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Repeated Games

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EC327 Game Theory

Outline

- So far, we have only seen games as either **one-shot** simultaneous or **finitely sequential**
- However, these representations can only do so much to represent the many complicated social interactions in which **repeated interactions** between the same players matter
- Specifically, in the Strategic Moves section, we discussed how reputation could play a role, but only allowed it to show up in the single-shot game through changing the payoffs

- In games with **finite** numbers of player actions, we can always use backwards induction to find equilibria
- But often players *do not know* when certain social interactions will end, and so it won't be reasonable to assume that they can backwards induct

- When games are repeated over time, we will use *discount rates* to represent how patient players are
- We can combine this with probability that a game will end at each stage in what we will call an effective rate of return

Trench Warfare as Repeated Prisonners' Dilemma

Trench Warfare in WWI

- On the Western Front, early advances ground to a halt and stagnated into trench warfare
- Technologies like artillery and machine guns made the war one of the bloodiest in human history

Trench Warfare in WWI

FIGURE 13.1 Trench Warfare Game

German soldiers

Allied soldiers		*Kill*	*Miss*
	Kill	2,2	6,0
	Miss	0,6	4,4

What's the **NE**?

Unexpected Truces Emerge

Christmas Day Truce, 1914:

Image Credit: Stephanie Lecocq/European Pressphoto Agency

Unexpected Truces Emerge

In one section [of the camp] the hour of 8 to 9 A.M. was regarded as consecrated to "private business," and certain places indicated by a flag were regarded as out of bounds by snipers on both sides.

So regular were [the Germans] in their choice of targets, times of shooting, and number of rounds fired, that, after being in the line one or two days, Colonel Jones had discovered their system, and knew to a minute where the next shell would fall. His calculations were very accurate, and he was able to take what seemed to uninitiated Staff Officers big risks, knowing that the shelling would stop before he reached the place being shelled.

I was having tea with A Company when we heard a lot of shouting and went out to investigate. We found our men and the Germans standing on their respective parapets. Suddenly a salvo arrived but did no damage. Naturally both sides got down and our men started swearing at the Germans, when all at once a brave German got on to his parapet and shouted out "We are very sorry about that; we hope no one was hurt. It is not our fault, it is that damned Prussian artillery."

The puzzle of trench truces

- How did cooperation between enemy armies achieved and sustained?
- One answer might be that in these parts of the front, interactions were **repeated** between the same units

Constructing a Repeated Game

- Suppose that Allied and German forces anticipate that they will play this game T times

FIGURE 13.1 **Trench Warfare Game**

		German soldiers	
		Kill	*Miss*
Allied soldiers	*Kill*	2,2	6,0
	Miss	0,6	4,4

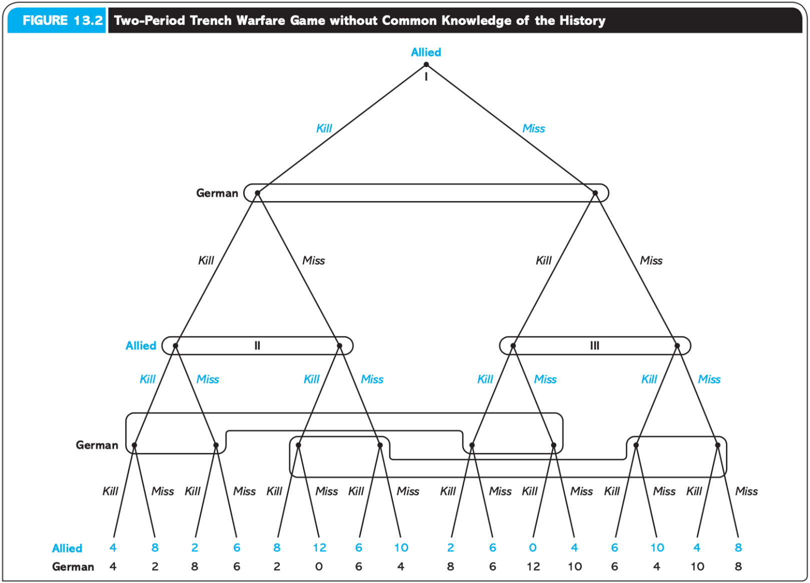
- A *strategy* will be made up of T *actions*; one for each time this stage game is played

