



Repeated Games

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EC327 Game Theory

- So far, games either **one-shot** simultaneous or **finitely sequential**
- However, how to represent situations in which **repeated interactions** matter?
 - how do **reputation**, **shared history**, etc matter for strategy?
 - power of **institutions** to coordinate cooperation

What do we mean by *cooperative* behavior?

- *altruism*
 - cooperation that ignores self-interest
- **cooperation**
 - cooperation achieved through self-interest
 - rational self-interest may still lead to pro-social behavior
 - more robust policy design to assume everyone is self-interested (but will respond to incentives) than to try to rely on pure altruism

Repeated Games Framework

A **repeated game** is one with potentially infinite subgames

- payoffs are realised in each stage
- each *stage* subgame is identical and itself is a *regular* subgame
- **perfect recall** - common knowledge of shared history
 - strategies may be contingent on observed history
- stage subgame is either repeated *infinitely* or *probabilistically*
- infinite strategies are complicated, so we need **present value** calculations

Trench Warfare as Repeated Prisonners' Dilemma

Trench Warfare in WWI

- On the Western Front, early advances ground to a halt and stagnated into trench warfare
- Technologies like artillery and machine guns made the war one of the bloodiest in human history

Trench Warfare in WWI

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	2, 2	6, 0
	Miss	0, 6	4, 4

What's the NE?

Unexpected Truces Emerge

Christmas Day Truce, 1914:



Image Credit: Stephanie Lecocq/European Pressphoto Agency

Unexpected Truces Emerge

- Despite the conditions of war, unexpected truces emerged
- Five months after hostilities began, soldiers from opposing sides ventured into no man's land to break bread, bury their dead, and exchange prisoners
- other examples of “[Live and let live](#)”
 - Some soldiers would shoot across the trenches at predictable intervals
 - refuse to shoot at supply lines
 - some soldiers refused to fire their weapons at all

The puzzle of trench truces

- How did cooperation between enemy armies achieved and sustained?
- One answer might be that in these parts of the front, interactions were **repeated** between the same units

Constructing a Repeated Game

Suppose that Allied and German forces anticipate that they will play this game T times

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	2, 2	6, 0
	Miss	0, 6	4, 4

- A *strategy* will be made up of T *actions*; one for each time this stage game is played

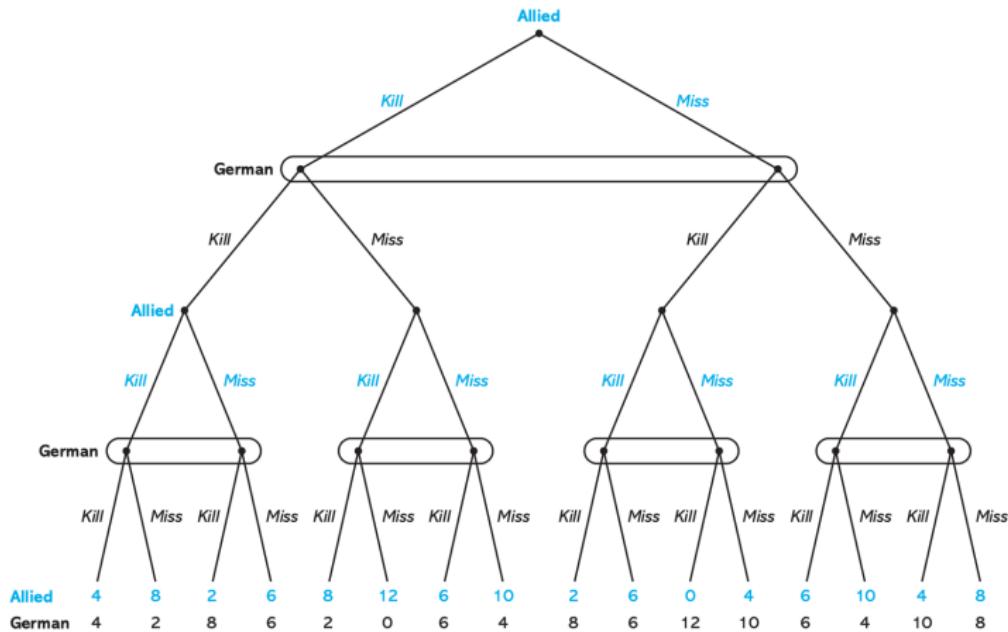
Constructing a Repeated Game

To represent this as an extensive form tree, let's suppose that $T = 2$: and suppose that the history of all past plays are **common knowledge**

- each player will have five info sets; one for day 1, and four in day 2
- What does the extensive form game look like?

2-period Prisoners' Dilemma

FIGURE 13.3 Two-Period Trench Warfare Game with Common Knowledge of the History



What is the SPNE?

T -period Prisoners' Dilemma

Suppose that we are already at the last period T of the T -period trench warfare game

Suppose that the Allies total payoff stream value so far is A^{T-1} and the Germans is G^{T-1}

FIGURE 13.7 Period T Subgame of the T -Period Trench Warfare Game

		German soldiers	
		<i>Kill</i>	<i>Miss</i>
Allied soldiers	<i>Kill</i>	$A^{T-1} + 2, G^{T-1} + 2$	$A^{T-1} + 6, G^{T-1}$
	<i>Miss</i>	$A^{T-1}, G^{T-1} + 6$	$A^{T-1} + 4, G^{T-1} + 4$

What will happen?

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	<i>Miss</i>	$A^{T-1}, G^{T-1} + 6$	$A^{T-1} + 4, G^{T-1} + 4$

What will happen?

