

# Computation of Option Pricing Models

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## Contents

<b>1 Fundamentals of Option Pricing</b>	<b>2</b>
1.1 Modelling a Risk-Free Asset . . . . .	2
1.2 Call-Put Parity . . . . .	3
1.3 Assumptions in Asset Pricing . . . . .	3
<b>2 Fundamentals of Probability Theory</b>	<b>4</b>
2.1 Measure Theoretic Foundations . . . . .	4
2.2 Expectation and Independence . . . . .	5
2.3 The Wiener Process . . . . .	7
<b>3 Modelling Asset Prices</b>	<b>8</b>
3.1 The Stochastic Differential Equation . . . . .	8
3.2 Itô's Lemma . . . . .	10
3.3 Solution to the Stochastic Differential Equation . . . . .	10



# 1 Fundamentals of Option Pricing

Recall that a European put option gives its holder (buyer) the right, but not the obligation, to *sell* a prescribed asset  $S$  to the writer (seller) of the option for a strike price  $K$  at the maturity date  $T$ . As such, the fair price  $P$  of a European put option at the maturity date is simply given by the payoff:

$$P_T = P(S_T, T) = \max(K - S_T, 0).$$

Similarly, a European call option gives its holder the right, but not the obligation, to *buy* a prescribed asset  $S$  from the writer of the option, and once again fair price  $P$  of a European call option at the maturity date is given by the payoff:

$$C_T = C(S_T, T) = \max(S_T - K, 0).$$

The objective of option pricing is to determine the value of *premium*:

$$P_0 = P(S_0, 0), \quad C_0 = C(S_0, 0)$$

building on the values of  $P_T$ . If  $P_0$  was simply set to 0 (no premium on the option), then a call option holder would never take on any risk, and never make a loss. On the other hand, the writer can never turn a profit. The price of the premium must be *fair* to both parties entering the contract.

An investor is said to have taken the *long position* if he/she has bought the option, and the *short position* if he/she has sold the option.

Our models pretend that we are in a risk-neutral world, where an investor wouldn't mind risk. In this hypothetical world, a risky stock is expected to grow at the exact same rate as a safe investment such as a government bond or a bank account. The *Risk-Neutrality Assumption* is as stated.

*At any time, the average return on a risky investment of an asset is equal to the return on a risk-free investment of that asset.*

Under this new assumption,

$$E(P_T) = P_0 e^{rT}.$$

And therefore the premium for a European call option would be

$$P_0 = e^{-rT} E(\max(S_T - K, 0)).$$

## 1.1 Modelling a Risk-Free Asset

A bond issued by the government, or accumulating interest in a bank can be regarded as a risk-free asset. If  $B_0$  is a risk-free investment at a time  $t = 0$ , the value of the investment after  $m$  years at a rate  $r$  would be

$$B_m = B_0(1 + mr)$$

if the interest is simple, and

$$B_m = (1 + r)^m B_0$$

if compounded annually. Building on the case of compound interest, consider  $N$  timestamps  $\{t_n : n \in 0 : N\}$  where  $t_n = n\delta t = n/N$  at which the interest compounds. Then,

$$\frac{B_{t_n} - B_{t_{n-1}}}{B_{t_{n-1}}} = r\delta t$$

for  $n \in 1 : N$ . Now,

$$B_T = B_{t_N} = (1 + r\delta t)^N B_0 = [(1 + r\delta t)^{1/\delta t}]^T B_0.$$

If the interest is compounded continuously, then

$$B_T = \lim_{\delta t \rightarrow 0} [(1 + r\delta t)^{1/\delta t}]^T B_0 = e^{rT} B_0.$$

## 1.2 Call-Put Parity

The call-put parity is a relation between a European call option, put option, and its underlying asset. Consider the following portfolio  $\Pi$ , consisting of one long put  $P$  and short call  $C$ , both with the same strike price  $K$  and expiration date  $T$  and one underlying asset  $S$ :

$$\Pi = S + P - C.$$

The payoff for this portfolio at maturity is always fixed:

$$S_T + \max(K - S_T, 0) - \max(S_T - K, 0) = K.$$

To determine how much one should pay for  $\Pi$ , we simply discount the risk-free interest (at rate  $r$ ) gained from  $K$ :

$$\Pi = Ke^{-r(T-t)} \Rightarrow S + P - C = Ke^{-r(T-t)}.$$

This final equation is called the call-put parity relation.

