Introduction to Game Theory

Sequential Games

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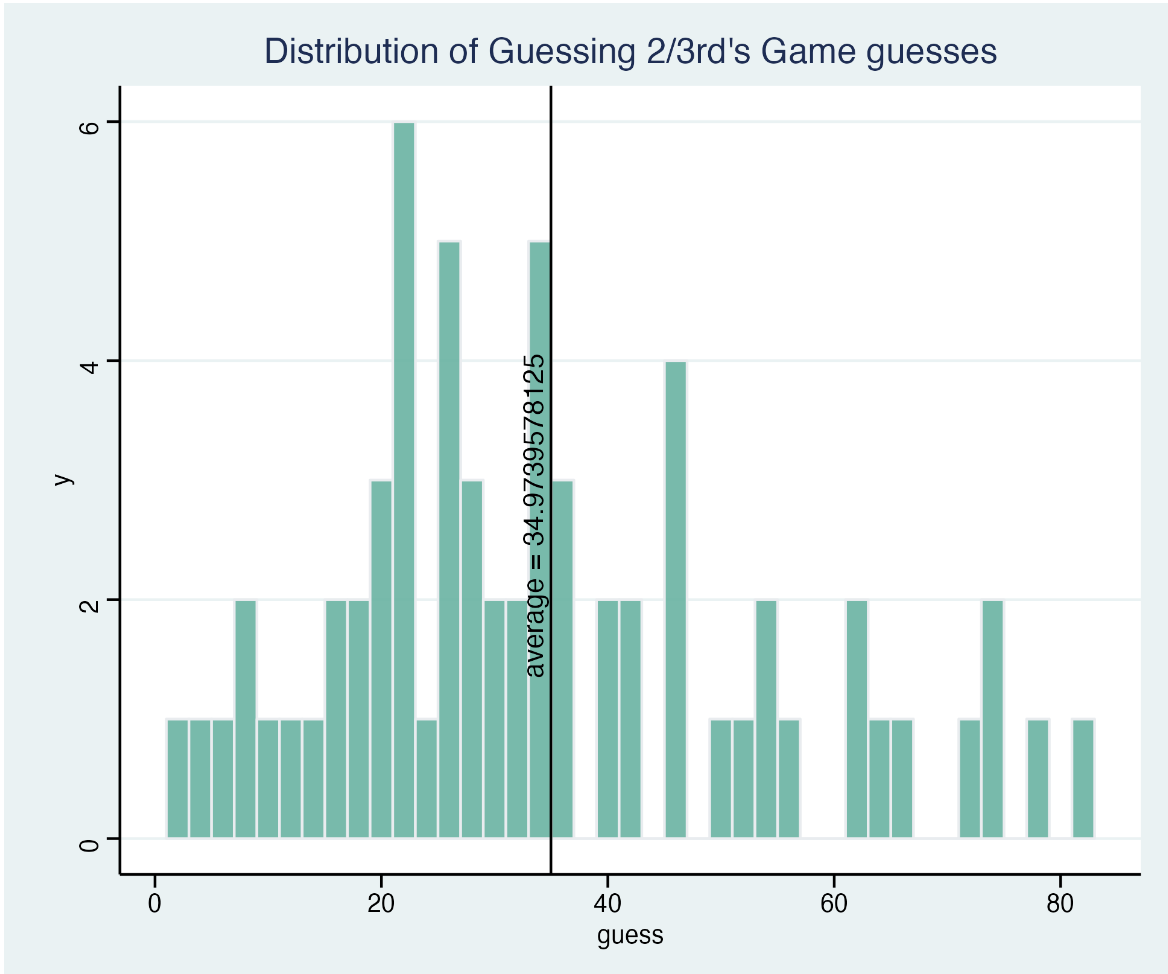
2024

Apologies to: 😓

Drew Rampelberg and Sean Loder



histogram

Outline

- Activity 2: *Survivor*
- Game trees
- Backwards Induction

Activity 2: Survivor Challenge

Watch the clip

https://youtu.be/aonCsvi0LKc

Split Into Groups

Group	Survivor 15	Survivor 16	Survivor 3	Survivor 10	Survivor 9	Survi 2
1	Zaki Al-Hardan	Geedi Ali	Charlie Anderson	Job Aquino-Rangel	Gabriel Arciga	Jenni Barn
2	Chay Bick	Cody Chase	Arthur Fargher	Mizuki Nakano	Eli Brenn	Isaac
3	Tobias Miller	Josh Gassner	Alden McVay	Oscar Oeding	Scott O'Meara	Matt Maba
4	Carson Powell	Deegan Smith	Justin White	Drew Rampelberg	Cara Reinke	Drew Mora

Set Up

- Find an opposing group
- Pick someone to record flags for each round
- Draw 21 flags on whiteboard
- Discuss strategy w/ your group for 1 minute
- Play the game!
    - Don't forget to record how many flags each round!!!

Modified Rules: choose number of flags to start

Find a new team to play against (losers move)

- This time, one team will choose the number of starting flags: between 15-25
- The other team will choose whether to allow max. of 2-5 flags per round
- Play and record!

Discuss

In your teams briefly discuss the following:

- Is there a correct way to play the game?
- How did you find out the best way to play the game?
- Is the game solved meaning that the winner can be correctly predicted from any point in the game?
  - If so, should the first or second team always win?
- Were there more errors in the first games played than the last games played?
  - Why?
- How was this game different from the guessing 2/3rds game?

Can You Solve The 21 Flags Game From Survivor?

Recap

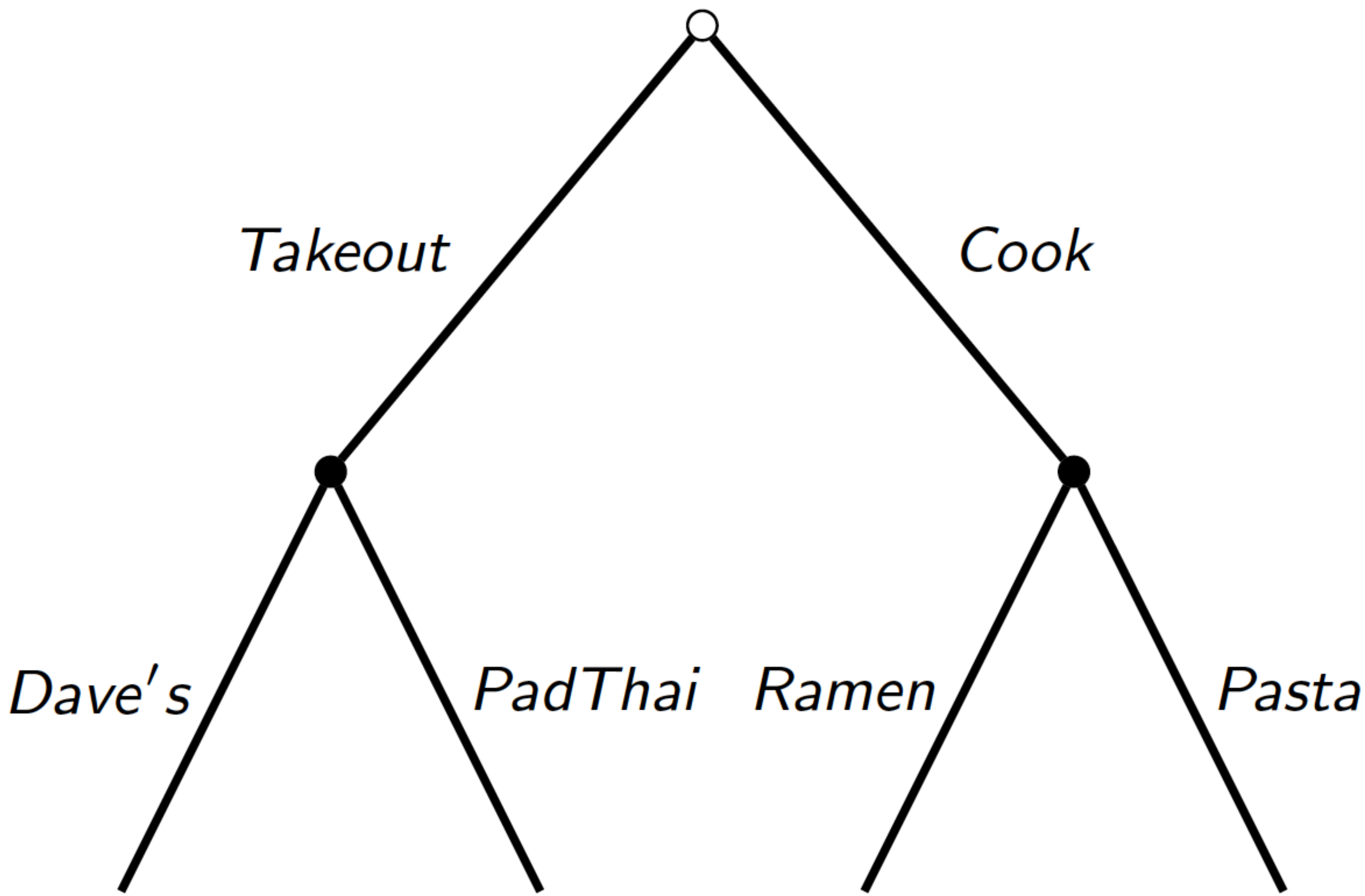
- There *is* a correct way to play: we will later call the *Nash equilibrium*
- You likely used some form of *backwards induction* or *rollback* reasoning
- This game was *sequential* unlike the *simultaneous* 2/3rds game

