Mixed Strategies

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Winter 2024

EC327 Game Theory

Mixed Strategies

Internal Uncertainty

- Now that we discussed *external uncertainty* over *states of nature*, let's talk about internal uncertainty.
- Internal uncertainty occurs when one or more players pick their strategies randomly.
- Picking a strategy at random is really just a different kind of strategy, called a mixed strategy.

Mixed Strategies

- When a player always does the same thing, it's called a pure strategy.
- A mixed strategy assigns a probability to each of a player's pure strategies.
  - Like a lottery, the probabilities in a mixed strategy must all be between 0 and 1,
  - and must sum to exactly 1.

Mixed Strategies

- A mixed strategy can assign 0 probability to a pure strategy.
- It can even assign probability 1 to a single pure strategy, and probability 0 to all others
  - this is still, technically, a mixed strategy, but it is a trivial one.
- When a player uses a mixed strategy, it turns the **other** player's payoffs into lotteries.

Mixed Strategies in the Deer Hunt

Consider the Deer Hunt:

		Ogg: *Deer*	Ogg: *Rabbit*
Igg	*Deer*	$2, 2$	$0, 1$
	Rabbit	$1, 0$	$1, 1$

- Suppose that Igg hunts Deer 3/4 of the time, and Rabbit 1/4 of the time.
- If Ogg *always hunts deer*; what is Ogg's *expected payoff*?

Mixed Strategies in the Deer Hunt: Generalizing

We can generalize this approach to calculate Ogg's expected payoffs from any strategy that Igg chooses to play:

- Suppose that Igg plays Deer with probability p, and Rabbit with probability 1 - p.
- Then Ogg's expected payoff from Deer is:

- Note that Ogg's expected payoff from Deer gets larger with p: the more likely Igg is to hunt Deer, the more attractive an option it becomes for Ogg.

When to Play a Mixed Strategy?

- It's possible for a mixed strategy to be a best response to the other player's strategy:
  - if and only if all of the mixed strategy's components (pure strategies that are assigned positive probability) are best responses too.
- Some intuition: If a strategy is not a best response, you should not play it—even as part of a mixed strategy.

When to play a Mixed Strategy?

If a player only has **two pure strategies**, it becomes simple to tell when a mixed strategy is a best response: the mixed strategy must be a mixture of those two pure strategies, and the only way that both of them are best responses is if they have equal expected payoffs.

- Taking the Deer Hunt as an example, the only way that it can be a best response for Ogg to play a mixed strategy is if Deer and Rabbit provide Ogg with equal expected payoffs: we must have $2p = 1$, or $p = \frac{1}{2}$.

Mixed-Strategy Nash Equilibrium

To solve for the Nash equilibria where players are allowed to use mixed strategies:

we need to look for the conditions under which a player would be willing to use a mixed strategy.

- This means that we're going to use one player's expected payoffs to solve for the **other** player's mixed strategy

MSNE in the Deer Hunt

- Returning to the Deer Hunt, let's say that Igg plays Deer with probability p and Rabbit with probability $1 - p$...
- While Ogg plays Deer with probability q and Rabbit with probability $1 - q$.
    - This is simply a framework for describing each player's mixed strategies: we're using placeholder variables for the players' mixed strategy probabilities

MSNE in the Deer Hunt

- We already saw that Ogg's expected payoffs from Deer and Rabbit are $2p$ and 1, respectively, so Ogg would only play a mixed strategy if $p = \frac{1}{2}$.
- Likewise, Igg's expected payoffs are $2q$ and 1, and Igg will play a mixed strategy if $q = \frac{1}{2}$.
- The MSNE in this game can be written as:

$$\{(1/2\ Deer, 1/2\ Rabbit)_{Ogg},\ (1/2\ Deer, 1/2\ Rabbit)_{Igg}\}$$

.

