Classical and Bayesian Statistics

Problems 3

Problem 3.1

(∗) Calculate p_2 so that the table below becomes a probability distribution.

x_k	-5	-4	1	3	6
p_k	0.3	p_2	0.1	0.2	0.3

Problem 3.2

The random variable X describes the number of household members in a sample and has the distribution:

k	1	2	3	4	5
$P(X = k)$	0.4	0.2	0.2	0.1	0.1

Describe the looked for probabilities in b) – e) in the form $P(\ldots)$. For example: $P(X \leq 5)$ or $P(3 \leq X \leq 5)$.

(∗) a) Does the table describe a probability distribution? Justify your answer.

(∗) b) Calculate the probability that a randomly selected household has between 2 and 4 members.

(∗∗) c) Calculate the probability that a randomly selected household has more than 2 members.

(∗) d) Calculate the probability that a randomly selected household has no more than 4 members.

(∗∗) e) Calculate the probability that a randomly selected household has more than one member.

Problem 3.3

Ein Multiple-Choice-Test besteht aus 15 Fragen, mit jeweils 5 Antwortmöglichkeiten, von denen genau eine richtig ist. Die Wahrscheinlichkeit dafür, eine Aufgabe richtig zu beantworten, ist also 0.2. Die Wahrscheinlichkeits- und Verteilungsfunktion sind gegeben durch:

k	8	9	10	11	12	13	14	15
$P(X \leq k)$	0.711	0.939	0.969	0.982	0.989	0.992	0.999	1

Beachten Sie: Es handelt sich hier um die *kumulierten* Wahrscheinlichkeiten $P(X \leq k)$ und nicht $P(X = k)$.

Beschreiben Sie die gesuchten Wahrscheinlichkeiten wieder in der Form $P(\ldots)$.

$(*)$ a) Die Wahrscheinlichkeit dafür, dass höchstens 13 Aufgaben richtig beantwortet sind.

$(**)$ b) Die Wahrscheinlichkeit dafür, dass mindestens 10 Aufgaben richtig sind.

$(**)$ c) Die Wahrscheinlichkeit dafür, dass genau 15 Aufgaben richtig beantwortet sind.

$(**)$ d) Die Wahrscheinlichkeit dafür, dass zwischen 9 und 12 Aufgaben richtig beantwortet sind.

Problem 3.4

We toss a coin three times. The random variable X indicates how many times "head" are tossed.

$(**)$ a) Set up the probability distribution of X as a table.

$(*)$ b) Calculate the probability that exactly 2 heads are tossed.

$(**)$ c) Calculate the probability that at least 2 heads are tossed.

$(**)$ d) Calculate the probability that no more than 1 head is tossed.

Problem 3.5

$(**)$ Calculate the expected value of the following probability distribution. Use **R**.

x_k	-5	-4	1	3	6
p_k	0.3	p_2	0.1	0.2	0.3

Problem 3.6

We roll a blue and a red dice together.

$(**)$ a) Determine the probability distribution of the sum of the eyes rolled.

$(**)$ b) Calculate the expected value and the standard deviation. Interpret these values.

Use **R** by creating two vectors `x` and `p`, multiplying the two and using the command `sum(...)`.

Classical and Bayesian Statistic

Sample solution for Problems 3

Solution 3.1

First we need to determine the value for p_2. Since the sum *has to be* equal 1, it follows that

$$p_2 = 1 - 0.3 - 0.1 - 0.2 - 0.3 = 0.1$$

Solution 3.2

a) Yes, because the probabilities add up to 1:

$$0.4 + 0.2 + 0.2 + 0.1 + 0.1 = 1$$

b) Wanted:

$$P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = 0.2 + 0.2 + 0,1 = 0.5$$

c) Sought:

$$P(X > 2) = P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) = 0.2 + 0.1 + 0.1 = 0.4$$

Or:

$$P(X > 2) = 1 - P(X \leq 2) = 1 - (P(X = 1) + P(X = 2)) = 1 - 0.4 - 0.2 = 0.4$$

d) Wanted:

$$P(X \leq 4) = 1 - P(X = 5) = 1 - 0.1 = 0.9$$

e) Sought:

$$P(X \geq 2) = 1 - P(X = 1) = 1 - 0.4 = 0.6$$

Solution 3.3

a) Gesucht:

$$P(X \leq 13) = 0.992$$

Kann direkt aus der Tabelle abgelesen werden.

b) Gesucht:

$$P(X \geq 10) = 1 - P(X \leq 9) = 1 - 0.939 = 0.061$$

c) Gesucht:

$$P(X = 15) = P(X \leq 15) - P(X \leq 14) = 1 - 0.999 = 0.001$$

d) Gesucht:

$$P(9 \leq X \leq 12) = P(X \leq 12) - P(X \leq 8) = 0.989 - 0.711 = 0.278$$

Solution 3.4

a) The sample space is (T: tails, H: heads)

$$\Omega = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$$

It follows that

$$P(X=0) = \frac{1}{8}, \qquad P(X=1) = \frac{3}{8}, \qquad P(X=2) = \frac{3}{8}, \qquad P(X=3) = \frac{1}{8}$$

So:

k	0	1	2	3
$P(X=k)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b) Sought:

$$P(X=2) = \frac{3}{8}$$

c) Sought:

$$P(X \geq 2) = P(X=2) + P(X=3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

d) Sought:

$$P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$$

Solution 3.5

See Problem 3.1

$$p_2 = 0.1$$

The expected value is determined by

$$\begin{aligned} \mathrm{E}(X) &= x_1p_1 + x_2p_2 + \ldots + x_5p_5 \\ &= -5 \cdot 0.3 + (-4) \cdot 0.1 + \ldots + 6 \cdot 0.3 = 0.6 \end{aligned}$$

We use **R**:

```
x <- c(-5, -4, 1, 3, 6)
p <- c(0.3, 0.1, 0.1, 0.2, 0.3)

sum(x*p)

[1] 0.6
```

Solution 3.6

a) Let X be the random variable for the eye sum thrown. Then

$$\Omega = \{2, 3, 4, \ldots, 12\}$$

This results in the following table for the probability distribution:

x_i	elementary event	abs. frequency	p_i
2	11	1	$\frac{1}{36}$
3	12,21	2	$\frac{2}{36}$
4	13,22,31	3	$\frac{3}{36}$
5	14,23,32,41	4	$\frac{4}{36}$
6	15,24,33,42,51	5	$\frac{5}{36}$
7	16,25,34,43,52,61	6	$\frac{6}{36}$
8	26,35,44,53,62	5	$\frac{5}{36}$
9	36,45,54,63	4	$\frac{4}{36}$
10	46,55,64	3	$\frac{3}{36}$
11	56,65	2	$\frac{2}{36}$
12	66	1	$\frac{1}{36}$

b) The expected value is determined by

$$\begin{aligned} \mathrm{E}(X) &= x_1 p_1 + x_2 p_2 + \ldots + x_{11} p_{11} \\ &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \ldots + 12 \cdot \frac{1}{36} \\ &= 7 \end{aligned}$$

We use **R** for the calculation:

```
x <- 2:12
p <- c(1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1) / 36

E <- sum(x*p)
E

[1] 7
```

If we roll the two dice a lot of times, the rolled mean of the eye sum is about 7. This was to be expected because the table above is symmetrical.

We calculate the standard deviation.

```
var_X <- sum((x-E)^2*p)
var_X

[1] 5.833333

sigma <- sqrt(var_X)
sigma

[1] 2.415229
```

The standard deviation is 2.42. If we roll the two dice a lot of times, the deviation from the expected value 7 is on "average" 2.42 .