Homework 4 Solutions

Problem 1

For each of the following, express the set in list or interval notation (whichever is appropriate) or as a union of such sets.

(a) $f[(-1,3)]$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ for all $x \in \mathbb{R}$.

(b) $g^{-1}[(0,2]]$, where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = |x-1| + |x+1|$ for all $x \in \mathbb{R}$.

(c) $h[\mathbb{R} \setminus \mathbb{Z}]$, where $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) = x^2$ for all $x \in \mathbb{R}$.

(d) (Extra practice, not to be submitted)
$p^{-1}[\{2,3\}]$, where $p : \mathbb{Z} \to \mathbb{Z}$ is defined by letting $p(n)$ be the remainder of n^2 when divided by 5 (for example, $7^2 = 49 = 9 \times 5 + 4$, so $p(7) = 4$).

Suggested solutions:

(a) $f[(-1,3)] = [0,9)$.
Let $y \in f[(-1,3)]$. Then $y = f(x)$ for some $x \in (-1,3)$. There are two cases: if $x \in (-1,0)$, then $x^2 \in (0,1) \subseteq [0,9)$. If $x \in [0,3)$, then $x^2 \in [0,9)$. Therefore, we must have $y = x^2 \in [0,9)$.
Let $y \in [0,9)$. We need to find $x \in (-1,3)$ such that $f(x) = y$. Define $x = \sqrt{y} \in [0,3) \subseteq (-1,3)$.

(b) $g^{-1}[(0,2]] = [-1,1]$.
Before proving the equality, we consider cases of x. When $x < -1$, $g(x) > |-1-1| + |x+1| = 2 + |x+1| \geq 2$ so $g(x) > 2$. When $x \in [-1,1]$, $g(x) = -(x-1) + (x+1) = 2$. When $x > 1$, $g(x) > |x-1| + |1+1| = |x-1| + 2 \geq 2$ so $g(x) > 2$.
Let $x \in [-1,1]$. The middle case of the previous paragraph has already showed that $g(x) = 2$ so $x \in g^{-1}[(0,2]]$.
Let $x \in g^{-1}[(0,2]]$. The case analysis above gives $x \in [-1,1]$ (if x is outside $[-1,1]$, $g(x) > 2$).

(c) $h[\mathbb{R} \setminus \mathbb{Z}] = \bigcup_{n\in\mathbb{N}} (n^2, (n+1)^2)$.
Let $y \in h[\mathbb{R} \setminus \mathbb{Z}]$. There is $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $y = h(x) = x^2$. Since $x \notin \mathbb{Z}$, there is $z \in \mathbb{Z}$ such that $x \in (z, z+1)$. If $z \geq 0$, then $y = x^2 \in (z^2, (z+1)^2)$ so $n = z$ witnesses $y \in \bigcup_{n\in\mathbb{N}} (n^2, (n+1)^2)$. If $z < 0$, then $y = x^2 \in ((z+1)^2, z^2) = ((-z-1)^2, (-z)^2)$ so $n = -z-1$ witnesses $y \in \bigcup_{n\in\mathbb{N}} (n^2, (n+1)^2)$.
Let $y \in \bigcup_{n\in\mathbb{N}} (n^2, (n+1)^2)$. There is $n \in \mathbb{N}$ such that $y \in (n^2, (n+1)^2)$. We need to find $x \in \mathbb{R} \setminus \mathbb{Z}$ such that $y = h(x) = x^2$. Define $x = \sqrt{y}$. Then $x \in (n, n+1) \subseteq \mathbb{R} \setminus \mathbb{Z}$.

(d) $p^{-1}[\{2,3\}] = \emptyset$.
Let $n \in \mathbb{Z}$. We can write $n = 5k + r$ for some $r \in \{0, 1, 2, 3, 4\}$. Consider $(5k + r)^2 = 25k^2 + 10kr + r^2 = 5(5k^2 + 2kr) + r^2$. Since the first term $5(5k^2 + 2kr)$ is divisible by 5, we know $p(n)$ is the remainder of r^2 when divided by 5. For $r \in \{0, 1, 2, 3, 4\}$, the remainder of r^2 when divided by 5 can only be 0, 1 or 4. Therefore it is impossible for $p(n)$ to be 2 or 3.

Problem 2

Let $f : X \to Y$ be a function. Suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Decide (with proof) whether each of the following is true or false. If the statement is false, prove which of the inclusions ($\subseteq$ or $\supseteq$) must be true and provide a counterexample for the other inclusion.

