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21-127 Test 2 Solutions

Wednesday, 22 March 2023

Please read the following instructions carefully before the test begins.

Before the test

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

During the test

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 12–13), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

After the test

- Stop working immediately when you are instructed to do so.
- Turn in all 14 pages of this test; if you tore out any pages, put them back in their correct positions.

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1. (a) Define what it means for a function to be surjective **[5]**

Suggested solution:

A function $f : X \to Y$ is surjective if for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

(b) For each of the following functions, decided if the given function is well-defined. If it is, no need for justification. If it is not, explain why it is not well-defined **[10]**

(i) $f : \mathscr{P}(\mathbb{Z}) \to \mathbb{Z}$, where $f(A) = \min(A)$, that is, $f(A)$ is the smallest element in A, for all $A \in \mathscr{P}(\mathbb{Z})$.

(ii) $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, where $g(m,n) = 6m - 3n$, for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$

Suggested solution:

f is not well-defined because $\mathbb{Z} \in \mathscr{P}(\mathbb{Z})$ does not have a smallest element. g is well-defined.

(c) For the well-defined function(s) above, decide if they are surjective **[5]**

Suggested solution:

g is not surjective becasue for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, $g(m,n) = 3(2m - n)$ is always divisible by 3. But 1 in the codomain $\mathbb{Z}$ is not.

More space for (Q1)

2. (a) Write the definition of a left inverse of a function **[5]**

Suggested solution:

Let $f: X \to Y$ be a function. g is the left inverse of f if $g \circ f = \mathrm{id}_X$.

(b) Let $f: X \to Y$ be a function. Prove that, if f has a left inverse, then f is injective. **[10]**

Suggested solution:

By assumption let g be the left inverse of f. We prove injectivity.
Let $a, b \in X$ such that $f(a) = f(b)$. Applying g to both sides to the equation, we have

$$a = g(f(a)) = g(f(b)) = b.$$

More space for (Q2)

3. (a) Explain in details the main difference between the following two proof strategies: **[5]**

- Proof by weak induction with one base case
- Proof by strong induction with one base case

Suggested solution:

Let $p(n)$ be the proposition to be proven. In the inductive assumption, weak induction assumes $p(n)$ for some $n \in \mathbb{N}$. In contrast, strong induction with one base case fixes some $n \in \mathbb{N}$ and assumes $p(k)$ for $k \leq n$.

(b) Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all $n \in \mathbb{N}$. **[10]**

Suggested solution:

Base case: $5^{2\cdot 0+1} + 2^{2\cdot 0+1} = 5 + 2 = 7 = 1 \cdot 7$ is divisible by 7.
Inductive assumption: assume $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for some $n \in \mathbb{N}$.
Inductive step: we need to show $5^{2(n+1)+1} + 2^{2(n+1)+1}$ is divisible by 7. By the inductive assumption, there is $r \in \mathbb{Z}$ such that $5^{2n+1} + 2^{2n+1} = 7r$. We rewrite this as $5^{2n+1} = 7r - 2^{2n+1}$. Now

$$\begin{aligned} 5^{2(n+1)+1} + 2^{2(n+1)+1} &= 5^2 \cdot 5^{2n+1} + 2^{2(n+1)+1} \\ &= 5^2(7r - 2^{2n+1}) + 2^{2(n+1)+1} \\ &= 7(25r) + 2^{2n+1}(-25+4) \\ &= 7(25r - 3 \cdot 2^{2n+1}). \end{aligned}$$

Therefore, $5^{2(n+1)+1} + 2^{2(n+1)+1}$ is divisible by 7.

More space for (Q3)

4. (a) Define what it means for a relation R on a set X to be reflexive **[5]**

(b) Give an example of an equivalence relation $\sim$ on $\mathbb{Z}$ with infinitely many equivalence classes, that is, $\mathbb{Z}/\sim$ is infinite. Prove your claim by proving the following: **[10]**

- Prove that your definition of $\sim$ is in fact an equivalence relation
- prove that $\mathbb{Z}/\sim$ is infinite

Suggested solutions:

(a) A relation R on a set X is reflexive if for any $x \in X$, xRx.

(b) An example is the equality relation $=$.
Reflexivity: for any $x \in \mathbb{Z}$, $x = x$.
Symmetry: for any $x, y \in \mathbb{Z}$, if $x = y$, then $y = x$.
Transitivity: for any $x, y, z \in \mathbb{Z}$, if $x = y$ and $y = z$, then $x = z$.
(These properties hold by the axioms of equality.)
For each $x \in \mathbb{Z}$, $[x]_= = \{x\}$. Thus $|\mathbb{Z}/=|$ is the same as $|\mathbb{Z}|$ which is infinite.

More space for (Q4)

5. (a) Let $f : X \to Y$ be a function and let $B \subseteq Y$. Define the preimage $f^{-1}[B]$ of B under f. **[5]**

Suggested solution: $f^{-1}[B] = \{x \in X \mid f(x) \in B\}$.

(b) True or false: $f^{-1}[Y] = X$? Prove the statement if it is true, otherwise, provide a counterexample to show that the statement is false. **[8]**

Suggested solution: True:

$$\begin{aligned} f^{-1}[Y] &= \{x \in X \mid f(x) \in Y\} \\ &= \{x \in X \mid \top\} \text{ because } f \text{ has codomain } Y \\ &= X \end{aligned}$$

(c) Let $f : X \to Y$ be a surjective function. Prove that $f[f^{-1}[B]] = B$, for all $B \subseteq Y$. **[7]**

Suggested solution:

Fix $B \subseteq Y$.

We first prove $f[f^{-1}[B]] \subseteq B$. Let $y \in f[f^{-1}[B]]$. There exists $x \in f^{-1}[B]$ such that $f(x) = y$. Since $x \in f^{-1}[B]$, $f(x) \in B$. But then $y = f(x) \in B$ as desired.

Conversely, we prove $B \subseteq f[f^{-1}[B]]$. Let $y \in B$. By surjectivity there exists $x \in X$ such that $f(x) = y$. Since $f(x) = y$ and $y \in B$, we have $f(x) \in B$. This implies $x \in f^{-1}[B]$. Combining this with $y = f(x)$, we can conclude that $y \in f[f^{-1}[B]]$.

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 12).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 13).

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