MAT327 Notes

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Contents

1 Topology and Topological Spaces

1. Metric Space: Let X be a set, a metric $d : X \times X \to \mathbb{R}_{\geq 0}$ such that
   - $\forall x, y \in X$, $d(x, y) = d(y, x)$
   - $\forall x, y \in X$ $d(x, y) = 0 \iff x = y$
   - $\forall x, y \in X$ $d(x, y) \leq d(x, z) + d(z, y)$
2. Examples
   - (Euclidean Metric) $d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$
   - (Taxicab Metric) $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ on $\mathbb{R}^n$
   - (Square Metric) $d_\infty(x, y) = max_{1 \leq i \leq n} |x_i - y_i|$
   - (Discrete Metric) $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$
   - (l^p-metirc) $d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$
3. Let (X, d_x) and (Y, d_y) be two metric spaces, $f : X \to Y$ is continuous iff
$$\forall x_0 \in X \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in X \; (d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon)$$
4. Open Ball:
$$B_d(x_0, r) = \{x \in X : d(x, x_0) < r\}$$
5. let (X, d) be metric space, define $U \subset X$ is open iff
$$U = \bigcup_{i \in I} B_d(x_i, r_i)$$
   Some examples
   - on $\mathbb{R}$, $(0, 1) = B_d(\frac{1}{2}, \frac{1}{2})$ is open
   - Any open ball is open
   - X , $\emptyset(= \bigcup_{i \in \emptyset} B(x_i, r_i))$ are open
   - To Show X is open, we fix $x \in X$ and show that $X = \bigcup_{r > 0} B(x, r)$ holds for all r.
6. Prop: let (X, d) be metric space, $U \subseteq X$ is open iff
$$\forall x \in U \, \exists r > 0 \, , \, B_d(x, r) \subseteq U$$
   Proof:
   - The only if ($\to$) is clear, following from the definition of open sets.
   - The if direction ($\leftarrow$), we fix x,
7. Let (X, d_x) and (Y, d_y) be two metric spaces, $f : X \to Y$ is continuous iff
$$\forall U \in \mathcal{T}_y \, , \, f^{-1}(U) \in \mathcal{T}_X$$
   Proof:
8. Topological spaces(Open sets in a metric space form a topology)

   Let (X, d) be metric space, $\mathcal{T}$ is a toplogoy on X(topological space) if
   - $\emptyset$ and X are open and in $\mathcal{T}$.

- U_i is open $\forall i \in I$ then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- For i = 1,2,...,n, U_i is open, then $\bigcap_{i\in I} U_i$ is open if $n < \infty$

  Counterexample
  In general the infinite intersection of open sets is not open, consider the case

  $$U_i = (-\frac{1}{i}, \frac{1}{i})$$

  where $\bigcap_{i\in I} U_i = \{0\}$ which is not open.

  More preciesly, $(X, \mathcal{T})$ is called topological space.

9. Open sets U, if $U \subseteq \mathcal{T}$
10. (Definition of Topology) Let X be a set. A topology on X is a collection Y of subsets of X satisfying:
    - $\emptyset$, $X \in \mathcal{T}$
    - If $U_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$ is open
    - If U_i is open, then $\bigcap_{i\in I} U_i \in Y$ for $i = 1, 2, ..., n$ (still only for finite intersection)

    Elements of $\mathcal{T}$ are callled open sets, where $(X, \mathcal{T})$ is a topological space
11. **Thoughts on the definition:**
    - $\mathcal{T}$ is a subset of X, equipped with specific requirements, $\mathcal{T}$ is a structure on the set X, then we have the topological space $(X, \mathcal{T})$.
12. Examples
    - Metric Topology
      $$\mathcal{T}_d = \{U \subseteq X : \forall x \in U \, \exists r > 0 \, B_d(x, r) \subseteq U\}$$
      (a) For $\mathbb{R}$ with usual metric, we get the usual topology on $\mathbb{R}$, where open sets are open intervals or union of open intervals.
      (b) $(X, \mathcal{T})$ is **metrizable** iff
      $$\exists d \,\text{on}\, X \;,\; \mathcal{T} = \mathcal{T}_d$$
    - Indiscrete Topology
      $$\mathcal{T}_{indiscrete} = \{\emptyset \,,\, X\}$$
    - Discrete Topology
      $$\mathcal{T}_{discrete} = \alpha(x)$$
      (all subsets of X)
      - $(X, \mathcal{T})$ is discrete, if $\forall x \in X$, $\{x\} \in \mathcal{T}$
      - Discrete Topology is metrizable by the discrete metric
        Proof
    - Co-finite Topology
      $$\mathcal{T}_{co-finite} = \{U \subseteq X : X \setminus U \,\text{is finite}\} \cup \{\emptyset\}$$
13. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X, if $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is coarser than $\mathcal{T}'$.(finer = bigger; coarser = smaller)
    - For any topology $\mathcal{T}$, $\mathcal{T}_{\text{indiscrete}} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\text{discrete}}$
    - $\mathcal{T}$ and $\mathcal{T}'$ are comparable if $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$

14. Let d and d' be two metrics, $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$ iff

$$\forall x \in X \, \forall \epsilon > 0 \; \exists \delta > 0 \, B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$$

Proof

15. Let d and d' be two metrics. If there is a constant $\alpha > 0$ such that for all $x, y \in X$,

$$d(x, y) \leq \alpha \cdot d'(x, y)$$

Then $\mathcal{T}_d \subset \mathcal{T}_{d'}$

16. $\mathcal{T}$ and $\mathcal{T}'$ are comparable iff

$$\mathcal{T} \subseteq \mathcal{T}' \text{ or } \mathcal{T}' \subseteq \mathcal{T}$$

In metric case, often compare (metric) topologies by using open balls.

17. A Template Problem(revisiting the properties of $\mathbb{R}$) Let $\mathbb{R}$ be the set of reals, decide which are topologies

- $\mathcal{T}_1$ consists of $\mathbb{R}$ and $\emptyset$ and every interval (a, b), for a and b any real numbers with $a < b$.

  $(1, 2) \cup (3, 4) \notin \mathcal{T}_1$; not closed under arbitrary union.
- $\mathcal{T}_2$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $(-r, r)$, for any positive reals.
- $\mathcal{T}_3$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $(-r, r)$, for any positive rationals.

  We can find a sequence of rationals, monotonically increasing and $(x_n) \to \sqrt{2}$, then

$$\bigcup_{i=1}^{n} (-x_i, x_i) = lim_{n\to\infty}(-x_n, x_n) = (-\sqrt{2}, \sqrt{2}) \notin \mathcal{T}_3$$

- $\mathcal{T}_4$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $[-r, r]$, for any positive rationals.

  The argument is similar to 3.
- $\mathcal{T}_5$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $(-r, r)$, for any positive irrationals.

  Similar to 3, we can find a sequence of irrationals $(x_n) \to 1$, then

$$\bigcup_{i=1}^{n} (-x_i, x_i) = lim_{n\to\infty}(-x_n, x_n) = (-1, 1) \notin \mathcal{T}_5$$

- $\mathcal{T}_6$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $[-r, r]$, for any positive irrationals.

  The argument is similar to 5.
- $\mathcal{T}_7$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $[-r, r)$, for any positive reals.

  We can take monotonic increasing sequence $(x_n) = 1 - \frac{1}{n} \uparrow 1$, then

$$\bigcup_{i=2}^{\infty} [-x_i, x_i) = (-1, 1) \notin \mathcal{T}_7$$

  The lower bound might not be attained.
- $\mathcal{T}_8$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $(-r, r]$, for any positive reals.

  The argument is similar to 7, where we can take $(x_n) = 1 + \frac{1}{n} \downarrow 1$, we come across the same issue, the upper-bound can't be attained.

- $\mathcal{T}_9$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $[-r, r]$ and every interval $(-r, r)$, for any positive reals.
- $\mathcal{T}_{10}$ consists of $\mathbb{R}$ and $\emptyset$ and every interval $[-n, n]$ and every interval $(-r, r)$, for positive integer n and positive reals r.

18. Since every element in $\mathcal{T}$ is open set, then to define a topology means to define open set.
19. Revisit Metric Space and metrizable
    - Distance in $\mathbb{R}^n$ is generalized from $\mathbb{R}$, where $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ define d(X,Y) as
    $$d(X, Y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$
    - Some Examples of Metric Space
        -
        $$d(x, y) = \begin{cases} 0 & \text{if} x = y \\ 1 & \text{if} x \neq y \end{cases}$$
        - $d_n(X, Y) = \sqrt[n]{\sum_{i=1}^{n} |x_i - y_i|^n}$
    - Metric Topology In metric space (X, d), we define the open ball $\mathcal{B} = \{x \in X : d(x, x_0) < r\}$
    - metrizable space is more a topological concept, focus on the properties of defining open sets.
    - If any singleton set is open set in some topological space, we can conclude the space is dicrete topology.

2 Basis for Topological Spaces

1. Basis for topology:
    - open balls are 'basic open sets'.
    - we can generate into arbitrary topological spaces, which provides a new way to show a set is open.

    Theorem 2.1. *In metric (X, α), $U \subseteq X$ is open iff*
    $$\forall x \in U \,,\, \exists r > 0 \; B(x, r) \subseteq U$$
2. **Thoughts on definitions:**
    - A set is open means the element which is contained in the topology(i.e. $\forall U \in \mathcal{T}$, U is open).
    - Defining an open set, we get a topology (broadly speaking).
    - Arbitrary union of open sets, $\emptyset$ and X are open, but only **finite** intersection of open set is open(countable/infinite is not enough). Take $U_n = (-\frac{1}{n}, \frac{1}{n})$
    - The complement of open set is **closed**, then we can get X and $\emptyset$ are both closed and open.
    - From above, we see that open and closed are not exclusions, but dual concepts.
3.

    Theorem 2.2. *Let X be a set and B be a collection of subsets of X satisfying:*
    - *($\mathcal{B}$ covers the set X)* $\forall x \in X \; \exists B \in \mathcal{B}$ *s.t.* $x \in B$
    - *(Intersection remains a basis)* $\forall B_1 \, B_2 \in \mathcal{B} \,,\, \forall x \in B_1 \cap B_2 \,,\, \exists B_3 \in \mathcal{B}$ *s.t.* $x \in B_3 \subseteq B_1 \cap \mathcal{B}_2$

    Then we can define $\mathcal{T}_{\mathcal{B}}$ as
    $$\mathcal{T}_{\mathcal{B}} = \{\bigcup B : B \subseteq \mathcal{B}\} \cup \{\emptyset\}$$
    Here we can say $\mathcal{T}_{\mathcal{B}}$ is generated by $\mathcal{B}$.

- ***Thoughts:*** *the union of basis of X must equal to X ($\bigcup_i B_i = X$), and we can always find a B that is the intersection of two other B's. If the Bs are non-disjoint, then similar to* ***Nested Interval Iheorem***, $|\mathcal{B}| = \infty$
- *If X is any set, then the collection of all one-point subsets of X is a base of $\mathcal{T}_{discrete}$*
- *(Show that a collection of set $\mathcal{B}$ is a basis of a given topology $\mathcal{T}$)*

(Step1:) *Show that $\mathcal{B}$ is a basis ((i) $\mathcal{B}$ covers X; (ii) The intersection of bases remians a basis).*

(Step2:) *Arbitrary open set $U \subseteq X$ and $\forall x \in U$, we can find a basis such that $x \in B_x \subseteq U$. (i.e. any open set in $\mathcal{T}$ is also open in the topology generated by et $\mathcal{B}$).*

4. **Thoughts on Basis of Topology**

   - Basis of a topology: A specific subset of $\mathcal{T}$ say $\mathcal{B}$, such that we can get(generate) all the elements in X from the elements in $\mathcal{B}$.
   - $\mathcal{B}$ is a basis if $\forall U \subseteq X$ and U is open, U can be written as the union of elements from $\mathcal{B}$.
   - An example that a subset of X is not a topology.
     Let X $= \{a, b, c\}$ and $\mathcal{B} = \{\{a\}, \{c\}, \{a, c\}, \{b, c\}\}$
     To see it's not a basis for any topology on X, we can assume it's a basis first and we get

     $$\mathcal{T} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$$

     But for the finite intersection $\{b\} = \{a, b\} \cap \{b, c\} \notin \mathcal{T}$, which shows it's not closed under finite intersection, then it's not a topology, then $\mathcal{B}$ is not a basis.
   - A set $\mathcal{B}$ is a basis of $\mathcal{T}$,
     - $\mathcal{B}$ can covers the entire set X
     - The intersection of elements from the basis can also be generated from the basis

     The reverse direction, remains sufficient to check the three properties for a subset of X to be a topology.
   - More neat to examine the property of a basis instead of the entire $\mathcal{T}$.

5. Some examples of bases of topology

   - X be a set, $\mathcal{B} = \{\{x\} : x \in X\}$, then $\mathcal{B}$ is a base on X and $\mathcal{T}_{\mathcal{B}}$ is Discrete Topology.
   - $\mathcal{A} = \{(a, \infty) \subseteq \mathbb{R} : a \in \mathbb{R}\}$ is a base for Upper-Ray Topology.
   - $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} : a < b\}$ is a base for Standard Topology on $\mathbb{R}$.
   - $\mathcal{B}_2 = \{B_\epsilon(x) \subseteq \mathbb{R}^2 : x \in \mathbb{R}^2$ and $\epsilon > 0\}$ is a base of Standard Topology on $\mathbb{R}^2$.
   - $\mathcal{T}$ itself on X is also a base.

6. A collection $\mathcal{B}$ of subsets of a set X satisfying the condition above is called a basis for a topology on X. Elements of $\mathcal{B}$ are called basic open sets. $\mathcal{T}_{\mathcal{B}}$ is called the topology generated by $\mathcal{B}$.

   Lemma 2.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases for topologies on X, the following are equivalent*

   - $\mathcal{T}_{\mathcal{B}_1} \subseteq \mathcal{T}_{\mathcal{B}_2}$
   - $\forall x \in X$, $\forall B_1 \in \mathcal{B}_1$, *($x \in B_1 \implies \exists B_2 \in \mathcal{B}_2$ s.t. $x \in B_2 \subseteq B_1$)*

   The second condition can be interpreted in this way, if x is in the base of the smaller topology, then we can find a base of the bigger topology that is a subset of the smaller base.

   Some Thoughts:

   - from here, we can get $\mathcal{T}_{\mathcal{B}} = \mathcal{T} \iff \mathcal{B}$ generates $\mathcal{T}$

   Lemma 2.2. *If $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}_{\mathcal{B}}$ is the collection of all unions of elements in $\mathcal{B}$.*

- But such expression for U is not unique.
- Thie lemma implies that every open set $U \subseteq X$, can be expressed as the union of some bases.

Lemma 2.3. *Let $(x, \mathcal{T})$ be a topological space. If $\mathcal{C}$ is a collection of subsets of X such that*

$$\forall U \in \mathcal{T}\, \forall x \in U\, \exists c \in \mathcal{C}\,,\, x \in c \subseteq U$$

Then $\mathcal{C}$ is a basis for $\mathcal{T}$ and $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$.

We still have to show $\mathcal{T}_{\mathcal{C}}$ is a topology on X.
where

$$\mathcal{T}_{\mathcal{C}} = \{U \subseteq X : \forall x \in U\,,\, \exists C \in \mathcal{C}\ s.t.\ x \in C \subseteq U\}$$

Proof:

- Clearly X and $\emptyset$ are in $\mathcal{T}_{\mathcal{C}}$.
- Let U_1 and U_2 be in $\mathcal{T}_{\mathcal{C}}$, then clearly $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{C}}$.
- To show arbitrary union is remains in the set.

7. Some examples
   - (Standard Topology) $\mathcal{B}$ is the collection of all the open intervals on the real-line (i.e. $\mathcal{B} = \bigcup(a, b)$).
   - (Lower Limit Topology $\mathbb{R}_l$) $\mathcal{B}$ is the collection of all the half-open intervals on the real-line (i.e. $\mathcal{B} = \bigcup[a, b)$).
   - (K - Topology $\mathbb{R}_K$) First define $K = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$, $\mathcal{B}''$ is the collection of form $(a, b) - K$.

   Lemma 2.4. *$\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than $\mathbb{R}_{std}$, but not comparable to each other.*

8. (Re-write Lemma 2.3) Let $(X, \mathcal{T})$ be topological space and $\mathcal{B}$ is a base of on X, then $\mathcal{B}$ generates X iff
   - $\mathcal{B} \subseteq \mathcal{T}$
   - $\forall U \in \mathcal{T}\,,\, x \in U,\, \exists B \in \mathcal{B}\ s.t.\ x \in B \subseteq U$

   Proof

   - (only if) Since $\mathcal{B}$ generates $\mathcal{T}$, then we have

     $$\mathcal{T}_{\mathcal{B}} = \mathcal{T}$$

     and

     $$\mathcal{B} \subseteq \mathcal{T}_{\mathcal{B}} = \mathcal{T}$$

     Let $U \in \mathcal{T}$ and $x \in U$, since $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$, we have

     $$C \subseteq \mathcal{B}\ s.t.\ U = \bigcup C$$

     then

     $$\exists x \in \mathcal{B}_x \in C\ s.t.\ \in \mathcal{B}_x \in \mathcal{C}$$

   - (if) Show the two-way inclusion
     - ($\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$) Since we have

       $$\mathcal{B} \subseteq \mathcal{T}$$

       and $\mathcal{T}$ is closed under arbitrary union, with $\mathcal{T}_{\mathcal{B}}$(collections of unions of elements from $\mathcal{B}$),we have

       $$\mathcal{T}_{\mathcal{B}} \subseteq \mathcal{T}$$

– ($\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$) Since $\mathcal{T}$ itself is a basis, by the assumption, we have

$$x \in \mathcal{T}_{\mathcal{T}} \subseteq \mathcal{B}$$

By lemma 2.1, we have

$$\mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$$

9. How to show $\mathcal{T}$ is a discrete topology on X(show any subset of X is open; or any subset of X is in $\mathcal{T}$).
   - $\forall S \subseteq X \implies S \in \mathcal{T}$
10. Revisit Euclidean Topology (Standard Topology)
    - Let $S \subseteq \mathbb{R}$, $\forall x \in S$, $\exists a < b$ $s.t.$ $x \in (a,b) \subseteq S$
    - In Euclidean Topology,
      - Open Set: (a,b), $(a,+\infty)$, $(-\infty,a)$, $(a,b) \cup (c,d)$
      - Close Set: $[c,d]$, $\{a\}$(Singleton Set), $\mathbb{Z}$, $(-\infty,a] \cup [b,+\infty)$(Complement of open set (a,b))
      - Neither close nor open: $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$
      - Close and open: $\mathbb{R}$, $\emptyset$
    - An open set can be written as a union of open sets
    - The union of all the open set is a basis of Euclidean Topology.
    - Show $D = \{(x,y) : x^2 + y^2 < 1\}$ is an open set in $\mathbb{R}^2$
      - To show the set is open in $\mathbb{R}^2$, remains sufficient to show that for any point in D, we can always find an open square $x \in (a,b) \times (c,d) \subseteq D$. After showing this, we can show $D = \bigcup D_x$, which is the union of open set.
    - Apart from open rectangle, the open circle D, is also a basis for Euclidean Topology on $\mathbb{R}^2$.
    - Moreover $\{(a,b) : a,b \in \mathbb{Q}\}$ form a basis of Euclidean Topology over $\mathbb{R}$.

3 LOTS, Product Topology and Subspace Topology

1. Check if two generates the same topology

   Lemma 3.1. *If ϕ is a collection of open subsets of X, with*

   $$\forall U \in \mathcal{T}, \forall x \in U, \exists C \in \phi \ s.t. \ x \in C \subseteq U$$

   Which is saying $\mathcal{T} \subseteq \mathcal{T}_\phi$ Then ϕ is a basis for topology on X and $\mathcal{T}_\phi = \mathcal{T}$

2. Examples of Topological Space
   - (Sargenfry Line) On $\mathbb{R}$, $\mathcal{B}_l = \{[a,b) : a,b \in \mathbb{R}$ and $a < b\}$, which is the basis for Lower Limit Topology($\mathcal{T}_l$) on $\mathbb{R}$
     - $\mathcal{T}_l$ is finer than $\mathcal{T}_{\text{usual}}$
     - $\mathbb{R}_l$ is not second countable, $\{[a,b] : a,b \in \mathbb{Q}\}$ generates a topology but not $\mathcal{T}_l$.
3. Let X be a set with simple order relation, assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types
   - All open intervals $(a,b) \in X$
   - All intervals of the form $[a_0,b)$, where a_0 is the smallest element (if exists) of X.
   - All intervals of the form $(a,b_0]$, where b_0 is the largest element of X.