Extensive Form

Game Trees/Extensive Form as a tool

- Before we learn how to solve a game, it will helpful to be able to visualize them
- Because of the ordered nature of sequential games, a **tree diagram** makes sense

A *Decision* Tree



- Is this a *strategic* decision?

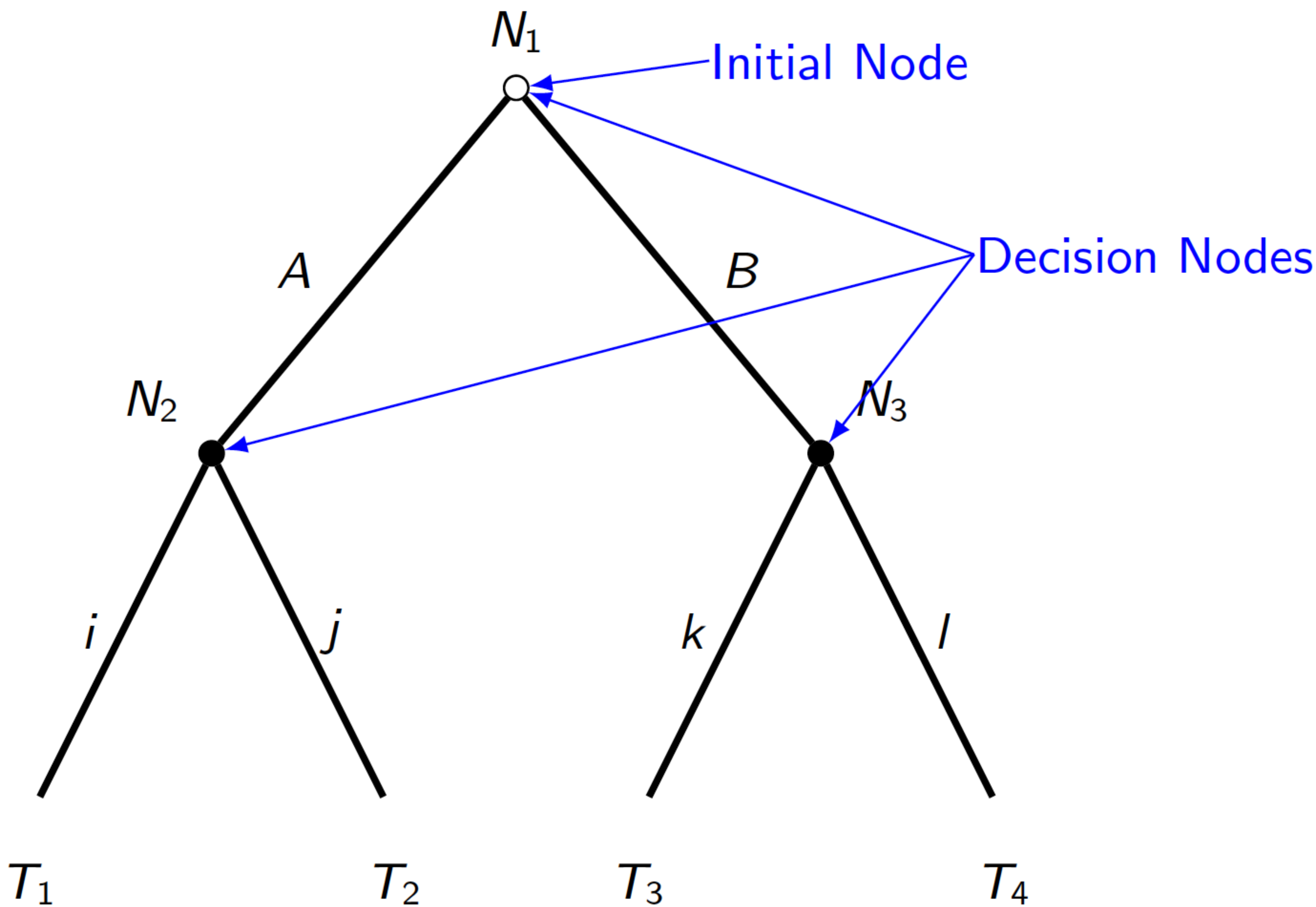
Extensive Form Definition

A **Tree Graph** consists of:

- Multiple **nodes** with an ordered hierarchy starting from one **initial node**
- **Branches** coming from each node which connect it to later nodes
- The tree ends in any of the multiple **terminal nodes**

> **Warning**
>
> Each (non-initial) terminal node may have multiple branches leading from it; but must only have *one* branch that *leads to it*.

Anatomy of a tree



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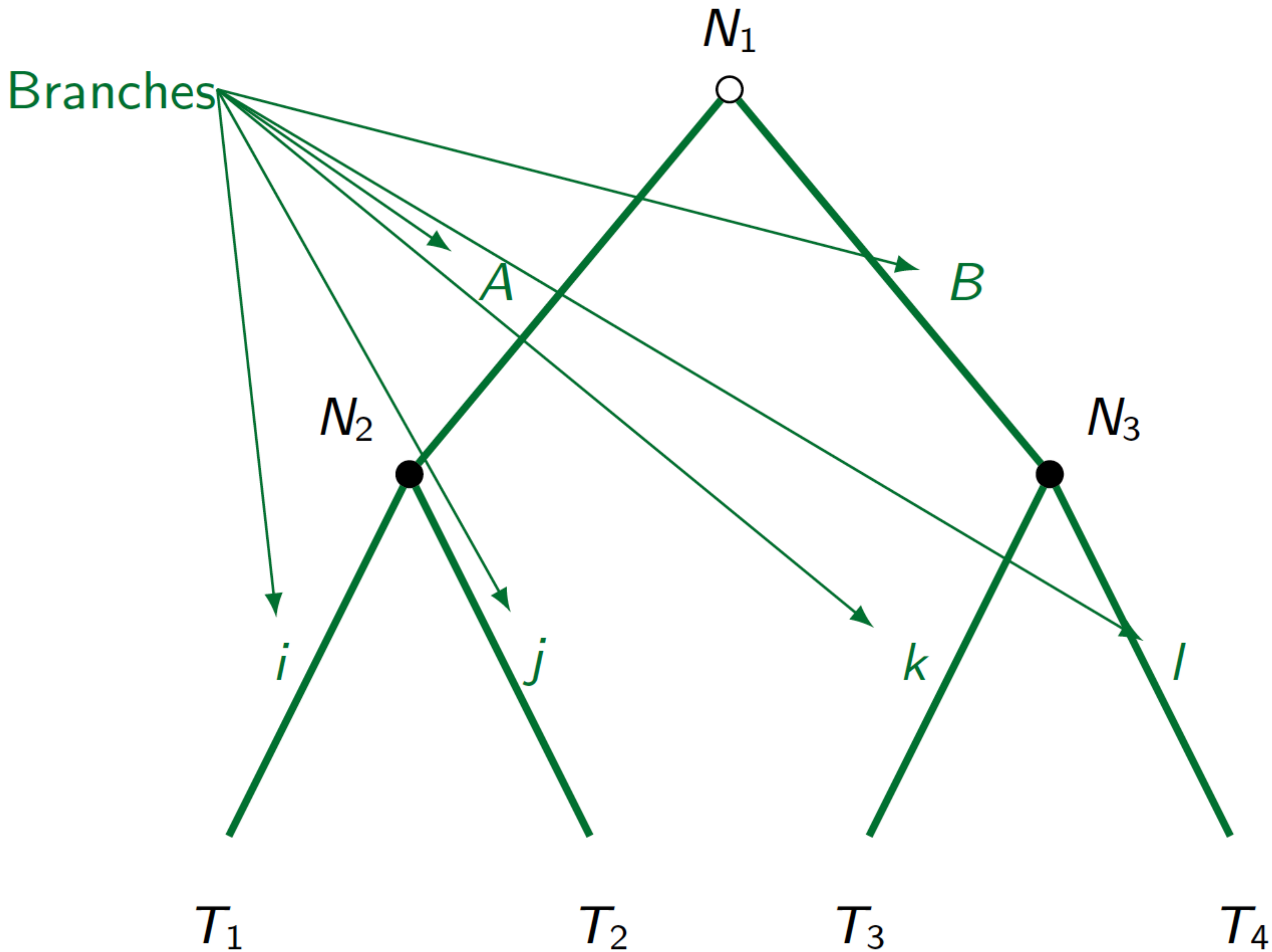

