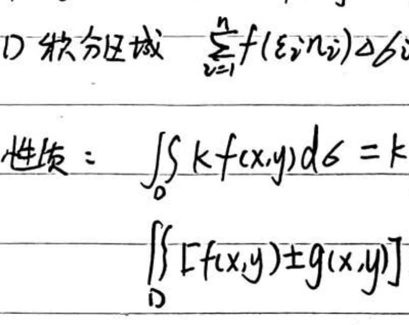
第九章

重积分

第九章 重积分

9.1 二重积分的概念与性质



$f(x,y)$ 有界函数 $\Delta\sigma_i$ 小区域面积，其中一点 $f(\xi_i, \eta_i)$，小区域直径最大值 λ

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\cdot\Delta\sigma_i = \iint_D f(x,y)d\sigma$$

$f(x,y)$ 被积函数　$f(x,y)d\sigma$ 被积表达式　$d\sigma$ 面积微元　x, y 积分变量

D 积分区域　$\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$ 积分和　$d\sigma = dx\,dy$

性质：$\iint_D k f(x,y)d\sigma = k\iint_D f(x,y)d\sigma$

$$\iint_D [f(x,y)\pm g(x,y)]d\sigma = \iint_D f(x,y)d\sigma \pm \iint_D g(x,y)d\sigma$$

$$\iint_D 1 d\sigma = \iint_D d\sigma = \sigma \quad (\text{高为1，体积=底面积×1})$$

$$\iint_D f(x,y)d\sigma = \iint_{D_1} f(x,y)d\sigma + \iint_{D_2} f(x,y)d\sigma$$

$\iint_D f(x,y)d\sigma \geq 0$　($f(x,y)\geq 0$ 时)　保号性

$g(x,y)\leq f(x,y)$　$\iint_D f(x,y)d\sigma \geq \iint_D g(x,y)d\sigma$

$$\left|\iint_D f(x,y)d\sigma\right| \leq \iint_D |f(x,y)|d\sigma$$

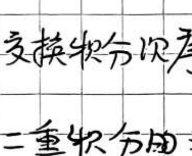
估值定理：M, m　$f(x,y)$ 在 D max, min

$$m\sigma \leq \iint_D f(x,y)d\sigma \leq M\sigma$$

中值定理，D 至少有一点 (ξ,η)，使 $\iint_D f(x,y)d\sigma = f(\xi,\eta)\cdot\sigma$

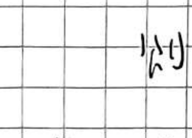
9.2 二重积分的计算

1. X型区域



$$\begin{cases} a\leq x\leq b \\ \varphi_1(x)\leq y\leq\varphi_2(x)\end{cases} \qquad \int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

2、Y型区域



$$\begin{cases} c\leq y\leq d \\ \psi_1(y)\leq x\leq\psi_2(y)\end{cases} \qquad \int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

3、交换积分次序

4、二重积分的对称性

积分区域 D 和被积函数 $f(x,y)$ 均为关于 x/y 的奇函数

则 $\iint_D f(x,y)d\sigma = 0$

积分区域 D 和被积函数 $f(x,y)$ 均为关于 x/y 的偶函数

则 $\iint_D f(x,y)d\sigma = 2\iint_{D_1} f(x,y)d\sigma$

xy 同时对称 $\iint_D f(x,y)d\sigma = 4\iint_{D_1} f(x,y)d\sigma$

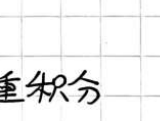
积分区域关于 $y=x$ 对称，则

$$\iint_D f(x)dxdy = \iint_D f(y)dxdy = \frac{1}{2}\iint_D [f(x)+f(y)]dxdy$$

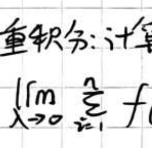
(二重积分的对换性/轮换对称性)

5、极坐标系下计算二重积分

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta\end{cases} \quad f(x,y)=f(\rho\cos\theta,\rho\sin\theta)$$



$$\begin{cases}\alpha\leq\theta\leq\beta \\ \rho_1(\theta)\leq\rho\leq\rho_2(\theta)\end{cases} \quad \iint_D f(x,y)d\sigma = \int_\alpha^\beta d\theta\int_{\rho_1\theta}^{\rho_2\theta} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$



$$\begin{cases}\alpha\leq\theta\leq\beta \\ 0\leq\rho\leq\rho(\theta)\end{cases} \quad \iint_D f(x,y)d\sigma = \int_\alpha^\beta d\theta\int_0^{\rho(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

$$\begin{cases}0\leq\theta\leq 2\pi \\ 0\leq\rho\leq\rho(\theta)\end{cases} \quad \iint_D f(x,y)d\sigma = \int_0^{2\pi} d\theta\int_0^{\rho\theta} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

9.3 三重积分

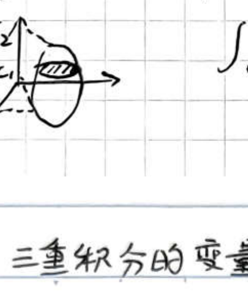
1、二重积分：计算密度均匀体积 → 三重积分：密度不均匀体积 $\Omega\Delta V_i$

