The Field Theoretic Renormalization Group in Fully Developed Turbulence

L.Ts. Adzhemyan, N.V. Antonov and A.N. Vasiliev
St Petersburg State University, Russia

Translated from the Russian by P. Millard

GORDON AND BREACH SCIENCE PUBLISHERS
Australia • Canada • China • France • Germany • India
Japan • Luxembourg • Malaysia • The Netherlands
Russia • Singapore • Switzerland

techniques of IR perturbation theory (see Sec. 2.6). It turned out that not only the power-law, but also the logarithmic IR divergences do not contribute at all to the static correlator and, therefore, do not affect the shape of the spectrum, which remains the Kolmogorov one. However, it should be recalled that this pertains to the nonstandard model [90,94] with the replacement $\nu_0 \to \nu_{\text{eff}} \sim k^{-4/3}$, the correctness of which is not certain.

The general conclusion is that the modern RG technique guarantees the presence of IR scaling for the $m \sim k \to 0$ asymptote of the static correlators, but does not yet allow exact solution of the second IR problem, that of determining the m dependence for $m/k \to 0$. The question should therefore be considered open.

Chapter 3

Multicharge Problems in the Stochastic Theory of Turbulence

From the viewpoint of the renormalization group, a charge is any parameter which can enter into the renormalization constants Z. The stochastic theory of turbulence discussed above is a single-charge theory. In this chapter we shall study more complicated models whose common feature is the presence of several parameters which act as charges.

3.1 Turbulent convection of a passive scalar

In [97,48–50,83,84,98] the RG method was used to study the turbulent convection of a passive scalar admixture. In this case, the equation

$$\nabla_t \theta = \nu_0' \Delta \theta, \qquad \nabla_t \equiv \partial_t + (\varphi \partial), \qquad \nu_0' = u_0 \nu_0 \tag{3.1}$$

is added to (1.1) (see [1], Secs. 21.6 and 24). The scalar field $\theta(t, \mathbf{x})$ can have various physical meanings, such as the temperature in heat transport problems, the concentration of impurity particles in a turbulent atmosphere, etc. In the first case ν_0' is the thermal conductivity, in the second it is the molecular diffusion coefficient, and so on. To be specific, we shall use the terminology of the second case. Instead of ν_0' it is more convenient to use the completely dimensionless positive parameter $u_0 \equiv \nu_0'/\nu_0$ (the inverse Prandtl number), which will also play the role of a second charge. The parameters of the original problem are assumed to be the bare parameters. They will be renormalized, and so the index 0 has been appended.

According to the general rules of Sec. 1.2, the stochastic problem (1.1), (3.1) reduces to the quantum-field model of four fields $\Phi \equiv \varphi$, φ', θ, and θ' with unrenormalized action

$$S(\Phi) = S(\varphi, \varphi') + \theta'[-\partial_t \theta + u_0 \nu_0 \Delta\theta - (\varphi\partial)\theta], \qquad (3.2)$$

where $S(\varphi, \varphi')$ is the functional (1.18) for the problem without the admixture. The elements of the diagrammatic technique (1.19), (1.20) are supplemented by the triple vertex $\sim \theta'\varphi\theta$ and lines of the type $\langle\theta\theta'\rangle_0 = \langle\theta'\theta\rangle_0^*$ of the admixed field, which correspond to the bare propagator

$$\langle\theta\theta'\rangle_0 = \langle\theta'\theta\rangle_0^* = (-i\omega + u_0\nu_0 k^2)^{-1} \qquad (3.3)$$

in the momentum–frequency representation. The propagators $\langle\theta'\theta'\rangle_0$ (and the corresponding lines in the graphs) are equal to zero according to the general rule of Sec. 1.2, and in this model also $\langle\theta\theta\rangle_0 = 0$ owing to the absence of a random force ("noise") in Eq. (3.1). Since the inclusion of the field θ does not affect Eq. (1.1), all the Green functions of the fields φ, φ' remain the same as in the model (1.18) (there are no new graphs involving the fields θ and θ'). This is the case of a "passive" admixture.

The basic action (Sec. 1.4) is obtained from (3.2) by replacing the bare parameters g_0, ν_0, and u_0 by their renormalized analogs $g\mu^{2\varepsilon}$, ν, and u, respectively (the latter should not be confused with the variable $u \equiv m/k$ introduced in Sec. 1.6; the "mass" m will not appear in this discussion). The new parameters are completely dimensionless (in momentum and frequency), and according to the form of the action (3.2) for the new fields θ, θ' we can define only the dimensions of the product $\theta\theta'$, but not those of each field separately:

$$d^k[\theta\theta'] = d[\theta\theta'] = d, \qquad d^\omega[\theta\theta'] = 0. \qquad (3.4)$$

Here d is the dimension of the space $\mathbf{x}$. This is sufficient, because in the model (3.2) only the Green functions with $n_\theta = n_{\theta'}$, i.e., with equal numbers of fields θ and θ', are nonzero (formally this follows from the invariance of the action under dilatations

