Homework 8 Solutions

(You must justify ALL your claims unless otherwise stated)

Problem 1

Let S be the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. That is,

$$S = \{f : \mathbb{N} \to \{0, 1\} : f \text{ is a function}\}$$

Define a bijection $\psi : \mathcal{P}(\mathbb{N}) \to S$ and prove that it is a bijection.

Suggested solution:

Let $A \in \mathcal{P}(\mathbb{N})$. We define $\psi(A) = f \in S$ via the following: for $x \in \mathbb{N}$, $x \in A$ iff $f(x) = 1$ (and $x \notin A$ iff $f(x) = 0$). ψ is injective: if $A, B \in \mathcal{P}(\mathbb{N})$ satisfies $\psi(A) = \psi(B)$. For $x \in \mathbb{N}$, we need to prove $x \in A$ iff $x \in B$. If $x \in A$, then $1 = \psi(A)(x) = \psi(B)(x)$ so $x \in B$. The other direction is symmetric, so $A = B$.

ψ is surjective: if $g \in S$, define $A \in \mathcal{P}(\mathbb{N})$ via the following: for $x \in \mathbb{N}$, $x \in A$ iff $g(x) = 1$. Then $\psi(A) = g$: Let $x \in \mathbb{N}$, if $x \in A$, then $\psi(A)(x) = 1$ by our definition of ψ. $g(x) = 1$ because $x \in A$. Similarly, if $x \notin A$, then $\psi(A)(x) = 0$ by our definition of ψ. $g(x) = 0$ because $x \notin A$. Therefore $\psi(A)(x) = g(x)$ for all $x \in \mathbb{N}$.

Problem 2

Suppose that A and B are sets with the same cardinality (that is, there is a bijection $f : A \to B$). Prove that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality by finding a function $F : \mathcal{P}(A) \to \mathcal{P}(B)$ and proving that it is bijective.

Suggested solution:

Let $f : A \to B$ be a bijection. We need to construct a bijection $\psi : \mathcal{P}(A) \to \mathcal{P}(B)$. Let $S \in \mathcal{P}(A)$. $S \subseteq A$. Define $\psi(S) = \{f(s) \mid s \in S\}$. We check injectivity: Let $S, T \in \mathcal{P}(A)$ and assume $\psi(S) = \psi(T)$. Then $\{f(s) \mid s \in S\} = \{f(t) \mid t \in T\}$. We need to check $S = T$. Let $x \in S$. $f(x) \in \{f(s) \mid s \in S\}$ so $f(x) \in \{f(t) \mid t \in T\}$. There is $y \in T$ such that $f(x) = f(y)$. By injectivity of f, $x = y$. Hence $x = y \in T$ and $x \in T$. The other direction is symmetric.

We check surjectivity: Let $U \in \mathcal{P}(B)$. $U \subseteq B$. Define $S = \{f^{-1}(b) \mid b \in B\}$. Then $\psi(S) = \{f(s) \mid s \in S\} = \{f(f^{-1}(b) \mid b \in B\} = B$.

Problem 3

Use the previous problem to prove that: For any set A, $\mathcal{P}(A)$ is either finite or uncountable.

Suggested solution: Let A be a set. If A is finite, then $|\mathcal{P}(A)| = 2^{|A|}$ which is also finite. Suppose A is not finite. Then A is either infinitely countable and uncountable. If A is infinitely countable, there is a bijection from $\mathbb{N}$ and A. From the previous problem, there is a bijection from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(A)$, which shows the latter is uncountable. If A is uncountable, the function $\phi : A \to \mathcal{P}(A)$ with $\psi(A) = \{A\}$ is injective, so $\mathcal{P}(A)$ is uncountable.

