Mixed Strategies

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EC327 Game Theory

Mixed Strategies

Internal Uncertainty

- Now that we discussed *external uncertainty* over *states of nature*, let's talk about internal uncertainty.
- Internal uncertainty occurs when one or more players pick their strategies randomly.
- Picking a strategy at random is really just a different kind of strategy, called a mixed strategy.

Mixed Strategies

- When a player always does the same thing, it's called a pure strategy.
- A mixed strategy assigns a probability to each of a player's pure strategies.
    - Like a lottery, the probabilities in a mixed strategy must all be between 0 and 1,
    - and must sum to exactly 1.

Mixed Strategies

- A mixed strategy can assign 0 probability to a pure strategy.
- It can even assign probability 1 to a single pure strategy, and probability 0 to all others
    - this is still, technically, a mixed strategy, but it is a trivial one.
- When a player uses a mixed strategy, it turns the **other** player's payoffs into lotteries.

Mixed Strategies in the Deer Hunt

Consider the Deer Hunt:

		Ogg: *Deer*	Ogg: *Rabbit*
Igg	*Deer*	$2, 2$	$0, 1$
	Rabbit	$1, 0$	$1, 1$

- Suppose that Igg hunts Deer 3/4 of the time, and Rabbit 1/4 of the time.
- If Ogg *always hunts deer*; what is Ogg's *expected payoff*?

Mixed Strategies in the Deer Hunt: Generalizing

We can generalize this approach to calculate Ogg's expected payoffs from any strategy that Igg chooses to play:

- Suppose that Igg plays Deer with probability p, and Rabbit with probability 1 - p.
- Then Ogg's expected payoff from Deer is:

- Note that Ogg's expected payoff from Deer gets larger with p: the more likely Igg is to hunt Deer, the more attractive an option it becomes for Ogg.

When to Play a Mixed Strategy?

- It's possible for a mixed strategy to be a best response to the other player's strategy:
  - if and only if all of the mixed strategy's components (pure strategies that are assigned positive probability) are best responses too.
- Some intuition: If a strategy is not a best response, you should not play it—even as part of a mixed strategy.

When to play a Mixed Strategy?

If a player only has **two pure strategies**, it becomes simple to tell when a mixed strategy is a best response: the mixed strategy must be a mixture of those two pure strategies, and the only way that both of them are best responses is if they have equal expected payoffs.

- Taking the Deer Hunt as an example, the only way that it can be a best response for Ogg to play a mixed strategy is if Deer and Rabbit provide Ogg with equal expected payoffs: we must have $2p = 1$, or $p = \frac{1}{2}$.

Mixed-Strategy Nash Equilibrium

To solve for the Nash equilibria where players are allowed to use mixed strategies:

we need to look for the conditions under which a player would be willing to use a mixed strategy.

- This means that we're going to use one player's expected payoffs to solve for the **other** player's mixed strategy

MSNE in the Deer Hunt

- Returning to the Deer Hunt, let's say that Igg plays Deer with probability p and Rabbit with probability $1 - p$...
- While Ogg plays Deer with probability q and Rabbit with probability $1 - q$.
    - This is simply a framework for describing each player's mixed strategies: we're using placeholder variables for the players' mixed strategy probabilities

MSNE in the Deer Hunt

- We already saw that Ogg's expected payoffs from Deer and Rabbit are $2p$ and 1, respectively, so Ogg would only play a mixed strategy if $p = \frac{1}{2}$.
- Likewise, Igg's expected payoffs are $2q$ and 1, and Igg will play a mixed strategy if $q = \frac{1}{2}$.
- The MSNE in this game can be written as:

$$\{(1/2\ Deer, 1/2\ Rabbit)_{Ogg},\ (1/2\ Deer, 1/2\ Rabbit)_{Igg}\}$$

.

Error-Checking

- Make sure that you're setting up the equations used to solve for a player's strategy correctly:
  - Remember that you are creating an equation to describe when a player is indifferent between their pure strategies: if you're trying to figure out when **Player 1** is indifferent, you need to use **Player 1's** payoffs.
  - However, when calculating expected payoffs, the probabilities will be based on the **other** player's mixed strategy: in a game with mixed strategies, the randomness a player deals with is created by the **other** player—not themselves.