- Allies will Shoot to **Kill**, Germans will Shoot to **Kill**

Backwards induction w/ T periods

Now that we know the T stage will end in $(Kill_T, Kill_T)$, we can look one period back to what will happen in $T - 1$:

FIGURE 13.8 Period $T - 1$ Subgame of the T -Period Trench Warfare Game

		German soldiers	
		<i>Kill</i>	<i>Miss</i>
Allied soldiers	<i>Kill</i>	$A^{T-2} + 4, G^{T-2} + 4$	$A^{T-2} + 8, G^{T-2} + 2$
	<i>Miss</i>	$A^{T-2} + 2, G^{T-2} + 8$	$A^{T-2} + 6, G^{T-2} + 6$

What will happen?

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		Kill	Miss
Allied soldiers	Kill	$A^{T-2} + 4, G^{T-2} + 4$	$A^{T-2} + 8, G^{T-2} + 2$
	Miss	$A^{T-2} + 2, G^{T-2} + 8$	$A^{T-2} + 6, G^{T-2} + 6$

What will happen?

- Both will shoot to **Kill** in $T - 1$, knowing they will both shoot to kill in T

Trench Game with Finite stages

By now, you should get the idea:

Insight

If the stage game has a unique NE, then any finitely repeated version will have a unique SPNE which is just the repetition of the single-stage NE. No cooperation is sustainable

So what was going on with those spontaneous truces?

Infinitely Repeated Trench Game

- The problem with that last equilibrium we found was that you have to know *exactly when the game will end* to use backwards induction
- But for World War I infantrymen, they didn't know how long it would be until the fronts shifted or their division was rotated out
- We will have to extend our models to allow for **indefinite horizons**

Repeated Prisoners' Dilemma with Uncertain Second Stage

Suppose that the first stage of the game is the Trenches Game:

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

But with probability p , the game repeats in the second round and with probability $1 - p$, it ends after the first round

Repeated Prisoners' Dilemma with Uncertain Second Stage

Consider the following strategy:

$$\begin{cases} \text{In stage 1} & : Miss_1 \\ \text{In stage 2} & : \begin{cases} Miss_2 \text{ if the other player Missed in stage 1} \\ Kill_2 \text{ if the other player Killed in stage 1} \end{cases} \end{cases}$$

Let's call this strategy *Punisher* because it starts off friendly, but will try to punish someone who defects in the first round by defecting in the second round.

Repeated Prisoners' Dilemma with Uncertain Second Stage

Suppose you are playing against a *Punisher* in this game.

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- What is your **expected utility** of playing $Kill_1$, $Kill_2$?

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		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- What is your **expected utility** of playing $Kill_1$, $Kill_2$?
- 8 in the first stage, 4 in the second stage,
- so $EU(Kill, Kill) = 8 + 4p$

Repeated Prisoners' Dilemma with Uncertain Second Stage

Suppose you are playing against a *Punisher* in this game.

		German soldiers	
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Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

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What is your **expected utility** of playing $Miss_1$, $Kill_2$?

- $6 + 8p$

What about from playing $Kill$, $Miss$?

Repeated Prisoners' Dilemma with Uncertain Second Stage

Suppose you are playing against a *Punisher* in this game.

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

What is your **expected utility** of playing $Miss_1$, $Kill_2$?

- $6 + 8p$

What about from playing $Kill$, $Miss$?

- $8 + 2p$

Would you rather defect earlier or later?

Repeated Prisoners' Dilemma with Uncertain Second Stage

Suppose you are playing against a *Punisher* in this game.

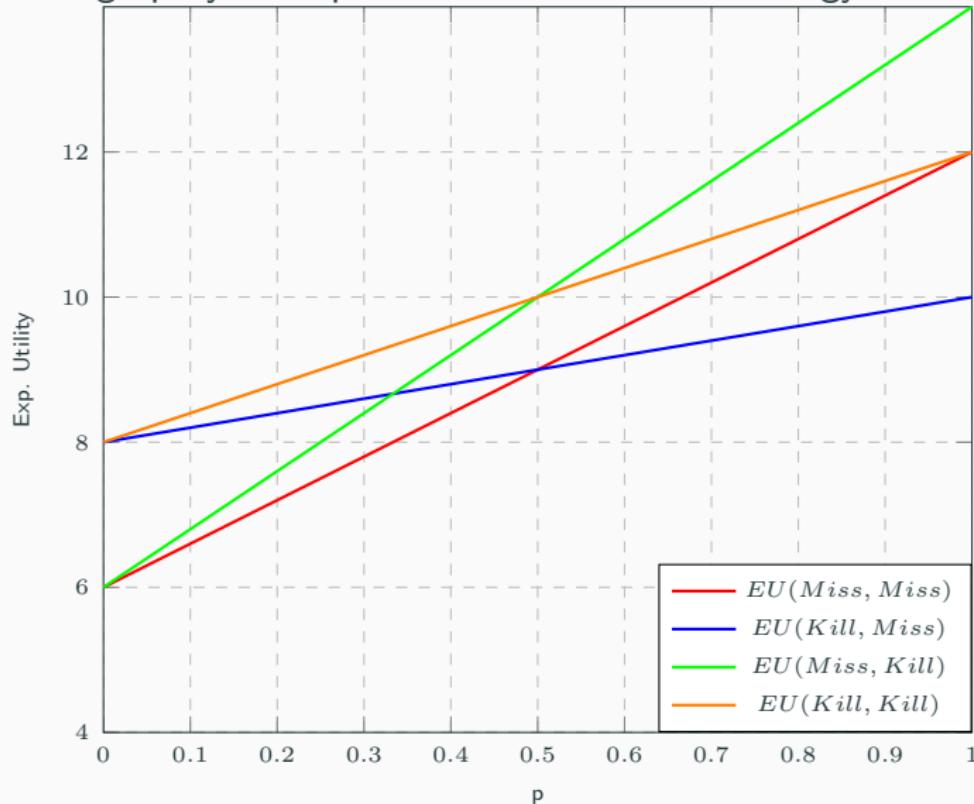
		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

What is your **expected utility** of playing *Miss, Miss*?

