Full name	Andrew ID	

21-127 Final (practice)

Friday, 5 May 2023

Please read the following instructions carefully before the test begins.

Before the test

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

During the test

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 18–21), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

After the test

- Stop working immediately when you are instructed to do so.
- Turn in all 22 pages of this test; if you tore out any pages, put them back in their correct positions.

Page 2 of 22

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1.	(a) Define what it means for a proposition to be a tautology					
	(b) Consider the following logical formula	[15]				
	$\varphi: \forall X, \forall Y, [(\exists n \in \mathbb{N}, X = n \land \exists f: X \to Y, f \text{ is injective}) \Rightarrow \exists m \in \mathbb{N}, Y = m]$ (i) Express the above statement in English.					
		[5]				
	(ii) Write the negation of the above logical formula in a maximally negated form	[5]				

(iii) Determine whether φ or $\neg \varphi$ is true. Justify your answer

[5]

Page 4 of 22 (Q1)

More space for (Q1)

2. (a) Write the set $\{X \in \mathcal{P}(\{1,2,3\}) : 3 \in X\}$ in list notation

(b) Let A and B be sets. Decide whether the following statement is true: [10]

$$(A \cap B) \times A = (A \times A) \cap (B \times A)$$

If it is true, prove it. Otherwise, provide a counterexample.

[5]

Page 6 of 22 (Q2)

More space for (Q2)

3. (a) Define composition of functions

[5]

[5]

- (b) Let $f: X \to Y$ and $g: Y \to Z$ be functions. Let B be a subset of Z. Consider the statement $f[(g \circ f)^{-1}[B]] = g^{-1}[B]$
 - (i) Find a counterexample to prove that the above statement is false

(ii) One of the set inclusions (⊆ or ⊇) for the above statement is always true. Determine which inclusion is true and prove it.[10]

Page 8 of 22 (Q4)

More space for (Q3)

4. (a) Prove that for all
$$n \in \mathbb{N}$$
, $4|5^n-1$ [10]

(b) Prove that for all
$$n \in \mathbb{N}$$
,
$$\sum_{k=0}^{n} 4^k = \frac{4^{n+1}-1}{3}$$
 [10]

Page 9 of 22 (Q4)

More space for (Q4)

5. (a) State the pigeonhole principle

[5]

(b) Let *S* be the set of all functions from [9] to [4] that send 2 and 3 to the same output and 1 [10] and 8 to the same output. That is

$$S = \{f : [9] \rightarrow [4] : f(2) = f(3) \land f(1) = f(8)\}$$

Find |S| and use a combinatorial argument to justify your answer

Page 11 of 22 (Q5)

More space for (Q5)

6. (a) Use the Euclidean Algorithm to decide whether 74 has a multiplicative inverse mod 383. If it does, use the Extended Euclidean Algorithm to find such an inverse

(b) Let p be a prime. Use Euclid's Lemma to prove that \sqrt{p} is irrational

Page 13 of 22 (Q7)

More space for (Q6)

7. (a) State the addition principle

[5]

(b) Let $n \ge 3$. Use the addition and the multiplication principles to prove that

$$\binom{n}{3} = \sum_{k=1}^{n-2} \frac{(n-k)(n-k-1)}{2}$$

Page 14 of 22 (Q7)

More space for (Q7)

8. (a) Let S be all the functions from [3] to [3]. Define the relation \leq on S as follows: [10]

$$\forall f, g \in S, (f \preccurlyeq g \Leftrightarrow \forall x \in [3], f(x) \le g(x)$$

Prove that (S, \preccurlyeq) is a poset

(b) Find an example of a subset *T* of *S* that has a infimum but does not have a least element. [5] Justify your answer

Page 16 of 22 (Q8)

More space for (Q8)

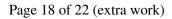
9. I	Decide if eac	h of the	following i	s true or	false by	circling T	or F . No	justification	needed.
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(a)
$$\mathbf{T} \quad \mathbf{F} \quad (\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$$
 [3]

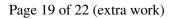
(b) **T F** Let
$$f: X \to Y$$
 be an injective function. Then for all $y \in Y$, $|f^{-1}[\{y\}]| = 1$ [3]

(c) **T F** Let
$$a,b,n$$
 be positive integers. Then
$$ab \equiv 0 \mod n \Rightarrow (a \not\equiv 0 \mod n) \lor (b \not\equiv 0 \mod n)$$
 [3]

- (d) \mathbf{T} \mathbf{F} The pigeonhole principle implies that if we place n+1 pigeons into n holes, then each hole will have at least one pigeon
- (e) **T F** Let $A \subseteq X$ where both A and X are infinite sets. Then, $|X \setminus A| < |X|$. [3]



If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 18).



If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 19).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 20).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 21).

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