第八章

多元函数微分学及其应用

8.1 平面点集与多元函数的基本概念

8.1.1 平面点集

$R^2 = R \times R = \{(x,y) \mid x, y \in R\}$

$E = \{(x,y) \mid (x,y)$ 满足条件 $T\}$

邻域 $U(P_0, \delta) = \{P \mid |PP_0| < \delta\} = \{(x,y) \mid \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\}$

去心邻域 $\mathring{U}(P_0, \delta) = \{P \mid 0 < |PP_0| < \delta\} = \{(x,y) \mid 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\}$

内点 $\exists U(p) \subset E$，P 为 E 的内点.

外点 $\exists U(p)$，使 $U(p) \cap E = \phi$，外点.

∂E 边界点 $\forall U(p)$ 内有同时在 E 内外的点、(P 本身可不属于 E)

聚点：对 $\forall \delta > 0$，使 $\mathring{U}(P, \delta) \cap E \neq \phi$，$P$ 为 E 聚点

内点一定是聚点、

孤立点：如果 $\exists \delta > 0$，使得 $U(P, \delta) \cap E = \{P\}$

开集：点集 E 的点都是 E 的内点、

闭集：开集连同其边界一起称闭集

连通集：点集 E 任两点可用一折线连通

开区域：连通的开集

闭区域：开区域 + 边界

有界集：如存在 $\delta > 0$，使 $E \subset U(O, \delta)$，E 为有界集

O 为坐标原点.

无界集：点集非有界集

D 为平面上的有界区域 $d = \max\limits_{P_1 P_2 \in D} \{|P_1 P_2|\}$ 为 D 的直径

8.1.2 n维空间

$R^n = \{(x_1, x_2 \cdots x_n) \mid x_i \in R, i = 1 \cdots n\}$

$P(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} = \|x - y\|$

$U(P_0, \delta) = \{P \mid |PP_0| < \delta, P \in R^n\}$

8.1.3 多元函数的概念

求多元函数的定义域和值域

8.1.4 多元函数的极限

二元函数的极限.

求极限不能用洛必达法则

极限存在：以任意方式逼近 $(x_0\ y_0)$

以不同方式逼近，极限不相等：极限不存在

8.1.5 多元函数的连续性

一切二元初等函数在其定义区域内都是连续的

由连续性 $\lim\limits_{P \to P_0} f(P) = f(P_0)$ （直接代值进去算）

不连续：间断点 → 间断线

多元函数的性质

$$\begin{tikzcd}
\text{连续} \arrow[rr, dotted, no head] & & \text{可偏导} \arrow[dl, dotted] \\
& \text{可微} \arrow[ul] \arrow[ur] \arrow[d, dotted] & \\
& \text{偏导数连续} \arrow[u] &
\end{tikzcd}$$

（其他均不成立）

非分段点：连续可偏导可微

分段点/边界点：

连续：$\lim\limits_{(x,y) \to (\text{分段})} f(x,y) \overset{?}{=} f(\text{分段})$

可偏导：$f_x(x_0 y_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x, y_0) - f(x_0 y_0)}{\Delta x}$ $\quad f_y(x_0, y_0) =$

可微：$\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta z - dz}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$ $\quad dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy$

8.3 全微分

1、偏增量 $\Delta z_x = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$

$\Delta z_y = f(x_0, y_0 + \Delta y) - f(x_0\ y_0)$

全增量 $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 y_0)$

2、全微分 若 $\Delta z = A \Delta x + B \Delta y + o(\rho)$ $\quad \rho = \sqrt{\Delta x^2 + \Delta y^2}$

（$A\ B$ 是 $x\ y$ 的函数，与 $\Delta x, \Delta y$ 无关）

则 $dz = A \Delta x + B \Delta y$ 为全微分 （$\Delta z \approx dz$）

定理：若 $z = f(x y)$ 在点 $(x_0 y_0)$ 可微，该函数在点 $(x_0 y_0)$ 的导数 $f_x(x_0 y_0)$ $f_y(x_0 y_0)$ 必存在，且 $df(x_0 y_0) = f_x(x_0 y_0) \Delta x + f_y(x_0 y_0) \Delta y$

