UNIVERSITY OF OREGON

Mixed Strategies

Dante Yasui
adapted from material by Zachary Kiefer

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EC327 Game Theory

Outline

Uncertainty

Thinking about Randomness

> *"I, at any rate, am convinced that [God] does not throw dice"*

- Albert Einstein [1]

Thinking about Randomness

> *"Rational decision makers are able to give reasons for each action they take; outside Las Vegas players do not spin roulette wheels"*

- Ariel Rubenstein [2]

Why should we use mixed strategies?!

- As we saw in Activity 2, rarely do real life data fit in completely deterministic models
- You can interpret the mixed strategy of one player as the beliefs of other players in equilibrium
- Learning some basics of probability theory will help you outside of this class

Lotteries

- In this class, any choices with *uncertain* payoffs will be called lotteries.
- A lottery doesn't have to only be about money
- For any set of outcomes in a lottery, there will be an associated probability: if a is a possible outcome, then $P(a)$ is the probability that it occurs.
  - All probabilities must be between 0 and 1: $0 \leq P(a) \leq 1$ for all possibilities a.
  - Additionally, the probabilities given in a particular lottery must sum to exactly 1: we will assume that one, and exactly one, outcome must actually occur.

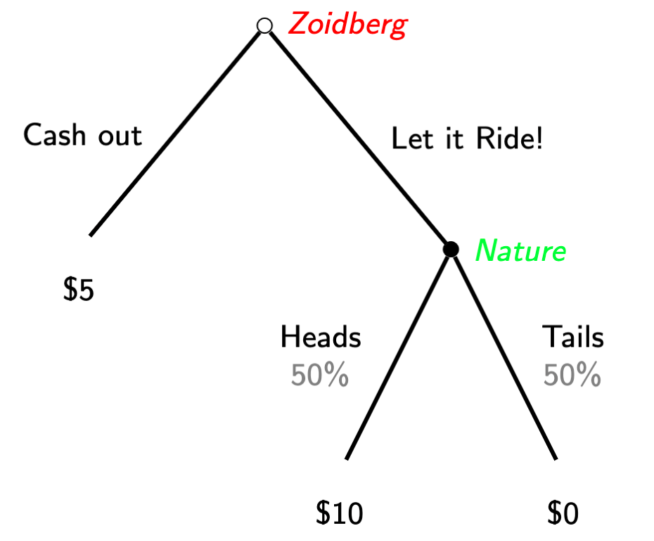
Expected Utility

- How can you know how much you like a given lottery if you don't even know what the outcome will be?
- A natural way would be to think of how happy it would make you *on average*.
- The economists' term for this idea is expected utility or expected payoffs.

Expected Payoffs

- You calculate an average by adding up the values of a lottery times the probability of how likely each is.
  - $U(a) = \mathbb{E}[u(a)] = \sum_a u(a)P(a)$
- This is a weighted average of the payoffs associated with each outcome a, with the weights being the probabilities of each one happening.

Double or Nothing 'game'



Expected Payoff Examples

- Zoidberg's expected payout if he Lets it Ride is $0.50(\$10) + 0.50(\$0) = \$5 - \$0 = \$5$.

- Let's face it, Zoidberg always has luck stacked against him, so suppose instead the coin has a 75% probability of tails,
- The expected payoff would then be $0.25(\$10) + 0.75(\$0) = \$2.50 + \$0 = \$2.50$.

Test Your Understanding

- Suppose that you like sunny weather, and so you get payoff 5 when it is sunny, and -5 when it is rainy. There is a 10% chance of rain and a 90% chance of sun tomorrow. What is the expected payoff of this lottery?
    1. -4.5
    2. -4
    3. 0
    4. 4
    5. 4.5

Test Your Understanding

- Suppose that in addition to a payoff of 5 when it is sunny and -5 when it is rainy, you get a payoff of 2 when it is overcast. There is a 40% chance of rain, a 40% chance of overcast, and a 20% chance of sun tomorrow. What is the expected payoff of this lottery?
  1. -1.2
  2. -0.2
  3. 0
  4. 0.2
  5. 1.2

Cardinal Payoffs

- Using expected payoffs implies that the payoffs of the outcomes are cardinal, not merely ordinal: this means that the payoffs can be used not just to rank, but to compare the relative "goodness" of outcomes.
  - i.e. an outcome with payoff 2 is actually twice as good as an outcome with payoff 1.