Constructing a Repeated Game

To represent this as an extensive form tree, lets suppose that $T = 2$:

- If neither side sees what their opponent's strategy was yesterday, each player has 3 info sets
- So each side has eight possible strategies:
    - ($Kill_1$, $Kill_2$ if $Kill_1$ or $Miss_1$)
    - ($Kill_1$, $Kill_2$ if $Kill_1$ else $Miss_2$ if $Miss_1$)
    - ($Kill_1$, $Miss_2$ if $Kill_1$ else $Kill_2$ if $Miss_1$)
    - ($Kill_1$, $Miss_2$ if $Kill_1$ or $Miss_1$)
    - ...

Constructing a Repeated Game

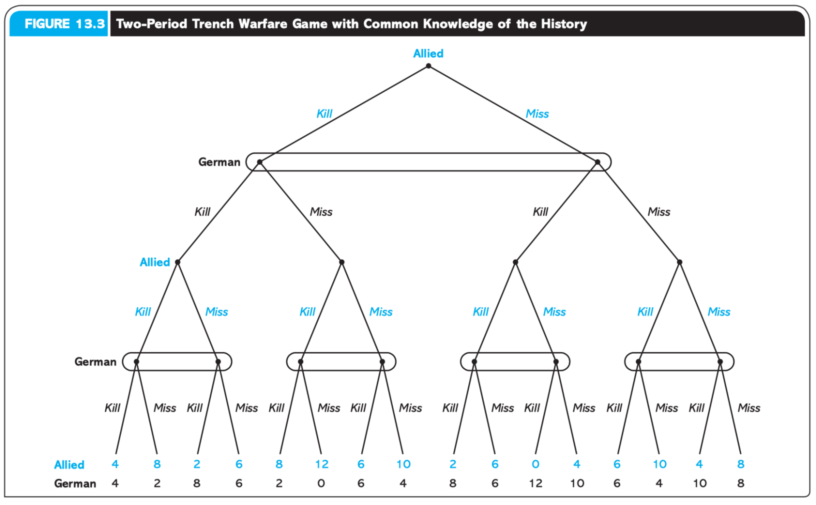


Constructing a Repeated Game

Now suppose that the history of all past plays are **common knowledge**

- Now each player will have five info sets; one for day 1, and four in day 2
- How many strategies does each player have now?

Constructing a Repeated Game



Constructing a Repeated Game

Let's generalize what a strategy in *any* finitely repeated game with common knowledge will look like:

- If a game has T periods, and each player has m actions at each stage,
- there is one initial info set, m^2 info sets in period 2, m^4 info sets in period 3, ..., $m^{2(T-1)}$ in the last period
- A complete strategy is made up of $1 + m^2 + m^4 + ... + m^{2(T-1)}$ actions

In an infinitely repeated game, there will be an infinite number of actions in each strategy

Constructing a Repeated Game

How to model streams payoffs over time?

- We could just add up all of the per-stage payoffs across an entire history
- But for infinitely-long histories, this sum would blow up and not make much sense
- Instead, we will use present value calculations

Present Values

Suppose that I have an income stream where I earn w_t dollars in every year t

- Suppose that there is a single discount factor δ which captures how much I value income tomorrow compared to income today
- My present value over my whole income stream is

$$w_1 + \delta w_2 + \delta^2 w_3 + \delta^3 w_4 + ... + \delta^{T-1} w_T$$

- It makes sense to assume that $0 < \delta < 1$ because I should probably care about tomorrow to some extent, but not as much as today

Present Values

What about calculating a present value of an infinte stream of payoffs?

