3.6 The RG and magnetic hydrodynamics. Calculation of the RG functions

Stochastic magnetic hydrodynamics (MHD) was first studied in [103] using the Wilson RG technique. Here we shall present the quantum-field formulation [104] with a few refinements.

The stochastic MHD equations for two transverse vector fields (φ is the velocity field and Θ is the magnetic field) are written as

$$\nabla_t \varphi_i = \nu_0 \Delta \varphi_i - \partial_i P + (\Theta\partial)\Theta_i + F_i^{\varphi} \tag{3.69}$$

and

$$\nabla_t \Theta_i = u_0 \nu_0 \Delta \Theta_i + (\Theta\partial)\varphi_i + F_i^{\Theta}, \tag{3.70}$$

where $\nabla_t \equiv \partial_t + (\varphi\partial)$, P is the pressure, $\Theta \equiv B/(4\pi\rho)^{1/2}$, $u_0\nu_0 \equiv \nu_0' = c^2/4\pi\sigma$, B is the magnetic induction, c is the speed of light, σ is the conductivity, ρ is the density of the medium, and u_0 is the inverse magnetic Prandtl number analogous to (3.1). The transverse random magnetic force F_i^{Θ} is the curl of the random current. The additional term in (3.69) is the Lorentz force, proportional to [curl $\mathbf{B} \times \mathbf{B}$] $= (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla(B^2/2)$, and the second term is included in the pressure. Equation (3.70) follows from the simplest form of Ohm's law for a moving medium $\mathbf{j} = \sigma(\mathbf{E} + [\boldsymbol{\varphi} \times \mathbf{B}]/c)$ and the Maxwell equations neglecting the displacement current. The field φ_i is a vector and Θ_i is a pseudovector. The canonical dimensions of these fields are the same, as are the dimensions of the constants ν_0 and ν_0'.

According to the general rules of Sec. 1.2, the stochastic problem (3.69), (3.70) corresponds to the quantum-field model with unrenormalized action

$$\begin{aligned} S(\Phi) &= \varphi' D^{\varphi\varphi}\varphi'/2 + \Theta' D^{\Theta\Theta}\Theta'/2 + \varphi' D^{\varphi\Theta}\Theta' + \\ &+ \varphi'[-\partial_t\varphi + \nu_0\Delta\varphi - (\varphi\partial)\varphi + (\Theta\partial)\Theta] + \\ &+ \Theta'[-\partial_t\Theta + u_0\nu_0\Delta\varphi - (\varphi\partial)\Theta + (\Theta\partial)\varphi]. \end{aligned} \tag{3.71}$$

Following [103,104], we shall study only the massless model analogous to (1.10), writing the matrix of correlators of the random forces F^{φ} and F^{Θ} as

$$D_{is}^{\varphi\varphi} = D_0^{\varphi} P_{is} k^{4-d-2\varepsilon}, \qquad D_{is}^{\Theta\Theta} = D_0^{\Theta} P_{is} k^{4-d-2\alpha\varepsilon},$$
$$D_{is}^{\varphi\Theta} = D_0^{\varphi\Theta} \varepsilon_{ism} k_m k^{3-d-\varepsilon-\alpha\varepsilon}, \tag{3.72}$$

with

$$D_0^{\varphi} = g_0\nu_0^3, \qquad D_0^{\Theta} = g_0'\nu_0^3, \qquad D_0^{\varphi\Theta} = \rho(D_0^{\varphi}D_0^{\Theta})^{1/2}, \qquad |\rho| < 1. \tag{3.73}$$

Here P_{is} is the transverse projector and ε_{ism} is the completely antisymmetric pseudotensor. The choice of the exponents of k in (3.72) was already explained in Sec. 3.3, and the inequality $|\rho| < 1$ in (3.73) is the necessary condition for the correlator matrix (3.72) to be positive-definite. The index structure of the correlators is determined by the requirements that the fields be transverse and spatial parity be conserved: the fields φ and φ' are true vectors and Θ and Θ' are pseudovectors, and so the mixed correlator is a pseudotensor. It is automatically transverse in the indices i and s.

In calculations performed in the spirit of dimensional regularization ([25], Ch. 4), the symbols δ_{is} and ε_{ism} can formally be used for arbitrary dimension d, but in the final expressions the symbol ε_{ism}, in contrast to δ_{is}, is meaningful only for the real dimension $d = 3$. In the other important case, $d = 2$, there is no pseudotensor which is transverse in both indices, and the mixed correlator in (3.72) must be taken to be zero. The mixed correlator was not introduced in [103], and the general case (3.72) was studied in [104]. The $d = 2$ problem was analyzed in both these studies, but there are errors: in [103] the need to renormalize the contributions with noise correlators (3.72), important for $d = 2$ (see below), was completely neglected, and in [104] this renormalization was done, but multiplicatively, which is also incorrect (see Sec. 3.10 below for a detailed analysis of the two-dimensional problem). We shall therefore not discuss the $d = 2$ case here, but consider only the real three-dimensional problem.