Von-Neumann Morgenstern Utility

There are a few other axioms beyond just *completeness* and *transitivity* which are needed when it comes to making rational choices over lotteries.

- Continuity: Small changes in lottery probabilities shouldn't make your ranking jump around.
- Independence: If you know which of two lotteries you prefer, when I add a little bit of another unrelated option into both, it shouldn't change your mind.

Because *expected utility* requires special assumptions beyond those of regular *utility*, it gets it's own special name: Von-Neumann Morgenstern utility function

Types of Uncertainty

- There are two main types of uncertainty that we'll cover in this course: external and internal.
- External uncertainty is the result of factors outside of the game, things that the players don't control, such as weather or other random events.
- Internal uncertainty is the result of the players' own actions inside of the game: it is caused whenever a player acts in a random way.

External Uncertainty: States of Nature

- The simplest form of uncertainty that we'll discuss is simple uncertainty, or uncertainty about the state of nature.
- Under this form of uncertainty, the world may be in one of multiple states of nature, with the payoffs from each strategy profile being different in each state.
- Neither of the players know which state of nature is the correct one—but they do know the probability associated with each state.
    - The probability of each state must be between 0 and 1, and the sum of the probabilities of every state must sum to exactly 1, much like with lotteries.

Example: A Card Game

- Consider the following game, which models a very simple card game.
- There are two players, Doc and Wyatt. Both are dealt a hand of cards: there is a 50% probability that Wyatt's hand is better, and a 50% probability that Doc's is. There are no ties.
- At the beginning of the game, both players must bet $1.
- After seeing his cards, Doc may either Stay, keeping the $1 bet, or may Raise, bringing his bet to $2.
- Doc simultaneously decides to either Match Doc's bet, whatever that may be, or to Quit and forfeit his $1.
- If Wyatt Matches, the player with the better hand wins all the money. If Wyatt Quits, Doc wins all the money by default.

Game Table: Doc's Hand is Better

- In the state of nature where Doc has the better hand, this is the game table:

		Wyatt	
		Match	*Quit*
Doc	*Stay*	$1, -1$	$1, -1$
	Raise	$2, -2$	$1, -1$

Game Table: Wyatt's Hand is Better

- On the other hand, if Wyatt has the better hand, this is the game table:

		Wyatt	
		$Match$	$Quit$
Doc	$Stay$	$-1, 1$	$1, -1$
	$Raise$	$-2, 2$	$1, -1$

Payoffs as Lotteries

- We could easily solve each of these game tables to find the NE in either state of nature—but that doesn't matter, because neither player knows which state really applies.
- Instead, to solve this game, we must approach the payoffs as lotteries:
  - Doc's payoff from strategy profile $(Stay,\ Match)$ is a lottery in which he gains 1 with probability 0.5, and loses 1 with probability 0.5.
  - Wyatt's payoff from the same strategy profile is a lottery in which he loses 1 with probability 0.5, and gains 1 with probability 0.5.
  - the expected payoffs of these lotteries are both 0.
- We can find the expected payoffs associated with each other strategy profile in the same way.

Game Table: Expected Payoffs

		Wyatt	
		$Match$	$Quit$
Doc	$Stay$	$0, 0$	$1, -1$
	$Raise$	$0, 0$	$1, -1$

- It is trivial to see that the Nash equilibria of this game under uncertainty are $(Stay,\ Match)$ and $(Raise,\ Match)$.