$\theta \to c\theta$, $\theta' \to c^{-1}\theta'$ with arbitrary coefficient c). It is these Green functions which are most interesting from the viewpoint of physics. In particular, the response function $\langle\theta(x)\theta'(x')\rangle$ has the meaning of the $\mathbf{x}$ distribution at time t of an admixed particle having coordinate $\mathbf{x}'$ at time $t' < t$. Higher-order functions of this type are simply related to the analogous multiparticle distribution functions $p_n(x_1...x_n; x_1'...x_n')$ ([1], Sec. 24) as

$$\langle\theta(x_1)...\theta(x_n)\theta'(x_1')...\theta'(x_n')\rangle = \sum_{\text{perms}} p_n(x_1...x_n; x_1'...x_n')$$

with summation over all $n!$ permutations of the arguments $x_1...x_n$ (or $x_1'...x_n'$, which is equivalent).

When analyzing the UV divergences, it should be remembered that since the field φ is transverse, the derivative ∂ at the vertex $\theta'(\varphi\partial)\theta$ in the action (3.2) can, if desired, be moved onto the field θ'. Therefore, in 1-irreducible graphs it is always possible to move the derivative ∂ onto any of the external lines θ, θ', which decreases the real degree of divergence δ' (Sec. 1.4): $\delta' = \delta - n_{\varphi'} - n_\theta - n_{\theta'}$ ($n_\theta = n_{\theta'}$). Then, using the general rules of Sec. 1.4 it can be verified that for $d > 2$ the contribution $\theta'\Delta\theta$ will be the only new counterterm in the model (3.2), and so the renormalized analog of (3.2) will be the functional

$$S_R(\Phi) = S_R(\varphi, \varphi') + \theta'[-\partial_t\theta + Z_2 u_0 \nu_0 \Delta\theta - (\varphi\partial)\theta] \qquad (3.5)$$

with $S_R(\varphi, \varphi')$ from (1.29) and a new renormalization constant Z_2. The action (3.5) is obtained from (3.2) by renormalization of the parameters (1.31) and

$$u_0 = uZ_u, \qquad Z_u = Z_2 Z_\nu^{-1}. \qquad (3.6)$$

Therefore, like (1.18), the model is multiplicatively renormalizable, and there is no renormalization of the fields ($Z_\Phi = 1$ for all Φ). The constant Z_ν is known from (1.37) (since the Green functions of the fields φ and φ' do not change). The new constant Z_2 in the one-loop approximation is determined by the graph of Fig. 3.1 and in the MS scheme is [97]

$$Z_2 = 1 - \frac{ga\alpha}{2\varepsilon u(u+1)}, \qquad \alpha \equiv \frac{2(d+2)}{d} \qquad (3.7)$$

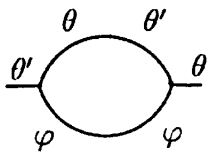

Figure 3.1.

with the constant a from (1.37).

The constant Z_2 involves the parameter u, so that from the viewpoint of the renormalization group it is also a charge, and the corresponding β function $\beta_u \equiv \tilde{D}_\mu u$ is given by the general rule (1.33). From the renormalization formula (3.6) we have

$$\beta_u = \tilde{D}_\mu u = -u\gamma_u, \qquad \gamma_u = \tilde{D}_\mu \ln Z_u. \tag{3.8}$$

The RG functions can be calculated from the renormalization constants using the identity following from the definitions (1.33) and (3.8):

$$\tilde{D}_\mu F(g,u) = [\beta_g \partial_g + \beta_u \partial_u]F = -[(2\varepsilon + \gamma_g)D_g + \gamma_u D_u]F \tag{3.9}$$

(we recall that $D_x \equiv x\partial_x$ for any parameter x). The RG functions γ_g and β_g are known from (1.34) and (1.36), and the analogous RG functions for the second charge u can be expressed in terms of the constant Z_u from (3.6), setting $F = \ln Z_u$ in (3.9). Taking into account the second equation in (3.8), we then obtain

$$\gamma_u = -\frac{(2\varepsilon + \gamma_g)D_g \ln Z_u}{1 + D_u \ln Z_u}, \tag{3.10}$$

and then we can find β_u from the first equation in (3.8). In the MS scheme from (3.10) we obtain a simple expression analogous to (1.36):

$$\gamma_u = -2D_g a_{1u}(g,u), \tag{3.11}$$

in which a_{1u} is the coefficient of $1/\varepsilon$ in the representation of the constant Z_u analogous to (1.30). In the one-loop approximation (1.37), (3.7) this calculation gives [97]

$$\beta_g = g[-2\varepsilon + 3ag], \qquad \beta_u = ag\left[u - \frac{\alpha}{u+1}\right] \tag{3.12}$$

with the constants a from (1.37) and α from (3.7).