   $\mathcal{B}$ is a basis of topology on X, called order topology.

4. **(***)** How to justify a basis $\mathcal{B}$ is a basis of the given topology $\mathcal{T}$
   - An example, on $\mathbb{R}$, $\{(a,b]\}$ form a basis of the topology (but not Euclidean Topology),to see why it's the case, (x_1, b) cannot be expressed as unions of $\{(a,b]\}$.
   - A lemma: Let $(X, \mathcal{T})$ is a topological space, $\mathcal{B}$ is a collection of the sets, $\mathcal{B}$ is a basis iff

$$\forall U\ ,\ \forall x \in U\ ,\ \exists B \in \mathcal{B}\ s.t.\ x \in B \subseteq U$$

     - $(\rightarrow)$ If $\mathcal{B}$ is a basis and U is open, then $U = \bigcup_i B_i$, clearly holds that $\forall x \in U$, where $x \in B_x = U = \bigcup_i B_i$
     - $(\leftarrow)$

   (the if direction checks if any set U is open under the topology)

5. **(***)** How to justify two bases generates the same topology $\mathcal{T}$
   - Two Topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent, if U is open in $\mathcal{T}_1$, then U is also open in $\mathcal{T}_2$; vice versa.

6. Rays
   - Open Rays $(a, +\infty)$, $(-\infty, a)$
   - Closed Rays $[a, +\infty)$, $(-\infty, a]$

   Open rays means open set in the order topology

7. Product Topology on $X \times Y$, has a basis $\mathcal{B}$ of all sets of the form $U \times V$, where U, V are open ssubsets of X and Y.

8.

   Theorem 3.1. *Suppose that for $i = 1, .., n$, (X_i, Y_i) is a topological space. Then*

$$\mathcal{B}_{prod} = \{\prod_{i=1}^{n} U_i : U_i \in \mathcal{T}_i\}$$

   is a basis for a topology on $\prod_{i=1}^{n} X_i$.

   Generalizing to infinite product will give a different topology.

9.

   Theorem 3.2. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

   is a basis for the topology of $X \times Y$

   Proof
   We want to show $\mathcal{D}$ is a basis for certain topology, we apply the lemma:

$$\text{Open sets } U\ ,\ \forall U \text{ and } \forall x \in U\ ,\ \exists D \in \mathcal{D}\ s.t.\ x \in D \subseteq U \implies \mathcal{D} \text{ is a basis}$$

   Let open set $W \subseteq X \times Y$ and a point $x \times y \in W$, then there is a basis element $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y, we can pick $B \in \mathcal{B}$ and $C \in \mathcal{C}$ satisfying $x \in B \subseteq U$ and $y \in C \subseteq V$.
   Then $x \times y \in B \times C \subseteq W$

10. (Subspace Topology) If $Y \subseteq X$ and $(X, \mathcal{T})$ be a topological space, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

   is a topology on Y.

11. Examples

    - $X = \mathbb{R}^2$ and $Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
    - The Basis of the open subsets of Y consists of arcs, since basis for $\mathbb{R}^2$ is open balls, that intersect an unit circle which gives the arc with open end points.

    Lemma 3.2. *If $\mathcal{B}$ is a basis for the topology of X then the collection*

    $$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

    is a basis for the subspace topology on Y.

12. The motivation behind subspace topology, is finding a smaller subset of the topology, that inherits all the properties of the original one.

    So if we want to define the open set in A, we can take an open subset in X, and intersect with A to define the open sets in A. (Refer to 10)

13.

    Theorem 3.3. *Let $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$, then $F \subseteq A$ is a closed set iff*

    $$\exists \text{ closed } C \subseteq X \text{ s.t. } F = C \cap A$$

14. **Notice:** Be careful when defining open/close sets, we define in the following ways

    - $U \in \mathcal{T}_Y \implies$ U is open in Y.
    - $U \in \mathcal{T}_X \implies$ U is open in X.

15. Prop: If Y is open in X, then every open subsets in Y is open in X.
    Proof:
    If V is open in Y, then $V = U \cap Y$ for some U open in X. Then V is also open in X.

    Lemma 3.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.($U \in \mathcal{T}_Y$ and $Y \in T_X \implies U \in \mathcal{T}_X$)*

16.

    Theorem 3.4. *If A is a subsapce of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

17. Partial order on a set, we define $< X, \leq >$ as a partially ordered set.

    Moreover, if any two elements in the partially ordered set are comparable, either $a \leq b$ or $b \leq a$, then we call the relation as a total order set.

18. (Linearly Ordered Topological Space) Let X be a set. A linear order on X is a binary relation satisfying:

    - (irreflexivity) $\forall x \in X$, $x \leq x$.
    - (comparability) $\forall x, y \in X$, $x = y$ or $x < y$ or $x > y$.
    - (Transitivity) $a \leq b$ and $b \leq c \implies a \leq c$

19. Examples

    - $(\mathbb{Z}, <)$ Every Singleton is open(i.e. $(\mathbb{Z}, <)$ is discrete topology)
    - $(\mathbb{R}, <)$ Usual Topology
    - $(\mathbb{N}, <)$ Since $\{0\} = (-\infty, 1)_<$, $(\mathbb{N}, <)$ is discrete topology

- ($\mathbb{N}$, $<_0$) [0 is the max(i.e. we arrange $\mathbb{N}$ as $1\,2\,3\ldots 0$)]where $n <_0 m$ iff ($n \neq 0$ and $m = 0$) or ($n, m \neq 0$ and $n < m$). $\{0\}$ is not open
- ($\mathbb{N}$, $<_1$) arrange the evens before the odds(i.e. $0\,2\,..\,1\,3\ldots$). $\{1\}$ not open, then not discrete.
- (Dictionary Order) order entry-by-entry.

20. **Theorem 3.5.** *If $(X, C)$is linearly ordered set with more than one point, then the collection $\mathcal{B}_<$ consisting of sets of the form*

    - $(a, b)_< = \{x \in X : a < x < b\}$
    - $(-\infty, b)_< = \{x \in X : x < b\}$
    - $(a, +\infty)_< = \{x \in X : a < x\}$

    is a basis for a topology on X.
    Notice: *The intersection of two is the remianing open set.*

21. A linearly ordered $(X, <)$ is well-ordered iff every nonempty subsets $S \subseteq X$ attains a minimum.

22. Examples

    - There are countable well-ordered sets, and also uncountable well-ordered sets in particular

      Theorem 3.6. *There exists a (unique) countable well-ordered set callled ω_1 with the following properties*

      - $\forall \alpha \in \omega_1$, $Pred(\alpha) = \{\beta \in \omega_1 : \beta < \alpha\}$ *is countable*
    - Elements of ω_1 are called countable ordinals.

23. Revisit Order Topology

    - Basis topology is generated by
      - $(a, b) \quad \forall a, b \in X$
      - $[a_0, b) \quad \forall b \in X$ and a_0 is the minimum if exists
      - $(a, b_0] \quad \forall a \in X$ and b_0 is the maximum if exists
    - Some examples
      - Standard topology on $\mathbb{R}$ is identical to the order topology on $\mathbb{R}$
      - On $\mathbb{N}$, the open sets are (a, b), $\{n\} = (n-1, n+1)$, $[0, 1) = \{0\}$, since every singleton set is open, it's the same as discrete topology
      - $\mathbb{R} \times \mathbb{R}$, with the dictionary order,

24. To Compare two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, if for all x, there exists a basis element of $\mathcal{B}_1$ of $\mathcal{T}_1$, such that we can find $B_2(x) \subseteq B_1(x) \subseteq E$, then we can say $\mathcal{T}_1$ is finer than $\mathcal{T}_2$

25. We first have to define metric or the space has to be metrizable, then we can talk about bounded or not of a set.

4 Closed Sets, Closure and Interior

1. (Closed Set) A is closed if $X \setminus A$ is open.

2. Examples

    - In $\mathbb{R}^2$, the set
      $$\{x \times y : x \geq 0 \text{ and } y \geq 0\}$$
      is closed since $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (b, +\infty)$ are both open in $\mathbb{R}^2$.

- In Discrete Topology, every set is closed

  (Since every subet of X is open, taking $X \setminus S$ where $S \subseteq X$ and S is open, which shows $\forall S \subseteq X$, S is both closed and open)
- In Indiscrete Topology, the only closed sets are $\emptyset$ and X.
- In $(X, \mathcal{T}_{\text{co-countable}})$, $A \subseteq X$ is closed iff A is countable.
- Let X be topological space, $Y \subseteq X$, $E \subseteq Y$ is closed in Y iff $E = C \cap Y$ for some $C \subseteq X$ is closed. (Theorem 4.2)
- In $\mathbb{R}_l$, let $a < b$, $[a, b]$ and $(-\infty, b]$ are both closed.

3. Closed and open are dual instead of opposite, compared to doors.

4. **Theorem 4.1.** *Let $(X, \mathcal{T})$ be topological space, the following hold:*

   - *X and $\emptyset$ are closed*
   - *Arbitrary intersections of closed sets are closed. ($\bigcap_{i \in I} A_i$ is closed if A_i is closed)*
   - *Finite unions of closed sets are closed*

   Closed set is closed under arbitrary intersection, but open set is open under arbitrary union. With the similar counterexample, take

   $$\bigcup_n A_n = \bigcup_n \left(-\frac{1}{n}, +\frac{1}{n}\right) = \{0\}$$

   Proof: By De Morgan's law:

   $$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i)$$

   If A_i is closed for all i, then $X \setminus A_i$ is open, then the arbitrary union of open set is open, then $\bigcap_{i \in I} A_i$ is closed.

5. Recall that $U \subseteq X$ is open iff $\forall x \in U$, $\exists r > 0$ *s.t.* $B_d(x, r) \subseteq U$
   A set $C \subseteq X$ is closed the following are equivalent:

   - $X \setminus C$ is open
   - $\forall x \in X$, $(x \notin C \implies \exists r > 0 \textit{ s.t. } B_d(x, r) \subseteq X \setminus C)$
   - $\forall x \in X$, $(\forall r > 0,\ B_d(x, r) \cap C \neq \emptyset \implies x \in C\ )$

6. To generalize the statements above

   Lemma 4.1. *Let X be topological space, $A \subseteq X$ is open iff*

   $$\forall x \in A\ \exists U \textit{ open s.t. } x \in U \subseteq A$$

   The second statement is saying $A \subseteq \bigcup_{U \subseteq A \text{ open}} U$.

7. To generalize more, an open set that contains x is called **open neighbourhood** of x.

   This motivates: Let $A \subseteq X$, $x \in X$ is a closure point of A iff

   $$\forall U \subseteq X \textit{ s.t. } x \in U\,,\, U \cap A \neq \emptyset$$

   where U is open in X.

   Or we say V is an open neighbourhood of x, if $\exists$ open U *s.t.* $x \in U \subseteq V$

8. All closure points of A is the closure of A, denote $Cl_{(X,\mathcal{T})}(A)$. (To check a point not in closure, sufficient to find an open set contains x and it's disjoint with the set).

9. **Theorem 4.2.** *Let $f : X \to Y$, f is continuous iff $f^{-1}(E)$ is closed in X, for all $E \subseteq Y$ closed in Y.*

10. **Theorem 4.3.** *Let Y be a subspace of X, then A is closed in Y iff A equals the intersection of a closed set of X with Y*

11. **Theorem 4.4.** *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

12. Let A is a subset of topological space X,
    - (Interior) The union of all open sets contained in A. ($IntA = \bigcup_i U_i$)
    - (Closure) The intersection of all closed sets contained in A.
      ($\overline{A} = \{C_1 \cap ... \cap C_n : n \geq 2 \text{ and } C_i \subseteq A\}$)
      Cl(A) is always closed
    - IntA $\subseteq$ A $\subseteq$ $\overline{A}$

13. **Theorem 4.5.** *Let X be topological space and $A \subseteq X$,*
    - *$Cl_X(A)$ is the smallest closed set that contains A.*
    - *A is closed iff*
      $$Cl_X(A) = A$$
      Proof:
      - *Assume A is closed, we have $A \subseteq Cl_X(A)$, we remians to show $Cl_X(A) \subseteq A$, let $x \in X \setminus A$, then x is in an open set, then $x \notin Cl(A)$, then $x \in X \setminus Cl(A)$, then $X \setminus A \subseteq X \setminus Cl(A)$, which is equivalent saying $Cl(A) \subseteq A$.*
      - *Assume $A = Cl(A)$, we show it by showing $X \setminus A$ is open, let $x \in X \setminus A = X \setminus Cl(A)$, since x is not in the closure of A, we can pick an open set U contained x such that $U_x \cap A = \emptyset \subseteq X \setminus A$, then for any x in $X \setminus A$, we can found U_x which is open in $X \setminus A$, then A is closed.*
    - *If $\mathcal{B}$ is a basis for X, then*
      $$Cl_X(A) = \{x \in X : \forall B \in \mathcal{B}\ (x \in B \implies B \cap A \neq \emptyset)\}$$

14. (***) To justify $x \in Cl_X(A)$ where $A \subseteq X$, we can use the fact that
    $$x \in Cl(A) \iff \forall U \subseteq X \text{ open neighbourhood around x },\ U \cap A \neq \emptyset$$

15. **Theorem 4.6.** *Let Y be a subspace of X, A be a subset of Y, $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.*

16. (Density) Let $(X, \mathcal{T})$ be a topological space. $D \subseteq X$ is dense if $Cl(D) = X$.
    (We can say D is dense if X is the closure of D.)
    Since we know $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $\mathbb{Q} \subseteq \mathbb{R}$, then $Cl(\mathbb{Q}) = \mathbb{R}$.
    The followings are equivalent,

- D is dense
- For every non-empty open $U \subseteq X$, $D \cap U \neq \emptyset$.
  Proof:

  - Suppose not the case, then $X \setminus Cl(D)$ is open, then $D \cap (X \setminus Cl(D)) \neq \emptyset$, since $D \subseteq Cl(D)$, then
  $$D \cap (X \setminus Cl(D)) \subseteq D \cap (X \setminus D) = \emptyset$$
  which leads to a contradiction, then $Cl(D) = X$
  - We can see that $\mathbb{Q} \subseteq \mathbb{R}$ is dense, but $\mathbb{Z} \subseteq \mathbb{R}$ is not dense (no integers between $(n, n+1)$)
  - Try to understand $D \subseteq X$ is dense as, no matter how small the subset of X is, there always part of it in D.

17. Ways to show $D \subseteq X$ is dense:

    - From definition, $Cl(D) = X$, if we have basis of X, we can try to show $B \in \mathcal{B} \cap D \neq \emptyset$
    - By the iff statement, take arbitrary nonempty open subtset $U \subseteq X$, show that $U \cap D \neq \emptyset$

18. A topological space $(X, \mathcal{T})$ is **separable** if
    $$\exists U \subseteq X \ s.t. \text{ U is countable , and } Cl(U) = X$$
    Examples

    - $\mathbb{R}^n_{\text{usual}}$ is separable, take $\mathbb{Q}^n$ as the dense set, every open set $U \subseteq \mathbb{R}^n$ contains a point in $\mathbb{Q}^n$.
    - (Lower Limit Topology $\mathbb{R}_l$) take the dense set $\mathbb{Q}$, for every $[a, b)$, we can find a rational that is included in the interval. So $\mathbb{R}_l$ is separable

19. A topological space $(X, \mathcal{T})$ is called **second countable** if
    $$\exists \mathcal{B} \ s.t. \ \mathcal{B} \text{ is a basis and it's countable}$$
    To show the lower-limit topology is not second countable,

    - Suppose not the case, where we construct a countable basis $\mathcal{B} = \{B_1, ..., B_n\}$
    - Consider the set $\{[x, x+1)\} \subseteq \mathbb{R}_l$, which is open in the topology, then $\exists B_{n_x} \in \mathcal{B} \ s.t. \ x \in B_{n_x} \subseteq [x, x+1)$
    - Clearly, for $x \neq y$, we have $[x, x+1) \cap [y, y+1) = \emptyset$, so we have $x \neq y \implies B_{n_x} \cap B_{n_y} = \emptyset$
    - Construct the index function $f : \mathbb{R} \to \mathbb{N}$, defined as $f(x) = n_x$, from above, we showed f is injective, which means $|\mathbb{N}| = |\mathbb{R}|$, leads to contradiction.

    A recap of the proof, to show 'second countable', we must have countable basis for the topology, then the index function has to be injective, which means $f : A \to \mathbb{N}$ is injective, i.e. $|\mathbb{N}| = |A|$.

20.

    Theorem 4.7. *Every second countable space is separable*

    Proof:

    - let $\mathcal{B}$ be a countable basis for X. For each basis $B \in \mathcal{B}$, pick $x_B \in B$, let $D = \{x_B : B \in \mathcal{B}\}$, which is countable.
    - Claim $Cl(D) = X$
      We will use (for every nonempty open subset U of X, $D \cap U \neq \emptyset$ is equivalent to D is dense in X)

      Let $U \subseteq X$ be open, then $U = \bigcup_{i \in I} B_i$, Since U is not empty, $\exists x_k \in B_k \subseteq U$ with $B_k \neq \emptyset$, clearly $x_k \in D$, therefore
      $$x_k \in D \cap U$$

But the converse is not true ($\mathbb{R}_l$ is separable but not second countable)

- In $\mathbb{R}_l$, $Cl_{\mathbb{R}_l}((a,b)) = [a,b)$.
- In (X,d) metric space, the set $\overline{B_d}(x_0,r) = \{x \in X : d(x,x_0) \leq r\}$ is closed. Moreover, $Cl_X(B_d(x_0,r)) = \overline{B_d}(x_0,r)$.

21. **Theorem 4.8.** *In (X,d) metric space, X is separable iff X is second countable*

22. Let A be any set in $\mathbb{R}^n$, then the interior of A, defined as

$$IntA = A^0 = \{x \in \mathbb{R}^n : \text{x is an interior point of A}\}$$

$$= \{x \in \mathbb{R}^n : \exists r > 0 \; s.t. \; B_d(x,r) \subseteq A\}$$

is open.
Proof:
Let $x \in A^0$, by definition of interior, $\exists r > 0 \; s.t. \; B_d(x,r) \subseteq A$.
It remains to show that $\forall y \in B_d(x,r), \; y \in A^0$
Take $r_0 = min(||x-y||, r - ||x-y||)$, then $B_d(y,r_0) \subseteq B_d(x,r) \subseteq A$
Since for any $x \in A^0$, we can find open set $B_d(x,r) \subseteq A^0$, then A^0 is open.

23. (Limit Point) $\forall U_p$, the nieghbourhood of p, $A \setminus \{p\} \subseteq U_p$.

24. A set A is closed if every limit point is contained in A.

25. Examples of Closed Set
   - $\{\frac{1}{n} : n \in \mathbb{N}_+\} \cup \{0\}$ is a closed set, since the limit point 0 is included.
   - $\{a\}$, singleton set is closed, since the limit point is empty set.
   - $\{1, 2, 3..., 10^{10}, .., n\}$, where $n < \infty$, is also closed set.
   - $\mathbb{R}^n$ is a closed set, since every limit point in $\mathbb{R}^n$ is also contained in $\mathbb{R}^n$. (is close and open)

26. Since we know in metric space, second countable iff separable
   - $\mathbb{R}_l$ is not second countable but separable, then the lower limit topology is not metrizable.
   - $S(I)$ is not second countable but separable, then $S(I)$ is not metrizable.