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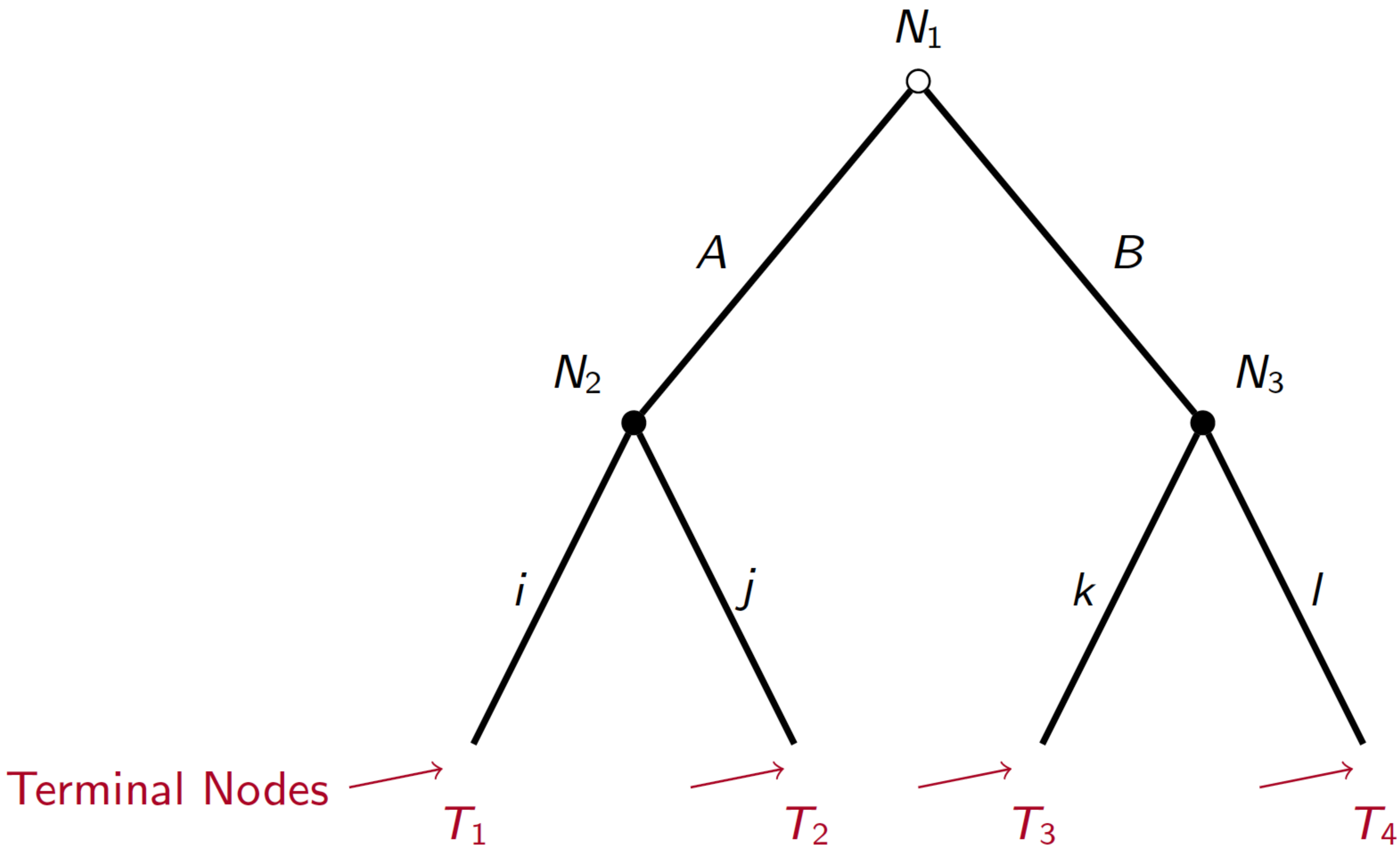

N_1
A
B
N_2
N_3
i
j
k
l
Terminal Nodes
T_1
T_2
T_3
T_4

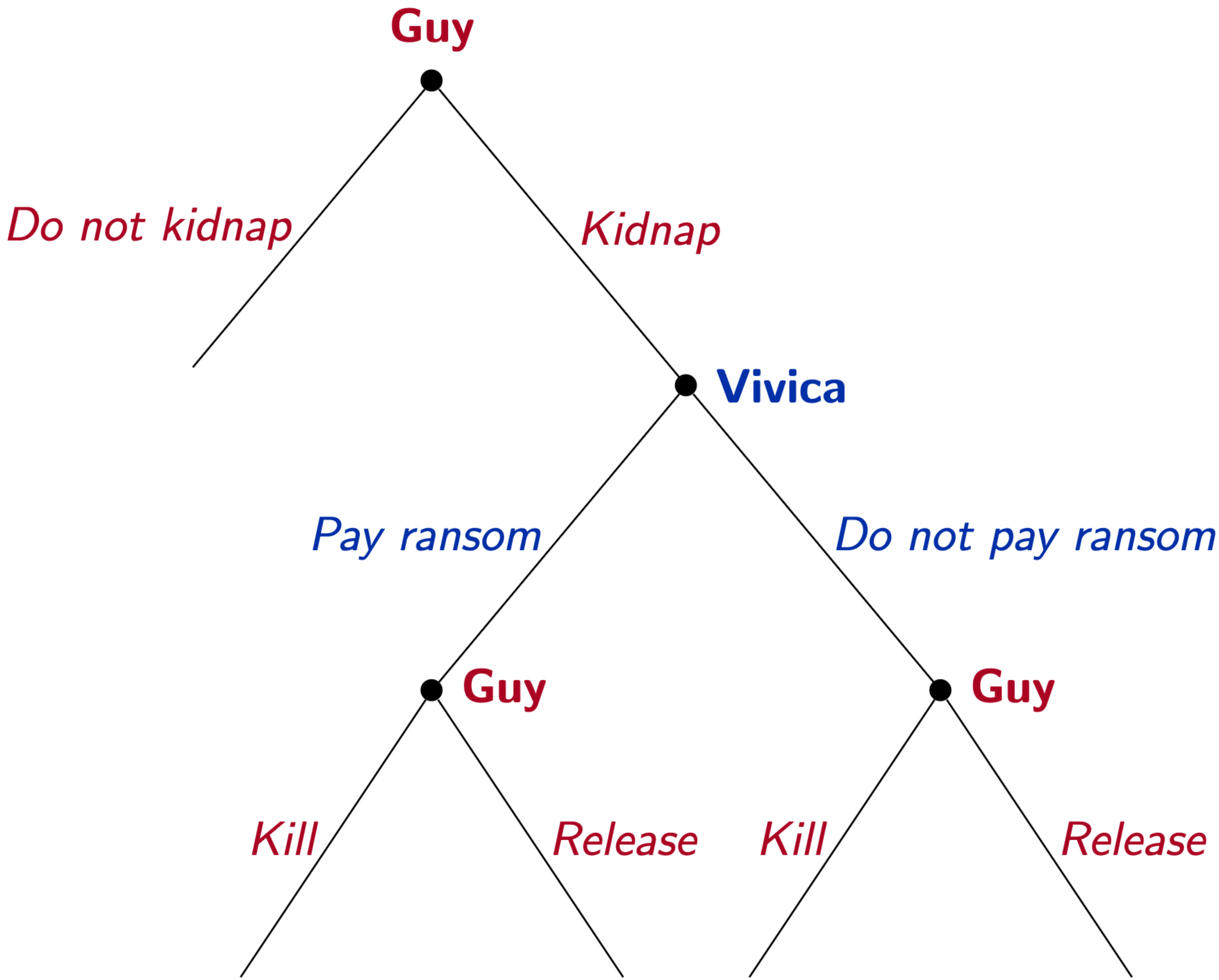
Kidnapping Game [1]

