Introduction to Game Theory

Sequential Games

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Outline

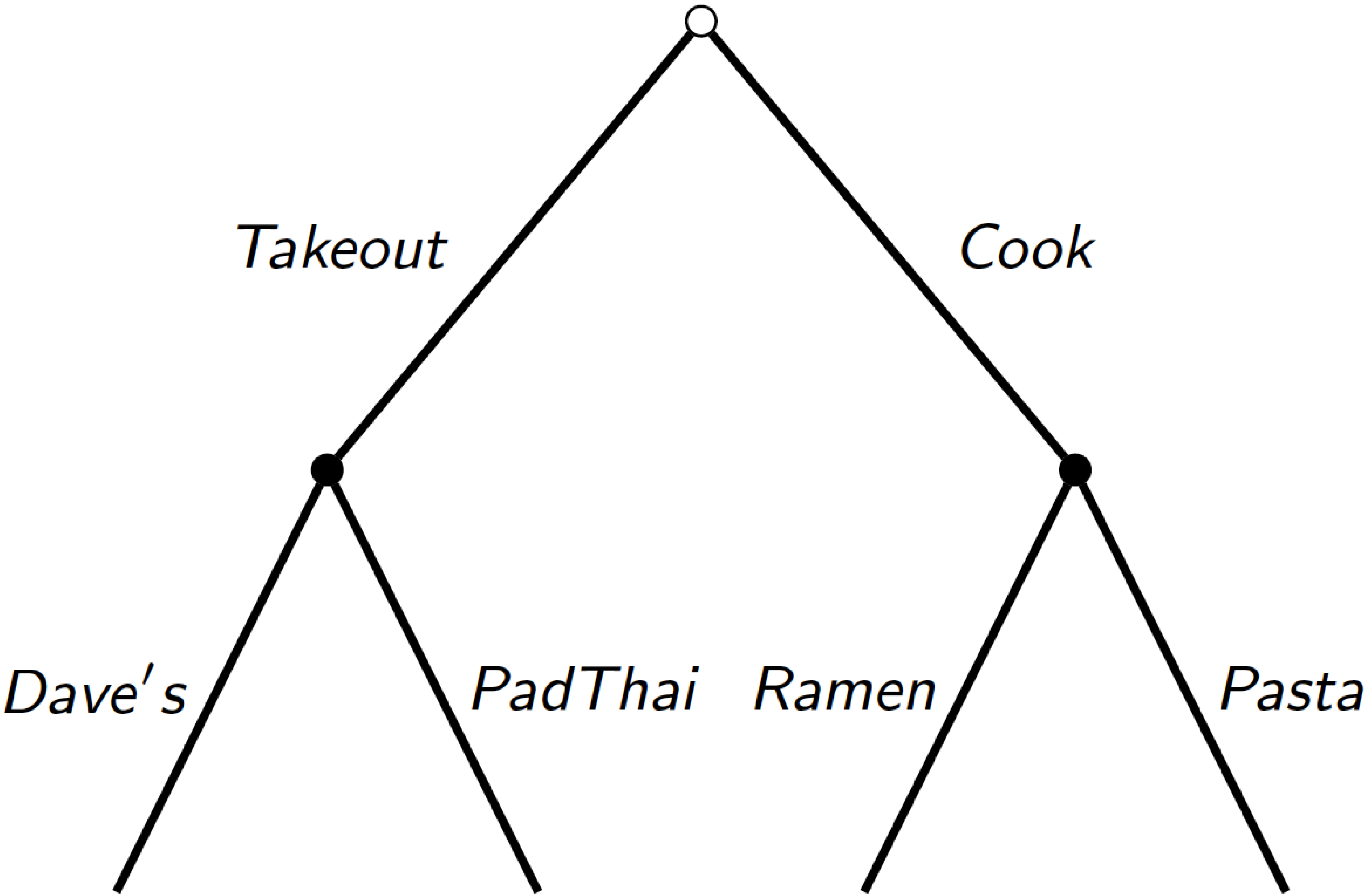
- Game trees
- Backwards Induction
- Efficiency

Extensive Form

Game Trees/Extensive Form as a tool

- Before we learn how to solve a game, it will helpful to be able to visualize them
- Because of the ordered nature of sequential games, a **tree diagram** makes sense

A *Decision* Tree



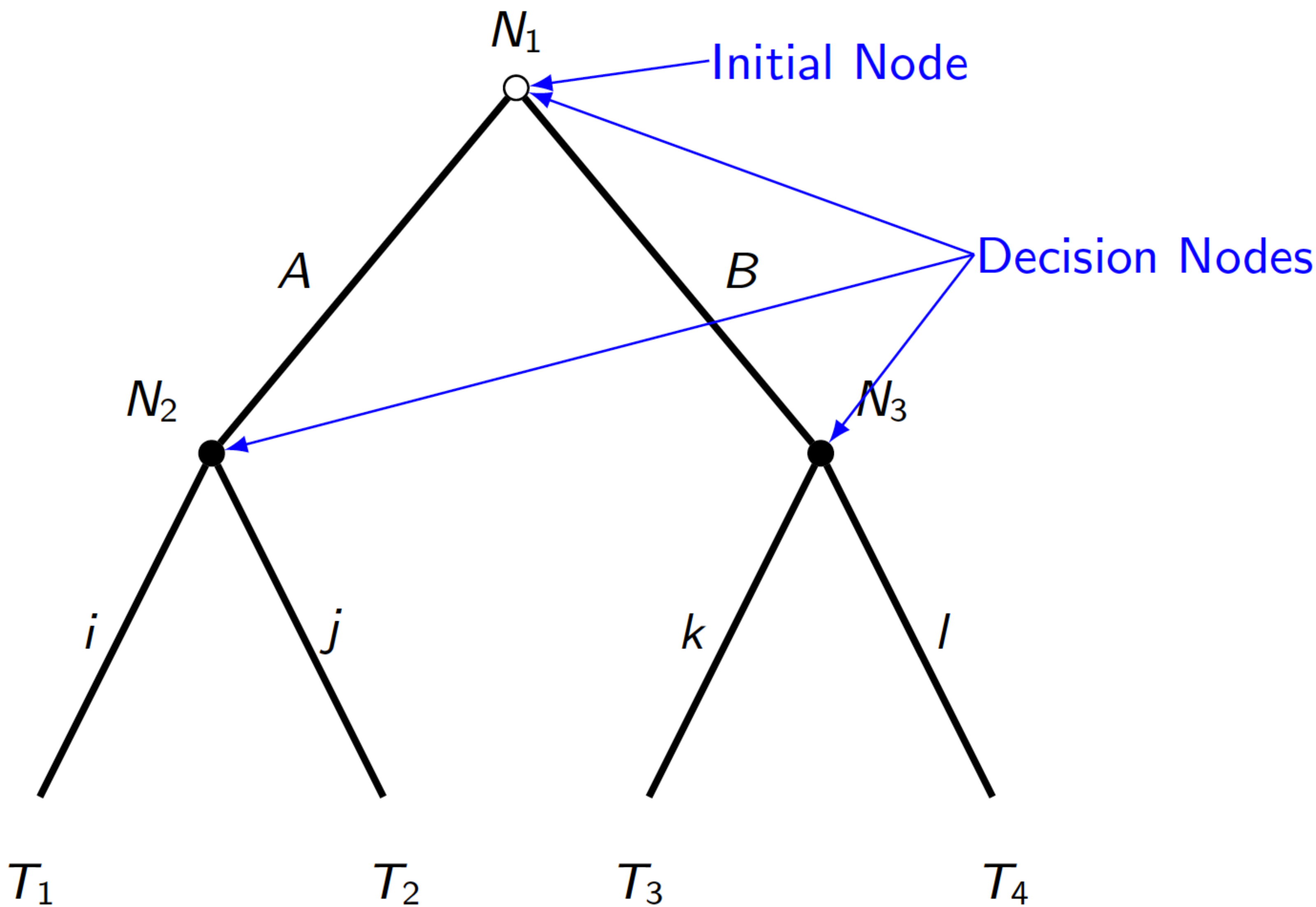
Extensive Form Definition

A **Tree Graph** consists of:

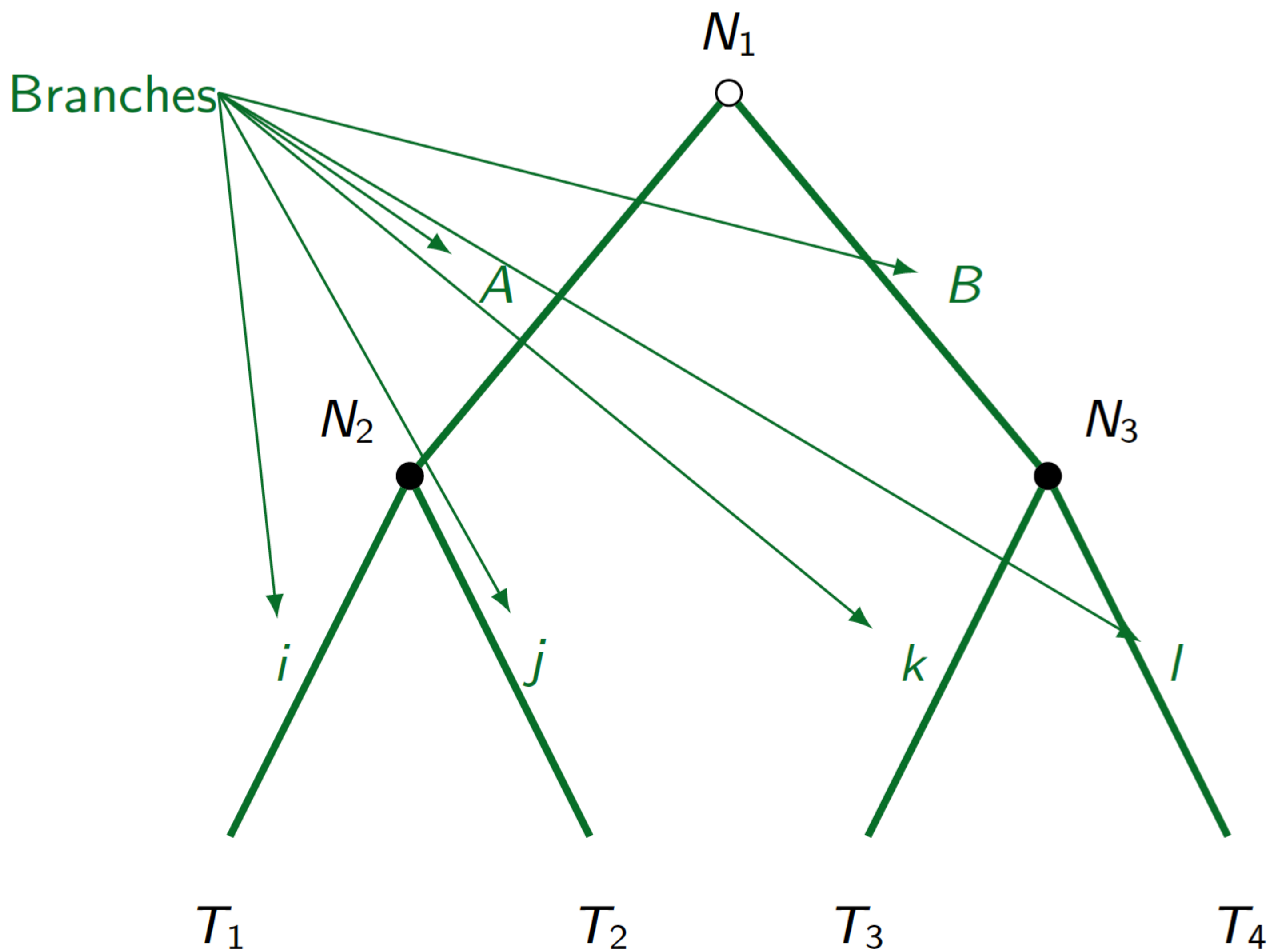
- Multiple **nodes** with an ordered hierarchy starting from one **initial node**
- **Branches** coming from each node which connect it to later nodes
- The tree ends in any of the multiple **terminal nodes**

> **Warning**
>
> Each (non-initial) terminal node may have multiple branches leading from it; but must only have *one* branch that *leads to it*.

