

# Effects of turbulent mixing on critical behaviour: Renormalization group analysis of the Potts model

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**Abstract.** Critical behaviour of a system, subjected to strongly anisotropic turbulent mixing, is studied by means of the field theoretic renormalization group. Specifically, relaxational stochastic dynamics of a non-conserved multicomponent order parameter of the Ashkin–Teller–Potts model, coupled to a random velocity field with prescribed statistics, is considered. The velocity is taken Gaussian, white in time, with correlation function of the form  $\propto \delta(t - t')/|\mathbf{k}_\perp|^{d-1+\xi}$ , where  $\mathbf{k}_\perp$  is the component of the wave vector, perpendicular to the distinguished direction (“direction of the flow”) — the  $d$ -dimensional generalization of the ensemble introduced by Avellaneda and Majda [1990 *Commun. Math. Phys.* **131** 381] within the context of passive scalar advection. This model can describe a rich class of physical situations. It is shown that, depending on the values of parameters that define self-interaction of the order parameter and the relation between the exponent  $\xi$  and the space dimension  $d$ , the system exhibits various types of large-scale scaling behaviour, associated with different infrared attractive fixed points of the renormalization-group equations. In addition to known asymptotic regimes (critical dynamics of the Potts model and passively advected field without self-interaction), existence of a new, non-equilibrium and strongly anisotropic, type of critical behaviour (universality class) is established, and the corresponding critical dimensions are calculated to the leading order of the double expansion in  $\xi$  and  $\varepsilon = 6 - d$  (one-loop approximation). The scaling appears strongly anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are essentially different.

PACS numbers: 05.10.Cc, 05.70.Jk, 64.60.ae, 64.60.Ht, 47.27.ef

## 1. Introduction

Numerous systems of very different physical nature reveal interesting singular behaviour in the vicinity of their critical points. Their correlation functions exhibit self-similar (scaling) behaviour with universal critical dimensions: they depend only on few global characteristics of the system (like symmetry or space dimension). Consistent qualitative and quantitative description of the critical behaviour is provided by the field theoretic renormalization group (RG). In the RG approach, possible types of critical regimes (universality classes) are associated with infrared (IR) attractive fixed points of renormalizable field theoretic models.

Most typical equilibrium phase transitions belong to the universality class of the  $O_n$ -symmetric  $\varphi^4$  model of an  $n$ -component scalar order parameter  $\varphi$ . Universal characteristics of the critical behaviour depend only on  $n$  and the space dimension  $d$  and can be calculated within systematic perturbation schemes, in particular, in the form of expansion in  $\varepsilon = 4 - d$ , the deviation of the space dimension from its upper critical value  $d = 4$ ; see the monographs [1, 2] and the literature cited therein.

Another important example is provided by the Ashkin–Teller–Potts (ATP) class of models [3]–[9]. In the continuous formulation, they are described by the effective Hamiltonian for the  $n$ -component order parameter with a trilinear interaction term, invariant under the hypertetrahedron symmetry group [5]–[9]. Such models have numerous physical applications: magnetic materials and solids with nontrivial symmetry, Edwards–Anderson spin-glass models within the replica formalism [10], and so on. In general, the ATP models describe systems which locally have  $n$  states, but the energy of any given configuration depends on whether pairs of neighboring sites are in the same state or not [3]. The case  $n = 2$  corresponds to nematic-to-isotropic transitions in the liquid crystals [11], while the formal limits  $n = 0$  and  $n = -1$  correspond to the percolation problem and the random resistor network, respectively [12]–[14]. Recently, models with trilinear interaction have attracted a new amount of interest due to their interesting formal properties [15, 16] and applications to the dynamics of first-order phase transitions [17]. The application of the cubic model to the Yang–Lee edge singularity has long been known [18].

The problem of the nature of the phase transition in the ATP model has a long and rather entangled history; see *e.g.* [4]–[8] and references therein. According to Landau’s phenomenological theory, existence of a trilinear term excludes the possibility of the second-order transition. On the contrary, exact two-dimensional results, numerical simulations and RG analysis suggest that for small  $n$ , the phase transition in the ATP model is of the second order, while for  $n$  large enough ( $n > 3$  in two dimensions [4] and  $n > 7/3$  in the vicinity of the upper critical dimension  $d = 6$  [7, 8]) the transition becomes a first-order one. In this paper, we accept the point of view that the existence of an IR attractive fixed point of the RG equations implies the existence of a self-similar (scaling) asymptotic regime and thus the existence of a kind of critical state.

It is well known that dynamical critical behaviour (critical singularities of relaxation

and correlation times, various kinetic and transport coefficients *etc.*) appears much richer, less universal and is comparatively less understood. Different nature of the order parameter (conserved or non-conserved), inclusion of additional slow modes (densities of entropy or energy) and interaction with hydrodynamical degrees of freedom produce different types of critical dynamics for the same static model [2, 19].

The behaviour of a real system near its critical point is extremely sensitive to external disturbances, gravity, geometry of the experimental setup, presence of impurities and so on; see the monograph [20] for the general discussion and the references. “Ideal” equilibrium critical behaviour of an infinite system can be obscured by limited accuracy of measuring the temperature, finite-size effects, finite time of evolution (ageing) and so on. In the presence of a distinguished direction, scaling behaviour of such systems can become strongly anisotropic, with different critical dimensions corresponding to different spatial directions. What is more, some disturbances (randomly distributed impurities in magnets and turbulent mixing of fluid systems) can change the type of the phase transition (first-order to second order one, and vice versa) and produce completely new types of critical behaviour (universality classes) with rich and rather exotic properties.

Investigation of the effects of various kinds of deterministic or chaotic flows (laminar shear flows, turbulent convection and so on) on the behaviour of the critical fluids (like liquid crystals or binary mixtures near their consolution points) has shown that the flow can destroy the usual critical behaviour: it can change to the mean-field behaviour or, under some conditions, to a more complex behaviour described by new non-equilibrium universality classes [21]–[32].

In this paper we study effects of turbulent mixing on the dynamical critical behaviour of the systems, described by the generalized ATP model, paying special attention to anisotropy of the flow. Bearing in mind application to liquid crystals or percolation in liquid media, we consider a purely relaxational stochastic dynamics of a non-conserved order parameter of the ATP model, coupled to a random velocity field with prescribed Gaussian statistics.

Recently, the models involving passive scalar fields advected by such synthetic velocity ensembles attracted a great deal of attention because of the insight they offer into the origin of intermittency and anomalous scaling in the real fluid turbulence; see the review paper [33] and references therein. The RG approach to the problem of passive advection is reviewed in [34]. In spite of their relative simplicity, such models reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. In the context of our study, it is especially important that they allow to easily model anisotropy of the flow, which in more realistic models would be introduced by the initial and/or boundary conditions. More specifically, we employ the  $d$ -dimensional generalization of a strongly anisotropic ensemble introduced in [35] and further discussed in a number of papers, *e.g.* [36]–[38], in connection with the passive scalar problem: the velocity field is oriented along a chosen direction  $\mathbf{n}$  and its correlation function depends only on the coordinates perpendicular to  $\mathbf{n}$ .

Although simplified, the model appears rather nontrivial and captures the main property of the problem: existence of a new, non-equilibrium and strongly anisotropic, universality class of scaling behaviour.

The plan of the paper is as follows. In section 2 we present the detailed description of the model and its field theoretic formulation. In section 3 we analyze the ultraviolet (UV) divergences, relaying upon the power counting and additional symmetry considerations. We show that the model, after proper extension, appears multiplicatively renormalizable. Thus we can derive the RG equations and introduce the RG functions ( $\beta$  functions and anomalous dimensions  $\gamma$ ) in the standard manner; see section 4.

In section 5 we show that, depending on the relation between the spatial dimension  $d$  and the exponent  $\xi$  in the velocity correlator, the model reveals four different types of critical behaviour, associated with four fixed points of the corresponding RG equations. Three fixed points correspond to known regimes: Gaussian or free field theory, non-interacting scalar field passively advected by the flow (the ATP nonlinearity in the original dynamical equations appears irrelevant), and the original critical behaviour of the model without mixing. The most interesting fourth point corresponds to a new full-scale non-equilibrium universality class, in which both the nonlinearity and turbulent mixing are relevant.