Example: The Deer Hunt with Randomness

- Consider a variant of the Deer Hunt game, in which the success of the hunt is random.
- Regardless of what Igg and Ogg decide to hunt, there is a 1/3% probability that there are simply no Deer or Rabbits to be caught, and a 1/6 probability that the Rabbits are four times as plentiful.
- This can be modeled as a game with three states of nature.

Game Table: Normal Wildlife

- In the normal state of nature, which occurs with probability 1/2, we have the normal Deer Hunt:

		Ogg	
		Deer	*Rabbit*
Igg	*Deer*	$2,2$	$0,1$
	Rabbit	$1,0$	$1,1$

Game Table: No Animals

- With 1/3 probability, there are no animals to catch:

		Ogg	
		Deer	*Rabbit*
Igg	*Deer*	$0, 0$	$0, 0$
	Rabbit	$0, 0$	$0, 0$

Game Table: Abundant Rabbits

- With 1/6 probability, there are very high payoffs from hunting Rabbits:

		Ogg	
		Deer	*Rabbit*
Igg	*Deer*	$2, 2$	$0, 4$
	Rabbit	$4, 0$	$4, 4$

Payoffs as Lotteries

- There is a convenient shorthand for finding the expected payoffs to the game:
  - (Deer, Deer): $1/2(2,2) + 1/3(0,0) + 1/6(2,2) = (1,1) + (0,0) + (1/3,1/3) = (4/3,4/3)$.
  - (Deer, Rabbit): $1/2(0,1) + 1/3(0,0) + 1/6(0,4) = (0,1/2) + (0,0) + (0,2/3) = (0,7/6)$.
  - (Rabbit, Deer): $1/2(1,0) + 1/3(0,0) + 1/6(4,0) = (1/2,0) + (0,0) + (2/3,0) = (7/6,0)$.
  - (Rabbit, Rabbit): $1/2(1,1) + 1/3(0,0) + 1/6(4,4) = (1/2,1/2) + (0,0) + (2/3,2/3) = (7/6,7/6)$.

Game Table: Expected Payoffs

		Ogg	
		Deer	*Rabbit*
Igg	*Deer*	$4/3, 4/3$	$0, 7/6$
	Rabbit	$7/6, 0$	$7/6, 7/6$

- Despite the introduction of randomness into this game, the Nash equilibria remain (Deer, Deer) and (Rabbit, Rabbit).

iClicker Q3

- Consider the following game with two states of nature. What are Player 1 and Player 2's expected payoffs from the strategy profile $(South, Split)$? (Choices on next slide.)

Sunny ($p = \frac{1}{3}$)

Player 2

Player 1		$Join$	$Split$
	$North$	$5, 2$	$1, 1$
	$South$	$0, 0$	$2, 5$

Rainy ($p = \frac{2}{3}$)

Player 2

Player 1		$Join$	$Split$
	$North$	$2, 5$	$0, 0$
	$South$	$1, 1$	$5, 2$

iClicker Q3

- What are Player 1 and Player 2's expected payoffs from the strategy profile $(South, Split)$?
    1. (3, 3)
    2. (3, 4)
    3. (4, 3)
    4. (4, 4)
    5. (5, 2)

Variation: Unknown Probabilities

- Suppose that the probability Doc has the better hand is p, and the probability Wyatt has the better hand is therefore $1 - p$. How large does p have to be before there is a Nash equilibrium where Wyatt Quits?
    - We can find the expected payoffs in terms of p:
    - (Stay, Match):
        - Doc's Expected Payoff:
          $1(p) + -1(1-p) = p - 1 + p = 2p - 1.$
        - Wyatt's Expected Payoff:
          $-1(p) + 1(1-p) = -p + 1 - p = 1 - 2p.$

Variation: Unknown Probabilities

- (Raise, Match):
  - Doc's Expected Payoff:
    $2(p) + -2(1-p) = 2p - 2 + 2p = 4p - 2$.
  - Wyatt's Expected Payoff:
    $-2(p) + 2(1-p) = -2p + 2 - 2p = 2 - 4p$.
- (Stay, Quit) and (Raise, Quit):
  - If Wyatt Quits, it doesn't matter who has the better hand: Wyatt loses the $1 he bet already, and Doc gets it. The expected payoffs here are just (1, -1).