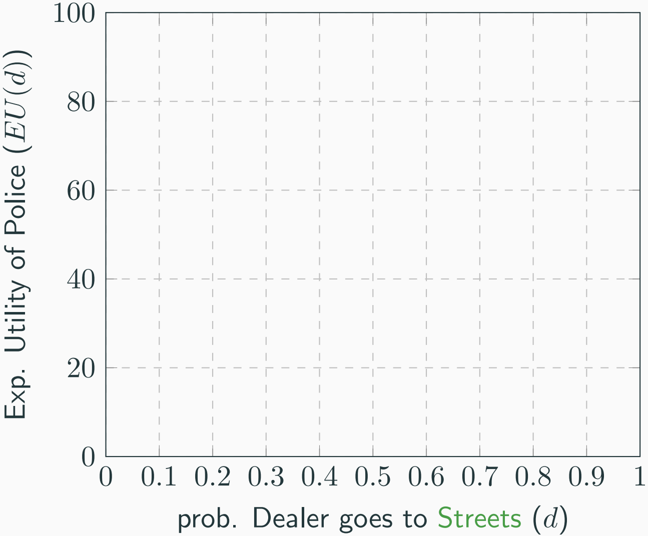
Another Example: Police Patrol and Drug Trade [1]

		Drug Dealer	
		Streets (d)	Park (1 - d)
Police	Streets (p)	80, 20	0,100
	Park (1 - p)	10,90	60,40

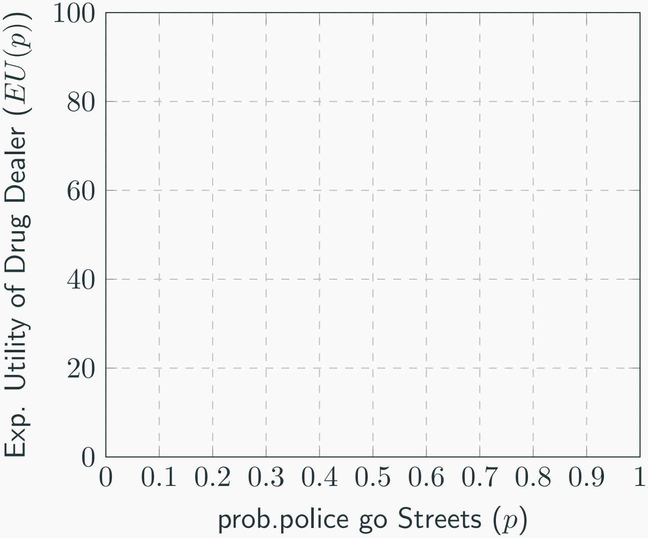
- Police Officer's expected utility:

- Drug Dealer's expected utility:

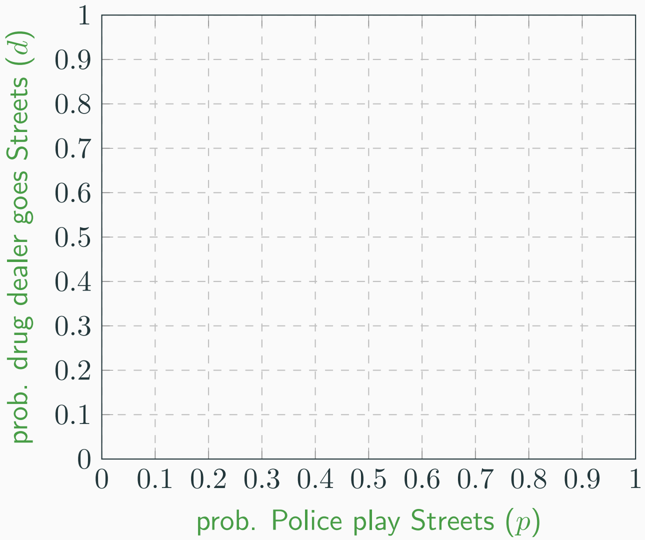
Graph Police Officer's expected utilities



Graph Drug Dealer's expected utilities



Graph Best Response functions



MSNE in Patrol and Trade game:

- When is the Police Officer indifferent between going to the Park and going to the Streets?

- When is the Drug Dealer indifferent between going to the Park and going to the Streets?

- What is the Mixed Strategies Nash equilibrium?

iClicker Q1

- Consider the following game table. What are Player 1's expected payoffs, given Player 2's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

(a) $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$

(b) $U_1(Up) = 3 - 5q, U_1(Down) = 6q - 1$

(c) $U_1(Up) = 5 - 7q, U_1(Down) = 1 - 6p$

(d) $U_1(Up) = 7p - 5, U_1(Down) = 1 - 4p$

(e) $U_1(Up) = 5 - 7p, U_1(Down) = 4p - 1$

iClicker Q2

- Consider the following game table. What are **Player 2's** expected payoffs, given Player 1's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