The diagrammatic technique for the action (3.71) is constructed following the standard Feynman rules (Sec. 1.2). The graphs will contain three types of vertex: $\varphi'\varphi\varphi$, $\varphi'\Theta\Theta$, and $\Theta'\varphi\Theta$, and five types of line (propagator): $\langle\varphi\varphi'\rangle_0$, $\langle\Theta\Theta'\rangle_0$, $\langle\varphi\varphi\rangle_0$, $\langle\Theta\Theta\rangle_0$, and $\langle\varphi\Theta\rangle_0$, where the last three are proportional to the corresponding correlators (3.72). The only pseudotensor among the propagators will be the mixed one $\langle\varphi\Theta\rangle_0$. The transversality of all the fields at the vertices allows the derivative ∂ to be moved onto the auxiliary fields φ' and Θ', and so the real degree of divergence δ' (see Sec. 1.4) is related to the formal one $\delta = d+2-\sum_i n_{\Phi_i} d_{\Phi_i}$ as $\delta' = \delta - n_{\varphi'} - n_{\Theta'}$. Using the data of Table 1.1 in Sec. 1.5 (the dimensions of the fields φ and Θ are identical), we find $\delta' = 2 - n_\varphi - n_\Theta + d(1 - n_{\varphi'} - n_{\Theta'})$. From this it follows that for $d > 2$ counterterms can be generated only by 1-irreducible Green functions of the type $\langle\varphi'\varphi\rangle$, $\langle\varphi'\Theta\rangle$, $\langle\Theta'\varphi\rangle$, and $\langle\Theta'\Theta\rangle$ (for all $\delta = 2$, $\delta' = 1$, $d^\omega = 1$) and $\langle\varphi'\varphi\varphi\rangle$, $\langle\varphi'\varphi\Theta\rangle$, $\langle\varphi'\Theta\Theta\rangle$, $\langle\Theta'\varphi\varphi\rangle$, $\langle\Theta'\varphi\Theta\rangle$, and $\langle\Theta'\Theta\Theta\rangle$ (for all $\delta = 1$, $\delta' = 0$, $d^\omega = 0$). For $d = 2$ additional divergences arise in the functions $\langle\varphi'\varphi'\rangle$ and $\langle\Theta'\Theta'\rangle$. We shall not consider this case here.

The finiteness of the number of needed counterterms implies that the model (3.71) is renormalizable in principle. However, to use the standard RG technique we need to have not simply renormalizability, but multiplicative renormalizability (see Sec. 1.4), i.e., the counterterms must be of the same form as the contributions of the unrenormalized action. This can be proved using the spatial parity of the fields, the Galilean invariance, and the transversality of the vertex $\Theta'\Theta\varphi$ in the index of the field Θ' [i.e., the transversality of the vector $(\Theta\partial)\varphi - (\varphi\partial)\Theta$, which is the coefficient of Θ' at the vertex]. In the end, only counterterms of the form $\varphi'\Delta\varphi$, $\Theta'\Delta\Theta$, and $\varphi'(\Theta\partial)\Theta$ are allowed, and so the renormalized action has the form

$$\begin{aligned} S_{\rm R}(\Phi) &= \varphi' D^{\varphi\varphi}\varphi'/2 + \Theta' D^{\Theta\Theta}\Theta'/2 + \varphi' D^{\varphi\Theta}\Theta' + \\ &+ \varphi'[-\partial_t\varphi + Z_1\nu\Delta\varphi - (\varphi\partial)\varphi + Z_3(\Theta\partial)\Theta] + \\ &+ \Theta'[-\partial_t\Theta + Z_2 u\nu\Delta\varphi - (\varphi\partial)\Theta + (\Theta\partial)\varphi] \end{aligned} \tag{3.74}$$

with the substitution $D_0^\varphi = g_0\nu_0^3 = g\nu^3\mu^{2\varepsilon}$ and $D_0^\Theta = g_0'\nu_0^3 = g'\nu^3\mu^{2\alpha\varepsilon}$ in the amplitudes (3.73). The parameter ρ in the mixed correlator is not renormalized; formally, $\rho_0 = \rho Z_\rho$ with $Z_\rho = 1$, as for m in (1.32).

The action (3.74) is obtained from (3.71) by the standard multiplicative renormalization $\Phi_i \to Z_{\Phi_i}\Phi_i$ of the fields $\Phi \equiv (\varphi, \varphi', \Theta, \Theta')$ and the parameters $\nu_0 = \nu Z_\nu$, $u_0 = uZ_u$, $g_0 = g\mu^{2\varepsilon}Z_g$, and $g_0' = g'\mu^{2\alpha\varepsilon}Z_{g'}$. These renormalization constants are expressed in terms of $Z_{1,2,3}$ in (3.74) as

$$\begin{gathered} Z_\varphi = Z_{\varphi'} = 1, \quad Z_\Theta = Z_{\Theta'}^{-1} = Z_3^{1/2}, \quad Z_\nu = Z_1, \quad Z_u = Z_2 Z_1^{-1}, \\ Z_g = Z_1^{-3}, \qquad Z_{g'} = Z_3 Z_1^{-3}, \end{gathered} \tag{3.75}$$

which is equivalent to the following relations between the RG functions $\gamma = \widetilde{D}_\mu \ln Z$:

$$\begin{gathered} \gamma_\varphi = \gamma_{\varphi'} = 0, \quad \gamma_\Theta = -\gamma_{\Theta'} = \gamma_3/2, \quad \gamma_\nu = \gamma_1, \quad \gamma_u = \gamma_2 - \gamma_1, \\ \gamma_g = -3\gamma_1, \qquad \gamma_{g'} = \gamma_3 - 3\gamma_1. \end{gathered} \tag{3.76}$$

In contrast to almost all the models studied earlier, here multiplicative renormalization of the fields Θ, Θ' is required, whereas φ and φ' are not renormalized. The renormalized charges are the parameters g, g', and u. Using the general definition (1.33) and the formulas for the parameter renormalization, the corresponding β functions are expressed in terms of the anomalous dimensions (3.76) as

$$\beta_u = u(\gamma_1 - \gamma_2), \quad \beta_g = g(-2\varepsilon + 3\gamma_1), \quad \beta_{g'} = g'(-2\alpha\varepsilon + 3\gamma_1 - \gamma_3). \tag{3.77}$$

