8$$

3 - Use BC to Find Demand of Other Good

$$\text{BC: } 5x + 2y = 70, \quad x^* = 8$$

$$\begin{aligned} 5x^* + 2y &= 70 \\ 5(8) + 2y &= 70 \\ 40 + 2y &= 70 \\ 2y &= 70 \\ y^* &= 35 \end{aligned}$$

Perfect Substitutes Constrained Optimization

Recall the functional form

$$U(x, y) = ax + by$$

- Mathematically, the MRS will just equal a constant $\left(\dfrac{a}{b}\right)$ so setting it equal to the price ratio doesn't really do much
- It will either be larger than, smaller than, or equal
- So what do we do? A graph will help us see

Perfect Substitutes Optimization Graph

$$\text{Let } U(x,y) = x + y,\ P_x = 10,\ P_y = 18,\ M = 90$$

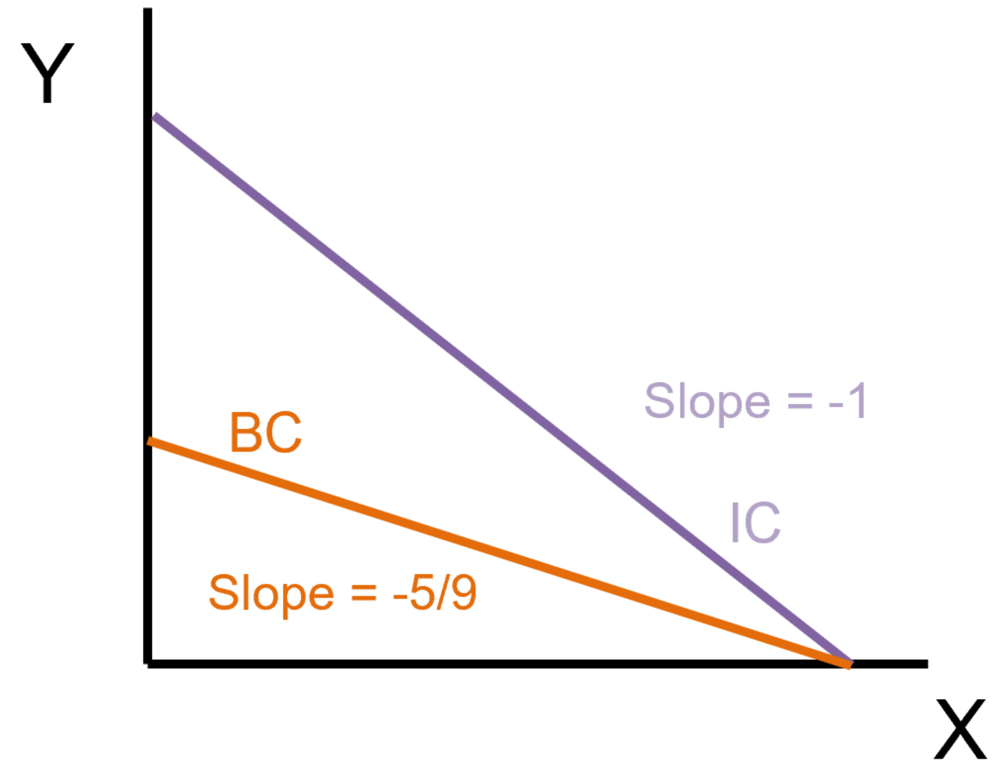
Find the MRS

$$MRS = \frac{1}{1} = 1$$

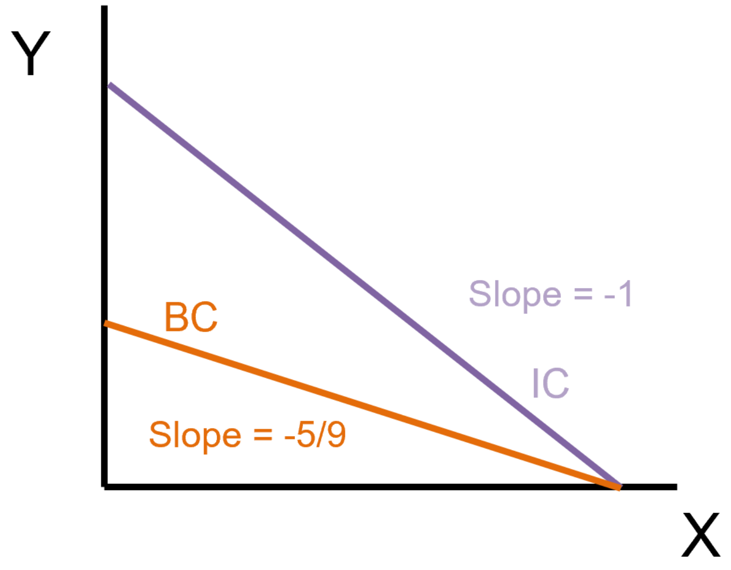
Find Price Ratio

$$\frac{P_x}{P_y} = \frac{10}{18} = \frac{5}{9}$$

Graph



Perfect Substitutes Optimization Graph



Here, the individual chooses to only consume good x and no y. Why?