The corresponding critical dimensions can be calculated as double expansions in two parameters:  $\xi$  and  $\varepsilon = 4 - d$ . The scaling behaviour appears strongly anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are essentially different. The practical calculation of the renormalization constants, RG functions, regions of stability and critical dimensions was performed in the leading order (one-loop approximation); some of the results, however, are exact (valid to all orders of the double  $\varepsilon$ - $\xi$  expansion). These issues are discussed in section 6, while section 7 is reserved for conclusion.

## 2. Description of the model and the field theoretic formulation

Relaxational dynamics of a non-conserved  $n$ -component order parameter  $\varphi_a(x)$  with  $x \equiv \{t, \mathbf{x}\}$  is described by a stochastic differential equation

$$\partial_t \varphi_a(x) = -\lambda_0 \frac{\delta \mathcal{H}(\varphi)}{\delta \varphi_a(x)} + \eta_a(x), \quad (2.1)$$

where  $\partial_t = \partial/\partial t$ ,  $\lambda_0$  is the (constant) kinetic coefficient, and  $\eta_a(x)$  is a Gaussian random noise with zero mean and the pair correlation function

$$\begin{aligned} \langle \eta_a(x) \eta_b(x') \rangle &= \delta_{ab} D_\eta(x - x'), \\ D_\eta(x - x') &\equiv 2\lambda_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \end{aligned} \quad (2.2)$$

$d$  being the dimension of the  $\mathbf{x}$  space. Near the critical point, the Hamiltonian  $\mathcal{H}(\varphi)$  of the ATP model is taken in the form [5, 6, 7]

$$\mathcal{H}(\varphi) = \int d\mathbf{x} \left\{ -\frac{1}{2} \varphi_a(\mathbf{x}) \partial^2 \varphi_a(\mathbf{x}) + \frac{\tau_0}{2} \varphi_a(\mathbf{x}) \varphi_a(\mathbf{x}) + \frac{g_0}{3!} R_{abc} \varphi_a(\mathbf{x}) \varphi_b(\mathbf{x}) \varphi_c(\mathbf{x}) \right\}, \quad (2.3)$$

where  $\partial_i = \partial/\partial x_i$  is the spatial derivative,  $\partial^2 = \partial_i \partial_i$  is the Laplacian,  $\tau_0 \propto (T - T_c)$  measures deviation of the temperature (or its analog) from the critical value and  $g_0$  is the coupling constant. Summations over repeated indices are always implied ( $a, b, c = 1, \dots, n$  and  $i = 1, \dots, d$ ); after the functional differentiation in (2.1) one has to replace  $\varphi(\mathbf{x}) \rightarrow \varphi(x)$ .

Following [9], we consider the generalized case of certain symmetry group  $\mathcal{G}$ , which has the only irreducible invariant third-rank tensor  $R_{abc}$ ; with no loss of generality it is assumed to be symmetric. In the original ATP model  $\mathcal{G}$  is the symmetry group of the hypertetrahedron in  $n$  dimensions. Then the tensor  $R_{abc}$  is conveniently expressed in terms of the set of  $n+1$  vectors  $e^\alpha$  which define its vertices [5, 6]:

$$R_{abc} = \sum_{\alpha} e_a^\alpha e_b^\alpha e_c^\alpha,$$

where the  $e_a^\alpha$  satisfy

$$\sum_{\alpha=1}^{n+1} e_a^\alpha = 0, \quad \sum_{\alpha=1}^{n+1} e_a^\alpha e_b^\alpha = (n+1) \delta_{ab}, \quad \sum_{a=1}^n e_a^\alpha e_a^\beta = (n+1) \delta^{\alpha\beta} - 1. \quad (2.4)$$

Using equations (2.4) all the contractions with the tensor  $R_{abc}$  can be calculated. For example, the following contractions of two and three tensors have the form

$$R_{abc} R_{abe} = R_1 \delta_{ce}, \quad R_{aec} R_{chb} R_{bfa} = R_2 R_{ehf}, \quad (2.5)$$

where

$$R_1 = (n+1)^2(n-1), \quad R_2 = (n+1)^2(n-2). \quad (2.6)$$

Coupling with the velocity field  $\mathbf{v} = \{v_i(x)\}$  is introduced by the replacement

$$\partial_t \rightarrow \nabla_t = \partial_t + v_i \partial_i, \quad (2.7)$$

where  $\nabla_t$  is the Lagrangian (Galilean covariant) derivative. For incompressible fluid, the velocity field  $\mathbf{v}$  is transverse due to the continuity relation:  $\partial_i v_i = 0$ . The velocity ensemble is defined as follows [35]. Let  $\mathbf{n}$  be a unit constant vector that determines some distinguished direction ('direction of the flow'). Then any vector can be decomposed into the components perpendicular and parallel to the flow, for example,  $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n} x_\parallel$  with  $\mathbf{x}_\perp \cdot \mathbf{n} = 0$ . The velocity field will be taken in the form

$$\mathbf{v} = \mathbf{n} v(t, \mathbf{x}_\perp),$$

where  $v(t, \mathbf{x}_\perp)$  is a scalar function independent of  $x_\parallel$ . Then the incompressibility condition is automatically satisfied:

$$\partial_i v_i = \partial_\parallel v(t, \mathbf{x}_\perp) = 0.$$

For  $v(t, \mathbf{x}_\perp)$  we assume a Gaussian distribution with zero mean and the pair correlation function of the form:

$$\begin{aligned} \langle v(t, \mathbf{x}_\perp) v(t', \mathbf{x}'_\perp) \rangle &= \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} \exp \{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')\} D_v(k) = \\ &= \delta(t - t') \int \frac{d\mathbf{k}_\perp}{(2\pi)^{d-1}} \exp \{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)\} \tilde{D}_v(k_\perp), \quad k_\perp = |\mathbf{k}_\perp| \end{aligned} \quad (2.8)$$

with the scalar coefficient functions

$$D_v(k) = 2\pi\delta(k_\parallel) \tilde{D}_v(k_\perp), \quad \tilde{D}_v(k_\perp) = D_0 k_\perp^{-d+1-\xi}. \quad (2.9)$$

Here and below  $d$  is the dimension of the  $\mathbf{x}$  space,  $D_0 > 0$  is a constant amplitude factor and  $\xi$  an arbitrary exponent. The IR regularization in (2.8) is provided by the cutoff  $k_\perp > m$  (by dimension,  $\tau_0 \propto m^2$ ). Precise form of the IR regularization is inessential; sharp cutoff is the most convenient choice from the calculational viewpoints. The natural interval for the exponent is  $0 < \xi < 2$ , when the so-called effective eddy diffusivity has a finite limit for  $m \rightarrow 0$ ; it includes the most realistic Kolmogorov value  $\xi = 4/3$ .

In order to ensure multiplicative renormalizability of the model, it is necessary to split the Laplacian in (2.1) into the parallel and perpendicular parts  $\partial^2 \rightarrow \partial_\perp^2 + f_0 \partial_\parallel^2$  by introducing a new parameter  $f_0 > 0$ . Here  $\partial_\perp^2$  is the Laplacian in the subspace orthogonal to the vector  $\mathbf{n}$  and  $\partial_\parallel = \partial/\partial x_\parallel$ . In the anisotropic case, these two terms will be renormalized in a different way; more detailed discussion of this point can be found in [31, 32]. Thus equation (2.1) becomes

$$\nabla_t \varphi_a = \lambda_0 (\partial_\perp^2 + f_0 \partial_\parallel^2 - \tau_0) \varphi_a - \frac{g_0 \lambda_0}{2} R_{abc} \varphi_b \varphi_c + \eta_a; \quad (2.10)$$

this completes formulation of the model.