Let us briefly explain how the RG analysis of Sec. 1.5 is generalized to the case of a theory like (3.2) with several charges $g \equiv \{g_1, g_2, ...\}$ [in the notation of (3.12), $g \equiv g_1$ and $u \equiv g_2$]. In the absence of renormalization of the fields Φ, any Green function $F = F_{\mathrm{R}}$ of these fields, expressed in terms of the renormalized parameters, satisfies the RG equation (1.24) with $\gamma_\Phi = 0$, i.e.,

$$D_{\mathrm{RG}} F_{\mathrm{R}}(k, \omega, g, \nu, \mu) = 0 \tag{3.13}$$

with RG operator of the type (1.25):

$$D_{\mathrm{RG}} = D_\mu + \sum_i \beta_{g_i} \partial_{g_i} - \gamma_\nu D_\nu. \tag{3.14}$$

From dimensional considerations we have (see Sec. 1.6)

$$F_{\mathrm{R}} = k^{d_F} \nu^{d_F^\omega} R(k/\mu, \omega/\nu k^2, g). \tag{3.15}$$

If there are several momenta, any of them can be taken as the "scale" k in (3.15). Then R will acquire an unimportant dependence on momentum ratios ("angles") which will not be written out explicitly. In models of the type (1.11) with a mass R will contain the additional argument m/k. It also does not affect the RG operator (3.14) because it does not involve D_m (the parameter m is not renormalized; see Sec. 1.5), and so it will be dropped.

The general solution of (3.13) for the function (3.15) (the RG representation) has the form

$$F_{\mathrm{R}} = k^{d_F} \overline{\nu}^{d_F^\omega} R(1, \omega/\overline{\nu}k^2, \overline{g}) \tag{3.16}$$

with invariant variable $\overline{\nu}$ from (1.51) and invariant charges $\overline{g}_i = \overline{g}_i(s,g)$. The latter are defined as the solutions of the following

Cauchy problem for a system of ordinary differential equations in the scaling variable $s \equiv k/\mu$:

$$D_s \overline{g}_i = \beta_i(\overline{g}), \qquad \overline{g}_i|_{s=1} = g_i \quad \text{for all } i. \tag{3.17}$$

Here and below $\beta_i \equiv \beta_{g_i}$. The fixed points $g_* \equiv \{g_{1*}, g_{2*}, ...\}$ of the system are given by the condition

$$\beta_i(g_*) = 0 \quad \text{for all } i. \tag{3.18}$$

In contrast to a single-charge theory, in a multicharge one there may be several such points with $g_* \sim \varepsilon$. The type of each point is determined by the eigenvalues ω_α of the corresponding stability matrix ω:

$$\omega_{ik} = \partial\beta_i(g)/\partial g_k|_{g=g_*}. \tag{3.19}$$

A point g_* will be IR-attractive (equivalently, IR-stable) only if the real parts of all the eigenvalues ω_α are strictly positive. For such a point, any phase trajectory [solution of the problem (3.17)] passing through the neighborhood of g_* will subsequently be attracted to that point, i.e., $\overline{g}(s,g) \to g_*$ for $s = k/\mu \to 0$, because in the neighborhood of g_* it follows from (3.17) that

$$\overline{g}_i(s,g) - g_{i*} \simeq \sum_\alpha c_{i\alpha} s^{\omega_\alpha} \tag{3.20}$$

with coefficients $c_{i\alpha}$ independent of s. Here $\overline{g}$ can approach g_* either along a certain direction (if all the ω_α are strictly positive), or along a shrinking spiral (if among the ω_α there are complex-conjugate pairs with Re $\omega_\alpha > 0$). In the last case the point is called an attractive focal point [the matrix (3.19) is real, and so complex ω_α can occur only in complex-conjugate pairs]. We note that the ω_α can be interpreted as corrections to the critical exponents, because (3.20) determines the value of the relative (compared to the leading contributions) corrections to IR-scaling formulas like (1.59)–(1.61) characterizing the nonexactness of reaching the limiting IR asymptote.

In multicharge problems it is also necessary to take into account the fact that even when an IR-attractive point g_* is present,

not every phase trajectory [solution of (3.17)] will necessarily reach it in the limit $s \to 0$. A trajectory might first pass outside the natural region of stability [for the system (3.12) this is the quadrant $g > 0$, $u > 0$], or go to infinity inside this region. Therefore, for each IR-stable point g_* it is possible to introduce the concept of its basin of attraction, defined as the set of all points g (initial data) which as starting points of trajectories guarantee that a trajectory reaches the point g_* in the limit $s \to 0$.

Let us now return to our problem with the two charges $g_1 \equiv g$, $g_2 \equiv u$. The matrix (3.19) for the β functions (3.12) is triangular, so that its eigenvalues ω_α are simply the diagonal elements. It is easily verified that the system of β functions (3.12) has an entire line ($g_* = 0$, u_* arbitrary) of IR-unstable fixed points and a pair of IR-stable points with $g = g_* = 2\varepsilon/3a$ [as in the model (1.27)] and $u = u_*$, where u_* is either root of the equation $u(u+1) = \alpha$. For physical reasons we are interested only in the quadrant $g > 0$, $u > 0$, in which there is only one IR-stable point:

$$g_* = \frac{2\varepsilon}{3a}, \qquad u_* = \frac{1}{2}(\sqrt{1+4\alpha} - 1) \tag{3.21}$$