- The **ICs** are steeper than the **BC**
- The intuition is saying that the **willingness to trade** y for x is always larger than the **ability to trade**
- This individual will get all the x they can!

Perfect Substitutes Optimization - Example

$$U(x, y) = 2x + 3y,\ P_x = 1,\ P_y = 1/2,\ M = 5$$

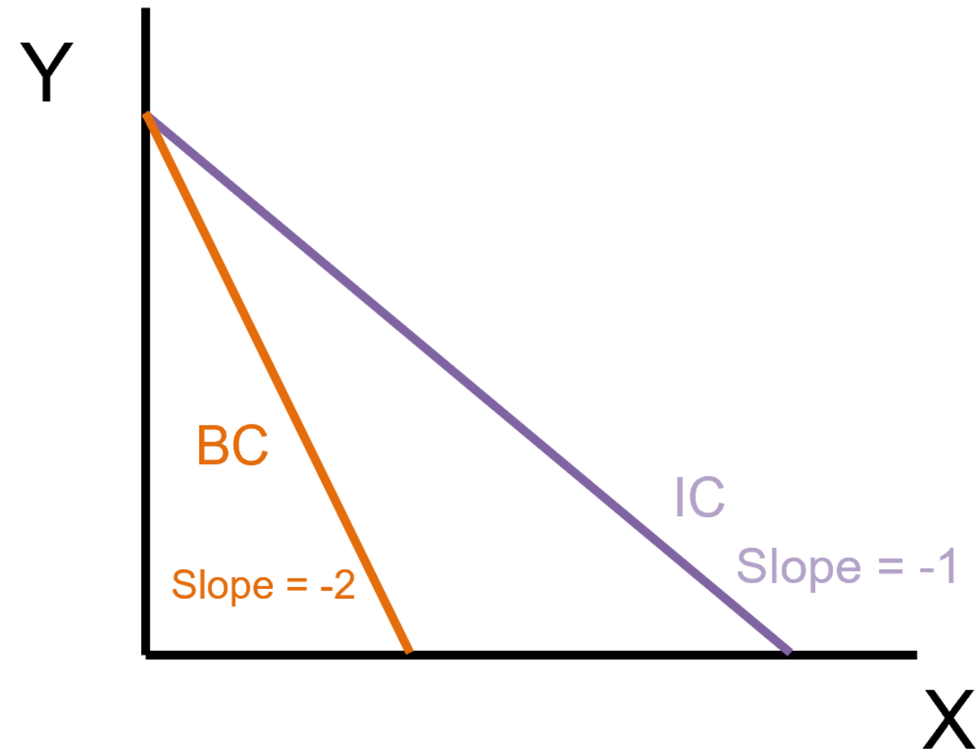
Find the MRS

$$MRS = \frac{2}{2} = 1$$

Find Price Ratio

$$\frac{P_x}{P_y} = \frac{1}{1/2} = 2$$

Graph



Perfect Substitutes Optimization

As we saw, maximization problems with P-Subs functional form have two predictable outcomes

- Either you consume all x and no y or all y and no x
  - It depends on the relationship between the MRS and the Price Ratio
- There is the unusual case where the slopes are the same
  - Graphically, this means that they will perfectly overlap
  - Mathematically, any bundle that meets the requirements will work

Perfect Substitutes Utility Constrained Optimization

The graphs we just saw tell us how to do the math:

1. Find the MRS and Price Ratio
2. Compare the MRS to the Price Ratio
    - If $\text{MRS} > \text{Price Ratio}$, consume all x and y = 0
    - If $\text{MRS} < \text{Price Ratio}$, consume all y and x = 0
3. Figure out how much x and y to consume using the BC

P-Subs Example (Mathematically)

$$U(x, y) = 3x + 2y,\ P_x = 1,\ P_y = 4,\ M = 60$$

Find MRS and Price Ratio

$$MRS = \frac{MU_x}{MU_y} = \frac{3}{2}$$
$$\text{Price Ratio} = \frac{P_x}{P_y} = \frac{1}{4}$$

Compare MRS to Price ratio

$$\frac{3}{2} > \frac{1}{4}$$
$$\text{MRS} > \text{Price Ratio}$$

Choose only x

Figure out how much x and y to consume using the BC

$$\begin{aligned} \text{BC: } x + 4y &= 60 \\ x + 4(0) &= 60 \\ x^* &= 60 \\ y^* &= 0 \end{aligned}$$

Perfect Substitutes Optimization Intuition

With Perfect Substitutes the goods do not interact at all

So the question really becomes:

- Which of these two goods is more cost-effective at generating utility?