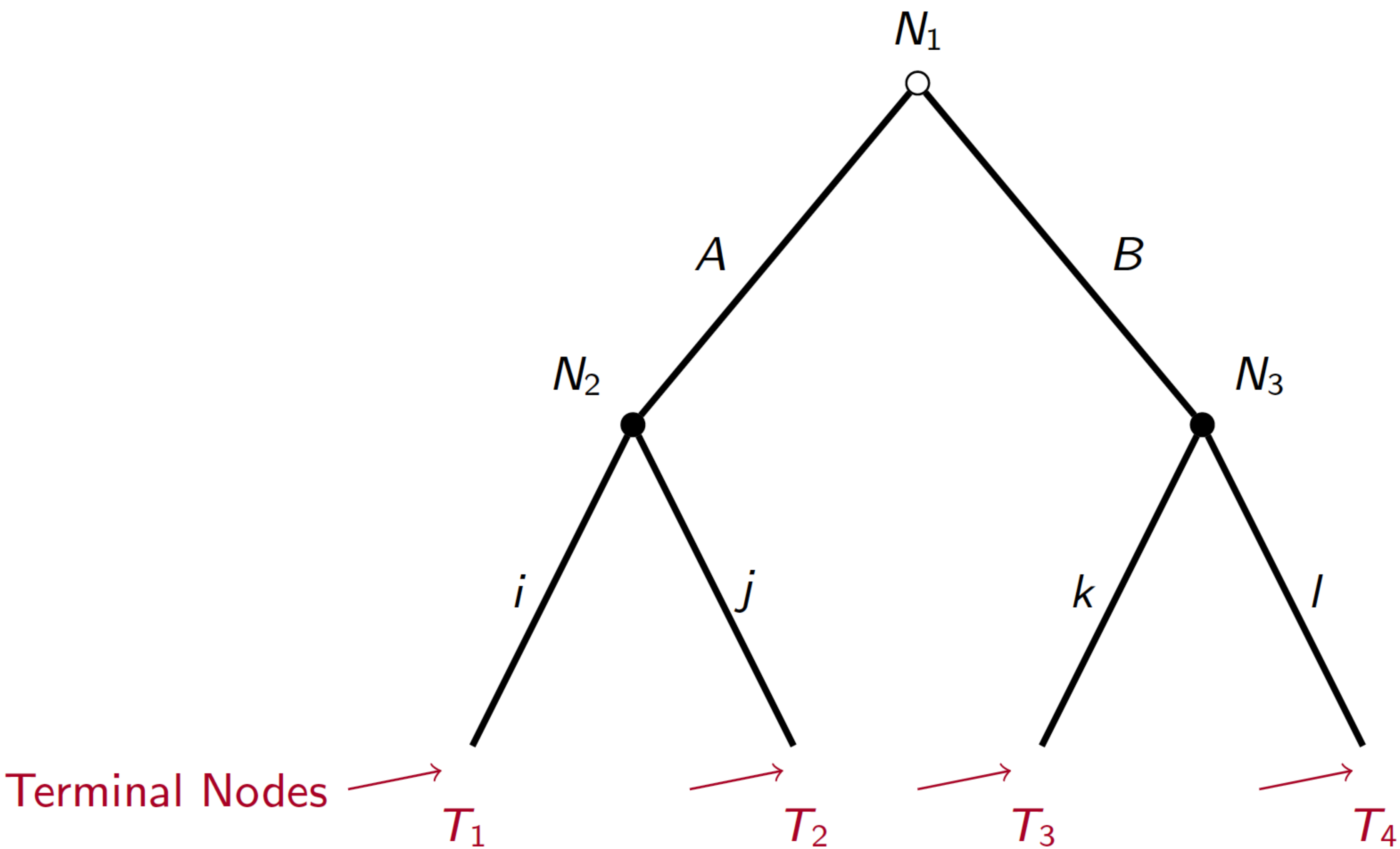
Anatomy of a tree



Anatomy of a tree



Anatomy of a tree



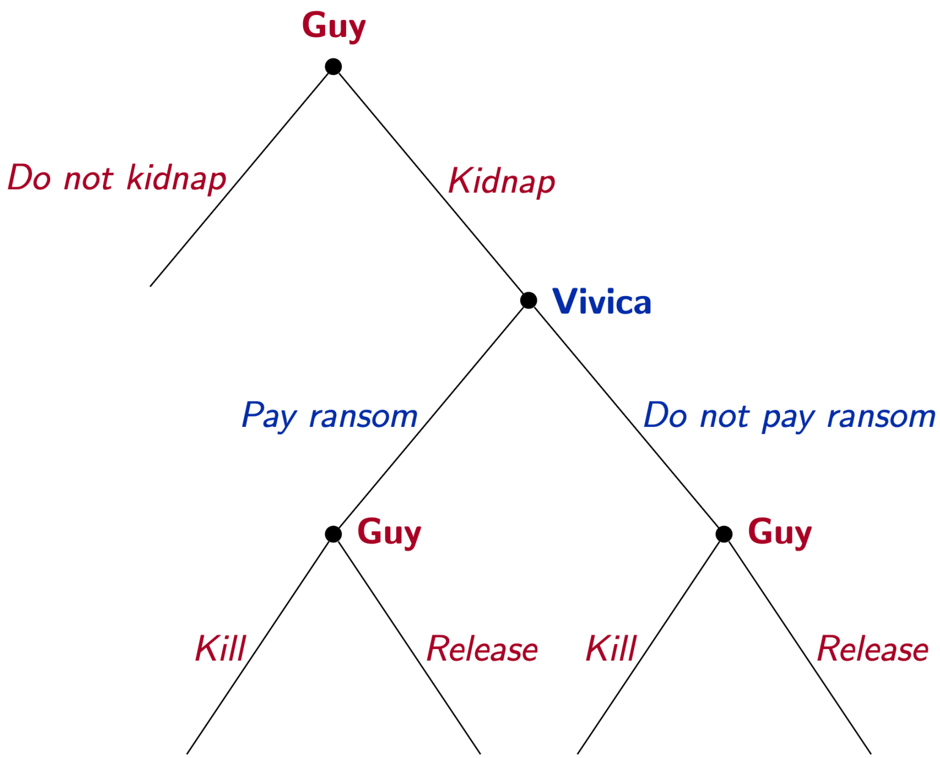
Kidnapping Game [1]

A kidnapper named **Guy** has contacted the victim's wife, named **Vivica**, to demand a ransom.

To predict what will happen to the victim, **Orlando**, we need to create a game theoretic model of the situation.

Let's use the language of the tree graph to visualize this game.

Kidnapping Game



- Who are the players?
- Where are the decisions?
- What are the branches? What do they represent?
- What do the terminal nodes represent?
- Is this a *complete* representation of a game? What's missing?

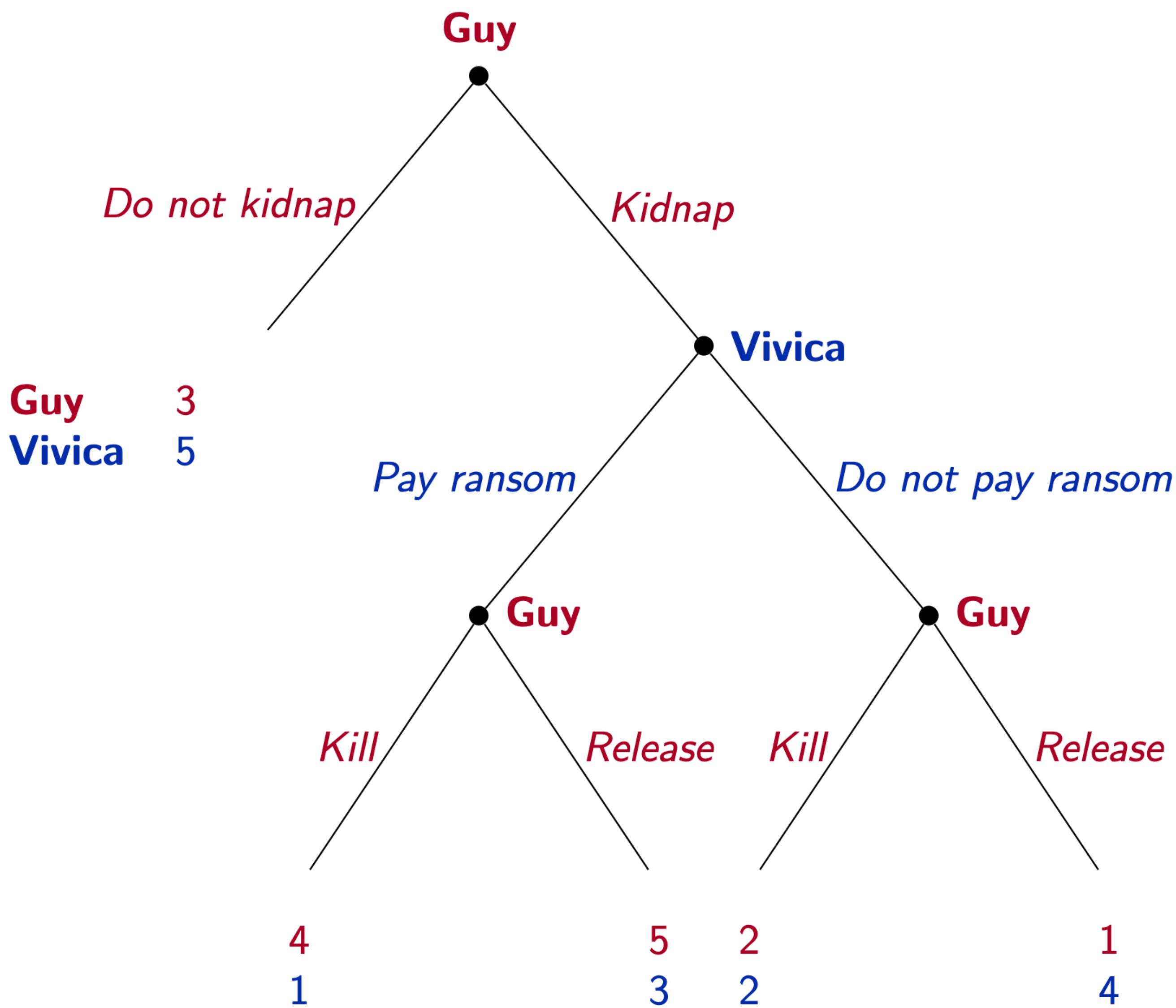
Kidnapping Game payoffs

Outcome	**Guy**
No kidnapping	3
Kidnapping, ransom paid, Orlando killed	4
Kidnapping, ransom paid, Orlando released	5
Kidnapping, no ransom paid, Orlando killed	2
Kidnapping, no ransom paid, Orlando released	1

Kidnapping Game payoffs

Outcome	Guy	Vivica
No kidnapping	3	5
Kidnapping, ransom paid, Orlando killed	4	1
Kidnapping, ransom paid, Orlando released	5	3
Kidnapping, no ransom paid, Orlando killed	2	2
Kidnapping, no ransom paid, Orlando released	1	4

Kidnapping game tree with payoffs



Predictions?

