Classical and Bayesian Statistics

Problems 11

Problem 11.1

a) Start with a prior distribution expressing some uncertainty that a coin is fair: $\text{Beta}(\theta \mid 4, 4)$.

   Flip the coin once. Suppose you get heads. What is the posterior distribution?

b) Use the posterior of the previous toss as the prior for the next toss. Suppose, You throw again and get heads. What is the new posterior now?

c) Using this posterior as the prior for the next throw, throw a third time and you get tails. What is the new posterior now?

d) Make the same three updates, but in the order H, T, T instead of T, T, H.

   Is the final posterior distribution the same for both orders of the toss results?

Problem 11.2

Suppose an election is coming up and you want to know whether the general population prefers candidate A or candidate B. There is a poll just published in the newspaper which says that out of 100 people randomly surveyed, 58 prefer candidate A and the rest prefer candidate B.

Note: To determine the HDI, use the command.

```
hdi <- function(a, b, prob = 0.95) {
    k <- 0
    x <- seq(0, 1, 0.001)
    y <- dbeta(x, a, b)
    while (TRUE) {
        k <- k + 0.001
        if (sum(y[y > k])/length(x) < prob) {
            break
        }
    }
    return(c(x[(y > k)][1], x[(y > k)][length(x[(y > k)])]))
}
```

You then only need to specify a and b:

```
hdi(5, 5)

## [1] 0.212 0.788
```

a) Assume that before the newspaper survey, your prior assumption was a uniform distribution. What is the 95 %-HDI for your beliefs after you learn about the newspaper survey results?

b) You would like to conduct a follow-up survey to narrow down your estimate of the narrow down your estimate of the population's preference.

   In your follow-up survey, you take a random sample of 100 more people and find that 57 people prefer candidate A and the rest prefer candidate B.

   Assuming that people's opinions have not changed between surveys, what is the 95 %-HDI for the posterior distribution?

Problem 11.3

Suppose you are training people in a simple learning experiment as follows: When people see the two words „radio“ and „ocean“ on the computer screen, they are to press the F key on the computer keyboard.

The subjects see several repetitions and learn the response well. Then introduce another relationship for them to learn: Whenever the words „radio“ and „mountain“ appear, have them press the J key on the computer keyboard.

You train the subjects until they know both relationships well. Now you check what they have learned by asking them to do two new test items. For the first test, show them the word „radio“ as such and instruct them to give the best answer (F or J) based on what they have learned before.

For the second test, show them the two words „ocean“ and „mountain“ and ask them to give the best answer. You carry out this procedure with 50 people.

Your data show that for „radio“ alone, 40 people chose F and 10 people chose J.

For the word combination „ocean“ and „mountain“, 15 people chose F and 35 people chose J.

Are people biased towards F or towards J for either sample type? To answer this question, assume a uniform prior and use a 95 %-HDI to decide which variance can be classified as credible.

Problem 11.4

Suppose we have a coin that we know comes from a trick toy shop.

We therefore believe that the coin shows either heads or tails, but we do not know which. Express this belief as a beta prior.

Now we toss the coin 5 times and heads comes out in 4 of the 5 tosses. What is the posterior distribution?

Problem 11.5

a) Suppose you have a coin that you know has been minted by the government and has has not been tampered with. Therefore, you have a strong belief that the coin is fair. You flip the coin 10 times and get 9 heads.

   What is your predicted probability of heads for the 11th toss? Explain your answer carefully; justify your choice of prior.

b) Now you have another coin, which is made of a strange material and is with the inscription (in small print) "Patent Pending, International Magic, Inc.". You flip the coin 10 times and get 9 heads. What is your predicted probability of heads on the 11th toss.

   Explain your answer carefully; justify your choice of prior. Hint: Use the prior from the from the task before.

Problem 11.6

We again assume a fair coin, i.e. $a = b$. We now want to determine a and b such that the HDI of the prior distribution has a certain width.

Again, use the `hdi` command from Task 2 and

```
a <- 2
hdi(a, a)[2] - hdi(a, a)[1]


## [1] 0.812
```

a) We are not very convinced that the coin is fair and assume that 95 % of the most credible parameters are in the range of 0.2 and 0.8. How should we choose the parameter $a = b$?

   Try values for a until you get approximately to the width of the interval.

b) The coin looks fair and we assume that 95 % of the most credible parameters are in the range 0.4 and 0.6. How should we choose the parameter $a = b$?

c) We have examined the coin closely and are very optimistic that the coin is fair and assume that 95 % of the most credible parameters are in the range 0.48 and 0.52. How should we choose the parameter $a = b$?

Classical and Bayesian Statistic

Sample solution for Problems 11

Solution 11.1

a) We denote by N the number of tosses and by z the number of tosses with heads.