Error-Checking

- Make sure that you're setting up the equations used to solve for a player's strategy correctly:
    - Remember that you are creating an equation to describe when a player is indifferent between their pure strategies: if you're trying to figure out when **Player 1** is indifferent, you need to use **Player 1's** payoffs.
    - However, when calculating expected payoffs, the probabilities will be based on the **other** player's mixed strategy: in a game with mixed strategies, the randomness a player deals with is created by the **other** player—not themselves.

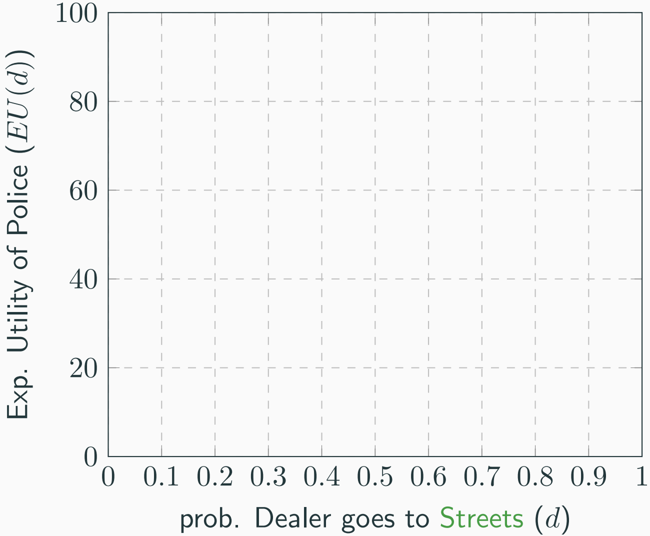
Another Example: Police Patrol and Drug Trade [1]

		Drug Dealer	
		Streets (d)	Park (1 - d)
Police	Streets (p)	80, 20	0,100
	Park (1 - p)	10,90	60,40

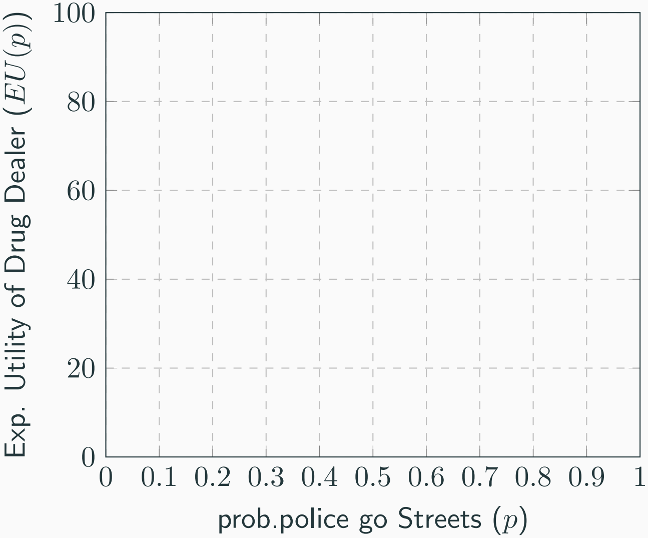
- Police Officer's expected utility:

- Drug Dealer's expected utility:

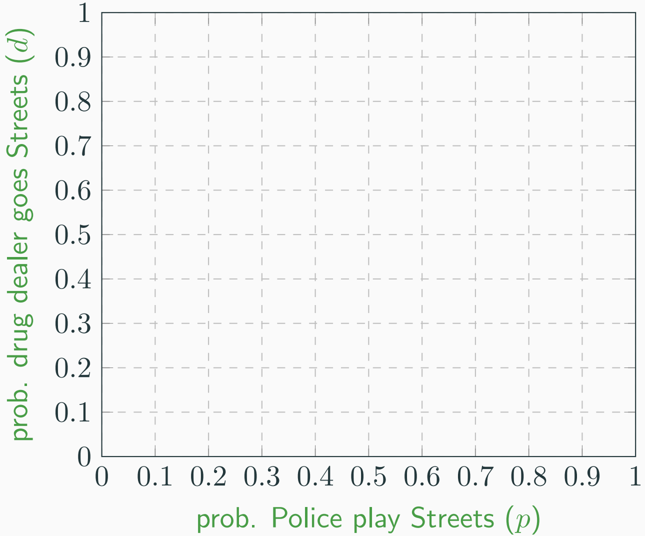
Graph Police Officer's expected utilities



Graph Drug Dealer's expected utilities



Graph Best Response functions



MSNE in Patrol and Trade game:

- When is the Police Officer indifferent between going to the Park and going to the Streets?