27. Some properties of Closure
   - $Cl(A) = Cl(Cl(A))$
   - $A \subseteq B \implies Cl(A) \subseteq Cl(B)$
   - $Cl(A \cup B) = Cl(A) \cup Cl(B)$
   - $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$

28. Revisit Separable
   - dense subset is countable, then the space is separable. In other words, for any point in the set, we can define a squence within the set such that it converges to the point.
   - A topological space is separable requires the space neither too fine nor too coarser. Using 'Countable' to control the number of open sets within the space, using 'Separable' to control or increase the number of open sets within the space.
   - If we want to examine limit within the topological space, we might want to refine the properties to C_1, C_2.
   -

Theorem 4.9. $C_2 \implies$ *Separable*

Proof:

Let $\mathcal{B} = \{B_n\}$ be a countable basis of X, fix every $x_n \in B_n$, then we take $A = \{x_n : n = 1, .., n\}$

Claim: $Cl(A) = X$

Proof:

$$\forall x \in X \ , \forall U(X) \ U(x) \cap A \neq \emptyset$$

Then exists $B_k(x) \subseteq U(x)$ such that

$$x_k \in B_k(x) \cap A \subseteq U(x)$$

Then we have

$$x_k \in U(x) \cap A$$

- Take the contrapositive, we have

$$\text{not } C_2 \implies \text{not Metrizable}$$

- Metrizable Space we have $C_2 \iff$ Separable
- To show a topological space is C_2,
  - The chosen collection of sets $\mathcal{B}$ is a basis
    Follow definition:
    * $\forall x \in X$, we can find a basis $B_x \in \mathcal{B}$ such that

      $$x \in B_x \subseteq X$$

    * for any B_1 and B_2, we can find $B_3 = B_1 \cap B_2 \in \mathcal{B}$

    Alternative:
    Show that for any open set $U \subseteq X$, and $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$
  - $\mathcal{B}$ is countable and open in X

5 Continuous Functions, Homeomorphism

1. Recall Continuous Function, Let $f : X \to Y$, if for any open set $U \subseteq Y$, we have

   $$f^{-1}(U) \subseteq X \text{ is open in X}$$

   Then we say f is continuous.

2. Some properties of f^{-1}
   - $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
   - $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
   - $f^{-1}(A^c) = (f^{-1}(A))^c$

3. Let f be continuous, iff
   - If U is open in Y, then $f^{-1}(U) \subseteq X$ is open in X
   - If C is closed in Y, then $f^{-1}(C) \subseteq X$ is closed in X
   - $\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

   The three statements are all equivalent.

4. Some Examples

   - (Identity) $id : X \to X$, where $x \mapsto x$ is continuous

     Let $U \subseteq X$ be open, then $id^{-1}(U) = U \subseteq X$ is open, then it's continuous.
   - (Constant) $f : X \to Y$, where $x \mapsto y_0 \in Y$ is continuous

     Let $U \subseteq Y$ be open, we have two cases to discuss,
     - $y_0 \in U$, $f^{-1}(y_0) = X$, which is open
     - $y_0 \notin U$, $f^{-1}(y_0) = \emptyset$, which is open
   - Let X be $\mathcal{T}_{\text{discrete}}$ and Y be $\mathcal{T}_{\text{trivial}}$, then

     $$f : X \to Y$$

     is continuous
     Proof:
     - Let X be discrete topology, then any $U \subseteq X$ is open, then for any $F \subseteq Y$ be open, $f^{-1}(F) \subseteq X$ is open.
     - Let Y be trivial topology, then $f^{-1}(Y) = X$ or $f^{-1}(\emptyset) = \emptyset$ open.

5. According to Munkres, we can consider number of sets contained in the topology as the roughness or fineness of a topology, for example, $\mathcal{T}_{\text{trivial}} = \{\emptyset, X\}$ can be regarded as the most rough topology, (or thinking it as a truck carrying only the entire stone as a whole), for $\mathcal{T}_{\text{discrete}}$, it's the union of singleton set, which are open, (can be considered as the smallest dust of stones), which is the finest topology.

6. If we want to make $f : X \to Y$ discontinuous, given f is continuous, we can make

   - X to be defined on a smaller topology
   - Y to be defined on a bigger topology

7. Composition of continuous functions is continuous
   (i.e. f,g are continuous, then $f\, o\, g$ is continuous)

8. - (Closed Map) $C \subseteq X$ is closed, then $f(C) \subseteq Y$ is closed.
   - (Open Map) $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.

9. The inverse of a continuous function not always continuous

   Take $f : X \to X$, where $x \mapsto x$, X defined on discrete topology and Y defined on trivial topology.
   Consider $f^{-1} : X \to X$, from trivial topology to discrete topology, but f^{-1} not continuous.
   Since $F \subseteq X$ be open in X and $F \neq \emptyset$ and $F \neq X$, the only open set in trivial topology are $\emptyset$ and X, then $f^{-1}(F) = F$ not open in trivial topology, then f^{-1} not continuous.

10. (**Homeomorphism**) Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological space, and $f : X \to Y$ satisfies

    - f is bijective
    - f is continuous
    - f^{-1} is continuous

    Then we say $f : X \to Y$ is a homeomorphism function, X and Y are homeomorphic, denote as $X \simeq Y$

11. Homeomorphism is an equivalent relation,

    - $X \simeq Y$
    - $X \simeq Y \implies Y \simeq X$

- $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$

12. Continuous funtion not necessary send open set to open set, we need bijection and the continuity of inverse.

    Consider $f : \mathbb{R} \to \mathbb{R}$, where $f(\mathbb{R}) = y_0$, which is continuous; but for any open set U, $f(U) = \{y_0\}$ which is not open, then not a homeomorphism.

13. An embedding is a function $f : X \to Y$ such that
$$f : X \to f(x)$$
    is a homeomorphism. Denote as $X \hookrightarrow Y$.

    In orther words, X is homeomrphic to a subspace of Y.

14. (***) To justify a given $f : X \to Y$ is an embedding
    - Show f is injective
    - Show f is continuous
    - Show $f^{-1} : f(X) \to X$ is continuous

15. Some Examples
    - $f : \mathbb{R}_l \to \mathbb{R}$, where $f(x) = x$, f is continuous but f^{-1} is not continuous.
      Let $U = [0, 1)$ open in $\mathbb{R}_l$, then
$$f^{-1}(f^{-1}(U)) = U = [0, 1)$$
      which is not open in $\mathbb{R}$.
    - $\mathbb{R} \hookrightarrow \mathbb{R}^2$, where $f(x) = (x, 0)$
    - $\mathbb{R}$ is homeomorphic to S^1 minus one point

16.

    Theorem 5.1. *Let X, Y, Z be topological spaces,*

    - *constant functions are continuous*
    - $i : A \to X$, *where* $A \subseteq X$ *and* $i(x) = x$ *is continuous (identity map with restricting the domain into the subspace)*
    - *Composition of two continuous functions is continuous*
    - *If* $f : X \to Y$ *be continuous and* $A \subseteq X$, *then* $f|_A : A \to Y$ *is continuous.*
    - *If* $f : X \to Y$ *is continuous, then extending or restricting the domain gives continuous functions.*
    - *If* $X = \bigcup_{\alpha \in \Gamma} U_\alpha$, *where* U_α *is open in X and* $f : X \to Y$ *is s.t.* $f|_{U_\alpha} : U_\alpha \to Y$ *is continuous, then* $f : X \to Y$ *is continuous.*
    - *If* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *are continuous, then* $f + g$ *and* $f \cdot g$ *are continuous and also* f/g *is continuous if* $g(x) \neq 0\ \forall x \in X$.
    - *The pojection maps* $\Pi_1 : X \times Y \to X$ *and* $\pi_2 : X \times Y \to Y$, *given by* $\pi_1(x, y) = x$ *and* $\pi_2(x, y) = y$
    - $f : X \to Y \times Z$ *is continuous iff* $\pi_1 o f$ *and* $\pi_2 o f$ *are continuous* $f = (f_1, f_2)$.

17. (**Pasting Lemma**)

    Lemma 5.1. *Suppose* $X = A \cup B$, *where A, B are closed in X.*
    Let $f : A \to Y$ *and* $g : B \to Y$ *be continuous functions.*
    Assume that $f(x) = g(x)$ *for all* $x \in A \cap B$, *then* $h : X \to Y$ *defined by* $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

We can write it as the following

$$\{A_1, ..., A_n\} \text{ is a closed set that covers X, If } f : A_i \to Y \text{ is continuous } \implies f \text{ is continuous}$$

Proof:
Let $F \subseteq Y$ be closed, WTS: $f^{-1}(F) \subseteq X$ is closed.
$f^{-1}(F) = \bigcup_{i=1}^{n}(f^{-1}(F) \cap A_i)$, since each A_i is closed in X.

We want to show $f^{-1}(F) \cap A_i$ is closed for $i = 1, .., n$
Notice $f^{-1}(F) \cap A_i = (f|_{A_i})^{-1}(F) \subseteq A_i$ is closed in X, since $f|_{A_i}$ is continuous, then $(f|_{A_i})^{-1}(F)$ is closed. Then

$$f^{-1}(F) \cap A_i$$

is closed.

18. If each A_i is open, the pasting lemma still works.

19. Some Examples about subspace topology and homeomorphism

    - $(a, b) \subseteq \mathbb{R}$, claim: $(a, b) \simeq \mathbb{R}_{\text{standard topology}}$
      Proof:
      We use the transitivity of homeomorphism
      - Claim: $(-\frac{\pi}{2}, +\frac{\pi}{2}) \simeq \mathbb{R}$ Take $f : (-\frac{\pi}{2}, +\frac{\pi}{2}) \to \mathbb{R}$, where $f(x) = tan(x)$ on $(-\frac{\pi}{2}, +\frac{\pi}{2})$
      - Claim: $(a, b) \simeq (-\frac{\pi}{2}, +\frac{\pi}{2})$ Take $f : (a, b) \to (-\frac{\pi}{2}, +\frac{\pi}{2})$, where $f(x) = \frac{\pi \cdot (2x-a-b)}{2 \cdot (b-a)}$

      Then we get

      $$(a, b) \simeq (-\frac{\pi}{2}, +\frac{\pi}{2}) \simeq \mathbb{R}$$

    - In $\mathbb{R}^n$,
      - (Disk) $D^n = \{x \in \mathbb{R}^n : ||x|| \leq r\}$
      - (Open Ball) $(D^n)^0 = \{x \in \mathbb{R}^n : ||x|| < r\} \subseteq \mathbb{R}^n$
      - (Sphere) $\partial D^n = \{x \in \mathbb{R}^n : ||x|| = r\}$, where we can define $S^1 = \partial D^2$ and $S^2 = \partial D^3$ and $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : ||x|| = r\}$
    - $S^1 = \{(x, y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ and X (the square centred at origin with side length 1/2) are homeomorphic
      Proof:
      Let $f : X \to S^1$, where $x \mapsto \frac{x}{||x||}$

20. (Sorgenfrey Line) On $\mathbb{R}$, $\mathcal{B} = \{[a, b) : a < b\,,\ a, b \in \mathbb{R}\}$
    Some properties of such topological space

    - $(\mathbb{R}_l, \mathcal{T})$ is coarser than $(\mathbb{R}, \mathcal{T}_{\text{standard}})$
      Proof: $[a, b)$ is open in $\mathbb{R}_l$, but not open in $\mathbb{R}$; $(a, b) = \bigcup_{n=1}^{+\infty}[a + \frac{1}{n}, b)$ which shows it's open in $\mathbb{R}_l$
    - $[a, b)$ is closed in $\mathbb{R}_l$ as well.
      Proof: Consider $[a, b)^c = (-\infty, a) \cup [b, +\infty) = (\bigcup_{n=1}^{+\infty}[-n, a)) \cup (\bigcup_{n=1}^{+\infty}[b, n))$, which is open in $\mathbb{R}_l$, then $[a, b)$ is closed in $\mathbb{R}_l$, Therefore it's clopen in $\mathbb{R}_l$.
    - $\mathbb{R}_l$ is separable
      Proof: Claim: $Cl(\mathbb{Q}) = \mathbb{R}_l$
      Let arbitrary $[a, b)$, $\exists \delta > 0$ such that

      $$\exists q \in \mathbb{Q} \ s.t. \ q \in (a + \delta, b) \subseteq [a, b)$$

      By the lemma that $Cl(Q) \subseteq \mathbb{R}$, we showed $\mathbb{R}_l$ is separable.

- $\mathbb{R}_l$ not second-countable
  Proof:
  Let $[x, y)$, $\exists \delta_x \ s.t. \ x < \delta_x < y \ s.t. \ x \in [x, \delta_x) \subseteq [x, y)$ so that $[x, \delta_x) \in \mathcal{B}$. But since x goes through all the reals, $\mathcal{B}$ at least contains uncountable sets.
- $\mathbb{R}_l$ is not metrizable **Proof:** $\mathbb{R}_l$ is metrizable but not C_2, then it's not metrizable.

21. In $\mathbb{R}^n$, we have $\overline{B(x_0, r)} = \overline{B} := \{x \in X : d(x, x_0) \le r\}$
    Not true in general, take $X = \mathbb{Z}$ with $\mathcal{T}_{\text{discrete}}$, where $B(0, 1) = \{0\}$, but $\overline{B}(0, 1) = \mathbb{Z}$, which is not true in general.

6 Sequence, C_1

1. Three Methods to show $f : X \to Y$ is continuous
   - (Pre-Image) If $f^{-1}(U)$ is open/closed in Y $\implies U \subseteq X$ is open/closed.
   - (Closure Sets) If $\forall A \subseteq X$, we have
     $$f(\overline{A}) \subseteq \overline{f(A)}$$
   - (Sequential Limit) If $\forall \{x_n\} \to x$ and $\{x_n\} \subseteq X$, then
     $$f(x_n) \to f(x)$$
     X metrizable is not required, it works for any topological space.
2. Goal: For what kind of Topological Space, we can have the statement (f is cont iff $\forall x \in X$, if x_n converges to x, then $f(x_n) \to f(x)$)
3. Let X be topological space, a sequence in X is $f : \mathbb{N} \to X$, defined as $f(n) = x_n$. We say $x_n \to x$ iff
   $$\forall \text{ open } U \ s.t. \ x \in U \, , \exists N \in \mathbb{N} \ \forall n > N \, , x_n \in U$$
   The tail is contained in the open set.
4. Remark: if (X, d) is metrizable, then $x_n \to x$ iff
   $$\forall \epsilon > 0 \, , \exists N \in \mathbb{N} \ s.t. \ \forall n > N \, , x_n \in B_d(x, \epsilon)$$
5. **Theorem 6.1.** *Let (X, d) be metric space and Y be any topological space. Then*
   $$f : X \to Y \text{ is continuous } \iff \forall x \in X \ (x_n \to x \implies f(x_n) \to f(x))$$
6. (Limit Point) x is a limit point of $A \subseteq X$if
   $$\forall U_x \, , U_x \cap (A \setminus \{x\}) \ne \emptyset$$
   In other words, x is a limit point of A if $x \in Cl(A - \{x\})$.
7. **Theorem 6.2.** *Let A be a subset of the topological space X, A' be the set of all the limit points of A. Then*
   $$\overline{A} = A \cup A'$$
   Recall that $Cl(A) = int_X(A) \cup \partial A$.

   Proof:
   pass

8. **Lemma 6.1.** *Let X, Y be topological space.*
   Assume that for all $A \subseteq X$, $\overline{A} = \{x \in X : \exists \{x_n\}_{n=1}^{\infty} \subseteq A \ , \ x_n \to x\}$

9. Let X be a topological space, and x in X. A local basis at x is a collection $\mathcal{B}$ of open sets containing x such that:
   for every oepn $U \subseteq X$, (if $x \in U$) $\to$ ($\exists B \in \mathcal{B}_x \ s.t. \ x \in B \subseteq U$)

   We say that X is first countable iff every $x \in X$ has a countable local basis.

10. Some Examples
    - Metric Spaces are first countable, since $\{B_d(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is countable local basis at x.
    - Second countable are first countable
      Proof: $\mathcal{B}$ is a countable basis of X, for all $x \in X$, consider $B_x = \{B \in \mathcal{B} : x \in B\}$
    - $\mathbb{R}_l$ is first countable and separable
    - ω_1 is first countable
    - $(X, \mathcal{T}_{\text{co-finite}})$ is uncountable and not C_1.

11. **Lemma 6.2.** *A subset of a topological space is closed iff it contains all the limit points*

    Proof:
    A is closed iff $Cl(A) = A$, then $A' \subseteq A$

12. (X is metrizable)
    $\to$ X is C_1
    $\to$ ($\forall A \subseteq X \ , \ x \in \overline{A}$ iff $\exists \{x_n\} \subseteq A \ s.t. \ \{x_n\} \to x$)
    $\to$ (for all topological space Y, $f : X \to Y$ is continuous iff for all $x \in X$, (if $x_n \to x$, then $f(x_n) \to f(x)$))

13. For Which Space we have sequence converges to a unique point? (Metric Space)

    Theorem 6.3. *Let (X, d) be metric space, then sequences converges to a unique point*

    Proof: Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence and $x_n \to x$ and $x_n \to y$ with $x \neq y$. Let $r = d(x, y) > 0$. Consider $U = B_d(x, \frac{r}{2})$ and $V = B_d(y, \frac{r}{2})$, notice we have $U \cap V = \emptyset$. Since converges, we have $\exists N_U \in \mathbb{N} \ s.t. \ \forall n > N_u \ , \ x_n \in U$ and $\exists N_V \in \mathbb{N} \ s.t. \ \forall n > N_V \ , \ y_n \in V$. Take $n = max(N_U, N_V)$, then $x_n \in U \cap V$ which leads to a contradiction.

14. **Lemma 6.3.** *If X is C_1, then for all $A \subseteq X \ s.t. \ \overline{A} = \{x \in X : \exists \{x_n\} \subseteq A \ , \ x_n \to x\}$*

15. Hausdorff Space
    - X is T_0 space if
    $$\forall x, y \in X \ , \ x \neq y \ , \text{either } \exists x \in U_x \text{ and } y \notin U_x \text{ or } \exists x \notin U_y \text{ and } y \in U_y$$
    - X is T_1 space if
    $$\forall x, y \in X \ , \ x \neq y \ \exists \text{ open } U \ s.t. \ \forall x \in U \ , \ y \notin U ; \ \exists \text{ open } V \ s.t. \ y \in V \ , \ x \notin V$$
    - X is T_2 space (Hausdorff Space) if
    $$\forall x, y \in X \ , \ x \neq y \ \exists \text{ open } U, V \ s.t. \ x \in U \ , \ y \in V \text{ and } U \cap V = \emptyset$$

16. Hausdorff Space is the most stricted space, then T_1, and T_0 is the least strict space.

    Lemma 6.4. *Hausdorff* $\rightarrow T_1 \rightarrow T_0$

17. Some Examples

    - Metric Spaces are Hausdorff Space
      Proof: Let X be metric space with metric d,

      $$\forall x, y \in X \ , \ d(x, y) > 0$$

      Since x and y are not equal. Take $U = B(x, \frac{d}{3})$ and $V = B(y, \frac{d}{3})$, then we have $U \cap V = \emptyset$. Then X is hausdorff space.
    - The reverse is not true. Take X to be co-finite, which is T_1 but not Hausdorff.
    - Linearly Ordered Topological Space is Hausdorff(ω_1 is Hausdorff)
    - Subspaces of Hausdorff are Hausdorff.

18. If X is a Hausdorff, then the sequence converges to a unique point.

19. Sequence Lemma

    Lemma 6.5. *Let X be a topological space with* $A \subseteq X$*. If* $\exists \{x_n\} \subseteq A$ *s.t.* $x_n \rightarrow x$*, then* $x \in Cl(A)$

    If X is metrizable, then the reverse holds as well. (i.e. X is metrizable, $x \in Cl(A)$ iff $\exists \{x_n\} \subseteq A \ s.t. \ x_n \rightarrow x$).