Variation: Unknown Probabilities

		Wyatt	
		$Match$	$Quit$
Doc	$Stay$	$2p-1, 1-2p$	$1, -1$
	$Raise$	$4p-2, 2-4p$	$1, -1$

- If Wyatt Quits, Doc doesn't care whether he Stays or Raises—he gets 1 either way. To have a NE where Wyatt Quits, we just need Wyatt to be happy with Quitting.
- $(Stay,\ Quit)$ is a NE if $-1 \geq 1-2p$, i.e. if $p \geq 1$.
- $(Raise,\ Quit)$ is a NE if $-1 \geq 2-4p$, i.e. if $p \geq \frac{3}{4}$.
- One interpretation of this is that it is only rational for Wyatt to Quit if he is very confident that Doc has the better hand.

Variation: Sequential Moves

- It's not realistic to have Doc and Wyatt move at the same time: in actual card games, Doc would decide whether to raise, and Wyatt would then see that move and decide whether to match or quit.
- So let's model this as an extensive-form game and make a game tree:

Sequential Card Game Tree

Information Sets in the Sequential Card Game

- The dashed lines in this game tree denote information sets, showing that neither player knows the state of nature—however, Wyatt does know whether Doc chose to Stay or Raise.
- When Doc and Wyatt reach one of their information sets, they don't know which of its nodes they're actually at, but in this case, because the node is determined by the state of nature, they know the probabilities of each node.
- Once again, we can find the expected payoffs associated with each strategy profile. In fact, they're the same payoffs that we had in the strategic-form game—we just need to fit them into a game tree.

Expected Payoff Game Tree

SPNE in the Sequential-Move Card Game

- Let's look for SPNEs in this game—they will depend on what p is.
- A good place to start is with indifference points, or values of p that make the players indifferent between their choices. Let's look at Wyatt's decisions:
  - In the left-hand subgame (after Doc Stays), Wyatt has the choice between $1 - 2p$ and -1. Indifference implies that $1 - 2p = -1$, or $p = 1$. Wyatt is only indifferent here if Doc is guaranteed to have the better hand; if $p < 1$, Wyatt will simply prefer to Match.
  - In the right-hand subgame (after Doc Raises), $2 - 4p = -1$ implies $p = \frac{3}{4}$. If $p < \frac{3}{4}$, Wyatt will prefer to Match; if $p > \frac{3}{4}$, Wyatt will prefer to Quit.
- It may help to plot this on a number line...

- Now let's consider Doc's decisions:
  - If Wyatt plays (Match, Match), implying that $p \leq \frac{3}{4}$, Doc's choice is between $2p - 1$ and $4p - 2$. Doc is indifferent when $p = \frac{1}{2}$; when p is smaller, he prefers to Stay, and when p is larger, he prefers to Raise.
  - If Wyatt plays (Match, Quit), implying that $\frac{3}{4} \leq p \leq 1$, Doc's choice is between $2p - 1$ and 1. Doc is indifferent if $p = 1$; for $\frac{3}{4} \leq p < 1$, Doc prefers to Raise.
  - If Wyatt plays (Quit, Quit), which he would only do if $p = 1$ exactly, Doc's choice is between -1 and -1: not really a choice at all. Doc is completely indifferent here.

