Homework 6 Solutions

(You must justify ALL your claims unless otherwise stated)

Problem 1

(a) Find the greatest integer, call it m, that can't be represented as the sum of multiples of 4 and 5.

(b) Prove that for all $n \geq m+1$, n can be represented as the sum of multiples of 4 and 5.

Suggested solutions:

(a) We claim that $m = 11$ is the greatest such integer. Here we show that 11 cannot be represented as the sum of multiples of 4 and 5. This will imply $m \geq 11$. In part (b) we will show that $m \leq 11$, and hence $m = 11$.
Suppose $4s + 5t = 11$ for some $s, t \in \mathbb{N}$. We know $s, t \leq 2$ because $4 \cdot 3$ and $5 \cdot 3$ are both greater than 11. We list all the remaining possibilities:

(i) $s = 0$ and $t = 0$: $4s + 5t = 0 < 11$.
(ii) $s = 1$ and $t = 0$: $4s + 5t = 4 < 11$.
(iii) $s = 2$ and $t = 0$: $4s + 5t = 8 < 11$.
(iv) $s = 0$ and $t = 1$: $4s + 5t = 5 < 11$.
(v) $s = 1$ and $t = 1$: $4s + 5t = 9 < 11$.
(vi) $s = 2$ and $t = 1$: $4s + 5t = 13 > 11$.
(vii) $s = 0$ and $t = 2$: $4s + 5t = 10 < 11$.
(viii) $s = 1$ and $t = 2$: $4s + 5t = 14 > 11$.
(ix) $s = 2$ and $t = 2$: $4s + 5t = 18 > 11$.

Therefore 11 cannot be represented as the sum of multiples of 4 and 5.

(b) We prove by induction that any $n \geq 12$ can be represented as the sum of multiples of 4 and 5. The base case is $12 = 4 \cdot 3 + 5 \cdot 0$.
Assume the claim is true for some $n \geq 12$, say $n = 4s + 5t$ for some $s, t \in \mathbb{N}$. We need to write $n + 1$ as the sum of multiples of 4 and 5. There are two cases:

(i) $s = 0$: $5t = n \geq 12$ so $t \geq 3$. Then $4 \cdot 4 + 5(t-3) = 16 + 5t - 15 = 5t + 1 = n + 1$.
(ii) $s \geq 1$: Then $4 \cdot (s-1) + 5 \cdot (t+1) = 4s + 5t + 1 = n + 1$.

Problem 2

Prove the following statement using weak induction:

$$\forall n \geq 1,\ \frac{1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots (2n+2)} \geq \frac{1}{2n+2}$$

Suggested solution:
Base case $n = 1$: LHS $= \frac{1\cdot 3}{2\cdot 4} = \frac{3}{8} \geq \frac{1}{4} =$ RHS.
Assume for some $n = 1$, we have

$$\frac{1 \cdots (2n+1)}{2 \cdots (2n+2)} \geq \frac{1}{2n+2}$$

Inductive step:

$$\begin{aligned}
\frac{1 \cdots (2n+1)(2n+3)}{2 \cdots (2n+2)(2n+4)} &= \frac{1 \cdots (2n+1)}{2 \cdots (2n+2)} \cdot \frac{2n+3}{2n+4} \\
&\geq \frac{1}{2n+2} \cdot \frac{2n+3}{2n+4} \\
&= \frac{2n+3}{2n+2} \cdot \frac{1}{2n+4} \\
&> 1 \cdot \frac{1}{2n+4} \\
&= \frac{1}{2(n+1)+2}
\end{aligned}$$

Problem 3

Prove the following statement using weak induction:

$$\forall n \in \mathbb{N},\ \sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

[Hint: Do we have a formula for $\sum_{k=1}^{n} k$ that we can use? You don't need to re-prove statements that we previously proved in class]
Suggested solution:
Base case $n = 0$: LHS $= \sum_{k=1}^{0} k^3 = 0 = 0^2 = (\sum_{k=1}^{0} k)^2 =$ RHS.
Assume for some $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