The constants $Z_{1,2,3}$ in (3.74) are calculated from the graphs of the corresponding basic theory (Sec. 1.4), obtained from (3.74) by the substitution $Z_i \to 1$. All the 1-irreducible graphs needed for the one-loop calculation are shown in Fig. 3.3. The results for arbitrary dimension $d > 2$ are

$$\begin{aligned} Z_1 &= 1 - \frac{gd(d-1)}{4B\varepsilon} - \frac{g'(d^2+d-4)}{4B\alpha\varepsilon u^2}, \\ Z_2 &= 1 - \frac{g(d+2)(d-1)}{2B\varepsilon u(u+1)} - \frac{g'(d+2)(d-3)}{2B\alpha\varepsilon u^2(u+1)}, \\ Z_3 &= 1 + \frac{g}{B\varepsilon u} - \frac{g'}{B\alpha\varepsilon u^2}, \end{aligned} \tag{3.78}$$

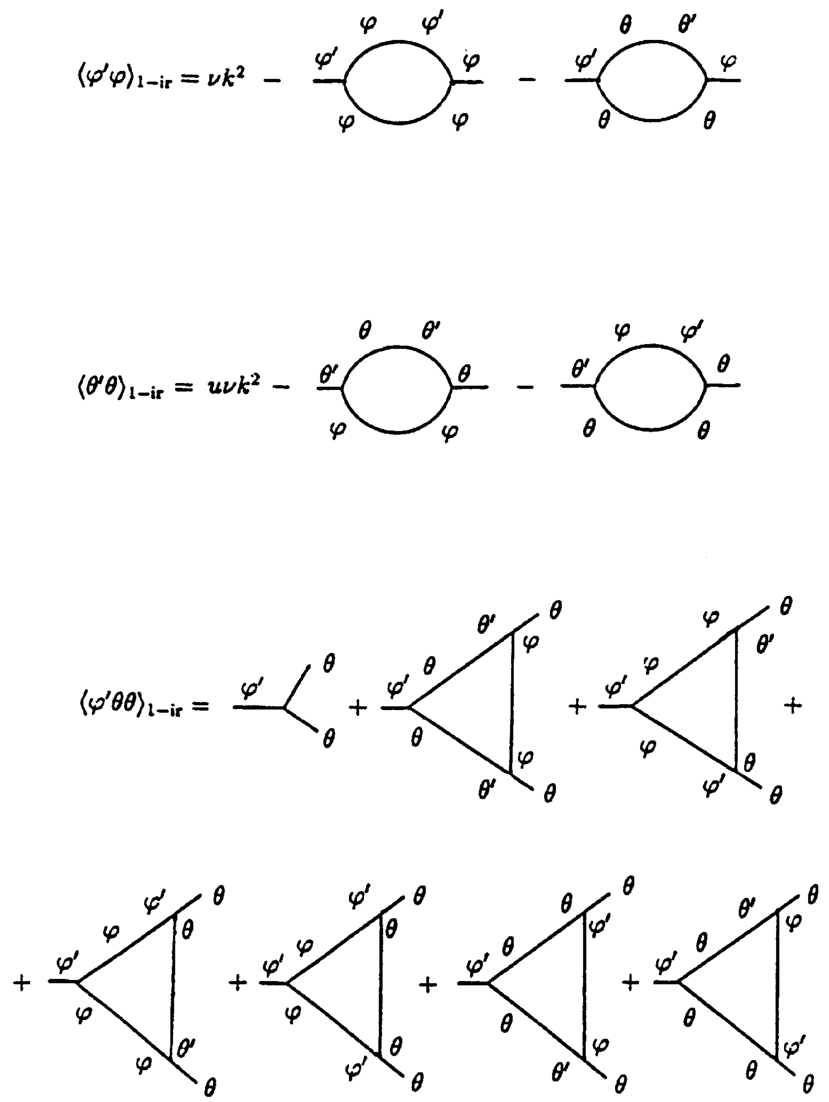

Figure 3.3.

where $B \equiv 2d(d+2)(2\pi)^d/S_d = d(d+2)(4\pi)^{d/2}\Gamma(d/2)$.

It is convenient to eliminate the ratios g/u and g'/u^2 by changing from g and g' to the new charges

$$g_1 \equiv g/Bu, \qquad g_2 \equiv g'/Bu^2, \tag{3.79}$$

for which from (3.75) and (3.77) we obtain

$$\beta_u = u(\gamma_1 - \gamma_2), \qquad \beta_{g_1} = g_1(-2\varepsilon + 2\gamma_1 + \gamma_2),$$

$$\beta_{g_2} = g_2(-2\alpha\varepsilon + \gamma_1 + 2\gamma_2 - \gamma_3). \tag{3.80}$$

Calculating the $\gamma_i = \widetilde{D}_\mu \ln Z_i$ with $\widetilde{D}_\mu = D_{\rm RG}$ (see Sec. 1.4) using the constants (3.78) in the new variables (3.79), for dimension $d = 3$ we find

$$\gamma_1 = 3g_1u + 4g_2, \quad \gamma_2 = \frac{10g_1}{(u+1)}, \quad \gamma_3 = -2g_1 + 2g_2, \tag{3.81}$$

from which using (3.80) we find the β functions in the one-loop aproximation:

$$\begin{aligned} \beta_{g_1} &= g_1[-2\varepsilon + 6g_1u + 8g_2 + 10g_1/(u+1)], \\ \beta_{g_2} &= g_2[-2\alpha\varepsilon + 3g_1u + 2g_1 + 2g_2 + 20g_1/(u+1)], \\ \beta_u &= u[3g_1u + 4g_2 - 10g_1/(u+1)]. \end{aligned} \tag{3.82}$$

We note that the parameter ρ from (3.73) does not enter into the one-loop β functions (3.82), because none of the graphs in Fig. 3.3 contains the mixed correlator $\langle\varphi\Theta\rangle_0$. However, ρ dependence appears in higher orders. The one-loop β functions (3.82) obtained in [104] coincide with the results of [103] up to notation (and misprints in the equations of [103]).