A kidnapper named **Guy** has contacted the victim's wife, named **Vivica**, to demand a ransom.

To predict what will happen to the victim, **Orlando**, we need to create a game theoretic model of the situation.

Let's use the language of the tree graph to visualize this game.

1. Harrington, pg 18


Guy
Do not kidnap
Kidnap
Vivica
Pay ransom
Do not pay ransom
Guy
Guy
Kill
Release
Kill
Release

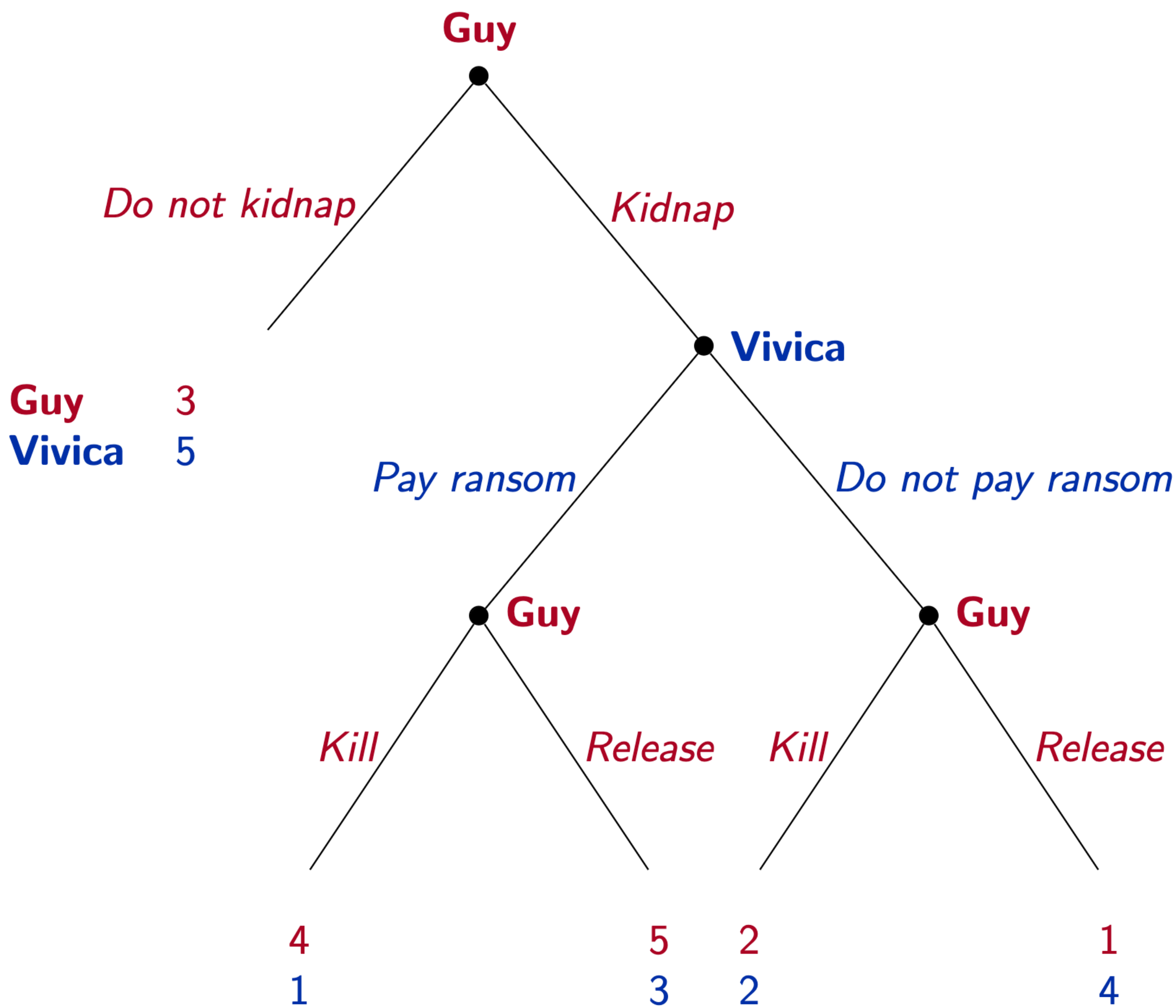
Applying the Extensive Form

- Who are the players?
- Where are the decisions?
- What are the branches? What do they represent?
- What do the terminal nodes represent?
- Is this a *complete* representation of a game? What's missing?

Kidnapping Game payoffs

Outcome	**Guy**	**Vivica**
No kidnapping	3	5
Kidnapping, ransom paid, Orlando killed	4	1
Kidnapping, ransom paid, Orlando released	5	3
Kidnapping, no ransom paid, Orlando killed	2	2
Kidnapping, no ransom paid, Orlando released	1	4

Kidnapping game tree with payoffs



Predictions?

Based on the extensive form game tree with payoffs, do you have any predictions for what strategies each player will choose?

a Definition of an Extensive Form Game: [1]

- A collection of decision-makers, called **players** or *agents*
- A set of **decision nodes**, each of which represents the information available to the player of that node
- **Branches** from each node which represent the possible actions available to the players
- The entire game tree serves as the **mapping** from intersections of players' strategy profiles to the outcomes at each **terminal nodes**

1. for finite, perfect information games

Strategies in Extensive Form Games

Definition [1]

A **strategy** for a player in a perfect-information game is a *list of choices*, one for *each decision node* of that player.

Warning

Be careful to distinguish between a ***strategy*** and a single *action/choice*

What's the difference?

1. Bonanno, pg 83

Apply this definition to the kidnapping game:

- How many choices does Guy make?

3

- Vivica?

1

- Write out a strategy list for each player:

Write out a strategy for each player

Guy:

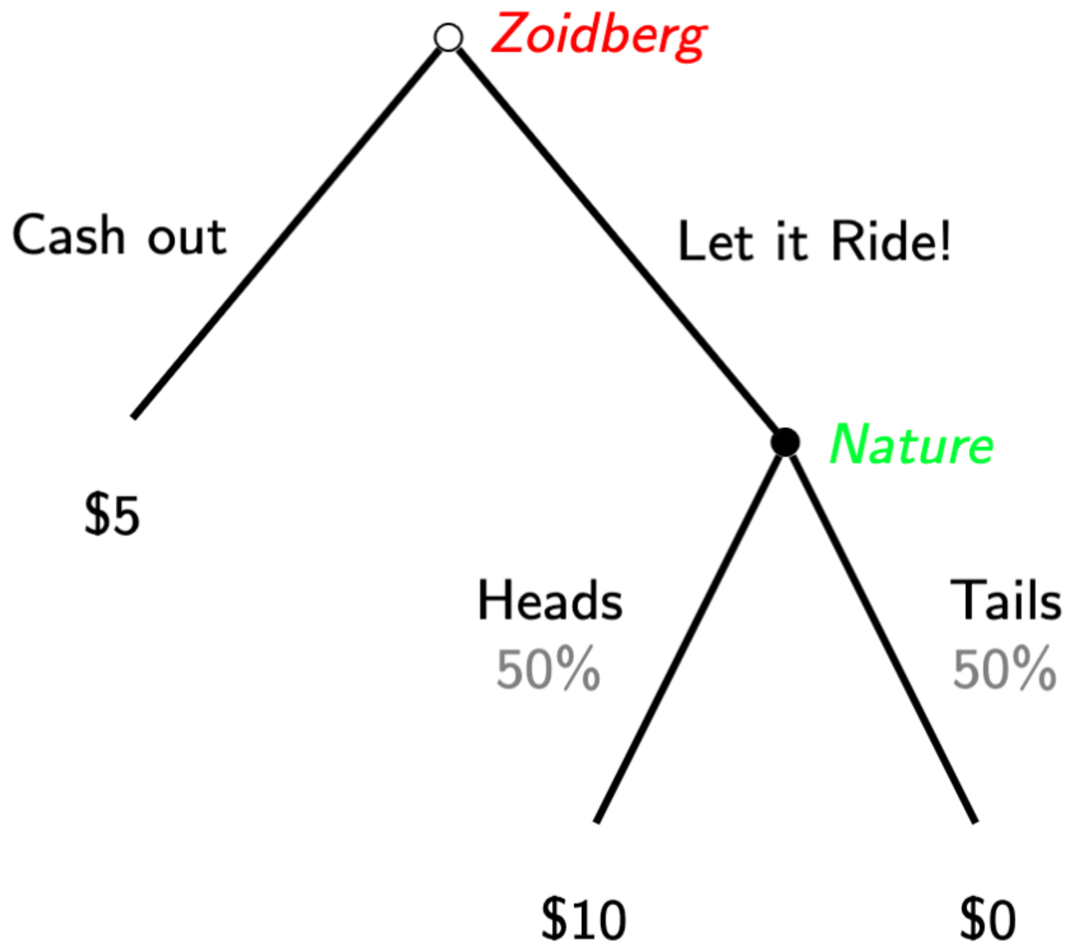
- ***{Kidnap, Kill*** if ransom paid***, Don't Kill*** if no ransom paid***}*** is *one* strategy
  - Guy has 8 total strategies:
  - **{*Kidnap*, *Kill*, *Kill*}**, **{*Kidnap*, *Kill*, *Don't*}**, **{*Kidnap*, *Don't*, *Kill*}**, **{*Kidnap*, *Don't*, *Don't*}**, **{*No Kidnap*, *Kill*, *Kill*}**, **{*No Kidnap*, *Kill*, *Don't*}**, **{*No Kidnap*, *Kill*, *Kill*}**, **{*Kidnap*, *Don't*, *Kill*}**

Vivica:

- Only two strategies: ***{Pay*** the ransom***}***, or ***{Don't*** pay***}***

Letting Nature take the wheel

One way to represent risk or uncertainty is to represent **Nature** as a 'player'.