[we recall that $\alpha \equiv 2(d+2)/d$ and numerically $u_* \simeq 1.393$ for $d = 3$], at which

$$\omega_g \equiv \omega_{11} = 2\varepsilon, \qquad \omega_u \equiv \omega_{22} = \frac{2\varepsilon(2u_*+1)}{3(u_*+1)}. \tag{3.22}$$

The initial data in the Cauchy problem (3.17) for the β functions (3.12) are specified in the band $u > 0$, $0 < g < g_*$ (see Sec. 1.6). It is easily verified that this entire band is contained in the basin of attraction of the point (3.21), i.e., $\{\overline{g}, \overline{u}\} \to \{g_*, u_*\}$ for $s \equiv k/\mu \to 0$ for any initial data of this type. At the UV asymptote $s \to \infty$ we will have

$$\overline{g} \simeq g_0 k^{-2\varepsilon} \to 0, \qquad \overline{u} \to u_0. \tag{3.23}$$

The first equation follows from (1.55) at the asymptote $\overline{g} \to 0$, where $Z(\overline{g}) \to 1$ [since $Z(g) = 1 + O(g)$], and the second follows from an expression analogous to (1.55):

$$\overline{u} Z_u(\overline{g}, \overline{u}) = u_0, \tag{3.24}$$

which is a special case of (3.16) for the function $F(s,g,u) = uZ_u(g,u) = u_0$ satisfying an equation like (3.13), i.e., $D_{\rm RG}F = \widetilde{D}_\mu u_0 = 0$, owing to the renormalization invariance of the bare parameter u_0 in (3.6).

The system (3.17) for the two charges $g_1 \equiv g$, $g_2 \equiv u$ with the β functions (3.12) contains two equations. The solution of the first is given by (1.50) and has the form (1.52) in the one-loop approximation. In the second equation $D_s\overline{u} = \beta_u(\overline{g},\overline{u})$ we can make the change of variable $s \to \overline{g}(s)$. Then in the one-loop approximation (3.12) we obtain an equation in separable variables, the general solution of which has the form

$$(\overline{u}-u_1)^{\lambda_1}(\overline{u}-u_2)^{\lambda_2} = C(\overline{g}-g_*) \tag{3.25}$$

with $u_1 \equiv u_*$ from (3.21), $u_2 \equiv -u_* - 1$, $\lambda_1 \equiv 3(1+u_*)/(1+2u_*)$, and $\lambda_2 \equiv 3u_*/(1+2u_*)$ [$u_{1,2}$ are the roots of the equation $u(u+1) = \alpha$]. The integration constant C in (3.25) can be found either from the normalization $\overline{g} = g$, $\overline{u} = u$ for $s \equiv k/\mu = 1$, which gives

$$\left(\frac{\overline{u}-u_1}{u-u_1}\right)^{\lambda_1}\left(\frac{\overline{u}-u_2}{u-u_2}\right)^{\lambda_2} = \frac{\overline{g}-g_*}{g-g_*} = \frac{g_*}{g_*+g(s^{-2\varepsilon}-1)} \tag{3.26}$$

[the second equation was obtained by substituting the explicit expression (1.52)], or from the normalization (3.23) for $k \to \infty$, which gives

$$\left(\frac{\overline{u}-u_1}{u_0-u_1}\right)^{\lambda_1}\left(\frac{\overline{u}-u_2}{u_0-u_2}\right)^{\lambda_2} = \frac{\overline{g}-g_*}{g_0k^{-2\varepsilon}-g_*} = \frac{g_*}{g_*+g_0k^{-2\varepsilon}}. \tag{3.27}$$

The critical dimensions $\Delta_t = -\Delta_\omega$, Δ_φ, and $\Delta_{\varphi'}$ at the fixed point (3.21) remain as before (1.44). The single new critical dimension $\Delta[\theta\theta']$ of the product $\theta\theta'$ is canonical owing to (3.4) and the general rule (2.3) [which is valid not only for composite operators, but also for the fields Φ themselves; cf. (1.44)]:

$$\Delta[\theta\theta'] = d^k[\theta\theta'] + \Delta_\omega d^\omega[\theta\theta'] = d. \tag{3.28}$$

As pointed out in [97], the result (3.28) corresponds to the phenomenological "four-thirds law" of Richardson ([1], Sec. 24)

for the spreading rate of a cloud of admixed particles in a turbulent atmosphere. Indeed, if the field $\theta(\mathbf{x},t)$ is the concentration of particles of the admixture, the effective radius R of a cloud of such particles at time $t > 0$ which started at time $t' = 0$ from the origin $\mathbf{x}' = 0$ is given by

$$R^2 = \int d\mathbf{x}x^2\langle\theta(\mathbf{x},t)\theta'(0,0)\rangle. \tag{3.29}$$

From (3.28) and (3.29) using $\Delta_{\rm R} = -1$ we find $\Delta[dR^2/dt] = -2 - \Delta_t = -4/3$ for $\Delta_t = -2/3$ ($\varepsilon = 2$) in (1.44), so that $dR^2/dt \sim R^{4/3}$, which is the statement of Richardson's law.