Based on the extensive form game tree with payoffs,

- Do you have any predictions for what strategies each player will choose?

a Definition of an Extensive Form Game: [1]

- A collection of decision-makers, called **players** or *agents*
- A set of **decision nodes**, each represents the information available to the player of that node
- Strategies for each player which list the **branches** from *each* node that represetn the actions a player would take if faced with that choice
- A **tree diagram** which maps the intersections of players' strategy profiles to the outcomes represented at each **terminal nodes**

Strategies in Extensive Form Games

Definition

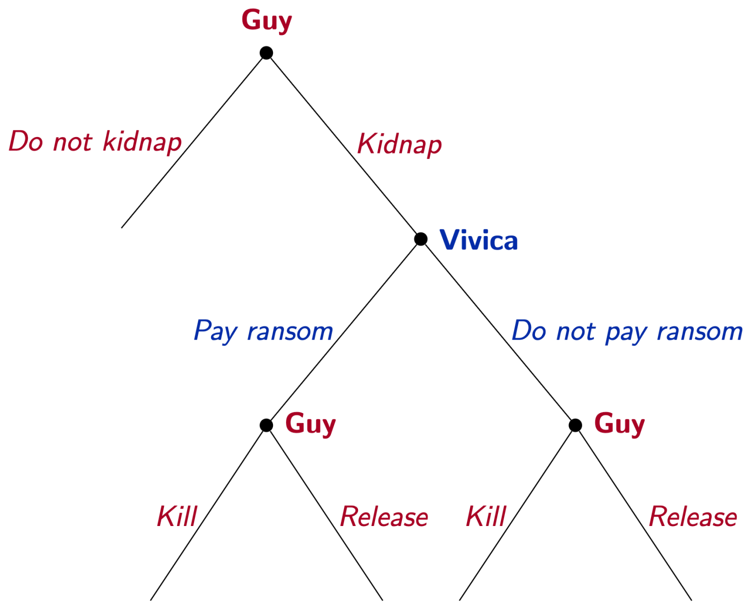
A **strategy** is a **complete plan of action** which assigns an action at *every* node where a player makes a decision

Warning

Be careful to distinguish between a ***strategy*** and a single *action/choice*

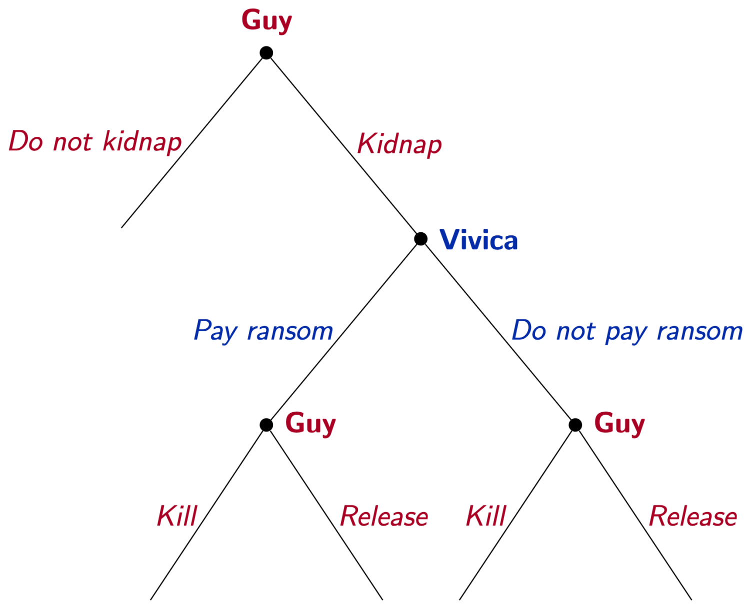
What's the difference?

Apply this definition to the kidnapping game:



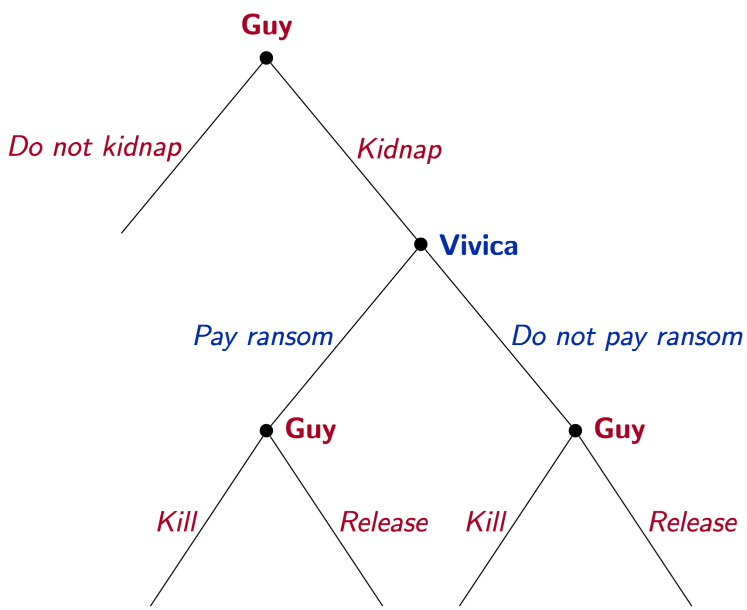
- How many decisions does Guy make?
  - 3
- How many decisions does Vivica make?
  - 1

Apply this definition to the kidnapping game:



- Write out a complete strategy for Vivica
  - Only two strategies:
    - *Pay* the ransom,
    - or *Don't* pay

Apply this definition to the kidnapping game:



- Write out at complete strategy for Guy
  - Let's give some shorter names for Guy's actions:
    - A ≡ Kidnap Orlando (Abduct)
    - I ≡ Don't kidnap Orlando (Ignore)
    - K ≡ Kill Orlando
    - L ≡ Let Orlando live

Apply this definition to the kidnapping game:

Guy has 8 total complete strategies:

If Guy Abducts	**If Guy Ignores**
$(\text{A}, \text{K}, \text{K})$	$(\text{I}, \text{K}, \text{K})$
$(\text{A}, \text{L}, \text{K})$	$(\text{I}, \text{L}, \text{L})$
$(\text{A}, \text{K}, \text{L})$	$(\text{I}, \text{K}, \text{L})$
$(\text{A}, \text{L}, \text{L})$	$(\text{I}, \text{L}, \text{L})$

Backwards Induction

Solving Sequential Games

Now that we have defined all the parts of what a sequential game is we can start to *solve* them.

