Homework 7

(You must justify ALL your claims unless otherwise stated)

Problem 1

Let $f : \mathbb{N} \to \mathbb{N}$ and suppose that, for all $n \in \mathbb{N}$, we have

$$f(n) = \begin{cases} 3f(\frac{n}{3}) & \text{if } n \equiv 0 \bmod 3 \\ f(n-1)+1 & \text{if } n \equiv 1 \bmod 3 \\ f(n-1)+3 & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

(a) Prove that $f(n) \equiv n^2 \bmod 3$ for all $n \in \mathbb{N}$.

(b) Prove that $f(n) \geq n$ for all $n \in \mathbb{N}$.

Problem 2

The *Tribonacci sequence* is the sequence $t_0, t_1, t_2, \ldots$ defined by

$$t_0 = 0, \quad t_1 = 1, \quad t_2 = 1 \quad \text{and} \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for all } n \geq 3$$

Prove that $t_n \equiv t_{n+8} \bmod 4$ for all $n \in \mathbb{N}$.

Problem 3

For each of the following relations, determine whether it is reflexive, whether it is symmetric, whether it is antisymmetric, whether it is transitive, whether it is connected, whether it is an equivalence relation, whether it is a partial order relation, and whether it is a total order relation.

(a) The relation $\updownarrow$ on $\mathbb{R}^2$ defined for all $(a,b),(c,d) \in \mathbb{R}^2$ by letting $(a,b) \updownarrow (c,d)$ if and only if either $a < c$, or $a = c$ and $b \leq d$.

(b) The relation $\pitchfork$ on $\mathcal{P}(\mathbb{R})$ defined for all $U, V \in \mathcal{P}(\mathbb{R})$ by letting $U \pitchfork V$ if and only if $U \subseteq V \cup \{0\}$.

(c) The relation $\bowtie$ on the set X of all functions $\mathbb{R} \to \mathbb{R}$ defined for all $f, g \in X$ by letting $f \bowtie g$ if and only if $f(x) - g(x) \in \mathbb{Q}$ for all $x \in \mathbb{R}$.

Problem 4

Let X be a set and let $\sim$ be an equivalence relation on X.

(a) Prove that the function $q : X \to X/{\sim}$ defined by $q(a) = [a]_\sim$ for all $a \in X$ is a surjection.

(b) A function $f : X \to Y$ is said to *respect* $\sim$ if, for all $a, b \in X$, if $a \sim b$, then $f(a) = f(b)$. Prove that for all $f : X \to Y$, f respects $\sim$ if and only if $f = g \circ q$ for some function $g : X/{\sim} \to Y$.

(c) Prove that the function g from part (b) is unique, in the sense that if $h : X/{\sim} \to Y$ is a function such that $f = h \circ q$, then $g = h$.

Problem 5

For each of the following sets X, partial orders $\preccurlyeq$ on X and subsets $A \subseteq X$, find $\sup_{\preccurlyeq}(A)$ if it exists (or prove that it doesn't), and find $\inf_{\preccurlyeq}(A)$ if it exists (or prove that it doesn't).

(a) $X = \mathbb{R}$, $\preccurlyeq \, = \, \leq$ and $A = [0, 1) \cap (\mathbb{R} \setminus \mathbb{Q})$;

(b) $X = \mathbb{N}$, $\preccurlyeq \, = \, |$ and $A = \{p \in \mathbb{N} \mid p \text{ is prime}\}$;

(c) $X = \{U \subseteq \{1, 2, 3, 4\} \mid |U| \text{ is even}\}$, $\preccurlyeq \, = \, \subseteq$, and $A = \{\{1, 2\}, \{2, 3\}\}$;

Problem 6

Assume A is a nonempty set and $(B, \preccurlyeq_B)$ is a nonempty poset. Let $\mathcal{F}$ be the set of all functions with domain A and codomain B. Define an $\preccurlyeq_{\mathcal{F}}$ on $\mathcal{F}$ to be the pointwise ordering. That is,

$$f \preccurlyeq_{\mathcal{F}} g \text{ iff } f(t) \preccurlyeq_B g(t) \text{ for all } t \in A.$$

Prove that $\preccurlyeq_{\mathcal{F}}$ is a partial order on $\mathcal{F}$.