Algebra I
Solutions to Tutorial Sheet 1

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1. To find all finite subgroups of $\mathbb{R}^*$, we must find all elements with a finite order. If a subgroup H is finite, all ements $x \in H$ must satisfy $x^n = 1$ for some $n \in \mathbb{N}$.

   $$x^n = 1 \Rightarrow x = 1, -1.$$

   As such, the only finite subgroups of $\mathbb{R}^*$ are $\{1\}$ and $\{1, -1\}$.

2. $\operatorname{ord}(a^d) = n/(n, d) = n/d$.

3. The six cyclic subgroups of $U(15)$ are:

   (a) $\{1\}$,

   (b) $\{1, 4\}$,

   (c) $\{1, 11\}$,

   (d) $\{1, 14\}$,

   (e) $\{1, 2, 4, 8\}$, and

   (f) $\{1, 4, 7, 13\}$.

4. *Proof.* Every element x in G either has an inverse that is distinct from itself ($x \neq x^{-1}$), or is its own inverse ($x = x^{-1} \Rightarrow x^2 = e \Rightarrow \operatorname{ord}(x) = 2$). Let the number of elements that have an inverse distinct from itself be k, and the set of elements that are their own inverses be S. Then,

   $$\operatorname{ord}(G) = |S| + 2k \Rightarrow |S| = \operatorname{ord}(G) - 2k.$$

   Since $\operatorname{ord}(G)$ is even by hypothesis, $|S|$ must also be even. Since $e^2 = e$, $e \in S$, which implies that $|S| \geq 1$. But since we know that $|S|$ is even, it must be the case that $|S| \geq 2$. As such, there exists a nonidentity element $x \in S$ such that $x^2 = e \Rightarrow \operatorname{ord}(x) = 2$. $\square$

5. Let $x \in G$ such that $\operatorname{ord}(x) = 3$. Then, $\langle x \rangle = \{e, x, x^2 = x^{-1}\}$. Note that $x \neq x^{-1}$, since that would imply that $\operatorname{ord}(x) = 2$. All such sets must be necessarily disjoint barring the identity element, and as such the number of subgroups of order 3 would be

   $$\frac{8}{3-1} = 4.$$

6. We need only find an element $x \in U(40)$ such that $x^4 \equiv 1 \pmod{40}$ different from the identity element 1. Consider $x = 3$: $3^4 = 81 \equiv 1 \pmod{40}$. The subgroup generated by 3 is given by
$$\langle 3 \rangle = \{1, 3, 9, 27\}.$$

7. Every nonidentity element of a noncyclic group of order 4 must have order 2. We may test and find that 11 and 21 are both of order 2. It is also necessary that the subgroup be closed under the operation: $11 \cdot 21 = 231 \equiv 31 \pmod{40}$. We require 31 to also be of order 2, which it is: $31^2 = 961 \equiv 1 \pmod{40}$. Therefore, our noncyclic order 4 subgroup is
$$\{1, 11, 21, 31\}.$$

8. $H \not\leq (\mathbb{C}, +)$ since it is not closed under the group operation: $(2+i)+(-1-5i) = 1-4i$.

9. H can be rewritten as $\{e^{it} : t \in \mathbb{R}\}$, which can be easily verified to be a group. Geometrically, these are all the points on the unit circle $|z| = 1$ on the complex plane.

10. (a) $\langle 8, 14 \rangle = \langle 2 \rangle = \{0, \pm 2, \pm 4, \dots\}$.

    (b) $\langle m, n \rangle = \langle (m, n) \rangle$.

    (c) $\langle 12, 18, 45 \rangle = \langle 3 \rangle = \{0, \pm 3, \pm 6, \dots\}$.

11. *Proof.* Since G has more than one element, there exists a nonidentity element $x \in G$. Let $n > 1$ be the order of x. We know that n must have atleast one prime divisor p, i.e. $n = pk$ for some integer k. We claim that $y = x^k$ has order p.
$$y^p = (x^k)^p = x^{kp} = x^n = e.$$
y cannot be the identity element since then $x^k = e$, which would imply that $n \mid k$, but since $n = pk > k$, this is not possible. $\square$

12. This proof has been omitted due to triviality.