- $6 + 6p$

Repeated Prisoners' Dilemma with Uncertain Second Stage

Let's graph your expected utilities of each strategy:



Repeated Prisoners' Dilemma with Uncertain Second Stage

With probabilistic 2nd stage:

- players are incentivized to *Miss* in the first stage if punishment is likely enough
- but still not sufficient to incentivize anyone to *Miss* in the second stage
- what if there are *infinite* stages?

Constructing a Repeated Game

Let's generalize what a strategy in *any repeated game* with **common knowledge** will look like:

- If a game has T periods, and each player has m actions at each stage,
- there is one initial info set, m^2 info sets in period 2, m^4 info sets in period 3, ..., $m^{2(T-1)}$ in the last period
- A complete strategy is made up of $1 + m^2 + m^4 + \dots + m^{2(T-1)}$ actions

In an **infinitely repeated game**, there will be an infinite number of actions in each strategy

Constructing a Repeated Game

How to model streams payoffs over time?

- We could just add up all of the per-stage payoffs across an entire history
- But for infinitely-long histories, this sum would blow up and not make much sense
- Instead, we will use **present value** calculations

Present Values

Suppose that I have an income stream where I earn w_t dollars in every year t

- The **interest rate** r is how much I could earn each year on a risk-free bond
- My present value over my whole income stream is

$$w_1 + \frac{w_2}{(1+r)} + \frac{w_3}{(1+r)^2} + \frac{w_4}{(1+r)^3} + \dots + \frac{w_T}{(1+r)^{T-1}}$$

- This represents the value I place on this income stream vs having the interest from the payment up front
- Money today is worth more than money tomorrow

Infinitely Repeated Games

Suppose the probability that at each stage, with probability p , the game continues and with $(1 - p)$, the game ends and you get $u = 0$

The **expected present value** of a stream of payoffs u_1, u_2, \dots is then:

$$V = u_1 + \frac{pu_2}{(1+r)} + \frac{p^2u_3}{(1+r)^2} + \dots = \sum_{t=1}^{\infty} \left(\frac{p}{(1+r)} \right)^{t-1} u_t$$

Infinitely Repeated Games

Now if we let $\delta = \frac{p}{(1+r)}$ represent the **subjective discount factor**

$$V = \sum_{t=1}^{\infty} (pd)^{t-1} u_t = \sum_{t=1}^{\infty} \delta^{t-1} u_t$$

- δ captures discounting from uncertainty and present value

Present Values

What about calculating a present value of an **infinte stream** of payoffs?

- It turns out:

$$x + \delta x + \delta^2 x + \delta^3 x + \dots + \delta^\infty x$$

- actually converges to $\frac{x}{1-\delta}$ as long as $\delta < 1$

SPNE in Repeated Games

A strategy profile is SPNE if and only if in each period and for each history, the prescribed action is optimal given:

- the other players act according to their strategies in the current period
- all players act according to their strategies in all future periods

Grim Trigger in the Trench Game

Consider the following strategy:

- In period 1, choose miss
- In period $t > 1$, choose miss if both chose miss in all past periods, else choose kill

This type of strategy is known as **Grim Trigger** because this type of player starts out cooperative, but if wronged once, they will always shoot to kill

Cooperative Equilibrium in the Trenches

Revisiting the Christmas Truce:

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- Suppose that the Allies play the Grim Trigger Strategy
- When will the Germans want to Miss?

Cooperative Equilibrium in the Trenches

Revisiting the Christmas Truce:

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- Suppose that the Allies play the Grim Trigger Strategy
- When will the Germans want to Miss?
 - when $pv(\text{Kill always}) < pv(\text{Miss always})$
 - $pv(\text{Kill always}) = 8 + 4\delta + 4\delta^2 + \dots$
 - $pv(\text{Miss always}) = 6 + 6\delta + 6\delta^2 + \dots$
 - So Cheat when $8 + \frac{4\delta}{1-\delta} < 6 + \frac{6\delta}{1-\delta}$
 - or when $\delta > \frac{1}{2}$

Tit for Tat in the Trench Game

- In period 1, choose miss
- In period $t > 1$, choose whatever other player chose in $t - 1$

This type of strategy is known as **Tit-for-Tat** because it copies the opponent's previous strategy

SPNE with Tit-for-Tat

Revisiting the Christmas Truce:

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- Suppose that the Allies play the Tit-for-Tat
- When will the Germans want to Kill once? (and then go back to Missing?)

