Full name	Andrew ID

21-127 Final (practice)

Friday, 5 May 2023

Please read the following instructions carefully before the test begins.

Before the test

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

During the test

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 18–21), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

After the test

- Stop working immediately when you are instructed to do so.
- Turn in all 22 pages of this test; if you tore out any pages, put them back in their correct positions.

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1. (a) Define what it means for a proposition to be a tautology **[5]**

(b) Consider the following logical formula **[15]**

$$\varphi : \forall X, \forall Y, [(\exists n \in \mathbb{N}, |X| = n \wedge \exists f : X \to Y, f \text{ is injective}) \Rightarrow \exists m \in \mathbb{N}, |Y| = m]$$

(i) Express the above statement in English. **[5]**

(ii) Write the negation of the above logical formula in a maximally negated form **[5]**

(iii) Determine whether φ or $\neg\varphi$ is true. Justify your answer **[5]**

More space for (Q1)

2. (a) Write the set $\{X \in \mathscr{P}(\{1,2,3\}) : 3 \in X\}$ in list notation **[5]**

(b) Let A and B be sets. Decide whether the following statement is true: **[10]**

$$(A \cap B) \times A = (A \times A) \cap (B \times A)$$

If it is true, prove it. Otherwise, provide a counterexample.

More space for (Q2)

3. (a) Define composition of functions **[5]**

(b) Let $f : X \to Y$ and $g : Y \to Z$ be functions. Let B be a subset of Z. Consider the statement **[15]**

$$f[(g \circ f)^{-1}[B]] = g^{-1}[B]$$

(i) Find a counterexample to prove that the above statement is false **[5]**

(ii) One of the set inclusions ($\subseteq$ or $\supseteq$) for the above statement is always true. Determine which inclusion is true and prove it. **[10]**

More space for (Q3)

4. (a) Prove that for all $n \in \mathbb{N}$, $4|5^n - 1$ **[10]**

(b) Prove that for all $n \in \mathbb{N}$, **[10]**

$$\sum_{k=0}^{n} 4^k = \frac{4^{n+1} - 1}{3}$$

More space for (Q4)

5. (a) State the pigeonhole principle **[5]**

(b) Let S be the set of all functions from $[9]$ to $[4]$ that send 2 and 3 to the same output and 1 and 8 to the same output. That is **[10]**

$$S = \{f : [9] \to [4] : f(2) = f(3) \land f(1) = f(8)\}$$

Find $|S|$ and use a combinatorial argument to justify your answer

More space for (Q5)

6. (a) Use the Euclidean Algorithm to decide whether 74 has a multiplicative inverse mod 383. If it does, use the Extended Euclidean Algorithm to find such an inverse **[10]**

(b) Let p be a prime. Use Euclid's Lemma to prove that $\sqrt{p}$ is irrational **[10]**

More space for (Q6)

7. (a) State the addition principle **[5]**

(b) Let $n \geq 3$. Use the addition and the multiplication principles to prove that **[10]**

$$\binom{n}{3} = \sum_{k=1}^{n-2} \frac{(n-k)(n-k-1)}{2}$$

More space for (Q7)

8. (a) Let S be all the functions from [3] to [3]. Define the relation $\preccurlyeq$ on S as follows: **[10]**

$$\forall f, g \in S, (f \preccurlyeq g \Leftrightarrow \forall x \in [3], f(x) \leq g(x)$$

Prove that $(S, \preccurlyeq)$ is a poset

(b) Find an example of a subset T of S that has a infimum but does not have a least element. **[5]**
Justify your answer

More space for (Q8)

9. Decide if each of the following is true or false by circling **T** or **F**. No justification needed.

(a) **T** **F** $(\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$ **[3]**

(b) **T** **F** Let $f : X \to Y$ be an injective function. Then for all $y \in Y$, $|f^{-1}[\{y\}]| = 1$ **[3]**

(c) **T** **F** Let a, b, n be positive integers. Then **[3]**

$$ab \equiv 0 \mod n \Rightarrow (a \not\equiv 0 \mod n) \vee (b \not\equiv 0 \mod n)$$

.

(d) **T** **F** The pigeonhole principle implies that if we place $n+1$ pigeons into n holes, then each hole will have at least one pigeon **[3]**

(e) **T** **F** Let $A \subseteq X$ where both A and X are infinite sets. Then, $|X \setminus A| < |X|$. **[3]**

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 18).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 19).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 20).

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