- When is the Drug Dealer indifferent between going to the Park and going to the Streets?

- What is the Mixed Strategies Nash equilibrium?

iClicker Q1

- Consider the following game table. What are Player 1's expected payoffs, given Player 2's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

(a) $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$

(b) $U_1(Up) = 3 - 5q, U_1(Down) = 6q - 1$

(c) $U_1(Up) = 5 - 7q, U_1(Down) = 1 - 6p$

(d) $U_1(Up) = 7p - 5, U_1(Down) = 1 - 4p$

(e) $U_1(Up) = 5 - 7p, U_1(Down) = 4p - 1$

iClicker Q2

- Consider the following game table. What are **Player 2's** expected payoffs, given Player 1's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

(a) $U_2(Up) = 5q - 3, U_2(Down) = 1 - 6q$

(b) $U_2(Up) = 3 - 5q, U_2(Down) = 6q - 1$

(c) $U_2(Up) = 5 - 7q, U_2(Down) = 1 - 6p$

(d) $U_2(Up) = 7p - 5, U_2(Down) = 1 - 4p$

(e) $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$

iClicker Q3

- The correct answers to the previous two questions were:
  - $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$.
  - $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$.
- Based on this, what are p and q in the MSNE of this game?
  (a) $p^* = 4/11,\ q^* = 5/11$
  (b) $p^* = 4/11,\ q^* = 6/11$
  (c) $p^* = 6/11,\ q^* = 4/11$
  (d) $p^* = 7/11,\ q^* = 5/11$
  (e) $p^* = 7/11,\ q^* = 6/11$

An MSNE With Only One Mixed Strategy

- Consider the following game table:

		Player 2	
		X (q)	Y $(1-q)$
Player 1	A (p)	2, 2	3, 2
	B $(1-p)$	4, 3	0, 0

- The players' expected payoffs are:
  - $U_1(A) = 2q + 3(1-q) = 2q + 3 - 3q = 3 - q.$
  - $U_1(B) = 4q + 0(1-q) = 4q.$
  - $U_2(X) = 2p + 3(1-p) = 2p + 3 - 3p = 3 - p.$
  - $U_2(Y) = 2p + 0(1-p) = 2p.$

An MSNE With Only One Mixed Strategy

- Based on this, the conditions under which each player will use a mixed strategy are:

$$
\begin{aligned}
&Player\ 1: && Player\ 2: \\
&3 - q = 4q && 3 - p = 2p \\
&3 = 5q && 3 = 3p \\
&q = 3/5 && p = 1
\end{aligned}
$$

- We've never seen anything like $p = 1$ in this context before...
- $p = 1$ tells us that Player 2 will only play a mixed strategy if Player 1 only play A, which isn't really a mixed strategy at all.
- This usually occurs when one strategy **weakly** dominates another.

An MSNE With Only One Mixed Strategy

- We can still approach this the same way that we have in the past:
- Suppose that in the MSNE, Player 1 plays a (non-trivial) mixed strategy. Then Player 2 must also play a mixed strategy, in which $q = 3/5$.
    - But Player 2 will only play a mixed strategy if Player 1 plays the mixed strategy where $p = 1$...which is a trivial mixed strategy. This is a contradiction, and it means that there is no MSNE where Player 1 plays a non-trivial mixed strategy.