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Backwards Induction

The smoking decision

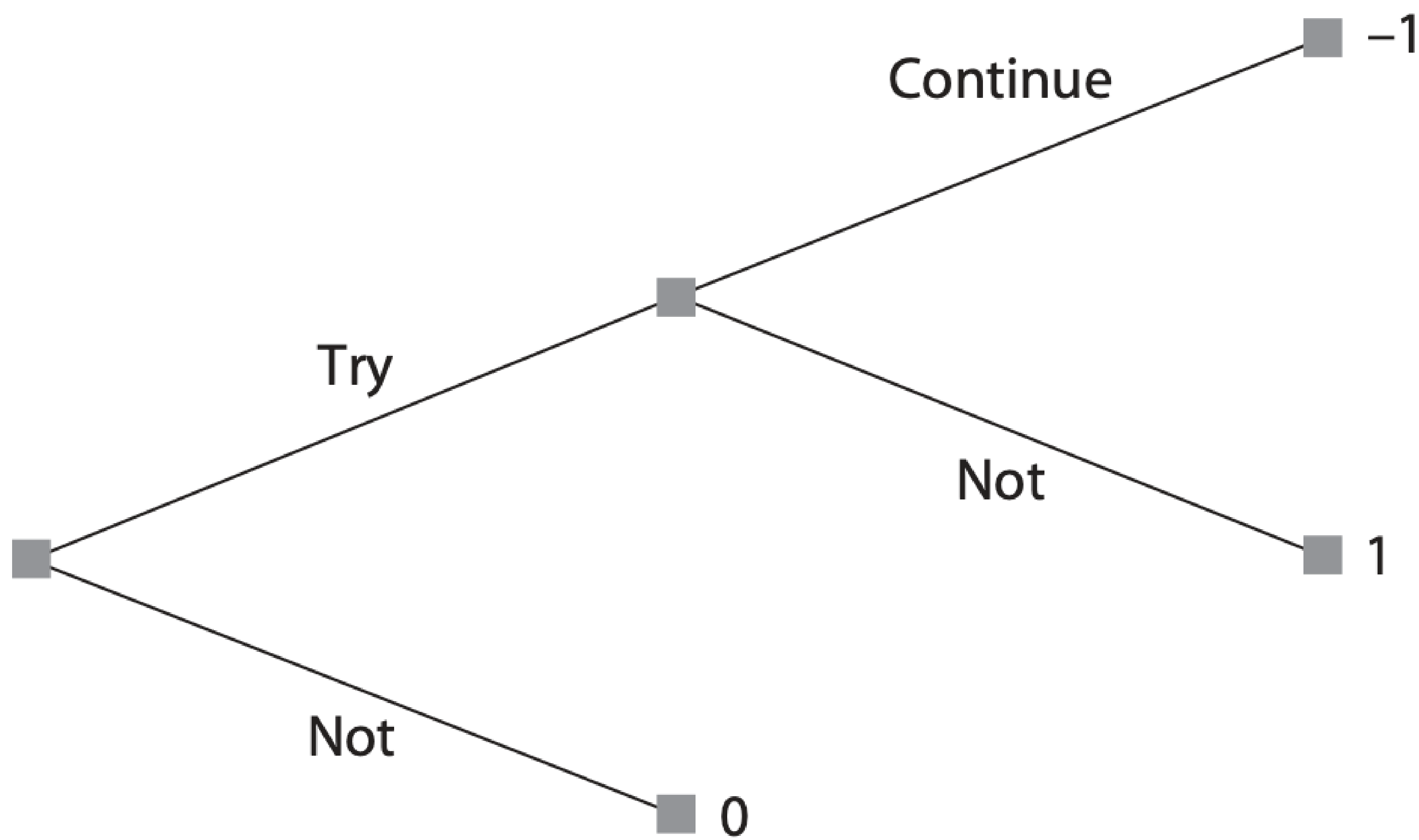

FIGURE 3.2 The Smoking Decision

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The smoking *game*

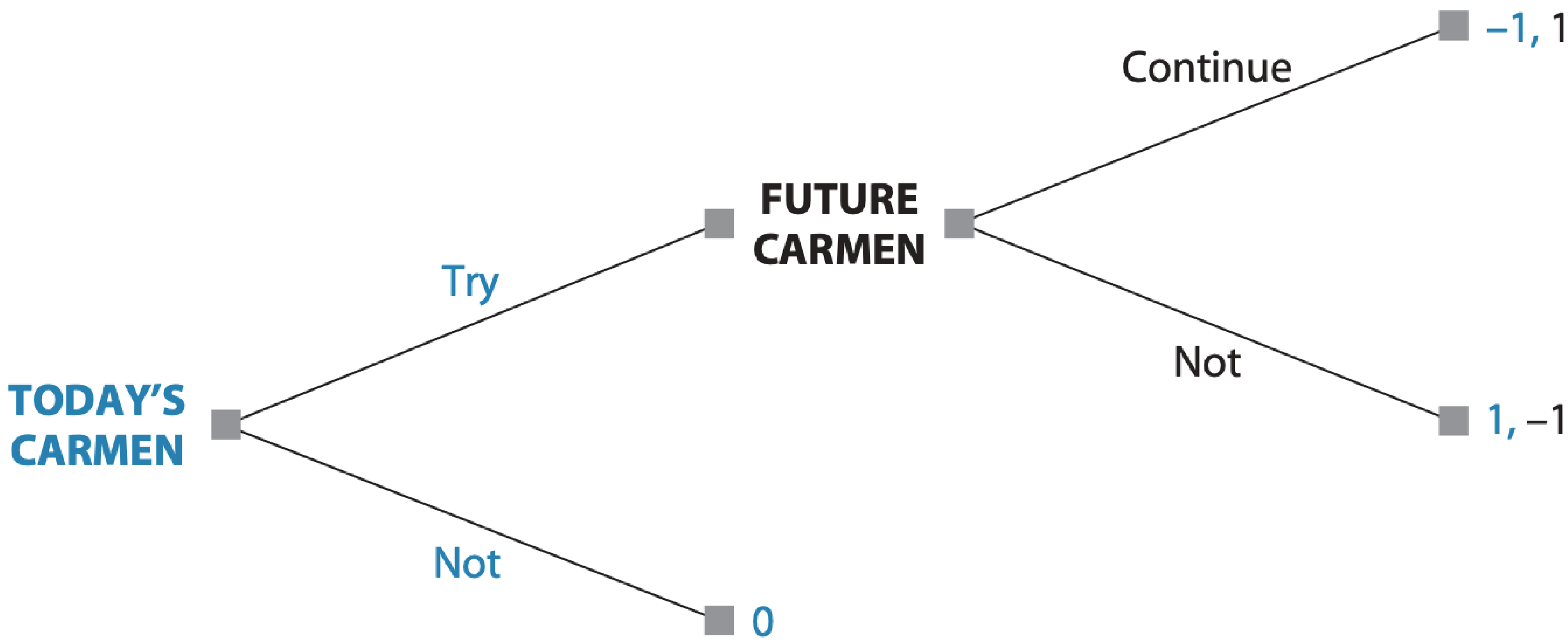

FIGURE 3.3 The Smoking Game

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'Pruning' branches

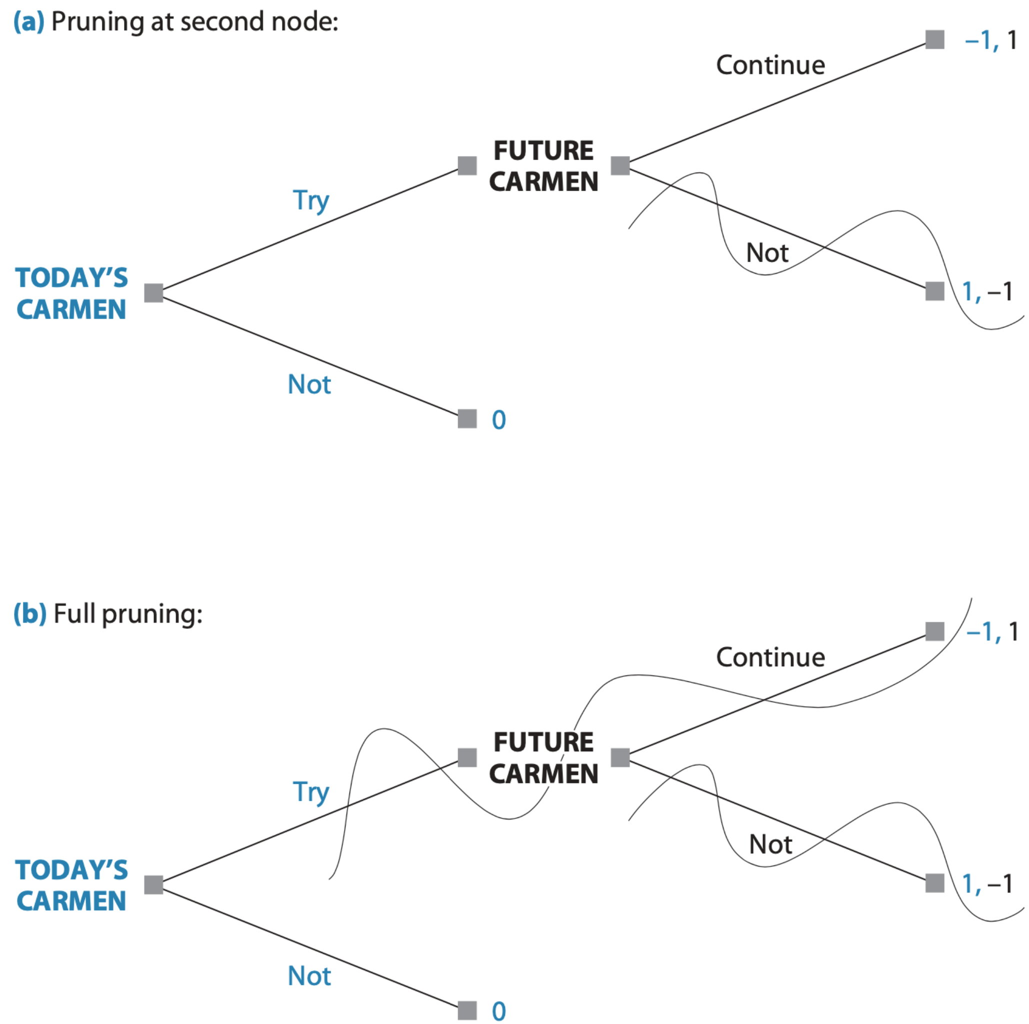

FIGURE 3.4 Pruning the Tree of the Smoking Game

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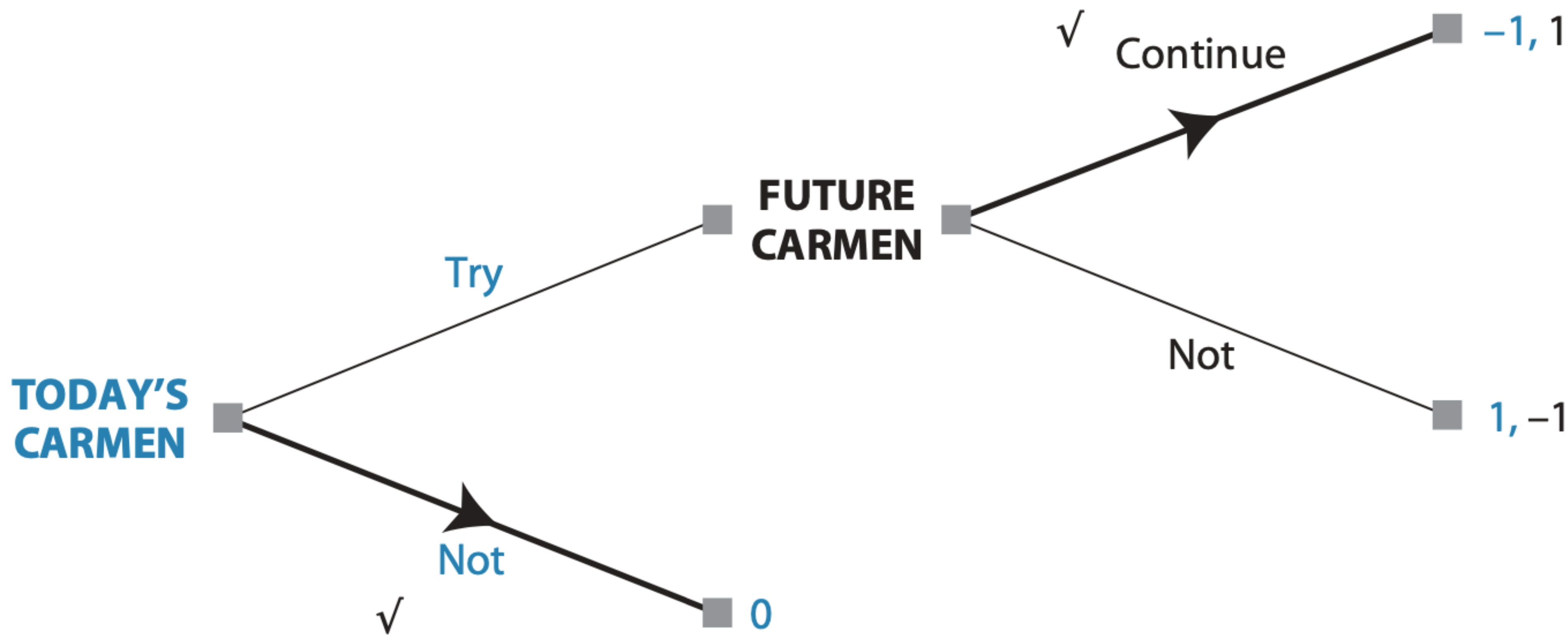

FIGURE 3.5 Showing Branch Selection on the Tree of the Smoking Game

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Backwards Induction defined

The method of looking at decisions in the future to decide what to do now is called **Backwards Induction** or **Rollback**

Definition [1]

When all players do *rollback analysis* to choose their optimal strategies, we call this set of strategies the ***rollback equilibrium***[2] of the game; the outcome that arises from playing these strategies is the ***rollback equilibrium outcome***

1. Dixit et al, pg 56
2. aka subgame perfect equilibrium

Group Exercise:

Consider the Flag game but instead of starting with 21 flags the game starts with 5 flags, and instead of being able to pick 1,2, or 3 flags teams can only pick 1 or 2 flags.

