1 Introduction

2 Description of the models. Field theoretic formulation

In the Langevin formulation the models are defined by stochastic differential equations for the order parameter $\psi = \psi(t, \mathbf{x})$:

$$\partial_t \psi_a = \lambda_0 \left\{ (-\tau_0 + \partial^2)\psi_a - V(\psi) \right\} + \zeta = 0, \tag{1}$$

where $\partial_t = \partial/\partial t$, ∂^2 is the Laplace operator, $\lambda_0 > 0$ is the kinematic (diffusion) coefficient and $\tau_0 \propto (T - T_c)$ is the deviation of the temperature (or its analog) from the critical value. The Gaussian random noise $\zeta = \zeta(t, \mathbf{x})$ with the zero mean is specified by the pair correlation function:

$$\langle \zeta(t, \mathbf{x}) \zeta(t', \mathbf{x}') \rangle = 2\lambda_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}') \tag{2}$$

d is the dimension of the $\mathbf{x}$ space. Here and below, the bare (unrenormalized) parameters are marked by the subscript "o". Their renormalized analogs (without the subscript) will appear later on. The nonlinearity has the form $V(\psi) = g_0 R_{abc} \psi_b(x) \psi_c(x)/2$; g_0 being the coupling constant. Summations over repeated indices are always implied. The tensor R_{abc} is expressed in the terms of the set of $n+1$ vectors e_α.

$$R_{abc} = \sum_\alpha e_a^\alpha e_b^\alpha e_c^\alpha, \tag{3}$$

where the e_a^α satisfy

$$\sum_{\alpha=1}^{n+1} e_a^\alpha = 0, \quad \sum_{\alpha=1}^{n+1} e_a^\alpha e_b^\alpha = (n+1)\delta_{ab}, \quad \sum_{\alpha=1}^{n} e_a^\alpha e_a^\beta = (n+1)\delta^{\alpha\beta} - 1. \tag{4}$$

Where n is dimension of hyperspace. Using (4) one can obtain that contraction of two and three tensors R have forms

$$R_{abc} R_{abd} = R_1 \delta_{cd}, \quad R_{abc} R_{cde} R_{efa} = R_2 R_{bdf}, \tag{5}$$

where

$$R_1 = (n+1)^2 (n-1), \quad R_2 = (n+1)^2 (n-2). \tag{6}$$

This stochastic problem (1), (2) can be reformulated as field theoretic model of the doubled set of fields $\Phi = \{\psi, \psi^\dagger\}$ with action functional

$$\mathcal{S}(\psi, \psi^\dagger) = \psi_a^\dagger \left(-\partial_t + \lambda_0 \partial^2 - \lambda_0 \tau_0 \right) \psi_a + \lambda_0 \psi_a^\dagger \psi_a^\dagger - g_0 R_{abc} \lambda \psi_a^\dagger \psi_b \psi_c / 2. \tag{7}$$

Here, $\psi^{\dagger} = \psi^{\dagger}(t, \mathbf{x})$ is the auxiliary "response field" and the integrations over the arguments of the fields are implied, for example

$$\psi^{\dagger}\partial_t\psi = \int dt \int d\mathbf{x}\psi^{\dagger}(t,\mathbf{x})\partial_t\psi(t,\mathbf{x}).$$

The field theoretic formulation means that the statistical averages of random quantities in the original stochastic problems can be represented as functional integrals over the full set of fields with weight $\exp\mathcal{S}(\Phi)$, and can therefore be viewed as the Green functions of the field theoretic models with actions (7). In particular, the linear response function of the problems (1), (2) is given by the Green function

$$G = \langle\psi^{\dagger}(t,\mathbf{x})\psi(t',\mathbf{x}')\rangle = \int \mathcal{D}\psi^{\dagger} \int \mathcal{D}\psi \;\; \psi^{\dagger}(t,\mathbf{x})\psi(t',\mathbf{x}') \, \exp\mathcal{S}(\psi,\psi^{\dagger}) \tag{8}$$

of the corresponding field theoretic model.

The model (7) corresponds to the standard Feynman diagrammatic technique with two bare propagators $\langle\psi\psi^{\dagger}\rangle_0$, $\langle\psi\psi\rangle_0$ and triple vertex $\sim \psi^{\dagger}\psi^2$. In the frequency-momentum representation the propagators have the forms

$$\langle\psi\psi^{\dagger}\rangle_0(\omega,k) = \frac{1}{-\mathrm{i}\omega + \lambda_0\,(k^2+\tau_0)},$$
$$\langle\psi\psi\rangle_0(\omega,k) = \frac{2\lambda_0}{\omega^2 + \lambda_0^2\,(k^2+\tau_0)^2} \tag{9}$$

The Galilean invariant coupling with the velocity field $\mathbf{v} = \{v_i(t,\mathbf{x})\}$ for the compressible fluid ($\partial_i v_i \neq 0$) can be introduced by the replacement

$$\partial_t\psi_b \to \partial_t\psi_b + a_0\,\partial_i(v_i\psi_b) + (a_0-1)(v_i\partial_i)\psi_b = \nabla_t\psi_b + a_0(\partial_i v_i)\psi_b \tag{10}$$

in (1). Here $\nabla_t \equiv \partial_t + v_i\partial_i$ is the Lagrangian derivative, a_0 is an arbitrary parameter and $\partial_i = \partial/\partial x_i$.

In the real problem, the field $\mathbf{v}(t,\mathbf{x})$ satisfies the Navier–Stokes equation. We will employ the rapid-change model [16], where the velocity obeys a Gaussian distribution with zero mean and the correlation function

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{x}')\rangle = \delta(t-t')\,D_{ij}(\mathbf{r}), \quad \mathbf{r} = \mathbf{x}-\mathbf{x}' \tag{11}$$

with

$$D_{ij}(\mathbf{r}) = \mathbf{D_0}\int_{\mathbf{k}>\mathbf{m}} \frac{\mathbf{dk}}{(\mathbf{2}\pi)^{\mathbf{d}}}\,\frac{\mathbf{1}}{\mathbf{k}^{\mathbf{d}+\xi}}\,\{\mathbf{P_{ij}}(\mathbf{k}) + \alpha\mathbf{Q_{ij}}(\mathbf{k})\}\,\exp(\mathrm{i}\mathbf{k}\mathbf{r}). \tag{12}$$

Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$ and $Q_{ij}(\mathbf{k}) = k_ik_j/k^2$ are the transverse and the longitudinal projectors, $k \equiv |\mathbf{k}|$ is the wave number, $D_0 > 0$ is an amplitude factor and

$\alpha > 0$ is an arbitrary parameter. The case $\alpha = 0$ corresponds to the incompressible fluid ($\partial_i v_i = 0$), while the limit $\alpha \to \infty$ at fixed αD_0 corresponds to the purely potential velocity field. The exponent $0 < \xi < 2$ is a free parameter which can be viewed as a kind of Hölder exponent, which measures "roughness" of the velocity field; the "Kolmogorov" value is $\xi = 4/3$, while the "Batchelor" limit $\xi \to 2$ corresponds to smooth velocity. The cutoff in the integral (12) from below at $k = m$, where $m \equiv 1/\mathcal{L}$ is the reciprocal of the integral turbulence scale $\mathcal{L}$, provides IR regularization. Its precise form is unimportant; the sharp cutoff is the simplest choice for the practical calculations.

The action functional for the full set of fields $\Phi = \left\{\psi, \psi^\dagger, \mathbf{v}\right\}$ become

$$\mathcal{S}(\Phi) = \psi_a^\dagger \left\{-\nabla_t + \lambda_0\left(\partial^2 - \tau_0\right) - a_0(\partial_i v_i)\right\}\psi_a + \\ +\lambda_0 \psi_a^\dagger \psi_a^\dagger - \frac{g_0\lambda_0}{2} R_{abc}\psi_a^\dagger \psi_b \psi_c + \mathcal{S}(\mathbf{v}) \tag{13}$$

This is obtained from (7), by the replacement (10) and adding the term corresponding to the Gaussian averaging over the field $\mathbf{v}$ with the correlator (11), (12):

$$\mathcal{S}(\mathbf{v}) = -\mathbf{\frac{1}{2}} \int \mathbf{dt} \int \mathbf{dx} \int \mathbf{dx' v_i(t,x) D_{ij}^{-1}(r) v_j(t,x')}, \tag{14}$$

where

$$D_{ij}^{-1}(\mathbf{r}) \propto \mathbf{D_0^{-1} r^{-2d-}}^{\xi}$$

is the kernel of the inverse linear operation for the function $D_{ij}(\mathbf{r})$ in (12).

In addition to (9), the Feynman diagrams for the model (14) involve the propagator $\langle vv\rangle_0$ specified by the relations (11), (12) and the new vertex

$$\psi^\dagger v_i V_i \psi \equiv -\psi^\dagger \left\{(v_i\partial_i)\psi + a_0(\partial_i v_i)\right\}\psi. \tag{15}$$

In the diagrams it corresponds to the vertex factor

$$V_i = -\mathrm{i}k_i - \mathrm{i}a_0 q_i, \tag{16}$$

where k_i is the momentum argument of the field ψ and q_i is the momentum of v_i.

Actual expansion parameter in the perturbation theory of the model is $u_0 = g_0^2$, so for the full model, the role of the coupling constants (expansion parameters in the perturbation theory) is played by the three parameters

$$u_0 = g_0^2 \sim \Lambda^{6-d}, \qquad w_0 = D_0/\lambda_0 \sim \Lambda^{\xi}, \qquad w_0 a_0 \sim \Lambda^{\xi}. \tag{17}$$

The last relations, following from the dimensionality considerations (more precisely, see the next section), define the typical UV momentum scale Λ. From the relations (17) it follows that the interactions $\psi^\dagger\psi_a\psi_a$ become logarithmic (the corresponding coupling constant u_0 becomes dimensionless) at $d=6$. Thus for the single-charge problems (7), the value $d=d_c=6$ is the upper critical dimension, and the deviation $\varepsilon = 4-d_c$ plays the part of the formal expansion parameter in the RG approach: the critical exponents are nontrivial for $\varepsilon>0$ and are calculated as series in ε.

The additional interactions $\sim \psi_a^\dagger v\partial\psi_a$ of the full model (14) become logarithmic at $\xi=0$. The parameter ξ is not related to the spatial dimension and can be varied independently. However, for the RG analysis of the full problems it is important that all the interactions become logarithmic at the same time. Otherwise, one of them would be weaker than the others from the RG viewpoint and it would be irrelevant in the leading-order IR behaviour. As a result, some of the scaling regimes of the full model would be overlooked. In order to study all possible scaling regimes and the crossovers between them, we need a genuine three-charge theory, in which all the interactions are treated on equal footing. Thus we will treat ε and ξ as small parameters of the same order, $\varepsilon\propto\xi$. Instead of the plain ε expansion in the single-charge models, the coordinates of the fixed points, critical dimensions and other quantities will be calculated as double expansions in the ε–ξ plane around the origin, that is, around the point in which all the coupling constants in (17) become dimensionless. Similar situation was encountered earlier in various models of turbulence and complex critical behaviour, e.g. [12, 13, 14, 15, 20].

3 Canonical dimensions, UV divergences and the renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions ("power counting"); see e.g. [1, 2]. Dynamical models of the type (7), in contrast to static ones, have two independent scales: the time scale T and the length scale L. Thus the canonical dimension of any quantity F (a field or a parameter) is characterized by two numbers, the frequency dimension d_F^ω and the momentum dimension d_F^k, defined such that $[F]\sim[T]^{-d_F^\omega}[L]^{-d_F^k}$. These dimensions are found from the obvious normalization conditions

$$d_k^k=-d_{\mathbf{x}}^k=1,\ d_k^\omega=d_{\mathbf{x}}^\omega=0,\ d_\omega^k=d_t^k=0,\ d_\omega^\omega=-d_t^\omega=1$$

and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F=d_F^k+2d_F^\omega$ (in the free theory, $\partial_t\propto\partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems; see Chap. 5 of [2]. The canonical dimensions for the models (14) are given in table 1, including renormalized parameters (without subscript "o"), which will be introduced soon.

Table 1: Canonical dimensions of the fields and parameters in the model (14).

F	ψ	$\psi^\dagger$	v	λ_0, λ	τ_0, τ	m, μ, Λ	g_0^2	ω_0	$g^2, \omega, \alpha, a_0, a$
d_F^k	$\frac{d-2}{2}$	$\frac{d+2}{2}$	-1	-2	2	1	$2-d$	ξ	0
d_F^ω	0	0	1	1	0	0	2	0	0
d_F	$\frac{d-2}{2}$	$\frac{d+2}{2}$	1	0	2	1	$6-d$	ξ	0

As already discussed in the end of the previous section, the full model is logarithmic (all the coupling constants are simultaneously dimensionless) at $d = 6$ and $\xi = 0$. Thus the UV divergences in the Green functions manifest themselves as poles in $\varepsilon = 6 - d$, ξ and, in general, their linear combinations.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_{1-\text{ir}}$ is given by the relation [2]

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi, \tag{18}$$

where $N_\Phi = \{N_\psi, N_{\psi^\dagger}, N_v\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ in logarithmic theory (that is, at $\varepsilon = \xi = 0$) is the formal index of the UV divergence $\delta_\Gamma = d_\Gamma|_{\varepsilon=\xi=0}$. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which δ_Γ is a non-negative integer.

From table 1 and (18) we find

$$\delta_\Gamma = 8 - 2N_\psi - 4N_{\psi^\dagger} - N_v. \tag{19}$$

In dynamical models, the 1-irreducible diagrams without the fields $\psi^\dagger$ vanish, and it is sufficient to consider the functions with $N_{\psi^\dagger} \geq 1$. In our model we have Galilean symmetry and symmetry of tensor $R_{acc} = 0$ from (3),(4). With these restrictions, the analysis of the expressions (19) shows that in model, superficial UV divergences can be present in the following 1-irreducible functions:

$$\langle \psi^\dagger \psi^\dagger \rangle \quad (\delta = 0) \quad \text{with the counterterms} \quad \psi^\dagger \psi^\dagger,$$

$$\langle \psi^\dagger \psi \rangle \quad (\delta = 2) \quad \text{with the counterterms} \quad \psi^\dagger \partial_t \psi,\ \psi^\dagger \partial^2 \psi,\ \psi^\dagger \psi,$$

$$\langle \psi^\dagger \psi \psi \rangle \quad (\delta = 0) \quad \text{with the counterterms} \quad \psi^\dagger \psi \psi,$$

$$\langle \psi^\dagger \psi v \rangle \quad (\delta = 1) \quad \text{with the counterterms} \quad \psi^\dagger (v\partial) \psi,\ \psi^\dagger (\partial v) \psi.$$

All the remaining terms are present in the corresponding action functional (14), so that our models are multiplicatively renormalizable. The Galilean symmetry also requires that the counterterms $\psi^\dagger\partial_t\psi$ and $\psi^\dagger(v\partial)\psi$ enter the renormalized action only in the form of the Lagrangian derivative $\psi^\dagger\nabla_t\psi$, imposing no restriction on the Galilean invariant term $\psi^\dagger(\partial v)\psi$.

We thus conclude that the renormalized actions can be written in the forms

$$\begin{aligned}\mathcal{S}^R(\Phi) = \psi_a^\dagger \left\{-Z_1\nabla_t + \lambda\left(Z_2\partial^2 - Z_3\tau\right) - aZ_6(\partial_i v_i)\right\}\psi_a + \\ +\lambda Z_5\psi_a^\dagger\psi_a^\dagger - gR_{abc}\mu^\varepsilon\lambda Z_4\psi_a^\dagger\psi_b\psi_c/2 + \mathcal{S}(\mathbf{v})\end{aligned} \tag{20}$$

for the model with $\mathcal{S}(\mathbf{v})$ from (14).

Here λ, τ, g, u and a are renormalized analogs of the bare parameters (with the subscripts "o") and μ is the reference mass scale (additional arbitrary parameter of the renormalized theory). The renormalization constants Z_i absorb the poles in ε and ξ and depend on the dimensionless parameters u, w, α and a. Expression (21) can be reproduced by the multiplicative renormalization of the fields $\psi \to \psi Z_\psi$, $\psi^\dagger \to \psi^\dagger Z_{\psi^\dagger}$ and the parameters:

$$\begin{aligned}g_0 = g\mu^{\varepsilon/2}Z_g, \quad u_0 = g\mu^\varepsilon Z_u, \quad w_0 = w\mu^\xi Z_w, \\ \lambda_0 = \lambda Z_\lambda, \quad \tau_0 = \tau Z_\tau, \quad a_0 = aZ_\tau.\end{aligned} \tag{21}$$

Since the last term $\mathcal{S}(\mathbf{v})$ given by (14) is not renormalized, the amplitude D_0 from (12) is expressed in renormalized parameters as $D_0 = w_0\lambda_0 = w\lambda\mu^\xi$, while the parameters m and α are not renormalized: $m_0 = m$, $\alpha_0 = \alpha$. Owing to the Galilean symmetry, the both terms in the covariant derivative ∇_t are renormalized with the same constant Z_1, so that the velocity field is not renormalized, either. Hence the relations

$$Z_w Z_\lambda = 1, \quad Z_m = Z_\alpha = Z_v = 1. \tag{22}$$

Comparison of the expressions (14) and (21) gives the following relations between the renormalization constants Z_1–Z_6 and (21):

$$\begin{aligned}Z_1 = Z_\psi Z_{\psi^\dagger}, \quad Z_2 = Z_1 Z_\lambda, \quad Z_3 = Z_2 Z_\tau, \\ Z_4 = Z_\lambda Z_{\psi^\dagger}^2, \quad Z_5 = Z_\lambda Z_g Z_\psi^3 Z_{\psi^\dagger}, \quad Z_6 = Z_1 Z_a\end{aligned} \tag{23}$$

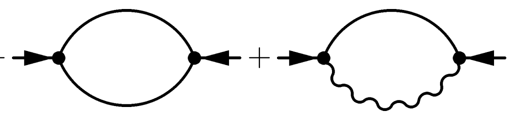
for the Potts model. Resolving these relations with respect to the renormalization constants of the fields and parameters gives

$$\begin{aligned}Z_\lambda = Z_1^{-1}Z_2, \quad Z_\tau = Z_2^{-1}Z_3, \quad Z_a = Z_1^{-1}Z_6, \quad Z_u = Z_1^{-1}Z_2^{-3}Z_4^2Z_5, \\ Z_\psi = Z_1^{1/2}Z_2^{1/2}Z_5^{-1/2}, \quad Z_\psi^\dagger = Z_1^{1/2}Z_2^{-1/2}Z_5^{1/2}\end{aligned} \tag{24}$$

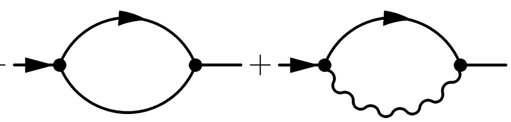
for the model, where we passed to the coupling constant $u = g^2$ with $Z_u = Z_g^2$.

The renormalization constants can be found from the requirement that the Green functions of the renormalized model (21), when expressed in renormalized variables, be UV finite (in our case, be finite at $\varepsilon \to 0$, $\xi \to 0$). The constants Z_1–Z_6 are calculated directly from the diagrams, then the constants in (21) are found from (25). In order to find the full set of constants, it is sufficient to consider the 1-irreducible Green functions which involve superficial divergences; In the one-loop approximation, they are given in this equations.

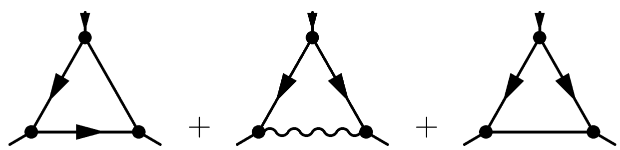
$$\langle\psi^\dagger\psi^\dagger\rangle_{1-ir} = 2\lambda Z_5 + \tag{25}$$



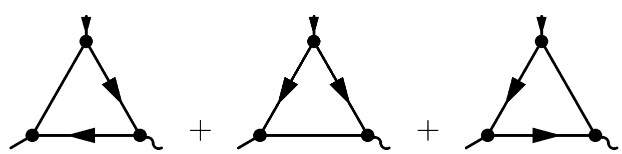
$$\langle\psi\psi^\dagger\rangle_{1-ir} = Z_1 i\omega - \lambda\left(Z_2 k^2 - Z_3\tau\right) + \tag{26}$$



$$\langle\psi\psi\psi^\dagger\rangle_{1-ir} = -g\lambda\mu^{\frac{\varepsilon}{2}} Z_4 + \tag{27}$$



$$\langle\psi\psi^\dagger v\rangle_{1-ir} = -a_0 k_i Z_6 + \tag{28}$$



The solid lines with arrows denote the propagator $\langle\psi\psi^\dagger\rangle_0$, the arrow points to the field $\psi^\dagger$. The solid lines without arrows correspond to the propagator $\langle\psi\psi\rangle_0$ and the wavy lines denote the velocity propagator $\langle vv\rangle_0$ specified in (11). The external ends with incoming arrows correspond to the fields $\psi^\dagger$, the ends without arrows correspond to ψ. The triple vertex with one wavy line corresponds to the vertex factor (16).

All the diagrammatic elements should be expressed in renormalized variables using the relations (21)–(24). In the one-loop approximation, the Z's in the bare terms should be taken in the first order in $u = g^2$ and w, while in the diagrams they should simply be replaced with unities, $Z_i \to 1$. Thus the passage to renormalized variables in the diagrams is achieved by the simple substitutions $\lambda_0 \to \lambda$, $\tau_0 \to \tau$, $g_0 \to g\mu^{\varepsilon/2}$ and $w_0 \to w\mu^\xi$.

In practical calculations, we used the minimal subtraction (MS) scheme, in which the renormalization constants have the forms $Z_i = 1+$ only singularities in ε and ξ, with the coefficients depending on the completely dimensionless renormalized parameters u, w, a and α. And results for z_i have forms:

$$Z_1 = 1 - \frac{uR_1}{2\varepsilon}, \quad Z_2 = 1 - \frac{uR_1}{3\varepsilon} - \frac{w}{6\xi}(5+\alpha), \quad Z_3 = 1 - \frac{2R_1 u}{\varepsilon},$$
$$Z_4 = 1 - \frac{2R_2 u}{\varepsilon} - \frac{w\alpha a^2}{\xi}, \quad Z_5 = 1 - \frac{uR_1}{2\varepsilon} - \frac{w}{\xi}\alpha(a-1)^2,$$
$$Z_6 = 1 - \frac{uR_1}{\varepsilon 2a}(4a-1). \tag{29}$$

To simplify the resulting expressions, we passed in (29) to the new parameters

$$u \to u/128\pi^3, \quad w \to w/64\pi^3;$$

here and below they are denoted by the same symbols u and w. The parameters R_1 and R_2 are related to the dimension n of the order parameter by the expression (6). Although we are especially interested in the cases $n = 0$ and $n = -1$, for completeness the coefficients R_1 and R_2 in what follows are assumed to be arbitrary.

These relations will play an important role in the analysis of the fixed points of the model (14) in the next sections.

4 RG functions and RG equations

Let us recall an elementary derivation of the RG equations; detailed exposition can be found in monographs [1, 2]. The RG equations are written for the renormalized Green functions $G^R = \langle \Phi \cdots \Phi \rangle_R$, which differ from the original (unrenormalized) ones $G = \langle \Phi \cdots \Phi \rangle$ only by normalization (due to rescaling of the fields) and choice of parameters, and therefore can equally be used for analyzing the critical behaviour. The relation $\mathcal{S}^R(\Phi, e, \mu) = \mathcal{S}(\Phi, e_0)$ between the bare (14) and renormalized (21) action functionals results in the relations

$$G(e_0, \dots) = Z_\psi^{N_\psi} Z_{\psi^\dagger}^{N_{\psi^\dagger}} G^R(e, \mu, \dots). \tag{30}$$

between the Green functions. Here, as usual, N_ψ and $N_{\psi^\dagger}$ are the numbers of corresponding fields entering into G (we recall that in our models $Z_v = 1$); $e_0 = \{\lambda_0, \tau_0, u_0, w_0, a_0, m_0, \alpha_0\}$ is the full set of bare parameters and $e = \{\lambda, \tau, u, w, a, m, \alpha\}$ are their renormalized counterparts (we recall that $\alpha_0 = \alpha$ and $m_0 = m$); the dots stand for the other arguments (times/frequencies and coordinates/momenta).

We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of the equation (30) with it. This gives the basic RG differential equation:

$$\left\{\mathcal{D}_{RG} + N_\psi\gamma_\psi + N_{\psi^\dagger}\gamma_{\psi^\dagger}\right\} G^R(e, \mu, \dots) = 0, \tag{31}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_u\partial_u + \beta_w\partial_w + \beta_a\partial_a - \gamma_\lambda\mathcal{D}_\lambda - \gamma_\tau\mathcal{D}_\tau. \tag{32}$$

Here we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, the anomalous dimensions γ are defined as

$$\gamma_F \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \tag{33}$$

and the β functions for the dimensionless couplings u, w and a are

$$\begin{aligned} \beta_u \equiv \widetilde{\mathcal{D}}_\mu u &= u\,(-\varepsilon - \gamma_u), \\ \beta_w \equiv \widetilde{\mathcal{D}}_\mu w &= w\,(-\xi - \gamma_w), \\ \beta_a \equiv \widetilde{\mathcal{D}}_\mu a &= -a\gamma_a, \end{aligned} \tag{34}$$

where the second equalities come from the definitions and the relations (21). The fourth β function

$$\beta_\alpha = \widetilde{\mathcal{D}}_\mu \alpha = -\alpha\gamma_\alpha \tag{35}$$

vanishes identically due to (41) and for this reason does not appear in the subsequent relations.

The anomalous dimension corresponding to a given renormalization constant Z_F is readily found from the relation

$$\gamma_F = (\beta_u\partial_u + \beta_w\partial_w + \beta_a\partial_a)\ln Z_F \simeq -\left(\varepsilon\mathcal{D}_u + \xi\mathcal{D}_w\right)\ln Z_F. \tag{36}$$

In the first relation, we used the definition (33), expression (32) for the operation $\widetilde{\mathcal{D}}_\mu$ in renormalized variables, and the fact that the Z's depend only on the completely dimensionless coupling constants u, w and a. In the second (approximate) relation, we only retained the leading-order terms in the β functions (34), which is sufficient for the first-order approximation. The leading-order expressions (29) for the renormalization constants have the form

$$Z_F = 1 + \frac{u}{\varepsilon}A_F(a, \alpha) + \frac{w}{\xi}B_F(a, \alpha). \tag{37}$$

Substituting (37) into (36) leads to the final UV finite expressions for the anomalous dimensions:

$$\gamma_F = -uA_F(a,\alpha) - wB_F(a,\alpha) \tag{38}$$

for any constant Z_F. This gives

$$\gamma_1 = R_1 u/2, \quad \gamma_2 = R_1 u/3 + w(5+\alpha)/6, \quad \gamma_3 = 2R_1 u,$$
$$\gamma_4 = 2R_2 u + w\alpha a^2, \quad \gamma_5 = uR_1/2 + w\alpha(a-1)^2, \quad \gamma_6 = uR_1(4a-1)/2a. \tag{39}$$

The multiplicative relations (24) between the renormalization constants result in the linear relations between the corresponding anomalous dimensions:

$$\gamma_\lambda = \gamma_2 - \gamma_1, \quad \gamma_\tau = \gamma_3 - \gamma_2,$$
$$\gamma_a = \gamma_6 - \gamma_1, \quad \gamma_u = -\gamma_1 - 3\gamma_2 + 2\gamma_4 + \gamma_5, \tag{40}$$
$$2\gamma_\psi = \gamma_1 + \gamma_2 - \gamma_5, \quad 2\gamma_\psi^\dagger = \gamma_1 - \gamma_2 + \gamma_5.$$

Along with (39), these relations give the final first-order explicit expressions for the anomalous dimensions of the fields and parameters. The exact relations (22) result in

$$\gamma_w = -\gamma_\lambda, \quad \gamma_m = \gamma_\alpha = \gamma_v = 0. \tag{41}$$

5 Attractors of the RG equations and scaling regimes for the Potts model

It is well known that possible asymptotic regimes of a renormalizable field theoretic model is determined by the asymptotic behaviour of the system of ordinary differential equations for the so-called invariant (running) coupling constants

$$\mathcal{D}_s \bar{g}_i(s,g) = \beta_i(\bar{g}), \quad \bar{g}_i(1,g) = g_i, \tag{42}$$

where $s = k/\mu$, k is the momentum, $g = \{g_i\}$ is the full set of coupling constants and $\bar{g}_i(s,g)$ are the corresponding invariant variables. As a rule, the IR ($s \to 0$) and UV ($s \to \infty$) behaviour of such system is determined by fixed points g_{i*}. The coordinates of possible fixed points are found from the requirement that all the β functions vanish:

$$\beta_i(g_*) = 0, \tag{43}$$

while the type of a given fixed point is determined by the matrix

$$\Omega_{ij} = \partial\beta_i/\partial g_j|_{g=g^*} : \tag{44}$$

for an IR attractive fixed points (which we are interested in here) the matrix Ω is positive, that is, the real parts of all its eigenvalues are positive. In our models, the fixed points for the full set of couplings u, w, a, α should be determined by the equations

$$\beta_{u,w,a,\alpha}(u_*, w_*, a_*, \alpha_*) = 0, \tag{45}$$

with the β functions defined in the preceding section. However, in our models the attractors of the system (42) involve, in general, two-dimensional surfaces in the full four-dimensional space of couplings. First, the function (35) vanishes identically, so that the equation $\beta_\alpha = 0$ gives no restriction on the parameter α. It is then convenient to consider the attractors of the system (42) in the three-dimensional space u, w, a; their coordinates, matrix (44) and the critical exponents will, in general, depend on the free parameter α. What is more, in this reduced space the attractors will be not only fixed points, but also lines of fixed points, which can be conveniently parametrized by the coupling a. Although the general pattern of the attractors appears rather similar for the both models, it is instructive to discuss them separately.

The one-loop expressions for the β functions in the model (21) are easily derived from the definitions (34), relations (41) and (41), and explicit expressions (39):

$$\begin{aligned} \beta_u &= u\left[-\varepsilon + Ru + w(5+\alpha)/2 - w\alpha f(a)\right], \\ \beta_w &= w\left[-\xi + R_1 u/6 + w(5+\alpha)/6\right], \\ \beta_a &= uR_1(1-3a)/2, \end{aligned} \tag{46}$$

where the function $f(a) = 2a^2 + (a-1)^2$ achieves the minimum value $f(1/3) = 2/3$ at $a = 1/3$. And $R = R_1 - 4R_2$. For the set (46) the equations (45) have the following four solutions:

(1) The line of Gaussian (free) fixed points: $u_* = w_* = 0$, a_* arbitrary.

(2) The point $w_* = 0$, $u_* = \varepsilon/R$, $a_* = 1/3$, corresponding to the pure Potts model (turbulent advection is irrelevant).

(3) The line of fixed points

$$u_* = 0, \quad w_* = 6\xi/(5+\alpha), \quad a_* \text{ arbitrary}, \tag{47}$$

corresponding to the passively advected scalar without self-interaction.

(4) The most nontrivial fixed point, corresponding to the new regime (universality class), both the advection and the self-interaction are relevant:

$$u_* = \frac{[\varepsilon(5+\alpha) - \xi(15-\alpha)]}{6\Delta}, \quad w_* = \frac{[-\varepsilon R_1/6 + R\xi]}{\Delta}, \quad a_* = 1/3. \tag{48}$$

Where $\Delta = [5(R + R_1/2) + \alpha(R - R_1/6)]/6$ We recall that α is treated as a free parameter, which the coordinates of the fixed points can depend on.

Admissible fixed point must be IR attractive and satisfy the conditions $u_* > 0$, $w_* > 0$, which follow from the physical meaning of these parameters.

For the point (1) we have

$$\Omega_u = -\varepsilon, \quad \Omega_w = -\xi, \quad \Omega_a = 0,$$

so it is admissible for $\varepsilon < 0$, $\xi < 0$ (region I in fig.[1-8]). Vanishing of the element Ω_a reflects the fact that the parameter a_* for the point I is arbitrary, or, in other words, the point is degenerate. The point (2) can be physical only if $R > 0, R_1 < 0$ (and any α) and is IR stable if

$$\varepsilon > 0, \quad \xi < -R_1\varepsilon/6R$$

(region II in fig.[1-8]).

The physical region of stability passive scalar point (region III in fig.[1-8]) can be described by this equations with restriction on α:

$$\begin{gathered} -(5+\alpha)\varepsilon + (15-\alpha)\xi > 0\xi > 0 \\ (a-1/3)^2 < \frac{-(5+\alpha)\varepsilon + (15-\alpha)\xi}{18+\alpha\xi}, \end{gathered} \tag{49}$$

Thus the both inequalities (21), (22) are satised in a sector in the upperleft quadrant; the lower bound is (21) and the upper bound is (22). From (23) it follows that, when changes from 0 to , the coecient D(C A)/A(D B) changes from R1 /6R to (R + R1 /6)/2R. The coecient D/B = (5 +)/(15) changes from R1 /6R to 1. Thus for = 0 the domain has zero width, and when grows it is getting wider.

The point (4), depending on the R, R_1, have three cases:

IV-1. $R > -R_1/2$, $R_1 < 0$

The point physical and IR attractive when

$$\xi > -\frac{R_1\varepsilon}{6R} \quad \xi < \frac{5+\alpha}{15-\alpha}\varepsilon \quad \alpha > 0. \tag{50}$$

So for small $\alpha < 15$, the physical region is the sectior in the upper-right quadrant in the ε–ξ plane, bounded by the ray $\xi = -R_1\varepsilon/6R$ from below and $\xi < (5+\alpha)/(15-\alpha)$ from above. When α grows, the upper ray $\xi < (5+\alpha)/(15-\alpha)$ rotates counter clockwise and moves to the upper-left quadrant. For case IV-1 α changes from 0 to ∞ and the ray changes from $\xi = \varepsilon/3$ to $\xi = -\varepsilon$ exactly like the boundary of the point III (fig.[1]).

IV-2. $-R_1/2 > R > 0$

The point describes by the same expressions (50), but α changes from α_0 to ∞,

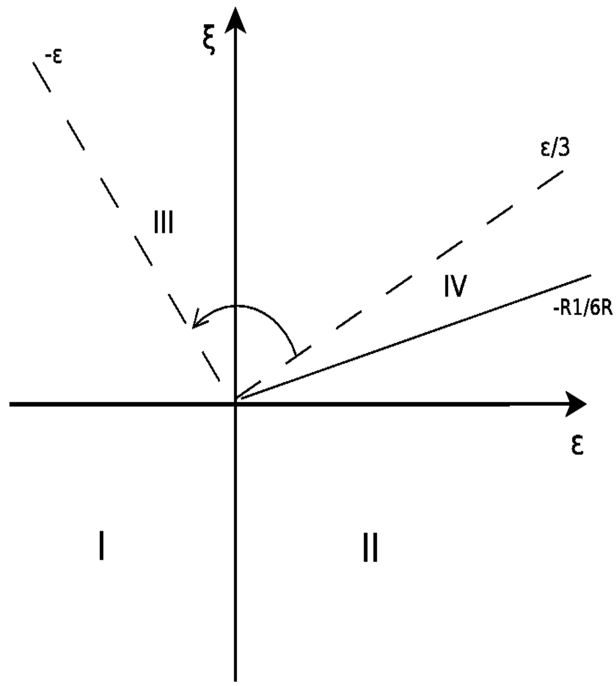


Figure 1: case:1

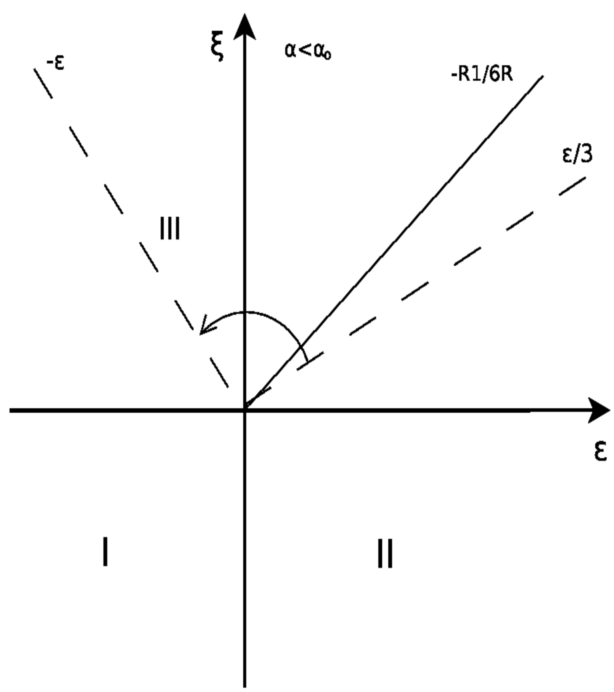


Figure 2: case:2

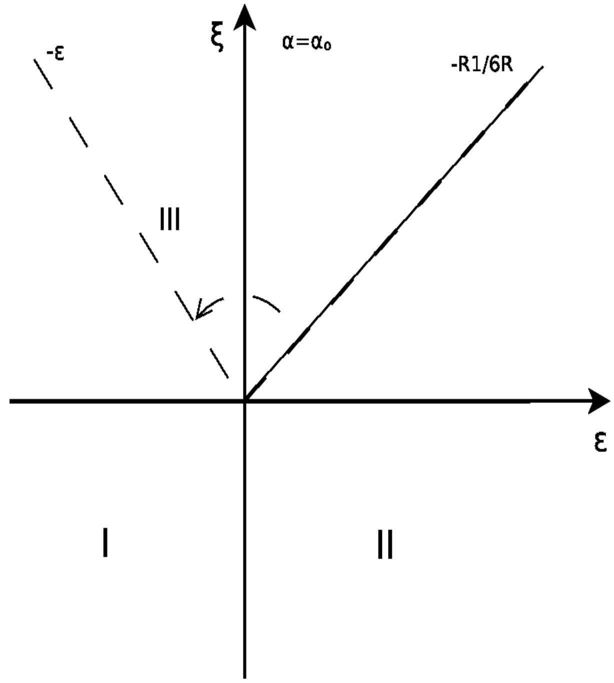


Figure 3: case:2

where $\alpha_0 = -5(R + R_1/2)/(R - R_1/6) > 0$. In this case $-R1/6R > 1/3$, and when $\alpha = 0$, regions II and III are crossed (fig.[2]).

When α grows boundaries of point II rotates counter clockwise and when $\alpha = \alpha_0$ boundaries of points II and III are coincides (fig.[3]).

For $\alpha > \alpha_0$ regions II and III aren't crossed and between them we have IV regime like in the case 1 (fig.[4]).

IV-3. $0 > R > R_1/6$

For this parameters we have no II regime. In this case regime of stability IV point describes by this inequalities:

$$\xi < \varepsilon(5 + \alpha)/(15 - \alpha), \tag{51}$$

$$\xi < \varepsilon\frac{(5 + \alpha)(R + R_1/6)}{(5 - \alpha)2R}, \tag{52}$$

$$\alpha > \alpha_0. \tag{53}$$

But now $\alpha_0 < 0$ so the both inequalities (51), (52) are satisfied in a sector in the upperleft quadrant; the lower bound is (51) and the upper bound is (52). When α changes from α to ∞, the coefficient in (51) changes from $R1/6R$ to $(R+R1/6)/2R$. The coefficient $(5 + \alpha)/(15 - \alpha)$ changes from $R1/6R$ to 1. Thus for $\alpha = \alpha_0$ the domain has zero width, and when α grows it is getting wider (fig.[5,6,7]).

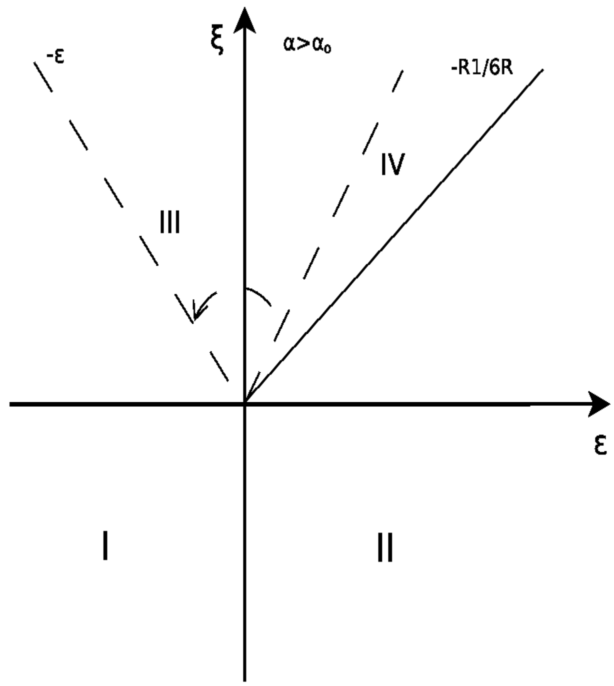


Figure 4: case:2

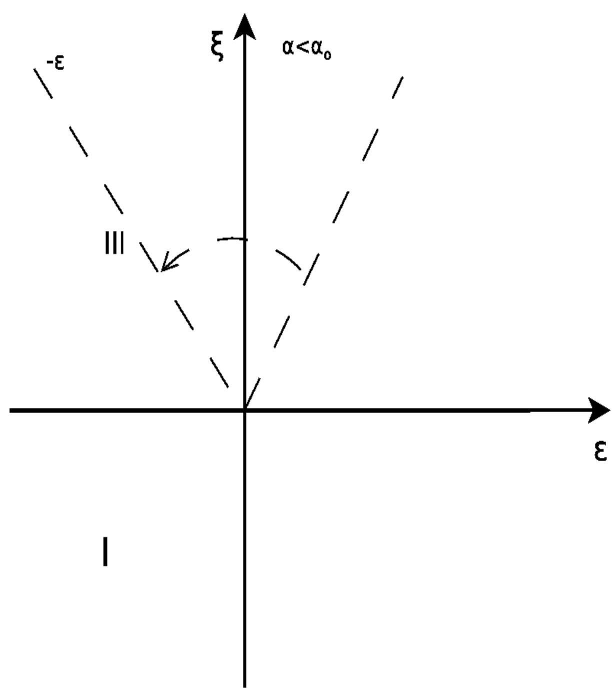


Figure 5: case3

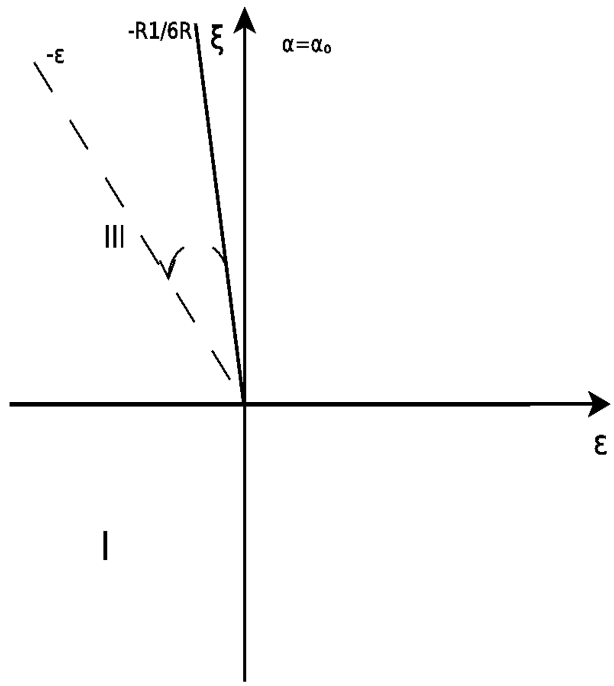


Figure 6: case:3

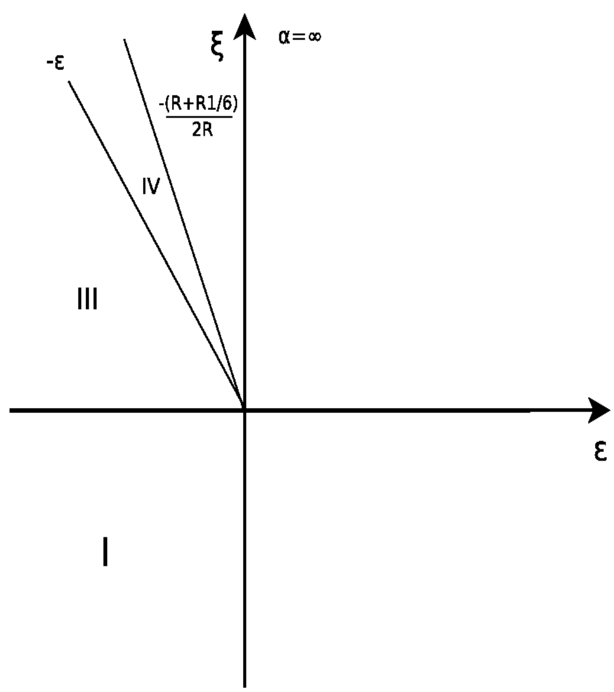


Figure 7: case:3

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