Plotting SPNEs

Summary of SPNE in Sequential-Move Card Game

- We can summarize all of the possible SPNEs as follows:
  - $\{$Stay, (Match, Match)$\}$ is an SPNE if $0 \leq p \leq \frac{1}{2}$.
  - $\{$Raise, (Match, Match)$\}$ is an SPNE if $\frac{1}{2} \leq p \leq \frac{3}{4}$.
  - $\{$Raise, (Match, Quit)$\}$ is an SPNE if $\frac{3}{4} \leq p \leq 1$.
  - $\{$Stay, (Match, Quit)$\}$, $\{$Stay, (Quit, Quit)$\}$, and $\{$Raise, (Quit, Quit)$\}$ are all SPNEs if $p = 1$.

Mixed Strategies

Internal Uncertainty

- Now that we have a good grasp of external uncertainty, in the form of states of nature, let's talk about internal uncertainty.
- Internal uncertainty occurs when the players themselves do something which creates uncertainty or randomness: this basically means that the players pick their strategies randomly.
- Picking a strategy at random is really just a different kind of strategy, called a mixed strategy.

Mixed Strategies

- The kind of strategy we've been working with up until now, in which the player always does the same thing, is called a pure strategy.
- A mixed strategy assigns a probability to each of a player's pure strategies. Much like a lottery, the probabilities in a mixed strategy must all be between 0 and 1, and must sum to exactly 1.
  - A mixed strategy can assign 0 probability to a pure strategy. It can even assign probability 1 to a single pure strategy, and probability 0 to all others; this is still, technically, a mixed strategy, but it is a trivial one.
- When a player uses a mixed strategy, it turns the **other** player's payoffs into lotteries.

Mixed Strategies in the Deer Hunt

- Consider the Deer Hunt:

		Ogg	
		Deer	*Rabbit*
Igg	*Deer*	$2, 2$	$0, 1$
	Rabbit	$1, 0$	$1, 1$

- Suppose that Igg hunts Deer 3/4 of the time, and Rabbit 1/4 of the time. (A mixed strategy assigning equal probability to both pure strategies.)
- If Ogg hunts Deer, then 3/4 of the time, the strategy profile that actually occurs will be (Deer, Deer), and the other 1/4 it will be (Rabbit, Deer).
- Ogg's expected payoff from playing Deer will be $0.75(2) + 0.25(0) = 1.5$.

Mixed Strategies in the Deer Hunt: Generalizing

- We can generalize this approach to calculate Ogg's expected payoffs from any strategy that Igg chooses to play:
- Suppose that Igg plays Deer with probability p, and Rabbit with probability 1 - p.
- Then Ogg's expected payoff from Deer is $2(p) + 0(1-p) = 2p$, and from Rabbit, it is $1(p) + 1(1-p) = 1$.
- Note that Ogg's expected payoff from Deer gets larger with p: the more likely Igg is to hunt Deer, the more attractive an option it becomes for Ogg.

When to Play a Mixed Strategy?

- It's possible for a mixed strategy to be a best response to the other player's strategy: this is the case if and only if all of the mixed strategy's components (pure strategies that are assigned positive probability) are best responses too.
- Some intuition: If a strategy is not a best response, you should not play it—even as part of a mixed strategy.
- If a player only has two pure strategies, it becomes simple to tell when a mixed strategy is a best response: the mixed strategy must be a mixture of those two pure strategies, and the only way that both of them are best responses is if they have equal expected payoffs.
- Taking the Deer Hunt as an example, the only way that it can be a best response for Ogg to play a mixed strategy is if Deer and Rabbit provide Ogg with equal expected payoffs: we must have $2p = 1$, or $p = \frac{1}{2}$.

What Mixed Strategy to Play

- However, if any mixed strategy is a best response, then **all** mixed strategies (with the same components) are also best responses.
- Intuitively, if the pure strategies going into a mixed strategy are just as good as each other, then it doesn't matter what proportions you mix them in.
- This means that, while it's easy to solve for **when** it's rational for a player to use a mixed strategy, there's no way to solve for a particular mixed strategy that the player **should** play.