The system of β functions (3.80) has the following fixed points:
(1) The line $g_{1*} = g_{2*} = 0$ with arbitrary u_*;
(2) $g_{2*} = u_* = 0$, $g_{1*} = \varepsilon/5$;
(3) $g_{1*} = u_* = 0$, $g_{2*} = \alpha\varepsilon$;
(4) $g_{2*} = 0$, $g_{1*} = \varepsilon(u_*+1)/15$, $u_* = (\sqrt{43/3} - 1)/2 = 1.393$;
(5) $u_* = 0$, $g_{1*} = \varepsilon(4\alpha - 1)/39$, $g_{2*} = \varepsilon(11 - 5\alpha)/39$;

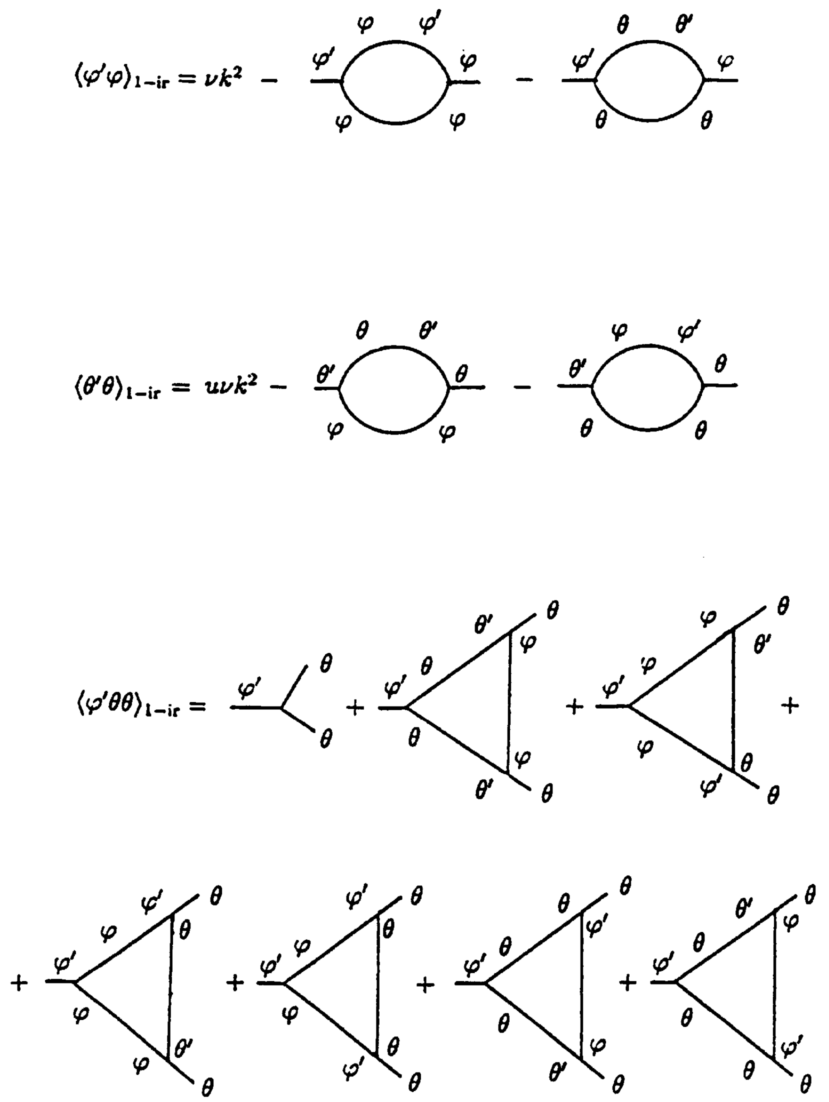

Figure 3.3.

where $B \equiv 2d(d+2)(2\pi)^d/S_d = d(d+2)(4\pi)^{d/2}\Gamma(d/2)$.

It is convenient to eliminate the ratios g/u and g'/u^2 by changing from g and g' to the new charges

$$g_1 \equiv g/Bu, \qquad g_2 \equiv g'/Bu^2, \tag{3.79}$$

for which from (3.75) and (3.77) we obtain

$$\beta_u = u(\gamma_1 - \gamma_2), \qquad \beta_{g_1} = g_1(-2\varepsilon + 2\gamma_1 + \gamma_2),$$

$$\beta_{g_2} = g_2(-2\alpha\varepsilon + \gamma_1 + 2\gamma_2 - \gamma_3). \tag{3.80}$$

Calculating the $\gamma_i = \widetilde{D}_\mu \ln Z_i$ with $\widetilde{D}_\mu = D_{\rm RG}$ (see Sec. 1.4) using the constants (3.78) in the new variables (3.79), for dimension $d = 3$ we find

$$\gamma_1 = 3g_1u + 4g_2, \quad \gamma_2 = \frac{10g_1}{(u+1)}, \quad \gamma_3 = -2g_1 + 2g_2, \tag{3.81}$$

from which using (3.80) we find the β functions in the one-loop aproximation:

$$\begin{aligned} \beta_{g_1} &= g_1[-2\varepsilon + 6g_1u + 8g_2 + 10g_1/(u+1)], \\ \beta_{g_2} &= g_2[-2\alpha\varepsilon + 3g_1u + 2g_1 + 2g_2 + 20g_1/(u+1)], \\ \beta_u &= u[3g_1u + 4g_2 - 10g_1/(u+1)]. \end{aligned} \tag{3.82}$$

We note that the parameter ρ from (3.73) does not enter into the one-loop β functions (3.82), because none of the graphs in Fig. 3.3 contains the mixed correlator $\langle\varphi\Theta\rangle_0$. However, ρ dependence appears in higher orders. The one-loop β functions (3.82) obtained in [104] coincide with the results of [103] up to notation (and misprints in the equations of [103]).