- A solution in our case will be a prediction of what rational agents would do in a sequential game

The smoking decision

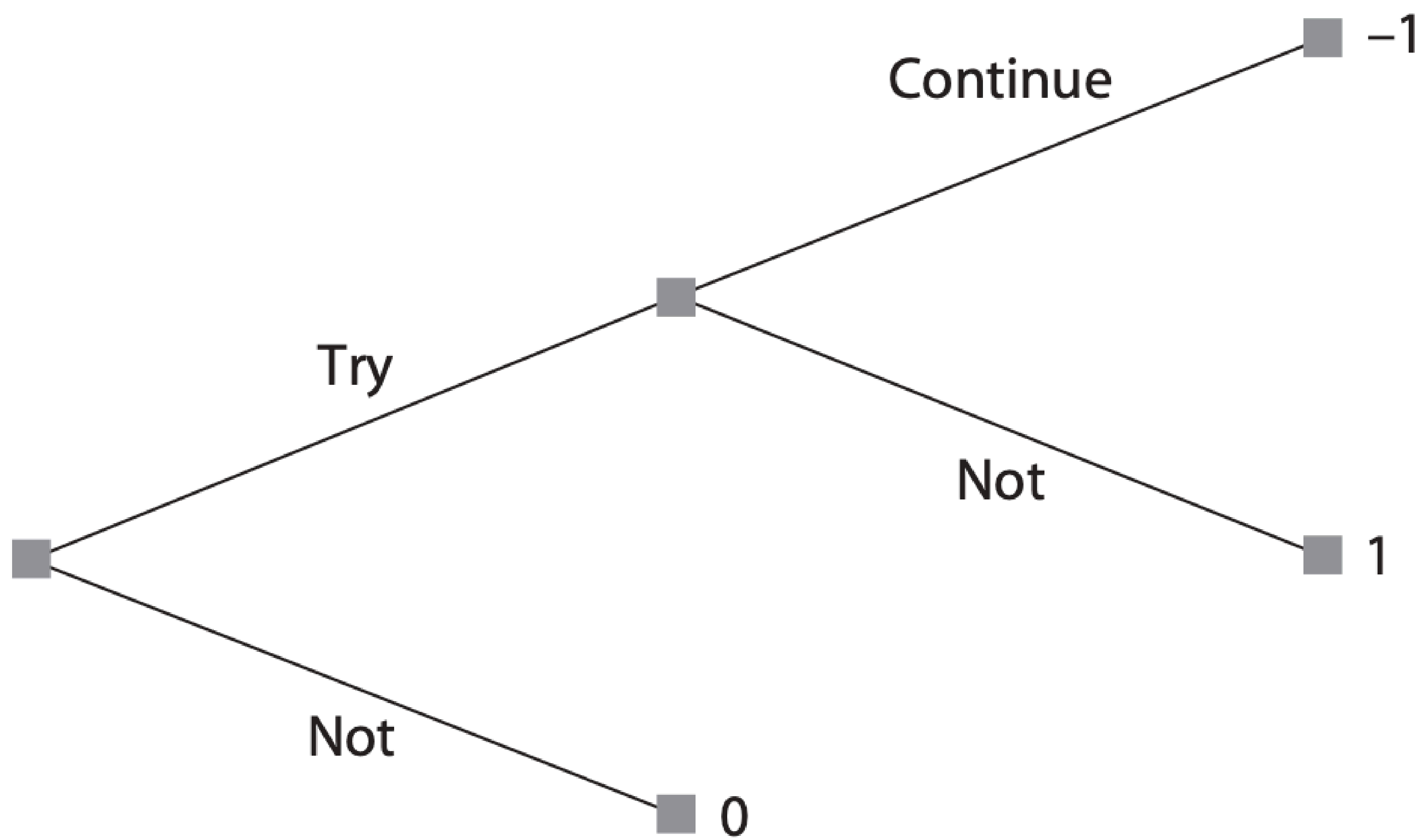


FIGURE 3.2 The Smoking Decision

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The smoking *game*

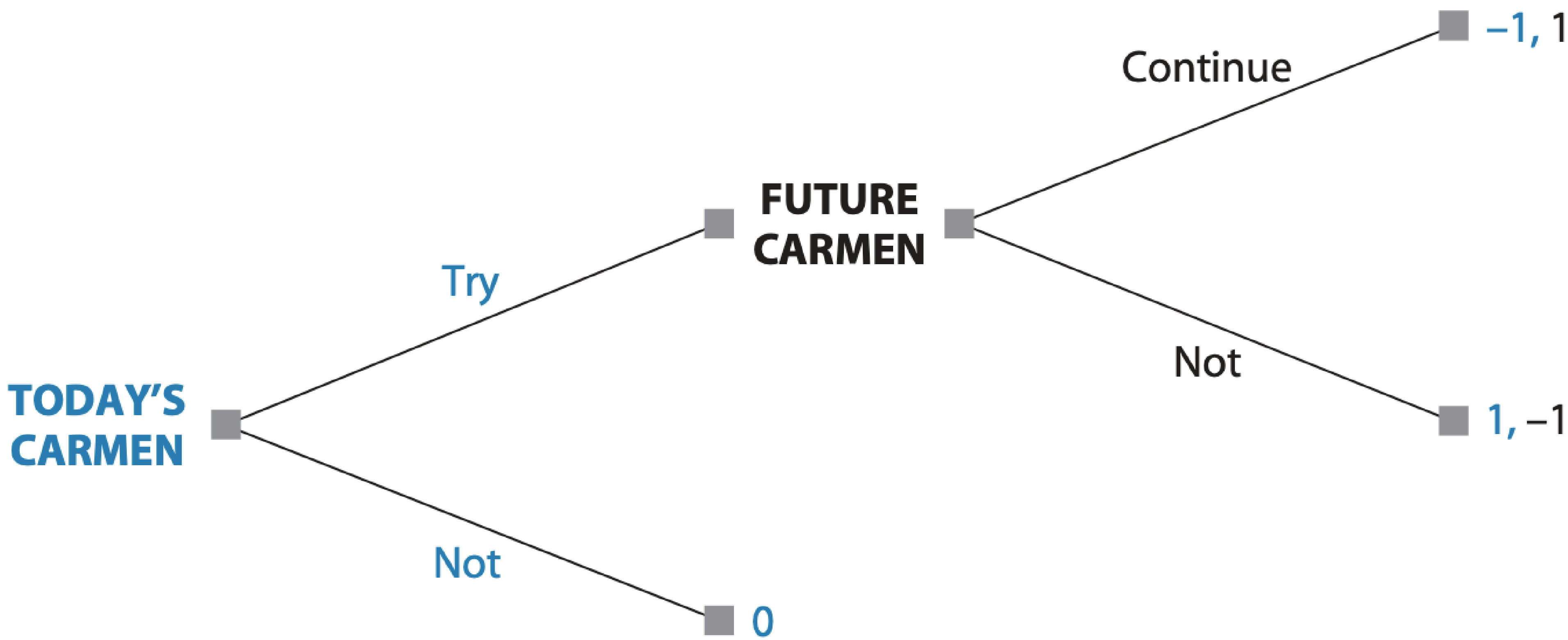

FIGURE 3.3 The Smoking Game

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‘Pruning’ branches

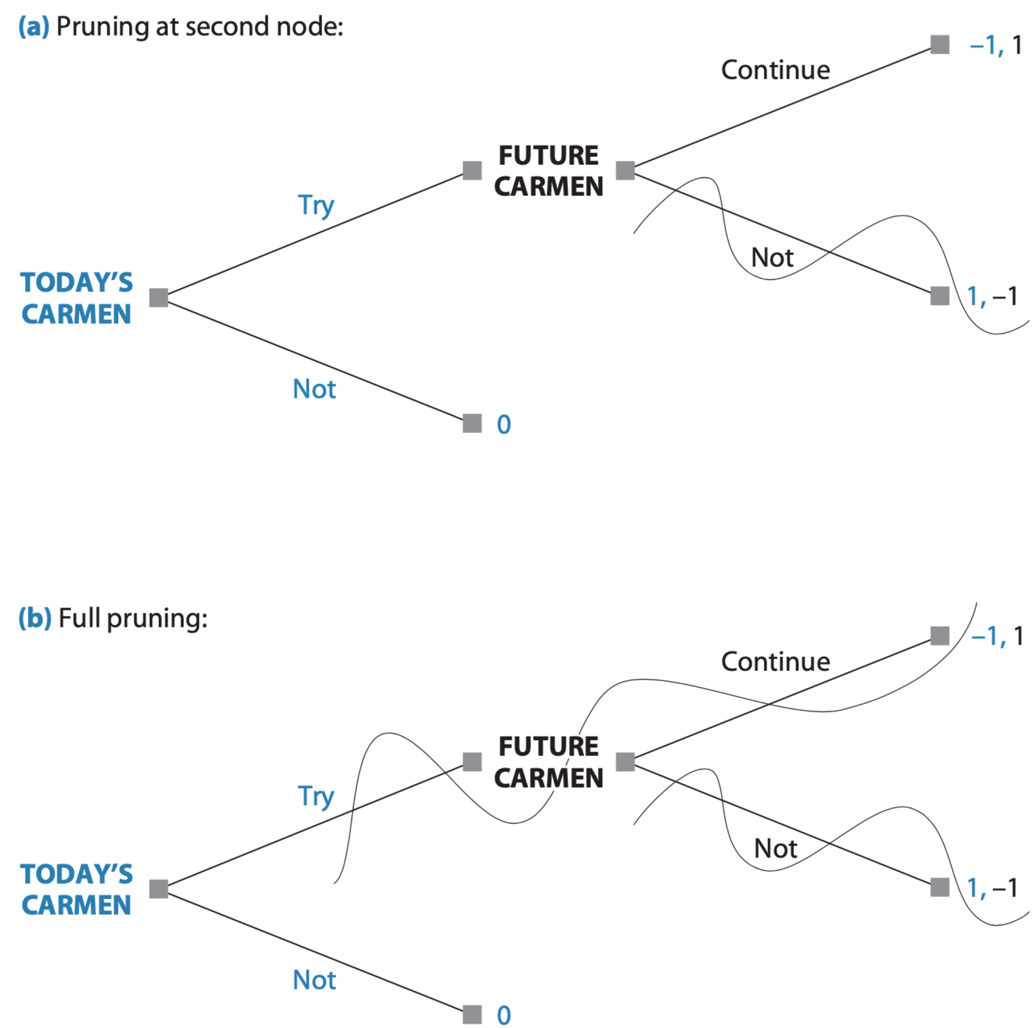

FIGURE 3.4 Pruning the Tree of the Smoking Game

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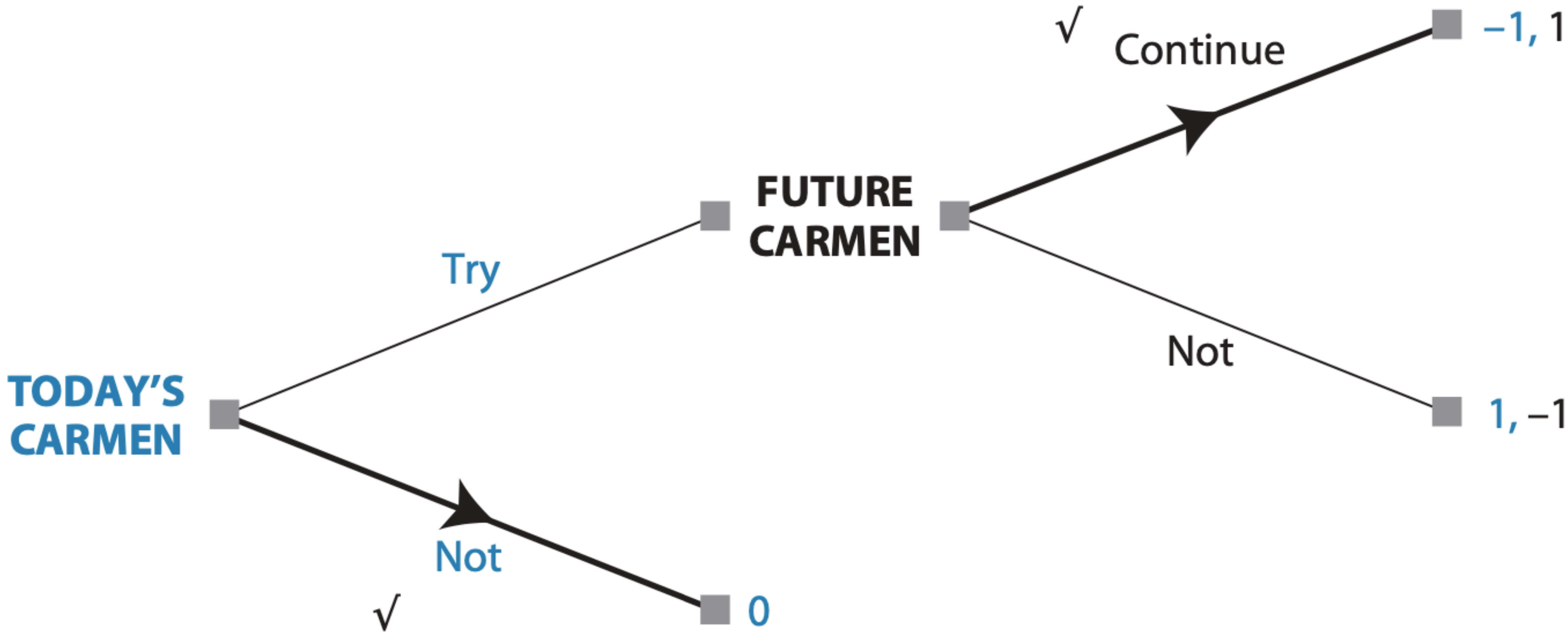

FIGURE 3.5 Showing Branch Selection on the Tree of the Smoking Game

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Backwards Induction defined

The method of looking at decisions in the future to decide what to do now is called **Backwards Induction** or **Rollback**

Definition [1]

When all players do *rollback analysis* to choose their optimal strategies, we call this set of strategies the ***rollback equilibrium***[2] of the game; the outcome that arises from playing these strategies is the ***rollback equilibrium outcome***

Group Exercise:

Consider the Flag game but instead of starting with 21 flags the game starts with 5 flags, and instead of being able to pick 1,2, or 3 flags teams can only pick 1 or 2 flags.

1. *Draw the extensive form game tree complete with all payoff for both teams.*
2. *How many total strategies are there for team 1?*
3. *Use pruning to eliminate actions to get to a rollback equilibrium. Who will win? What is the winning strategy?*

Adding more players

We can start to add more complexity with more than two players

3-player planting game

- **Emily**, **Nina**, and **Talia** are roommates who want to get a start on their communal garden.
- They like to enjoy the benefits of fresh produce and green space, but it is costly for them to put the work in.
- **2 or 3 people** working is enough to keep the garden healthy, but if **1 or 0** work, then the garden will die.