    - **Proof:**
      Let $\{x_n\} \subseteq A \ s.t. \ x_n \rightarrow z$. Take open set $x \in U_x$, $\exists N \in \mathbb{N} \ s.t. \ \forall n > N$, $x_n \in U_x \cap A$, then

      $$x \in Cl(A)$$

    - Why metrizable allows the inervse is true?
      Assume X is metrizable and take open ball $B_d(x, \frac{1}{n})$ (take 1/n as radius due to the ball is decreasing)
      then we can construct $\{x_n\} \in B_d(x, \frac{1}{n}) \cap A$, we can always find such x_n since x is in the closure and $x_n \rightarrow x$.
      Take arbitrary open ball U_x, $\exists \epsilon > 0, B(x, \epsilon) \subseteq U_x$, $\exists N \in \mathbb{N} \ \frac{1}{N} < \epsilon$, then $B_d(x, \frac{1}{N}) \subseteq B_d(x, \epsilon) \subseteq U_x$, then

      $$x_n \in U_x$$

    What if X is not metrizable? The key part of the proof relies on countable many basis B_n, which forces $\{B_n\}$ monotonically decreasing.

20. With the observation above, we develop Local Basis to address the property that any x we have an open basis around it.

    - (Local Basis) $\mathcal{U}$ is a collection of local basis if for any open neighbourhood V around x, $\exists U_0 \in \mathcal{U}$ such that $x \in U_0 \subseteq V$. Then we say $\mathcal{U}$ is a local basis.
    - C_1 (first countable) means every point in X, has countable local basis.
    - Local basis always exists at any point, otherwise we can't construct a basis for the space.

21. Metrizable Spaces are C_1 spaces. (Metrizable $\rightarrow C_1$)
    Proof: The intuition behind it, is that in any metric space, we can define a set of decreasing open balls with radius $\frac{1}{n}$, $(\{B_d(x, \frac{1}{n})\})$ which makes the basis countable at every point.

22. Sequence Lemma holds under C_1 space.
Proof:
To show the inverse direction, for any point x, we can pick $B_n = \{B_d(x, \frac{1}{N})\} \subseteq B_N(x) \subseteq U_x$ (By the definition of local basis)

23. $f : X \to Y$, and X is C_1, then

$$\text{f is continuous} \iff \forall\{x_n\} \subseteq X \text{ with } x_n \to x \text{ we have } f(x_n) \to f(x)$$

Proof: Show the reverse direction
Let closed set $B \subseteq Y$. By C_1, we apply the Sequence Lemma, WTS: $x \in f^{-1}(B)$.
Since $x_n \in f^{-1}(B)$, then $f(x_n) \in B$ and $f(x) \in B$. Then

$$f^{-1}(B)$$

is closed.

we have the following review:

- f is continuous $\to$ $(f(x_n))$ the sequence of f, is continuous.
- $(f(x_n) \to f(x))$ and $(C_1) \to$ f is continuous.

24. The property that continuous function preserves sequential limit always true, not related to properties of the space (i.e. f is cont and $x_n \to x$, then $f(x_n) \to f(x)$).
Proof:
Take arbitrary open neighbourhood V of $f(x)$, where $f(x) \in V$. Then $x \in f^{-1}(V)$ is open in X. Then

$$\exists N \ s.t. \ \forall n > N \, , \, x_n \in f^{-1}(V)$$

Then

$$f(x_n) \in V$$

which shows $f(x_n) \to f(x)$

25. Prove or disprove a space is C_1

- To show C_1, we have to construct a set of countable local basis.
- To disprove, we find an x such that any local basis is not countable.

26. Thoughts on C_1 and C_2

- C_1: for any point $x \in X$, there are countable local basis; C_2: the entire space has countable basis.
- $C_2 \implies C_1$ (C_1 is a more strict condition, that requires local basis countable at any point).
- Show that the discrete topology on $\mathbb{R}$ is not C_2
(The intuition behind is that every open set contained in X, there must be a basis contained within the open set, so if i want to disprove C_2, we can try to construct a collection of disjoint uncountable open set, then each open set contains a basis which shows that the basis is uncountable)
- The standard topology on $\mathbb{R}$ is C_2.
Proof:
Take $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$, clearly $\mathcal{B}$ is countable.
To show such set is a basis
  - for every $x \in \mathbb{R}$, we can find rationals such that $a < x < b$
  - Assume $a' = max(a, c)$ and $b' = min(b, d)$, then $a' < x < b'$, then (a', b') is a basis.
  - The topology generated by $\mathcal{B}$ is the standard topology
Claerly, any element in $\mathcal{B}$ is open in standard topology.
Since $\mathbb{R}$ is dense, for any x, we can find

$$x \in (a, b) \subseteq B(x, \delta) \subseteq E$$

- Show $\mathcal{T} = \{(-\infty, a) : a \in [-\infty, +\infty]\}$ on $\mathbb{R}$ is C_2
  Proof: (i) Take $\mathcal{B} = \{(-\infty, a) : a \in \mathbb{Q}\}$
- Dictionary Order on $\mathbb{R} \times \mathbb{R}$
  - C_1, for any $x \times y \in \mathbb{R} \times \mathbb{R}$, take $\mathcal{B} = \{(x \times (y - q), x \times (y + q)) : q \in \mathbb{Q}\,,\, q < \epsilon\}$
  - It's not C_2 Take $B_x \subseteq \{x \times (a, b) : x \in \mathbb{R}\}$, we have found a collection of disjoint uncountable sets, then they contained at least one basis in each, then we showed there are uncountable many basis, which is not C_2.

7 Product and Box Topology

1. (Product Topology) Let $U \subseteq X$ and $V \subseteq Y$ be open, then the topology generated by $\{U \times V\}$ is the product topology.
2. Intuition behind product of two spaces
   - Consider X, Y be the two sides of a rectangle, U and V be section of the side, then the cross-closed area (open rectangle) is a open set that generates the product topology.
   - Alternatively, we can consider the two open spaces in $X \times Y$, $U \times Y = \pi_1^{-1}(U)$ and $X \times V = \pi_2^{-1}(V)$, then take $\{U \times Y\} \cup \{X \times V\}$ as a basis, which generates the same topology as prior.
   - we can use the projection onto one of the side (set) and take the intersection to get the box topology. (i.e. $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$)
   - In general, box topology is finer than product topology.
3. Basis in two spaces
   - (Box Topology) $\Pi_{\lambda \in \Lambda} U_\lambda$
   - (Product Topology) $\pi_\lambda^{-1}(U_\lambda)$

   They are equivalent (equal)
   Proof:
   - $(LHS \subseteq RHS)$ $\pi_{\lambda_0}^{-1}(U_{\lambda_0}) = U_{\lambda_0} \times \Pi_{\lambda \in \Lambda \setminus \{\lambda_0\}} X_\lambda$
   - $(RHS \subseteq LHS)$ $\Pi_{\lambda \in \Lambda} U_\lambda = \bigcap_{\lambda \in \Lambda} \pi_\lambda^{-1}(U_\lambda)$
     Since finite intersection of open set is open, we know that if we take finite product of space, we get product topology equals to box topology.
4. Given $A^n = A \times A \times ... \times A = \{(a_1, ..., a_n) : a_i \in A\}$, we can regard it as $f : \{1, 2, ..., n\} \mapsto A$.
5. (Cartesian Product) $A_1 \times A_2 \times ... \times A_n = \{f : \{1, 2, .., n\} \to \bigcup_{i=1}^{n} A_i : f(i) \in A_i\}$
6. An analog of box and product topology

$$\begin{tikzcd}
\text{Product} \arrow[r, leftrightarrow] & P(x) \\
\text{Box Topology} \arrow[u] \arrow[r, leftrightarrow] & \sum_{n=1}^{\infty} a_n x^n \arrow[u, "\text{Finite}"']
\end{tikzcd}$$

7. Consider $f : A \to \Pi_{\lambda \in \Lambda} X_\lambda$, with $f_\lambda = \pi_\lambda o f$, then

$$\text{f is continuous} \iff f_\lambda \text{ is continuous}$$

8. Some Examples
   - Define $\mathbb{R}^\omega = \Pi_{n=1}^{+\infty} \mathbb{R} = \{(x_1, x_2, ..., x_n, ...) : x_i \in \mathbb{R}\}$
     Take countable n and $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times ... = \Pi_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n})$, it's open in box topology, but not open in product topology.
     But B shows that it's not continuous in box topology.
     Define $f : \mathbb{R} \to \mathbb{R}^\omega$, with $t \mapsto (t, t, ..., t, ...)$ and $f_n : \mathbb{R} \to \mathbb{R}$, with $t \mapsto t$.
   - Consider $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$, with $a_n > 0$, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$

9. We can derive a commutative graph

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f_\lambda"'] & \Pi_{\lambda\in\Lambda}X_\lambda \arrow[d, "\Pi_\lambda"] \\ & X_\lambda \end{tikzcd}$$

10. Projections are functions but take in the vector and print out the element lies in the projected set. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, defined as $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$

11. Some properties:

    - Surjective/Injective
    - Continuous
    - If $U \subseteq X$ is open, then $\pi^{-1}(U) = U \times Y$ open in $X \times Y$, since the projection is surjective.

12. Box & Product Topology
    Let $X = \Pi_{i\in I}X_i$

    - (Definition)
      Box Topology is generated by $\Pi_{i\in I}U_i$, where $U_i \subseteq X_i$ open for each i;
      Product Topology defines a projection, $\forall i \in I$, $\pi_i : X \to X_i$ is continuous and $\pi_i^{-1}(U_i)$ open if $U_i \subseteq X_i$ is open.
      Here we can see Box topology is defined from the basis that generates it, but product topology refers more to projection onto each component space.
    -

      Theorem 7.1. *Let $X = \Pi_{j\in J}X_j$ on product topology, $f : Y \to \Pi_{j\in J}X_j$, then*

      $$f \text{ is continuous iff } \Pi_j \, o \, f : Y \to X_i \text{ is cont on } \ X$$

      The only if part is obvious, since the composition of two continuous functions remains continuous
      The if direction we need to refer to the basis of product topology $\Pi_i U_i$ where $U_i = X_i$ for all i in I, but I is finitely many.
      WTS: $U \subseteq \Pi_{i\in I}X_i$ open, then $f^{-1}(U) \subseteq Y$ is open
      Let $B \subseteq \Pi_i X_i$ be open, then by definition of product topology, $B = \Pi_i U_i$, where $U_i \subseteq X_i$ open for all $i \in I$.
      There exists finite $I_0 \subseteq I$ such that $U_i = X_i$, then

      $$f^{-1}(B) = f^{-1}(\Pi_{i\in I}U_i)$$

      $= \{y \in Y : f(y) \in \Pi_{i\in I}U_i\}$

      $= \{y \in Y : \forall i \in I_0 \, , \, \pi_i(f(y)) \in U_i\}$

      $= \bigcap_{i\in I_0}\{y \in Y : \pi_i(f(y)) \in U_i\}$

      $= \bigcap_{i\in I_0}(\pi_i \, o \, f)^{-1}(U_i)$

      Notice that only the i's who are in I_0 have the property that the projection is in U_i.
      This motivates that $f(y)$ is in $\Pi_i U_i \subseteq \Pi_i X_i$, so only the ones in finite I_0, we have $\pi_{i\in I_0}(f(y)) \in U_i$, this doesn't hold under infinite case. Open set under finite intersection remains open, and here is when over box topology we can't find such I_0 to be finite.

8 Compactness

1. (Compactness) Links finite and infinite
   What properties make EVT and uniform continuity thm hold?
   (a) $[a, b]$ is closed and bounded.
   (b) Bolzano-Weirstrass: Every sequence in $[a, b]$ has a convergent subsequence.
   (c) Every infinite subset of $[a, b]$ has a limit point.

2. **Theorem 8.1.** *For $K \subseteq \mathbb{R}^n$, the following are equivalent,*
   (i) K is closed and bounded
   (ii) For every collection $\{U_\alpha : \alpha \in \Lambda\}$ of open sets in $\mathbb{R}^n$ s.t. $K \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$, there is a finite $\Lambda_0 \subseteq \Lambda$ s.t. $K \subseteq \bigcup_{\alpha\in\Lambda}$

   This theorem provides a way to show K is closed and bounded, by showing it's contained in a finite open cover.
   In Euclidean space, since compact iff (closed & bounded), it provides a way showing K is compact in $\mathbb{R}^n$.
   In general metric space, we only know compact implies (closed & bounded); a counterexample is discrete metric, where the entire space is closed and bounded but not compact.
   In Hausdorff Space, compact implies closed (boundedness not well defined without metric).

   To write the proof, we need to let arbitrary open cover, and then find a finite open cover with respect to the arbitrary open cover we picked.

3. X is compact iff for every collection $\{U_\alpha : \alpha \in \Lambda\}$ of open sets in X such that $Y \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$
   X is compact if it can be covered by finitely many open set.

4. Pasting Lemma only requires the domain to be open or closed, doesn't require the domain to be compact.

5. Some Examples
   (1) $\mathbb{R}$ is not compact, $\{(n, n+2) : n \in \mathbb{Z}\}$ is an infinite open cover of $\mathbb{R}$
   (2) $A = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$ is compact
   Let $\alpha_0 \in \Lambda$ such that $0 \in U_{\alpha_0}$. Given $\frac{1}{n} \to 0$, $\exists N \in \mathbb{N}$ *s.t.* $\forall n > N$, $\frac{1}{n} \in U_{\alpha_0}$.
   For n between 1 and N, which are finite, we can take $\alpha_n \in \Lambda$ *s.t.* $\frac{1}{n} \in U_{\alpha_n}$, then
   $$A \subseteq \bigcup_{i=0}^{N} U_{\alpha_i}$$
   a finite cover of the space.
   (3) On $\mathbb{R}$, $(0, 1]$ is not compact, but $[0, 1]$ is compact.

6. Properties of Compactness
   - Every closed subspace of compact space is compact
     (i.e. Let X be compact and $Y \subseteq X$ be closed, then Y is compact)
     Proof:
     Suppose X is compact and $y \subseteq X$ is closed. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a collection of open sets in X such that $y \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.
     Note that $X \setminus Y$ is open in X, so $X = (X \setminus y) \cup \bigcup_{\alpha\in\Lambda} U_\alpha$.
     By compactness of X, there is finite $\Lambda_0 \subseteq \Lambda$ such that either
     $$X = (X \setminus y) \cup \bigcup U_\alpha$$

or

$$X = \bigcup_{\alpha \in \Lambda_0} \bigcup U_\alpha$$

But in any case, $y \subseteq \bigcup_{\alpha \in \Lambda_0} U_\alpha$
Why 'closed' is crucial here?
Since $X \setminus Y$ is open, it allows removing Y from X, the rest remains an open cover of X.

- A is a compact subspace of Hausdorff space X, if $x \notin A$, then $\exists U_x \subseteq X$ open such that $U_x \cap A = \emptyset$
  Proof:
  Let $y \in A$ and fix $x \in X \setminus A$
  Since X is Hausdorff, we can find open neighbourhoods $x \in U_y$ and $y \in V_y$ such that $U_y \cap V_y = \emptyset$
  Then $\{V_y : y \in A\}$ is an open cover on A
  Take $\bigcap_{y \in A} U_y$ and $\bigcup_{y \in A} V_y$ (The motivation behind is we want to find disjoint open cover), but here if without compact, the finite intersection of open not necessarily be open
  Since A is compact, it has $\{V_i\}$ finite open cover and open neighbourhood at x $\{U_i\}$, repeat previous steps, we get desired.

- Every compact subspace of Hausdorff Space is closed
  (i.e. X is hausdorff, $Y \subseteq X$ is compact, then Y is closed)
  Proof:
  Let $Y \subseteq X$ be compact, WTS: $X \setminus Y$ is open,
  then we want to show for every $x \in X \setminus Y$, there is an open set $x \in U_x \subseteq X$ and $U_x \cap Y = \emptyset$
  Let $x \in X \setminus Y$, since X is hausdorff, $\forall y \in Y$, $\exists V_y \subseteq Y$ open such that

  $$V_y \cap U_x = \emptyset$$

  Note that $Y \subseteq \bigcup_{y \in Y} V_y$ bc Y is compact.
  There is finite $Y_0 \subseteq Y$ such that $Y \subseteq \bigcup_{y \in Y_0} V_y$. Take $U = \cap_{x \in Y_0} U_x$, then

  $$U \cap Y = \emptyset$$

  Notice that
  (i) taking intersection of U allows U is disjoint with any segment of Y.
  (ii) U is the open neighbourhood.

  The proof of this type relies a lot on taking the intersection and union of a collection of sets. And to show the space or set is compact, more often we show such space is a subset of a finite open cover.

- Continuous image of a compact space is compact
  (i.e. $f : X \to Y$ is continuous, and $E \subseteq X$ is compact, then $f(E) \subseteq Y$ is compact)
  Proof:
  Let $\{V_\alpha : \alpha \in \Lambda\}$ be the collection of open sets in Y such that $f(X) \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$
  By continuity, $f^{-1}(V_\alpha) \subseteq X$ is open for each α
  Note that $X \subseteq \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$
  By compactness of X, $\exists \Lambda_0 \subseteq \Lambda$ finite such that

  $$X \subseteq \bigcup_{\alpha \in \Lambda_0} f^{-1}(V_\alpha)$$

  Thus $f(X) \subseteq \bigcup_{\alpha \in \Lambda_0} V_\alpha$

7. A continuous bijection from compact to Hausdorff is Homeomorphism. (It's the generalization of continuity of inverse)
   (i.e. $f : X \to Y$, if X is compact and Y is hausdorff and f is bijective & continuous, then f is a homeomorphism)
   It remains to show $f^{-1} : Y \to X$ is continuous.

Proof:
Let $E \subseteq X$ is closed. Then E is compact (closed subspace of compact space remains compact)
then $f(E) \subseteq Y$ is compact (Image of the compact space of a continuosu function is compact)
then $f(E)$ is closed (compact subspace of hausdorff is closed)

8. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a collection of open sets in X such that $X = \bigcup_{\alpha\in\Lambda} U_\alpha$
Let $F_d = X \setminus U_\alpha \quad \forall \alpha \in \Lambda$. Taking complements we get,
$$\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$$
X is compact iff for every collection $\{F_\alpha : \alpha \in \Lambda\}$ of closed sets in X if $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$, then $\exists \Lambda_0 \subseteq \Lambda$ finite *s.t.* $\bigcap_{\alpha\in\Lambda_0} F_\alpha \neq \emptyset$

9. (FIP) Finite Intersection Property $\quad \forall \Lambda_0 \subseteq \Lambda$ finite, $\bigcap_{\alpha\in\Lambda_0} F_\alpha \neq \emptyset$
Using FIP, refers to characterize compactness by closed sets.

FIP means every finite intersection is not empty.

10. X is compact iff

(i) For all collection of $\{F_\alpha : \alpha \in \Lambda\}$ of closed sests, if $\bigcap F_\alpha = \emptyset$, then $\exists \Lambda_0 \subseteq \Lambda$ finite such that $\bigcap_{\alpha\in\Lambda} F_\alpha = \emptyset$

(ii) for all collection of $\{F_\alpha : \alpha \in \Lambda\}$, $\forall \Lambda_0 \subseteq \Lambda$ finite, if $\bigcap_{\alpha\in\Lambda_0} F_\alpha$, then $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

(i) Says that X is compact iff for arbitrary closed cover for the space, if the infinite intersection is empty, then we can find a finite intersection that is empty. (compact if and only if infinite empty implies finite empty)
(ii) Says the converse of statement (i), which is X is compact iff for arbitrary finite closed intersection is not empty, then the infinite intersection is not empty. (compact if and only if finite not empty implies infinite not empty)

11.

Theorem 8.2. *X is compact iff every collection of closed sets with FIP has nonempty intersection*

12.

Lemma 8.1. *If X is compact and $\{F_n : n \in \mathbb{N}\}$ is a collection of closed non-empty sets such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$. Then*
$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset$$
Notice this theorem is similar to nested-interval theorem

9 Limit Point Compact & Seq Compact

1. Develop more properties of $[a, b]$ which are related to compactness.

2. X is limit-point compact iff every infinite subset has a limit point.
Two Strategies to show it,
(i) A is infinite $\rightarrow$ A contains a limit point;
(ii) A doesn't contain limit point $\rightarrow$ A is finite.