Let us end this section with a brief historical discussion. The RG technique was first applied to the problem (3.1) in [31], however, not for a turbulent medium, but for a liquid in a state of thermal equilibrium with injection $d_F \sim k^2$ instead of (1.10). This corresponds to the boundary $2\varepsilon + d = 2$ in the plane of the parameters ε and d, where the value $2\varepsilon = 2 - d$ on this line plays the role of the parameter describing the deviation from logarithmic behavior (see Sec. 3.10 for more details). The above RG analysis of the model of turbulence (3.2) with injection (1.10) in the one-loop approximation was first performed in [97]. These results were then reproduced and augmented by explicit expressions of the type (3.27) for $\overline{u}$ (not given in [97]) in [48–50,98]. The slight difference in the expression in [50] for $\overline{u}$ is related to a different choice of normalization ($\overline{u} = u_0$ for $k = \Lambda$). The author of [98] obtained a numerical solution of Eq. (3.27) for the special case $u_0 = 1/7$ which illustrates its general features: $\overline{u} \to u_*$ for $k \to 0$ and $\overline{u} \to u_0$ for $k \to \infty$. We note that in [48–50] the expression (3.21) for u_* contains an error [in the notation of [48–50], the replacement $A_3 \to A_d$ should be made].

3.2 A passive admixture with noise. Calculation of the Batchelor constant

Some studies (see below for references) have considered a generalization of the model (1.1), (3.1) in which a scalar random force

(noise) F^θ is introduced into (3.1) in addition to the random force F_i in (1.1). It is assumed that $\langle F^\theta F_i\rangle = 0$ (transversality of F_i), and the correlator $D^{\theta\theta} \equiv \langle F^\theta F^\theta\rangle$ is given by the same expression (1.2) but without the transverse projector and with the injection function

$$d_F^\theta = D_0^\theta k^{4-d-2\varepsilon}, \tag{3.30}$$

which differs from (1.10) by only the amplitude factor. The unrenormalized action for this model will be the functional

$$S'(\Phi) = S(\Phi) + \frac{1}{2}\theta' D^{\theta\theta}\theta', \tag{3.31}$$

where $S(\Phi)$ is the action (3.2). In the graphs, the addition in (3.31) generates the additional bare propagator

$$\langle\theta\theta\rangle_0 = \frac{d_F^\theta(k)}{|i\omega + u_0\nu_0 k^2|^2}, \tag{3.32}$$

which was not present in the model (3.2). Taking into account the form of the vertex $\theta'(\varphi\partial)\theta$, it is easily checked that the appearance of the new line (3.32) does not generate new graphs in the Green functions with identical numbers of fields θ and θ', $n_\theta = n_{\theta'}$. However, now the Green functions with $n_\theta - n_{\theta'} = 2l = 2, 4, 6, \ldots$ become nonzero, and all the graphs of a function with a given value of $2l$ contain exactly l lines (3.32). Therefore, the entire dependence on the amplitude D_0^θ in (3.30) factors out in the form of a coefficient $(D_0^\theta)^l$ [this is most easily seen by noting that the parameter D_0^θ can be completely removed from the action (3.31) by the replacement $\theta \to (D_0^\theta)^{1/2}\theta$, $\theta' \to (D_0^\theta)^{-1/2}\theta'$]. The addition in (3.31) is nonlocal, and so the parameter D_0^θ is not renormalized and can be considered UV-finite. Therefore, this addition does not generate new renormalization constants, and so the entire RG analysis of the preceding section remains valid. The new renormalization-invariant parameter D_0^θ enters into the new objects, the Green functions with $n_\theta - n_{\theta'} = 2l > 0$, only in the form of factors $(D_0^\theta)^l$. In particular, from dimensional considerations for the static correlator $\langle\theta\theta\rangle$ we have

$$D_{\rm st}^\theta = D_0^\theta \nu^{-1} k^{2-d-2\varepsilon} R^\theta(k/\mu, g, u). \tag{3.33}$$

In lowest order in g from (3.32) we obtain $R^\theta(k/\mu, g, u) = 1/2u$.

The function (3.33) satisfies the RG equation (3.13), from which we find a representation like (3.16), namely,

$$D_{\rm st}^\theta = D_0^\theta \overline{\nu}^{-1} k^{2-d-2\varepsilon} R^\theta(1, \overline{g}, \overline{u}). \tag{3.34}$$

At the IR asymptote $s \to 0$ taking into account (1.61) we obtain

$$\begin{aligned} D_{\rm st}^\theta &= D_0^\theta \overline{\nu}_*^{-1} k^{2-d-2\varepsilon} R^\theta(1, g_*, u_*) \\ &= D_0^\theta (g_*/D_0)^{1/3} k^{2-d-4\varepsilon/3} R^\theta(1, g_*, u_*). \end{aligned} \tag{3.35}$$

The quantity studied is usually not the correlator (3.33) itself, but a quantity similar to (1.66): the spectrum

$$E^\theta(k) = \frac{1}{2(2\pi)^d} S_d k^{d-1} D_{\rm st}^\theta(k) \tag{3.36}$$