According to the general rule (see *e.g.* chap. 5 of the monograph [2]), our stochastic problem is equivalent to the field theoretic model of the extended set of fields  $\Phi = \{\varphi'_a, \varphi_a, \mathbf{v}\}$  with the action functional

$$\begin{aligned} \mathcal{S}(\Phi) &= \frac{\varphi'_a D_\eta \varphi'_a}{2} + \varphi'_a \{ -\nabla_t + \lambda_0 (\partial_\perp^2 + f_0 \partial_\parallel^2) - \lambda_0 \tau_0 \} \varphi_a - \\ &- \frac{\lambda_0 g_0 f_0^{1/4}}{2} R_{abc} \varphi'_a \varphi_b \varphi_c + \mathcal{S}_v(\mathbf{v}), \end{aligned} \quad (2.11)$$

where we segregated the factor  $f_0^{1/4}$  from the charge  $g_0$ . The first few terms represent the De Dominicis–Janssen action functional for the stochastic problem (2.1), (2.2) at fixed  $\mathbf{v}$ ; it involves the auxiliary scalar response field  $\varphi'_a(x)$ . All the required integrations over  $x = \{t, \mathbf{x}\}$  and summations over the vector indices are implied, for example,

$$\varphi'_a \partial_\perp^2 \varphi_a = \sum_{a=1}^n \int dt \int d\mathbf{x} \varphi'_a(x) \partial_\perp^2 \varphi_a(x).$$

The last term in (2.11) corresponds to the Gaussian averaging over  $\mathbf{v}$  with the correlator (2.8) and has the form

$$\mathcal{S}_v(\mathbf{v}) = -\frac{1}{2} \int dt \int d\mathbf{x}_\perp d\mathbf{x}'_\perp v(t, \mathbf{x}_\perp) \tilde{D}_v^{-1}(\mathbf{x}_\perp - \mathbf{x}'_\perp) v(t, \mathbf{x}'_\perp),$$

**Table 1.** Canonical dimensions of the fields and parameters in the model (2.11)

$F$	$\varphi'_a$	$\varphi_a$	$\mathbf{v}$	$\lambda_0, \lambda$	$f_0, f$	$\tau_0, \tau$	$\mu$	$g_0$	$w_0$
$d_F^\omega$	0	0	1	1	0	0	0	0	0
$d_F^\parallel$	1/2	1/2	-1	0	-2	0	0	0	0
$d_F^\perp$	$(d+1)/2$	$(d-3)/2$	0	-2	2	2	1	$(6-d)/2$	$\xi$

where

$$\tilde{D}_v^{-1}(\mathbf{r}_\perp) \propto D_0^{-1} r_\perp^{2(1-d)-\xi}$$

is the kernel of the inverse linear operation  $D_v^{-1}$  for the correlation function  $D_v$  in (2.9).

This formulation means that statistical averages of random quantities in the original stochastic problem coincide with the Green functions of the field theoretic model with the action (2.11), given by functional averages with the weight  $\exp \mathcal{S}(\Phi)$ . This allows one to apply the standard Feynman diagrammatic technique, the field theoretic renormalization theory and renormalization group to our stochastic problem.

### 3. Canonical dimensions, UV divergences and renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions; see *e.g.* [1, 2]. In general, dynamic models have two scales: canonical dimension of some quantity  $F$  (a field or a parameter in the action functional) is completely characterized by two numbers, the frequency dimension  $d_F^\omega$  and the momentum dimension  $d_F^k$ ; see *e.g.* chap. 5 in [2]. They are determined such that  $[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}$ , where  $L$  is some length scale and  $T$  is the time scale.

Our strongly anisotropic model, however, has two independent momentum scales, related to the directions perpendicular and parallel to the vector  $\mathbf{n}$ , and requires a more detailed specification of the canonical dimensions. Namely, one has to introduce two independent momentum canonical dimensions  $d_F^\perp$  and  $d_F^\parallel$  so that

$$[F] \sim [T]^{-d_F^\omega} [L_\perp]^{-d_F^\perp} [L_\parallel]^{-d_F^\parallel},$$

where  $L_\perp$  and  $L_\parallel$  are (independent) length scales in the corresponding subspaces. The dimensions are found from the obvious normalization conditions  $d_{k_\perp}^\perp = -d_{\mathbf{x}_\perp}^\perp = 1$ ,  $d_{k_\perp}^\parallel = -d_{\mathbf{x}_\perp}^\parallel = 0$ ,  $d_{k_\parallel}^\omega = d_{k_\parallel}^\omega = 0$ ,  $d_\omega^\omega = -d_t^\omega = 1$ , and so on, and from the requirement that each term of the action functional (2.11) be dimensionless (with respect to all the three independent dimensions separately).

The canonical dimensions of the model (2.11) are given in table 1, including renormalized parameters, which will be introduced a bit later. From table 1 it follows that the model is logarithmic (the coupling constants  $g_0$  and  $w_0$  are simultaneously dimensionless) at  $d = 6$  and  $\xi = 0$ , so that the UV divergences in the correlation functions manifest themselves as poles in  $\varepsilon \equiv 6 - d$ ,  $\xi$  and their linear combinations.

The total canonical dimension can be found from the relation  $d_F = d_F^\perp + d_F^\parallel + 2d_F^\omega$  (in the free theory,  $\partial_t \propto \partial_\perp^2 \propto \partial_\parallel^2$ ). In the renormalization theory it plays the same part as

the conventional (momentum) dimension does in one-scale static problems: superficial UV divergences, whose removal requires counterterms, can be present only in those 1-irreducible Green functions, for which the total canonical dimension at the logarithmic values  $\varepsilon = \xi = 0$  (formal index of divergence) is a non-negative integer.

The careful analysis of the table 1, augmented by symmetry considerations, shows that all the counterterms needed to cancel the UV divergences in our model are present in the action (2.11). Here, important role is played by the Galilean symmetry and the invariance with respect to the group  $\mathcal{G}$ . For example, the function  $\langle \varphi' \varphi v v \rangle$  can be omitted from consideration because the corresponding counterterm  $\varphi'_a \varphi_a v^2$  is forbidden by the Galilean invariance. A similar situation occurs for the functions  $\langle \varphi' v \rangle$  and  $\langle \varphi' v v \rangle$ . For the first function, a counterterm necessarily contains a spatial derivative, whose vector index is contracted with the velocity field; this gives zero in view of the transversality condition (2.8). For the second function, the possible counterterms  $\varphi'_a (\partial_i v_k) (\partial_k v_i)$  and  $\varphi'_a (\partial_i v_k) (\partial_i v_k)$  are forbidden by the symmetry with respect to  $\mathcal{G}$ . Furthermore, the Galilean symmetry requires that the counterterms  $\varphi'_a \partial_t \varphi_a$  and  $\varphi'_a (v_i \partial_i) \varphi_a$  enter the renormalized action only in the form of the Lagrangian derivative  $\varphi'_a \nabla_t \varphi_a$ . In turn, owing to the symmetry with respect to  $\mathcal{G}$ , the trilinear term in (2.11) is renormalized as a single entity.

Thus our model appears multiplicatively renormalizable with the renormalized action of the form

$$\begin{aligned} \mathcal{S}_R(\Phi) = & Z_1 \frac{\varphi'_a D_\eta \varphi'_a}{2} + \varphi'_a \{ -Z_2 \nabla_t + \lambda (Z_3 \partial_\perp^2 + Z_4 f \partial_\parallel^2) - Z_5 \lambda \tau \} \varphi_a - \\ & - Z_6 \frac{\lambda g \mu^{\varepsilon/2} f^{1/4}}{2} R_{abc} \varphi'_a \varphi_b \varphi_c + \mathcal{S}_v(\mathbf{v}). \end{aligned} \quad (3.1)$$

Here  $\lambda$ ,  $\tau$ ,  $f$ ,  $g$  and  $w$  are renormalized analogs of the bare parameters (with the subscripts “0”) and  $\mu$  is the reference mass scale (additional arbitrary parameter of the renormalized theory). The renormalization constants capture all the divergences at  $\varepsilon$ ,  $\xi = 0$  so that the correlation functions of the renormalized model (3.1) have finite limits for  $\varepsilon$ ,  $\xi \rightarrow 0$ , when expressed in renormalized parameters  $\lambda$ ,  $\mu$  and so on.

Simple analysis shows that the Feynman diagrams needed for the calculation of the renormalization constants in (3.1) are the same as in the one-component  $\varphi^3$  theory (model 2 from [32]), multiplied with the appropriate tensor contractions. Moreover, in the one-loop approximation only contractions (2.5) appear. Therefore, one can easily generalize the results of [32] to the case at hand with the aid of (2.5).

In [32] the minimal subtraction (MS) renormalization scheme was employed. In the MS scheme the renormalization constants have the forms  $Z_i = 1 +$  only singularities in  $\varepsilon$  and  $\xi$ , with the coefficients depending on the two completely dimensionless parameters — renormalized coupling constants  $g$  and  $w$ . The one-loop results for the renormalization constants in (3.1) are the following

$$\begin{aligned} Z_1 = Z_2 = 1 - \frac{u R_1}{2\varepsilon}, \quad Z_3 = 1 - \frac{u R_1}{3\varepsilon}, \quad Z_4 = 1 - \frac{u R_1}{3\varepsilon} - \frac{w}{\xi}, \\ Z_5 = 1 - \frac{2u R_1}{\varepsilon}, \quad Z_6 = 1 - \frac{2u R_2}{\varepsilon}. \end{aligned} \quad (3.2)$$



Here we have passed to more convenient coupling constants  $u \rightarrow g^2/128\pi^3$  and  $w \rightarrow w/24\pi^3$ .

The parameters  $R_1$  and  $R_2$  are related to the dimension  $n$  of the order parameter by the expression (2.6). Although we are especially interested in the cases  $n = 0$  and  $2$ , for completeness the coefficients  $R_1$  and  $R_2$  in what follows are assumed to be arbitrary.

Expression (3.1) is equivalent to the multiplicative renormalization of the fields  $\varphi_a \rightarrow \varphi_a Z_\varphi$ ,  $\varphi'_a \rightarrow \varphi'_a Z_{\varphi'}$  and the parameters:

$$\begin{aligned} \lambda_0 &= \lambda Z_\lambda, \quad \tau_0 = \tau Z_\tau, \quad f_0 = f Z_f, \\ g_0 &= g \mu^{\varepsilon/2} Z_g \quad (u_0 = u \mu^\varepsilon Z_u), \quad w_0 = w \mu^\xi Z_w \end{aligned} \quad (3.3)$$

(no renormalization of the velocity field is needed:  $Z_v = 1$ ). The constants in Eqs. (3.1) and (3.3) are related as follows:

$$\begin{aligned} Z_1 &= Z_\lambda Z_{\varphi'}^2, \quad Z_2 = Z_{\varphi'} Z_\varphi, \quad Z_3 = Z_{\varphi'} Z_\lambda Z_\varphi, \\ Z_4 &= Z_{\varphi'} Z_\lambda Z_f Z_\varphi, \quad Z_5 = Z_{\varphi'} Z_\lambda Z_\tau Z_\varphi, \quad Z_6 = Z_\lambda Z_u^{1/2} Z_f^{1/4} Z_{\varphi'} Z_\varphi^2. \end{aligned} \quad (3.4)$$

Since the last term  $\mathcal{S}_v(\mathbf{v})$  is not renormalized, the amplitude  $D_0$  is expressed in renormalized parameters as

$$D_0 = w_0 f_0 \lambda_0 = w f \mu^\xi \lambda,$$

which leads to the relation

$$Z_w Z_f Z_\lambda = 1 \quad (3.5)$$

between the renormalization constants.

#### 4. RG equations and RG functions

Let us recall an elementary derivation of the RG equations; more detailed discussion can be found in monographs [1, 2]. The RG equations are written for the renormalized correlation functions  $G_R = \langle \Phi \cdots \Phi \rangle_R$ , which differ from the original (unrenormalized) ones  $G = \langle \Phi \cdots \Phi \rangle$  only by normalization and choice of parameters, and therefore can equally be used for analyzing the critical behaviour. The relation  $\mathcal{S}_R(Z_\Phi \Phi, e, \mu) = \mathcal{S}(\Phi, e_0)$  between the functionals (2.11) and (3.1) results in the relations

$$G(e_0, \dots) = Z_{\varphi'}^{N_{\varphi'}} Z_\varphi^{N_\varphi} G_R(e, \mu, \dots) \quad (4.1)$$

between the correlation functions. Here,  $N_{\varphi'}$  and  $N_\varphi$  are the full numbers of corresponding fields entering into  $\Gamma$  (we recall that in our model  $Z_v = 1$ );  $e_0 = \{\lambda_0, \tau_0, f_0, u_0, w_0\}$  is the full set of bare parameters and  $e = \{\lambda, \tau, f, u, w\}$  are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta *etc.*).

We use  $\tilde{\mathcal{D}}_\mu$  to denote the differential operation  $\mu \partial_\mu$  for fixed  $e_0$  and operate on both sides of the equation (4.1) with it. This gives the basic RG differential equation:

$$\{\mathcal{D}_{RG} + N_\varphi \gamma_\varphi + N_{\varphi'} \gamma_{\varphi'}\} G^R(e, \mu, \dots) = 0, \quad (4.2)$$

where  $\mathcal{D}_{RG}$  is the operation  $\tilde{\mathcal{D}}_\mu$  expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_u \partial_u + \beta_w \partial_w - \gamma_f \mathcal{D}_f - \gamma_\lambda \mathcal{D}_\lambda - \gamma_\tau \mathcal{D}_\tau. \quad (4.3)$$

Here we have written  $\mathcal{D}_x \equiv x \partial_x$  for any variable  $x$ , the anomalous dimensions  $\gamma$  are defined as

$$\gamma_F \equiv \tilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \quad (4.4)$$

and the  $\beta$  functions for the two dimensionless couplings  $u$  and  $w$  are

$$\beta_u \equiv \tilde{\mathcal{D}}_\mu u = u [-\varepsilon - \gamma_u], \quad \beta_w \equiv \tilde{\mathcal{D}}_\mu w = w [-\xi - \gamma_w], \quad (4.5)$$

where the second equalities come from the definitions and the relations (3.3).

Equations (3.4) result in the following relations between the anomalous dimensions

$$\begin{aligned} \gamma_1 &= \gamma_\lambda + 2\gamma_{\varphi'}, & \gamma_2 &= \gamma_{\varphi'} + \gamma_\varphi, & \gamma_3 &= \gamma_{\varphi'} + \gamma_\lambda + \gamma_\varphi, \\ \gamma_4 &= \gamma_f + \gamma_3, & \gamma_5 &= \gamma_\tau + \gamma_3, & \gamma_6 &= \gamma_\lambda + \gamma_u/2 + \gamma_f/4 + \gamma_{\varphi'} + 2\gamma_\varphi, \end{aligned} \quad (4.6)$$

while from (3.5) one obtains

$$\gamma_f + \gamma_w + \gamma_\lambda = 0. \quad (4.7)$$

The dimensions  $\gamma_1$ – $\gamma_6$  are calculated from the corresponding renormalization constants using the definition (4.4):

$$\begin{aligned} \gamma_1 &= \gamma_2 = \frac{uR_1}{2}, & \gamma_3 &= \frac{uR_1}{3}, & \gamma_4 &= \frac{uR_1}{3} + w, \\ \gamma_5 &= 2uR_1, & \gamma_6 &= 2uR_2, \end{aligned} \quad (4.8)$$

with the corrections of the order  $u^2$ ,  $w^2$ ,  $uw$  and higher.

The RG functions entering the equation (4.3) are easily found from the relations (3.2), (4.6) and (4.7):

$$\begin{aligned} \gamma_{\varphi'} &= \gamma_2 - \gamma_3/2 = uR_1/3, & \gamma_\varphi &= \gamma_3/2 = \frac{uR_1}{6}, \\ \gamma_f &= \gamma_4 - \gamma_3 = w, & \gamma_\tau &= \gamma_5 - \gamma_3 = 5uR_1/3, \\ \gamma_\lambda &= \gamma_3 - \gamma_2 = -uR_1/6, & \gamma_w &= \gamma_2 - \gamma_4 = uR_1/6 - w, \\ \gamma_u &= 2\gamma_6 - 5\gamma_3/2 - \gamma_4/2 = (4R_2 - R_1)u - w/2, \end{aligned} \quad (4.9)$$

with the higher order corrections.

From the definitions (4.5), relations (4.9) and explicit expressions (4.8) for the anomalous dimensions we derive the following leading-order expressions for the  $\beta$  functions of our model:

$$\beta_u = u [-\varepsilon + Ru + w/2], \quad \beta_w = w [-\xi - uR_1/6 + w], \quad (4.10)$$

where we have introduced a new convenient parameter  $R = R_1 - 4R_2$ .

## 5. Fixed points and scaling regimes

It is well known that possible large-scale scaling regimes of a renormalizable model are associated with IR attractive fixed points of the corresponding RG equations. In our model, the coordinates  $u_*$ ,  $w_*$  of the fixed points are found from the equations

$$\beta_u(u_*, w_*) = 0, \quad \beta_w(u_*, w_*) = 0, \quad (5.1)$$

with the  $\beta$  functions given in (4.5). The type of a fixed point is determined by the matrix

$$\Omega = \{\Omega_{ij} = \partial\beta_i/\partial u_j\}, \quad (5.2)$$

where  $\beta_i$  denotes the full set of the  $\beta$  functions and  $u_j = \{u, w\}$  is the full set of couplings. For IR stable fixed points the matrix  $\Omega$  is positive, *i.e.*, the real parts of all its eigenvalues are positive. This condition defines the regions of IR stability for the corresponding scaling regimes.

The couplings  $u$  and  $w$  should be non-negative (by definition,  $u \propto g^2 \geq 0$  and  $w \propto D_0/f\lambda \geq 0$ ), and in the following we will be interested only in the “good” (admissible from the physics viewpoints) fixed points, which satisfy the conditions

$$u_* \geq 0, \quad w_* \geq 0 \quad (5.3)$$

and can be IR attractive for some values of the model parameters.

In order to give the complete picture of possible scaling regimes, it is instructive to discuss at first a more general case, specified by the  $\beta$  functions of the form

$$\beta_u = u[-\varepsilon + au + bw], \quad \beta_w = w[-\xi + cu + dw], \quad (5.4)$$

with arbitrary real coefficients  $a$ – $d$ .

From equations (5.1) and (5.4) we can identify four different fixed points. For the first three points the matrix  $\Omega$  appears to be triangular, so that its eigenvalues (and hence the regions of IR stability of the corresponding scaling regimes) are simply determined by the diagonal elements:  $\Omega_u = \partial\beta_u/\partial u > 0$  and  $\Omega_w = \partial\beta_w/\partial w > 0$ .

1. Gaussian (free) fixed point:  $u_* = w_* = 0$ ;  $\Omega_u = -\varepsilon$ ,  $\Omega_w = -\xi$ .

2.  $u_* = 0$  (exact result to all orders),  $w_* = \xi/d$ ;  $\Omega_u = -\varepsilon + b\xi/d$ ,  $\Omega_w = \xi$ . This point can be “good” only if  $d > 0$ ; otherwise the conditions  $\Omega_w > 0$  and  $w_* > 0$  cannot be simultaneously satisfied.

3.  $w_* = 0$  (exact result to all orders),  $u_* = \varepsilon/a$ ;  $\Omega_u = \varepsilon$ ,  $\Omega_w = -\xi + c\varepsilon/a$ . Similarly to the case 2, this point can be “good” only if  $a > 0$ .

The last, fully nontrivial, fixed point requires more detailed discussion.

4. The coordinates of this point are

$$u_* = (d\varepsilon - b\xi)/\Delta, \quad w_* = (a\xi - c\varepsilon)/\Delta, \quad \Delta = ad - bc, \quad (5.5)$$

while the matrix  $\Omega$  can be written in the form

$$\Omega = \begin{pmatrix} au_* & bu_* \\ cw_* & dw_* \end{pmatrix}. \quad (5.6)$$

(it is useful not to substitute explicit expressions (5.5) into (5.6) for a while).

The necessary and sufficient condition for the IR stability of this point can be restated as the fulfillment of two inequalities:

$$\det \Omega > 0, \quad \text{tr} \Omega > 0. \quad (5.7)$$

From (5.6) one obtains

$$\det \Omega = u_* w_* \Delta > 0, \quad (5.8)$$

which along with (5.3) shows that this point can be “good” only if  $\Delta > 0$ .

For the trace of  $\Omega$  we obtain:

$$\text{tr} \Omega = au_* + dw_* > 0. \quad (5.9)$$

There are three possibilities:

1)  $a > 0, d > 0$ . In this case the inequality (5.9) is an automatic consequence of (5.3). Four regions of stability of the fixed points 1–4 divide the  $\varepsilon$ – $\xi$  plane without “gaps” or overlaps. This is the most typical situation, realized for the  $\varphi^4$  model or the Gribov process in various kinds of random flows [30]–[32].

2)  $a < 0, d < 0$ . Then (5.9) contradicts to (5.3) and this point can never be “good.” Such a situation was not yet encountered.

3) The parameters  $a$  and  $d$  are opposite in sign:  $ad < 0$ . For definiteness, we assume that  $a < 0, d > 0$ . We will see in short that this situation can realize for the ATP model. In this case one obtains from (5.9):

$$w_* > -au_*/d > 0, \quad (5.10)$$

where the last inequality follows from  $a < 0, u_* > 0$ . The second inequality in (5.3) is implied by the (5.10) and thus becomes superfluous. The region where the fixed point is IR attractive and positive is given by the two inequalities

$$u_* > 0, \quad au_* + dw_* > 0. \quad (5.11)$$

Summing up, we conclude that the scaling regime corresponding to the fixed point 4 exists if  $\Delta > 0$  and at least one of the two parameters  $a$  and  $d$  is positive. The region where the fourth fixed point is “good” is determined by the inequalities (5.3) if  $a$  and  $d$  are simultaneously positive, and by the conditions of the type (5.11) if  $a$  and  $d$  are opposite in sign.

Let us turn to our specific model with the  $\beta$  functions (4.10). Identifying them with (5.4) gives

$$a = R_1 - 4R_2 \equiv R, \quad b = \frac{1}{2}, \quad c = -\frac{R_1}{6}, \quad d = 1; \\ \Delta = R + R_1/12 \equiv \Delta^R. \quad (5.12)$$

The coordinates of the four possible fixed points are obtained by substituting the expressions (5.12) into the general results.

In the scaling regime corresponding to the fixed point 2, the nonlinearity  $\varphi^2$  in the stochastic equation (2.1) becomes irrelevant in the sense of Wilson due to the exact relation  $u_* = 0$ . Thus we arrive at the linear advection-diffusion equation for a passive scalar field  $\varphi$ . In turn, the effects of the velocity field become irrelevant in the third regime (fixed point 3) due to the exact relation  $w_* = 0$ . The isotropy violated by the velocity ensemble is restored and the leading terms of the IR behaviour coincide with those of the equilibrium dynamic model ATP. Finally, the fixed point 4 corresponds to a new nontrivial IR scaling regime, in which the both nonlinearities in the stochastic equation for  $\varphi$  are important; the corresponding critical dimensions reveal strong anisotropy, depend essentially on the both RG expansion parameters  $\varepsilon$  and  $\xi$ , and are calculated as double series in those parameters; see section 6.

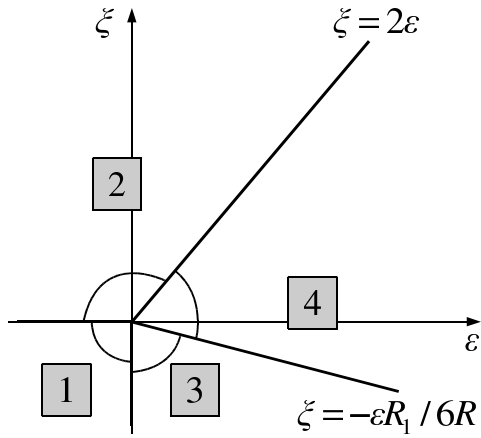
The regions of IR stability for all possible fixed points in the  $\varepsilon$ - $\xi$  plane for different values of the parameters  $R_1$  and  $R_2$  are shown on figures 1–5. In the one-loop approximation, all the boundaries of the regions are given by straight lines.

The figures 3 and 5 show that there are overlaps between the IR stability regions of fixed points 2 and 3 and points 1 and 4, respectively. This means that for the values of  $\varepsilon$  and  $\xi$ , corresponding to the region of an overlap, the system has two variants of the IR scaling behaviour. Which one of them is realized depends on the initial data for the parameters  $u$  and  $w$  in the RG equations.

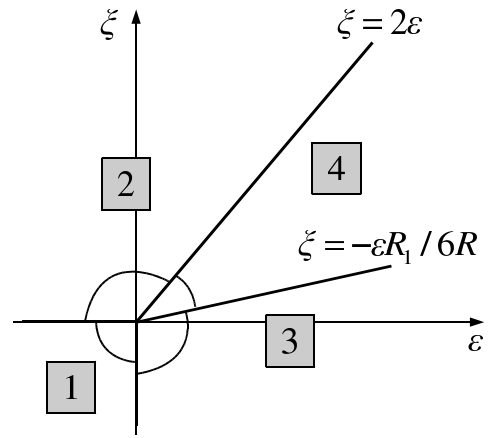
On the other hand, from figure 4 one can see that if the parameters  $R_1$  and  $R_2$  are such that  $\Delta_R < 0$  and  $R < 0$ , there is a gap. The system does not exhibit scaling behaviour for corresponding values of  $\varepsilon$  and  $\xi$ , which can be interpreted as existence of a first-order phase transition.

It is interesting to note that for  $a < 0$ , the original static model has no “good” fixed point (which is usually interpreted as a first-order transition), but a “good” point of the type (5.5) can appear in the full dynamic model with two couplings, as illustrated by figure 5. One can say that the phase transition changes its type and becomes a second-order one owing to the turbulent mixing.

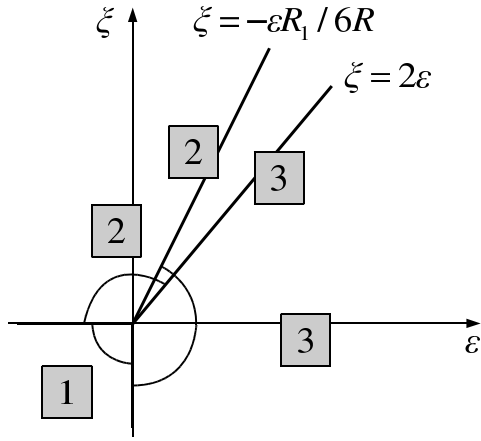
Let us briefly discuss the pattern of the RG flows in the plane of couplings  $u$ – $w$  for two special cases. Consider first the simplest (and the most typical) situation, illustrated by figure 2. The schematic picture of the fixed points and RG flows is sketched on figure 6 for the case, when the values of the parameters  $\varepsilon$  and  $\xi$  lie in the region 4 on figure 2. Then all the fixed points lie in the physical domain  $u \geq 0$ ,  $w \geq 0$ : the point 4 is IR attractive, point 1 is repulsive and points 2 and 3 are saddle points. If the parameters  $\varepsilon$  and  $\xi$  are changing such that the corresponding point in the



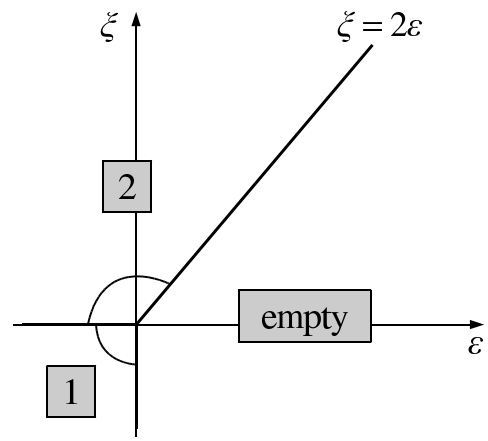
**Figure 1.**  $\Delta^R > 0$ ,  $R > 0$ ,  $R_1 > 0$ .



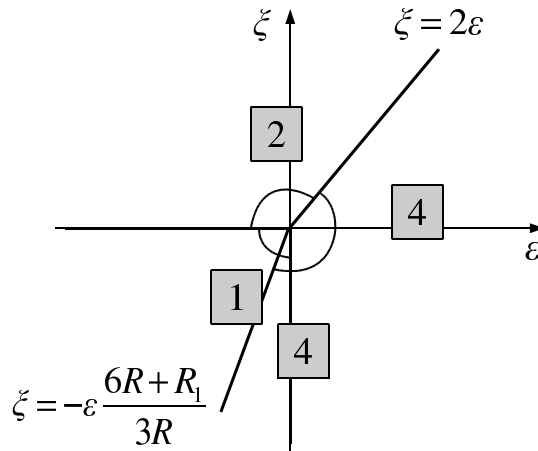
**Figure 2.**  $\Delta^R > 0$ ,  $R > 0$ ,  $R_1 < 0$ .



**Figure 3.**  $\Delta^R < 0$ ,  $R > 0$ ,  $R_1 < 0$ .

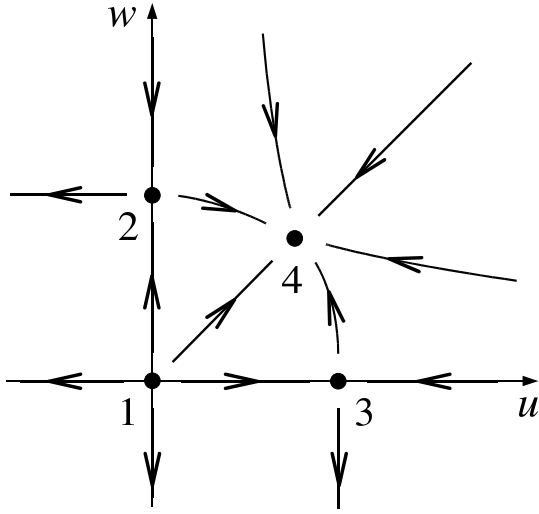


**Figure 4.**  $\Delta^R < 0$ ,  $R < 0$ ,  $R_1$  arbitrary.

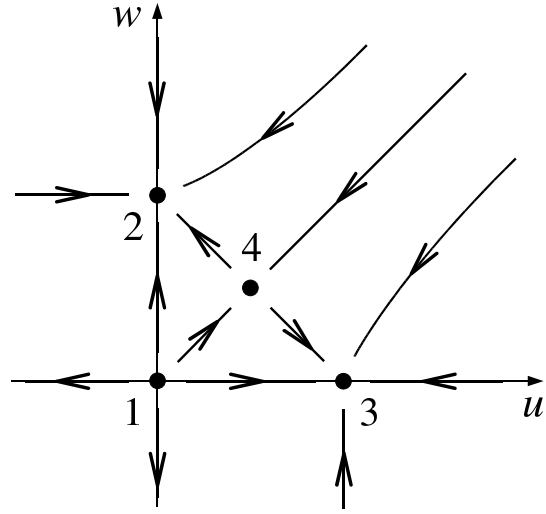


**Figure 5.**  $\Delta^R > 0$ ,  $R < 0$ ,  $R_1 > 0$ .

Regions of stability of the fixed points in the model (2.11).



**Figure 6.** RG flows for the case shown in figure 2.



**Figure 7.** RG flows for the case shown in figure 3.

$\varepsilon$ - $\xi$  plane crosses the boundary  $\xi = 2\varepsilon$  and moves into the region 2, the fixed point 4 on figure 6 crosses the ray  $u = 0$ ,  $w \geq 0$ , going through the point 2, and moves into the unphysical domain  $u < 0$ . The point 2 becomes IR attractive. If the point in the figure 2 crosses the boundary  $\xi = -\varepsilon R_1/6R$ , moving from region 4 to 3, the fixed point 4 crosses the ray  $u \geq 0$ ,  $w = 0$ , going through the point 3, and moves into the unphysical domain  $w < 0$ ; the point 3 becomes IR attractive. Although this picture is based on the explicit one-loop expressions for the  $\beta$  functions and regions of stability, it appears robust with respect to the higher-order corrections. It is crucial here that the functions  $\beta_u$  at  $w = 0$  and  $\beta_w$  at  $u = 0$  coincide with the  $\beta$  functions of the corresponding single-charge problems: the ATP model and the passive scalar case, respectively. The latter are supposed to have a unique nontrivial fixed point each: for the passive scalar, the  $\beta$  function is known exactly, why for the ATP model (with  $R > 0$ ) this is true at least within the  $\varepsilon$  expansion. Thus we may conclude that the coincidence of the fixed points 2 and 4 or 3 and 4 takes place simultaneously with the changeover in their type of stability. In turn, this means that, although the boundaries between the regions 2 and 4 or 3 and 4 in the  $\varepsilon$ - $\xi$  plane can become curved beyond the one-loop approximation due to the higher-order corrections to the  $\beta$  functions in (5.4), no gaps nor overlaps can appear between those regions. This is equally true for the boundaries between the regions 1 and 2 or 1 and 3 in figure 2, but there the boundaries are not affected by the higher-order corrections due to the simple exact expressions for the eigenvalues  $\Omega_{u,w}$  at the Gaussian fixed point.

The RG flows for another interesting situation, illustrated by figure 3, are depicted on figure 7 for the case when the values of the parameters  $\varepsilon$  and  $\xi$  lie in the overlap of the regions 2 and 3. Then the both fixed points 2 and 3 are “good” and the asymptotic behaviour of a flow depends on the initial data for  $u$  and  $w$ . The point 4 is unphysical:

although it lies in the domain  $u \geq 0$ ,  $w \geq 0$ , inspection of the explicit expressions (5.5) and (5.8) shows that it is a saddle point for this case. It is also worth noting that, for all possible situations, the RG flow with the initial data in the physical domain  $u \geq 0$ ,  $w \geq 0$  can never leave it, because the function  $\beta_u$  vanishes for  $u = 0$  and arbitrary  $w$ , while  $\beta_w$  vanishes for  $w = 0$  and arbitrary  $u$ .

Let us conclude this section with a brief discussion of the most interesting physical cases of the original ATP model. Then from (2.6) and (5.12) we obtain  $R = (n+1)^2(7-3n)$  and  $\Delta^R = (n+1)^2(83-35n)/12$ .

The case  $n = 2$  corresponds to the nematic-to-isotropic transition in a liquid crystal. Then

$$R = R_1 = 9, \quad R_2 = 0, \quad \Delta^R = 39/4, \quad (5.13)$$

and the case represented by figure 1 is realized. One can see that the interval of the most realistic values of the model parameters,  $d = 3$  and  $0 < \xi < 2$ , belongs completely to the region of stability of the most nontrivial fixed point 4.

The second case of interest is the critical behaviour of bond percolation, obtained in the limit  $n = 0$ . Strictly speaking, more realistic description of *dynamical* percolation is given by a special model with a nonlocal in-time interaction [39], but it is interesting to look at the dynamics of the ATP model with  $n = 0$  as some kind of approximation. For  $n = 0$  relations (2.6) give

$$R_1 = -1, \quad R_2 = -2, \quad R = 7, \quad \Delta^R = 83/12. \quad (5.14)$$

Thus, we arrive at the case shown on figure 2. Again, the most realistic values of the model parameters ( $d = 3$  and  $\xi = 4/3$ ) belong to the stability region of the new anisotropic scaling regime, corresponding to the fixed point 4.

For  $n \geq 3$  the case shown on figure 4 is realized. Now the values  $d = 3$  and  $\xi = 4/3$  lie in the “empty” region where none of the fixed points are “good.” This fact is usually interpreted as a first-order transition; the turbulent mixing does not change its type. The interesting situation illustrated by figure 5, where the mixing gives rise to the changeover in the type of the phase transition to the second-order one, is realized for the interval  $7/3 < n < 83/35$ , which does not contain integer values and has no documented physical interpretation.

## 6. Critical scaling and critical dimensions

We recall the definition of generalized homogeneity. Let  $F$  be a function of  $n$  independent arguments  $\{x_1, \dots, x_n\}$  that satisfies the dimensional relation

$$F(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^{\alpha_F} F(x_1, \dots, x_n) \quad (6.1)$$

with a certain set of constant coefficients (scaling dimensions)  $\{\alpha_1, \dots, \alpha_n, \alpha_F\}$  and an arbitrary positive parameter  $\lambda > 0$ . Differentiating relation (6.1) with respect to  $\lambda$  and then setting  $\lambda = 1$ , we obtain a first-order differential equation with constant coefficients

$$\sum_{i=1}^n \alpha_i \mathcal{D}_i F(x_1, \dots, x_n) = \alpha_F F(x_1, \dots, x_n), \quad \mathcal{D}_i = x_i \partial / \partial x_i. \quad (6.2)$$



Its general solution has the form

$$F(x_1, x_2, \dots, x_n) = x_1^{\alpha_F/\alpha_1} \tilde{F}\left(\frac{x_2}{x_1^{\alpha_2/\alpha_1}}, \dots, \frac{x_n}{x_1^{\alpha_n/\alpha_1}}\right),$$

where  $\tilde{F}$  is an arbitrary function of  $(n-1)$  arguments. Obviously, the dimensions are defined up to a common factor (this can be seen by replacing  $\lambda \rightarrow \lambda^a$  in (6.1) or multiplying equation (6.2) by  $a$ ); this arbitrariness can be eliminated, for example, if we set  $\alpha_1 = 1$ . If  $\alpha_i = 0$  for some  $x_i$ , this variable is not dilated in (6.1), and the corresponding derivative is absent from (6.2).

It is well known that the leading terms, determining the asymptotic behaviour of (renormalized) correlation functions at large distances, satisfy the RG equation (4.2), in which the renormalized coupling constants are replaced with their values at the fixed points. In our case, this leads to the equation

$$\{D_\mu - \gamma_f^* D_f - \gamma_\lambda^* D_\lambda - \gamma_\tau^* D_\tau + \gamma_\varphi^* N_\varphi + \gamma_{\varphi'}^* N_{\varphi'}\} G^R = 0, \quad (6.3)$$

where  $\gamma^* = \gamma(u = u_*, w = w_*)$  for all the anomalous dimensions.

We are interested in the critical scaling behaviour, that is, behaviour of the type (6.1) in which all the IR relevant parameters (momenta/coordinates, frequencies/times, deviation of the temperature from its critical value  $\tau \propto (T - T_c)$ ) are dilated, while the IR irrelevant parameters (those which remain finite at the fixed point:  $\lambda$ ,  $\mu$  and  $f$ ) are fixed [1, 2]. Thus we combine the equation (6.3) with the analogous equations, corresponding to the canonical scale invariance (see section 3), so that the derivatives with respect to the IR irrelevant parameters are eliminated; this gives the desired equation which describes the critical scaling behaviour (for more details, see *e.g.* [31, 32]):

$$\{D_\perp + \Delta_\parallel D_\parallel + \Delta_\omega D_\omega + \Delta_\tau D_\tau - N_\Phi \Delta_\Phi\} G_{N_\Phi} = 0,$$

where  $D_\perp = k_\perp \partial / \partial k_\perp$  and  $D_\parallel = k_\parallel \partial / \partial k_\parallel$ . Here,  $\Delta_\perp = 1$  is the normalization condition, and the critical dimension of any IR-relevant parameter  $F$  is given by the general expression

$$\Delta_F = d_F^\perp + \Delta_\parallel d_F^\parallel + \Delta_\omega d_F^\omega + \gamma_F^*, \quad (6.4)$$

with canonical dimensions from Table 1 and the relations

$$\Delta_\omega = 2 - \gamma_\lambda^*, \quad \Delta_\parallel = (2 + \gamma_f^*) / 2. \quad (6.5)$$

We are in a position to write the final one-loop results for the critical dimensions. Substituting (4.9) into the general formulae (6.4) and (6.5) gives

$$\begin{aligned} \Delta_\omega &= 2 + \frac{u_* R_1}{6}, \quad \Delta_\parallel = 1 + \frac{w_*}{2}, \quad \Delta_\tau = 2 + \frac{5u_* R_1}{3} \\ \Delta_{\varphi'} &= \frac{d}{2} + 1 + \frac{u_* R_1}{3} + \frac{w_*}{4}, \quad \Delta_\varphi = \frac{d}{2} - 1 + \frac{u_* R_1}{6} + \frac{w_*}{4}. \end{aligned}$$

By inserting the explicit expressions for the fixed point coordinates and taking the equality  $d = 6 - \varepsilon$  into account one obtains the leading-order expressions for the critical dimensions. The results for all scaling regimes are summarized in Table 2.

**Table 2.** Critical dimensions of the fields and parameters in the model (2.11)

	FP1	FP2	FP3	FP4
$\Delta_\omega$	2	2	$2 + \frac{R_1 \varepsilon}{6R}$	$2 + \frac{R_1(2\varepsilon - \xi)}{12\Delta^R}$
$\Delta_\parallel$	1	$1 + \frac{\xi}{2}$	1	$1 + \frac{6R\xi + R_1\varepsilon}{12\Delta^R}$
$\Delta_\tau$	2	2	$2 + \frac{5R_1\varepsilon}{3R}$	$2 + \frac{5R_1(2\varepsilon - \xi)}{6\Delta^R}$
$\Delta_{\varphi'}$	$4 - \frac{\varepsilon}{2}$	$4 - \frac{\varepsilon}{2} + \frac{\xi}{4}$	$4 + \frac{(2R_1 - 3R)\varepsilon}{6R}$	$4 + \frac{(9R_1 - 12\Delta^R)\varepsilon + (6R - 4R_1)\xi}{24\Delta^R}$
$\Delta_\varphi$	$2 - \frac{\varepsilon}{2}$	$2 - \frac{\varepsilon}{2} + \frac{\xi}{4}$	$2 + \frac{(R_1 - 3R)\varepsilon}{6R}$	$2 + \frac{(5R_1 - 12\Delta^R)\varepsilon + (6R + 2R_1)\xi}{24\Delta^R}$

The expressions for the first and second fixed points are exact. Other dimensions have corrections, given by higher powers of  $\varepsilon$  for the third fixed point and higher powers of  $\varepsilon$  and  $\xi$  for the fourth one. The critical dimensions for the models of a liquid crystal and bond percolation are derived from the general results by substituting the expressions (5.13) and (5.14), respectively.

Let us discuss the consequences of the general scaling relations for the most interesting special case of the pair correlation function. They result in the scaling expression

$$\langle \varphi_a(\mathbf{x} + \mathbf{r}, t + t') \varphi_b(\mathbf{x}, t') \rangle = \delta_{ab} r_\perp^{-2\Delta_\varphi} \mathcal{F} \left( \tau_0 r^{\Delta_\tau}, t/r_\perp^{\Delta_\omega}, r_\parallel/r_\perp^{\Delta_\parallel} \right), \quad (6.6)$$

where  $r_\perp = |\mathbf{r}_\perp|$ ,  $r_\parallel = |\mathbf{r}_\parallel|$  and  $\mathcal{F}$  is some scaling function. This representation is valid in the symmetric phase ( $\tau_0 \geq 0$ ), where the tensor structure is simply given by the  $\delta$  symbol. It is natural to assume that  $\mathcal{F}$  has a finite limit for  $\tau_0 = 0$  (that is, exactly at the critical point) and/or for  $t = 0$  (equal-time correlation function). Then from (6.6) one obtains

$$\langle \varphi_a(\mathbf{x} + \mathbf{r}, t) \varphi_b(\mathbf{x}, t) \rangle = \delta_{ab} r_\perp^{-2\Delta_\varphi} \tilde{\mathcal{F}} \left( r_\parallel/r_\perp^{\Delta_\parallel} \right)$$

with another nontrivial function  $\tilde{\mathcal{F}}(x) = \mathcal{F}(0, 0, x)$ .

The two last arguments in the scaling representation (6.6) can also be chosen in the form  $r_\perp/L_\perp(t)$  and  $r_\parallel/L_\parallel(t)$  with two different characteristic length scales

$$L_\perp(t) \sim t^{\alpha_\perp}, \quad L_\parallel(t) \sim t^{\alpha_\parallel}, \quad \alpha_\perp = 1/\Delta_\omega, \quad \alpha_\parallel = \Delta_\parallel/\Delta_\omega. \quad (6.7)$$

For the most realistic values  $\varepsilon = 3$  ( $d = 3$ ) and  $\xi = 4/3$  (Kolmogorov spectrum of the velocity) and for the  $n = 2$  case of our model (liquid crystals) the explicit results from Table 2 and the expressions (5.13) give

$$\Delta_\omega \approx 2.359, \quad \Delta_\parallel \approx 1.846, \quad \alpha_\perp \approx 0.424, \quad \alpha_\parallel \approx 0.783, \quad \Delta_\varphi \approx 1.487,$$

while for the percolation limit  $n = 0$  from (5.14) one obtains

$$\Delta_\omega \approx 1.944, \quad \Delta_\parallel \approx 1.639, \quad \alpha_\perp \approx 0.514, \quad \alpha_\parallel \approx 0.843, \quad \Delta_\varphi \approx 0.731.$$

Existence of two different length scales (6.7) with power-law dependence on the time was established earlier in a number of studies within numerical simulations [26], approximate analytical solutions [27], RG analysis [31, 32] and exactly soluble simplified models [28]. It is interesting to note that the inequality  $\alpha_\parallel > \alpha_\perp$  also holds for all those cases.

## 7. Conclusion

We studied effects of turbulent mixing on the critical behaviour of the system, described by a relaxational dynamics of a non-conserved order parameter of the ATP model. The mixing was modelled by a Gaussian statistics with vanishing correlation time and strongly anisotropic correlation function  $\propto \delta(t - t')/k_{\perp}^{d-1+\xi}$ ; see equations (2.8), (2.9). Such ensembles were employed earlier in [35]–[37] in the analysis of the two-dimensional passive turbulent advection (linear equation for the scalar field).

The model, originally described by stochastic differential equations (2.1)–(2.3), (2.7), can be reformulated as a multiplicatively renormalizable field theory with the action (2.11), which allows one to employ the field theoretic RG to study its critical behaviour. The model reveals four different IR scaling regimes, related with the four different fixed points of the RG equations. Their regions of stability in the  $\varepsilon$ – $\xi$  plane were identified in the leading order of the double expansion in  $\varepsilon$  and  $\xi$  and are shown on figures 1–5. These regimes correspond to:

- (1) Gaussian (free) model;
- (2) Linear passive scalar advection (the self-action term in the ATP Hamiltonian (2.3) is irrelevant in the sense of Wilson);
- (3) Equilibrium critical dynamics of the ATP model (interaction with the velocity field is irrelevant); and
- (4) The full-fledged strongly anisotropic scaling regime in which the both interactions are important; it corresponds to a new non-equilibrium universality class.

It was shown that the equilibrium critical regimes for the both physically interesting cases (liquid crystals and percolation process) become unstable for the realistic range of parameters  $d = 3$  and  $0 < \xi \leq 2$ , which includes the Kolmogorov spectrum ( $\xi = 4/3$ ) and the Batchelor limit ( $\xi \rightarrow 2$ ) and is replaced with the new non-equilibrium regime. The corresponding critical dimensions were calculated to first order of the corresponding RG expansion, which in this case takes on the form of the double expansion in  $\varepsilon$  and  $\xi$ ; explicit expressions are given in Table 2.

Those results were derived within the leading (one-loop) approximation, that is, in the leading order of the double expansion in  $\varepsilon$  and  $\xi$ , and their validity for finite physical values of these parameters can be called in question (especially because of large physical values of  $\varepsilon = 6 - d$ ). Careful analysis of this problem requires calculation of the higher-order corrections and applying some kind of summation procedure to the results obtained, as was done *e.g.* in [9] for the scalar static  $\varphi^3$  model. Such analysis goes far beyond the scope of the present paper, and we hope to address it in the future. Nevertheless, the discussion of the RG flows, given in section 5, suggests that the pattern of fixed points (and thus of critical regimes), obtained within the one-loop approximation, appears robust with respect to higher-order corrections and is preserved for finite values of  $\varepsilon$  and  $\xi$ .

Thus we hope that our simplified model of a non-conserved order parameter and Gaussian velocity ensemble captures the most important features of the full-fledged

problem: emergence of a new non-equilibrium universality class with a new set of critical exponents, completely different from those of the classical ATP model; existence (for a strongly anisotropic velocity ensemble) of two different length scales (with a power law time dependence), and so on. Further investigation should take into account conservation of the order parameter, compressibility of the fluid, non-Gaussian character and finite correlation time of the velocity field, and so on. This work is partly in progress and partly remains for the future.

## Acknowledgments

The authors thank L Ts Adzhemyan, Michal Hnatich, Juha Honkonen and Paolo Muratore Ginanneschi for discussions. AVM was supported in part by the Dynasty Foundation.

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