(a) $f[A \cap B] = f[A] \cap f[B]$

(b) $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$

(c) (Extra practice, not to be submitted) $f[A \cup B] = f[A] \cup f[B]$

(d) (Extra practice, not to be submitted) $f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D]$

Suggested solutions:

(a) Only $\subseteq$ is true: let $y \in f[A \cap B]$. There is $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$, $x \in A$ and $x \in B$. Therefore, $x \in A$ and $f(x) = y$. This implies $y \in f[A]$. Similarly, $x \in B$ and $f(x) = y$. This implies $y \in f[B]$. Therefore $y \in f[A] \cap f[B]$.

We show that $f[A] \cap f[B] \not\subseteq f[A \cap B]$: let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $A = \{x \in \mathbb{R} \mid x > 0\}$ and $B = \{x \in \mathbb{R} \mid x < 0\}$. We have $f[A \cap B] = f[\emptyset] = \emptyset$ which cannot contain $f[A] \cap f[B] = A \cap A = A \neq \emptyset$.

(b) True: let $x \in f^{-1}[C \cap D]$. $f(x) \in C \cap D$. Hence $f(x) \in C$ and $f(x) \in D$. Since $f(x) \in C$, $x \in f^{-1}[C]$. Since $f(x) \in D$, $x \in f^{-1}[D]$. Therefore, $x \in f^{-1}[C] \cap f^{-1}[D]$.

Let $x \in f^{-1}[C] \cap f^{-1}[D]$. $x \in f^{-1}[C]$ and $x \in f^{-1}[D]$. Thus $f(x) \in C$ and $f(x) \in D$. These give $f(x) \in C \cap D$. $x \in f^{-1}[C \cap D]$.

(c) True: let $y \in f[A \cup B]$. There is $x \in A \cup B$ such that $f(x) = y$. Case (1): $x \in A$. Since $y = f(x)$, we can conclude that $y \in f[A]$. Hence $y \in f[A] \cup f[B]$. Case (2): $x \in B$. Since $y = f(x)$, we can conclude that $y \in f[B]$. Hence $y \in f[A] \cup f[B]$. Therefore, we can conclude $y \in f[A] \cup f[B]$.

Let $y \in f[A] \cup f[B]$. Case (1): $y \in f[A]$. There is $x \in A$ such that $y = f(x)$. Since $x \in A$, $x \in A \cup B$. Therefore $y \in f[A \cup B]$. Case (2): $y \in f[B]$. There is $x \in B$ such that $y = f(x)$. Since $x \in B$, $x \in A \cup B$. Therefore $y \in f[A \cup B]$.

(d) True: let $x \in f^{-1}[C \cup D]$. $f(x) \in C \cup D$. Case (1): $f(x) \in C$. Then $x \in f^{-1}[C]$. Case (2): $f(x) \in D$. Then $x \in f^{-1}[D]$. Therefore, $x \in f^{-1}[C] \cup f^{-1}[D]$.

Let $x \in f^{-1}[C] \cup f^{-1}[D]$. Case (1): $x \in f^{-1}[C]$. $f(x) \in C$. Thus $f(x) \in C \cup D$. $x \in f^{-1}[C \cup D]$. Case (2): $x \in f^{-1}[D]$. $f(x) \in D$. Thus $f(x) \in C \cup D$. $x \in f^{-1}[C \cup D]$. Therefore, $x \in f^{-1}[C \cup D]$.

Problem 3

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Suppose that $g \circ f$ is bijective. Decide (with proof) if each of the following must be true, otherwise, provide a counterexample.

(a) f is injective.

(b) f is surjective.

(c) g is injective.

(d) g is surjective.

Suggested solutions:

(a) True: suppose $f(a) = f(b)$ for some $a, b \in X$. Then $g(f(a)) = g(f(b))$. Since $g \circ f$ is injective, $a = b$.

(b) False: let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 2x$ and $g : \mathbb{N} \to \mathbb{N}$ be defined by

$$g(y) = \begin{cases} y/2 & y \text{ is even} \\ 0 & y \text{ is odd.} \end{cases}$$

f is not surjective because the range does not contain 3. $g \circ f$ is the identity function so it is bijective.

(c) False: using the same example in (b), g is not injective because $g(1) = 0 = g(3)$.

(d) True: let $z \in Z$. Since $g \circ f$ is surjective, we can find $x \in X$ such that $g(f(x)) = z$. Since $f : X \to Y$, $f(x) \in Y$. Therefore we can choose $y = f(x)$ to be the input of g such that $g(y) = z$.

