The fixed points are physical iff they are IR stable and $u \geq 0$, $w \geq 0$.

The β functions:

$$\beta_u = u\left\{-\varepsilon + Ru + \frac{5+\alpha}{2}w - \alpha f(a)w\right\}, \tag{1}$$

$$\beta_w = w\left\{-\xi - R_1 u/6 + \frac{5+\alpha}{6}w\right\}, \tag{2}$$

$$\beta_a = uR_1(1-3a)/2. \tag{3}$$

The stability matrix:

$$\Omega_{ik} = \partial\beta_i/\partial g_k, \qquad \Omega_i = \Omega_{ii}, \tag{4}$$

where $i = u, w, a$ (no summation over i).

Points with $u = 0$.

Gaussian point (Point I): $u = w = 0$, a arbitrary. The only nonzero off-diagonal element is Ω_{au}; so the matrix Ω is triangular. Eigenvalues: $-\varepsilon$, $-\xi$, 0.

Passive scalar point (Point III):

$$u = 0, \quad w = \frac{6\xi}{5+\alpha}, \quad a \text{ arbitrary}. \tag{5}$$

Now $\Omega_{ua} = \Omega_{wa} = 0$, so that Ω_a decouples. The eigenvalue is $\Omega_a = 0$.

$\Omega_{uw} = 0$, so that the remaining 2×2 matrix is triangular, and Ω_u, Ω_w are eigenvalues:

$$\begin{aligned} \Omega_w &= \xi, && (6) \\ \Omega_u &= -\varepsilon + 3\xi - f(a)\frac{6\alpha\xi}{5+\alpha}. && (7) \end{aligned}$$

$\Omega_w > 0$ also gives $w > 0$.

The function $f(a)$ has a minimum $f(1/3) = 2/3$. So in Ω_u we can write $f(a) = f(1/3) + [f(a) - f(1/3)]$. This gives

$$\Omega_u = \Omega_u|_{a=1/3} - \frac{6\alpha\xi}{5+\alpha}[f(a) - f(1/3)] > 0 \tag{8}$$

The second term is negative, so (8) can be satisfied only if $\Omega_u|_{a=1/3} > 0$. This gives

$$-(5+\alpha)\varepsilon + (15-\alpha)\xi > 0. \tag{9}$$

This is the domain in the ξ–ε plane where the point can be stable. Now (8) gives the restriction for a:

$$6\alpha\xi[f(a) - f(1/3)] < -(5+\alpha)\varepsilon + (15-\alpha)\xi, \tag{10}$$

which gives

$$(a - 1/3)^2 < \frac{-(5+\alpha)\varepsilon + (15-\alpha)\xi}{18\alpha\xi}. \tag{11}$$

Conclusion: the coordinates of the point are given by (5) and (11). The region of IR stability in the ξ–ε plane is given by (9) and $\xi > 0$. Then $w > 0$ automatically.

For $\alpha = 0$ the boundary is $\xi = \varepsilon/3$, when α grows, it rotates counter clockwise, and for $\alpha \to \infty$ tends to $\xi = -\varepsilon$. This is very similar to ϕ^4 model.

This is all about points with $u = 0$.

Now let $u > 0$. Then from $\beta_a = 0$ we have $a = 1/3$. Then $\Omega_{ua} = \Omega_{wa} = 0$, and Ω_a decouples again. The eigenvalue is $\Omega_a = -3R_1u/2$. This is positive only if $R_1 < 0$. So in the following we assume $R_1 < 0$ (the case $R_1 > 0$ requires $a \to \infty$ and will be discussed later).

We put $a = 1/3$ in $\beta_{u,w}$ and obtain a closed system of two β functions. Such systems we discussed earlier. In general,

$$\beta_u = u[-\varepsilon + Au + Bw], \quad \beta_w = w[-\xi + Cu + Dw]. \tag{12}$$

Now

$$A = R, \quad B = (15 - \alpha)/6, \quad C = -R_1/6 > 0, \quad D = (5 + \alpha)/6 > 0 \tag{13}$$

and

$$\Delta = AD - BC = \frac{5}{6}(R + R_1/2) + \frac{\alpha}{6}(R - R_1/6). \tag{14}$$

Pure Potts point (Point II). Here $w = 0$, $u = \varepsilon/R$. Now $\Omega_{wu} = 0$ and the matrix is triangular. Then the point is IR stable for $\varepsilon > 0$ (so that $R > 0$ if we want $u > 0$) and $-\xi + Cu > 0$. The last relation gives $\xi < -R_1\varepsilon/6R$ (because $A/C > 0$).

Thus the point

$$u = \varepsilon/R, \quad w = 0, \quad a = 1/3$$

can be physical only if $R > 0$, $R_1 < 0$ (and any α) and is IR stable if

$$\varepsilon > 0, \quad \xi < -R_1\varepsilon/6R. \tag{15}$$

Full-scale point (Point IV).

$$u = (D\varepsilon - B\xi)/\Delta, \quad w = (A\xi - C\varepsilon)/\Delta, \tag{16}$$

with Δ from (14). This point can be physical only if $\Delta > 0$. Since $D > 0$, $A = R$ can be of either sign.

There are four cases (we recall that $R_1 < 0$):

Case IV-1. $R > -R_1/2$. Then authomatically $R > 0$, $R - R_1/6 > 0$. Thus $\Delta > 0$ for all α. The conditions that the point is IR attractive coincide with the conditions that its coordinates (16) are positive:

$$u > 0, \quad w > 0. \tag{17}$$

Case IV-2. $-R_1/2 > R > 0$. Then $R - R_1/6 > 0$. Thus $\Delta < 0$ for small α, but becomes positive for $\alpha > \alpha_0$, where

$$\alpha_0 = -5\frac{(R + R_1/2)}{(R - R_1/6)} > 0. \tag{18}$$

The conditions that the point is IR attractive are again (17).

