

# Numerical Optimization

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## 1 Review

### 1.1 Inner Product Spaces

**Definition 1.1.** Let  $V$  be a real vector space. A function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  is called a *real inner product* if it satisfies the following three properties for  $x, y, z \in V$  and  $c \in \mathbb{R}$ :

- (a)  $\langle x, y \rangle = \langle y, x \rangle$ ,
- (b)  $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ ,
- (c)  $\langle cx, y \rangle = c\langle x, y \rangle$ , and
- (d)  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ .

The inner product is the generalization of the dot product on Euclidean vector spaces: for  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  where  $x_i, y_i \in \mathbb{R}$  for  $i \in 1 : n$ ,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

It may also be defined on  $\mathbb{C}$ -spaces, replacing (a) with conjugate symmetry:  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ , and adding conjugate linearity in the second argument:  $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$ .

**Definition 1.2.** Let  $V$  be a real vector space and  $\langle \cdot, \cdot \rangle$  an inner product. Then,  $(V, \langle \cdot, \cdot \rangle)$  is called a *real inner product space*.

For brevity, we may simply state that  $V$  is an inner product space, with the notation for the inner product being implicit.

### 1.2 Normed Linear Spaces

**Definition 1.3.** Let  $V$  be a real vector space. A function  $\|\cdot\| : V \rightarrow \mathbb{R}$  is called a *norm* if it satisfies the following properties for  $x, y \in V$  and  $c \in \mathbb{R}$ :

- (a)  $\|cx\| = |c|\|x\|$ ,
- (b)  $\|x\| \geq 0$  and  $\|x\| = 0 \Leftrightarrow x = 0$ , and

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(c)  $\|x + y\| \leq \|x\| + \|y\|.$

This last property is referred to as the *triangle inequality*. The norm assigns a notion of length to vectors, and generalizes the standard formula for the length of a vector in a Euclidean vector space: for  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  where  $x_i \in \mathbb{R}$  for  $i \in 1 : n$ .

$$\|\mathbf{x}\| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}.$$

**Definition 1.4.** Let  $V$  be a vector space and  $\|\cdot\|$  a norm. Then  $(V, \|\cdot\|)$  is called a *normed linear space*.

Once again, we may conventionally omit the norm from notation when defining a new normed linear space.

**Definition 1.5.** Let  $p \geq 1$ . The  $p$ -norm (or  $\ell^p$ -norm) of a vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  where  $x_i \in \mathbb{R}$  for  $i \in 1 : n$  is

$$\|\mathbf{x}\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

For  $p = 1$ , we get the *taxicab* or *Manhattan* norm, for  $p = 2$ , we get the standard Euclidean norm, and for  $p \rightarrow \infty$ , the  $p$ -norm approaches the *infinity* or *maximum* norm:

$$\|\mathbf{x}\|_\infty := \max_i |x_i|.$$

For  $p \in (0, 1)$ , the triangle inequality does not hold, and the resulting functions are called *quasinorms*.