Problem 4

Prove that the unit circle $\mathcal{C}$ is uncountable. Recall

$$\mathcal{C} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

(Hint: Can you find an interval (a, b) and a bijective function between that interval and a part of $\mathcal{C}$? Can you find a bijection between (a, b) and $(0, 1)$?)
Suggested solution:

Define an injection $g : (0, 1) \to \mathcal{C}$ via $g(x) = (x, \sqrt{1 - x^2})$ (the latter is an ordered pair). The function is well-defined because for any $x \in (0, 1)$, $x^2 + (\sqrt{1 - x^2})^2 = 1$. It is injective: for $x_1, x_2 \in (0, 1)$, if $g(x_1) = g(x_2)$, then their first coordinates are the same, so $x_1 = x_2$.

Since $(0, 1)$ is uncountable, $\mathcal{C}$ is also uncountable.

Problem 5

Show that the set of all polynomials $\mathbb{Z}[x]$ with integer coefficients is countable by proving the following statements:

(a) For $n \in \mathbb{N}$, let $P_n[\mathbb{Z}]$ be the set of all polynomials of degree n with integer coefficients. Prove that $P_n[\mathbb{Z}]$ is countable.

(b) Prove that

$$\mathbb{Z}[x] = \bigcup_{n \in \mathbb{N}} P_n[\mathbb{Z}]$$

(c) Prove that $\mathbb{Z}[x]$ is countable.

Suggested solutions:

(a) Let $n \in \mathbb{N}$. We find a bijection $\psi : \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\}) \to P_n)[\mathbb{Z}]$. Since $\mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})$ is countable, we can conclude $P_n[\mathbb{Z}]$ is also countable. Let $(a_0, \ldots, a_n) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})$. Set $\psi(a_0, \ldots, a_n) = a_0 + a_1 x + \cdots + a_n x^n$.

(b) Let $n \in \mathbb{N}$, $f \in P_n[\mathbb{Z}]$. f is a polynomial of degree n with integer coefficients. In particular, it is a polynomial with integer coefficients, so $f \in \mathbb{Z}[x]$. Let $g \in \mathbb{Z}[x]$. g is a polynomial with integer coefficients. Let k be the degree of g (the maximum k such that the coefficient of x^k is nonzero). Then $g \in P_n[\mathbb{Z}]$.

(c) Since each $P_n[\mathbb{Z}]$ is countable and $\mathbb{Z}[x]$ is a countable union of these $P_n[\mathbb{Z}]$, $\mathbb{Z}[x]$ is also countable.

Problem 6

Find a set S of subsets of $\mathbb{R} \times \mathbb{R}$ (That is, $S \subseteq \mathcal{P}(\mathbb{R} \times \mathbb{R})$) that satisfies that following properties:

(a) S is countable.

(b) For every $(x, y) \in \mathbb{R} \times \mathbb{R}$, there exists a set $A \in S$ and there exists $(a, b) \in A$ such that the distance between (x, y) and (a, b) is less than $\frac{1}{2}$.
(You can use the fact that for every real number $r \in \mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$, with $|r - q| < \frac{1}{2}$. This is still true if $\frac{1}{2}$ is replaced by any other positive number).

You must prove that the set S you define does in fact satisfy conditions (a) and (b).
Suggested solution:

Define $S = \{\{(q_1, q_2)\} \mid (q_1, q_2) \in \mathbb{Q} \times \mathbb{Q}\}$. S is countable because $\mathbb{Q} \times \mathbb{Q}$ is countable and the function $\psi : \mathbb{Q} \times \mathbb{Q} \to S$ via $\psi(q_1, q_2) = \{(q_1, q_2)\}$ is bijective. Let $(x, y) \in \mathbb{R} \times \mathbb{R}$. Since $x \in \mathbb{R}$, there is $q_1 \in \mathbb{Q}$ such that $|x - q_1| < 0.1$. Since $y \in \mathbb{R}$, there is $q_2 \in \mathbb{Q}$ such that $|y - q_2| < 0.1$. Then $(q_1, q_2) \in \{(q_1, q_2)\} \in S$ satisfies $|(q_1, q_2) - (x, y)| = |(q_1 - x, q_2 - y)| = \sqrt{(q_1 - x)^2 + (q_2 - y)^2} \le \sqrt{0.1^2 + 0.1^2} < 1/2$.