3.

Theorem 9.1. *Every compact space is limit-point compact*

Proof:
Let $A \subseteq X$ be compact, suppose not the case, $A' = \emptyset$.
Then $\forall x \in A$, $\exists$ open $U_x \subseteq A$ such that $U_x \cap A = \{x\}$.
Then $A \subseteq \bigcup_{x \in A} U_x$, also A is closed. Then A is compact.
By compactness, let A_0 be finite such that $A \subseteq \bigcup_{x \in A_0} U_x$, then

$$A = \bigcup_{x \in A_0} (U_x \cap A) = \bigcup_{x \in A_0} \{x\} = A_0$$

where LHS is infinite, but RHS is finite which leads to a contradiction.
Notice the converse is false, take ω_1 as an example, which is not compact but is limit-point compact.

4. X is sequentially compact iff every sequence has a convergent subsequence.
(i.e. if $\{x_n\}_{n=1}^{+\infty} \in X$, then there exists increasing n_k such that $\{x_{n_k}\}$ converges in X)
For example, $[a, b]$ where a,b are reals, is sequential compact.

5.

Theorem 9.2. *(Lebesgue Number Lemma) Let* (X, d) *be a metric sequentially compact space. If* $\{U_\alpha : \alpha \in \Lambda\}$ *is an open cover of X, then there is* $\delta > 0$ *such that for all* $A \subseteq X$*, if* $diam(A) = sup\{d(x, y) : x, y \in A\} < \delta$*, then there is* $\alpha \in \Lambda$ *such that* $A \subseteq U_\alpha$

6. The intuition behind the lemma is we take the diameter of each open ball, denoted as $r(x_i, U_i)$
We take $r(x) = sup(r(x_1, U_1), ...,)$, which is a continuous positive function on X.
If X is metrizable and compact, then r attains minimum on X, such minimum is the lebesgue number; however, if X is not compact, such sequence of numbers $(r(x_n))$ might tend to 0, not bounded from below above 0.
See also here.

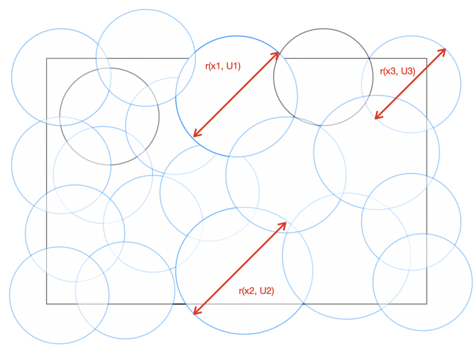


Figure 1: The Intuition Behind

Proof:
Suppose not the case.
For any $n \in \mathbb{N}$, we can find $A_n \subseteq X$ with diam(A_n) ¡ $\frac{1}{n}$ and there is no $\alpha \in \Lambda$ such that $A_n \subseteq U_\alpha$
Pick $x_n \in A_n$, by sequential compactness, $\exists\{x_{n_k}\} \to x \in X$.
Let $\alpha \in \Lambda$ such that $x \in U_\alpha$.
Let $\epsilon > 0$ be such that

$$B_d(x, \epsilon) \subseteq U_\alpha$$

We can find large enough n so that $d(x_{n_k}, x) < \frac{\epsilon}{2}$ and $\frac{1}{n_k} < \frac{\epsilon}{2}$
Claim: $A_{n_k} \subseteq B_d(x, \epsilon)$. Let $y \in A_{n_k}$, $d(y, x_{n_k}) < \frac{1}{n_k} < \frac{\epsilon}{2}$
By triangle inequality, $d(x, y) \leq d(x, x_{n_k}) + d(x_{n_k}, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
which leads to a contradiction that it converges to a positive number$\rightarrow\leftarrow$

7. Let (X, d) be a metric space, then the following are equivalent:
   (i) X is compact
   (ii) X is limit-point compact
   (iii) X is sequential compact
8. We only have Compact $\implies$ Limit-Point Compact and Seq Compact $\implies$ Limit-Point Compact
   Other directions are all false.
9. Revisit Compactness
   (1) In $\mathbb{R}^n$, there are three types of special sets:
   (i) Bounded and closed set
   (ii) Compact set: Any open cover(open sets) has a finite subcover
   (iii) Sequential Compact Set: Any sequence has a convergent subsequence
   the subsequence should convergent to the set.
   (2) If X is compact, $A \subseteq X$, then A is compact **Proof:** Take the finite open subcover and intersect with A, we get a finite open subcover for A
   (3) Consider $[a, b] \in \mathbb{R}$, for any continuous function f on $[a, b]$, it always attains min and max, since $\{U_x : x \in [a, b]\}$, we can always find $supM_x$
   (4) Finite union of compact set is compact
   Proof:
   Sufficient to show it holds for 2. Let $E, F \subseteq X$ be compact sets.
   Set $E \cup F$ be an open cover $\{U_\lambda\}_{\lambda \in \Lambda}$
   Since E and F are compact, then we can construct finite open cover for each set, $\{U_i\}$ and $\{U_{i'}\}$
   Then take the union of two finite open cover, which is finite clearly, it's an open cover for $E \cup F$
   (5) Though in $\mathbb{R}^n$, compact and closed sets have similar properties, but in general space, it's not the case
   (6) In Hausdorff Space, compact set must be closed
   (7) In Hausdorff, infinite intersection of compact set is compact **Proof:** Compact in Hausdorff is closed, then infinite intersection of closed is closed, then it's compact, since it's the intersection of compact sets, then it's compact.
   (8) In Hausdorff, disjoint compact has disjoint neighbouthood **Proof:** Let E, F be two disjoint compact sets, fix $x \in F$, then $x \in V_x$ is disjoint with open neighbourhood U_x.
   Take the intersection of U_x and the union of V_x, similar to the process above, we show there exists disjoint open neighbourhood.
   (9) Closed and Bounded vs Compact
   - compact $\rightarrow$ closed (if Hausdorff)
   - compact $\rightarrow$ bounded (if in metric space)
   - In metric space, compact space is bounded
     Proof:
     Let $A \subseteq X$ be compact. Take $\{B(x, n) : n \in \mathbb{N}_+\}$ be an open cover for A
     $Y = max\{n_1, .., n_k\}$, then $B(x, n_k) \subseteq B(x, r)$, so $A \subseteq B(x, r)$, then A is bounded
   (10) Recall that on $\mathbb{R}$, for continuous $f : [a, b] \rightarrow \mathbb{R}$, the following holds:
   (i) Bounded
   (ii) Attains min and max
   (iii) Intermediate Value Thm
   (iv) Uniformly Continuous
   (v) f cont iff f^{-1} cont if exists
   So we want to generalize to general topological space, except (iii), the properties hold on compact space; (iii) relies on connectness

(11) If X is seq compact space and $f : X \to \mathbb{R}$ is continuous, then f bounded and attains min and max
Proof:
Suppose not the case, there exists $\{x_n\}$ such that $f(x_n) \to +\infty$
By seq compact, $\exists\,\{x_n\}$ such that $x_n \to x_0$
Since f is continuous, then $f(x_n) \to f(x_0)$, which leads to a contradiction.
suffices to show it attains max. Set $M = sup_{x\in X} f(x)$
$\exists x_n$ such that $f(x) \in (M - \frac{1}{n}, M]$, we get $\{x_n\} \subseteq X$, then $f(x_n) \to M$
By seq compact, there exists $x_{n_k} \to z_0$, since f is continuous, then $f(x_{n_k}) \to f(x_0)$, then $f(x_0) = M$, which shows M is attainable.

(12) Compact vs Seq Compact

- Compact + $C_1 \to$ Seq Compact
  Proof:
  Let $\{x_n\}$ be an infinite sequence
  Claim: $\exists x \in X$ such that for all open neighbourhood at x, there exists infinite terms of the sequence
  Suppose not the case, then $\forall x \in X\ \exists U_x\ s.t.$ only finite terms of the seqeunce is in it
  Then $\{U_x : x \in X\}$ is an open cover of X, with finite many cover, then the union of U_x covers the sequence, but leads to a contradiction, since infinite terms can't be covered by finitely many covers.
  Since X is C_1, we can find a countable decreasing open neighbourhood at x $\{V_n\}$
  Then $x_{n_k} \to x$
- Metrizable Space Seq compact implies Compact

10 Axiom of Choice, Zorn's Lemma, Tychonoff's Theorem

1. Let A be a set, the following are equivalent:
   (i) There exists an injection function $f : \mathbb{Z}_+ \to A$
   (ii) There exists a bijection of A with proper subset of itself
   (iii) A is infinite
2. (Axiom of choice)
   Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is , a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.
3.

   Lemma 10.1. *Given a collection $\mathcal{B}$ of nonempty sets (not necessarily disjoint), there exists a function*

   $$c : \mathcal{B} \to \bigcup_{B\in\mathcal{B}} B$$

   such that $c(B)$ is an element of B, for each $B \in \mathcal{B}$.
4. Well-Ordering Theorem

   Theorem 10.1. *Every set can be well-ordered*

   Well-Ordered refers to any nonempty subset attains a least element under such ordering.
5. Partial Order on a set P is that
   (i) (Reflexive) $\forall p \in \mathcal{P}$, $p \leq p$
   (ii) (Antisymmetric) $\forall p, q \in \mathcal{P}$, ($p \leq q$ and $q \leq p$, then $p = q$)

(iii) (Transitive) $\forall p, q, r \in \mathcal{P}$, ($p \leq q$ and $q \leq r$, then $p \leq r$)

We say $(\mathcal{P}, \leq)$ is a partially ordered set

6. The Maximum Theorem

Theorem 10.2. *Let A be a set and $\prec$ be a strict partial order on A, then there exists a simply order subset B of A*

7. Let $(\mathcal{P}, S)$ be partially ordered set
   - $\mathcal{C} \subseteq \mathcal{P}$ is a chain iff $\forall p, q \in \mathcal{C}$, $p \leq q$ or $q \leq p$
     Chain means every element is comparable.
   - $p \in \mathcal{P}$ is a maximal iff $\forall q \in \mathcal{P}$, $p \leq q \implies p = q$

8. Zorn's Lemma

Lemma 10.2. *Let $(\mathcal{P}, \leq)$ be a non-empty partially ordered set. If every chain(simply ordered set) in $\mathcal{P}$ is bounded above, then $\mathcal{P}$ has a maximal element*

9. The Following are equivalent
   (1) Axiom of Choice
   (2) Well-Ordering Theorem
   (3) Zorn's Lemma

10. Countable Chain Condition (ccc) If every pairwise disjoint collection of open sets is countable

11. ω_1 does not have ccc (i.e. there exists an uncountable collection of pairwise disjoint open sets)
Proof:
Let $\mathcal{P}$ be the set of all collections of pairwise disjoint countable non-empty open sets. Ordered by $\subseteq$, then apply Zorn's Lemma

   - ($\mathcal{P} \neq \emptyset$) Let $\alpha \in \omega_1$, $\{(-\infty, \alpha + 1)_{<}\} \in \mathcal{P}$, where $\alpha + 1 = min\{\beta \in \omega_1 : \beta > \alpha\}$
   - (Every Chain is bounded above) Fix $\mathcal{C} \subseteq \mathcal{P}$ a chain. $\bigcup_{c \in \mathcal{C}} c$ is an upper bound of $\mathcal{C}$, we just need to show $\bigcup_{c \in \mathcal{C}} c \in \mathcal{P}$
     It's suffices to show that $\bigcup_{c \in \mathcal{C}} c$ is pairwise disjoint
     Let $U, V \in \bigcup_{c \in \mathcal{C}} c$ with $U \neq V$ and $c_1, c_2 \in \mathcal{C}$ such that $U \in c_1$ and $V \in c_2$.
     Since $\mathcal{C}$ is a chain, the WLOG assume $c_1 \subseteq c_2$, then $U, V \in c_2$.
     Since c_2 is pairwise disjoint, we have $U \cap V = \emptyset$
     By Zorn's Lemma, let $c \in \mathcal{P}$ be maximal
     Claim: c is uncountable
     Proof:
     Suppose not the case. Then $\bigcup_{U \in c} U$ is a countable open set. SInce every countable subset of ω_1 is bounded above, then let $\alpha \in \omega_1$ be such that $\forall \beta \in \bigcup_{U \in cU}$, $\beta \leq \alpha$
     Consider $V := \{\alpha + 1\}$, $c \subseteq c \cup \{V\} \subseteq \mathcal{P}$.
     By maximal, $c = c \cup \{v\}$, so $V \in c$, whcih leads to a contradiciton

12. Finite product of compact sets is compact Under the finite case, we don't have to refer to Axiom of Choice, we use Tube Lemma and basic tools to show the finite product is compact
Since the index is finite, we can do induction on the index of the space, so it remains sufficies to show the case of 2
(i.e. If X and Y are compact, then $X \times Y$ is compact)

Some Intuition Behind
With the first attempt, the black lattices creates an open cover for $X \times Y$, the blue blocks are open

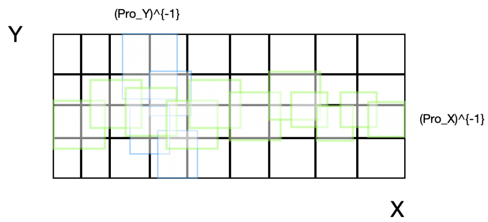


Figure 2: illustration of taking open cover for each and then take the product

sets in $X \times Y$ project onto Y and the green ones are project onto X.
we try to take finite open cover for X and Y, the problem is we only have a row or a column of open sets, which clearly couldn't cover the entire $X \times Y$

The next motivation is, what if we slice Y, since we know it's compact, and then project each thick

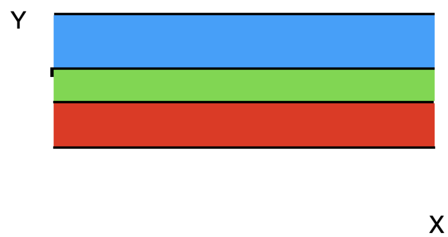


Figure 3: Slice Y

slice onto X to get something useful

Lemma 10.3. ***Tube Lemma***
Let $A \subseteq X$ be compact, $y \in Y$, which is in $X \times Y$, W is an open neighbourhood contains $A \times \{y\}$. Then exists open neighbourhood $U(A)$ and $V(y)$ such that

$$U \times V \subseteq W$$

In particular, If X is compact, W be an open neighbourhood of $X \times \{y\}$, then exists open nhood $V \subseteq Y$ such that

$$X \times Y \subseteq W$$

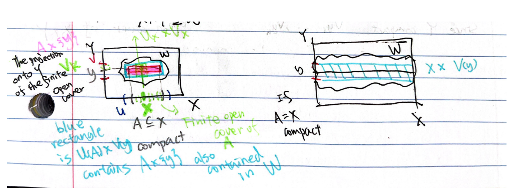


Figure 4: Tube Lemma

Some Insights
The key is to fix a y and then show there is an open neighbourhood in Y satisfies the condition
First we fix a y in Y, and take an open set A in X, the cartesian product gives the pink line segment in the graph, which is $A \times \{y\}$

Then we can take an open neighbourhood of such line segment, which is the black boundary in the graph
We then can construct blue open interval $U \subseteq X$ and red open interal $V \subseteq Y$ at y.
The blue rectangle contained in W, is the open interval $A \times \{y\} \subseteq U \times V \subseteq W$
Now we use the fact that A and Y are compact, then we can create finite open covers, which are the green intervals in the graph, U_k and correspond V_k
Then we have the green rectangles $U_k \times V_k$ as finite open cover for the pink line segment $A \times \{y\}$.

The graph on the right shows when X is compact, such A can be extend to the entire space
We can see the black line segment can travels along X.
The blue rectangle is $X \times V(y)$

Proof:
Let $\forall x \in A$, then $(x, y) \in A \times \{y\} \in W$ and $(x, y) \in U_x \times V_x$
$U_x \times V_x : x \in A$ is an open cover of $A \times \{y\}$
Consider $\pi_x : U_x \times V_x \to X$, so $\{U_x\}$ is an open cover of A
Since A is compact, we have finite open cover $\{U_k\}$ and $\{V_k\}$
Set $U = \bigcup_{i=1}^n U_i$ and $V = \bigcap_{i=1}^n V_i$
We take the intersection on Y since we want to be as precise as possible at point y; taking the union on A since we want to extend as far as possible along the set X
(Notice V remains open relies on A is compact, so finite intersection of closed set is closed)
Then $U \times V \subseteq \bigcup_{i=1}^n (U_i \times V_i) \subseteq W$ (each $U_i \times V_i$ refers to the little green rectangle)

13. Tychonoff's Theorem

Theorem 10.3. *Product of compact spaces is compact*

Proof1: (Closed formulation of compactness and Zorn's lemma)
Motivations

(1) Consider the case X_1 and X_2 are compact sets, we want to use closed formulation to show $X_1 \times X_2$ is compact
Define $\pi_i : X_1 \times X_2 \to X_i$
Since X_i is compact, $\{\overline{\pi_i(A)} : A \in \mathcal{A}\}$ has finite intersection property, then $\overline{\pi(A)}$ is not empty.
Take $x_1 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_1(A)}$ and $x_2 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_2(A)}$
If we can show $x_1 \times x_2 \in \bigcap_{A \in \mathcal{A}} A$, then we are done, by showing the intersection of such collection is not empty

But It's not true Here we fix two foci, define A as areas bounded with ellipses. Eventually,

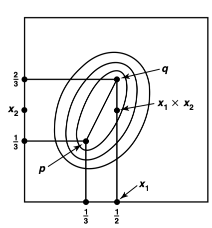


Figure 5: Why it fails

collection of such closed ellipses will shrink to the line segment between the two points.
This way of construction fails because we can freely choose the points within intersection, then what if we refine a bit more and expand the collection $\mathcal{A}$ to $\mathcal{D}$, but still follows finite intersection property In this illustration, we fix one of the foci, but the second foci starts sliding towards $(\frac{1}{3}, \frac{1}{3})$

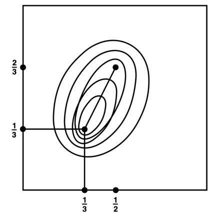


Figure 6: Refines the intersection but still fails

as we keep shrink the ellipses, and still defines the bounded area as previous
But in general how do we guarantee the big collection $\mathcal{D}$ is always correct, since we want it to be as large as possible.
Here come's Zorn's Lemma, which ensures the existence of such 'large' collection, so it remains to show such $\mathcal{D}$ forces the proper choice of the intersection.

(2) We want to show that $X = \Pi_j \in JX_j$ and $\mathcal{F}$ be a collection of closed sets in X with FIP, WTS: $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$

(3)

Lemma 10.4. *Let X be a set, $\mathcal{A}$ be a collection of subsets of X with FIP. Then there exists a collection $\mathcal{D} \subseteq X$ such that $\mathcal{A} \subseteq \mathcal{D}$ and $\mathcal{D}$ has FIP and no collection of subsets of X that properly contains $\mathcal{D}$ has this property*

i.e. $\mathcal{F} \subseteq \mathcal{D}$with FIP such that

$$\forall A \subseteq X \ \ A \cap D \neq \emptyset \implies A \in \mathcal{D}$$

The property in the lemma ensures such $\mathcal{D}$ is the largest candidate
Proof:
Let $\mathbb{P}$ be the set of all collections $\mathcal{F} \subseteq \mathcal{D}$ with FIP, ordered by $\subseteq$
Notice that $\mathbb{P} \neq \emptyset$, since $\mathcal{F} \in \mathcal{P}$
Every chain is bounded above: Fix $\mathcal{C} \subseteq \mathbb{P}$ be a chain,
If $\mathbb{C} = \emptyset$, then $\mathcal{F}$ is an upper bound
If $\mathbb{C} \neq \emptyset$, then we check $\bigcup_{\mathcal{D}\in\mathbb{C}} \mathcal{D} \in \mathbb{P}$
Let $\mathcal{D}_0 \in \mathbb{C}$, then $\mathcal{F} \subseteq \mathcal{D}_0 \subseteq \bigcup_{\mathcal{D}\in\mathbb{C}} D$
Since it has FIP, let $\{D_i\}_{i=1}^n \in \bigcup_{\mathcal{D}\in\mathbb{C}} D$
Let $\mathcal{D}_i \in \mathbb{C}$ such that $D_i \in \mathcal{D}_i$
Since $\mathbb{C}$ is a chain, $\bigcup_{i=1}^n \mathcal{D}_i = \mathcal{D}_j$
Then $D_1, ..., D_n \in \mathcal{D}_|$, so $\bigcap_{i=1}^n D_i \neq \emptyset$ as $\mathcal{D}_j$ has FIP.