Case IV-3. $0 > R > R_1/6$. Then $R - R_1/6 > 0$, and the point can be physical for $\alpha > \alpha_0$ with the same α_0. Now $A < 0$, and the conditions that the point is physical are given by the inequalities

$$u > 0, \quad Au + Dw > 0. \tag{19}$$

Case IV-4. $R < R_1/6$. Then $\Delta < 0$ for all α, unphysical case.

Now let us write the stability condition (17) for cases IV-1 and IV-2 in detail:

$$(A\xi - C\varepsilon) > 0, \quad (D\varepsilon - B\xi) > 0. \tag{20}$$

Since $R = A > 0$, the first is $\xi > (C/A)\varepsilon$.

Since $B > 0$ for $\alpha < 15$, the second inequality is $\xi < (D/B)\varepsilon$. It is also important that

$$\frac{C}{A} - \frac{D}{B} = \frac{-\Delta}{AB} < 0$$

so that

$$\frac{C}{A} < \frac{D}{B}.$$

Also note that (C/A) and D/B are positive. For $\alpha > 15$, we have $B < 0$ and $\xi > (D/B)\varepsilon$, and now $D/B < 0$.

Thus for $\alpha < 15$, the physical region is the sector in the upper-right quadrant in the ε–ξ plane, bounded by the ray $\xi = (C/A)\varepsilon$ from below and $\xi = (D/B)\varepsilon$ from above.

When α grows, the upper ray $\xi = (D/B)\varepsilon$ rotates counter clockwise and moves to the upper-left quadrant. For case IV-1 α changes from 0 to ∞ and the ray changes from $\xi = \varepsilon/3$ to $\xi = -\varepsilon$ (exactly like the boundary (9) of the point III).

For case IV-2 α changes from α_0 to ∞ and the ray changes from $\xi = -\varepsilon R_1/6R$ to $\xi = -\varepsilon$. Note that $1/3 < -R_1/6R$ (because $R + R_1/2 < 0$). Also note that for $\alpha = \alpha_0$ the two boundaries of the point IV coincide with each other and with the boundary (15) of the point II. Also note that

$$\alpha_0 - 15 = \frac{-20R}{(R - R_1/6)} < 0.$$

Case IV-3. The first condition $u > 0$ in (19) is

$$B\xi < \varepsilon D.$$

Now

$$\alpha_0 - 15 = -20\frac{R}{(R - R_1/6)} > 0$$

so that $B < 0$ and we have

$$\xi > \varepsilon D/B, \quad \text{where } D/B < 0. \tag{21}$$

The second condition $Au + Dw > 0$ in (19) is

$$\xi A(D - B) > \varepsilon D(C - A)$$

where $A = R < 0$ and $D - B = (\alpha - 5)/3 > 0$. The last inequality holds because $\alpha > \alpha_0$ and $\alpha_0 - 5 > 0$:

$$\alpha_0 - 5 = -10\frac{(R + R_1/6)}{(R - R_1/6)} > 0.$$

Thus the second inequality is

$$\xi < \varepsilon\frac{D(C - A)}{A(D - B)}, \tag{22}$$

where

$$\frac{D(C - A)}{A(D - B)} = \frac{(5 + \alpha)(R + R_1/6)}{2R(5 - \alpha)} < 0. \tag{23}$$

It is also important that

$$\frac{D}{B} > \frac{D(C - A)}{A(D - B)} \tag{24}$$

because

$$\frac{D(C - A)}{A(D - B)} - \frac{D}{B} = \frac{-\Delta D}{AB(D - B)} < 0.$$

Thus the both inequalities (21), (22) are satisfied in a sector in the upper-left quadrant; the lower bound is (21) and the upper bound is (22).

From (23) it follows that, when α changes from α_0 to ∞, the coefficient $D(C - A)/A(D - B)$ changes from $-R_1/6R$ to $-(R + R_1/6)/2R$. The coefficient $D/B = (5 + \alpha)/(15 - \alpha)$ changes from $-R_1/6R$ to -1. Thus for $\alpha = \alpha_0$ the domain has zero width, and when α grows it is getting wider.

Now let $R_1 > 0$. We are interesting only in points with $u \neq 0$. We pass to the new couplings $b = 1/a$ and $v = wa^2$ with the β functions

$$\beta_b = (-1/a^2)\beta_a, \quad \beta_v = a^2\beta_w + 2aw\beta_a. \tag{25}$$

Then

$$\beta_b = -uR_1 b(b-3)/2. \tag{26}$$

The relevant fixed point is $b = 0$ with $\Omega_{bv} = \Omega_{bu} = 0$, so that $\Omega_b = 3uR_1/2 > 0$ (for $u, R_1 > 0$) is the eigenvalue, and b decouples. We put $b = 0$ in the other functions and again obtain a closed system of the type (12):

$$\beta_u = u[-\varepsilon + Ru - 3\alpha v], \quad \beta_v = v[-\xi - (19/6)uR_1]. \tag{27}$$

We immediately see that $\Delta = -(19/2)\alpha R_1 < 0$, so that the full-scale point $u \neq 0$, $w \neq 0$ cannot be physical.

The other point is $v = 0$, $u = \varepsilon/R$, with the inequalities

$$\varepsilon > 0, \quad \xi < -\varepsilon(19/6)R_1/R$$

so that $R > 0$. The sector of stability is in the lower right quadrant. The point is similar to II because $v = 0$ and the advection is irrelevant.