Inductive step:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \left(\sum_{k=1}^{n} k\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{2^2} + \frac{4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2+4(n+1))}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\sum_{k=1}^{n+1} k\right)^2
\end{aligned}$$

Problem 4

Recall that the Fibonacci numbers are defined by:

$$f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3$$

Show that: $f_1 + f_3 + f_5 + ... + f_{2n-1} = f_{2n}$ for all $n \geq 1$.
Suggested solution:
Base case $n = 1$: LHS $= f_1 = 1 = f_2 =$ RHS.
Assume for some $n \geq 1$, we have

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

Inductive step:

$$\begin{aligned}
f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} &= (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} \\
&= f_{2n} + f_{2n+1} \\
&= f_{2n+2}
\end{aligned}$$

Problem 5

Use strong induction to prove that: $\forall n \in \mathbb{N}$, $12 | (n^4 - n^2)$.
[Hint: In your IS, write $n+1$ as $m+6$, where $m = n-5$. This means that the $n+1^{st}$ step uses the $n-5^{th}$ step. How many base cases will you need?]
Suggested solution:
Base cases:

(i) $n = 0$: $0^4 - 0^2 = 0$ is divisible by 12.

(ii) $n = 1$: $1^4 - 1^2 = 0$ is divisible by 12.

(iii) $n = 2$: $2^4 - 2^2 = 12$ is divisible by 12.

(iv) $n = 3$: $3^4 - 3^2 = 72 = 12 \cdot 6$ is divisible by 12.

(v) $n = 4$: $4^4 - 4^2 = 240 = 12 \cdot 20$ is divisible by 12.

(vi) $n = 5$: $5^4 - 5^2 = 600 = 12 \cdot 50$ is divisible by 12.

Let $n \in \mathbb{N}$. Assume the claim is true for $n, n+1, n+2, n+3, n+4$ and $n+5$. For the n case, there is $k \in \mathbb{N}$ such that $n^4 - n^2 = 12k$. We need to prove that $12|(n+6)^4 - (n+6)^2$.

$$\begin{aligned}
(n+6)^4 - (n+2)^2 &= n^4 + 24n^3 + 216n^2 + 864n + 2196 - n^2 - 12n - 36 \\
&= n^4 - n^2 + 24n^3 + 216n^2 + 864n + 2196 - 12n - 36 \\
&= 12k + 24n^3 + 216n^2 + 852n + 2160 \\
&= 12(k + 6n^3 + 18n^2 + 71n + 180)
\end{aligned}$$

Problem 6

Prove that:

$$\forall n \geq 1, \prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}$$

Suggested solution:
Base cases $n = 1$: LHS $= 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} =$ RHS. Assume for some $n \geq 1$, we have

$$\prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}$$

Inductive step:

$$\begin{aligned}
\prod_{k=1}^{n+1} \left(1 - \frac{1}{2^k}\right) &= \prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right) \cdot \left(1 - \frac{1}{2^{n+1}}\right) \\
&\geq \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)\left(1 - \frac{1}{2^{n+1}}\right) \\
&= \frac{1}{4} + \frac{1}{2^{n+1}} - \frac{1}{4 \cdot 2^{n+1}} - \frac{1}{2^{2n+2}} \\
&= \frac{1}{4} + \frac{3}{4 \cdot 2^{n+1}} - \frac{1}{2^{2n+2}} \\
&= \frac{1}{4} + \frac{1}{2 \cdot 2^{n+1}} + \frac{1}{4 \cdot 2^{n+1}} - \frac{1}{2^{2n+2}} \\
&= \frac{1}{4} + \frac{1}{2^{n+2}} + \left(\frac{1}{2^{n+3}} - \frac{1}{2^{2n+2}}\right) \\
&\geq \frac{1}{4} + \frac{1}{2^{n+2}} + \left(\frac{1}{2^{2n+2}} - \frac{1}{2^{2n+2}}\right) \text{ since } n \geq 1 \\
&= \frac{1}{4} + \frac{1}{2^{n+2}}
\end{aligned}$$