By Zorn's Lemma, let $\mathcal{D} \in \mathbb{P}$ be the maximal with FIP
First, we show if $\{D_i\} \in \mathcal{D}$, then $\bigcap_{i=1}^n D_i \in \mathcal{D}$
Let $D_i \in \mathcal{D}$, note that $\mathcal{D} \cup \{\bigcap_{i=1}^n D_i\}$ has FIP. So $\mathcal{D} \cup \{\bigcap_{i=1}^n D_i\} \in \mathbb{P}$
Since we know $\mathcal{D}$ is the maximum, then

$$\mathcal{D} = \mathcal{D} \cup \{\bigcap_{i=1}^n D_i\}$$

So $\{\bigcap_{i=1}^n D_i\} \in \mathcal{D}$
Second, we show such $\mathcal{D}$ satisfy the property
Let $A \subseteq X$ and assume $A \cap D \neq \emptyset$ for all $D \in \mathcal{D}$
Since $\mathcal{D} \cup \{\bigcap_{i=1}^n D_i\}$ has FIP. Let $\{D_i\} \in \mathcal{D}$, then

$$D := \bigcap_{i=1}^n D_i \in \mathcal{D}$$

By assumption, $A \cap \bigcap_{i=1}^{n} D_i = A \cap D \neq \emptyset$
By maximum,

$$\mathcal{D} = \mathcal{D} \cup \{A\}$$

So $A \in \mathcal{D}$

Lemma 10.5. $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$

Proof:
Consider $\pi_j : X \to X_j$
First, show $\mathcal{D}_j$ ahs FIP, where $\mathcal{D}_j = \{\pi_j(D) : D \in \mathcal{D}\}$
Let $x \in \bigcap_{i=1}^{n} D_i$, since $\mathcal{D}$ has FIP, then $\pi_j(x) \in \bigcap_{i=1}^{n} \pi_j(D_i)$
Then we have $\{\overline{\pi_j(D)} : D \in \mathcal{D}\}$ has FIP
BY compactness of X_j, we can pick $x_j \in \bigcap_{D \in \mathcal{D}} \overline{\pi_j(D)}$, by Axiom of Choice
Let $X = (x_j)_{j \in J} \in X$
It's suffices to check that $x \in \overline{D}$ for arbitrary $D \in \mathcal{D}$ Second, we show $\forall j \in J$ and $U_j \subseteq X_j$ open, if $x \in \Pi_j^{-1} U_j$, then $\Pi_j^{-1} U_j \cap D \neq \emptyset$ for all $D \in \mathcal{D}$
Therefore, $\Pi_j^{-1} U_j \in \mathcal{D}$
Let $j \in J$ and $U_j \subseteq X_j$ open such that $x \in \Pi_j^{-1} U_j$. Fix $D \in \mathcal{D}$
SInce $x_j \in \overline{\Pi_j(D)}$ and $x_j = \Pi_j(x) \in U_j$, then $U_j \cap \Pi_j(D) \neq \emptyset$
Then $\Pi_j^{-1} U_j \cap D \neq \emptyset$ Third, we show for all $D \in \mathcal{D}$, $x \in \overline{D}$
Let $D \in \mathcal{D}$, we already show that for every basic open $B \subseteq X$ such that $x \in B$ and $B \cap D \neq \emptyset$
If B is a basic open set in τ_{prod}, then $B = \bigcap_{i=1}^{n} \Pi_j^{-1}(U_{j_i})$, for some $j_1, ..., j_n \in J$ and $U_{j_i} \in X_{j_i}$ open
Then,

$$\Pi_j^{-1}(U_{j_i}) \in \mathcal{D}$$

Then $B = \bigcap_{i=1}^{n} \Pi_j^{-1}(U_{j_i}) \in \mathcal{D}$
SInce $\mathcal{D}$ has FIP, then $B \cap D \neq \emptyset$

(4) Revisit the Proof
We start with an arbitrary collection $\mathcal{A}$ of subsets of $X = \Pi_{i=1}^{\infty} X_i$ and we want to show X is compact
Guided by the compact defined with closed sets who have FIP, we are aiming to consturct a collection of all the closed sets in X with FIP and the intersection is non-empty
By Zorn's Lemma, we are able to extend such arbitrary collection $\mathcal{A}$ to $\mathcal{D}$, the largest collection of subsets of X.
$\mathcal{D}$ is the largest, or we can understand it as the big guy contains everything
By the two lemma, $\mathcal{D}$ has two properties:
(1) (Closed under finite intersection) $\bigcap_{i=1}^{n} D_i \in \mathcal{D}$
(2) (The Biggest Guy) $\forall A \subseteq X$, if $A \cap \mathcal{D} \neq \emptyset$, then $A \subseteq \mathcal{D}$

But keep in mind, we need to collect the ingredients of the proof via closed sets with FIP.
So we have to find a collection of closed sets (It should be large enough that contains all the closed sets, so the subcollection in it represents any collection of closed sets), where for each particular collection of closed sets, every closed set in such collection has FIP, and eventually, the intersection of every collection has to be not empty.
With the set-up lemma, we only have to show $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$
We concetrate on the closure of D because it's unclear whether D_i is closed or not
The D here can be analog as open neighbourhood of the horizontal tube in $X \times Y$ case
In order to make full use of the compactness of each X_i, we consider the projection $\pi_i : X \to X_i$
After the decomposition of the candidate point $(x_i)_{i \in I}$ (this is the point in the intersection, showing it's not empty)
It remains to show $(x_i)_{i \in I} \in \overline{D}$ for every $D \in \mathcal{D}$
Since we want to show a point belongs to a closure of a set, we find arbitrary open neighbourhood at x and then argue the intersection of such set with the target set is not empty

Since we know for each i, $\bigcap_{D\in\mathcal{D}} \overline{\pi_i(D)} \neq \emptyset$
By **Axiom of Choice**, we can pick $x_i \in \bigcap_{D\in\mathcal{D}} \overline{\pi_i(D)}$
Take $U_i \subseteq X_i$ be the open neighbourhood at x_i
Then we have $\pi_i^{-1}(U_i) \cap D \neq \emptyset$ for all $D \in \mathcal{D}$, because we have $x_i \in \overline{\pi_i(D)}$
So $\exists y \in D$ such that

$$\pi_i(y) \in U_i$$

Then $y \in \pi_i^{-1}(U_i)$
Therefore we have

$$y \in D \cap \pi_i^{-1}(U_i) \neq \emptyset$$

Then we show $(x_i)_{i\in I} \in \overline{D}$ for every $D \in \mathcal{D}$

Proof2: (Applying the filters)

11 One-Point Compactification

1. (Locally Compact) For some compact K at x, there exists an open neighbourhood U_x such that $x \in U_x \subseteq K$

2. Some Examples

   (1) $\mathbb{R}$ is locally compact
   For any real x, we can find a closed interval on $\mathbb{R}$ such that $\forall x \in [a,b]$. We know $[a,b]$ is compact, and we always have $x \in (a,b) \subseteq [a,b]$.

   (2) $\mathbb{Q}$ is not locally compact
   We argue that at 0, it's not locally compact
   Fix $a \in (-\epsilon, +\epsilon)$ and $\{x_n\} \in \mathbb{Q}$ with $x_n \to x$ only in reals, then $\{x_n\}$ is an infinite sequence in $\mathbb{Q}$, then $[-\epsilon, +\epsilon] \cap \mathbb{Q}$ is not compact. (Since we know compact implies limit point compact, by showing the contrapositive, we show the interval is not compact)
   Suppose at 0 is locally compact, then we can find an open interval and a compact set such that

   $$0 \in U \subseteq K \subseteq \mathbb{Q}$$

   Set $U = [-\epsilon, +\epsilon] \cap \mathbb{Q}$, since it's a closed subset of a compact space K, then it's compact, which leads to a contradiction $\rightarrow\leftarrow$

   (3) Similarly $\mathbb{R}^n$ is locally compact by considering $\Pi_i^n (a_i, b_i)$
   But $\mathbb{R}^\omega$ is not locally compact, by similar argument, we can find $B = (a_1, b_1) \times \ldots \times (a_n, b_n) \times \mathbb{R} \ldots \times \ldots$ is contained in a compact subspace, then $\overline{B}$ is compact but it's not.

   (4) Every simpliy ordered set X having the least upper bound property is locally compact
   For any given basis for X, we can find a closed interval in X contained such basis, which shows X is locally compact.

3. X is locally compact Hausdorff if and only if there exists Y such that

   (1) X is a subspace of y

   (2) $T \setminus X$ is singleton

   (3) Y is compact Hausdorff

4. Let X be a topological space. A compactification of X is a map $\phi : X \to Y$ such that

   (i) Y is compact space

   (ii) ϕ is an embedding ($\phi : X \to \phi(X)$ is homeomorphism)

   (iii) $\phi(x)$ is dense in Y

A compactification is a way converting general space into a compact space

5. If additionally Y is Hausdorff, then it's Hausdorff compactification

6. Moreover, we say it's a Hausforff one-point compactification (unique if exists) if $Y \setminus \phi(X)$ is a singleton (co-domain minus the image is singleton)
Proof:
Let X be a topological space and $\phi_1 : X \to Y_i$ be Hausdorff one-point compactification for all i.
Let p_i be such that $\{p_i\} = Y_i \setminus \phi_i(X)$.
Define $\phi(y) = \begin{cases} \phi_2(\phi_1^{-1}(y))\&\text{if } y \in \phi_1(X) \\ p_2\&\text{if } y = p_1 \end{cases}$ Clearly ϕ is a bijection and a map from compact space to Hausdorff space, then it's sufficies to check such mapping is continuous.
Claim: ϕ is continuous
Proof:
Fix $U \subseteq Y_2$ open, we split into two cases:

   (1) $p_2 \notin U$ so $U \subseteq p_2(X)$. Since U is open in Y_2, it's open in $\phi_2(x)$.
   Then $\phi^{-1}(U) = \phi_1(\phi_2^{-1}(U))$ is open in $\phi_1(x)$. But $\phi_1(x)$ is open in Y_1, since $Y_1 \setminus \phi_1(x) = \{p_1\}$ is closed in Y_1 (because Y_1 is Hausdorff), therefore

   $$\phi^{-1}(U) \text{ is open in } Y_1$$

   (2) $p_2 \in U$, then $F = Y_2 \setminus U$ is closed in Y_2 and $F \subseteq \phi_2(x)$, so F is compact.
   $\phi^{-1}(F) = \phi_1(\phi_2^{-1}(F))$ is compact subset of Y_1
   $\phi^{-1}(U) = Y_1 \setminus \phi^{-1}(F)$ is open in Y_1.

7. Suppose $\phi : X \to Y$ is a Hausdorff one-point compactification of X, then

   (1) X is Hausdorff

   (2) X is non-compact, otherwise $\phi(x)$ is compact so $\phi(x) \subseteq Y$ is closed. But the contradicts with $\overline{\phi(x)} = Y$ and $Y \setminus \phi(x) \neq \emptyset$

   (3) $\forall x \in X$, $\exists k \subseteq X$ compact, $\exists U \subseteq k$ open in X such that $x \in U \subseteq l$

8. X is locally compact iff every $x \in X$ has a compact neighbourhood

9.

   Theorem 11.1. *A topological space X has a unique Hausdorff one-point compactification iff X is locally compact, non-compact,Hausdorff*

10. Revisit Locally Compact

   - Compact $\implies$ locally compact, since we can take the compact set to be the whole set
   - Some Examples of locally Compact but not compact
   (1) $\mathbb{R}^n$ We can take a closed ball $B_d(x, \epsilon) \subseteq \mathbb{R}^n$, which is compact neighbourhood
   (2) Open subset of $\mathbb{R}^n$, for any x, we can find an open ball, we can also find a smaller closed ball, which is the compact neighbourhood
   - (Compactification) Convert a general space to a compact space
     - $\mathbb{R}$ is not compact since $\{(n, n+2) : n \in \mathbb{N}\}$ is an open cover that doesn't have a finite subcover of $\mathbb{R}$
     What if we add one point at $\mathbb{R}$ the real line, which makes it can be covered by a finite open cover (one-point compactification)
     - Let (X, τ) is a non-compact, Hausdorff space. Adding a new element Ω, we get the set X_0. Define the subset of X_*

     $$\tau_* = \tau \cup \{X_*\} \cup \{X_* \setminus K : \text{ K is a compact subset of X}\}$$

Then we have the following:
(1) τ_* is the topology on X_*, and the induced subspace topology of τ_* on X is τ
We verify the following
(a) $X_*, \emptyset \in \tau_*$
(b) Let $\{E_\mathcal{I}\}$, $\{X_* \setminus E_\mathcal{I}\}$
(2) X is a dense subset of (X_*, τ_*)
(3) (X_*, τ_*) is compact, define as one-point compactification
(4) If (X, τ) is locally compact Hausdorff space, then (X_*, τ_*) is Hausdorff space

12 Quotient Topology

1. Show $(X_\infty, \mathcal{T}_\infty)$ is compact Hausdorff
   Quotient Topology

2. Let X and Y be topological spaces and $p : X \to Y$ be a surjection. We say that $P : X \to Y$ is a quotient map iff
$$\forall U \subseteq Y \text{ (U is open in Y } \iff p^{-1}(U) \subseteq X \text{ is open)}$$

3. Let X be a topological space and Y a set. If $p : X \to Y$ is surjective the quotient topology on Y is
$$\mathcal{T} := \{U \subseteq Y : p^{-1}(U) \subseteq X \text{ is open}\}$$
   Typical Situation when this occurs is when we identify point in a space
   - If $p : X \to Y$ is surjective, then $\{p^{-1}(\{y\}) : y \in Y\}$ is a partition of X
     This partition induces an equivalence relation on X, given by
$$x \sim y \iff p(x) = p(y)$$
   - is an equivalence relation on X and $X\setminus$ is the set of equivalence classes, then the map $p : X \to X\setminus$ is surjective, defined as $x \mapsto [x]_\sim$
     $U \subseteq X\setminus$ is open iff $\bigcup_{x \in p^{-1}(U)} [x]_\sim = p^{-1}(U) \subseteq X$ is open

4. Alternative definition of quotient map
   Let $\mathcal{C}$ be an equivalent relation on $X \times Y$, the map $p : X \to X \setminus \mathcal{C}$ such that $p(x) = [x]_\sim$, where $[x]_\sim$ is an equivalence relation is defined to be a quotient map
   Naturally, we can define quotient topology as well, where $U \in X \setminus \mathcal{C}$ is open iff $p^{-1}(U) \subseteq X$ is open
   In the first way of defining quotient, the surjection map holds the property that
$$x \sim y \iff f(x) = f(y)$$
   which means they are in the same equivalent classes Y

5. The intuition behind quotient map is pasting the original space, so the quotient topology focuses on the topology on the space after pasting

6. (Pasting sets) Let X be a set, we have $x_1, .., x_6$, we define a pasting $\sim$, where $x_1 \sim x_2$ and $x_3 \sim x_5$
   Know we define the new set $X/\sim$ be the set containing elements after the pasting, we consider $\{x_1, x_2\}$ and $\{x_4\}$ and $\{x_6\}$ and $\{x_3, x_5\} \subseteq X/\sim$
   $\sim$ is a equivalence relation, can be interpreted as 'pasting-operators', after $\sim$, they turn into an equivalent class.
   $\pi : X \to X/\sim$, which maps x_1 and x_2 to $\{x_1, x_2\}$ and so on

7. Consider X a topological space, we use $\sim$ to do the pasting and get the new space $X/\sim$, and define the topology $\overline{\tau}$ on $X/\sim$,
$$\overline{\tau} = \{V \subseteq X/\sim : \pi^{-1}(V) \in \tau\}$$
   Then we say $< X/\sim, \overline{\tau} >$ is a quotient space of $< X, \tau >$ with equivalence relation $\sim$

8. Consider $X = [0,1] \subseteq \mathbb{R}$, where $\sim$ defines as pasting 0 and 1, where $\pi(0) = \pi(1) = \{0,1\}$, then we get $X/\sim = \{\{0,1\}\} \cup (X \setminus \{0,1\})$
Consider the open sets in this example

9. Quotient Topology $\pi : X \to X/\sim$

10.

Lemma 12.1. *X, Y be topological space, define $\sim$ be an equivalence relation and define $g : X/\sim \to Y$, then*

$$g \text{ is continuous} \iff g \text{ o } \pi \text{ is continuous}$$

Proof:
(i) The only if casse is obvious, where g and π are continuous, then the composition is continuous
(ii) Assume g o π is continuous, then for any $V \subseteq Y$ open, $\pi^{-1}(g^{-1}(V)) = (g\, o\, \pi)^{-1}(V)$ is an open set in X, since $(g\, o\, \pi)^{-1} = \pi^{-1}\, o\, g^{-1}$
By definition of quotient topology, $g^{-1}(V) \subseteq X/\sim$ is open

11. Some properties of π
(1) continuous
(2) π is surjective, since every element in $X/\sim$ we can find the ingredient in X
(3) Not necessary to be open map nor closed map
(4) Let $V \subseteq X/\sim$, if $\pi^{-1}(V) \subseteq X$ is open, then $V \subseteq X/\sim$ is open
It works for closed set as well, since $\pi^{-1}(A^c) = (\pi^{-1}(A))^c$

12. Generalize the pasting map, we get the quotient map, if it satisfies the above, which is
If $f : X \to Y$ satisfies

    (1) f is continuous

    (2) f is surjective

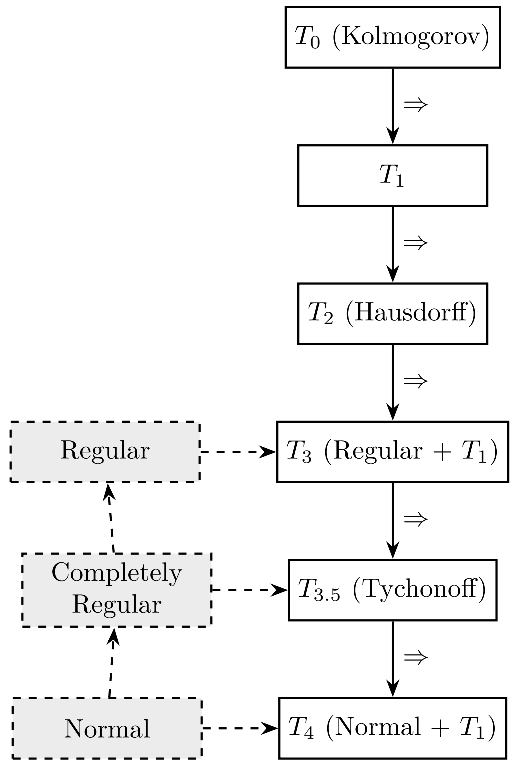
    (3) Let $B \subseteq Y$. If $f^{-1}(B) \subseteq X$ is open, then $B \subseteq Y$ is open

    Then we say f is a quotient map

13. The image of any quotient map is homeomorphic to a quotient space of X
Proof:
Consider $f : X \to Y$, define equivalence relation $\sim$, where $x \sim x' \iff f(x) = f(x')$
WTS: $(X/\sim) \simeq Y$
Define $g : X/\sim \to Y$, with $x \mapsto f(\pi^{-1}(x))$

13 Separation Axioms

1. Overview



2. T_0 Space

   (a) For every $x \neq y \in X$, $\exists U \subseteq X$ open such that ($x \in U$ and $y \notin U$) or ($x \notin U$ and $y \in U$)

3. T_1 Space

   (a) For every $x \neq y \in X$, $\exists U, V \subseteq X$ open such that ($x \in U$ and $y \notin U$) or ($x \notin V$ and $y \in V$)

   (b) In this case, we don't require them to be disjoint open sets, they only have to exclude each point.