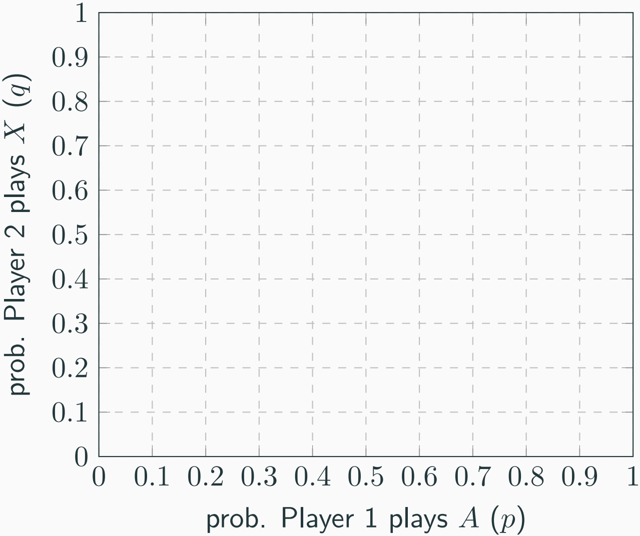
An MSNE With Only One Mixed Strategy

- Approach it the other way next: Suppose Player 2 plays a non-trivial mixed strategy. Then Player 1 must play A as a pure strategy.
    - Player 2 will play A if $3 - q \geq 4q$, i.e. if $3/5 \leq q$.
- This lets Player 2 play a non-trivial mixed strategy! There is no contradiction here.

An MSNE with only one mixed strategy

- There are a range of MSNEs here: all strategy profiles of the form {(1, 0), (q, 1 - q)}, in which $0 < q \leq 3/5$, are MSNEs.
- There are also two trivial MSNEs, {(1, 0), (0, 1)} and {(0, 1), (1, 0)}, which are really just the pure-strategy Nash equilibria (A, Y) and (B, X) expressed in the form of an MSNE.
- It will help to understand what's going on with the Best Responses graph:

Graph Best Responses



Absence of MSNEs

- Let us return to the Prisoner's Dilemma and check for MSNEs:

		Luca	
		$Testify\ (q)$	$Keep\ Quiet\ (1-q)$
Guido	$Testify\ (p)$	$-10, -10$	$0, -20$
	$Keep\ Quiet\ (1-p)$	$-20, 0$	$-1, -1$

- Guido and Luca's expected payoffs are:
  - $U_G(Testify) = -10q + 0(1-q) = -10q$.
  - $U_G(KeepQuiet) = -20q + (-1)(1-q) = -1 - 19q$.
  - $U_L(Testify) = -10p + 0(1-p) = -10p$.
  - $U_L(KeepQuiet) = -20p + (-1)(1-p) = -1 - 19p$.

Absence of MSNEs

- Guido will play a mixed strategy if:

$$
\begin{aligned}
-10q &= -1 - 19q \\
9q &= -1 \\
q &= -1/9
\end{aligned}
$$

- But **-1/9 is not a valid probability!**
- We could also note that if $q \in [0, 1]$, which is the range for valid probabilities, $-10q$ is always greater than $-1 - 19q$. In other words, as we saw weeks ago, $Testify$ strictly dominates $Keep\ Quiet$...so why would Guido mix between the two of them?

Getting Bad Probabilities

- If you've set up the expected-payoff equation, and solved for a player's mixed strategy, and you find that the probability is less than 0, or more than 1...
- **It means something is wrong.** Probability can only be between 0 and 1 (inclusive).
- First of all, double-check your math—it could be an algebra error.
- But if you're confident in your math, this means that there is **no way that the player would ever play a mixed strategy**: in fact, they have a strictly dominated strategy.
- There will be no MSNE where this player uses a mixed strategy—but there might be MSNEs where the other player does, so you should still check that.

Existence of Nash Equilibria

What is the *pure strategy* Nash Equilibria of the game Rock, Paper, Scissors?

What is the **Mixed Strategy Nash Equilibria** of the game Rock, Paper, Scissors?