(a) $U_2(Up) = 5q - 3, U_2(Down) = 1 - 6q$

(b) $U_2(Up) = 3 - 5q, U_2(Down) = 6q - 1$

(c) $U_2(Up) = 5 - 7q, U_2(Down) = 1 - 6p$

(d) $U_2(Up) = 7p - 5, U_2(Down) = 1 - 4p$

(e) $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$

iClicker Q3

- The correct answers to the previous two questions were:
  - $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$.
  - $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$.
- Based on this, what are p and q in the MSNE of this game?
  (a) $p^* = 4/11,\ q^* = 5/11$
  (b) $p^* = 4/11,\ q^* = 6/11$
  (c) $p^* = 6/11,\ q^* = 4/11$
  (d) $p^* = 7/11,\ q^* = 5/11$
  (e) $p^* = 7/11,\ q^* = 6/11$

An MSNE With Only One Mixed Strategy

- Consider the following game table:

		Player 2	
		X (q)	Y $(1-q)$
Player 1	A (p)	2, 2	3, 2
	B $(1-p)$	4, 3	0, 0

- The players' expected payoffs are:
  - $U_1(A) = 2q + 3(1-q) = 2q + 3 - 3q = 3 - q$.
  - $U_1(B) = 4q + 0(1-q) = 4q$.
  - $U_2(X) = 2p + 3(1-p) = 2p + 3 - 3p = 3 - p$.
  - $U_2(Y) = 2p + 0(1-p) = 2p$.

An MSNE With Only One Mixed Strategy

- Based on this, the conditions under which each player will use a mixed strategy are:

$$\begin{aligned} &Player\ 1: & &Player\ 2: \\ &3-q=4q & &3-p=2p \\ &3=5q & &3=3p \\ &q=3/5 & &p=1 \end{aligned}$$

- We've never seen anything like $p = 1$ in this context before...
- $p = 1$ tells us that Player 2 will only play a mixed strategy if Player 1 only play A, which isn't really a mixed strategy at all.
- This usually occurs when one strategy **weakly** dominates another.

An MSNE With Only One Mixed Strategy

- We can still approach this the same way that we have in the past:
- Suppose that in the MSNE, Player 1 plays a (non-trivial) mixed strategy. Then Player 2 must also play a mixed strategy, in which $q = 3/5$.
    - But Player 2 will only play a mixed strategy if Player 1 plays the mixed strategy where $p = 1$...which is a trivial mixed strategy. This is a contradiction, and it means that there is no MSNE where Player 1 plays a non-trivial mixed strategy.

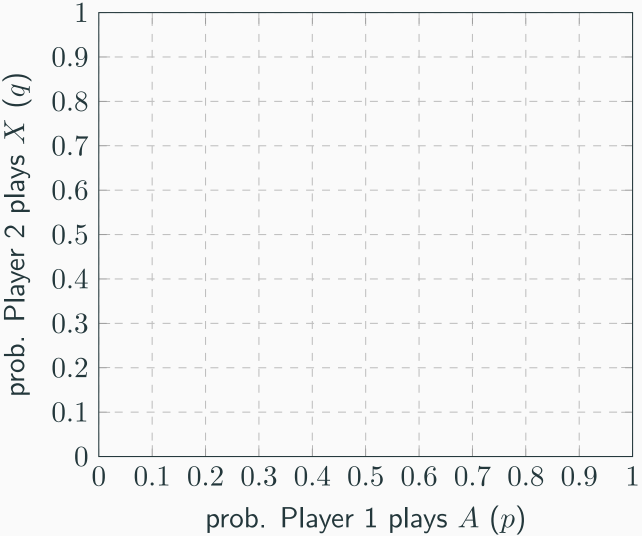
An MSNE With Only One Mixed Strategy

- Approach it the other way next: Suppose Player 2 plays a non-trivial mixed strategy. Then Player 1 must play A as a pure strategy.
    - Player 2 will play A if $3 - q \geq 4q$, i.e. if $3/5 \leq q$.
- This lets Player 2 play a non-trivial mixed strategy! There is no contradiction here.