If one were to introduce a stock that pays dividends  $D$ , the relation becomes

$$S + P = C + D + Ke^{-r(T-t)}.$$

## 1.3 Assumptions in Asset Pricing

The basic principle of asset pricing is that the value of an asset cannot be predicted, since it is a measure of investors' confidence, and as such, is strongly dependent on news, rumours, speculation, and general human behaviour. For the sake of mathematical simplification, we assume that the market responds instantaneously to external influence. This is captured in the *Efficient Market Hypothesis* (EMH) described below.

1. *The past history of an asset is fully reflected in its present price.*

2. *The market responds immediately to any new information about an asset.*

Based on the first principle, asset prices are modelled with Markov processes.

In addition to the EMH, a number of assumptions are made about the asset, as listed.

- The asset price may take any nonnegative value.
- Buying and selling an asset may take place at any time  $t \in [0, T]$
- It is possible to buy and sell any amount of the asset.
- The bid-ask spread is zero: the price for buying an asset equals the price for selling it.
- There are no transaction costs.
- There are no dividends or stock splits.
- Short-selling is permitted: it is possible to hold a negative amount of the asset.
- There is a single, constant, risk-free interest rate that applies to any amount of money borrowed from or deposited in a bank.

## 2 Fundamentals of Probability Theory

### 2.1 Measure Theoretic Foundations

**Definition 2.1.** A nonempty collection  $\mathcal{A}$  of subsets of a nonempty set  $\Omega$  is said to be a  $\sigma$ -algebra on  $\Omega$  if it satisfies the following assumptions.

1.  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ .
2. If  $\{A_i\}$  is a sequence of sets in  $\mathcal{A}$ , then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

It follows from these axioms that  $\Omega, \emptyset \in \mathcal{A}$ , and  $\mathcal{A}$  is closed under countable intersections as well due to De Morgan's rules:

$$\bigcap_{i=1}^{\infty} A_i = \left( \bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{A}$$

for some sequence  $\{A_i\}$  in  $\mathcal{A}$ .

The pair  $(\Omega, \mathcal{A})$  is called a *measurable space*.

**Definition 2.2.** The Borel  $\sigma$ -algebra of  $\mathbb{R}$  denoted  $\mathcal{B}(\mathbb{R})$  is the smallest  $\sigma$ -algebra that contains all open sets in  $\mathbb{R}$ . All intervals, singleton subsets, and countable sets are contained in the Borel  $\sigma$ -algebra, and are called *Borel sets*.

**Definition 2.3.** A set function  $P : \mathcal{A} \rightarrow [0, 1]$  is known as a *probability measure* on a measurable space  $(\Omega, \mathcal{A})$  if it satisfies the following assumptions.

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1.  $P(\emptyset) = 0$  and  $P(\Omega) = 1$ .
  2. For a sequence of sets  $\{A_i\}$  which are pairwise disjoint,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Now the tuple  $(\Omega, \mathcal{A}, P)$  is called a *probability space*.

We may now define random variables and stochastic processes.

**Definition 2.4.** A real valued *random variable*  $X$  on a probability space  $(\Omega, \mathcal{A}, P)$  is a  $\mathcal{A}$ -measurable real valued function on  $\Omega$ . That is, for every  $B \in \mathcal{B}(\mathbb{R})$ , it follows that  $X^{-1}(B) \in \mathcal{A}$ .

A random variable can be viewed as a useful tool for transitioning from an abstract probability space  $(\Omega, \mathcal{A}, P)$ , to a more tangible one on the real numbers:  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ . Here  $P_X$  is called a *push-forward measure*. For any  $B \in \mathcal{B}(\mathbb{R})$ ,

$$P_X(B) = P(X^{-1}(B)).$$

**Definition 2.5.** A real valued *stochastic process*  $\{X_t\}$  on a probability space  $(\Omega, \mathcal{A}, P)$  assigns to each time  $t$  from a set  $T$ , a random variable  $X_t$  on  $(\Omega, \mathcal{A}, P)$ . The process  $X_t$  has *a.s.* continuous sample paths  $t \mapsto X_t(\omega)$ .

We will drop the braces around  $\{X_t\}$ , and refer to  $X_t$  as a stochastic process for brevity.

