高数 第四章

不定积分

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4.1 不定积分的概念与性质

$F(x)$ 有 $F'(x)=f(x)$ 或 $dF(x)=f(x)dx$

称 $F(x)$ 为 $f(x)$ 在区间I上的一个原函数

∴连续函数必定存在原函数

不定积分：$F(x)$ 是 $f(x)$ 在区间I上的一个原函数，称 $F(x)+C$ 为 $f(x)$ 在区间I上的不定积分，记作 $\int f(x)dx$，即

$$\int f(x)dx = F(x)+C$$

$f(x)$ 被积函数　x 积分变量　如：$\int x^2 dx = \frac{1}{3}x^3+C$

$\int$ 积分号　$f(x)dx$ 被积表达式

$$\frac{d}{dx}\left[\int f(x)dx\right]=f(x) \qquad d\left[\int f(x)dx\right]=f(x)dx$$

$$\int F'(x)dx = F(x)+C \qquad \int dF(x)=F(x)+C$$

性质：

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

积分公式：

(13) $\int \frac{1}{1+x^2}dx = \arctan x + C$　(14) $\int \operatorname{sh}x\,dx = \operatorname{ch}x + C$

(1) $\int k\,dx = kx + C$（k为常数）　(2) $\int x^\mu dx = \frac{x^{\mu+1}}{1+\mu}+C$　(15) $\int \operatorname{ch}x\,dx = \operatorname{sh}x + C$

(2) $\int \frac{1}{x}dx = \ln|x|+C$　(4) $\int e^x dx = e^x + C$

(5) $\int a^x dx = \frac{a^x}{\ln a}+C$ $(a>0, a\neq 1)$　(6) $\int \cos x\,dx = \sin x + C$

(7) $\int \sin x\,dx = -\cos x + C$　(8) $\int \sec^2 x\,dx = \int \frac{1}{\cos^2 x}dx = \tan x + C$

~~(9) $\int \csc^2 x\,dx = \int \frac{1}{\sin^2 x}dx = -\cot x + C$~~　(10) $\int \sec x\tan x\,dx = \sec x + C$

(11) $\int \csc x\cot x\,dx = -\csc x + C$　(12) $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

4.2 不定积分的换元积分法

$\int 1\,dx = x+C$　$\int 1\,du = u+C$　$\int 1\,dF(u) = F(u)+C$

令 $F(\mu)=F(\varphi(x))$　$\int 1\,dF(\varphi(x)) = F(\varphi(x))+C$

$\because F'(\varphi(x)) = f(\varphi(x))\varphi'(x)$　$\therefore \int f(\varphi(x))\varphi'(x)dx = F(\varphi(x))+C$

第一类换元积分法 $\int f(\mu)d\mu = F(\mu)+C$，$\mu=\varphi(x)$

凑微分法：把dx前面某一部分，求原函数，拿到d的里面去

例 $\int 2\cos 2x\,dx$：把2的原函数2x凑到d里，$\int \cos 2x\,d2x = \sin 2x + C$

d的里面可以任加减常数

例 $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = -\int (1-\cos^2 x)d\cos x = -\cos x + \frac{\cos^3 x}{3} + C$

例 $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

表 4-2-1　常用凑微分形式

$dx = \frac{1}{a}d(ax), a\neq 0$	$xdx = \frac{1}{2}d(x^2)$
$x^2dx = \frac{1}{3}d(x^3)$	$x^n dx = \frac{1}{n+1}d(x^{n+1})$
$\frac{1}{x^2}dx = -d\left(\frac{1}{x}\right)$	$\frac{1}{\sqrt{x}}dx = 2d(\sqrt{x})$
$x^\mu dx = \frac{1}{\mu+1}d(x^{\mu+1}), \mu\neq -1$	$\frac{1}{x}dx = d(\ln\lvert x\rvert)$
$\sin x dx = -d(\cos x)$	$e^x dx = d(e^x)$
$\cos x dx = d(\sin x)$	$\frac{1}{\sqrt{1-x^2}}dx = d(\arcsin x) = -d(\arccos x)$
$\sec^2 x dx = d(\tan x)$	$\frac{1}{1+x^2}dx = d(\arctan x) = -d(\operatorname{arccot} x)$
$\csc^2 x dx = -d(\cot x)$	$\left(1-\frac{1}{x^2}\right)dx = d\left(x+\frac{1}{x}\right)$
$\sec x\tan x dx = d(\sec x)$	$\left(1+\frac{1}{x^2}\right)dx = d\left(x-\frac{1}{x}\right)$
$\csc x\cot x dx = -d(\csc x)$	$\frac{x}{\sqrt{1-x^2}}dx = -d(\sqrt{1-x^2})$

(16) $\int \tan x\,dx = -\ln|\cos x| + C$

(17) $\int \cot x\,dx = \ln|\sin x| + C$

(18) $\int \sec x dx = \ln|\sec x + \tan x| + C;$

(19) $\int \csc x dx = \ln|\csc x - \cot x| + C;$

(20) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C(a\neq 0);$

(21) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C;$

(22) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C(a\neq 0);$

(23) $\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C(a\neq 0);$

(24) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln(x+\sqrt{x^2+a^2}) + C;$

(25) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C.$

第二类换元积分法　(1) $x=\varphi(t)$，积出来 (2) 换回x　求 $\int \sqrt{\ }dx$

$\int f(x)dx$, $x=\varphi(t)$ $= \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt = F[\varphi^{-1}(x)]+C$