   (c) TFAE :

   – X is T_1
   – Every singleton is closed
   – Every finite set is closed
   – For all $A \subseteq X$, A is the intersection of all open sets containing A.

4. T_2 Space

   (a) For every $x \neq y \in X$, $\exists U, V \subseteq X$ open with $U \cap V = \emptyset$ such that $x \in U$ and $y \in V$

   (b) Distinct points can be separated by disjoint open sets.

   (c) In Hausdorff space, any convergent sequence can at most converges to one limit.

   (d) Subspace of Hausdorff space is Hausdorff and closed under (infinite) product

   (e) In first countable, Hausdorff space, any sequence can converge to at most one point.

   (f) Let $f, g : X \to Y$ be continuous, assume Y is Hausdorff, then $\{x : f(x) = g(x)\} \subseteq X$ is closed.
   Pf: Denote $A = \{x : f(x) = g(x)\}$. Let $x \in X \setminus A$, then $f(x) \neq g(x)$.
   Given Y is Hausdorff, there exists disjoint open sets $f(x) \in O_1$ and $g(x) \in O_2$, then we have

$$x \in f^{-1}(O_1) \cap g^{-1}(O_2) \subseteq X \setminus A$$

   Clearly $f^{-1}(O_1) \cap g^{-1}(O_2) \subseteq X$ is open, therefore we show A is closed.

5. $T_{2.5}$ Space
   (a)
6. T_3 Space
   (a) X is T_3 space if X is T_1 and Regular
   (b) Regular Space if $\forall x \in X, C \subseteq X$ closed with $x \notin C$, there exists disjoint open sets O_1, O_2 such that
   $$x \in O_1 \text{ and } C \subseteq O_2$$
   (c) X is T_0, then X is regular if and only if $\forall x \in X, \exists U \subseteq X$ with $x \in U$ such that
   $$\exists x \in V\ s.t.\ x \in V \subseteq \overline{V} \subseteq U$$
   Proof:
   ($\rightarrow$) Suppose X is regular, let $x \in X$ and U be an open nbhd.
   Let $B = X \setminus U$, then B is closed. Given X is regular, there exists disjoint open sets O_1, O_2 such that
   $$x \in O_1\ ,\ B \subseteq O_2$$
   Notice $\overline{O_1}$ is disjoint from B and O_1 is an open mbhd of x
   Therefore, $x \in O_1 \subseteq \overline{O_1} \subseteq U$
   (d) Subspace of regular space is regular and finitely productive.
   (e) X is regular, then $\forall x \neq y \in X$, there exists open nbhd $x \in O_1$ and $y \in O_2$ with $O_1 \cap O_2 = \emptyset$ such that
   $$\overline{O_1} \cap \overline{O_2} = \emptyset$$
   (f) Every Order Topology(LOTS) is regular
   Here we provide a proof for LOTS are regular
   Let $(X, <)$ be linearly ordered space
   Let $x \in X$ and $C \subseteq X$ be closed with $x \notin C$, then $X \setminus C$ is open
   $\exists a, b \in X$ such that $x \in (a, b) \subseteq X \setminus C$, so $(a, b) \cap C = \emptyset$
   Let $V = (-\infty, a) \cup (b, +infty)$, then $C \subseteq V$
   Take $U := (a, b)$, we have $x \in U \subseteq X$ open and $C \subseteq V \subseteq X$, then $U \cap V = \emptyset$
7. $T_{3.5}$ Space
   (a)
8. T_4 Space
   (a) X is T_4 space if X is T_1 and Normal
   (b) Normal Space if $\forall C_1, C_2 \subseteq X$ closed with $C_1 \cap C_2 = \emptyset$, there exists disjoint open sets O_1, O_2 such that
   $$C_1 \subseteq O_1 \text{ and } C_2 \subseteq O_2$$
   (c) Closed subspace of normal space is normal
   (d) Normal space is not closed under product $\quad \mathbb{R}_l$ is normal, but $\mathbb{R}_l \times \mathbb{R}_l$ is not normal.
   (e) Let $p : X \to Y$ be a closed continuous onto map, if X is normal, then Y is normal.
   (f)
   Theorem 13.1. *Regular* + $C_2 \implies$ *Normal*
   (g) Every metrizable space is normal
   (h) Every Compact Hausdorff space is normal

(i) Every WO set X is normal in order topology

Claim: Well-Ordered Spaces are normal
Claim1: Every interval of the form $(x, y]$ is open in X
We split into two cases y is the maximum element or not.
If y is the largest element, then $(x, y]$ is the basis at y
If y is not, it has an immediate successor $y' = y + 1$, where $(x, y] = (x, y')$

Let A,B be two closed sets in X, we try to construct two open covers and argue A,B are disjoint
(1) Assume neither A nor B contains the smallest element of X
For each $a \in A$, there exists basis element at a disjoint from B, which contains some interval of form $(x, a]$
Pick for each $a \in A$ such that $(x_a, a] \cap B = \emptyset$
Similarly, for each $b \in B$, we have $(y_b, b] \cap A = \emptyset$
Take $U = \bigcup_{a \in A}(x_a, a]$ and $V = \bigcup_{b \in B}(y_b, b]$ are open sets containing A,B.

Claim2: $U \cap V = \emptyset$
Suppose $U \cap V \neq \emptyset$, let $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$
Assume $a < b$
Then if $y_b \geq a$, they are disjoint, if $a > y_b$, we have $a \in (y_b, b]$ contradict with $A \cap (y_b, b] = \emptyset$
Assume $b < a$
If $x_a \geq b$, they are disjoint, if $b > x_a$, then $b \in (x_a, a]$ contradict with $B \cap (x_a, a] = \emptyset$
Assume A,B are disjoint closed sets in X.
(2) WLOG: A contains smallest element a_0 of X.
The set $\{a_0\}$ is clopen in X. Then exists disjoint open sets U,V containing $A \setminus \{a_0\}$ and B.
Then $U \cup \{a_0\}$ and V are disjoint open sets containing A, B respectively.

(j) If ΠX_α is normal, and X_α not empty for all α, then X_α is normal.

(k)

Theorem 13.2. *If J is uncountable, then $\mathbb{R}^J$ is not normal.*

(l) If X is a topological space where every basic open set is closed, then X is regular

9. Some Counterexamples

(1) T_0 but not T_1
(Sierpinski Space) Let $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$

(2) T_1 but not T_2
Consider $(\mathbb{R}, \tau_{\text{co-finite}})$ Any two non-empty open sets intersect, hence not hausdorff.

(3) T_2 but not $T_{2.5}$
Consider $\mathbb{R}_K$ where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$

(4) $T_{2.5}$ but not T_3
(Rational Seq Topology)

(5) T_2 but not T_3 $\mathbb{R}_K$.
Clearly $\mathbb{R}_K$ is Hausdorff, since any distinct points on real line can be separated by open intervals, but unfortunately we can't separate 0 and $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ which is closed.
To see why, notice $\frac{1}{n} \to 0$ as $n \to +\infty$

(6) T_3 but not $T_{3.5}$
(Spiral Staircase)

(7) $T_{3.5}$ but not T_4

(8) T_3 but not T_4
(Variation of Tychonoff's Plank) Let $X = \mathbb{N} \cup \{\infty\}$, D be the reals with discrete topology. Let

$Y = D \cup \{\infty'\}$
Define $A = \mathbb{N} \times \{\infty'\} \subseteq X \times Y \setminus \{(\infty, \infty')\}$ is closed.
$B = \{infty\} \times D \subseteq X \times Y \setminus \{(\infty, \infty')\}$
Then A,B can't be separated by disjoint open sets.

(9) Subspace of normal space that is not normal
(Tychonoff's Plank) Consider $X = (\omega_1 + 1) \times (\omega + 1)$ and $Y = X \setminus \{(\omega_1, \omega)\}$, here X is normal but Y is not normal

10. Completely regular space if X is T_1 and for each point x_0 and closed set $x_0 \notin A$, there exists a continuous $f : X \to I$ such that

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \in A \end{cases}$$

14 Urysohn Lemma, Urysohn Metrization Theorem and Tietze's Extension Theorem

1. Urysohn Lemma

(a) Let X be a normal space, $A, B \subseteq X$ be closed and disjoint, let $I = [0, 1] \subseteq \mathbb{R}$ be a closed interval on the real line. Then there exists a continuous map

$$f : X \to I$$

such that $f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$

(b) (Urysohn's Lemma) X is normal iff any two disjoint closed sets can be separated by a continuous function

(c) Proof:
Fix $E, F \subseteq X$ closed disjoint.
($\leftarrow$) This direction is trivial, we can consider open nbhds $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$, which are disjoint open sets contained disjoint closed sets.

($\rightarrow$) Step 1: Construct a sequence of open sets $\{U_p : p \in \mathbb{Q} \cap [0, 1]\}$ such that

$$p < q \implies \overline{U_p} \subseteq U_q$$

Enumerate rational points in $[0, 1]$ as $\{p_n : n \in \mathbb{N}\}$. Assume that $p_0 = 1, p_1 = 0$
$U_{p_0} = U_1 = X \setminus F$
Since $E \subseteq X \setminus F$, X is normal, there is $U_{p_1} = U_0$ such that

$$E \subseteq U_0 \subseteq \overline{U_0} \subseteq X \setminus F = U_p$$

This shows the base case holds, then we will construct such open sets with rational indices inductively.

Suppose we have $U_{p_0}, ..., U_{p_n}$ for some $n \geq 1$.

Let p be the largest in $\{p_0, ..., p_n\}$ such that $p < p_{n+1}$;
q be smallest in $\{p_0, ..., p_n\}$ such that $p_{n+1} < q$

By hypothesis, $\overline{U_p} \subseteq U_q$ by X is normal, $\exists U_{p+1}$ such that

$$\overline{U_p} \subseteq U_{p+1} \subseteq \overline{U_{p+1}} \subseteq U_q$$

We define $U_p = \begin{cases} \emptyset & \text{if p} < 0 \\ X & \text{if p} > 1 \end{cases}$
Define $x \in X,\ f(x) = \inf\{p \in Q : x \in U_p\},\ f(x) \in [0,1]$

Claim1: $f(x) = \begin{cases} 0 & x \in E \\ 1 & x \in F \end{cases}$
If $x \in E$, then $x \in U_0$. Then $f(x) \leq 0 \implies f(x) = 0$
(By assumption $E \subseteq U_0$, $U_0 = U_{p_1}$, then the infimum is 0)
If $x \in F$, $x \notin X \setminus F = U_1$, so $x \notin U_p$ for all $p \leq 1$
If $x \in U_p$, then $p > 1$, then $f(x) \geq 1$, so $f(x) = 1$

Claim2:
(1) If $x \in \overline{U_p}$, then $f(x) \leq p$
(2) If $x \notin U_p$, then $f(x) \geq p$
For (1), if $x \in \overline{U_p}$, then $x \in U_q$, for all $q > p$, then $(p, +\infty) \subseteq \inf\{p \in \mathbb{Q} : x \in U_p\}$, so $f(x) \leq p$
For (2), if $x \notin U_p$, then $x \notin U_q$ for all $q \leq p$, then $\inf\{p \in \mathbb{Q} : x \in U_p\} \subseteq (p, +\infty)$, so $p \leq f(x)$

Claim3: f is continuous
Fix $U = (a,b) \cap [0,1]$ WTS: $f^{-1}(U)$ is open
Fix $x \in f^{-1}(U)$ WTS: $V \subseteq X$ open such that $x \in V \subseteq f^{-1}(U)$
We know $a < f(x) < b$, pick rationals a,b such that $a < p < f(x) < q < b$ (The contrapositive of Claim2)
Then $x \notin \overline{U_p}$, then $x \in U_q \setminus \overline{U_p} := V \subseteq X$
We claim $x \in V \subseteq f^{-1}(U)$, remians to show $V \subseteq f^{-1}(U)$.
Fix $y \in V$, so $y \in U_q \setminus \overline{U_p}$
Then $y \in U_q \implies y \in \overline{U_q} \implies f(y) \leq q < b$
$y \notin \overline{U_p} \implies a < p \leq f(y)$. Thus $f(y) \in U$

2. Urysohn Metrization Theorem

   (a) Every second countable regular space X is metrizable

3. Tietze's Extension Theorem
   X be normal, $A \subseteq X$ is closed

   (a) If $f : A \to [a,b]$ is continuous, then $\exists \hat{f}$ such that
   $$f(x) = \hat{f}(x) \quad \forall x \in A$$

   (b) $f : A \to \mathbb{R}$, then $\exists \hat{f}$ such that
   $$f(x) = \hat{f}(x) \quad \forall x \in A$$

15 Useful Maps

1. Continuity

   (a) Sending open/closed map back to domain.
   $f : X \to Y$ is continuous if $\forall$ open/closed $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open/closed.

   (b) X is first countable space, if $((x_n) \to x \implies f(x_n) \to f(x))$, then f is continuous.
   Notice: for any space X, we have f is continuous implies $((x_n) \to x \implies f(x_n) \to f(x))$
   Clearly we have X is first countable,
   $$\text{f is continuous} \iff ((x_n) \to x \implies f(x_n) \to f(x))$$

(c) In metric space, the situation is similar to analysis.
Let $\epsilon > 0$
f is continuous at point y if

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

(d) Pasting Lemma
We construct a continuous function from two related continuous functions, which is often used in pasting paths ot homotopies.

If $f : A \to Y$ and $g : B \to Y$ are continuous, and $f(x) = g(x)$ for $x \in A \cap B$, and $X = A \cup B$, then $h : X \to Y$ is continuous defined as

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

(e) If $f = \Pi_{i\in I} f_i$, then f is continuous if and only if f_i is continuous $\forall i \in I$.

(f) An extension of (a)
Let τ_1, τ_2 be two topologies on X, if $\tau_2 \subseteq \tau_1$, then the identity map

$$id : (X, \tau_1) \to (X, \tau_2)$$

is continuous.

Since τ_1 is finer than τ_2, it contains more open sets in the topology compared to τ_2, then any open sets contained in τ_2 must be open in τ_1, then the identity map is continuous.

2. Homeomorphism
$f : X \to Y$ is continuous, bijective and $f^{-1} : Y \to X$ is continuous, then f is a homeomorphism.

We say X is homeomorphic to Y, which means X and Y have same topological property.
We can regard homeomorphism as the isomorphism but between topological spaces.

(a) $f : X \to Y$ is continuous and X is compact and Y is Hausdorff, then f is homeomorphism.

(b) Bijective and local homeomorphism $\implies$ homeomorphism.
Then we can deduce, if a covering map $p : E \to X$ is injective, then p is homeomorphism.

(An example) Let (X, τ) be a compact Hausdorff space, τ' is either $\tau' \subsetneq \tau$ or $\tau \subsetneq \tau'$, then

$$(X, \tau') \text{ is not compact Hausdorff space.}$$

The reason is when constructing $id : (X, \tau) \to (X, \tau')$ or $id : (X, \tau') \to (X, \tau)$, both cases lead to contradiction, since identity map is homeomorphism.

3. Projection Map
Let $X = \Pi_{i\in I} X_i$, define the projection map onto space X_i as $\pi_{X_i} : X \to X_i$.

Notice such projection map is surjective and continuous open map, but it's not guaranteed to be a closed map.
Recall that if $U_i \subseteq X_i$ be the basic open set for each X_i, then $\Pi_{i\in I} U_i$ is a basic open set in X with the condition that $U_i = X_i$ for all but finitely many i.

A case when π is closed map.
Define $\pi_1 : X \times Y \to X$, if Y is compact, then π_1 is closed map.
To show this is true, we apply Tube Lemma to the space $X \setminus \pi_1(C)$.

4. Onto and Into Map
   Consider $f : X \to Y$
   Injective
   (a) $\forall x, y \in X$, if $f(x) = f(y)$, then $x = y$.
   (b) if there exists g such that $gof = id_X$, then f is injective (exists left inverse)
   (c) If A is a deformation retract of X, then the induced map $i_* : \pi_1(A) \to \pi_1(X)$ is isomorphism, hence injective.
   (d) If f is injective, then $|X| \leq |Y|$.

   (c), (d) are often used to construct the contradiction for whether there exists a retraction between A and X.

   Surjective
   (a) For any $y \in Y$, there exists $x \in X$ such that $f(x) = y$
   (b) if there exists h such that $foh = id_Y$, i.e. f has right inverse.
   (c) if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective.
   To show this is true, we can apply Borsuk-Ulam Theorem to $f : S^2 \to R^2$, defined as $f = iog$, where $i : S^2 \setminus s_0 \to R^2$ is a homeomorphism, which will lead to a contradiction.
   (d) Another frequently used approach is to suppose f is not surjective, and create a contradiction using the property of the space why, to see things will go wrong if $\exists y \in Y$ such that $y \notin f(X)$.

   Show X is comapct and $f : (X, d) \to (X, d)$, defined as $(x, y) \mapsto (f(x), f(y))$ is surjective.
   Suppose f is not surjective, then $\exists x_0 \in X$ such that $x_0 \notin f(X)$
   Define the sequence as $x_{n+1} = f(x_n)$. Let $n > m$, consider $d(x_m, x_n)$, which will create a sequence whose subsequences are all divergent, contradicts with in metric space, compact if and only if sequentially compact.

5. Property preserved by continuous map
   (a) X is compact/seq-compact, then $f(X)$ is compact/seq-compact, if f is continuous.
   (b) X is path-connected, then $f(X)$ is path-connected.
   (c) X is separable, then $f(X)$ is separable.
   (d) Separation properties are not preserved via continuous map.
   (e) If X is C_1/C_2 and f is continuous open surjection, then Y is C_2/C_1.
   This is a slightly stronger version for $C_2 \implies C_1$.
   (f) Ways to show X is second-countable(C_2)
      (i) X has countable basis
      Mostly, we have to contruct countable collection of sets, and then argue that such collection is a basis.
      (ii) Compact + Metric $\implies C_2$
      (iii) Metric + Separable $\implies C_2$
   (g) In metric space, the following are equivalent
      (i) Compact $\iff$ Seq-compact $\iff$ Limit-point compact
      (ii) $C_2 \iff$ Separable (In general space, we only have $C_2 \implies$ Separable)
   (h) An example
   Show there is no continuous surjection map from $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ to $\mathbb{R}^2$.
   Suppose not the case, let $g : B \to \mathbb{R}^2$ be the continuous surjection.
   Notice that B is closed and bounded subset of $\mathbb{R}^2$, which is compact, then $|g(B)| \geq |\mathbb{R}^2|$ and $g(B)$ is comapct, but it's impossible, since $\mathbb{R}^2$ is closed subspace of compact space $g(B)$ which is compact but it's not in fact.

6. Quotient Map ($q : X \to X/\sim$)
Quotient Map can be visualized as pasting the particular points in the domain together, the process of pasting refers to $x \sim y$, then x and y are in the same equivalent class in $X/\sim$.

   (a) $q : X \to Y$ is quotient map if $U \subseteq Y$ is open if and only if $q^{-1}(U) \subseteq X$ is open.
   (b) Alternatively we can regard quotient map as continuous open surjection.
   (c) Retraction is a quotient map.
   Let $r : X \to A$ be a retraction, clearly it's continuous. Let $U \subseteq A$ such that $q^{-1}(U)$ is open in X, by the property of retraction, we have
   $$U = A \cap q^{-1}(U)$$
   which is clearly open in A.
   (d) Building The Torus:
      (i) Consider $[0,1]^2$ and the quotient map that identified $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$.
      Then $q : [0,1]^2 \to T^2$ is the quotient map, T^2 is compact since $[0,1]^2$ is compact in $\mathbb{R}^2$.
      (ii) Consider the torus as the product of two S^1, then $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \mathbb{Z}^2$.
      Here we can explicitly write the map as $(x,y) \mapsto (e^{2\pi ix}, e^{2\pi iy})$ from $[0,1]^2$ to $S^1 \times S^1$.