**Definition 2.6.** A *discrete random variable*  $X$  takes values from a finite set of numbers  $\{x_1, x_2, \dots, x_m\}$  and associates these with probabilities  $\{p_1, p_2, \dots, p_m\}$  such that  $x_i$  occurs with probability  $p_i$ . Symbolically,

$$P(X = x_i) = p_i.$$

## 2.2 Expectation and Independence

**Definition 2.7.** Let real valued  $X$  be a random variable on a probability space  $(\Omega, \mathcal{A}, P)$ . The *expected value* or *expectation* of  $X$  is given by

$$E(X) = \int_{\Omega} X(\omega) dP(\omega).$$

The definition of expectation provided above is the most general one, and from it one can derive the familiar definitions from elementary probability. For the case of a discrete random variable  $X$  taking on values  $\{x_1, x_2, \dots, x_n\}$ ,

$$X(\omega) = \sum_{i=1}^n x_i \mathbf{1}_{A_i}(\omega)$$

where  $A_i = \{\omega \in \Omega : X(\omega) = x_i\}$ . Plugging this into our definition for expected value,

$$E(X) = \int_{\Omega} \left( \sum_{i=1}^n \mathbf{1}_{A_i}(\omega) \right) dP(\omega) = \sum_{i=1}^n x_i \int_{\Omega} \mathbf{1}_{A_i}(\omega) dP(\omega) = \sum_{i=1}^n x_i P(X = x_i).$$

In the case that  $X$  is a continuous random variable, we may switch to integrating over  $\mathbb{R}$  instead:

$$E(X) = \int_{\Omega} X(\omega) dP(\omega) = \int_{\mathbb{R}} x dP_X(x).$$

Using the Radon-Nikodym theorem with the Lebesgue measure, we know that there exists a derivative function  $f$  such that  $dP_X = f dx$ . As such,

$$\int_{\mathbb{R}} x dP_X(x) = \int_{\mathbb{R}} xf(x) dx.$$

The expectation of  $g(X)$  for any function  $g$  that is Borel measurable and absolutely integrable with respect to  $f$  is given by

$$E(g(X)) = \int_{\mathbb{R}} g(x)f(x) dx.$$

Note that  $P(X \in A) = E(\mathbf{1}_A(X))$ , therefore:

$$P(X \in A) = \int_{\mathbb{R}} \mathbf{1}_A(x)f(x) dx = \int_A f(x) dx.$$

For the case that  $A$  is an interval  $(a, b]$ :

$$P(a < X \leq b) = \int_a^b f(x) dx.$$

The operator  $E$  is linear. That is, for two random variables  $X$  and  $Y$ , and  $a \in \mathbb{R}$ ,

$$E(X + aY) = E(X) + aE(Y).$$

We now look at another important property: independence.

**Definition 2.8.** Two random variables  $X$  and  $Y$  are said to be *independent* if for all Borel measurable functions  $g, h \in \mathbb{R}^{\mathbb{R}}$ ,

$$E(g(X)h(Y)) = E(g(X))E(h(Y)).$$

**Definition 2.9.** A set of random variables  $\{X_1, X_2, \dots, X_n\}$  are said to be *independently and identically distributed* or i.i.d. if

- (a) for discrete random variables, every  $X_i$  for  $i \in 1 : n$  has the same set of possible outcomes and same probability for each outcome, and the same density for continuous random variables, and

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- (b) the outcome of one variable provides no information on the others.

We now proceed to define variance, a measure of the amount of variation around the mean.

**Definition 2.10.** For a random variable  $X$  such that  $E(X^2) < \infty$ , its *variance* is defined as

$$\text{Var}(X) = E((X - E(X))^2).$$

It can be expanded as

$$\text{Var}(X) = E(X^2) - E(X)^2.$$

It also holds that  $\text{Var}(aX) = a^2 \text{Var}(X)$  for all  $a \in \mathbb{R}$ , and the *standard deviation* is defined as the square root of the variance:  $\text{std}(X) = (\text{Var}(X))^{1/2}$ .

### 2.3 The Wiener Process

We now proceed to define the normal distribution, which is central to our study.

**Definition 2.11.** If  $X$  is a continuous random variable with density function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where  $\mu, \sigma \in \mathbb{R}$  with  $\sigma > 0$ , then we say that  $X$  has a *normal distribution* or is *normally distributed*. The parameters  $\mu$  and  $\sigma^2$  are the mean and variance of the distribution respectively, and we symbolically write

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

We recover the standard normal distribution for  $\mu = 0$  and  $\sigma = 1$ .