Planting Game payoffs

outcome:	utlity:
I don't contribute, but garden lives	4
I contribute, and get garden.	3
I don't contribute, and garden dies	2
I contribute, but garden dies	1

Planding Game Tree

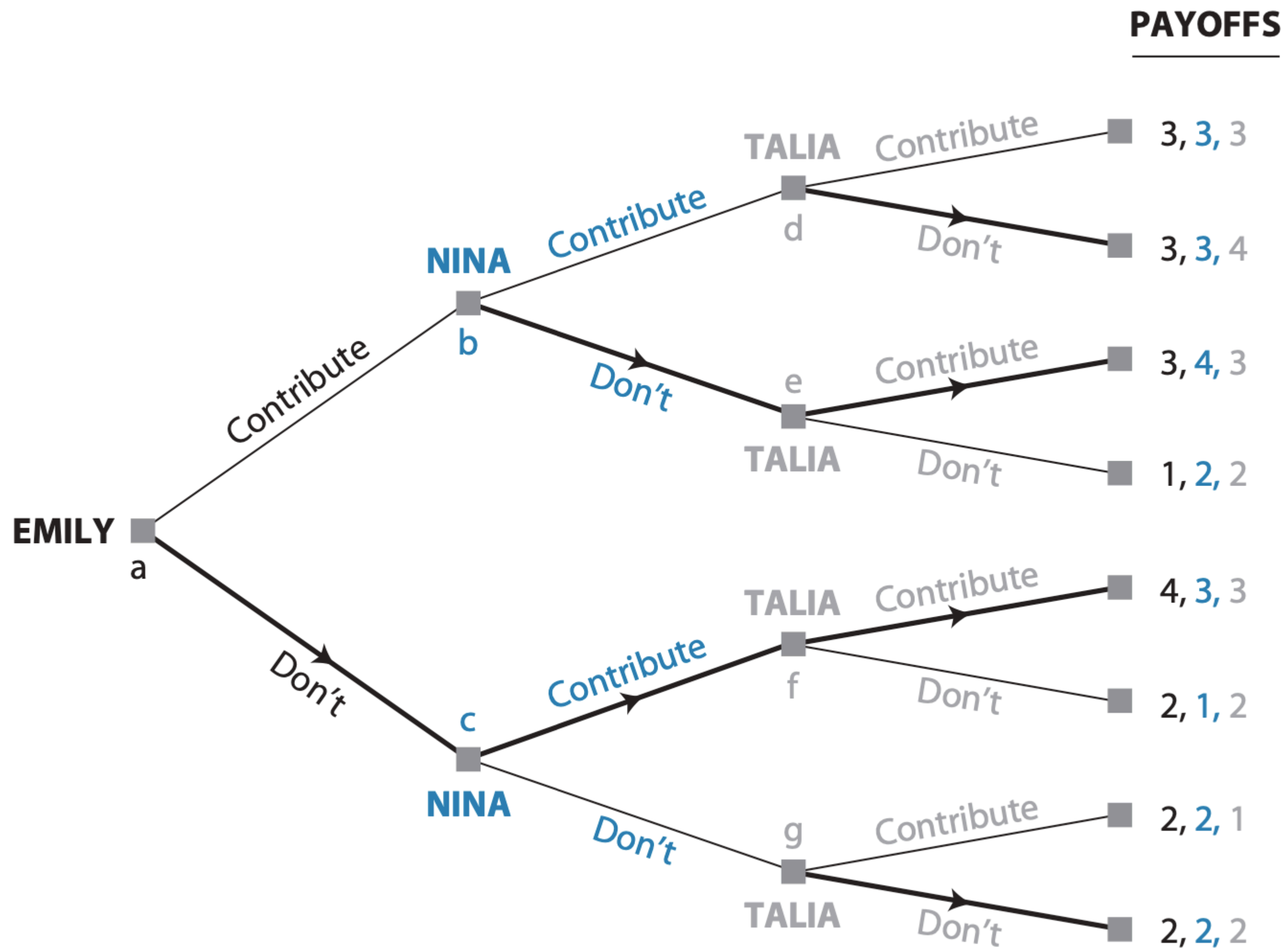

FIGURE 3.6 The Street–Garden Game

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Equilibrium Path of Play

Note that there is one continuous path we traced from the initial node to a final equilibrium outcome.

However, we couldn't have gotten their without the other arrows paths **even though they are never reached** in equilibrium.

Recall that a **strategy** is a collection of choices at **every** decision node.

Equilibrium Strategies

Even though the players available actions are all called the same (Contribute or Don't), this tree provides labels of each decision node so we can say something like:

> "Nina's **strategy** in the rollback equilibrium is { ***Don't Contribute*** at **b**, ***Contribute*** at **c** }".

- To make it even shorter, let's call this strategy **DC**.

How many strategies does Talia have?

- CCCC, CCCD, CCDC, CCDD, CDCC, CDCD, CDDC, CDDD, DCCC, DCCD, DCDC, DCDD, DDCC, DDCD, DDDC, DDDD
- 16 total strategies

Rollback Equilibrium Strategies

The equilibrium is:

- { $\mathbf{D}^{1}$, $\mathbf{DC}^{2}$, $\mathbf{DCCD}^{3}$ }

Adding More Moves

Even a simple game get complicated fast

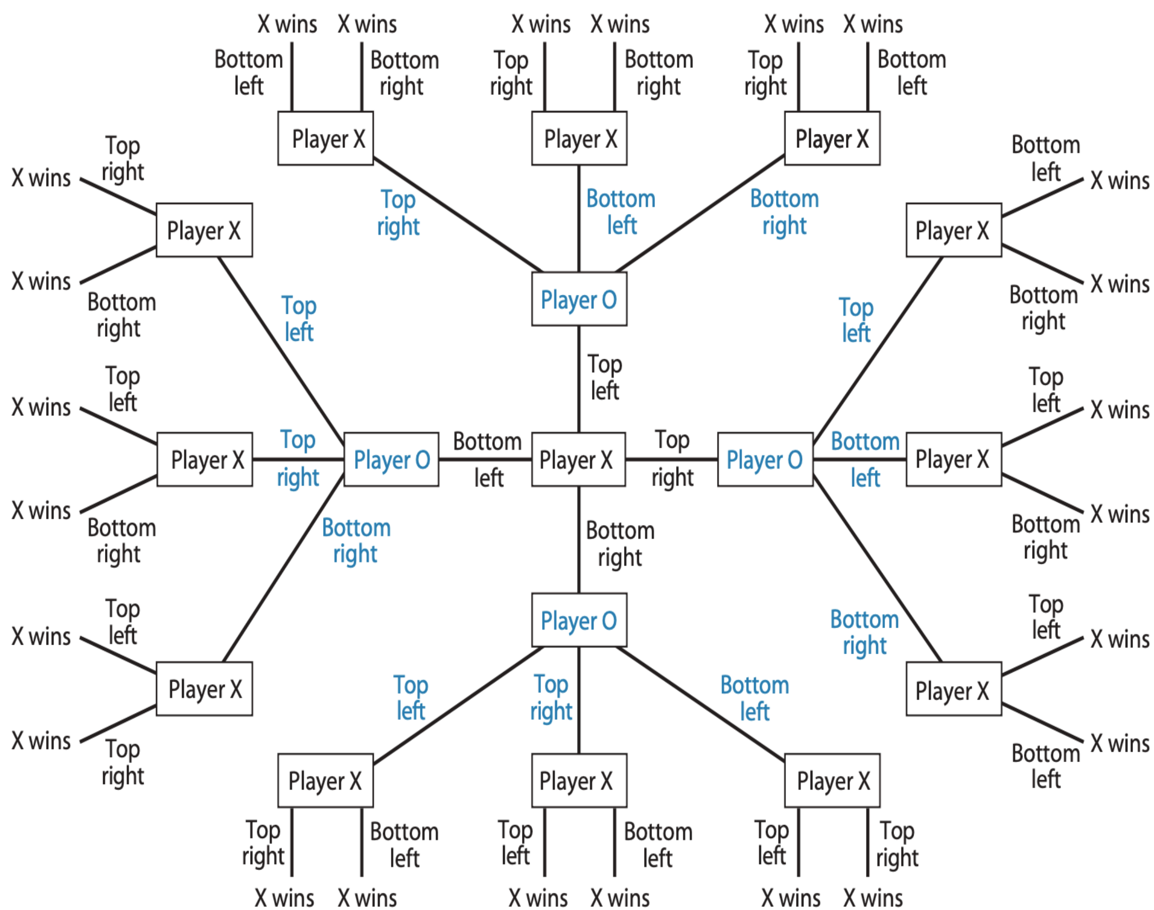

FIGURE 3.7 The Complex Tree for Simple Two-by-Two Tic-Tac-Toe

Tic-Tac-Toe

- Even though it looks complicated, the main branches are really just copies of each other
- Most people probably figure out the rollback equilibrium after playing it enough
- Insert relevant xkcd here: https://xkcd.com/832/

Chess

- What about more complicated games like chess?
  - technically rollback solvable, but with 10^{120} possible moves, it hasn't been solved by either human or machine
- Players of complicated sequential games often implement some **intermediate valuation function** to assign payoffs to non-terminal nodes.

Welfare and Efficiency

What are the **good** outcomes in the planting game?

Can we rank outcomes by collective welfare?

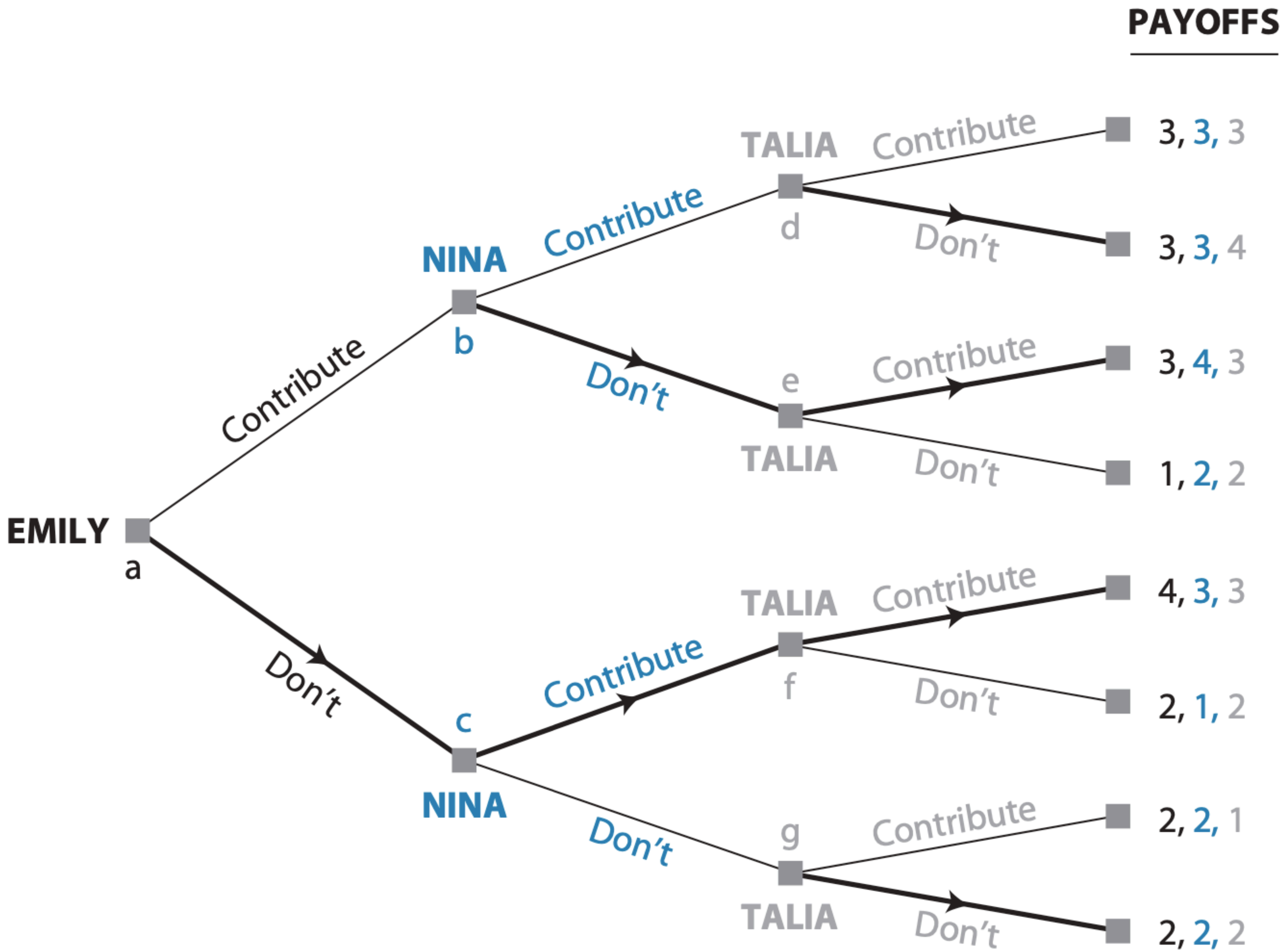

FIGURE 3.6 The Street–Garden Game

Pareto Dominance

Pareto optimality (or efficiency) is econonomists' best shot at coming up with a ranking of which outcomes are objectively 'better'

- For any two outcomes (🎉, 🎊), 🎉 is **Pareto dominated** by 🎊 if both:
  1. No one strictly prefers 🎉 to 🎊 - U👤(🎊) ≥ U👤(🎉) ∀👤 ∈ {👨‍🦲, 👦, 💆, 👰, 👦, 😺, . . . }
  2. At least one person strictly🎉 prefers 🎉 to 🎊 - ∃👤 such that U👤(🎊) > U👤(🎉)

Pareto Improvement

The move from a policy y to an alternative policy x is a **Pareto improvement** if x Pareto dominates y.

- Such a policy change should reasonably be seen as unambiguously good
- Another perspective is that *no-one would veto* a pareto improvement

Pareto Efficiency

An outcome is **Pareto Efficient** (Optimal) if no other outcome Pareto dominates it.

An outcome is **Pareto Inefficient** if at least one other outcome Pareto dominates it.

Ranking the Planting Payoffs

Compare (4,3,3) to (1,2,2)

- Which one is Pareto dominating?

Ranking the Planting Payoffs

Now compare (4,3,3) to (3,4,3) or (3,3,4)

- Which one is Pareto dominating?

Is the rollback equilibrium outcome a *Pareto efficient* one?

Discussion: Efficiency vs other social comparisons

- How useful is Pareto Efficiency in the real world?
- How else could we group outcomes?
- We might address this later in the class with what is known as *Cooperative Game Theory*