Existence of Nash equilibria

Any with game with a *finite* set of moves will have at least one **Nash equilibrium** when allowing for *mixed strategies*.

The true power of this concept and the most important contribution of its namesake, John Nash, is that it is a simple concept which has universal application.

Advanced Mixed Strategies

MSNE in a Larger Game

- Suppose that we have this 3×2 game:

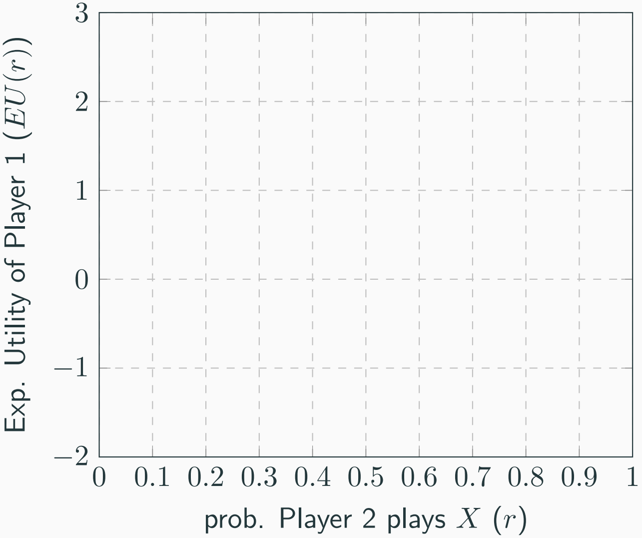
		Player 2	
		X (r)	Y (1 - r)
Player 1	A (p)	2, 1	0, 1
	B (q)	1, 2	2, 0
	C (1 - p - q)	0, 0	3, 2

- Player 1's mixed strategy uses probabilities p, q, and 1 - p - q, since they have three pure strategies.

MSNE in a Larger Game

- Algebraically:
  - $U_1(A) = 2r + 0 = 2r$.
  - $U_1(B) = 1r + 2(1 - r) = 2 - r$.
  - $U_1(C) = 0 + 3(1 - r) = 3 - 3r$.

Graph Player 1's expected utilities



When will Player 1 mix?

- What it would take to get Player 1 to mix different pairs of strategies:
  - A and B: $2r = 2 - r \implies r = \frac{2}{3}$.
  - A and C: $2r = 3 - 3r \implies r = \frac{3}{5}$.
  - B and C: $2 - r = 3 - 3r \implies r = \frac{1}{2}$.
- Note that there is no intersection between all three lines simultaneously
- This means that Player 1 will never mix between all three strategies

MSNE in a Larger Game

- Let's check Player 2's expected payoffs next:
  - $U_2(X) = 1p + 2q + 0$.
  - $U_2(Y) = 1p + 0 + 2(1 - p - q)$.
- So Player 2 will play a mixed strategy if

$$p + 2q = p + 2(1 - p - q)$$

$$\implies q = 1 - p - q$$

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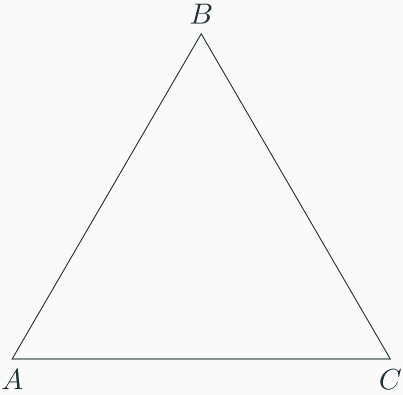
  - Recall that q was the probability we put on Player 2 playing B,
  - and $1 - p - q$ was the probability they play C.

visualizing Player 2's Best Responses



When will Player 2 mix?

- We found they are indifferent between X and Y when

$$q = 1 - p - d$$

.

- There are two ways that this can be true:
    - Either Player 1 plays B and C with equal probability (and we know from earlier that they would **only** be playing these two, not A),
    - or Player 1 plays A only, and B and C not at all.