**Definition 2.12.** If  $X$  is a continuous random variable such that  $\log X \sim \mathcal{N}(\mu, \sigma^2)$  for constants  $\mu, \sigma \in \mathbb{R}$  where  $\sigma > 0$ , then  $X$  is said to be *lognormally distributed* or have a *lognormal distribution*.

**Definition 2.13.** Given a density function  $f$  for a continuous random variable  $X$ , the *cumulative distribution function* (cdf)  $F : \mathbb{R} \rightarrow \mathbb{R}$  of  $X$  is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

The cdf of the standard normal distribution is of interest to us, and we denote it with  $\Phi$ :

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

We may now proceed to define a Wiener process.

**Definition 2.14.** A *Wiener process* or *Brownian motion* in one dimension is a stochastic process  $W_t$  where  $t \geq 0$  with the following properties.

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1.  $W_0 = 0$  a.s.
  2.  $W_t$  has independent increments: for every  $t > 0$  and  $\Delta t \geq 0$ ,  $\Delta W_t = W_{t+\Delta t} - W_t$  is independent of  $W_s$  for every  $s < t$ .
  3.  $W_t$  has Gaussian increments: for every  $t > 0$  and  $\Delta t \geq 0$ ,  $\Delta W_t = W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t)$ .
  4.  $W_t$  is a.s. continuous in  $t$ .

We will reserve the notation  $W_t$  for a Weiner process henceforth, without explicit declaration.

The following properties hold for a Wiener process  $W_t$ . For  $\Delta W_t = W_{t+\Delta t} - W_t$  where  $t > 0$  and  $\Delta t \geq 0$ ,

- (a)  $E(\Delta W_t) = 0$ , and
- (b)  $\text{Var}(\Delta W_t) = \Delta t$  implying that  $\text{std}(\Delta W_t) = \sqrt{\Delta t}$ .

In fact,  $\Delta W_t = \varepsilon \sqrt{\Delta t}$  where  $\varepsilon$  is drawn from a standard normal distribution. The heuristic that  $dW_t \approx \sqrt{dt}$  may be drawn from this. We define a generalized Weiner process  $\tilde{W}_t$  with a drift rate  $r$  and volatility  $\sigma$ :

$$d\tilde{W}_t = r dt + \sigma dW_t \Rightarrow \tilde{W}_t = \tilde{W}_0 + rt + \sigma W_t.$$

This should be viewed as a specific case of an Itô drift-diffusion process  $X_t$  that satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

## 3 Modelling Asset Prices

### 3.1 The Stochastic Differential Equation

Asset prices follow stochastic processes. Let  $\mu$  be the expected rate of return of a risk free ( $\sigma = 0$ ) asset  $S$ . The asset price can be modelled as

$$\frac{dS_t}{S_t} = \mu t.$$

This is a deterministic differential equation with solution

$$S_t = S_0 e^{\mu t}.$$

We now attempt to model assets that have risk involved ( $\sigma > 0$ ). Returns from the asset come from two sources: the risk free rate interest rate governed by  $\mu$ , and the random changes due to the EMH, which we model with a Weiner process on some probability space  $(\Omega, \mathcal{A}, P)$ . As such, we arrive at the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

This is called a *geometric Brownian motion*. The notation is mathematically meaningless when written with differentials ( $dW_t = W'_t dt$  doesn't really exist) and is better written as in integral equation, and we switch notation from  $S_t$  to  $S(t)$  to emphasize the path function for a fixed  $\omega \in \Omega$ .

$$S(t) = S_0 + \mu \int_0^t S(\tau) d\tau + \sigma \int_0^t S(\tau) dW_\tau.$$

These integrals exist *a.s.* with the first one being a standard Riemann integral, and the second integral being a well-defined *Itô integral*.

**Example 3.1.** For an asset  $S$  that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding interest, determine the return  $\Delta S/S$  after a week (0.0192 years, since a working year is 250 days). Also find the mean and variance of  $\Delta S$  when the asset price is 100.