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(3) $g_{1*} = u_* = 0$, $g_{2*} = \alpha\varepsilon$;

(4) $g_{2*} = 0$, $g_{1*} = \varepsilon(u_*+1)/15$, $u_* = (\sqrt{43/3} - 1)/2 = 1.393$;

(5) $u_* = 0$, $g_{1*} = \varepsilon(4\alpha - 1)/39$, $g_{2*} = \varepsilon(11 - 5\alpha)/39$;

(6) $g_{1*} = \varepsilon(u_* + 1)/15$, $g_{2*} = \varepsilon[10 - 3u_*(u_* + 1)]/60$, where u_* is the positive root of the equation $3u^2 + 7u + 54 = 60\alpha$, which exists for $\alpha \geq 0.9$.

The IR-stable fixed points are those for which all the $\mathrm{Re}\,\omega_\alpha > 0$ for the matrix (3.19) of the three charges $g_i = \{g_1, g_2, u\}$. In the physical region of nonnegative $g_{1,2}$ and u, only points 3 and 4 (the magnetic and kinetic points in the terminology of [103]) can be stable: the first is stable for any $\alpha \geq 0.25$, and the second is stable for any $\alpha \leq 1.16$. In the intermediate region $0.25 \leq \alpha \leq 1.16$ both points are stable, and the critical regime depends on which of their basins of attraction the initial data of the RG equations (3.17) for the invariant charges are located in. In this sense the critical behavior is nonuniversal. The basins of attraction (see Sec. 3.1) of these points for the real case $\alpha = 1$ are studied in [105]. However, the results quoted there cannot be considered reliable, as the authors of [105] did not take into account the renormalization of the vertex $\varphi'\Theta\Theta$, erroneously assuming that it is an effect of higher order in the charges. It should also be noted that the analysis of the d-dimensional problem indicates that the magnetic regime vanishes for $d < 2.8$ [103].

3.7 Critical dimensions in magnetic hydrodynamics

If the critical dimensions Δ_F of the various quantities F in the model (3.71) are calculated using the standard rule (2.3) and the RG functions from (3.76) and (3.80), for $d = 3$ we obtain

$$\Delta_\omega = -\Delta_t = 2 - \gamma_1^*, \quad \Delta_\varphi = 1 - \gamma_1^*, \quad \Delta_\Theta = 1 - \gamma_1^* + \gamma_3^*/2,$$

$$\Delta_\varphi + \Delta_{\varphi'} = \Delta_\Theta + \Delta_{\Theta'} = d = 3. \tag{3.83}$$

At the kinetic point (fixed point 4 in the previous section) we have $\gamma_1^* = \gamma_2^* = 2\varepsilon/3$ and $\gamma_3^* = -4\varepsilon/9u_* + \ldots$ with $u_* = 1.393$, which for ω, φ, and φ' gives the previous results (1.44). For the field Θ we find

$$\Delta_\Theta = 1 - 2\varepsilon/3 - 2\varepsilon/9u_* + \ldots\,, \tag{3.84}$$

where the ellipsis everywhere denotes possible corrections $\sim \varepsilon^2$, ε^3, and so on.

At the magnetic fixed point (fixed point 3 in the previous section) we have

$$\gamma_1^* = 4\alpha\varepsilon + \ldots\,, \qquad \gamma_2^* = 0 + \ldots\,, \qquad \gamma_3^* = 2\alpha\varepsilon + \ldots\,, \tag{3.85}$$

which when substituted into (3.83) gives the desired Δ_F.

The same conclusion regarding the critical dimensions in the two possible regimes was arrived at in [104]. However, as was later shown in [102], it is actually incorrect because we are dealing with a special case in which the standard rule (2.3) is inapplicable. The authors of [103] avoided these problems, since they made no use at all of the concept of critical dimensions and did not discuss IR scaling as a common property of all the Green functions, but studied only the pair correlators $\langle\varphi\varphi\rangle$ and $\langle\Theta\Theta\rangle$.

To understand the arguments of [102], let us consider an RG representation like (3.16) of an arbitrary renormalized Green function in some dynamical model like (1.13) with charges $g = \{g_i\}$ (in MHD, $g = \{g_1, g_2, u\}$). The representation (3.16) was obtained assuming that there is no renormalization of the fields and the Green functions. In the model (3.71) the fields Θ and Θ' are renormalized, and so for the Green functions we have $F = Z_F F_{\mathrm{R}}$ with $Z_F = (Z_\Phi)^{n_\Phi}$. This generates the additional term $\gamma_F = \widetilde{D}_\mu \ln Z_F$ in the operator D_{RG} in (3.13), and that equation becomes

$$[D_{\mathrm{RG}} + \gamma_F]F_{\mathrm{R}} = 0, \qquad \gamma_F = \sum_\Phi \gamma_\Phi n_\Phi, \tag{3.86}$$

along with the additional factor R_{γ_F} in the RG representation (3.16):

$$F_{\mathrm{R}} = k^{d_F}(\overline{\nu})^{d_F^\omega} R(1, \omega/\overline{\nu}k^2, \overline{g}) R_{\gamma_F}, \tag{3.87}$$

$$R_{\gamma_F} = \exp\left\{\int_1^s \frac{ds'}{s'}\gamma_F(\overline{g}(s', g))\right\}, \tag{3.88}$$

where $\overline{g}(s, g)$ is the solution of the Cauchy problem (3.17) with $s = k/\mu$. In the special case of a single-charge model, (3.88) can be rewritten as

$$R_{\gamma_F} = \exp\left\{\int_g^{\overline{g}} dg' \frac{\gamma_F(g')}{\beta(g')}\right\}. \tag{3.89}$$

At the IR asymptote $s \to 0$ in (3.87) we have $\overline{g} \to g_*$, $\overline{\nu} \to \overline{\nu}_*$ (see Sec. 1.6), and

$$R_{\gamma_F} \to \text{const} \cdot (k/\mu)^{\gamma_F^*}, \qquad \gamma_F^* \equiv \gamma_F(g_*), \tag{3.90}$$

which with the additional assumption that the scaling function R is finite in the IR regime leads to (2.3) for the critical dimensions Δ_F.