Mixed-Strategy Nash Equilibrium

- We still have a way to solve for the Nash equilibria when the players are allowed to use mixed strategies: we will solve for the conditions under which a player would be willing to use a mixed strategy.
- This means that we're going to use one player's expected payoffs to solve for the **other** player's mixed strategy: it's a little bit different from what we've done before.

MSNE in the Deer Hunt

- Returning to the Deer Hunt, let's say that Igg plays Deer with probability p and Rabbit with probability $1 - p$...
- While Ogg plays Deer with probability q and Rabbit with probability $1 - q$.
    - This is simply a framework for describing each player's mixed strategies: we're saying that Igg and Ogg each play Deer some of the time (p or q) and Rabbit the rest of the time (1 - p or 1 - q).
- We already saw that Ogg's expected payoffs from Deer and Rabbit are $2p$ and 1, respectively, and that Ogg would only play a mixed strategy if $p = \frac{1}{2}$.
- Likewise, Igg's expected payoffs are $2q$ and 1, and Igg will play a mixed strategy if $q = \frac{1}{2}$.
- The MSNE in this game can be written as $\{$(p, 1 - p), (q, 1 - q)$\}$ $=$ $\{$(1/2, 1/2), (1/2, 1/2)$\}$.

Error-Checking

- As mentioned, this way of solving for MSNEs is unlike what we've done before—and it can be counterintuitive at first.
- It's important to make sure that you're setting up the equations used to solve for a player's strategy correctly:
  - Remember that you are creating an equation to describe when a player is indifferent between their pure strategies: if you're trying to figure out when **Player 1** is indifferent, you need to use **Player 1's** payoffs.
  - However, when calculating expected payoffs, the probabilities will be based on the **other** player's mixed strategy: in a game with mixed strategies, the randomness a player deals with is created by the **other** player—not themselves.

Another Example: Bach or Stravinsky

		Stravinsky Fan	
		Bach (q)	Strav. (1 - q)
Bach Fan	Bach (p)	3, 2	0,0
	Strav. (1 - p)	0,0	2, 3

- To begin with, we're going to set up the players' mixed strategies the same way as in the Deer Hunt (as shown in the table).
- The Bach Fan's expected payoffs will be $U_B(Bach) = 3q + 0(1-q) = 3q$ and $U_B(Strav.) = 0q + 2(1-q) = 2 - 2q$.
- Likewise, the Stravinsky Fan will have payoffs $U_S(Bach) = 2p$ and $U_S(Strav.) = 3 - 3p$.

MSNE in Bach or Stravinsky

- The Bach Fan will be indifferent between Bach and Stravinsky (and thus willing to play a mixed strategy) when $3q = 2 - 2q$, or when $q = \frac{2}{5}$.
- The Stravinsky Fan will be indifferent when $2p = 3 - 3p$, or when $p = \frac{3}{5}$.
- Thus, the MSNE in this game will be $\{$(0.6, 0.4), (0.4, 0.6)$\}$.
- Note that the players' asymmetric preferences result in each of them buying a ticket for their more preferred concert most of the time in this MSNE.
- If we gave them stronger preferences (i.e. increased the amount by which they prefer their favorite composer), it would amplify this effect in the MSNE.

iClicker Q1

- Consider the following game table. What are Player 1's expected payoffs, given Player 2's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

1. $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$
2. $U_1(Up) = 3 - 5q, U_1(Down) = 6q - 1$
3. $U_1(Up) = 5 - 7q, U_1(Down) = 1 - 6p$
4. $U_1(Up) = 7p - 5, U_1(Down) = 1 - 4p$
5. $U_1(Up) = 5 - 7p, U_1(Down) = 4p - 1$

iClicker Q2

- Consider the following game table. What are **Player 2's** expected payoffs, given Player 1's mixed strategy?