1. *Draw the extensive form game tree complete with all payoff for both teams.*
2. *How many total strategies are there for team 1?*
3. *Use pruning to eliminate actions to get to a rollback equilibrium. Who will win? What is the winning strategy?*

Adding more players

We can start to add more complexity with more than two players

3-player planting game

- **Emily**, **Nina**, and **Talia** are roommates who want to get a start on their communal garden.
- They like to enjoy the benefits of fresh produce and green space, but it is costly for them to put the work in.
- **2 or 3 people** working is enough to keep the garden healthy, but if **1 or 0** work, then the garden will die.

Planting Game payoffs

outcome:	**utlity:**
I don’t contribute, but garden lives	4
I contribute, and get garden.	3
I don’t contribute, and garden dies	2
I contribute, but garden dies	1

Planding Game Tree

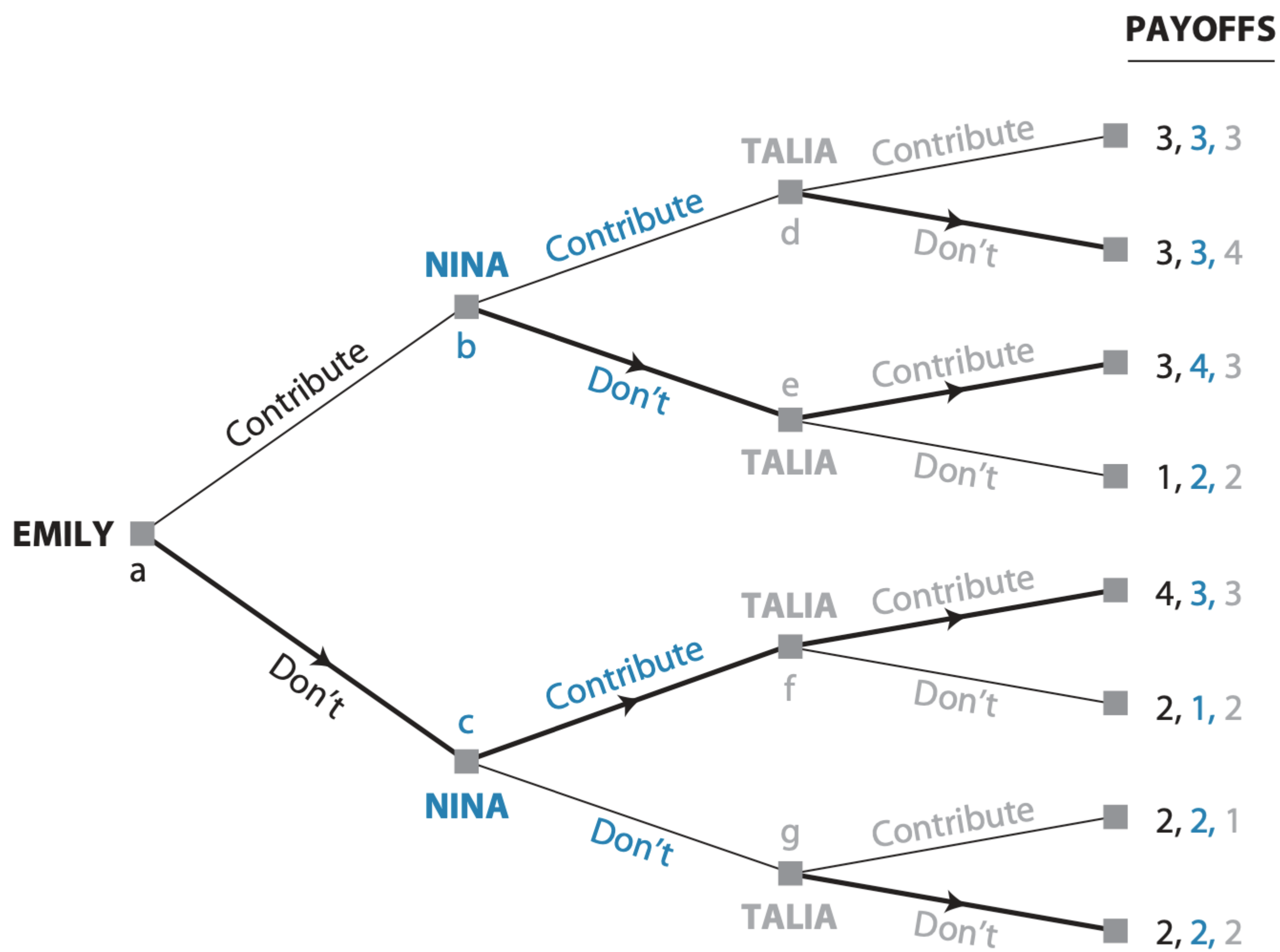

FIGURE 3.6 The Street–Garden Game

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Equilibrium Path of Play

Note that there is one continuous path we traced from the initial node to a final equilibrium outcome.

However, we couldn't have gotten their without the other arrows paths **even though they are never reached** in equilibrium.

Recall that a **strategy** is a collection of choices at **every** decision node.

Equilibrium Strategies

Even though the players available actions are all called the same (Contribute or Don't), this tree provides labels of each decision node so we can say something like:

> "Nina's **strategy** in the rollback equilibrium is { ***Don't Contribute*** at **b**, ***Contribute*** at **c** }".

- To make it even shorter, let's call this strategy **DC**.

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How many strategies does Talia have?

- CCCC, CCCD, CCDC, CCDD, CDCC, CDCD, CDDC, CDDD, DCCC, DCCD, DCDC, DCDD, DDCC, DDCD, DDDC, DDDD
- 16 total strategies

Rollback Equilibrium Strategies

The equilibrium is:

- { $\mathbf{D}^1$, $\mathbf{DC}^2$, $\mathbf{DCCD}^3$ }

1. Emily
2. Nina
3. Talia

Adding More Moves

Even a simple game get complicated fast

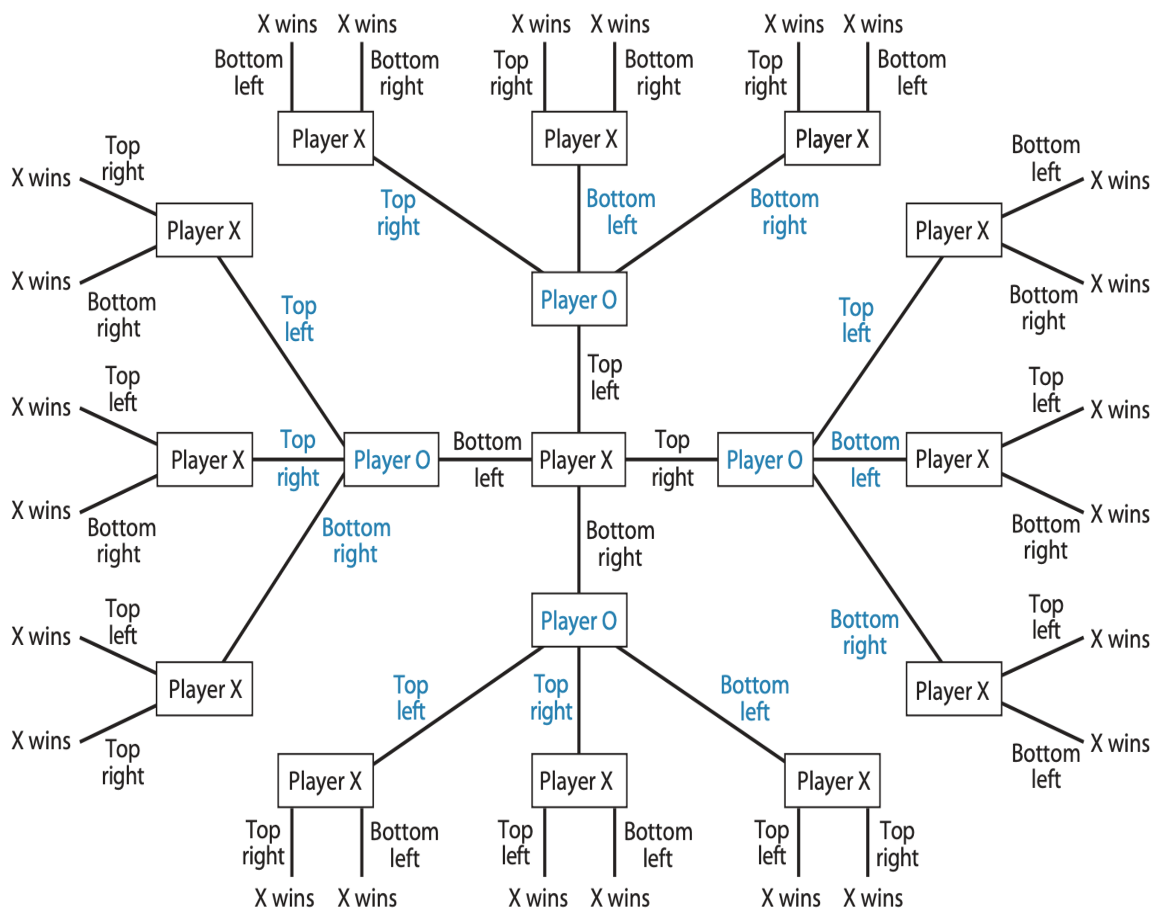

FIGURE 3.7 The Complex Tree for Simple Two-by-Two Tic-Tac-Toe

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Tic-Tac-Toe

- Even though it looks complicated, the main branches are really just copies of each other
- Most people probably figure out the rollback equilibrium after playing it enough
- Insert relevant xkcd here: https://xkcd.com/832/

Chess

- What about more complicated games like chess?
  - technically rollback solvable, but with 10^{120} possible moves, it hasn't been solved by either human or machine
- Players of complicated sequential games often implement some **intermediate valuation function** to assign payoffs to non-terminal nodes.

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Welfare and Efficiency

What are the **good** outcomes in the planting game?

Can we rank outcomes by collective welfare?

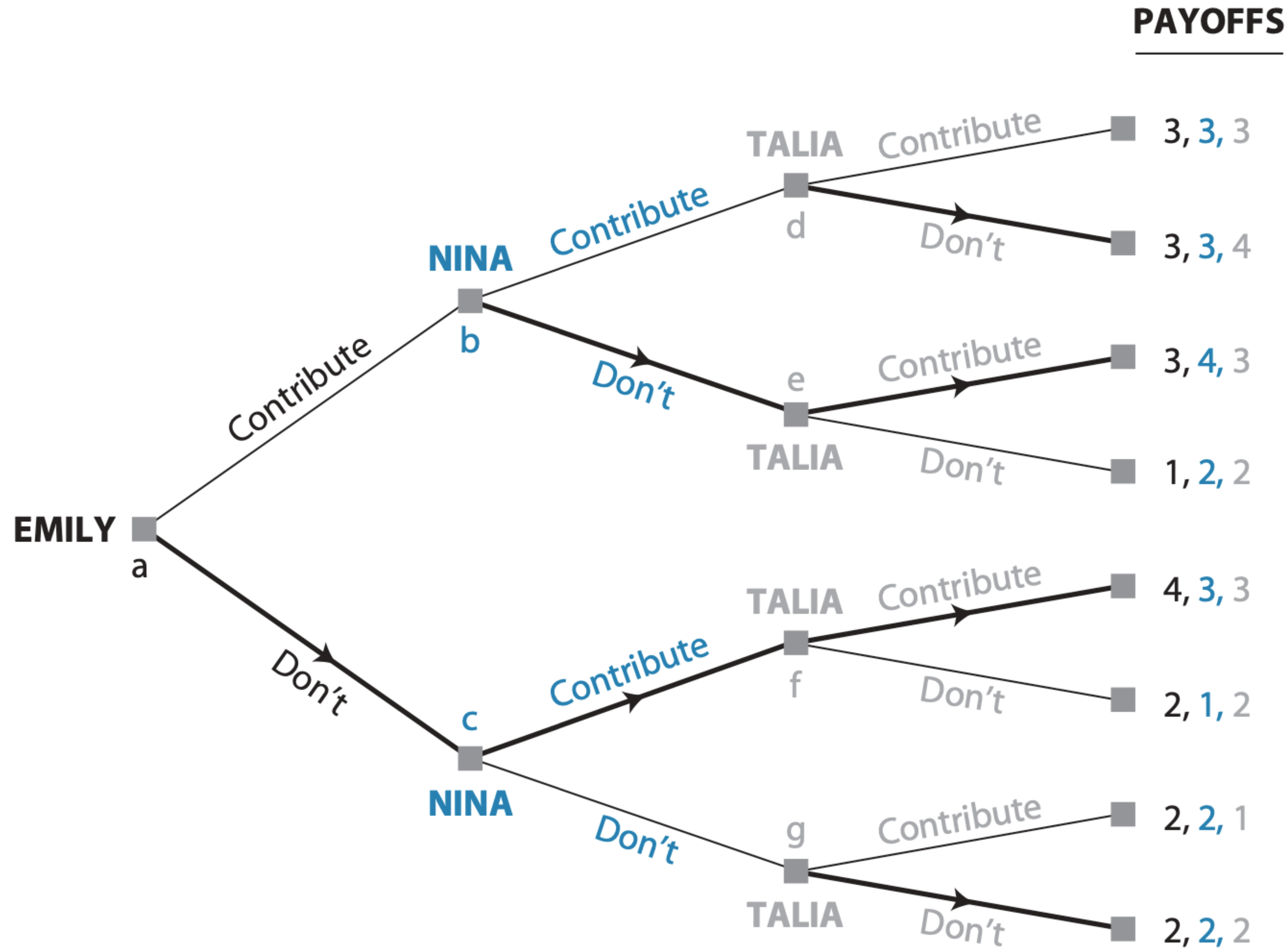

FIGURE 3.6 The Street–Garden Game

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Pareto Dominance

Pareto optimality (or efficiency) is econonomists' best shot at coming up with a ranking of which outcomes are objectively 'better'

- For any two outcomes (🎉, 🎊), 🎉 is **Pareto dominated** by 🎊 if both:
  1. No one strictly prefers 🎉 to 🎊 - U👤(🎊) ≥ U👤(🎉)
     ∀👤 ∈ {👨‍🦲, 👦, 💆, 👰, 👦, 😺, . . .}
  2. At least one person strictly🎉 prefers 🎉 to 🎊 - ∃👤 such that U👤(🎊)
     > U👤(🎉)

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Pareto Improvement

The move from a policy y to an alternative policy x is a **Pareto improvement** if x Pareto dominates y.

- Such a policy change should reasonably be seen as unambiguously good
- Another perspective is that *no-one would veto* a pareto improvement

Pareto Efficiency

An outcome is **Pareto Efficient** (Optimal) if no other outcome Pareto dominates it.

An outcome is **Pareto Inefficient** if at least one other outcome Pareto dominates it.

Ranking the Planting Payoffs

Compare (4,3,3) to (1,2,2)

- Which one is Pareto dominating?

Ranking the Planting Payoffs

Now compare (4,3,3) to (3,4,3) or (3,3,4)

- Which one is Pareto dominating?

Is the rollback equilibrium outcome a *Pareto efficient* one?