$\therefore A = f_x(x_0 y_0) \quad B = f_y(x_0 y_0)$

由 $dz \approx \Delta z$

$\therefore f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0 y_0) + f_x'(x_0 y_0) \Delta x + f_y'(x_0 y_0) \Delta y$

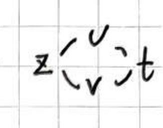
3.

$$\begin{tikzcd}
\text{连续} \arrow[r, shift left, "\times"] \arrow[rr, bend right, "\times"'] & \text{可偏导} \arrow[l, shift left, "\times"] \arrow[r, shift left, "\times"] & \text{可微} \arrow[l, shift left] \arrow[r, shift left, "\times"] \arrow[ll, bend left] & \text{偏导数连续} \arrow[l, shift left]
\end{tikzcd}$$

4、若 $z = f(x,y)$ 处处可微，则 $dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy$

8.4 多元复合函数的求导法则

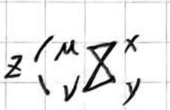
① $z = f(u, v)$ $\ u = \varphi(t)$ $\ v = \psi(t)$ $\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial u} \cdot \dfrac{du}{dt} + \dfrac{\partial z}{\partial v} \dfrac{dv}{dt}$



② $z = f(u, v)$ $\ u = \varphi(x, y)$ $\ v = \psi(x, y)$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \dfrac{\partial v}{\partial x}$

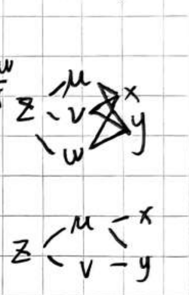
$\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v} \dfrac{\partial v}{\partial y}$

右边有1分支，求导，>1 求偏导



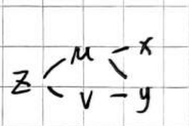
③ $z = f(u\,v\,w)$ $\ u = \varphi(x, y)$ $\ v = \psi(x, y)$ $\ w = w(x, y)$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \dfrac{\partial v}{\partial x} + \dfrac{\partial z}{\partial w} \dfrac{\partial w}{\partial x}$

$\dfrac{\partial z}{\partial y} = \text{〃}$



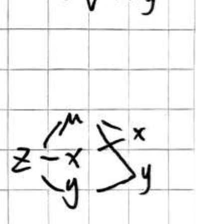
④ $z = f(u, v)$ $\ u = \varphi(x, y)$ $\ v = \psi(y)$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \dfrac{\partial u}{\partial x}$

$\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v} \dfrac{dv}{dy}$



⑤ $z = f(u, x, y)$ $\ u = \varphi(x, y)$ $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial x} \left(\dfrac{dx}{dx}\right)$

$\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u} \dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial y} \left(\dfrac{dy}{dy}\right)$



例：$z = f(x, y, u, v, w)$ $\ u = g(x, y)$ $\ v = h(y)$ $\ w = w(x)$

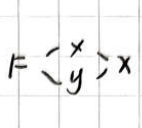
$\dfrac{\partial H}{\partial x} = \dfrac{\partial f}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial w} \dfrac{dw}{dx} + \dfrac{\partial f}{\partial x}$

例 $u = f(x, y, z) = e^{x^2+y^2+z^2}$ $\quad z = x^2 \sin y$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z} \dfrac{\partial z}{\partial x} = 2x e^{x^2+y^2+z^2} + 2z e^{x^2+y^2+z^2} \cdot 2x \sin y = \cdots$