[the additional factor of $(d-1)$ in (1.66) and (1.38) comes from the trace of the transverse projector, which is not present in (3.33)]. From (3.35) and (3.36) we find the Kolmogorov law $E^\theta \sim k^{-5/3}$ for the IR asymptote of the spectrum at real $\varepsilon = 2$. It is usually written as [1]

$$E^\theta(k) = \text{Ba}\ N W^{-1/3} k^{-5/3}, \tag{3.37}$$

where Ba is the Batchelor constant and

$$N \equiv \frac{u_0\nu_0}{(2\pi)^d} \int d\mathbf{k} D_{\rm st}^\theta(k) k^2 \tag{3.38}$$

is the dissipation rate of the passive admixture. Equating it to the corresponding injection power [the analog of (1.3)]

$$W^\theta = \frac{1}{(2\pi)^d} \int d\mathbf{k} d_F^\theta(k)/2, \tag{3.39}$$

we obtain a relation similar to (1.71) between the parameters $N = W^\theta$ and D_0^θ:

$$D_0^\theta = 4(2\pi)^d S_d^{-1}(2-\varepsilon) N \Lambda^{2\varepsilon-4}. \tag{3.40}$$

Comparison of the expression for the Batchelor constant obtained from (3.35)–(3.40) with the analogous expressions for the Kolmogorov constant C_k (Sec. 2.10) gives the relation between them:

$$\frac{C_k}{\mathrm{Ba}} = \frac{R(1,g_*)}{g_* R^\theta(1,g_*,u_*)}. \tag{3.41}$$

In lowest order in ε we have $R(1,g_*) = g_*/2$ (see Sec. 1.7) and $R^\theta(1,g_*,u_*) = 1/2u_*$ (see above), from which we find

$$\frac{C_k}{\mathrm{Ba}} \simeq u_*, \tag{3.42}$$

i.e., numerically $C_k/\mathrm{Ba} \simeq 1.393$ for $d = 3$. Both the scaling functions R and R^θ in (3.41) must have a singularity $(2-\varepsilon)^{-2/3}$ (see the discussion in Sec. 2.10). In the ratio (3.41) the singularity cancels, and so the ε expansion can be constructed using the first term (3.42).

We have presented the elementary calculation of the ratio (3.41) using the quantum-field RG technique in a form which has not been published before. In [50] the Batchelor constant was calculated in a more complicated manner by analogy with the calculation of the Kolmogorov constant by the same authors (see Sec. 2.10). There the noise F^θ was not introduced explicitly into (3.1), but the correlator $\langle\theta\theta\rangle$ was assumed to be nonzero owing to the assumed nonstationarity of the problem [50]. The numerical result obtained in [50] for the ratio C_k/Ba at $d = 3$ is practically the same as (3.42). Equation (3.42) was obtained by yet another method in [84] directly for $d = 3$, $\varepsilon = 2$. The authors of that study think that Eq. (3.42) is exact, but in view of (3.41) this is unlikely.

Equation (3.1) with the noise F^θ was used in [83] to describe a specific experiment on convective turbulence in a Bénard cell (see the literature cited in [83]). It was assumed that random forces with the injection functions (1.10), (3.30) model the real conditions of the problem (the finite volume, the finite temperature difference between the upper and lower walls of the cell, the gravitational force, and so on), at least in the central part of

the cell where the turbulence can be assumed to be developed. Although this assumption seems rather bold, the agreement with experiment for $\varepsilon = 2$ is quite good [83]. In [83] it was noted that it can be improved somewhat by choosing a "real value" of $\varepsilon < 2$ close to $\varepsilon = 2$, but the fundamental objection to this choice (violation of the second Kolmogorov hypothesis; see Sec. 2.11 for more details) remains.

3.3 Turbulent convection of a chemically active scalar admixture

The RG method was used in [99] to study the problem of the turbulent convection of a chemically active scalar admixture ([1], Sec. 14.5). The noise term F^θ and the nonlinear term $-\lambda_0\theta^n/n!$ with coefficient $\lambda_0 > 0$ and any integer $n \geq 1$ (an admixture of the nth kind) are added to the right-hand side of (3.1). In [99] it was assumed that $n > 1$, and the parameter λ_0 was eliminated by a suitable rescaling of θ, which leads to the equation

$$\nabla_t\theta = u_0\nu_0\Delta\theta - \frac{\theta^n}{n!} + F^\theta \tag{3.43}$$

corresponding to the unrenormalized action functional

$$S''(\Phi) = S'(\Phi) - \frac{\theta'\theta^n}{n!} \tag{3.44}$$

with $S'(\Phi)$ from (3.31). The injection function d_F^θ in the noise correlator $D^{\theta\theta} \equiv \langle F^\theta F^\theta\rangle$ was chosen to have the form [99]

$$d_F^\theta = g_0'\nu_0^{1+2/(n-1)}k^{2+4/(n-1)-d-2\alpha\varepsilon}. \tag{3.45}$$