An MSNE with only one mixed strategy

- There are a range of MSNEs here: all strategy profiles of the form $\{$(1, 0), (q, 1 - q)$\}$, in which $0 < q \leq 3/5$, are MSNEs.
- There are also two trivial MSNEs, $\{$(1, 0), (0, 1)$\}$ and $\{$(0, 1), (1, 0)$\}$, which are really just the pure-strategy Nash equilibria (A, Y) and (B, X) expressed in the form of an MSNE.
- It will help to understand what's going on with the Best Responses graph:

Graph Best Responses



Absence of MSNEs

- Let us return to the Prisoner's Dilemma and check for MSNEs:

		Luca	
		$Testify\ (q)$	$Keep\ Quiet\ (1-q)$
Guido	$Testify\ (p)$	$-10, -10$	$0, -20$
	$Keep\ Quiet\ (1-p)$	$-20, 0$	$-1, -1$

- Guido and Luca's expected payoffs are:
  - $U_G(Testify) = -10q + 0(1-q) = -10q$.
  - $U_G(KeepQuiet) = -20q + (-1)(1-q) = -1 - 19q$.
  - $U_L(Testify) = -10p + 0(1-p) = -10p$.
  - $U_L(KeepQuiet) = -20p + (-1)(1-p) = -1 - 19p$.

Absence of MSNEs

- Guido will play a mixed strategy if:

$$\begin{aligned} -10q &= -1 - 19q \\ 9q &= -1 \\ q &= -1/9 \end{aligned}$$

- But **-1/9 is not a valid probability!**
- We could also note that if $q \in [0,1]$, which is the range for valid probabilities, $-10q$ is always greater than $-1 - 19q$. In other words, as we saw weeks ago, $Testify$ strictly dominates $Keep\ Quiet$...so why would Guido mix between the two of them?

Getting Bad Probabilities

- If you've set up the expected-payoff equation, and solved for a player's mixed strategy, and you find that the probability is less than 0, or more than 1...
- **It means something is wrong.** Probability can only be between 0 and 1 (inclusive).
- First of all, double-check your math—it could be an algebra error.
- But if you're confident in your math, this means that there is **no way that the player would ever play a mixed strategy**: in fact, they have a strictly dominated strategy.
- There will be no MSNE where this player uses a mixed strategy—but there might be MSNEs where the other player does, so you should still check that.

MSNE in a Larger Game

- Suppose that we have this 3×2 game:

		Player 2	
		X (r)	Y (1 - r)
	A (p)	2, 1	0, 1
Player 1	B (q)	1, 2	2, 0
	C (1 - p - q)	0, 0	3, 2

- First, note that in this game, Player 1's mixed strategy uses probabilities p, q, and 1 - p - q, since they have three pure strategies.
- As a rule of thumb, a player's mixed strategy will need one variable less than their number of strategies.

Existence of Nash Equilibria

What is the *pure strategy* Nash Equilibria of the game Rock, Paper, Scissors?

Existence of Nash Equilibria

What is the **Mixed Strategy Nash Equilibria** of the game Rock, Paper, Scissors?

Existence of Nash equilibria

Any with game with a *finite* set of moves will have at least one **Nash equilibrium** when allowing for *mixed strategies*.

The true power of this concept and the most important contribution of its namesake, John Nash, is that it is a simple concept which has universal application.

MSNE in a Larger Game

- To begin with, let's put together Player 1's expected payoffs, of which there will be three:
  - $U_1(A) = 2r + 0 = 2r$.
  - $U_1(B) = 1r + 2(1 - r) = 2 - r$.
  - $U_1(C) = 0 + 3(1 - r) = 3 - 3r$.
- Next, let's see what it would take to get Player 1 to mix different pairs of strategies:
  - A and B: $2r = 2 - r \implies r = \frac{2}{3}$.
  - A and C: $2r = 3 - 3r \implies r = \frac{3}{5}$.
  - B and C: $2 - r = 3 - 3r \implies r = \frac{1}{2}$.
- Note that each pair of strategies requires a different value of r: there is no mixed strategy for Player 2 that would make Player 1 willing to mix all three of their pure strategies.

MSNE in a Larger Game

- Let's check Player 2's expected payoffs next:
  - $U_2(X) = 1p + 2q + 0$.
  - $U_2(Y) = 1p + 0 + 2(1 - p - q)$.
- So Player 2 will play a mixed strategy if $p + 2q = p + 2(1 - p - q) \implies q = 1 - p - q$.
- There are two ways that this can be true: Either Player 1 plays B and C with equal probability (and we know from earlier that they would **only** be playing these two, not A), or Player 1 plays A only, and B and C not at all.

MSNE in a Larger Game

- So, one type of MSNE is where Player 1 only plays A: this requires $2r \geq 2 - r$ and $2r \geq 3 - 3r$, which imply that $r \geq \frac{2}{3}$ and $r \geq \frac{3}{5}$.
  - MSNE: {(1, 0, 0), (r, 1 - r)}, where $r \geq \frac{2}{3}$.
- And the other type of MSNE is where Player 1 plays B and C with equal (1/2) probability, and Player 2 plays X and Y with equal (1/2) probability.
  - MSNE: {(0, 1/2, 1/2), (1/2, 1/2)}