When you say that the MRS is less than the Price Ratio, you are also saying that **you get fewer units of utility per dollar spent on x than per dollar spent on y**. Therefore, you should only consume y.

> **Note: This logic only works for Perfect Substitutes. This is because the MRS is always a constant**

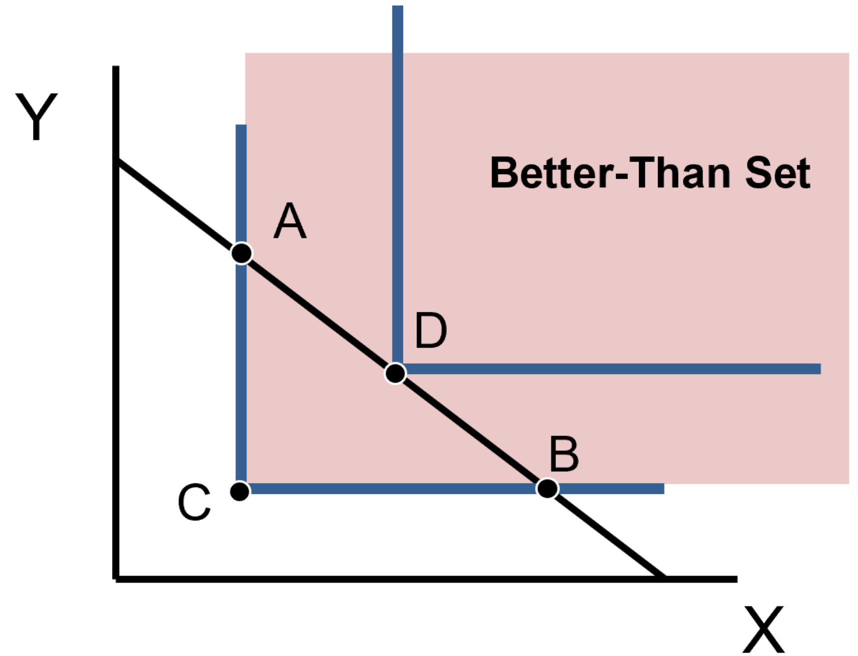
Perfect Complements Utility Constrained Optimization

Recall: We cannot solve this utility form like the others because there is no MRS

So let's see what a graph can tell us

$$U(x, y) = \min\{2x, 3y\},\ P_x = 2,\ P_y = 2,\ M = 40$$

Perfect Complements Utility Constrained Optimization - Graph



Hidden **Point D** is the actual maximizing point

Remember that there are is always a **Better-Than Set** where other ICs exist

Perfect Complements Graph Intuition

The bundles on both the IC and the BC involve "wasting" one good or the other

- Recall that the **"No-Waste Condition"** says that you should achieve a certain utility level with the minimum units of x and y possible
  - In other words, you want to be at the **"kink"** of the function

The previous graph showed us that **Point D** is the utility maximizing point, now let's learn how to find it

Perfect Complements Contrained Optimization

I will be bold and say that this is one of the simpler utility functions to maximize

Just be sure to follow the steps:

1. Solve the **“No-Waste Condition”** for one of the goods (Does not matter which one)
2. Plug the optimality condition into the budget constraint and find the demand of one Good
3. Use either the BC or the **“No-Waste Condition”** to solve for the demand of the other good

1 - Solve the No-Waste Condition for One of the Goods

$$U(x,y) = \min\{\frac{x}{2}, 2y\},\ P_x = 1,\ P_y = 3,\ M = 56$$

Solve the No-Waste Condition

Solving for x^*

$$\frac{x}{2} = 2y$$
$$x^* = 4y$$

Solving for y^*

$$\frac{x}{2} = 2y$$
$$y^* = \frac{x}{4}$$

2 - Plug the Optimality Condition into the BC to Find Demand of One good

$$x^* = 4y \text{ or } y^* = \frac{x}{4} \quad , \quad \text{BC: } x + 3y = 56$$

$$\begin{aligned} x + 3y &= 56 \\ (4y) + 3y &= 56 \\ 7y &= 56 \\ y^* &= 8 \end{aligned}$$

3 - Use BC or "No-Waste Condition" to Find Demand for Other Good

$$y^* = 8 \text{ and No-Waste Condition: } \frac{x}{2} = 2y$$

$$\begin{aligned} \frac{x}{2} &= 2y \\ x^* &= 4y \\ x^* &= 4(8) \\ x^* &= 32 \end{aligned}$$

So our optimal bundle is:

$$x^* = 32 \;\; \& \;\; y^* = 8$$

Utility Maximization Summary

This lecture has taught you how to solve the 4 different utility functional forms constrained maximization problems graphically and mathematically

With this in our toolbox, we can find the individual demand of goods.

The next step is to find Demand Functions for the entire market