Homework 5 Solutions

Problem 1

For each of the following functions, find its inverse if it has one, or prove that it does not have an inverse. [If the function does not have an inverse by showing that it is not bijective.

(a) $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = \dfrac{n + |n|}{2}$ for all $n \in \mathbb{Z}$.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, $h(A) = \mathbb{N} \setminus A$ for all $A \in \mathcal{P}(\mathbb{N})$.

Suggested solutions:

(a) The inverse $f^{-1} : \mathbb{Z} \to \mathbb{N}$ is defined by

$$f^{-1}(z) = \begin{cases} 2z & z \geq 0 \\ -2z - 1 & z < 0 \end{cases}$$

We compute $f^{-1} \circ f$:

(i) If n is even, then $f(n) = n/2$ and $f^{-1} \circ f(n) = f^{-1}(n/2) = 2(n/2) = n$.

(ii) If n is odd, then $f(n) = -(n+1)/2$ and $f^{-1} \circ f(n) = f^{-1}(-(n+1)/2) = -2[-(n+1)/2] - 1 = n$.

We compute $f \circ f^{-1}$: Let $r \in \mathbb{Z}$. If $r \geq 0$, $f \circ f^{-1}(r) = f(2r) = r$. If $r < 0$, $f \circ f^{-1}(r) = f(-2r - 1) = r$.

(b) g does not have an inverse. We show that it is not injective:

$$g(0) = \frac{0 + |0|}{2} = 0 = \frac{-1 + |-1|}{2} = g(-1).$$

(c) The inverse of h is itself. To verify this, it suffices to prove for all $A \in \mathcal{P}(\mathbb{N})$, $h(h(A) = \mathbb{N} \setminus (\mathbb{N} \setminus A) = A$. Let $x \in \mathbb{N} \setminus (\mathbb{N} \setminus A)$. Then $x \in \mathbb{N}$ and $x \notin \mathbb{N} \setminus A$. The latter implies $\neg(x \in \mathbb{N} \wedge x \notin A)$. Since we know $x \in \mathbb{N}$, we have $\neg(x \notin A)$ which is $x \in A$.

Conversely, let $x \in A$. We need to check $x \in \mathbb{N} \setminus (\mathbb{N} \setminus A)$. Since $A \subseteq \mathbb{N}$, we have $x \in \mathbb{N}$. It remains to check $x \notin \mathbb{N} \setminus A$, equivalently

$$\begin{aligned} &\neg(x \in \mathbb{N} \setminus A) \\ \equiv\ &\neg(x \in \mathbb{N} \wedge x \notin A) \\ \equiv\ &(x \notin \mathbb{N} \vee x \in A) \\ \equiv\ &x \in A \text{ (since the first disjunct is always false)} \end{aligned}$$

which is true by assumption.

Problem 2

Let $f : X \to Y$ be a function. Let A and B be subsets of X. If f is injective, can we conclude that $f[A \setminus B] = f[A] \setminus f[B]$? Prove your answer.
Suggested solution:

Yes: Let $y \in f[A \setminus B]$. There is $x \in A \setminus B$ such that $y = f(x)$. Since $x \in A$ and $y = f(x)$, we know $y \in f[A]$. Suppose $y \in f[B]$. Then there is $z \in B$ such that $y = f(z)$. But $f(x) = y = f(z)$. By injectivity $x = z$. This contradicts $x \notin B$. Therefore $y \notin f[B]$. We can conclude that $y \in f[A] \setminus f[B]$.

Conversely, let $y \in f[A] \setminus f[B]$. Since $y \in f[A]$, there is $x \in A$ such that $y = f(x)$. It suffices to check that this particular x is not in B. Otherwise, $x \in B$ and $y = f(x)$. Then $y \in f[B]$ contradiction. Therefore $x \notin B$. Combining with $x \in A$, we have $x \in A \setminus B$. Since $y = f(x)$, $y \in f[A \setminus B]$.