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i = \iiint_\Omega f(x,y,z)dv = \iiint_\Omega f(x,y,z)dxdydz$$

性质与二重积分相同

2、直角坐标系下的三重积分计算

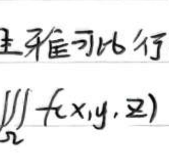
投影法(先一后二)



$$\iint_{D_{xy}}\Big[\underbrace{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz}_{\text{关于}z\text{一重积分}}\Big]dxdy$$

$$=\int_a^b dx\int_{y_1(x)}^{y_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz$$

截面法(先二后一)



$$\int_{c_1}^{c_2} dz\iint_{D_z} f(x,y,z)dxdy$$

3、三重积分的变量代换

设函数 $x=\varphi(u,v,w)$　$y=\psi(u,v,w)$　$z=\chi(u,v,w)$，有一阶偏导数

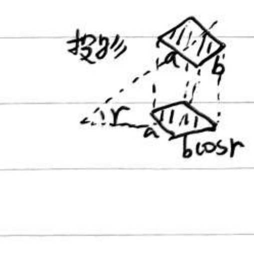
且雅可比行列式 $J(u,v,w)\neq 0$，则变换是 uvw 到 xyz 对应变换

$$\iiint_\Omega f(x,y,z)dv = \iiint_{\Omega'} f[\varphi(u,v,w),\psi(u,v,w),\chi(u,v,w)]|J|dudvdw$$

4、柱面坐标系下三重积分的计算　(极坐标)

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z\end{cases} \quad \iiint_\Omega f(x,y,z)dxdydz = \iiint_{\Omega'} f(\rho\cos\theta,\rho\sin\theta,z)\underline{\underline{\rho}}d\rho d\theta dz$$

5、球面坐标系下三重积分的计算

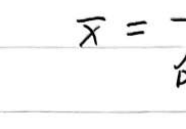


$$\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi\end{cases} \quad \begin{cases} 0\leq r\leq+\infty \\ 0\leq\varphi\leq\pi \\ 0\leq\theta\leq 2\pi\end{cases}$$

$$\iiint_\Omega f(x,y,z)dxdydz = \iiint_\Omega f(\rho\cos\theta,\rho\sin\theta,z)\rho d\rho d\theta dz$$

6、三重积分的应用(求曲面的面积)

投影



$$dA = \frac{d\sigma}{\cos r} \qquad \cos r = \frac{1}{\sqrt{1+(f_x')^2+(f_y')^2}}$$

$$\therefore dA = \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,d\sigma$$

9.4 重积分的应用

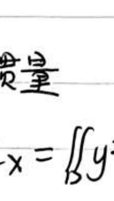
1、求质心

$$\bar{x} = \frac{\iint_D x\mu(x,y)d\sigma}{\iint_D \mu(x,y)d\sigma} \qquad \bar{y} = \frac{\iint_D y\mu(x,y)d\sigma}{\iint_D \mu(x,y)d\sigma}$$

若平面薄片均匀分布　$\bar{x} = \frac{1}{A}\iint_D x d\sigma$　$\bar{y} = \frac{1}{A}\iint_D y d\sigma$

A 为面积，D 为面积区域

例：



$\rho=2\sin\theta$
$\rho=4\sin\theta$

$$\bar{y} = \frac{1}{A}\iint_D y d\sigma = \frac{1}{3\pi}\int_0^\pi d\theta\int_{2\sin\theta}^{4\sin\theta}\rho\sin\theta\rho d\rho$$

三维时　$\bar{x} = \dfrac{\iiint_\Omega x\mu(x,y,z)dv}{\iiint_\Omega \mu(x,y,z)dv}$　$\bar{z} = \dfrac{\iiint_\Omega z\mu(x,y,z)dv}{\iiint_\Omega \mu(x,y,z)dv}$

均匀分布：$\bar{x} = \frac{1}{A}\iiint_\Omega x dv$　$\bar{z} = \frac{1}{A}\iiint_\Omega z dv$

2、求转动惯量

$$I_x = \iint_D y^2\mu(x,y)d\sigma \qquad I_y = \iint_D x^2\mu(x,y)d\sigma$$

三维：$I_x = \iiint_\Omega (y^2+z^2)\mu(x,y,z)$

$I_z = \iiint_\Omega (x^2+y^2)\mu(x,y,z)$

3、求引力

$$F_x = \iiint_\Omega Gm\frac{\mu(x,y,z)(x-x_0)}{r^3}dv$$

$$F_y = \iiint_\Omega Gm\frac{\mu(x,y,z)(y-y_0)}{r^3}dv$$

$$F_z = \iiint_\Omega Gm\frac{\mu(x,y,z)(z-z_0)}{r^3}dv$$

物体对 (x_0, y_0, z_0)

$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$

$F = (F_x, F_y, F_z)$