SPNE with Tit-for-Tat

Revisiting the Christmas Truce:

		German soldiers	
		Kill	Miss
Allied Soldiers	Kill	4, 4	8, 2
	Miss	2, 8	6, 6

- Suppose that the Allies play the Tit-for-Tat
- When will the Germans want to Kill once? (and then go back to Missing?)
 - when $pv(Cheat) < pv(Coop)$
 - $pv(Cheat) = 8 + 2\delta + 6\delta^6 + \dots$
 - $pv(Coop) = 6 + 6\delta + 6\delta^2 + \dots$
 - So Kill once when
$$8 + 2\delta + 6 \sum_{t=3}^{\infty} \delta^{t-1} < 6 + 6\delta + 6 \sum_{t=3}^{\infty} \delta^{t-1}$$
 - or when $\delta > \frac{1}{2}$

Cooperative Equilibrium in the Trenches

Revisiting the Christmas Truce:

- SPNE is achieved using both strategy (Grim Trigger or Tit-for-Tat)
- as long as player have discount rate $\delta > \frac{1}{2}$
- interpret this as more patient \rightarrow more likely to cooperate
- alternatively, the more likely is future repeated play
 - what happened in the trenches as both sides started to cycle troops more often?
 - when will cooperative repeated games equilibria break down?

General Repeated Prisoners' Dilemma

General Prisoners' Dilemma

		Column	
		Defect	Cooperate
Row	Defect	D, D	H, L
	Cooperate	L, H	C, C

What ordering of payoffs D , H , L , and C make this a **Prisoners' Dilemma**?

- a) $C > D > H > L$
- b) $H > D > C > L$
- c) $H > C > D > L$
- d) $C > H > L > D$

General Prisoners' Dilemma

		Column	
		Defect	Cooperate
Row	Defect	D, D	H, L
	Cooperate	L, H	C, C

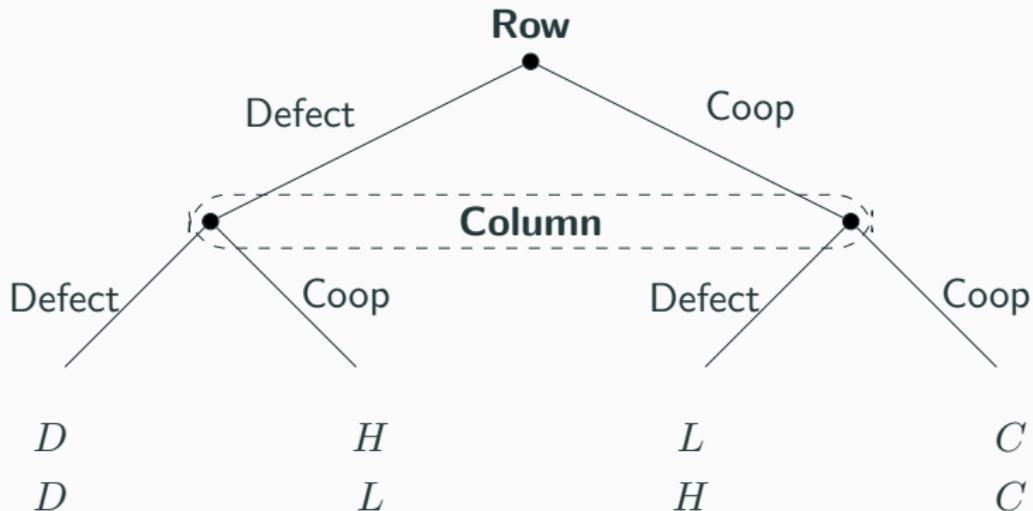
What ordering of payoffs D , H , L , and C make this a **Prisoners' Dilemma**?

- a) $C > D > H > L$
- b) $H > D > C > L$
- c) $H > C > D > L$
- d) $C > H > L > D$

Answer: (c)!

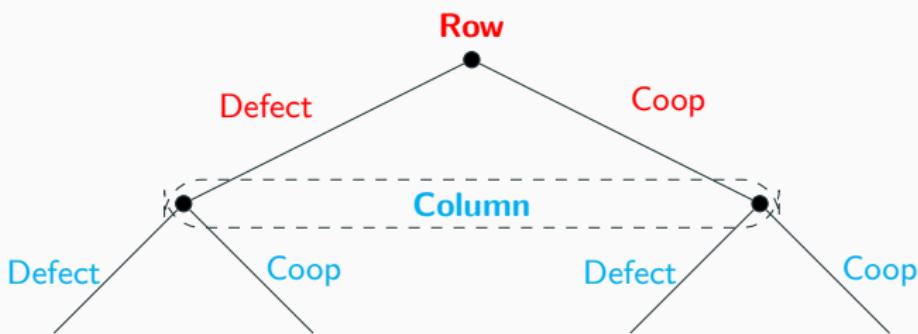
General Prisoners' Dilemma

A single-stage prisoners' dilemma in extensive form:



General Prisoners' Dilemma

A two-stage prisoners' dilemma in mixed extensive form:



	Def.	Coop.									
Def.	$D + \delta D,$ $D + \delta D$	$D + \delta H,$ $D + \delta L$	Def.	$H + \delta D,$ $L + \delta D$	$H + \delta H,$ $L + \delta L$	Def.	$L + \delta D,$ $H + \delta D$	$L + \delta H,$ $H + \delta L$	Def.	$C + \delta D,$ $C + \delta D$	$C + \delta H,$ $C + \delta L$
Coop.	$D + \delta L,$ $D + \delta H$	$D + \delta C,$ $D + \delta C$	Coop.	$H + \delta L,$ $L + \delta H$	$H + \delta C,$ $L + \delta C$	Coop.	$L + \delta L,$ $H + \delta H$	$L + \delta C,$ $H + \delta C$	Coop.	$C + \delta L,$ $C + \delta H$	$C + \delta C,$ $C + \delta C$