MSNE in a Larger Game

Case 1: Player 1 only plays A:

- this requires $2r \geq 2 - r$ and $2r \geq 3 - 3r$,
- which imply that $r \geq \frac{2}{3}$ and $r \geq \frac{3}{5}$.
- **MSNE 1:** $\{$(1, 0, 0), (r, 1 - r)$\}$, where $r \geq \frac{2}{3}$.

MSNE in a Larger Game

Case 2: Player 1 plays B and C with equal probability

- then Player 2 plays X and Y with equal (1/2) probability.
- **MSNE**: $\{$(0, 1/2, 1/2), (1/2, 1/2)$\}$

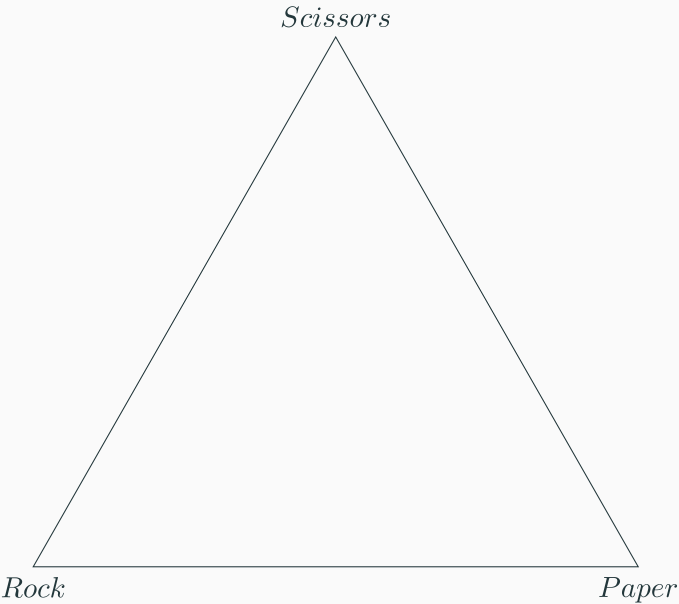
What about a 3x3 game?

		Player 2		
		Rock (r_2)	Paper (p_2)	Scissors ($1 - r_2 - p_2$)
	Rock (r_1)	0, 0	-1, 1	1, -1
Player 1	Paper (p_1)	1, -1	0,0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

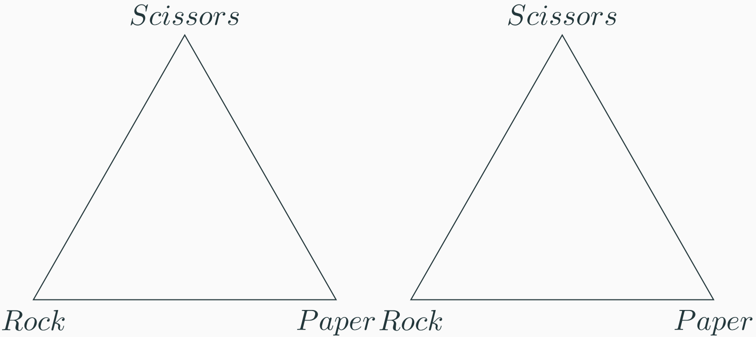
What about a 3x3 game?

- Rock, Paper, Scissors is a *symmetric* game, so let's just pay attention to Player 1's utility
- $EU_1(Rock|r_2, p_2) =$
- $EU_1(Paper|r_2, p_2) =$
- $EU_1(Scissors|r_2, p_2) =$

visualizing Player 1's Best Responses



Finding MSNE in 3x3 game



Finding MSNE in 3x3 game

So the results from our math confirm our intuition that the stable strategies in equilibrium are:

- Player 1 plays Rock with $r = 1/3$, Paper with $p = 1/3$, and Scissors with $1 - p - r = 1/3$
- Player 1 plays Rock with $r = 1/3$, Paper with $p = 1/3$, and Scissors with $1 - p - r = 1/3$

Another 3x3 game

		Player 2		
		Left	Center	Right
	Top	2, 2	0, 0	1, 3
Player 1	Middle	1, 3	3, 0	1, 0
	Bottom	3, 1	2, 3	2, 2

Solving for 3-strategy MSNE

Step 1: Define Mixed Strategies

- Player 1's mixed strategy: Let $\sigma_1 = (t, m, b)$
- Player 2's mixed strategy: Let $\sigma_2 = (\ell, c, r)$

Note that the lowercase letters represent the probabilities played on the uppercase pure strategies.