Since the function R is finite in the IR regime, it can be assumed to have zero critical dimension. This is the usual situation which occurs in most models (we recall that it is the requirement that the argument $\omega/\overline{\nu}k^2$ of the function R be finite that fixes the relative rate of falloff of the variables ω and k at the IR asymptote). However, in the model (3.71) the assumption that R is finite is not valid, which leads to violation of the rule (2.3). Below we shall consider the two possible critical regimes separately.

(1) The kinetic point. At this point $g_{2*} = 0$, and the problem is that for Green functions vanishing for $g_2 = 0$ with $n_\Theta > n_{\Theta'}$ the function R in (3.87) contains factors $\overline{g}_2 \to 0$. In particular, a single factor of g_2 is isolated from R for the correlator $F = \langle\Theta\Theta\rangle$. Denoting $R = g_2\widetilde{R}$, at the IR asymptote $s \equiv k/\mu \to 0$ we obtain (we explicitly indicate only the charge dependence)

$$R \equiv \overline{g}_2\widetilde{R}(\overline{g}_1, \overline{g}_2, \overline{u}) \sim s^{\omega_2}\widetilde{R}(g_{1*}, 0, u_*), \tag{3.91}$$

where ω_2 is the correction exponent characterizing the rate of falloff as $\overline{g}_2 \to 0$ (see Sec. 3.1):

$$\omega_2 = \partial_{g_2}\beta_{g_2}|_{g_*} = 2\varepsilon - 2\alpha\varepsilon - \gamma_3^* \tag{3.92}$$

[this was obtained from (3.80) using $\gamma_1^* = \gamma_2^* = 2\varepsilon/3$ for the kinetic point].

Therefore, in this case the function R for the correlator $\langle\Theta\Theta\rangle$ turns out to have nonzero critical dimension $\Delta[R] = \omega_2$, which corresponds to the addition of $\omega_2/2$ to the critical dimension of the field Θ.

Denoting the new (correct) dimensions by $\widetilde{\Delta}_F$, we have

$$\widetilde{\Delta}_\Theta = \Delta_\Theta + \omega_2/2 = 1 - 2\varepsilon/3 + \varepsilon(1-\alpha) \tag{3.93}$$

without corrections $\sim \varepsilon^2, \varepsilon^3, \dots$ since the contribution of γ_3^* to (3.93) cancels.

The dimension (3.93) is determined from the correlator $\langle\Theta\Theta\rangle$, but it can be shown that for all the Green functions in this regime there is IR scaling with critical dimensions $\widetilde{\Delta}_F$ coinciding with the earlier expressions (1.44) for $F = \omega, \varphi$, and φ' and with the new dimensions $\widetilde{\Delta}_\Theta$, $\widetilde{\Delta}_{\Theta'} = d - \widetilde{\Delta}_\Theta$ $(d = 3)$ for the fields Θ and Θ'. For this it is necessary to make a change of variable in (3.74), going to the new fields $\widetilde{\Theta}$ and $\widetilde{\Theta}'$ [102]:

$$\widetilde{\Theta}(x) = \frac{\Theta(x)}{\sqrt{g_2}}, \qquad \widetilde{\Theta}'(x) = \Theta'(x)\sqrt{g_2}, \tag{3.94}$$

without changing φ and φ'. For the Green functions of the new fields in Φ the quantity γ_F in (3.86) acquires the additional term $(n_\Theta - n_{\Theta'})\beta_{g_2}/2g_2$, which in the critical regime we are considering is equivalent to the addition of $\pm\omega_2/2$ to the critical dimensions of the fields Θ and Θ'. Now the critical dimensions can be calculated using the usual rule (2.3), since the problem of the zero in the R functions disappears after the substitution (3.94). In fact, the substitution (3.94) moves the charge g_2 from the bare propagators to the vertex $\varphi'\Theta\Theta$, so that this vertex can simply be neglected when calculating an asymptote of the type (3.91) in the regime $\overline{g}_2 \to 0$. The Green functions remain finite, and the magnetic field plays the role of a passive (vector) admixture. For $\alpha = 1$ and $\varepsilon = 2$, the true dimensions $\widetilde{\Delta}_{\Theta,\Theta'}$, in contrast to $\Delta_{\Theta,\Theta'}$, coincide with the Kolmogorov values (1.44) for φ and φ'.