Recall that δ is the subjective discount rate from stage to stage

\mathbb{T} -stage repeated Prisoners' Dilemma

A complete strategy in a \mathbb{T} -stage repeated game will look like:

$$S_{t=1}^{\mathbb{T}} = \begin{cases} \text{In stage } t = 1 & \text{take action } A_0 \\ \text{In stage } t > 1 & \begin{cases} \text{If history so far was } h_t, \text{ take action } A_t(h_t) \\ \text{Else if history was } h'_t, \text{ take action } A_t(h'_t) \\ \dots \end{cases} \end{cases}$$

We can see that the number of possible strategies increases exponentially as \mathbb{T} gets larger

\mathbb{T} -stage repeated Prisoners' Dilemma

Suppose that \mathbb{T} is a very large number, but we have played to the very last stage of a repeated Prisoners' Dilemma with that many stages:

		Column	
		Defect	Cooperate
Row	Defect	$\text{Tot.}^R + D, \text{Tot.}^C + D$	$\text{Tot.}^R + H, \text{Tot.}^C + L$
	Cooperate	$\text{Tot.}^R + L, \text{Tot.}^C + H$	$\text{Tot.}^R + C, \text{Tot.}^C + C$

Let Tot.^R and Tot.^C represent the total payoffs that both players have earned over stages 0 to $\mathbb{T} - 1$

\mathbb{T} -stage repeated Prisoners' Dilemma

		Column	
		Defect	Cooperate
Row	Defect	$\text{Tot.}^R + D, \text{Tot.}^C + D$	$\text{Tot.}^R + H, \text{Tot.}^C + L$
	Cooperate	$\text{Tot.}^R + L, \text{Tot.}^C + H$	$\text{Tot.}^R + C, \text{Tot.}^C + C$

Notice that the equilibrium of this subgame is still *Defect, Defect* because Tot.^R and Tot.^C are already decided by prior actions.

Back to the General Form Prisoners' Dilemma

		Column	
		Defect	Cooperate
Row	Defect	D, D	H, L
	Cooperate	L, H	C, C

Now let's suppose that this game is repeated for an **infinite number of stages**.

Extending Plays to Infinity

Suppose the game is in the ‘good’ equilibrium where all players always play *Cooperate*.

- What is **present value** from this equilibrium?

$$\begin{aligned}pv(\{C\}_{t=1}^{\infty}) &= C + \delta C + \delta^2 C + \delta^3 C + \dots \\&= C \sum_{t=1}^{\infty} \delta^{t-1} \\&= C \frac{1}{1 - \delta}\end{aligned}$$

Extending Plays to Infinity

Let's extend the *Punisher* strategy we had from the two-stage game into the *Grim Trigger strategy* of the general infinite horizon game:

$$\begin{cases} \text{In stage 1} & : \text{Cooperate} \\ \text{In stage } t \geq 2 & : \begin{cases} \text{Cooperate if only cooperation has happened so far} \\ \text{Defect if anyone has ever Defected in the past} \end{cases} \end{cases}$$

Grim Trigger SPNE in Repeated PD

Is both players playing *Grim Trigger* stable?

- Does a player have an incentive to *Defect* against *Grim Trigger*:

$$pv(\text{Always Coop}) \geq pv(\text{Defect once})$$

$$C + \delta C + \delta^2 C + \dots \geq H + \delta D + \delta^2 D + \dots$$

$$C + C \sum_{t=2}^{\infty} \delta^t \geq H + D \sum_{t=2}^{\infty} \delta^t$$

$$C + C\delta \sum_{t=2}^{\infty} \delta^{t-1} \geq H + D\delta \sum_{t=2}^{\infty} \delta^{t-1}$$

$$C + \frac{\delta C}{1 - \delta} \geq H + \frac{\delta D}{1 - \delta}$$

$$\frac{\delta}{1 - \delta} \geq \frac{H - C}{C - D}$$

Grim Trigger SPNE in Repeated PD

How do we interpret this statement:

$$\text{Cooperation is stable when } R \geq \frac{C - D}{H - C}$$

- Recall that the definition of the Prisoners' Dilemma was that $H > C > D > L$
- So this means $\frac{H-C}{C-D}$ is positive and less than 1
- As the $H - C$, the relative benefit of defecting increases, it gets harder to sustain cooperation
- It also gets harder to sustain cooperation as the relative penalty of defecting, $H - D$, shrinks

Other Strategies in Repeated Games

So far we've only looked at one example of a type of strategy in repeated game, *Grim Trigger*.

- Can you think of some others?
 - Recall that a complete strategy for a repeated game needs:
 - An initial move at $t = 1$
 - A plan of action for *every* possible history in *every* later stage $t \geq 2$
 - Ideally you would be able to tell a computer how to implement your strategy

Other Strategies in Repeated Games

Telling a computer how to implement strategies is exactly what Robert Axelrod did in a famous tournament in 1980.

- He invited people to submit their programs which would play 200 rounds of the prisoners' dilemma against each other
- The winning program was the one which had the highest total score after playing 200 rounds against all other programs
- What types of strategies do you think would succeed?

An Unexpected Winner

The winning program was named TIT FOR TAT

Surprisingly, it was fairly simple:

$$\begin{cases} \text{In stage 1} & : \text{Cooperate} \\ \text{In stage } t \geq 2 & : \left\{ \begin{array}{l} \text{repeat what the other player did in } t - 1 \end{array} \right. \end{cases}$$

Tit-for-Tat

Like *Grim Trigger*, *Tit-for-Tat* can punish other players for defecting.