We have $N = 1$, $z = 1$, $a = b = 4$ and use the formula

$$p(\theta \mid z, N) = \text{Beta}(\theta \mid z + a, N - z + b)$$

If we put in all the values, we get

$$p(\theta \mid 1, 1) = \text{Beta}(\theta \mid 5, 4)$$

b) Now $\text{Beta}(\theta \mid 5, 4)$ is the new prior distribution: again we have $N = 1$, $z = 1$, but $a = 5$ and $b = 4$:

$$p(\theta \mid 1, 1) = \text{Beta}(\theta \mid 6, 4)$$

c) Now $\text{Beta}(\theta \mid 6, 4)$ is the new prior distribution: we have $N = 1$, $z = 0$, but $a = 6$ and $b = 4$:

$$p(\theta \mid 1, 1) = \text{Beta}(\theta \mid 6, 5)$$

d) The result is the same. This has to do with the fact that the throws are stochastically independent and thus the permutation does not matter.

Solution 11.2

a) We have $N = 100$, $z = 58$ and $a = b = 1$. The posterior distribution is:

$$p(\theta \mid 58, 100) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 59, 43)$$

The HDI is

```
hdi(59, 43)

## [1] 0.483 0.673
```

The HDI is $(0.483, 0.673)$. Has a width of approximately 0.2. The HDI describes the range where 95 % of the most likely values are. So we have got a rough sense of what the election result will be. However, there is still a probability that candidate B will win.

b) We take as a new prior distribution $\text{Beta}(\theta \mid 59, 43)$ and get $N = 100$ and $z = 57$ with $a = 59$ and $b = 43$

$$p(\theta \mid 57, 100) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 116, 86)$$

The HDI is

```
hdi(116, 86)

## [1] 0.507 0.642
```

The HDI has shrunk, the width is now only about 0.14. Most importantly, the interval has shifted further to the right, so the chances of candidate B winning have worsened. These values are no longer in the HDI at all.

Solution 11.3

We again take as prior distribution a beta distribution with $a = 1$ and $b = 1$ with $N = 50$ and $z = 10$ in the 1st case. We obtain

$$p(\theta \mid 10, 50) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 11, 41)$$

With the HDI

```
hdi(11, 41)


## [1] 0.107 0.323
```

If the combinations are equally probable, then 0.5 "must" be in the HDI. This is not the case, so we can assume that one of the samples has a higher probability of being chosen.

We take as prior distribution again a beta distribution with $a = 1$ and $b = 1$ with $N = 50$ and $z = 15$ in the 2nd case. We get

$$p(\theta \mid 15, 50) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 16, 36)$$

With the HDI

```
hdi(16, 36)


## [1] 0.187 0.433
```

If the combinations are equally probable, then 0.5 "must" be in the HDI. This is not the case, so we cannot assume that the deviation is rather random.

Solution 11.4

For the prior distribution, we choose a beta distribution that is very low in the middle at $\theta = 0$ and very high at both ends $\theta \approx 0$ and $\theta \approx 1$. Such a beta distribution has about $a = b = 0.1$.

We have $N = 5$ and $z = 4$ and therefore

$$p(\theta \mid 4, 5) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 4.1, 1.1)$$

Solution 11.5

a) We are very sure that the coin is fair, so we take a beta distribution with large a and b but equal, so for example $a = b = 100$. Now we have $N = 10$ and $z = 9$, which actually contradicts a fair coin. We calculate the posterior distribution.

$$p(\theta \mid 9, 10) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 109, 101)$$

The distribution has shifted slightly, but by how much?

One possibility is that we calculate the mode. This is calculated by

$$\omega = \frac{a - 1}{a + b - 2} = \frac{109 - 1}{109 + 101 - 2} = \frac{108}{208} \approx 0.519$$

Note that we use the new a and b of the posterior distribution to calculate the mode.

The new probability that we still throw heads is now about 0.52, which is still very close to 0.5.

b) We choose the prior distribution from task 4 with $a = b = 0.1$ and $N = 10$ and $z = 9$

$$p(\theta \mid 9, 10) = \text{Beta}(\theta \mid z + a, N - z + b) = \text{Beta}(\theta \mid 9.1, 1.1)$$

We calculate again the mode

$$\omega = \frac{a - 1}{a + b - 2} = \frac{9.1 - 1}{9.1 + 1.1 - 2} = \frac{8.1}{8.2} \approx 0.988$$

Here we get a clear shift towards the data.

Solution 11.6

a) The width of the interval is 0.6 and with a little trial and error we get $a \approx 4$ or 5.

b) The width of the interval is 0.2 and with a little trial and error you get $a \approx 50$.

c) The width of the interval is 0.04 and with a little trial and error we get $a \approx 1000$.

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