(2) The magnetic point. At this point $u_* = 0$, $g_{1*} = 0$, and $g_{2*} = \alpha\varepsilon + \dots$, and the variables $\overline{u}$ and $\overline{g}_1$ fall off for $s \to 0$ according to a power law:

$$\overline{u} \sim s^{\omega_u} \to 0, \qquad \overline{g}_1 \sim s^{\omega_1} \to 0 \tag{3.95}$$

with exponents

$$\omega_u = \gamma_1^* - \gamma_2^* = 4\alpha\varepsilon + \dots\,, \quad \omega_1 = -2\varepsilon + 2\gamma_1^* - \gamma_2^* = -2\varepsilon + 8\alpha\varepsilon + \dots \tag{3.96}$$

For $g_1 = 0$, $u \neq 0$, and $g_2 \neq 0$ only the Green functions with odd value of the sum $n_\Theta + n_{\Theta'}$ vanish (they can be generated only by graphs involving the mixed correlator $\langle\varphi\Theta\rangle_0$), and all the others remain finite. Excluding for now the special case of odd $n_\Theta + n_{\Theta'}$, we conclude that for the other Green functions there is no problem of a zero for $\overline{g}_1 \to 0$ in the R functions. However, there is a second problem, that of singularities for $\overline{u} \to 0$ (in contrast to the charges $g_{1,2}$, the parameter u can also occur in denominators, and so instead of the term "zero" we use the broader term "singularity"). This problem can be solved [102] for dynamical Green functions with the charges in (3.79) using a variable substitution $\Phi(x) \to \widetilde{\Phi}(x)$ analogous to (3.94) in the functional (3.74):

$$\varphi(x) = u\widetilde{\varphi}(\widetilde{x}),\ \ \varphi'(x) = \widetilde{\varphi}'(\widetilde{x}),\ \ \Theta(x) = \sqrt{u}\widetilde{\Theta}(\widetilde{x}),\ \ \Theta'(x) = \frac{\widetilde{\Theta}'(\widetilde{x})}{\sqrt{u}}, \tag{3.97}$$

where $x \equiv \{t, \mathbf{x}\}$ and $\widetilde{x} \equiv \{ut, \mathbf{x}\}$. The RG equation for the new fields has the form (3.86) with the additional term $[D_t + n_\varphi + n_\Theta/2 - n_{\Theta'}/2]\beta_u/u$ in the RG operator. In the IR regime under consideration we have $\beta_u/u = \omega_u = \gamma_1^* - \gamma_2^*$, and, when the RG equation is combined with scale transformations (see Sec. 1.5), this leads to the following new values of the critical dimensions ($d = 3$):

$$\widetilde{\Delta}_\omega = -\widetilde{\Delta}_t = 2 - \gamma_2^*, \qquad \widetilde{\Delta}_\varphi = 1 - \gamma_2^*, \qquad \widetilde{\Delta}_{\varphi'} = 2 + \gamma_1^*,$$
$$\widetilde{\Delta}_\Theta = 1 - \alpha\varepsilon + \gamma_2^*/2, \qquad \widetilde{\Delta}_\Theta + \widetilde{\Delta}_{\Theta'} = d = 3, \tag{3.98}$$

with the values of γ_i^* from (3.85). After changing to the new variables $\widetilde{\Phi}$ and charges (3.79) in (3.74), the parameter u appears as a factor only in the terms $\widetilde{\varphi}'\partial_t\widetilde{\varphi}$ and $\widetilde{\varphi}'(\widetilde{\varphi}\partial)\widetilde{\varphi}$. These contributions to the action vanish at the asymptote (3.95), but the remaining contributions are sufficient for ensuring the finiteness of the limit

of dynamical Green functions with even value of $n_\Theta + n_{\Theta'}$. For the new fields $\widetilde{\Phi}$ the bare propagators have the form

$$\langle\widetilde{\varphi}\widetilde{\varphi}'\rangle_0 = (-iu\omega + \nu k^2)^{-1}, \qquad \langle\widetilde{\Theta}\widetilde{\Theta}'\rangle_0 = (-i\omega + \nu k^2)^{-1},$$
$$\langle\widetilde{\varphi}\widetilde{\varphi}\rangle_0 = \frac{g_1\nu^3\mu^{2\varepsilon}k^{4-d-2\varepsilon}}{B|-iu\omega + \nu k^2|^2},$$
$$\langle\widetilde{\Theta}\widetilde{\Theta}\rangle_0 = \frac{g_2\nu^3\mu^{2\alpha\varepsilon}k^{4-d-2\alpha\varepsilon}}{B|-i\omega + \nu k^2|^2}, \tag{3.99}$$

with the constant B from (3.78). For $g_1 \to 0$ and $u \to 0$ the propagator $\langle\widetilde{\varphi}\widetilde{\varphi}\rangle$ vanishes, and the dependence on the frequency ω drops out of $\langle\widetilde{\varphi}\widetilde{\varphi}'\rangle$.

The critical dimensions (3.98) determine the IR asymptote of the dynamical Green functions and also of the corresponding static objects if the needed integrals over frequencies in expressions like (1.48) exist in the limit theory with $g_1 = 0$ and $u = 0$. This condition of convergence in ω is satisfied for all the correlation functions of the fields $\widetilde{\varphi}$ and $\widetilde{\Theta}$ except the correlators $\langle\widetilde{\varphi}\widetilde{\varphi}\rangle$ and $\langle\widetilde{\varphi}\widetilde{\varphi}\widetilde{\varphi}\rangle$. According to the standard rules of Sec. 1.7 [see (1.62)], for the static correlator $\langle\widetilde{\Phi}\widetilde{\Phi}\rangle$ of any of the fields $\widetilde{\Phi} = \widetilde{\varphi}, \widetilde{\Theta}$ we must obtain

$$\langle\widetilde{\Phi}\widetilde{\Phi}\rangle_{\text{stat}} \sim k^{-d+2\widetilde{\Delta}_\Phi}, \tag{3.100}$$

where $\widetilde{\Delta}_\Phi$ is the corresponding dimension (3.98). In this case the representation (3.98), (3.100) is valid for the correlator $\langle\widetilde{\Theta}\widetilde{\Theta}\rangle$ [since the frequency dependence in the magnetic propagators (3.99) is normal], but it is not valid for the correlator $\langle\widetilde{\varphi}\widetilde{\varphi}\rangle$ owing to the anomalous behavior in ω of the propagators (3.99) involving φ, φ' in the limit theory with $u = 0$.