7. Inclusion and Identity Map
The key difference is id is homeomorphism but inclusion map is not, by retricting the domain of inclusion map, we get identity map.
Moreover, inclusion map can be explained as an embedding which embeds A into X.

8. Retraction
continuous $r : X \to A$ is a retraction if $r(a) = a$ for all a in A.

   (a) Let $i : A \to X$ be the inclusion map, then $roi = id_A$, and the induced map $i_* : \pi_1(A) \to \pi_1(X)$ is injective, since it has left inverse r_*.
   The main usage is arguing whether there exists a retraction between A and X.
   (b) Examples
      (i) There is no retraction from $X = S^1 \times D^2$ to $A = S^1 \times S^1$.
      Suppose not, then $i_* : \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times D^2)$ is injective, but it's impossible since $|\mathbb{Z}^2| > |\mathbb{Z}|$
      (ii) There is no retraction between $X = R^3$ and $A = S^1$
      Suppose not, then $i_* : \pi_1(S^1) \to \pi_1(R^3)$ is injective, but it's impossible since the only injective map to trivial group is trivial group, clearly $\pi_1(S^1) = \mathbb{Z} \neq 0$
      (iii) There is no retraction between X (D^2 with two boundary points are identified) and A (∂D which is homeomorphic to $S^1 \vee S^1$)
      We first reparametrize points on the disc as (r,θ), with $(1,0) \sim (1,\pi)$.
      Consider $\hat{H}((r,\theta),t) = ((1-t)r + t, \theta)$, which is a homotopy between id_X and A, then $\pi_1(X) = \pi_1(\mathbb{R}^2 \setminus 0) = \mathbb{Z}$. But $\pi_1(A) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.
      There can't be an injective from $\pi_1(A)$ to $\pi_1(X)$.
   (c) Notice the product of retraction is retraction and the coposition of retraction is retraction.

9. Path (Visualizing path will be connect the two points with a continuous curve in the space)

   (a) continuous $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ is a path between x and y.
   (b) In convex set X, any two points are path-connected, because we can directly define $\gamma : [0,1] \to X$ as
   $$\gamma(t) = (1-t)x + ty$$
   This only works when X is convex, since at any time t, $\gamma(t)$ is the line segment contained in X, which guarantees continuity.

(c) An example when X is not convex
Claim: $\mathbb{R}^2 \setminus \{0\}$ is path connected.
Define $\gamma(t) = (cos(\pi t), sin(\pi t))$, which satisfies $\gamma(0) = (1, 0)$ and $\gamma(1) = (-1, 0)$
We can see that when X is not convex, we often try to use curve instead of straight line path.

(d) Some properties

(1) Path-connected $\implies$ connected
To see this we show by contradiction, suppose not fix $x_0 \in X$ and define A be the set that captures all the points path-connected to x_0 and B captures all the points not reachable by x_0.

(2) Product of path-connected spaces is path-connected, but not inheritary, $\mathbb{R}$ is connected but $\mathbb{Q} \subseteq \mathbb{R}$ is not connected, since $\mathbb{Q} = ((-\infty, \sqrt{2}) \cap \mathbb{Q}) \cup ((\sqrt{2}, +\infty) \cap \mathbb{Q})$

(3) $A \subseteq \mathbb{R}$ is connected if and only if A is path-connected

(4) $\bigcup_{n\in\mathbb{N}} A_n$ is connected if A_n is connected and $A_n \cap A_{n+1} \neq \emptyset$

(5) Any subset of $\mathbb{R}^n$ is path-connected, S^n is path-connected, $\mathbb{R}^n \setminus A$ is path-connected, where A contains countably many points in $\mathbb{R}^n$.

The reason why remove countable many points in a convex set remains path-connected, pick arbitrary points p,q in $\mathbb{R}^n \setminus A$, there are uncountably many lines passing p but only countably many points removed, hence with similar construction, we can form $L = L_1 * L_2$ to be the desired path that connects p and q.

Notice S^n is homeomorphic to $\mathbb{R}^{n+1} \setminus 0$. To see this, consider continuous $f : \mathbb{R}^{n+1} \to S^n$, defined as

$$f(v) = \frac{v}{||v||}$$

Deforming all the vectors back to the unit sphere in $\mathbb{R}^n$.

(6) If $U \subseteq \mathbb{R}^n$ is open and connected, then U is path-connected.

10. Path-Homotopy and Homotopy
Path-Homotopy

(a) Consider paths $\gamma_1, \gamma_2 : [0, 1] \to X$ in X, if there exists continuous $F : I \times I \to X$ such that
(i) $F(s, 0) = \gamma_1(s)$ for all s
(ii) $F(s, 1) = \gamma_2(s)$ for all S
(iii) $F(0, t) = x$ and $F(1, t) = y$ for all $t \in I$
(This is the crucial property of path-homotopy, two paths must share endpoints)
denote as $\gamma_1 \simeq_p \gamma_2$.

(b) Let $A \subseteq \mathbb{R}^n$ is convex, then any two paths who share the same endpoints are path-homotopic. The reason why refers to the space is simply connected.

(c) Path-Homotopy Class
$\simeq_p$ is an equivalence relation, hence path-homotopy class is an equivalence class.

(i) (Reflexive) Define $F(s, t) = \gamma(s)$ for all t and all s, then we can check that endpoints shared.

(ii) (Transitive) Let γ_1 be a path from x to y and γ_2 be a path from y to z. Define

$$\gamma = \begin{cases} \gamma_1(2s) & 0 \leq s \leq 0.5 \\ \gamma_2(2s-1) & 0.5 \leq s \leq 1 \end{cases}$$

is a path from x to z, but it travels twice faster on γ_1 and twice faster on γ_2 so that the total traveling remains the same.

(iii) (Symmetric) Let $\gamma_1(s)$ be a path from x to y, consider the reverse path $\overline{\gamma_1}(s) = \gamma_1(1-s)$, which is the path from y to x.
The Symmetric property shows that the orientation of paths doesn't matter, since we can always reverse the time to travel backwards.

Homotopy

(a) Let $f, g : X \to Y$ be continuous, if there exists continuous $F : X \times I \to Y$ such that

$$F(s, 0) = f(s)$$

and

$$F(s, 1) = g(s)$$

Then f is homotopic to g, denote as

$$f \simeq g$$

(b) A very crucial visualization is that f is homotopic to g means f can be continuously deform to g.

(c) $\simeq$ is an equivalence relation, the proof is similar to above.

(d) We can easily see that path-homotopy implies homotopy, which is a stronger condition compared to homotopy.

11. Reparametrization and Algebraic Operations
Reparametrization

(a) Reparametrization can be visualized as we change the speed while traveling along the fixed path γ, we have to notice that such change has to be a continuous change on the speed.

(b) Let $\phi : [0, 1] \to [0, 1]$ continuous with $\phi(0) = \gamma(0)$ and $\phi(1) = \gamma(1)$, then

$$\gamma \simeq_p \gamma\, o\, \phi$$

To see why it's path-homotopic, we define $F : I \times I \to X$ as

$$F(s, t) = \gamma((1 - t)s + t\phi(s))$$

A more intuitive understanding is regard the path-homotopy as interpolated schedule where s is the old one, and $\phi(s)$ is the reparametrized one.

Algebraic Operations

(a) Let γ_1, γ_2 be two paths, define $\gamma_1 * \gamma_2$ as

$$\gamma_1 * \gamma_2 = \begin{cases} \gamma_1(2s) & 0 \le s \le 0.5 \\ \gamma_2(2s - 1) & 0.5 \le s \le 1 \end{cases}$$

(b) Operation on Loops (loop is defined as a path with $\gamma(0) = \gamma(1)$), we have the following equality

$$[\gamma_0 * \gamma_1] = [\gamma_0] * [\gamma_1]$$

(c) Consider paths $\gamma_0 \simeq_p \gamma_0'$ and $\gamma_1 \simeq_p \gamma_1'$, then

$$\gamma_0 * \gamma_1 \simeq_p \gamma_0' * \gamma_1'$$

(d) The Left and right identity (Constant Maps e_{x_0} and e_{x_1})
We can visualize constant maps as single points in the space discretely.

Let $\gamma : I \to X$ be a path from x_0 to x_1, then we have

$$e_{x_0} * \gamma \simeq \gamma$$

and

$$\gamma * e_{x_1} \simeq \gamma$$

(e) Moving the equivalence class of loops along the path preseves the property, more precisely, such operation is an isomorphism.
Let α be a path from x_0 to x_1 in X. Consider $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, defined as

$$\hat{\alpha}([\gamma]) = [\overline{\alpha} * \gamma * \alpha]$$

Since it's a group isomorphism, we have to show bijective and homomorphism.
Via the commutative diagram, we can see that problems studying homeomorphism between topological spaces can be converted to algebraic probelms showing group homomorphisms.

(f) If X is path-connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic for any $x_0, x_1 \in X$.

(g) It's natural to wonder how the basepoints affect the homotopy class of loops in space X, since each loop is determined by the basepoint.

X is simply connected if and only if X is path-connected and $\pi_1(X) = 0$, which means every loop can be continuous deformed into another loop, even though they don't have the same basepoints, since we can use path-connectness to connect the basepoints via a path.
More precisely, all paths in simply connected space X are all homotopic and all loops in X are homotopic to a point (e_{x_0} the constant map in X).

12. Covering Map

(a) $p : E \to X$ is a covering map if p is surjective and continuous and for all $x_0 \in X$, there exists open neighbourhood U_x at x such that

$$p^{-1}(U_x) = \bigsqcup_{j \in J} S_j$$

where $p|_{S_j} : S_j \to U_x$ is homeomorphism. The crucial property of covering map is local homeomorphism and surjective continuous open map.

(b) We can visualize covering map as for any x in X, $p^{-1}(x) = \{e_\alpha\}_{\alpha \in \Lambda}$ and for each e_α, there exists $S_{j(\alpha)}$ such that

$$e_\alpha \in S_{j(\alpha)}$$

and only contains one e_α and the open covers in E are disjoint. We call each $S_{j(\alpha)}$ as sheet/ filters.

(c) Lifting and Path-Lifting
Let $p : E \to X$ be a covering map and continuous $f : Y \to X$, then there exists $\hat{f} : Y \to E$ such that

$$f = p \, o \, \hat{f}$$

Similarly we have path-lifting
Let $\alpha : I \to X$ be a path and $p : E \to X$ be a covering map such that $p(e_0) = x_0$, then there exists a path $\hat{\alpha} : I \to E$ starting at point e_0 such that

$$\alpha = p \, o \, \hat{\alpha}$$

Unique Path-Lifting Property
Let $p : E \to X$ be a covering map with $p(e_0) = x_0$, $F : I \times I \to X$ be a path-homotopy, then there exists unique $\hat{F} : I \times I \to X$ such that

$$\hat{F}(0, t) = e_0$$

Notice that $\gamma_0 \simeq_p \gamma_1$, then $\hat{\gamma}_0 \simeq_p \hat{\gamma}_1$

(d) Sepration Properties that can be pulled back via a covering map

Consider $p : E \to X$ be a covering map, we have the following statements:

(i) If X is Hausdorff, then E is Hausdorff
(ii) If X is regular, then E is regular
(iii) If X is compact and $p^{-1}(x)$ is finite for all x, then E is comapct

(e) Let $p_0 : E_0 \to X_0$ and $p_1 : E_1 \to X_1$ be covering maps, then $p : E_0 \times E_1 \to X_0 \times X_1$ defined as $p(x, y) = (p_0(x), p_1(y))$ is a covering map.
The key is to argue that $p|_{U_\alpha \times V_\alpha} = p_0|_{U_\alpha} \times p_1|_{V_\alpha}$ which is a product of homeomorphism, hence homeomorphism.

13. Fixed Point and Antipodal
Fized Point

(a) (IVT on $\mathbb{R}$ or I) If $f : [-1, 1] \to [-1, 1]$ is continuous, then there exists a point x such that $f(x) = 0$.
If we set $g(x) = f(x) - x$, we can see g has a fixed point on $[-1, 1]$

(b) (Brower Fixed Point Theorem on D^2) If continuous $f : D^2 \to D^2$, then there exists $x \in D^2$ such that
$$f(x) = x$$
Similarly, the fized point theorem still works for $D^2 \setminus [0, 1]$.
Consider $f : S^1 \to \mathbb{R}$, defined as $f(x) = f(-x)$, then f has a fixed point on S^1.
The reason why this holds is because S^1 is homeomorphic to I and S^1 is path-connected.

Antipodal Points

(a) (Antipodal Map) $a : S^n \to S^n$ defined as for all $x \in S^n$,
$$a(x) = -x$$

(b) (Antipodal-Preserving Map) continuous $f : S^n \to S^n$, for all $x \in S^n$, we have
$$f(-x) = -f(x)$$

(c) (Non-vanishing Map) $\forall v$, we have $f(v) \neq 0$

(d) Let $v : D^2 \to \mathbb{R}^2 \setminus \{0\}$ is a nonvanishing map, then for $k > 0$,
$$\exists x \in S^1 \text{ s.t. } v(x) = k \cdot x$$
and for k <0,
$$\exists x \in S^1 \text{ s.t. } v(x) = k \cdot x$$
We can understand it as for nonvanishing map, there are always a pair of eigenvectors on ∂D^2 which is S^1, who has positive eigenvalue and negative eigenvalue respectively.

(e) If continuous $f : S^1 \to S^1$ and f is antipodal preserving map, then h is not nullhomotopic.

(f) There is no continuous map from S^2 to S^1 such that f is anotipodal-preserving.

Borsuk-Ulam Theorem

Theorem 15.1. *If $f : S^2 \to \mathbb{R}^2$ is continuous then there exists $x \in S^2$ sucht that*
$$f(x) = f(-x)$$

14. Deformation Retraction

(a) Deformation Retraction describes the homotopy between retraction map and identity map, which is a stricter case of retraction.
If A is a deformation retract of X, then A can be attained via continuous deformation from X (like shrinking from the boundary to a single point at the center etc).

(b) Let $A \subseteq X$, if there exists continuous $F : X \times I \to X$ such that
$F(s, 0) = id_X(s)$
$F(s, 1) = r(s) \in A$
$F(a, t) = a$ for any $a \in A$, at any time $t \in I$

Then we say A is a deformation retract of X and F is the deformation retraction.

(c) If $A \subseteq X$ is a deformation retract of X, then the induced map $i : \pi_1(A) \to \pi_1(X)$ is isomorphism.

(d) Some Examples (Explicit/ Geometric Intuition)

(1) This graph shows the geometric motivation behind $\mathbb{R}^3 \setminus S^1$

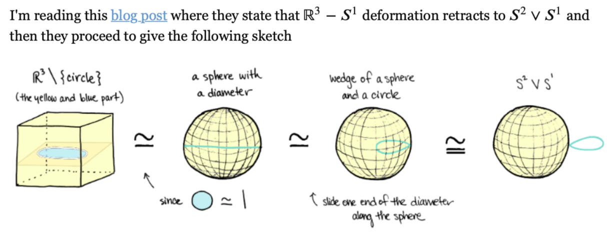


Figure 7: $\pi_1(\mathbb{R}^3 \setminus S^1)$

15. Nullhomotopic and Contractible
A main difference between nullhomotopic and contractible is that nullhomotopic refers to relation between maps and constant map, but contractible refers to space property that $id : X \to X$ is nullhomotopic.

Nullhomotopic

(a) $f : X \to Y$ is nullhomotopic if $f \simeq e_{x_0}$

(b) To visualize nullhomotopic, if f is nullhomotopic, then we can shrink f all the way to $f(x_0)$

(c) Let $h : S^1 \to X$, TFAE:
(i) h is nullhomotopic
(ii) there exists continuous $k : D^2 \to X$ such that

$$k|_{S^1} = h$$

(iii) The induced map $h_* : \pi_1(S^1) \to \pi_1(X)$ is trivial homomorphism
To see why (iii) is true we can consider the following,
Let $[\gamma]$ be an equivalence class, then

$$h_*([\gamma]) = [h \, o \, \gamma] = [e_{x_0}]$$

Since h is nullhomotopic gives us $h \simeq e_{x_0}$, then $(h \, o \, \gamma) \simeq (e_{x_0} \, o \, \gamma) \simeq e_{x_0}$

Contractible

(a) X is contractible if $id : X \to X$ satisfies $id \simeq e_{x_0}$

(b) If Y is contractible, X is arbitrary topological space, then the homotopy class $[X, Y]$ has single element.
Let H be the homotopy between e_{y_0} and id_Y, define $F(s,t) = H(f(s), t)$.
We check that $F(s, 0) = H(f(s), 0) = f(s)$ and $F(s, 1) = H(f(s), 1) = y_0$

(c) If X is contractible, Y is path-connected, then $[X, Y]$ has single element.
Let H be the homotopy between $e_{f(0)}$ and f, define $F(s,t) = f(H(s,t))$
we check that $F(s, 0) = f(H(s, 0)) = f(s)$ and $F(s, 1) = f(H(s, 1)) = f(0)$, then we have

$$f \simeq e_{f(0)}$$

By Y is path-connected, then $e_{y_0} \simeq e_{f(0)}$, hence

$$f \simeq e_{f(0)} \simeq e_{y_0}$$

(d) Any contractible space is path-connected.
Let X be contractible space, H be the homotopy between id_X and e_{x_0}.
Let a,b be two points in X, we show a and x_0; b and x_0 are path-connected respectively.

Claim: a and x_0 are path-connected.
Consider $\gamma_1(t) = H(a,t)$, then we check that $\gamma_1(0) = H(a,0) = a$ and $\gamma_1(1) = H(a,1) = x_0$

Claim: b and x_0 are path-connected.
Consider $\gamma_2(t) = H(b, 1-t)$, then we check that $\gamma_2(0) = H(b,1) = x_0$ and $\gamma_2(1) = H(b,0) = b$.
Define $\gamma : I \to X$ as

$$\gamma = \begin{cases} \gamma_1(2t) & 0 \le t \le 0.5 \\ \gamma_2(2t-1) & 0.5 \le t \le 1 \end{cases}$$

we check the following:
(1) $\gamma(0) = \gamma_1(0) = a$
(2) $\gamma(1) = \gamma_2(1) = b$
(3)when $t = 0.5$, $\gamma_1(1) = \gamma_2(0) = x_0$, by pasting lemma, γ is continuous.

16. Computing Fundamental Group
Two main Strategies:
(1) Find the deform retract of the target space, which if luckily enough, easy to compute.
(2) Using algebraic operations such as product, to decompose the space into several well-known spaces.

Well-Known Ones

(1) $\pi_1(S^1) = \mathbb{Z}$
(2) $\pi_1(\mathbb{R}^n) = \pi_1(D^2) = \pi_1(S^n)(\text{for } n \ge 2) = 0$
(3) $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \pi_1(S^1) = 0$

Examples for (1)

(1) Let X be $\mathbb{R}^3 - \{\text{z-axis}\}$
Conider $A = \mathbb{R}^2 \setminus \{0\}$, define $F : X \times I \to X$ as

$$F((x,y,z),t) = (x, y, (1-t)z)$$

It can be easily checked that such F is a deformation retraction from X to A.
Then we have $\pi_1(X) \simeq \pi_1(A) = \mathbb{Z}$

(2) Similarly we can argue that $\mathbb{R}^3$ minus arbitrary line through the origin is homeomorphic to $\mathbb{R}^3 - \{\text{z-axis}\}$ which is $\mathbb{Z}$.

(3) Find $\pi_1(D^2 \setminus \{0\})$
We claim that $\mathbb{R}^2 \setminus \{0\}$ is a deformation retract of X. Define $F : X \times I \to X$ as

$$F((x,y),t) = \frac{(x,y)}{(1-t)||(x,y)|| + tr}$$

Examples for (2)

(1) $\pi_1(\mathbb{R}^n) = \Pi_{i=1}^n \pi_1(\mathbb{R}) = 0$
(2) $\pi_1(S^1 \times D^2) = \pi_1(S^1) \times \pi_1(D^2) = \mathbb{Z}$
(3) $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$
(4) $\pi_1(S^1 V S^1) = \pi_1(S^1) \,\mathrm{V}\, \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$