- If a player plays *Defect*, it will copy them with *Defect* next round

But unlike *Grim Trigger* it has a short memory; or is very forgiving

- If the player who defected goes back to playing cooperatively, *Tit-for-Tat* will go back to cooperating too

Axelrod's Tournament

If you want to learn more:

- Read the original paper:

Axelrod, Robert; Hamilton, William D. (27 March 1981), "The Evolution of Cooperation" (PDF), *Science*, 211 (4489): 1390–96

- The 1984 Book *The Evolution of Cooperation*, Basic Books
- Run the tournament yourself in python!

<https://github.com/Axelrod-Python/Axelrod>

- Play this fun and short web game!

<https://ncase.me/trust/>

Other Repeated Games

A More Complicated Game

		Player j		
		x	y	z
Player i	x	5, 5	2, 7	1, 3
	y	7, 2	3, 3	0, 1
	z	3, 1	1, 0	2, 2

What are the **pure strategy Nash equilibria** of the *one-shot* game?

A More Complicated Game

		Player j		
		x	y	z
Player i	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

What are the **pure strategy Nash equilibria** of the *one-shot* game?

A More Complicated Game

		Player j		
		x	y	z
Player i	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

What are the **pure strategy Nash equilibria** of the *one-shot* game?

- (y,y) and (z,z)

A More Complicated Game

		Player j		
		x	y	z
Player i	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

What are the **pure strategy Nash equilibria** of the *one-shot* game?

- (y,y) and (z,z)
- Any other strategy profile is not *stable* in the one-shot game
 - for example, (x,x) ; either player would deviate to y

Repeated Game with 3 strategies per period

Now suppose that this game is played repeatedly an infinite number of times.

		Player j		
		x	y	z
Player i	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

- Can we do better than the single period equilibrium?
- Are there any *Pareto improvements* to be made?

Grim Trigger

Player i

$$\begin{cases} t = 1 & \text{Play } x \\ t > 1 & \begin{cases} \text{Play } x \text{ only if } s_j^1 \dots s_j^{t-1} = x \\ \text{Play } y \text{ if anything other than } x \text{ has been played} \end{cases} \end{cases}$$

Player j

- $EV_{Coop} = \frac{5}{1-\delta}$
- $EV_{Cheat} = 7 + \frac{3\delta}{1-\delta}$

Grim Trigger

Solve for the value of δ for which this is a **SPNE**

Grim Trigger

Solve for the value of δ for which this is a **SPNE**

$$\begin{aligned}\frac{5}{1-\delta} &\geq 7 + \frac{3\delta}{1-\delta} \\ 5 + \frac{5\delta}{1-\delta} &\geq 7 + \frac{3\delta}{1-\delta} \\ \frac{(5-3)\delta}{1-\delta} &\geq 7 - 5 \\ \delta &\geq \frac{1}{2}\end{aligned}$$

Grim Trigger with harsher punishment

		Player j		
		x	y	z
Player i	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

The fallback strategy y is not the harshest punishment.

- z is still *credible* because it is another NE of the stage game

Grim Trigger with harsher punishment

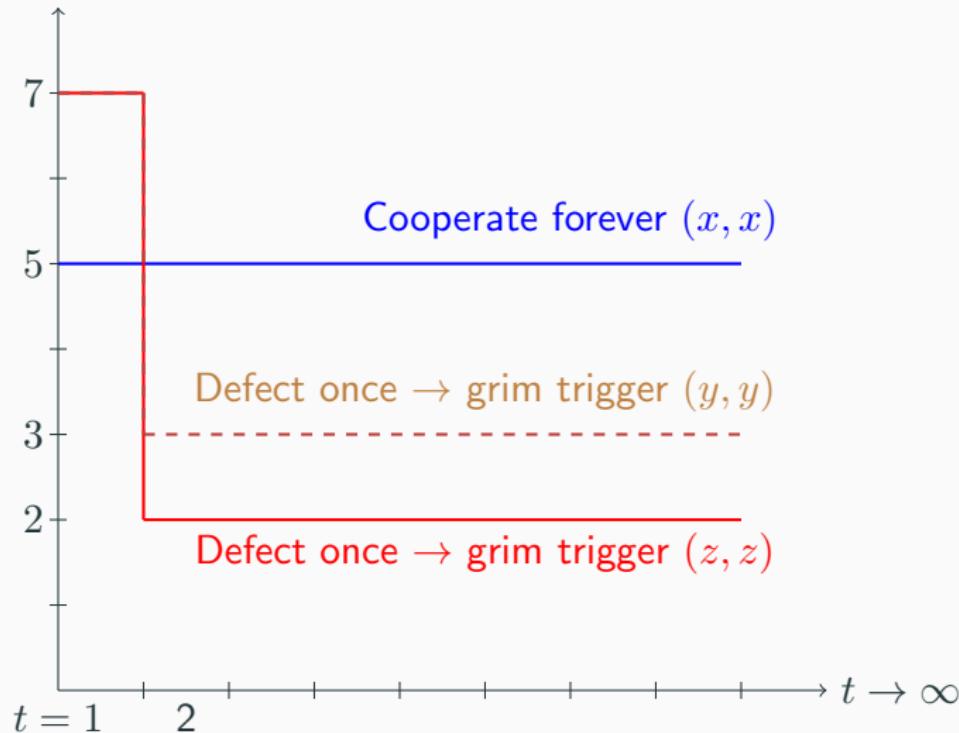
With z as the fallback punishment strategy:

$$\begin{cases} t = 1 & \text{Play } x \\ t > 1 & \begin{cases} \text{Play } x \text{ if only } x \text{ has been played previously} \\ \text{Play } z \text{ if anything other than } x \text{ has been played} \end{cases} \end{cases}$$

$$EU_{\text{Coop}} = 5 + 5\delta + 5\delta^2 \dots = 5 + \frac{5\delta}{1 - \delta}$$

$$EU_{\text{Cheat}} = 7 + 2\delta + 2\delta^2 + \dots = 7 + \frac{2\delta}{1 - \delta}$$

