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21-127 Test 1 SOLUTIONS

Wednesday, 15 February 2023

Please read the following instructions carefully before the test begins.

Before the test

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

During the test

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 12–13), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

After the test

- Stop working immediately when you are instructed to do so.
- Turn in all 14 pages of this test; if you tore out any pages, put them back in their correct positions.

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1. (a) Let A be a set. Write the definition of the power set of A using set-builder notation. **[5]**

(b) For each of the following, name one element from the set. If it is not possible, **prove** that it is the empty set. **[10]**

(i) $\mathscr{P}(\mathscr{P}(\mathbb{R}))$

$\varnothing, \{\{\pi\}\}, \{\{1,2\}, \{-38,-1,0,2\pi\}\} \in \mathscr{P}(\mathscr{P}(\mathbb{R}))$

(ii) $(\{0\} \times \mathbb{N}) \cap (\{1\} \times \mathbb{Z})$

The given set is empty. Assume seeking a contradiction that there exists $x \in (\{0\} \times \mathbb{N}) \cap (\{1\} \times \mathbb{Z})$. This means that $x \in \{0\} \times \mathbb{N}$ and $x \in \{1\} \times \mathbb{Z}$. Therefore, $x = (a,b)$, for some $a \in \{0\}$ and $b \in \mathbb{N}$. Similarly, $x = (c,d)$, for some $c \in \{1\}$ and $d \in \mathbb{Z}$. This implies that $a = 0$ and $c = 1$. But we have that $x = (a,b) = (c,d)$, which means that $a = c$, a contradiction. This shows that the given set in empty.

(ii) $\{n \in \mathbb{Z} : |5n| < 4\}$

$0 \in \{n \in \mathbb{Z} : |5n| < 4\}$

More space for (Q1)

2. (a) State De Morgan's laws for sets (Make sure to include both laws) **[5]**

Let A, B, C be sets. Then

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$
$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

(b) Let A, B, C be three arbitrary sets. Prove that $(A \setminus B) \setminus C \subseteq A \setminus C$ **[10]**

Let x be an arbitrary element. Suppose that $x \in (A \setminus B) \setminus C$. Then $x \in A \setminus B$ and $x \notin C$. This implies that $x \in A$, $x \notin B$, and $x \notin C$. Since $x \in A$ and $x \notin C$, then, by definition of relative difference, $x \in A \setminus C$. This shows that $(A \setminus B) \setminus C \subseteq A \setminus C$.

(c) Prove that: if $B \subseteq C$, then the other containment holds. That is, $A \setminus C \subseteq (A \setminus B) \setminus C$ **[10]**

Let x be an arbitrary element. Assume that $B \subseteq C$. To show that $A \setminus C \subseteq (A \setminus B) \setminus C$, suppose that $x \in A \setminus C$. This means that $x \in A$ and $x \notin C$. But since $B \subseteq C$, then x can not be an element of B, because, otherwise, it would be an element of C. Therefore, $x \in A$, $x \notin B$, and $x \notin C$, hence $x \in (A \setminus B) \setminus C$

More space for (Q2)

3. (a) Define divisibility of integers. **[5]**

$$\forall a, b \in \mathbb{Z}, (a \mid b \Leftrightarrow \exists q \in \mathbb{Z}, b = qa)$$

(b) Let m be an integer. Prove that **[10]**

$$m \text{ is odd} \quad \text{if and only if} \quad 4 \mid m^2 + 2m + 1$$

Let m be odd. Then $m = 2q + 1$, for some $q \in \mathbb{Z}$. Therefore we have

$$m^2 + m + 1 = (2q+1)^2 + 2(2q+1) + 1 = 4q^2 + 4q + 1 + 4q + 2 + 1 = 4(q^2 + 2q + 1)$$

Since $q^2 + 2q + 1$ is an integer, this shows that $m^2 + m + 1$ is a multiple of 4, hence $4 \mid m^2 + 2m + 1$.

Conversely, suppose that $4 \mid m^2 + 2m + 1$. This means that $m^2 + 2m + 1 = 4r$, for some $r \in \mathbb{Z}$.Assume seeking a contradiction that m in even. That is, $m = 2s$, for some $s \in \mathbb{Z}$. We have,

$$m^2 + 2m + 1 = 4s^2 + 4s + 1 = 2(2s^2 + 2s) + 1$$

The above equation shows that $m^2 + 2m + 1$ is odd. On the other hand, $m^2 + 2m + 1 = 4r$, which means that $m^2 + 2m + 1$ is even. This is a contradiction. This shows that m is odd, as desired.

More space for (Q3)

4. Give an example of logical formulae $p(x), q(x)$ and a set S such that the following are not logically equivalent. Justify your answer with a proof. **[10]**

- $\forall x \in S, (p(x) \vee q(x))$
- $(\forall x \in S, p(x)) \vee (\forall x \in S, q(x))$

Let $S = \mathbb{N}$, let $p(x)$ be the predicate "x is even", and let $q(x)$ be the predicate "x is odd".

Since every natural number is either even or odd, then $\forall x \in S, (p(x) \vee q(x))$ is true. Whereas, both the statements "every natural number is even" and "every natural number is odd" are false, hence $(\forall x \in S, p(x)) \vee (\forall x \in S, q(x))$ is false.

More space for (Q4)

5. Consider the proposition $\varphi : \exists x \in \mathbb{R}, \forall n \in \mathbb{Q}, (x \neq n \wedge \exists m \in \mathbb{Z}, x = \sqrt{m})$

(a) Determine whether φ is True or False. Justify your answer with a proof. **[7]**

The statement is true. Let $x = \sqrt{5}$. Since $\sqrt{5}$ is not a rational number, then $\forall n \in \mathbb{Q}, x \neq n$ is true. Also, since 5 is an integer and $x = \sqrt{5}$, then the statement $\exists m \in \mathbb{Z}, x = \sqrt{m}$ is true. This shows that φ is true.

(b) Write $\neg\varphi$ in maximally negated form. **[7]**

$$\neg\varphi \equiv \forall x \in \mathbb{R}, \exists n \in \mathbb{Q}, (x = n \vee \forall m \in \mathbb{Z}, x \neq \sqrt{m})$$

(c) (Free Point) True or false: Concepts is the most amazing course you ever took! **[1]**

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 12).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 13).

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