Homework 4

Problem 1

For each of the following, express the set in list or interval notation (whichever is appropriate) or as a union of such sets.

(a) $f[(-1, 3)]$, where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ for all $x \in \mathbb{R}$.

(b) $g^{-1}[(0, 2]]$, where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = |x - 1| + |x + 1|$ for all $x \in \mathbb{R}$.

(c) $h[\mathbb{R} \setminus \mathbb{Z}]$, where $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) = x^2$ for all $x \in \mathbb{R}$.

(d) (Extra practice, not to be submitted)
$p^{-1}[\{2, 3\}]$, where $p : \mathbb{Z} \to \mathbb{Z}$ is defined by letting $p(n)$ be the remainder of n^2 when divided by 5 (for example, $7^2 = 49 = 9 \times 5 + 4$, so $p(7) = 4$).

Problem 2

Let $f : X \to Y$ be a function. Suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Decide (with proof) whether each of the following is true or false. If the statement is false, prove which of the inclusions ($\subseteq$ or $\supseteq$) must be true and provide a counterexample for the other inclusion.

(a) $f[A \cap B] = f[A] \cap f[B]$

(b) $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$

(c) (Extra practice, not to be submitted) $f[A \cup B] = f[A] \cup f[B]$

(d) (Extra practice, not to be submitted) $f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D]$

Problem 3

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Suppose that $g \circ f$ is bijective. Decide (with proof) if each of the following must be true, otherwise, provide a counterexample.

(a) f is injective.

(b) f is surjective.

(c) g is injective.

(d) g is surjective.

Problem 4

Let $f : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R} \times \mathbb{R})$ be defined by

$$f(A, B) = A \times B.$$

Decide (with proof) whether each of the following is true or false.

(a) f is injective.

(b) f is surjective.

Problem 5

For each of the following functions, determine whether it is injective, surjective, bijective, or neither injective nor surjective.

(a) $f : [0, 1] \to [a, b]$, $f(x) = a + x(b - a)$ for all $x \in [0, 1]$, where $a, b \in \mathbb{R}$ with $a < b$.

(b) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(x, y) = (x + y, x - y, x^2 - y^2)$ for all $(x, y) \in \mathbb{R}^2$.

(c) $h : \mathcal{P}(\mathbb{R})^2 \to \mathcal{P}(\mathbb{R})$, $h(A, B) = A \cup B$ for all $(A, B) \in \mathcal{P}(\mathbb{R})^2$.

Problem 6

(a) Find functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that $f \circ g = \mathrm{id}_{\mathbb{N}}$ but $g \circ f \neq \mathrm{id}_{\mathbb{N}}$.

(b) Find functions $h : \mathbb{Z} \to \mathbb{Q}$ and $k : \mathbb{Q} \to \mathbb{Z}$ such that $k \circ h = \mathrm{id}_{\mathbb{Z}}$ but $h \circ k \neq \mathrm{id}_{\mathbb{Q}}$.