Problem 3

The *Fibonacci Numbers* is a sequence of natural numbers defined by:

(i) $f_0 = 0$

(ii) $f_1 = 1$

(ii) $\forall n \in \mathbb{N}, f_{n+2} = f_{n+1} + f_n$

Use induction to prove that 3 divides f_{4n+3}, for all $n \in \mathbb{N}$.
Suggested solution:

The base case is true because $f_{4\cdot 0} = f_0 = 0$ is divisible by 3. Assume f_{4n} is divisible by 3 for some $n \in \mathbb{N}$. We can write $f_{4n} = 3k$ for some $k \in \mathbb{Z}$. We need to prove that $f_{4(n+1)} = f_{4n+4}$ is also divisible by 3.

$$\begin{aligned} f_{4n+4} &= f_{4n+2} + f_{4n+3} \\ &= (f_{4n} + f_{4n+1}) + (f_{4n+1} + f_{4n+2}) \\ &= f_{4n} + 2(f_{4n+1}) + f_{4n+2} \\ &= f_{4n} + 2(f_{4n+1}) + (f_{4n} + f_{4n+1}) \\ &= 2f_{4n} + 3f_{4n+1} \\ &= 2(3k) + 3f_{4n+1} \\ &= 3(2k + f_{4n+1}) \end{aligned}$$

Therefore f_{4n} is also divisible by 3.

Problem 4

Prove that for all $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Suggested solution:

The base case is true because $2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$ is divisible by 3. Assume $2^{2n} - 1$ is divisble by 3 for some $n \in \mathbb{N}$. We can write $2^{2n} - 1 = 3k$ for some $k \in \mathbb{Z}$. We need to prove that $2^{2(n+1)} - 1$ is also divisible by 3.

$$\begin{aligned}
2^{2(n+1)} - 1 &= 2^{2n+2)} - 1 \\
&= 4 \cdot 2^{2n} - 1 \\
&= 3 \cdot 2^{2n} + 2^{2n} - 1 \\
&= 3 \cdot 2^{2n} + 3k \\
&= 3(2^{2n} + k)
\end{aligned}$$

Therefore, $2^{2(n+1)} - 1$ is also divisible by 3. By induction, $2^{2n} - 1$ is divisble by 3 for all $n \in \mathbb{N}$.

Problem 5

A sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = 5 \quad \text{and} \quad a_{n+1} = 3a_n - 8 \text{ for all } n \in \mathbb{N}$$

Find an expression for a general term a_n in terms of $n \in \mathbb{N}$ and prove your formula by weak induction. [Hint: Try to find a formula for $a_{n+1} - 4$ in terms of a_n]

Suggested solution:

Compute $a_1 = 7$, $a_2 = 13$, $a_3 = 31$. Observe that $a_0 - 4 = 1$, $a_1 - 4 = 3$, $a_2 - 4 = 9$, $a_3 - 4 = 27$. We conjecture that $a_n - 4 = 3^n$ for $n \in \mathbb{N}$. The base case is verified already ($a_0 - 4 = 3^0$). Assume $a_n - 4 = 3^n$ for some $n \in \mathbb{N}$. We show that $a_{n+1} - 4 = 3^{n+1}$.

$$\begin{aligned}
a_{n+1} - 4 &= (3a_n - 8) - 4 \\
&= 3a_n - 12 \\
&= 3(3^n + 4) - 12 \\
&= 3^{n+1} + 12 - 12 \\
&= 3^{n+1}
\end{aligned}$$

By induction, $a_n - 4 = 3^n$ for all $n \in \mathbb{N}$, so $a_n = 3^n + 4$.

Problem 6

The operators of *indexed conjunction* $\bigwedge_{i=1}^{n}$ and *indexed disjunction* $\bigvee_{i=1}^{n}$ are defined by recursion on $n \in \mathbb{N}$ as follows:

- $\bigwedge_{i=1}^{0} p_i = \top$ and $\bigwedge_{i=1}^{n+1} p_i = \left(\bigwedge_{i=1}^{n} p_i\right) \wedge p_{n+1}$, for all $n \in \mathbb{N}$;
- $\bigvee_{i=1}^{0} p_i = \bot$ and $\bigvee_{i=1}^{n+1} p_i = \left(\bigvee_{i=1}^{n} p_i\right) \vee p_{n+1}$, for all $n \in \mathbb{N}$.

where $\top$ represents the true proposition '$0 = 0$', and $\bot$ represents the false proposition '$0 = 1$'.