The quantity $\alpha > 0$ was treated [99] as a new independent parameter, and $g_0' > 0$ is the third [in addition to g_0 and u_0 in (3.31)] bare charge. Let us explain the choice (3.45). From (3.43) we easily find the canonical dimensions of θ and F^θ, and then from the contribution $\theta'\partial_t\theta$ in (3.44) we find the dimension of θ', which gives

$$d_\theta^k = 0, \quad d_\theta^\omega = \frac{1}{n-1}, \quad d_\theta = \frac{2}{n-1}, \quad d_{\theta'}^k = d,$$

$$d^{\omega}_{\theta'} = -\frac{1}{n-1}, \qquad d_{\theta'} = \frac{d-2}{n-1} \tag{3.46}$$

and $d^{\omega}[F^{\theta}] = 1 + d^{\omega}_{\theta}$, $d[F^{\theta}] = 2 + d_{\theta}$. From the last expressions and the analog of (1.2) for F_{θ} we find the dimensions of the injection function (3.45):

$$d^{\omega}[d^{\theta}_{F}] = 2d^{\omega}[F^{\theta}] - 1 = 1 + \frac{2}{n-1},$$

$$d[d^{\theta}_{F}] = 2d[F^{\theta}] - 2 - d = 2 + \frac{4}{n-1} - d.$$

The exponent of ν_0 in (3.45) is $d^{\omega}[d^{\theta}_{F}]$, which ensures $d^{\omega}[g'_0] = 0$, and the exponent of k is $d[d^{\theta}_{F}] - 2\alpha\varepsilon$, from which we find $d[g'_0] = d^{k}[g'_0] = 2\alpha\varepsilon$. For this choice of exponents the charge g'_0 becomes completely dimensionless for $\varepsilon = 0$, along with g_0. In other words, both interactions become logarithmic at the same time (for $\varepsilon = 0$), which is necessary if we want to have a genuine three-charge theory. Otherwise, one of the two interactions would be weaker than the other from the viewpoint of the RG and would determine only the corrections to scaling, which can be neglected in leading order of the IR asymptote. These arguments determine the form of the exponent of k in (3.45) up to a contribution which vanishes for $\varepsilon = 0$. In (3.45) it was taken to be linear.

A detailed analysis of the UV divergences of the basic theory corresponding to the action (3.44) was performed in [99] following the general rules described in Sec. 1.4. According to the results of that study, in the general case of arbitrary n and d the model is not multiplicatively renormalizable owing to the appearance of counterterms of a new structure absent in the functional (3.44). Therefore, it cannot be subjected to the standard RG analysis (see Sec. 1.3). However, multiplicative renormalizability does occur for certan n and d, in particular, for $d = 3$ and any $n \geq 3$ except $n = 5$ (when an additional counterterm $\sim \theta'\theta'$ is required), and for $n = 3$, $d > 4$. Specific calculations using the standard quantum-field RG technique (Sec. 1.5) were performed in [99] only for the case $n = 3$, $d = 3$ in the one-loop approximation. The β functions of the charges g and u coincide with the expressions

in (3.12), and the new objects are the β function of the charge g' and the anomalous dimensions γ_{Φ} of the fields θ, θ', defined by the general expressions (1.33), which now [in contrast to the model (3.31)] are also renormalized. In [99] these were found to be (up to notation)

$$\beta_{g'} = g'[-2\alpha\varepsilon + 2ag + 3bg'/u^2], \tag{3.47}$$

$$\gamma_{\theta} = -\gamma_{\theta'} = -\frac{3g'}{2bu^2} \tag{3.48}$$

with the constant a from (1.37) and $b \equiv S_d/4(2\pi)^d$ with $d = 3$.

Among the set of fixed points of the system (3.12) and (3.47), only the two points with g_* and u_* from (3.21) and $g'_* = 0$ for the first and $g'_* = 2\varepsilon(\alpha - 2/3)u_*^2/3b$ for the second can be IR-attractive. For $\alpha < 2/3$ only the first point is IR-attractive, and for $\alpha > 2/3$ only the second one is. The critical dimensions of the fields φ, φ', and $\Delta_t = -\Delta_{\omega}$ remain unchanged (1.44), while for $F = \theta$, θ' they are determined according to the general rule (2.3), which gives [99]

$$\Delta_{\theta} = 1 - \varepsilon/3, \quad \Delta_{\theta'} = d - 1 + \varepsilon/3 \quad \text{for } \alpha < 2/3 \tag{3.49a}$$

and

$$\Delta_{\theta} = 1+(1/3-\alpha)\varepsilon, \quad \Delta_{\theta'} = d-1+(\alpha-1/3)\varepsilon \quad \text{for } \alpha > 2/3. \tag{3.49b}$$

In both cases corrections $\sim \varepsilon^2$, ε^3 and higher are not present. The equation $\Delta[\theta\theta'] = d$ in (3.28) remains valid, and so the author of [99] concludes that the four-thirds law of Richardson (see Sec. 3.1) is satisfied for the real value $\varepsilon = 2$ in their case of a chemically active admixture with $n = 3$, $d = 3$.