8.5 隐函数的求导公式

1、二元方程的情形 $F(x,y) = 0$ $\quad \dfrac{dy}{dx} = \dfrac{-F'(x)}{F'(y)}$



$\underbrace{\dfrac{\partial F}{\partial x}}_{F'(x)} + \underbrace{\dfrac{\partial F}{\partial y}}_{F'(y)} \dfrac{dy}{dx} = 0$

2、三元方程的情形 $F(x, y, z) = 0$ $\quad \dfrac{\partial z}{\partial x} = -\dfrac{F'(x)}{F'(z)}$ $\quad \dfrac{\partial z}{\partial y} = -\dfrac{F'(y)}{F'(z)}$

3、方程组的情形

$\begin{cases} F(x, y, u, v) \\ G(x, y, u, v) \end{cases}$

雅可比行列式 $J = \dfrac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$

$\dfrac{\partial u}{\partial x} = -\dfrac{1}{J} \dfrac{\partial(F, G)}{\partial(x, y)} = -\dfrac{\begin{vmatrix} F_x' & F_v' \\ G_x' & G_v' \end{vmatrix}}{\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}}$

例：$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ $u.v.$ 对 x 求偏导 $\begin{cases} u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} = 0 \\ y \frac{\partial u}{\partial x} + v + x \frac{\partial v}{\partial x} = 0 \end{cases}$

$\begin{cases} x u_x' - y v_x' = -u \\ y u_x' + x v_x' = -v \end{cases}$ 行列式，得 $\dfrac{\partial u}{\partial x} = \dfrac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}$ $\quad \dfrac{\partial v}{\partial x} = \dfrac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}$

另一解法

$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ → $y \dfrac{yv}{x} + xv = 1$ $\quad y^2 v + x^2 v = x$ $\quad v = \dfrac{x}{x^2+y^2}$

$u = \dfrac{y}{x} \dfrac{x}{x^2+y^2} = \dfrac{y}{x^2+y^2}$ $\quad \dfrac{\partial u}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2}$ $\quad \dfrac{\partial u}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

8.6 多元函数微分学的几何应用

1、一元向量值函数及其导数

$\vec{f}(t) = f_1(t) \vec{i} + f_2(t) \vec{j} + f_3(t) \vec{k} = (f_1(t), f_2(t), f_3(t))$

极限：$\lim\limits_{t \to t_0} f(t) = \vec{r}_0$ $\quad \lim\limits_{t \to t_0} \vec{f}(t) = (\lim\limits_{t \to t_0} f_1(t), \lim\limits_{t \to t_0} f_2(t), \lim\limits_{t \to t_0} f_3(t))$

连续：$\lim\limits_{t \to t_0} f(t) = f(t_0)$ $\quad$ 导数 $\lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$

$f'(t_0) = (f_1'(t_0), f_2'(t_0), f_3'(t_0))$

求导法则：与普通的求导方法相同

其中 $\dfrac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$

$\dfrac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$

2、空间曲线的切线与法平面

① 空间曲线 Γ 的方程为参数式的情形

$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $(\alpha \le t \le \beta)$ $\quad$ 切向量 $(x'_{t_0}\ y'_{t_0}\ z'_{t_0})$

Γ 在 P_0 处的切线方程为 $\dfrac{x - x_0}{x'(t_0)} = \dfrac{y - y_0}{y'(t_0)} = \dfrac{z - z_0}{z'(t_0)}$

$x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0$ 法平面方程

② 空间曲线 Γ 的方程为一般式的情形

1° $\begin{cases} y = \varphi(x) \\ z = \psi(x) \end{cases}$ → $\begin{cases} x = x \\ y = \varphi(x) \\ z = \psi(x) \end{cases}$

2° $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$

例：$\begin{cases} x^2 + y^2 + z^2 = 6 \\ x + y + z = 0 \end{cases}$ 过点 $(1, -2, 1)$ 的切线和法平面

