Practice Problems for Lectures 9-11 Solutions

(not to be submitted)

Problem 1

Which one of the following statements is the true one? Prove your answer.

(a) $\mathscr{P}(\mathbb{N}) \subseteq \bigcup_{n\in\mathbb{N}} \mathscr{P}([n])$

(b) $\bigcup_{n\in\mathbb{N}} \mathscr{P}([n]) \subseteq \mathscr{P}(\mathbb{N})$

Suggested solutions:

(a) False: $\mathbb{N} \in \mathscr{P}(\mathbb{N}) \setminus \bigcup_{n\in\mathbb{N}} \mathscr{P}([n])$. $\mathbb{N} \in \mathscr{P}(\mathbb{N})$ because $\mathbb{N} \subseteq \mathbb{N}$. Let $n \in \mathbb{N}$. Any subset of $[n]$ does not contain $n+1 \in \mathbb{N}$. Therefore $\mathbb{N}$ cannot be a subset of $[n]$. Hence $\mathbb{N} \notin \mathscr{P}([n])$. Since this is true for any $n \in \mathbb{N}$, $\mathbb{N} \notin \bigcup_{n\in\mathbb{N}} \mathscr{P}([n])$.

(b) True: let $A \in \bigcup_{n\in\mathbb{N}} \mathscr{P}([n])$. There is $n \in \mathbb{N}$ such that $A \in \mathscr{P}([n])$. This implies $A \subseteq [n]$. Since $A \subseteq [n]$ and $[n] \subseteq \mathbb{N}$, we have $A \subseteq \mathbb{N}$ (For any $x \in A$, by $A \subseteq [n]$ we have $x \in [n]$. By $[n] \subseteq \mathbb{N}$ we have $x \in \mathbb{N}$). This means $A \in \mathscr{P}(\mathbb{N})$.

Problem 2

Let A be a set. Evaluate the following:

(a) $\bigcup_{X\in\mathscr{P}(A)} X$

(b) $\bigcap_{X\in\mathscr{P}(A)} X$

Suggested solutions:

(a) We claim that $\bigcup_{X\in\mathscr{P}(A)} X = A$. Since $A \in \mathscr{P}(A)$, we have $A \subseteq \bigcup_{X\in\mathscr{P}(A)} X$. On the other hand, if $x \in \bigcup_{X\in\mathscr{P}(A)} X$, there is $X \in \mathscr{P}(A)$ such that $x \in X$. Since $x \in X$ and $X \subseteq A$, we have $x \in A$. Therefore, $\bigcup_{X\in\mathscr{P}(A)} X \subseteq A$.

(b) We claim that $\bigcap_{X\in\mathscr{P}(A)} X = \emptyset$. Suppose there is $x \in \bigcap_{X\in\mathscr{P}(A)} X$. Then x is in every $X \in \mathscr{P}(A)$. In particular, it is in $\emptyset$ (which is a subset of A), contradiction.

Problem 3

For each of the following, decide if the given description is a well-defined function. Justify your answer.

(a) $f : \mathbb{Q} \to \mathbb{Q}$, defied by $f(x) = \frac{1}{x+3}$, for all $x \in \mathbb{Q}$.

(b) $g : \mathbb{R} \to \mathbb{R}$, defied by: for $x \in \mathbb{R}$

$$g(x) = \begin{cases} x^2 + 1, & \text{if} \quad x \geq -2 \\ x + 5, & \text{if} \quad x < 0 \end{cases}$$

.

(c) $h : \mathbb{R} \times \mathbb{R} \to \{-2, -1, 0, 1, 2\}$, defied by: for $(x, y) \in \mathbb{R} \times \mathbb{R}$

$$h(x, y) = \begin{cases} 1, & \text{if} \quad x > y \\ -1, & \text{if} \quad y > x \\ 0, & \text{if} \quad x = y \end{cases}$$

.

Suggested solutions:

(a) No, it fails totality when $x = -3$.

(b) No, it fails uniqueness at $x = -2$. The first case gives $g(-2) = (-2)^2 + 1 = 5$ while the second case gives $g(-2) = -2 + 5 = 3$.

(c) Yes, the cases given in the definition of $h(x, y)$ are disjoint and exhaustive (over all possible pairs $(x, y) \in \mathbb{R} \times \mathbb{R}$). Also, the defined values 1, -1 and 0 are within the codomain.

Problem 4

Let $f : \mathbb{N} \to \mathbb{Q}$, $g : \mathbb{Q} \to \mathbb{R}$, $h : \mathbb{N} \to \mathbb{Z}$, and $k : \mathbb{Z} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x^2 + 1}, \text{ for all } x \in \mathbb{N}$$

$$g(x) = \begin{cases} \frac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}, \text{ for all } x \in \mathbb{Q}$$

$$h(x) = x^4, \text{ for all } x \in \mathbb{N}$$

$$k(x) = \begin{cases} \sqrt{x} + 1, & \text{if} \quad x \geq 0 \\ \sqrt{-x} + 1, & \text{if} \quad x < 0 \end{cases}, \text{ for all } x \in \mathbb{Z}$$

Prove that $g \circ f = k \circ h$, using the extensionality axiom for functions.

Suggested solution:

Both composite functions have domain $\mathbb{N}$ and codomain $\mathbb{R}$. Let $x \in \mathbb{N}$. We check that $g \circ f(x) = k \circ h(x)$.

$$\begin{aligned}
g \circ f(x) &= g(f(x)) \\
&= g(1/(x^2+1)) \\
&= 1/(1/(x^2+1)) \text{ since } 1/(x^2+1) \neq 0 \\
&= x^2+1 \\
k \circ h(x) &= k(h(x)) \\
&= k(x^4) \\
&= \sqrt{x^4}+1 \text{ since } x^4 \text{ is nonnegative} \\
&= x^2+1
\end{aligned}$$