Solving for 3-strategy MSNE

Step 2: Solve for Expected Utilities

- Player 1:
  - $EU_1(T, \sigma_2) =$
  - $EU_1(M, \sigma_2) =$
  - $EU_1(B, \sigma_2) =$
- Player 2:
  - $EU_2(L, \sigma_1) =$
  - $EU_2(C, \sigma_1) =$
  - $EU_2(R, \sigma_1) =$

Solving for 3-strategy MSNE

Step 3: Find Indifference Conditions

- When will Player 1 mix between 2 pure strategies?
  - When does $EU_1(Top, \sigma_2) = EU_1(Middle, \sigma_2)$:
  - When does $EU_1(Top, \sigma_2) = EU_1(Bottom, \sigma_2)$:
  - When does $EU_1(Middle, \sigma_2) = EU_1(Bottom, \sigma_2)$:

Solving for 3-strategy MSNE

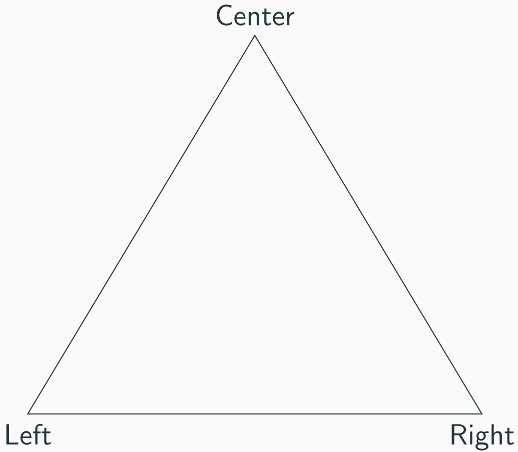
Step 3: Find Indifference Conditions

- When will Player 2 mix between 2 pure strategies?
  - When does $EU_2(Left, \sigma_1) = EU_2(Center, \sigma_1)$:
  - When does $EU_2(Left, \sigma_1) = EU_2(Right, \sigma_1)$:
  - When does $EU_2(Center, \sigma_1) = EU_2(Right, \sigma_1)$:

Step 4.a: Graph Indifference Points on Number Lines for Player 1

Solving for 3-strategy MSNE

Step 4.b: Combine Number Lines into Player 1's BR Triangle

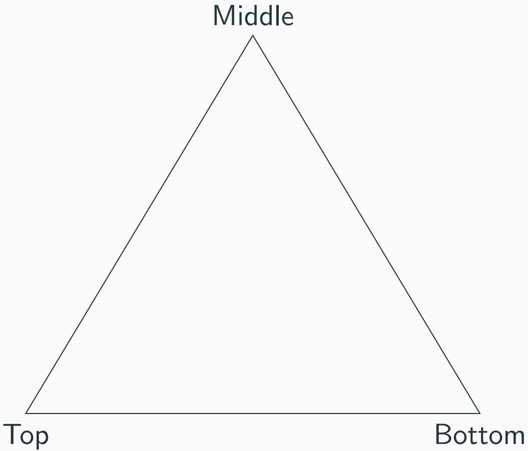


Solving for 3-strategy MSNE

Step 4.c: Graph Indifference Points on Number Lines for Player 2

Solving for 3-strategy MSNE

Step 4.d: Combine Number Lines into Player 2's BR Triangle



Solving for 3-strategy MSNE

Step 5: Check Cases for possible Nash Equilibria: