1. Description of the NS model

The stochastic NS equation with the active field θ:

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \wp + w_0 (\partial_i \theta) \partial^2 \theta + f_i, \tag{1.1}$$

where ∇_t is the Lagrangian derivative,

$$\nabla_t \equiv \partial_t + (v_k \partial_k), \tag{1.2}$$

$\wp$ and f_i are the pressure and the transverse random force per unit mass. We assume for f a Gaussian distribution with zero mean and correlation function

$$\langle f_i(x) f_j(x') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int_{k \geq m} d\mathbf{k}\, P_{ij}(\mathbf{k})\, d_f(k)\, \exp\left[i\mathbf{k}\,(\mathbf{x}-\mathbf{x}')\,\right], \tag{1.3}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $d_f(k)$ is some function of $k \equiv |\mathbf{k}|$ and model parameters. The momentum $m = 1/L$, the reciprocal of the integral scale L related to the velocity, provides IR regularization.

The standard RG formalism is applicable to the problem (1.1), (1.3) if the correlation function of the random force is chosen in the power form

$$d_f(k) = D_0\, k^{4-d-y}, \tag{1.4}$$

where $D_0 > 0$ is the positive amplitude factor and the exponent $0 < y \leq 4$ plays the role of the RG expansion parameter. The most realistic value of the exponent is $y = 4$: with an appropriate choice of the amplitude, the function (1.4) for $y \to 4$ turns to the delta function, $d_f(k) \propto \delta(\mathbf{k})$, which corresponds to the injection of energy to the system owing to interaction with the largest turbulent eddies.

The advection-diffusion equation for the active scalar field $\theta(t, \mathbf{x})$:

$$\nabla_t \theta = \kappa_0 \partial^2 \theta + \eta, \qquad \nabla_t \equiv \partial_t + (v_k \partial_k), \tag{1.5}$$

where ∇_t is the same Lagrangian derivative, κ_0 is the diffusivity, ∂^2 is the Laplace operator and $\eta(x)$ is a Gaussian stirring force with zero mean and covariance

$$\langle \eta(x) \eta(x') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int_{k \geq m} d\mathbf{k}\, d_\eta(k)\, \exp\left[i\mathbf{k}\,(\mathbf{x}-\mathbf{x}')\,\right], \tag{1.6}$$

where

$$d_\eta(k) = K_0\, k^{2-d-z}. \tag{1.7}$$

There are "too many" parameters and the canonical dimensions are ambiguous. One can set $K_0 = 1$ or $w_0 = 1$ by rescaling the fields.

According to the general theorem the full-scale stochastic problem is equivalent to the field theoretic model of the doubled set of fields $\Phi = \{v, v', \theta, \theta'\}$ with the action functional

$$\mathcal{S}(\Phi) = \mathcal{S}_v(\mathbf{v}', \mathbf{v}, \theta) + \theta' D_\theta \theta'/2 + \theta' \left\{ -\nabla_t + \kappa_0 \partial^2 \right\} \theta, \tag{1.8}$$

where D_θ is the correlation function (1.6) of the random noise η and S_v is the action for the problem (1.1)–(1.4):

$$\mathcal{S}_v(\mathbf{v}', \mathbf{v}) = v' D_v v'/2 + v' \left\{ -\nabla_t + \nu_0 \partial^2 \right\} v + w_0 v'_i (\partial_i \theta) \partial^2 \theta, \tag{1.9}$$

where D_v is the correlation function (1.3) of the random force f_i. All the integrations over $x = \{t, \mathbf{x}\}$ and summations over the vector indices are understood. The auxiliary vector field v' is also transverse, $\partial_i v'_i = 0$, which allows to omit the pressure terms on the right-hand sides of expressions (1.8), (1.9), as becomes evident after the integration by parts. For example,

$$\int dt \int d\mathbf{x}\ v'_i \partial_i \wp = -\int dt \int d\mathbf{x}\ \wp(\partial_i v'_i) = 0.$$

The role of the coupling constants is played by the parameters $g_0 = D_0/\nu_0^3$, $u_0 = \kappa_0/\nu_0$ and w_0. The model is logarithmic at $y = z = 0$.