A Discrete Choice Model with Residual Uncertainty

1 Framework

Assume some universal partitioning of the unit interval into W disjoint intervals with cutoffs denoted by the (W+1)-dimensional vector $\boldsymbol{\pi} \in \Delta^{W-1}$. Let $\boldsymbol{\alpha}$ denote the partial sums of $\boldsymbol{\pi}$, so that

$$\boldsymbol{\alpha} = \{0, \quad \pi_1, \quad \pi_1 + \pi_2, \quad \dots, \quad \underbrace{\sum_{i=1}^w \pi_i}_{\alpha_w}, \quad \dots, \quad 1\}$$
 (1)

$$\mathcal{W}_w = [\alpha_{w-1}, \alpha_w) \tag{2}$$

$$\bigcup_{i=1}^{W} \mathcal{W}_i = [0, 1) \tag{3}$$

Consumer i's utility for good j is given by

$$U_{ij} = X_j' \beta_i + \eta_{ij}, \quad \eta_{ij} \sim \text{Gumbel}(0, 1)$$
 (4)

where X is the design matrix of good characteristics and X^j denotes the j^{th} row of X. The "zero"-th good is special and is associated with a zero vector of good characteristics. Consumers observe X and η for all goods $j \in \mathcal{J}$, but they do not observe η_{i0} .

First, consumers report their preferred good in $\mathcal{J} = \{1, 2, ..., J\}$, which is given by

$$t_i = e_{j_i^*} \text{ s.t. } j_i^* = \arg\max_{j \in \mathcal{I}} U_{ij}.$$

$$\tag{5}$$

Let V_i denote the utility of good j_i^* for consumer i.

Second, consumers report the probability that they prefer good j_i^* to the outside good 0. Consumers know V_i , but do not know their draw of η_{i0} , and thus this probability is given by

$$p_i = \mathbb{P}(\eta_{i0} < V_i). \tag{6}$$

Each consumer reports the interval \mathcal{W}_w into which p_i falls

$$w_i = w \text{ s.t. } p_i \in [\alpha_{w-1}, \alpha_w). \tag{7}$$

2 Deriving the Likelihood

As is well known, V_i follows a Gumbel distribution with location parameter $\overline{\mu}_i$ and scale parameter 1, where

$$\overline{\mu}_{i} = \ln \left(\sum_{j \in \mathcal{J}} \exp \left(X_{j}^{'} \beta_{i} \right) \right). \tag{8}$$

Under these assumptions, we have

$$p_i = F_{\text{Gumbel}(0,1)}(V_i) \tag{9}$$

The consumer's report of w_i is therefore equivalent to reporting that

$$V_i \in \left[F_{\text{Gumbel}(0)}^{-1} \left(\alpha_{w(i)-1} \right), F_{\text{Gumbel}(0)}^{-1} \left(\alpha_{w(i)} \right) \right), \tag{10}$$

^{1.} This can be motivated by a framework in which $U_0=0$ and $U_{ij}=X_j{'}\beta_i+\eta_{ij}-\eta_0$ for $j\in\mathcal{J}$. Here, η_0 captures the consumer's uncertainty about their future tastes. Given that utilities are ordinal and η_0 is a common shock, it plays no role in the choice among the most preferred good $j^*\in\mathcal{J}$.

^{2.} For example, see McFadden (1981) and Cardell (1997).

where we omit the common scale parameter for brevity.

This occurs with probability

$$\mathbb{P}\left(V_{i} \in \mathcal{W}_{w(i)}\right) = F_{\mathrm{Gumbel}(\overline{\mu})}\left(F_{\mathrm{Gumbel}(0)}^{-1}\left(\alpha_{w(i)}\right)\right) - F_{\mathrm{Gumbel}(\overline{\mu})}\left(F_{\mathrm{Gumbel}(0)}^{-1}\left(\alpha_{w(i)-1}\right)\right) \tag{11}$$

$$= \left(\alpha_{w(i)}\right)^{\exp(\overline{\mu})} - \left(\alpha_{w(i)-1}\right)^{\exp(\overline{\mu})}. \tag{12}$$

This is $p(w_i \mid \alpha, \beta)$, the conditional likelihood of w_i .³

3 Sketch of Estimation

- Specify hyper-parameters governing the prior distribution of (π, β) . There is a relatively large amount of flexibility in the prior distribution over β . $\pi \sim$ Dirichlet deterministically maps to α .
- (First Branch) Given draws of (α, β) , the probability that consumer i reports j_i^* is given by a softmax

$$p(j_i^* \mid \alpha, \beta) = p(j_i^* \mid \beta) = \frac{\exp\left(X_j^{\prime} \beta_i\right)}{\sum_{j' \in \mathcal{J}} \exp\left(X_{j'}^{\prime} \beta_i\right)}$$
(13)

• (Second Branch) Given (α, β) , the probability that consumer i reports w_i is given by

$$p(w_i \mid \alpha, \beta, j_i^*) = p(w_i \mid \alpha, \beta) = \left(\alpha_{w(i)}\right)^{\exp(\overline{\mu})} - \left(\alpha_{w(i)-1}\right)^{\exp(\overline{\mu})}$$

$$\tag{14}$$

where $\overline{\mu}_i(\beta_i)$ is a function of consumer i's tastes β_i and the design matrix X provided in Equation 8.⁴

Thus the overall likelihood (conditional on some draw of parameters) looks like

$$p((j_i^*, w_i) \mid \alpha, \beta) = p(w_i \mid \alpha, \overline{\mu}_i(\beta)) \times p(j^* \mid \beta). \tag{15}$$

Comment: Relation to the Brazell et al. (2006) Dual Response Model

Another common framework when soliciting consumer preferences is to directly ask consumers if $V_i \geq 0$. This implicitly assumes that $\eta_{i0} = 0$, so that consumers deterministically know whether or not the "inside" good j_i^* is preferred to the outside good. As we have already noted, our framework can nest that standard case by assuming that, rather than following a standard Gumbel distribution, η_0 is instead a degenerate distribution. In such a case, all but 2 of the W partitions are empty, and the non-empty partitions are $\mathcal{W}_0 = \{0\}$ and $\mathcal{W}_W = \{1\}$, as $p_i \in \{0,1\}$.

We acknowledge that this requires a slight abuse of notation, as \mathcal{W}_w was defined above using left-closed intervals. These have the advantage of being invertible under the inverse-CDF mapping. In the degenerate case, we instead have $p(w_i = W \mid \alpha, \beta) = \mathbb{P}(\eta_{i0} < V_i) = \mathbb{P}(V_i > 0) = 1 - \exp{(-\exp{(\overline{\mu})})}$.

^{3.} While the parametrization of $\eta_0 \sim \text{Gumbel}(0,1)$ preserves symmetry among the J+1 goods and is thus a natural choice, the framework can easily accommodate an alternative distribution for η_0 . For example, one could use an affine function of individual characteristics to accommodate individual-level variation in the propensity to prefer the outside good.

^{4.} Note that the observed choice j_i^* is irrelevant for this likelihood through "Gumbel distribution magic". Specifically, it is the "memory-less" property of the Gumbel that yields this property.

References

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