Let us consider the behavior of the static correlator $\langle\widetilde{\varphi}\widetilde{\varphi}\rangle$ in more detail, following [103]. The graphs needed for the analysis are shown in Fig. 3.4. All the other possible one-loop graphs contain vertices $u\widetilde{\varphi}'(\widetilde{\varphi}\partial)\widetilde{\varphi}$ proportional to the small factor u (see above), and so they are not important in this regime, as shown in [103]. Multiloop graphs with vertices $\widetilde{\varphi}'\widetilde{\Theta}\widetilde{\Theta}$ were not discussed in [103], but it can be shown that they give contributions of the same type as the one-loop graph in Fig. 3.4.

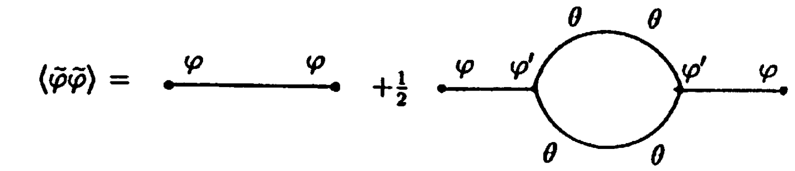

Figure 3.4.

When calculating the scaling function R using the graphs of Fig. 3.4, in a representation like (3.87) for $F_R = \langle\tilde\varphi\tilde\varphi\rangle$ the parameters ν, u, and g_i are replaced by the corresponding invariant variables. The contribution of the bare graph vanishes in the limit (3.95), and the ω dependence drops out in the external lines $\langle\tilde\varphi\tilde\varphi'\rangle$ of the one-loop graph. It was shown in [103] that the internal 1-irreducible block of this graph with lines $\langle\tilde\Theta\tilde\Theta\rangle$ has asymptote $\sim \omega^{-1/2}$ for $\omega \to \infty$ in the dynamical theory, and so its integral over ω diverges. This means that it is impossible to go directly from dynamical objects to static ones in the limit theory. The asymptotic expression must be analyzed more accurately *before* taking the limit, rather than at the limit (3.95). It is then found that the integral over ω is cut off by the factor $iu\omega$ in the propagators $\langle\tilde\varphi\tilde\varphi'\rangle$, and so the formal "divergence" $\int d\omega\ \omega^{-1/2} \sim \omega^{1/2}$ is actually realized as an additional [compared to the normal representation (3.100) valid when there is no divergence] factor $(\overline{u})^{-1/2} \sim s^{-2\alpha\varepsilon}$, according to (3.96). This should be compared to the analogous factor $\overline{g}_1\overline{u}^{-1} \sim s^{-2\varepsilon+4\alpha\varepsilon}$ arising from the contribution of the bare graph in Fig. 3.4 (g_1 enters explicitly into the propagator, and u^{-1} appears after the integration over ω).

Therefore, owing to the divergence of the integrals over ω, in the limit theory with $u = 0$ the one-loop graph generates an extra factor $s^{-2\alpha\varepsilon}$ in (3.100), and the bare factor is $s^{-2\varepsilon+4\alpha\varepsilon}$.

For $\alpha > 1/3$ with $s \equiv k/\mu \to 0$ the loop contribution is more important, and for $\alpha < 1/3$ the bare one is. The addition of these factors to (3.100) with $\widetilde{\Delta}_\Phi = \widetilde{\Delta}_\varphi$ from (3.98) and (3.85) leads to the final result [103] for the static correlator:

$$\langle\tilde\varphi\tilde\varphi\rangle|_{\text{stat}} \sim \begin{cases} k^{-d+2-2\alpha\varepsilon} & \text{for } \alpha > 1/3 \\ k^{-d+2-2\varepsilon+4\alpha\varepsilon} & \text{for } \alpha < 1/3. \end{cases} \tag{3.101}$$

The most realistic value $\alpha = 1$ (see the discussion in Sec. 3.3) corresponds to the first version in (3.101).

In conclusion, we add that the Green functions with odd $n_\Theta + n_{\Theta'}$ excluded from our consideration so far are generated only by the mixed correlator (3.72) with the amplitude factor from (3.73). Therefore, in the regime (3.95) they all involve the additional small factor $\overline{g}_1^{1/2} \sim s^{\omega_1/2}$ compared to those studied above. This is also a violation of the universality of IR scaling in the magnetic regime.

3.8 Anisotropy and critical regimes in magnetic hydrodynamics

Let us briefly present the main results of [102] regarding the effect of an anisotropy of the type (3.51) on the possible critical regimes in magnetic hydrodynamics. The analysis was carried out following the general scheme presented in Sec. 3.4 with the approximation of linearity in the set of four anisotropy parameters ρ ($\rho_{1,2}$ in (3.51) for the correlator $D^{\varphi\varphi}$ and $\rho_{3,4}$ in the analogous expression for $D^{\Theta\Theta}$; the mixed correlator $D^{\varphi\Theta}$ was not introduced in [102]). As the needed counterterms, the anisotropy generates three additional contributions of the anisotropic viscosity type, which are introduced into the functional (3.52) (they correspond to the renormalized charges $u_{1,2,3}$), and three analogous contributions to the magnetic part of the action with charges $u_{4,5,6}$. The new feature compared to the analysis of Sec. 3.4 is the generation of the three additional vertices $\varphi'(n\partial)\Theta(n\Theta)$, $(n\varphi')(n\partial)(\Theta\Theta)$, and $(n\varphi')(\Theta\partial)(n\Theta)$. The corresponding charges are denoted as $g_{3,4,5}$ [in addition to $g_{1,2}$ in (3.79)]. Therefore, in the complete