关于x　$t=\varphi^{-1}(x)$　$F(t)+C$ 关于t　关于x

① 被积中有 $\sqrt{a^2-x^2}$　$x=a\sin t$　$\sqrt{a^2-a^2\sin^2 t}=\sqrt{a^2\cos^2 t}$　$(-\frac{\pi}{2}<t<\frac{\pi}{2})$

② 被积中有 $\sqrt{x^2-a^2}$　$x=a\sec t$　$\sqrt{a^2\sec^2 t-a^2}=\sqrt{a^2\tan^2 t}$　$(0<t<\frac{\pi}{2})$

③ 被积有 $\sqrt{x^2+a^2}$　$x=a\tan t$　$\sqrt{a^2\tan^2 t+a^2}=\sqrt{a^2\sec^2 t}$　$(-\frac{\pi}{2}<t<\frac{\pi}{2})$

① 可改成 $a\cos t$　② 可改成 $a^2\csc^2 t$　③ 可改成 $a\cot t$

④ 被积有 $\sqrt{\alpha x^2+\beta x+r}$ 先配方，转为 ①②③

⑤ 当被积函数的分母次幂较高时，倒代换 $x=\frac{1}{t}$

⑥ 含有 $\sqrt[n]{ax+b}$ $\sqrt[n]{\frac{ax+b}{cx+d}}$ 等无理根式时 作根式代换，将无理函数转换为有理函数来积分 $\sqrt[n]{ax+b}=t$

4.3 不定积分的分步积分法

$(\mu v)' = \mu'v + \mu v'$　$\int \mu dv = \mu v - \int v d\mu$

$\mu v' = (\mu v)' - \mu' v$　前 后　前 后　后 前　1°) e^x

2°) $\sin x$ $\cos x$

$\int \mu v' dx = \int (\mu v)'dx - \int \mu' v dx$　① 谁做 μ，谁做 v　3°) x^n

$\int \mu dv = \mu v - \int v d\mu$　② 优先级：选 μ 的顺序 反<对<幂<三<指

③ 要求的东西又出现了

$$\int \ln x\,dx = x\ln x - \int x\,d\ln x = x\ln x - x + C$$

4.4 简单有理函数的积分

有理分式 $\frac{P(x)}{Q(x)}$　$P(x)$、$Q(x)$ 为多项式　$\frac{P(x)}{Q(x)}$　m次 n次

1°) $m\geq n$　假分式：用多项式除法，化为真分式

2°) $m<n$　真分式：

① $\int \frac{1}{2x+1}dx = \frac{1}{2}\int \frac{1}{2x+1}d2x+1 = \frac{1}{2}\ln|2x+1|+C$

② $\int \frac{1}{x^2-3x+2}dx = \int \frac{1}{(x-1)(x-2)}dx = \int \left(\frac{1}{x-2}-\frac{1}{x-1}\right)dx$

③ $\int \frac{1}{x^2-2x+4}dx = \int \frac{1}{(x-1)^2+3}dx-1 = \int \frac{1}{\sqrt{3}} \frac{1}{(\frac{x-1}{\sqrt{3}})^2+1}d\left(\frac{x-1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}\arctan\frac{x-1}{\sqrt{3}}+C$

④ $\int \frac{1}{x^2-2x-3}dx = \int \frac{1}{(x-1)^2-4}dx \overset{t=x-1}{=\!=} \int \frac{1}{t^2-4}dt = \frac{1}{4}\int \left(\frac{1}{(t-2)}-\frac{1}{t+2}\right)dt$

⑤ $\int \frac{x}{x^2+2x+2}dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+2}dx = \frac{1}{2}\int \frac{d(x^2+2x+2)}{x^2+2x+2} - \frac{1}{2}\int \frac{2}{x^2+2x+2}dx$

⑥ $\int \frac{1}{(1+2x)(1+x^2)}dx = \int \left(\frac{A}{1+2x}+\frac{Bx+C}{1+x^2}\right)dx = \frac{2}{5}\int \frac{d(2x+1)}{1+2x} - \frac{1}{5}\int \frac{2x-1}{x^2+1}dx$

⑦ $\int \frac{x^2+1}{(x+2)(x+1)^2}dx = \int \left(\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\right)dx$

$A(x+1)^2+B(x+2)(x+1)+C(x+2)=x^2+1$　代入 $x=-1$，$x=-2$，$x=0$ …

⑧ $\int \frac{\sqrt{x-1}}{x}dx$　$t=\sqrt{x-1}$　$x=t^2+1$　$dx=2tdt$　$= \int \frac{t}{t^2+1}2tdt = 2\int \left(1-\frac{1}{t^2+1}\right)dt$

⑨ $\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$　$t=\sqrt[6]{x}$　$dx=6t^5dt$　$= \int \frac{6t^5}{(1+t^2)t^3}dt$

⑩ 令 $\tan\frac{x}{2}=t$　$\sin x = \frac{2t}{1+t^2}$　$\cos x = \frac{1-t^2}{1+t^2}$

$\int \frac{dx}{4+5\cos x} = \int \frac{1}{4+5\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt = -2\int \frac{1}{(t-3)(t+3)}dt$

本笔记在https://github.com/dydcyy-gh/study-notes开源