Payoff



Grim Trigger with harsher punishment

Harsher punishments make cooperation easier to sustain:

$$\begin{aligned}\frac{5}{1-\delta} &\geq 7 + \frac{2\delta}{1-\delta} \\ 5 + \frac{5\delta}{1-\delta} &\geq 7 + \frac{2\delta}{1-\delta} \\ \frac{(5-2)\delta}{1-\delta} &\geq 7 - 5 \\ \delta &\geq \frac{2}{5}\end{aligned}$$

Tit-for-Tat

Player i

$$\begin{cases} t = 0 & \text{Play } x \\ t > 0 & \text{Play Player } j \text{'s strategy from } t - 1 \end{cases}$$

Player j

- In period $t = 1$, cheat by playing y to get payoff of 7
- In period $t > 1$, go back to playing cooperatively
- $EV_{\text{Coop}} = 5 + 5\delta + \dots \frac{5}{1-\delta}$
- $EV_{\text{Cheat once}} = 7 + 2\delta + 5\delta^2 + 5\delta^3 + \dots = 7 + 2\delta + \frac{5\delta^2}{1-\delta}$

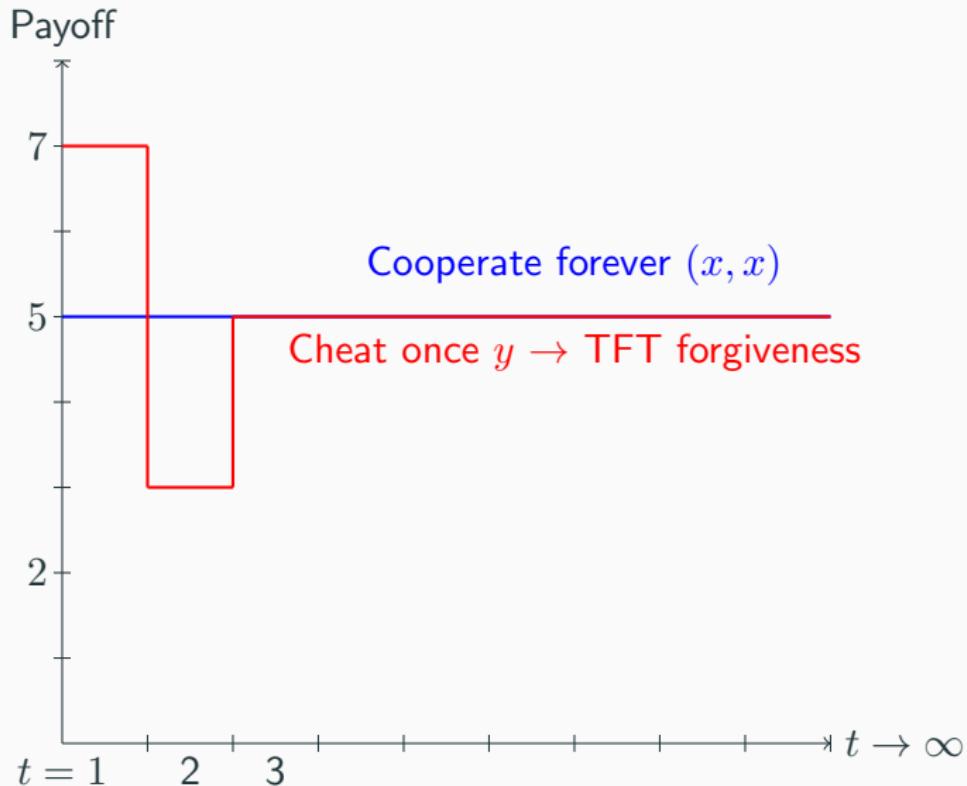
Tit-for-Tat

Solve for the value of δ for which this is a **SPNE**

$$5 + 5\delta + \frac{5\delta^2}{1 - \delta} \geq 7 + 2\delta + \frac{5\delta^2}{1 - \delta}$$

$$5 + 5\delta \geq 7 + 2\delta$$

$$\delta \geq \frac{2}{3}$$



A reciprocating cooperation strategy

Player i

$$\begin{cases} t = 0 & \text{Play } y \\ t > 0 & \begin{cases} \text{Play } y \text{ if } t \text{ is even} \\ \text{Play } x \text{ if } t \text{ is odd} \\ \text{Play } z \text{ forever if P2 played } y \text{ when } t \text{ is even} \end{cases} \end{cases}$$

Player j

$$\begin{cases} t = 0 & \text{Play } x \\ t > 0 & \begin{cases} \text{Play } y \text{ if } t \text{ is odd} \\ \text{Play } x \text{ if } t \text{ is even} \\ \text{Play } z \text{ forever if P2 played } y \text{ when } t \text{ is odd} \end{cases} \end{cases}$$

SPNE with reciprocation

What if Player j defects in period 1?