*Solution.* Given that  $\mu = 0.15$ , and  $\sigma = 0.30$ , the return would be

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\Delta W.$$

We know that  $\Delta W = \varepsilon\sqrt{\Delta t}$  where  $\varepsilon \sim \mathcal{N}(0, 1)$ , and  $\Delta t = 0.0192$ . Therefore, the return would be

$$\frac{\Delta S}{S} = 0.00288 + 0.041\varepsilon.$$

Solving for  $\Delta S$  at  $S = 100$ , we have

$$\Delta S = 0.288 + 4.16\varepsilon.$$

Therefore,

$$E(\Delta S) = 0.288 \quad \text{and} \quad \text{Var}(\Delta S) = 4.16^2 = 17.31.$$

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An asset's expected return rate and volatility is rarely known to us, and must be estimated. Given an asset price  $S$  at  $n+1$  equal time steps  $\Delta t$   $\{S_0, S_1, \dots, S_n\}$  in chronological order, we may estimate the expected rate of return

$$\bar{\mu} = \frac{1}{n\Delta t} \sum_{i=0}^n \frac{S_{i+1} - S_i}{S_i},$$

and the historical volatility

$$\bar{\sigma}^2 = \frac{1}{(n-1)\Delta t} \sum_{i=0}^{n-1} \left( \frac{S_{i+1} - S_i}{S_i - \bar{\mu}} \right)^2.$$

## 3.2 Itô's Lemma

Itô's lemma can be viewed as the stochastic equivalent of the chain rule from naive calculus. We explore a specific case of it here, relevant to option pricing.

Let  $S_t$  be a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and  $f : [0, T) \times \mathbb{R}^+ \rightarrow \mathbb{R}$  a twice-differentiable scalar function. Using its Taylor series expansion, we have

$$\begin{aligned} df &= f(t + dt, S + dS) - f(t, S) \\ &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \left( \frac{\partial^2 f}{\partial t^2}(dt)^2 + 2 \frac{\partial^2 f}{\partial t \partial S} dt dS + \frac{\partial^2 f}{\partial S^2}(dS)^2 \right) + \dots \end{aligned}$$

From the Itô drift-diffusion equation, we know that

$$(dS)^2 = (\mu S_t dt + \sigma S_t dW_t)^2 = \mu^2 S_t^2 (dt)^2 + 2\mu\sigma S_t^2 dt dW_t + \sigma^2 S_t^2 (dW_t)^2.$$

We may now use the heuristic  $dW_t \approx \sqrt{dt}$ . Retaining terms that are atmost  $O(dt)$  (if any larger, they shrink too fast to be nonnegligible as  $dt \rightarrow 0$ ), we obtain

$$df = \left( \mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW_t.$$

This statement is the version of Itô's lemma we will be using.

Itô's lemma can be extended to higher dimensions. If  $\mathbf{S}_t = [S_t^1 \ S_t^2 \ \dots \ S_t^n]^\top$  is a vector of Itô drift diffusion processes, then

$$d\mathbf{S}_t = \boldsymbol{\mu}_t dt + \mathbf{G}_t d\mathbf{W}_t$$

for a vector  $\boldsymbol{\mu}_t$  and matrix  $\mathbf{G}_t$ . This version of Itô's lemma is called the Kunita-Watanabe lemma:

$$df = \left( \frac{\partial f}{\partial t} + (\nabla_{\mathbf{S}} f)^\top \boldsymbol{\mu}_t + \frac{1}{2} \text{tr}(\mathbf{G}_t^\top (H_{\mathbf{S}}(f)) \mathbf{G}_t) \right) dt + (\nabla_{\mathbf{S}} f)^\top \mathbf{G}_t d\mathbf{W}_t.$$

## 3.3 Solution to the Stochastic Differential Equation

Armed with Itô's lemma, we may now attempt to solve the stochastic differential equation that models an asset's price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Apply Itô's lemma on  $f(t, S) = \log S$ :

$$\begin{aligned} d(\log S_t) &= \left( \mu S \left( \frac{1}{S} \right) + 0 + \frac{1}{2} \sigma^2 S^2 \left( -\frac{1}{S^2} \right) \right) dt + \sigma S \left( \frac{1}{S} \right) dW_t \\ &= \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t. \end{aligned}$$



This is an equation representing the previously described generalized Brownian motion; the solution to which can be obtained by “integrating”:

$$\log S_t = \log S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t.$$

Exponentiating we get,

$$S_t = S_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right).$$

Since  $W_t \sim \mathcal{N}(0, t)$ , it follows that

$$\log S_t \sim \mathcal{N} \left( \log S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right).$$

As such,  $S_t$  is lognormally distributed.