| Full name | Andrew ID |
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# 21-127 Test 2 (Practice)

Wednesday, 22 March 2023

Please read the following instructions carefully before the test begins.

**Before the test**

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

**During the test**

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 12–13), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

**After the test**

- Stop working immediately when you are instructed to do so.
- Turn in all 14 pages of this test; if you tore out any pages, put them back in their correct positions.

**Do not write on this page**

**1.** (a) Write the definition of an injective function **[5]**

(b) For each of the following functions, decided if the given function is well-defined. If it is, no need for justification. If it is not, explain why it is not well-defined **[10]**

(i) $f : \mathscr{P}(\mathbb{N}) \to \mathbb{N}$, where $f(A) = |A|$, for all $A \in \mathscr{P}(\mathbb{N})$.

(ii) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $g(m,n) = 4m + 5n$, for all $(m,n) \in \mathbb{N} \times \mathbb{N}$

(c) For the well-defined function(s) above, decide if they are injective **[5]**

**Suggested solutions:**

(a) A function $f : X \to Y$ is injective if the following holds: For any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(b) $f$ is not well-defined because $f(\mathbb{N})$ is infinite, not a natural number.
$g$ is well-defined.

(c) $g$ is not injective because $g(5,0) = 20 = g(0,4)$.

More space for (Q1)

**2.** (a) Write the definition of a right inverse of a function **[5]**

(b) Let $f : X \to Y$ be a function. Prove that, if $f$ has a right inverse, then $f$ is surjective. **[10]**

**Suggested solutions:**

(a) Let $f : X \to Y$ be a function. A right inverse $g : Y \to X$ of $f$ satisfies $f \circ g = \mathrm{id}_Y$ (or $f(g(y)) = y$ for all $y \in Y$).

(b) Let $y \in Y$. We need to find $x \in X$ such that $f(x) = y$. Since $f$ has a right inverse, say $g$, we know $f(g(y)) = y$. Therefore $x = g(y)$ witnesses that $f$ is surjective.

More space for (Q2)

**3.** (a) Write down the recursive definition of $n!$, for all $n \in \mathbb{N}$ **[5]**

(b) Prove that $n^3 + 2n$ is divisible by 3 for all integers $n \in \mathbb{N}$. **[10]**

**Suggested solutions:**

(a) $0! = 1$ and $(n+1)! = (n!) \cdot (n+1)$ for $n \in \mathbb{N}$.

(b) We prove by weak induction. Base case: $0^3 + 2n = 0 = 0 \cdot 3$ is divisible by 3. Assume $n^3 + 2n$ is divisible by 3 for some $n \in \mathbb{N}$. We can write $n^3 + 2n = 3r$ for some $r \in \mathbb{N}$. Consider

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= n^3 + 2n + 3n^2 + 3n + 3 \\ &= 3r + 3n^2 + 3n + 3 \\ &= 3(r + n^2 + n + 1)\end{aligned}$$

Therefore, $(n+1)^3 + 2(n+1)$ is also divisible by 3.

More space for (Q3)

**4.** (a) Define what it means for a relation to be antisymmetric **[5]**

(b) Let $X = \{1, 2, 3\}$. **[10]**

- Give the graph of a nonempty relation $R$ on $X$ that is transitive and symmetric but not reflexive. If no such relation exists, prove that it is not possible
- Give the graph of a nonempty relation $S$ on $X$ that is reflexive and symmetric but not transitive. If no such relation exists, prove that it is not possible

**Suggested solutions:**

(a) A relation $R$ on $X$ is antisymmetric if the following holds: For any $x_1, x_2 \in X$, if $x_1 R x_2$ and $x_2 R x_1$, then $x_1 = x_2$.

(b) Transitive, symmetric but not reflexive: $\mathrm{Gr}(R) = \{(2,2)\}$. (Not reflexive becuase (1,1) is not in the graph.)
Reflexive, symmetric but not transitive: $\mathrm{Gr}(S) = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$.
(Not transitive because (1,3) is not in the graph but (1,2), (2,3) are.)

More space for (Q4)

**5.** (a) Let $f : X \to Y$ be a function and let $A \subseteq X$. Define the image of $A$ under $f$. **[5]**

(b) Give an example of a function $f : X \to Y$ and a subset $A \subseteq X$ such that $f^{-1}[f[A]] \neq A$. **[7]**

(c) Let $f : X \to Y$ be an injective function. Prove that $f^{-1}[f[A]] = A$, for all $A \subseteq X$. **[8]**

**Suggested solutions:**

(a) $f[A] = \{y \in Y \mid \exists x \in A, y = f(x)\}$.

(b) $X = \{1,2\}$, $Y = \{3\}$, $f(1) = f(2) = 3$, $A = \{1\}$. Then $f^{-1}[f[A]] = f^{-1}[\{3\}] = \{1,2\} \neq A$.

(c) We first prove $A \subseteq f^{-1}[f[A]]$ (this does not use injectivity). Let $x \in A$. We need to check that $x \in f^{-1}[f[A]]$. Equivalently we need to check $f(x) \in f[A]$ but this is true because $x \in A$. Conversely, let $u \in f^{-1}[f[A]]$. $f(u) \in f[A]$. There is $a \in A$ such that $f(u) = f(a)$. By injectivity $u = a \in A$. Therefore $u \in A$.

More space for (Q5)

**If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 12).**

**If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 13).**

**Do not write on this page**