		Player 2	
		$Up(q)$	$Down(1-q)$
Player 1	$Up(p)$	2, -2	-3, 3
	$Down(1-p)$	-5, 5	1, -1

1. $U_2(Up) = 5q - 3, U_2(Down) = 1 - 6q$
2. $U_2(Up) = 3 - 5q, U_2(Down) = 6q - 1$
3. $U_2(Up) = 5 - 7q, U_2(Down) = 1 - 6p$
4. $U_2(Up) = 7p - 5, U_2(Down) = 1 - 4p$
5. $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$

iClicker Q3

- The correct answers to the previous two questions were:
  - $U_1(Up) = 5q - 3, U_1(Down) = 1 - 6q$.
  - $U_2(Up) = 5 - 7p, U_2(Down) = 4p - 1$.
- Based on this, what are p and q in the MSNE of this game?
  1. $p = 4/11, q = 5/11$
  2. $p = 4/11, q = 6/11$
  3. $p = 6/11, q = 4/11$
  4. $p = 7/11, q = 5/11$
  5. $p = 7/11, q = 6/11$

An MSNE With Only One Mixed Strategy

- Consider the following game table:

		Player 2	
		X (q)	Y $(1-q)$
Player 1	A (p)	2, 2	3, 2
	B $(1-p)$	4, 3	0, 0

- The players' expected payoffs are:
  - $U_1(A) = 2q + 3(1-q) = 2q + 3 - 3q = 3 - q$.
  - $U_1(B) = 4q + 0(1-q) = 4q$.
  - $U_2(X) = 2p + 3(1-p) = 2p + 3 - 3p = 3 - p$.
  - $U_2(Y) = 2p + 0(1-p) = 2p$.

An MSNE With Only One Mixed Strategy

- Based on this, the conditions under which each player will use a mixed strategy are:

$$\begin{aligned} &Player\ 1: && Player\ 2: \\ &3 - q = 4q && 3 - p = 2p \\ &3 = 5q && 3 = 3p \\ &q = 3/5 && p = 1 \end{aligned}$$

- We've never seen anything like $p = 1$ in this context before...
- $p = 1$ tells us that Player 2 will only play a mixed strategy if Player 1 plays the mixed strategy where p = 1...in other words, if they only play A, which isn't really a mixed strategy at all.

An MSNE With Only One Mixed Strategy

- We can still approach this the same way that we have in the past:
- Suppose that in the MSNE, Player 1 plays a (non-trivial) mixed strategy. Then Player 2 must also play a mixed strategy, in which $q = 3/5$.
  - But Player 2 will only play a mixed strategy if Player 1 plays the mixed strategy where $p = 1$...which is a trivial mixed strategy. This is a contradiction, and it means that there is no MSNE where Player 1 plays a non-trivial mixed strategy.

An MSNE With Only One Mixed Strategy

- Approach it the other way next: Suppose Player 2 plays a non-trivial mixed strategy. Then Player 1 must play A as a pure strategy.
  - Player 2 will play A if $3 - q \geq 4q$, i.e. if $3/5 \geq q$.
- This lets Player 2 play a non-trivial mixed strategy! There is no contradiction here.
- There are a range of MSNEs here: all strategy profiles of the form $\{$(1, 0), (q, 1 - q)$\}$, in which $q \in (0, 3/5]$, are MSNEs.
- There are also two trivial MSNEs, $\{$(1, 0), (0, 1)$\}$ and $\{$(0, 1), (1, 0)$\}$, which are really just the pure-strategy Nash equilibria (A, Y) and (B, X) expressed in the form of an MSNE.
- We will cover a more methodical way to find MSNEs like this later.

Absence of MSNEs

- Let us return to the Prisoner's Dilemma and check for MSNEs:

		Luca	
		$Testify\ (q)$	$Keep\ Quiet\ (1-q)$
Guido	$Testify\ (p)$	$-10, -10$	$0, -20$
	$Keep\ Quiet\ (1-p)$	$-20, 0$	$-1, -1$

- Guido and Luca's expected payoffs are:
  - $U_G(Testify) = -10q + 0(1-q) = -10q$.
  - $U_G(KeepQuiet) = -20q + (-1)(1-q) = -1 - 19q$.
  - $U_L(Testify) = -10p + 0(1-p) = -10p$.
  - $U_L(KeepQuiet) = -20p + (-1)(1-p) = -1 - 19p$.