Problem 4

Let $f : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R} \times \mathbb{R})$ be defined by

$$f(A, B) = A \times B.$$

Decide (with proof) whether each of the following is true or false.

(a) f is injective.

(b) f is surjective.

Suggested solutions:

(a) False: $f(\{1\}, \emptyset) = \emptyset = (\emptyset, \{1\})$.

(b) False: we claim that there are no $A, B \in \mathscr{P}(\mathbb{R})$ such that $f(A, B) = C = \{(0, 0), (0, 1), (1, 0)\}$. Since $(0, 0)$ and $(1, 0)$ are in C, A must contain 0 and 1. Similarly, since $(0, 0)$ and $(0, 1)$ are in C, B must contain 0 and 1. Hence $A \times B$ contain $\{0, 1\} \times \{0, 1\}$ as a subset. However C does not contain $(1, 1)$.

Problem 5

For each of the following functions, determine whether it is injective, surjective, bijective, or neither injective nor surjective.

(a) $f : [0, 1] \to [a, b]$, $f(x) = a + x(b - a)$ for all $x \in [0, 1]$, where $a, b \in \mathbb{R}$ with $a < b$.

(b) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(x, y) = (x + y, x - y, x^2 - y^2)$ for all $(x, y) \in \mathbb{R}^2$.

(c) $h : \mathcal{P}(\mathbb{R})^2 \to \mathcal{P}(\mathbb{R})$, $h(A, B) = A \cup B$ for all $(A, B) \in \mathcal{P}(\mathbb{R})^2$.

Suggested solutions:

(a) It is bijective. Injectivity: let $x_1, x_2 \in [0, 1]$ be such that $f(x_1) = f(x_2)$. Solving $a + x_1(b - a) = a + x_2(b - a)$ gives $x_1 = x_2$. Surjectivity: let $y \in [a, b]$. Define $x = (y - a)/(b - a) \in [0, 1]$. Then $y = f(x)$.

(b) Injective: let $x_1, y_1, x_2, y_2 \in \mathbb{R}$ such that

$$\begin{aligned} g(x_1, y_1) &= g(x_2, y_2) \\ (x_1 + y_1, x_1 - y_1, x_1^2 - y_1^2) &= (x_2 + y_2, x_2 - y_2, x_2^2 - y_2^2) \\ x_1 + y_1 = x_2 + y_2 &\wedge x_1 - y_1 = x_2 - y_2 \text{ (we ignore the last coordinate)} \\ x_1 = x_2 &\wedge y_1 = y_2 \end{aligned}$$

Not surjective: consider $(1, 1, 2) \in \mathbb{R}^3$. Suppose $g(x, y) = (1, 1, 2)$ for some $x, y \in \mathbb{R}$.

$$x^2 - y^2 = (x + y)(x - y) = 1 \cdot 1 = 1 \neq 2$$

contradiction.

(c) Not injective: $h(\{1\}, \{2\}) = \{1, 2\} = h(\{2\}, \{1\})$.

Surjective: let $C \in \mathcal{P}(\mathbb{R})$. We can choose $A = B = C$. Then $h(A, B) = h(C, C) = C \cup C = C$.

Problem 6

(a) Find functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g = \mathrm{id}_{\mathbb{N}}$ but $g \circ f \neq \mathrm{id}_{\mathbb{N}}$.

(b) Find functions $h : \mathbb{Z} \to \mathbb{Q}$ and $k : \mathbb{Q} \to \mathbb{Z}$ such that $k \circ h = \mathrm{id}_{\mathbb{Z}}$ but $h \circ k \neq \mathrm{id}_{\mathbb{Q}}$.

Suggested solutions:

(a) Let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(x) = 2x$ and $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(x) = \begin{cases} x/2 & x \text{ is even} \\ 0 & x \text{ is odd.} \end{cases}$$

$g(f(1)) = g(0) = 0 \neq 1$ so $g \circ f$ is not an identity function.

(b) Let $h : \mathbb{Z} \to \mathbb{Q}$ be defined by $h(x) = x$ for $x \in \mathbb{Z}$. Let $k : \mathbb{Q} \to \mathbb{Z}$ be defined by

$$k(x) = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Q} \setminus \mathbb{Z} \end{cases}$$

$h(k(1/2)) = h(0) = 0 \neq 1/2$ so $h \circ k$ is not an identity function.