Prove by induction that $\left(\bigvee_{i=1}^{n} p_i\right) \Rightarrow q \equiv \bigwedge_{i=1}^{n} (p_i \Rightarrow q)$ for all $n \in \mathbb{N}$, where $p_1, p_2, \ldots$ and q are propositional variables.

Suggested solution: Base case: we need to prove

$$\left(\bigvee_{i=1}^{0} p_i\right) \Rightarrow q \equiv \bigwedge_{i=1}^{0} (p_i \Rightarrow q).$$

LHS$\equiv \bot \Rightarrow q \equiv \top \equiv \bigwedge_{i=1}^{0} (p_i \Rightarrow q) \equiv$RHS. The second $\equiv$ is because a false antecedent always gives a true implication.

Assume $(\bigvee_{i=1}^{n} p_i) \Rightarrow q \equiv \bigwedge_{i=1}^{n} (p_i \Rightarrow q)$ for some $n \in \mathbb{N}$. We consider the inductive case:

$$\begin{aligned}
\left(\bigvee_{i=1}^{n+1} p_i\right) \Rightarrow q &\equiv \left(\bigvee_{i=1}^{n} p_i\right) \vee p_{n+1} \Rightarrow q \\
&\equiv \left[\left(\bigvee_{i=1}^{n} p_i\right) \Rightarrow q\right] \wedge (p_{n+1} \Rightarrow q) \text{ (*)} \\
&\equiv \left[\bigwedge_{i=1}^{n} (p_i \Rightarrow q)\right] \wedge (p_{n+1} \Rightarrow q) \text{ by IH on the first conjunct} \\
&\equiv \bigwedge_{i=1}^{n+1} (p_i \Rightarrow q)
\end{aligned}$$

We justify (*) above: let r, s, q be propositional variables. We show that

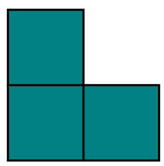
$$(r \vee s) \Rightarrow q \equiv (r \Rightarrow q) \wedge (s \Rightarrow q)$$

The truth values of LHS and RHS are the same. In (*) we substitute $r = \bigvee_{i=1}^{n} p_i$, $s = p_{n+1}$ and $q = q$.

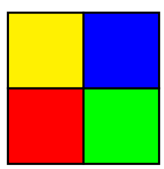
r	s	q	$r \vee s$	LHS	$r \Rightarrow q$	$s \Rightarrow q$	RHS
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Problem 7 (Extra practice, not to be submitted)

Consider a chessboard of size $2^n \times 2^n$ for some arbitrary positive integer n. Remove any square from the board. Is it possible to tile the remaining squares with L-shaped triominoes (showed below)? If your answer is Yes, prove it. If your answer is No, provide a counterexample.



Suggested solution: We prove by induction on $n \geq 1$. The base case is trivial because removing a square from 2×2 gives an L-shaped triominoes. Assume the proposition is true for some $n \geq 1$. Now we are given a $2^{n+1} \times 2^{n+1}$ chessboard with one square removed. Consider the $2^{n+1} \times 2^{n+1}$ chessboard as four square blocks of size $2^n \times 2^n$. We color them by yellow, blue, red and green.



The removed square is in one of the blocks. Without loss of generality assume it is in the yellow block. By the inductive assumption, it is possible to cover the yellow block without the removed square by L-shaped triominoes. It remains to cover the blue, red and green blocks as a whole by L-shaped triominoes. Remove the bottom-left corner of the blue block, the top-left corner of the green block and the top-right corner of the red block. We obtain three blocks each with a missing square. By the inductive assumption again we can cover each block by L-shaped triominoes. Finally we have to add back the three squares we just removed. But it is an L-shaped triomino itself.