- $EV_2(\text{Defect}) = 3 + 2\delta + 2\delta^2 + \dots = 3 + \frac{2\delta}{1-\delta}$
- $EV_2(\text{Coop}) = 2 + 7\delta + 2\delta^2 + \dots = \frac{2+7\delta}{1-\delta^2}$

$$\frac{2+7\delta}{1-\delta^2} \geq 3 + \frac{2\delta}{1-\delta}$$

.....

$$\delta^2 + 5\delta - 1 \geq 0$$

$$\delta > \approx 0.65$$

When can cooperation be achieved?

With all of these different ways of achieving repeated cooperation, you might be wondering if there is a way to tell what strategies can actually work

- Punishment outcome must be BR in stage game
- Punishment strategies can't be exploitable (only credible threats)
- Different coordinating strategies are sustainable for different discount factors

Folk Theorem

Any strategy is a potential SPNE for a **repeated** stage game if:

- Both agents are sufficiently patient and far-sighted (high enough δ)
- The payoffs from the cooperative strategy profile satisfy the two properties:
 - **Individually Rational:** the payoffs to each agent (weakly) exceed their minimax payoffs in the stage game
 - **Feasibility:** the payoffs are weighted averages of the payoffs found in the stage game

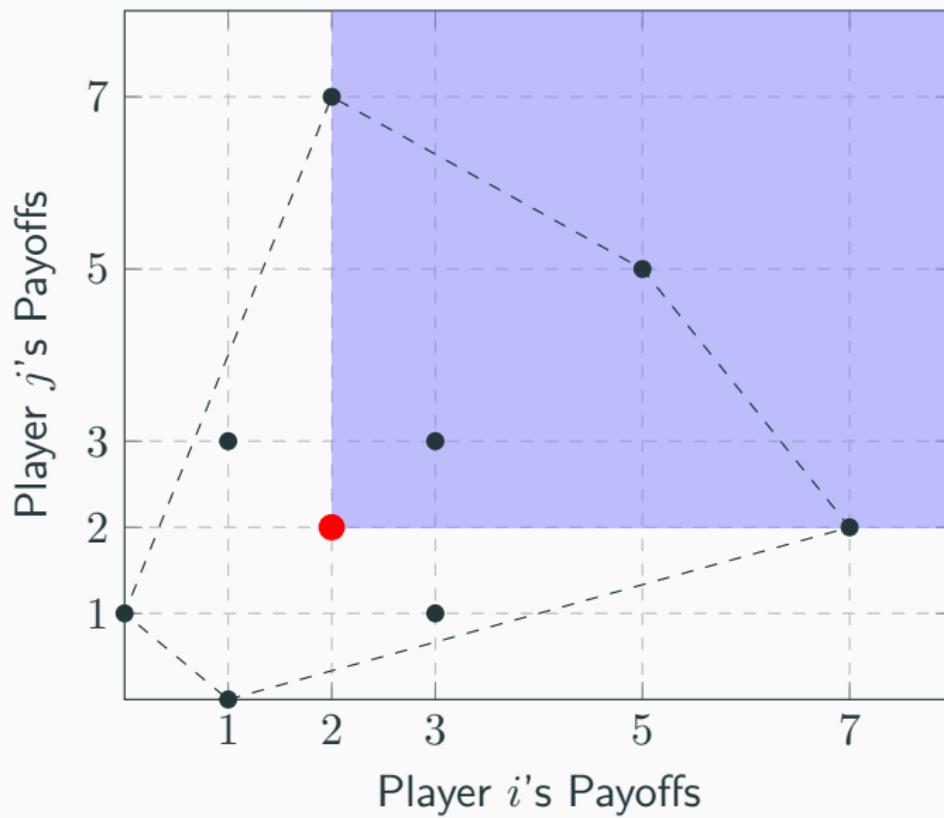
Folk Theorem with 3x3 Repeated Game Example

		Player <i>j</i>		
		x	y	z
Player <i>i</i>	x	5, 5	2, <u>7</u>	1, 3
	y	<u>7</u> , 2	<u>3</u> , <u>3</u>	0, 1
	z	3, 1	1, 0	<u>2</u> , <u>2</u>

The **Minimax** equilibrium is (z,z)

- it *minimizes* the *maximum* payoff that your opponent could get
- The Minimax payoffs in this stage game are (2, 2)
- Intuitively, this is the *safe* option: you can always fall back on it if cooperation fails

Incentive Compatible and Individually Rational conditions in 3x3 repeated game



Folk Theorem with 3x3 Repeated Game Example

- The shaded region of the graph shows us all of the strategy profiles which could be sustained by the **Folk Theorem**
- This shows us why that strategy profile of alternating between (x, y) and (y, x) worked:
 - even though getting 2 on even or odd periods was no better than the Minimax payoffs, because you could alternate with the higher payoff of 7 you could do better as long as you are patient enough
 - this mix between $(2, 7)$ and $(7, 2)$ is *within the convex hull* of feasible payoffs

Importance of the Folk Theorem

Why does this matter for real life?

- Most strategic interactions in your life are repeated
- Even when you don't repeatedly interact with the same exact people, you still see cooperative outcomes
- **Institutions, Reputations, and Social Structures** all serve to allow for past interactions to be common knowledge
- The history of humanity is built on how we arrange our strategic interactions in ways so that people are incentivized to play nice with others

Importance of the Folk Theorem

Some caveats:

- People can't know exactly when the game will end
 - Institutions have to *seem* like they are infinitely lived (compared to finitely lived humans)
- Cooperative equilibria must beat outside options
 - If you make your institution too costly for people to engage with, they will opt out
- People need to be patient enough
- Punishments must be credible