Absence of MSNEs

- Guido will play a mixed strategy if:

$$
\begin{aligned}
-10q &= -1 - 19q \\
9q &= -1 \\
q &= -1/9
\end{aligned}
$$

- But -1/9 is not a valid probability...
- We could also note that if $q \in [0, 1]$, which is the range for valid probabilities, $-10q$ is always greater than $-1 - 19q$. In other words, as we saw weeks ago, $Testify$ strictly dominates $Keep\ Quiet$...so why would Guido mix between the two of them?
- The same logic applies to Luca as well: neither Guido or Luca

Getting Bad Probabilities

- If you've set up the expected-payoff equation, and solved for a player's mixed strategy, and you find that the probability is less than 0, or more than 1...
- **It means something is wrong.** Probability can only be between 0 and 1 (inclusive).
- First of all, double-check your math—it could be an algebra error.
- But if you're confident in your math, this means that there is **no way that the player would ever play a mixed strategy**: in fact, they have a strictly dominated strategy.
- There will be no MSNE where this player uses a mixed strategy—but there might be MSNEs where the other player does, so you should still check that.

MSNE in a Larger Game

- Suppose that we have this 3×2 game:

		Player 2	
		X (r)	Y (1 - r)
Player 1	A (p)	2, 1	0, 1
	B (q)	1, 2	2, 0
	C (1 - p - q)	0, 0	3, 2

- First, note that in this game, Player 1's mixed strategy uses probabilities p, q, and 1 - p - q, since they have three pure strategies.
- As a rule of thumb, a player's mixed strategy will need one variable less than their number of strategies.

MSNE in a Larger Game

- To begin with, let's put together Player 1's expected payoffs, of which there will be three:
  - $U_1(A) = 2r + 0 = 2r$.
  - $U_1(B) = 1r + 2(1 - r) = 2 - r$.
  - $U_1(C) = 0 + 3(1 - r) = 3 - 3r$.
- Next, let's see what it would take to get Player 1 to mix different pairs of strategies:
  - A and B: $2r = 2 - r \implies r = \frac{2}{3}$.
  - A and C: $2r = 3 - 3r \implies r = \frac{3}{5}$.
  - B and C: $2 - r = 3 - 3r \implies r = \frac{1}{2}$.
- Note that each pair of strategies requires a different value of r: there is no mixed strategy for Player 2 that would make Player 1 willing to mix all three of their pure strategies.

MSNE in a Larger Game

- Let's check Player 2's expected payoffs next:
  - $U_2(X) = 1p + 2q + 0$.
  - $U_2(Y) = 1p + 0 + 2(1 - p - q)$.
- So Player 2 will play a mixed strategy if $p + 2q = p + 2(1 - p - q) \implies q = 1 - p - q$.
- There are two ways that this can be true: Either Player 1 plays B and C with equal probability (and we know from earlier that they would **only** be playing these two, not A), or Player 1 plays A only, and B and C not at all.

MSNE in a Larger Game

- So, one type of MSNE is where Player 1 only plays A: this requires $2r \geq 2 - r$ and $2r \geq 3 - 3r$, which imply that $r \geq \frac{2}{3}$ and $r \geq \frac{3}{5}$.
    - MSNE: $\{$(1, 0, 0), (r, 1 - r)$\}$, where $r \geq \frac{2}{3}$.
- And the other type of MSNE is where Player 1 plays B and C with equal (1/2) probability, and Player 2 plays X and Y with equal (1/2) probability.
    - MSNE: $\{$(0, 1/2, 1/2), (1/2, 1/2)$\}$