This brief discussion of the calculation of [99] requires some commentary. In our opinion, the result (3.49a) for $\alpha < 2/3$ is incorrect, because the author of [99] did not take into account the fact that in this model the case $g'_* = 0$ is special and the dimensions of θ and θ' cannot be calculated using the simple rule (2.3). The point is that the correlator $\langle\theta\theta\rangle$ contains an overall factor of g'. In RG representations of the type (1.47) this becomes

the corresponding invariant charge $\overline{g}'$, which tends to g'_* at the IR asymptote (see Sec. 1.6). If, as usual, $g'_* \neq 0$, this factor does not affect the critical dimensions. However, this is not so in the special case $g'_* = 0$, because then the factor $\overline{g}'_* \to 0$ produces an additional suppression at the IR asymptote and generates an addition to the critical dimensions which was not included in the general expression (2.3). It is very simple to correctly calculate the critical dimensions in this case. The problem of $g'_* = 0$ can be eliminated by a suitable scaling of the fields θ and θ' in (3.44). The factor $g'_0\nu_0^{\cdots}$ can thus be removed from the injection function (3.45), leaving only a power of the momentum, which will directly ensure the absence of renormalization of the field θ' and therefore also θ owing to the absence of the counterterm $\theta'\partial_t\theta$ assumed in [99]. Instead, we obtain a numerical coefficient (the charge) at the $\theta'\theta^n$ vertex while the other contributions in (3.44) remain unchanged. The critical dimensions of the fields θ and θ' are not changed in this procedure (because the scaling parameters have zero critical dimension), and for the canonical dimensions instead of (3.46) we obtain ($d = 3$)

$$d_\theta^\omega = -\frac{1}{2}, \quad d_\theta^k = 2 - \alpha\varepsilon, \quad d_\theta = 1 - \alpha\varepsilon, \quad d_{\theta'}^\omega = \frac{1}{2},$$

$$d_{\theta'}^k = d - 2 + \alpha\varepsilon, \qquad d_{\theta'} = d - 1 + \alpha\varepsilon. \tag{3.50}$$

Since in this scheme the fields θ and θ' are not renormalized (see above), the desired critical dimensions are determined by (2.3) (which now is certainly correct) with $\gamma_F^* = 0$ according to the canonical dimensions (3.50). This leads to the result (3.49*b*) for any $\alpha > 0$, in spite of the statement made in [99].

We should also mention the choice of α in (3.45). The value $\varepsilon = 2$ for massless injection (1.10) is assumed to be the real value on the basis of (2.75). The same will be true for the second injection function (3.45) in this problem only when the exponent of k in (3.45) becomes $-d$ at $\varepsilon = 2$. Then α is determined uniquely: $\alpha = (n+1)/2(n-1)$, in particular, $\alpha = 1$ for $n = 3$. We note that for $\alpha = 1$ and $\varepsilon = 2$ the dimensions (3.49*b*) coincide with the Kolmogorov values (1.44) for φ and φ'.

Let us conclude with a brief discussion of the recent paper [150]. There it was shown that the proof [99] that the counterterm $\theta'\partial_t\theta$ is absent is in fact incorrect, and that this counterterm actually appears in the two-loop approximation. However, this does not lead to any changes in the above discussion of the existence and stability of the fixed points. The result (3.49*b*) also remains exact for the fixed point with $g'_* = 0$, while for the point with $g'_* \neq 0$ it acquires a correction of order ε^2, which leads to a small numerical deviation from the 4/3 exponent in Richardson's law.

3.4 Anisotropic turbulence

One step toward the construction of a more realistic model of turbulence is the inclusion of the anisotropy present in most real turbulent flows. Anisotropic turbulent systems with distinguished direction $\mathbf{n}$ were first studied using the RG approach in [100]. The generalization to the case of anisotropic turbulence with a passive admixture was made in [101,133], and to the case of non-isotropic magnetic hydrodynamics in [102] (see Secs. 3.5 and 3.8).

The stochastic hydrodynamics of an anisotropic turbulent system is described by (1.1). The existence of a distinguished direction $\mathbf{n}$ is manifested only in the noise correlator (1.2), in which the transverse projector $P_{ij}(\mathbf{k})$ is replaced by the expression $c_1 P_{ij} + c_2 \widetilde{n}_i \widetilde{n}_j$ with transverse vector $\widetilde{n}_i = P_{ij} n_j$ and coefficients c_1, c_2 depending only on the angular variable $\cos^2\theta = (\mathbf{n}\cdot\mathbf{k})^2/k^2$. In [100] the anisotropy was assumed to be small, and the following expression was used as the correlator $D_{ij}^{\varphi\varphi} \equiv \langle F_i F_j \rangle$:

$$D_{ij}^{\varphi\varphi} = d_F(k)\{[1 + \rho_1(k_\parallel^2/k^2)]P_{ij} + \rho_2 \widetilde{n}_i \widetilde{n}_j\} \tag{3.51}$$

with the function d_F from (1.10) and the new constants $\rho_{1,2}$, the anisotropy parameters. In (3.51) and everywhere below we use the notation $k_\parallel^2 \equiv (\mathbf{n}\cdot\mathbf{k})^2$ and $\partial_\parallel^2 \equiv (\mathbf{n}\cdot\boldsymbol{\partial})^2$, and $\partial^2 \equiv \Delta$ is the Laplace operator. The parameters $\rho_{1,2}$ were assumed to be small in [100], and all the actual calculations were performed only