

Numerical Optimization

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1 Preliminaries

1.1 Inner Product Spaces

Definition 1.1. Let V be a real vector space. A function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ is called a *real inner product* if it satisfies the following three properties for $x, y, z \in V$ and $c \in \mathbb{R}$:

- (a) $\langle x, y \rangle = \langle y, x \rangle$,
- (b) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$,
- (c) $\langle cx, y \rangle = c\langle x, y \rangle$, and
- (d) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

The inner product is the generalization of the dot product on Euclidean vector spaces: for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ where $x_i, y_i \in \mathbb{R}$ for $i \in 1 : n$,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i.$$

It may also be defined on \mathbb{C} -spaces, replacing (a) with conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$, and adding conjugate linearity in the second argument: $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$.

Definition 1.2. Let V be a real vector space and $\langle \cdot, \cdot \rangle$ an inner product. Then, $(V, \langle \cdot, \cdot \rangle)$ is called a *real inner product space*.

For brevity, we may simply state that V is an inner product space, with the notation for the inner product being implicit, and all future inner product spaces map into \mathbb{R} unless stated otherwise.

1.2 Normed Linear Spaces

Definition 1.3. Let V be a real vector space. A function $\|\cdot\| : V \rightarrow \mathbb{R}$ is called a *norm* if it satisfies the following properties for $x, y \in V$ and $c \in \mathbb{R}$:

- (a) $\|cx\| = |c|\|x\|$,



(b) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$, and

(c) $\|x + y\| \leq \|x\| + \|y\|$.

This last property is referred to as the *triangle inequality*. The norm assigns a notion of length to vectors, and generalizes the standard formula for the length of a vector in a Euclidean vector space: for $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ where $x_i \in \mathbb{R}$ for $i \in 1 : n$.

$$\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}.$$

Definition 1.4. Let V be a vector space and $\|\cdot\|$ a norm. Then $(V, \|\cdot\|)$ is called a *normed linear space*.

Once again, we may conventionally omit the norm from notation when defining a new normed linear space.

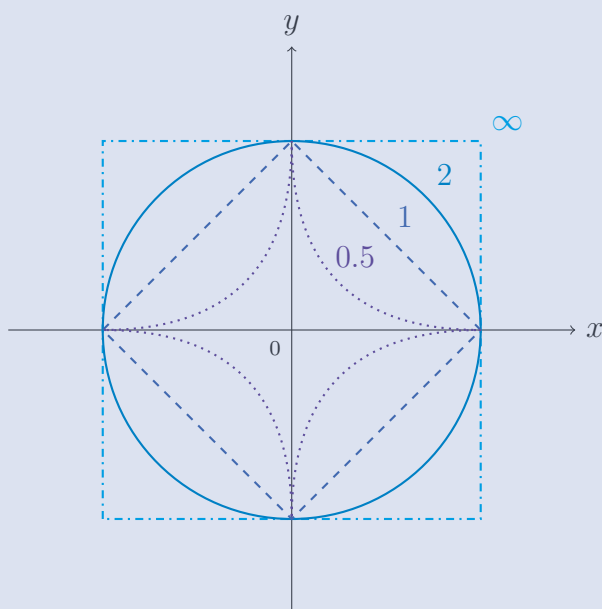
Definition 1.5. Let $p \geq 1$. The p -norm (or ℓ^p -norm) of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ where $x_i \in \mathbb{R}$ for $i \in 1 : n$ is

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

For $p = 1$, we get the *taxicab* or *Manhattan* norm, for $p = 2$, we get the standard Euclidean norm, and for $p \rightarrow \infty$, the p -norm approaches the *infinity* or *maximum* norm:

$$\|\mathbf{x}\|_\infty := \max_i |x_i|.$$

For $p \in (0, 1)$, the triangle inequality does not hold, and the resulting functions are called *quasinorms*. Pictured below are unit circles in the 0.5-quasinorm, and p -norms for $p \in \{1, 2, \infty\}$.





Lemma 1.6. *Let V be a normed linear space. Then,*

$$|||x| - |y||| \leq \|x - y\|$$

for every $x, y \in V$.