对 x 求得 $\begin{cases} y y_x' + z z_x' = -x \\ y_x' + z_x' = -1 \end{cases}$

由克莱姆法则 $D = \begin{vmatrix} y & z \\ 1 & 1 \end{vmatrix}$ $\quad D_1 = \begin{vmatrix} -x & z \\ -1 & 1 \end{vmatrix}$ $\quad D_2 = \begin{vmatrix} y & -x \\ 1 & -1 \end{vmatrix}$

$\therefore y_x' = \dfrac{-x + z}{y - z}$ $\quad z_x' = \dfrac{-y + x}{y - z}$ $\quad \therefore T(1, 0, -1)$

3、空间曲面的切平面和法线

① $F(x, y, z) = 0$

$\vec{n} = (F_x(x_0 y_0 z_0), F_y(x_0 y_0 z_0), F_z(x_0 y_0 z_0))$

切平面方程 $F_x(x_0 y_0 z_0)(x - x_0) + F_y(x_0 y_0 z_0)(y - y_0) + F_z(x_0 y_0 z_0)(z - z_0) = 0$

法线方程 $\dfrac{x - x_0}{F_x(x_0 y_0 z_0)} = \dfrac{y - y_0}{F_y(x_0 y_0 z_0)} = \dfrac{z - z_0}{F_z(x_0 y_0 z_0)}$

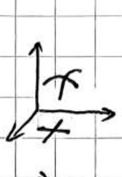
② $z = f(x, y)$ $\quad \vec{n} = (f_x(x_0 y_0), f_y(x_0 y_0), -1)$ $\quad$ 切平面方程 $f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0) - (z - z_0) = 0$

法线方程 $\dfrac{x - x_0}{f_x(x_0 y_0)} = \dfrac{y - y_0}{f_y(x_0 y_0)} = \dfrac{z - z_0}{-1}$

8.7 方向导数和梯度

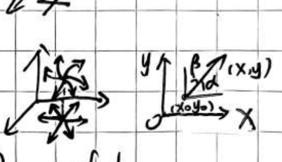
1、方向导数

偏导数 $f_x'(x_0 y_0)$ $\ f_y'(x_0 y_0)$



$(x_0 y_0)$ 处 x 方向 y 方向斜率

方向导数 射线 $\begin{cases} x = x_0 + t\cos\alpha \\ y = y_0 + t\cos\beta \end{cases}$



$\lim\limits_{t \to 0^+} \dfrac{f(x_0 + t\cos\alpha, y_0 + t\cos\beta) - f(x_0 y_0)}{t} = \left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)}$

(是 0^+ !)

特殊：当 $e_l = (1, 0)$ $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = f_x'(x_0 y_0)$

当 $e_l = (0, 1)$ $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = f_y'(x_0 y_0)$

方向导数存在，偏导数不一定存在 （0^+ 的问题，只有一边）

定理：$f(x y)$ 在 $(x_0 y_0)$ 可微，方向导数存在，且 $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = f_x'(x_0 y_0) \cos\alpha + f_y'(x_0 y_0) \cos\beta$

求法：求 f_x' f_y' 方向向量 e_l 代入得

多元函数：偏导存在 $\not\to$ 连续

可微 ⟶ 偏导存在

偏导存在且连续 → 可微

可微 ⟶ 连续

可微 ⟶ 方向导数存在

（多元函数可微是关键）

（一元函数可导是关键）

2、梯度

方向导数 $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = f_x'(x_0 y_0) \cos\alpha + f_y'(x_0 y_0) \cos\beta$

$= (f_x'(x_0 y_0)\ f_y'(x_0 y_0))(\cos\alpha \cos\beta)$ （是数）

梯度 $\mathrm{grad} f(x_0 y_0) = \nabla f(x_0 y_0) = (f_x'(x_0 y_0), f_y'(x_0 y_0))$ （是向量）

梯度为 xoy 投影向量

由梯度可知 $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = |\mathrm{grad} f(x_0 y_0)| |e_l| \cos\theta$