- It turns out:

$$x + \delta x + \delta^2 x + \delta^3 x + ... + \delta^\infty x$$

- actually converges to $\frac{x}{1-\delta}$ as long as $\delta < 1$

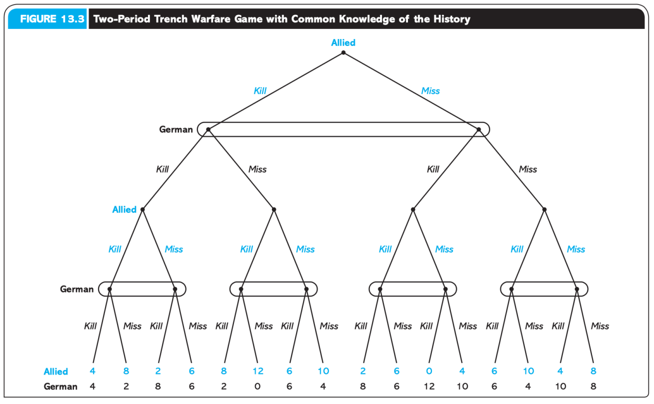
Check Your Understanding

Suppose you are deciding between three different payoff streams:

Period	Stream A	Stream B	Stream C
1	15	25	5
2	15	15	10
3	15	10	20
4	15	5	30

Which has the **highest present value** when $\delta = 0.8$?

Going back to the trenches



This was our extensive form game for only 2 periods

Going back to the trenches

Now suppose that we have a potentially very large T

How can we find a SPNE?

Going back to the trenches

Suppose that we are already at the last period T of the T-period trench warfare game

Suppose that the Allies total payoff stream value so far is A^{T-1} and the Germans is G^{T-1}

FIGURE 13.7 **Period *T* Subgame of the *T*-Period Trench Warfare Game**

		German soldiers	
		Kill	*Miss*
Allied soldiers	*Kill*	$A^{T-1}+2, G^{T-1}+2$	$A^{T-1}+6, G^{T-1}$
	Miss	$A^{T-1}, G^{T-1}+6$	$A^{T-1}+4, G^{T-1}+4$

What will happen?

Going back to the trenches

Now that we know the T stage will end in $(Kill,\ Kill)$, we can look one period back to what will happen in $T-1$:

FIGURE 13.8 **Period $T-1$ Subgame of the T-Period Trench Warfare Game**

		German soldiers	
		Kill	*Miss*
Allied soldiers	*Kill*	$A^{T-2}+4, G^{T-2}+4$	$A^{T-2}+8, G^{T-2}+2$
	Miss	$A^{T-2}+2, G^{T-2}+8$	$A^{T-2}+6, G^{T-2}+6$

What will happen?

Trench Game with Finite stages

By now, you should get the idea:

Insight

If the stage game has a unique NE, then any finitely repeated version will have a unique SPNE which is just the repetition of the single-stage NE. No cooperation is sustainable

So what was going on with those spontaneous truces?

Infinitely Repeated Trench Game

- The problem with that last equilibrium we found was that you have to know *exactly when the game will end* to use backwards induction
- But for World War I infantrymen, they didn't know how long it would be until the fronts shifted or their division was rotated out
- We will have to extend our models to allow for indefinite horizons

Suppose the probability that at each stage, with probability p, the game continues and with $(1-p)$, the game ends and you get $u=0$

The **expected present value** of a stream of payoffs $u_1, u_2, ...$ is then:

$$V = u_1 + pdu_2 + p^2d^2u_3 + ... = \sum_{t=1}^{\infty}(pd)^{t-1}u_t$$

Infinitely Repeated Games

Now if we let $\delta = pd$ represent the discount factor from both time preferences and the likelihood of the game terminating:

$$V = \sum_{t=1}^{\infty} (pd)^{t-1} u_t = \sum_{t=1}^{\infty} \delta^{t-1} u_t$$

Which is exactly the same as the expected present value of a stream of **infinite** payments

SPNE in Repeated Games

Subgame Perfect Nash Equilibrium for a Repeated Game

A strategy profile is SPNE iff in each period and for each history, the prescribed action is optimal given:

1. the other players act according to their strategies in the current period
2. all players act according to their strategies in all future periods

Grim Trigger in the Trench Game

Consider the following strategy:

- In period 1, choose miss
- In period $t > 1$, choose miss if both chose miss in all past periods, else choose kill

This type of strategy is known as Grim Trigger because this type of player starts out cooperative, but if wronged once, they will always shoot to kill