

Numerical Optimization

Dhyan Laad
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1 Review

1.1 Inner Product Spaces

Definition 1.1. Let V be a real vector space. A function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ is called a *real inner product* if it satisfies the following three properties for $x, y, z \in V$ and $c \in \mathbb{R}$:

- (a) $\langle x, y \rangle = \langle y, x \rangle$,
- (b) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$,
- (c) $\langle cx, y \rangle = c\langle x, y \rangle$, and
- (d) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

The inner product is the generalization of the dot product on Euclidean vector spaces: for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ where $x_i, y_i \in \mathbb{R}$ for $i \in 1 : n$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

It may also be defined on \mathbb{C} -spaces, replacing (a) with conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$, and adding conjugate linearity in the second argument: $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$.

Definition 1.2. Let V be a real vector space and $\langle \cdot, \cdot \rangle$ an inner product. Then, $(V, \langle \cdot, \cdot \rangle)$ is called a *real inner product space*.

For brevity, we may simply state that V is an inner product space, with the notation for the inner product being implicit.

1.2 Normed Linear Spaces

Definition 1.3. Let V be a real vector space. A function $\|\cdot\| : V \rightarrow \mathbb{R}$ is called a *norm* if it satisfies the following properties for $x, y \in V$ and $c \in \mathbb{R}$:

- (a) $\|cx\| = |c|\|x\|$,
- (b) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$, and



$$(c) \|x + y\| \leq \|x\| + \|y\|.$$

This last property is referred to as the *triangle inequality*. The norm assigns a notion of length to vectors, and generalizes the standard formula for the length of a vector in a Euclidean vector space: for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{R}$ for $i \in 1 : n$.

$$\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}.$$

Definition 1.4. Let V be a vector space and $\|\cdot\|$ a norm. Then $(V, \|\cdot\|)$ is called a *normed linear space*.

Once again, we may conventionally omit the norm from notation when defining a new normed linear space.

Definition 1.5. Let $p \geq 1$. The *p-norm* (or *ℓ^p -norm*) of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{R}$ for $i \in 1 : n$ is

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

For $p = 1$, we get the *taxicab* or *Manhattan* norm, for $p = 2$, we get the standard Euclidean norm, and for $p \rightarrow \infty$, the *p-norm* approaches the *infinity* or *maximum* norm:

$$\|\mathbf{x}\|_\infty := \max_i |x_i|.$$

For $p \in (0, 1)$, the triangle inequality does not hold, and the resulting functions are called *quasinorms*.