Theorem 1.7. *Let $(V, \|\cdot\|)$ be a normed linear space. Then $\|\cdot\| : V \rightarrow \mathbb{R}$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$, and choose $\delta = \varepsilon$. If $\|x - y\| < \delta$, then by Lemma 1.6,

$$|||x| - |y||| \leq \|x - y\| < \delta = \varepsilon.$$

Therefore, $|||x| - |y||| < \varepsilon$. □

Theorem 1.8. *Every inner product space V is naturally a normed linear space. For $x \in V$ define*

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

However, the converse is not always true.

Theorem 1.9. *A normed linear space V is an inner product space iff the norm satisfies the parallelogram law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The inner product is given by

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

for all $x, y \in V$.

We now look at a fundamental theorem in linear algebra.

Theorem 1.10 (Cauchy-Schwarz Inequality). *Let V be an inner product space with an induced norm. Then for all $x, y \in V$*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Proof. We utilize the discriminant of a quadratic equation and the nonnegativity of the norm. Firstly, if $y = 0$, then the inequality holds trivially. For $y \neq 0$, consider the vector $x + ty$ for some $t \in \mathbb{R}$. Then

$$0 \leq \|x + ty\|^2 = \langle x + ty, x + ty \rangle = \|y\|^2 t^2 + 2\langle x, y \rangle t + \|x\|^2.$$

Since the quadratic in t is nonnegative, its discriminant must be nonpositive, giving us

$$4\langle x, y \rangle^2 - 4\|x\|^2 \|y\|^2 \leq 0 \Rightarrow |\langle x, y \rangle| \leq \|x\| \|y\|.$$

□



1.3 Eigenvalues and Eigenvectors

Definition 1.11. Let A be a square matrix of order n . Then, λ is said to be an *eigenvalue* of A if there exists nonzero $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Here, \mathbf{x} is called an *eigenvector* corresponding to λ .

To find the eigenvalues of an arbitrary matrix, consider the following:

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

Since \mathbf{x} is nonzero, we require the matrix $A - \lambda I$ to be singular. That is,

$$\det(A - \lambda I) = 0.$$

The left side of the equation is a polynomial in λ , and is called the *characteristic polynomial*. Therefore the eigenvalues of a matrix A are the *roots* to its characteristic polynomial.

Theorem 1.12. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Definition 1.13. A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

It turns out that in Definition 1.13, the entries of D along the diagonal are the eigenvalues of A , and P contains corresponding eigenvectors in the respective columns of the eigenvalues in D .

1.4 Quadratic Forms

Certain quadratic functions of several variables may be represented using matrices.

Definition 1.14. A *quadratic form* is a homogenous polynomial of degree 2 in multiple variables.

A quadratic form in n variables may be written with a symmetric matrix A of order n and a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ of unknowns:

$$Q(x) = \langle A\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^\top A\mathbf{x} = \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n a_{ij}x_i x_j$$

where a_{ij} is the (i, j) th entry of A .

Definition 1.15. Let A be a square matrix of order n , and let $p = \langle A\mathbf{x}, \mathbf{x} \rangle$. If for all nonzero $\mathbf{x} \in \mathbb{R}^n$



- (a) $p > 0$, then A is said to be *positive definite*,
- (b) $p \geq 0$, *positive semidefinite*,
- (c) $p < 0$, *negative definite*, and
- (d) $p \leq 0$, *negative semidefinite*.

Otherwise, A is said to be *indefinite*.

The definiteness of a matrix may also be realized through its principal minors.

Definition 1.16. Let A be a square matrix of size n . A *principal minor of order k* of A may be computed by the following.

1. Choose any set of k indices $\{i_1, i_2, \dots, i_k\}$ from $\{1, 2, \dots, n\}$.
2. Keep the rows and columns corresponding to these indices, creating a $k \times k$ submatrix.
3. Take the determinant of this resulting submatrix to yield a principal minor of order k .

If the first k indices $\{1, 2, \dots, k\}$ are chosen, then the resulting principal minors called *leading principal minors of order k* .

Theorem 1.17. A matrix is

- (a) *positive definite if all leading principle minors are positive*,
- (b) *positive semidefinite if all principal minors are nonnegative*,
- (c) *negative definite if the k th leading principal minor has the sign of $(-1)^k$, and*
- (d) *negative semidefinite if all principal minors of order k either have the sign of $(-1)^k$ or are 0.*

A matrix that does not fall into the aforementioned categories is *indefinite*.