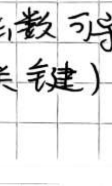
(1) $\theta = 0$ 时 $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0 y_0)} = |\mathrm{grad} f(x_0 y_0)|$ 方向导数取最大值

$\theta = 180$ 时，方向导数最小 $\quad \theta = 90$ 时，方向导数为 0

$z = f(x, y)$ 为等值线（等高线）

$\dfrac{(f_x'(x_0 y_0), f_y'(x_0 y_0))}{\sqrt{[f_x'(x_0 y_0)]^2 + [f_y'(x_0 y_0)]^2}}$ 为 $P_0(x_0 y_0)$ 处的单位法向量

梯度方向为等值线在此处的法线方向



8.9 多元函数的极值及其求法

1、多元函数的极值及最大值最小值

① 极值存在的必要条件：

$(x_0 y_0)$ 处有偏导数，有极值，则 $f_x(x_0 y_0) = 0$ $f_y(x_0 y_0) = 0$

驻点：$f_x = 0$ $f_y = 0$ 同时成立 $\quad$ 极值 + 可偏导 → 驻点，反之不成立

② 极值存在的充分条件：（判断驻点是否为极值点）

$z = f(x, y)$ 在 $(x_0 y_0)$ 有2阶偏导，且 $f_x = 0$ $f_y = 0$

$f_{xx}(x_0 y_0) = A$ $\quad f_{xy}(x_0 y_0) = B$ $\quad f_{yy}(x_0 y_0) = C$

$\begin{cases} AC - B^2 > 0 & \text{极值} \quad A < 0 \text{ 极大值} \quad A > 0 \text{ 极小值} \\ AC - B^2 < 0 & \text{无极值} \\ AC - B^2 = 0, & \text{无法判断} \end{cases}$

③ 最值：可能在驻点、偏导不存在，端点、

2、条件极值，拉格朗日乘数法

$z = f(x, y)$，约束条件 $\phi(x, y) = 0$，计算知

$f_x(x_0 y_0) - f_y(x_0 y_0) \dfrac{\phi_x(x_0 y_0)}{\phi_y(x_0 y_0)} = 0$

易知 $\dfrac{f_x(x_0 y_0)}{\phi_x(x_0 y_0)} = \dfrac{f_y(x_0 y_0)}{\phi_y(x_0 y_0)} = -\lambda_0$

$\therefore z = f(x, y)$ 在 $\phi(x, y) = 0$ 条件下在 $P_0(x_0 y_0)$ 取极值的必要条件

$\begin{cases} f_x(x_0 y_0) + \lambda_0 \phi_x(x_0 y_0) = 0 \\ f_y(x_0 y_0) + \lambda_0 \phi_y(x_0 y_0) = 0 \\ \phi(x_0 y_0) = 0 \end{cases}$

做题：$z = f(x, y)$ $\ \phi(x, y) = 0$

构造拉格朗日辅助函数 $F(x, y, \lambda) = f(x, y) + \lambda \phi(x, y)$

$\begin{cases} L_x' = 0 \\ L_y' = 0 \end{cases}$ $\quad \begin{cases} f_x' + \lambda \phi_x' = 0 \\ f_y' + \lambda \phi_y' = 0 \\ \phi(x, y) = 0 \end{cases}$ $\quad$ 对 x, y 求偏导

当2个约束时 $u = f(x\,y\,z\,t)$ $\quad \varphi(x\,y\,z\,t) = 0$ $\quad \psi(x, y, z, t) = 0$

$L = f(x, y, z, t) + \lambda \varphi(x, y, z, t) + \mu \psi(x, y, z, t)$

$\begin{cases} L_x' = 0 \quad L_y' = 0 \quad L_z' = 0 \quad L_t' = 0 \\ \varphi(x, y, z, t) = 0 \quad \psi(x, y, z, t) = 0 \end{cases}$

本笔记在https://github.com/dydcyy-gh/study-notes开源