Homework 10 Solutions

(You must justify ALL your claims unless otherwise stated)

Problem 1

An organization consists of 13 men and 12 women.

(a) The organization wishes to form a 10 person committee consisting of 5 men and 5 women. However, there are 2 men, call them Adam and Bob, who refuse to work together and thus cannot be on the committee together. How many such committees can be formed?

(b) The entire organization wishes to form a line. How many ways can this be done in which at least one woman is standing next to another?

Suggested solutions:

(a) The total number of 10-person committees is $\binom{13}{5}\binom{12}{5}$ (choosing 5 men first then 5 women). The number of committees containing both Adam and Bob is $\binom{11}{3}\binom{12}{5}$. The desired number of committees is $\binom{13}{5}\binom{12}{5} - \binom{11}{3}\binom{12}{5}$.

(b) The number of ways of forming a line is $25!$. The number of ways where no woman is next to another is $\binom{14}{12}13!12!$ (first decide the locations of the women relative to the men, then rearrange among the men and among the women). So the desired number is $25! - \binom{14}{12}13!12!$.

Problem 2

3. Let a, b, k be three positive integers with $a + b \geq k$. Prove that

$$\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i}$$

by counting in two ways argument. Use the exact form given. Do not simplify algebraically.

Suggested solution:

Suppose there are two groups of people A and B. LHS counts the number of k-person committees out of $A \cup B$.

RHS counts the following: pick i members from A then $k - i$ members from B (the total will be k). Apply the addition principle for i between 0 and k. (There might be invalid choices where $i > a$ or $k - i > b$. It does not affect the formula because $\binom{a}{i}$ and $\binom{b}{k-i}$ are defined to be zero in those cases.)

Problem 3

If I flip a coin 20 times, I get a sequence of Heads H and tails T.

(a) How many different sequences of heads and tails are possible?

(b) How may different sequences of heads and tails have exactly five heads?

(c) How many different sequences have at most 2 heads?

(d) How many different sequences have at least 3 heads?

Suggested solutions:

(a) 2^{20} (two choices for each flip, 20 flips in total).

(b) $\binom{20}{5}$ (decide which flips give H).

(c) $\binom{20}{0}+\binom{20}{1}+\binom{20}{2}$ (count the number with exactly 0 heads, 1 head and 2 heads separately).

(d) $2^{20} - (\binom{20}{0} + \binom{20}{1} + \binom{20}{2})$ (the complement is at most 3 heads).

Problem 4

Let X be a finite set, let $|X| = n$. Define $E, O \subseteq \mathcal{P}(X)$ as follows:

$$E = \{U \subseteq X : |U| \text{ is even}\} \text{ and } O = \{U \subseteq X : |U| \text{ is odd}\}$$

(a) Use the addition principle to prove that

$$|E| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$

(b) Use the addition principle to prove that

$$|O| = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$$

(c) (c) Use parts (a) and (b) to conclude that, for all $n > 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Suggested solutions:

(a) Let $U \in E$. Then $s = |U|$ is even. $s = 0, 2, 4, \dots$. For a fixed s, there are $\binom{n}{s}$ choices to pick U. By the addition principle, there are

$$|E| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$

choices (it does not matter if $s > n$ because $\binom{n}{s}$ is defined to be zero in that case).

(b) Let $U \in O$. Then $t = |U|$ is even. $t = 1, 3, 5, \ldots$. For a fixed t, there are $\binom{n}{t}$ choices to pick U. By the addition principle, there are

$$|O| = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

choices (it does not matter if $t > n$ because $\binom{n}{t}$ is defined to be zero in that case).

(c) Since there is a bijection between E and O, we have

$$\begin{aligned} |E| &= |O| \\ \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots &= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots \\ \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} &= 0 \\ \sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \text{ (terms are zero if } k > n) \end{aligned}$$

Problem 5

Use the multiplication principle to prove the following:

(a) The number of ordered pairs (A, a) such that $A \subseteq [n]$ and $a \in A$ is equal to $n \cdot 2^{n-1}$ for all $n > 0$.

(b) The number of injections $[k] \to [n]$ is equal to $\binom{n}{k} \cdot k!$, for all $n, k \in \mathbb{N}$.

Suggested solutions:

(a) We find pick a ($\binom{n}{1} = n$ choices) then pick $A \subseteq [n]$ which contains a. For A, it suffices to decide whether each element $x \in [n] \setminus \{a\}$ is in A or not. There are 2 choices for each element (yes or no) and there are $n - 1$ such elements. Therefore there are 2^{n-1} choices to form $A \ni a$. In total there are $n \cdot 2^{n-1}$ choices for (A, a).

(b) To decide an injection $f : [k] \to [n]$, we first decide the range C of f. Since f is injective, C is of size k. Write $C = \{x_1, \ldots, x_k\}$. There are $\binom{n}{k}$ choices for C. Next we decide the values $f(x)$ for each $x \in [k]$. It suffices to decide what the subscript i is when we say $f(x) = x_i$. Since f is injective, the output subscripts will be a permutation of $[k]$ (if $k = 3$, $f(1) = x_2$, $f(2) = x_1$, $f(3) = x_3$, then the permutation is $(1, 2, 3) \to (2, 1, 3)$). The number of permutations is $k!$. Therefore there are $\binom{n}{k} \cdot k!$ choices.

Problem 6

Use the addition and multiplication principles to prove that, for all $n > 0$, the number of surjections $[n] \to [3]$ is equal to $3^n - 3 \cdot 2^n + 3$.

Suggested solution:

We first compute the number c of surjections $[n] \to [2]$. The total number of functions $[n] \to [2]$ is 2^n (each input from [n] has 2 choices). Non-surjections must has a singleton range. There are 2 choices to decide such range. Therefore $c = 2^n - 2$.

The total number of functions $[n] \to [3]$ is 3^n because each input has three possible values. We need to subtract this number by functions f whose range has at most 2 elements (then they cannot be surjective).

- In the case that the range is a singleton, we just need to decide the common value of all the inputs. There are 3 choices.
- In the case that the range is a 2-element subset, we first decide which element x to be outside of the range. There are 3 choices. Then we make sure that $f : [n] \to ([3] \setminus \{x\})$ is a surjection: otherwise the range might end up being a singleton. By the first paragraph, there are $c = 2^n - 2$ such surjections. Therefore there are $3 \cdot (2^n - 2)$ choices.

The desired number of surjections $[n] \to [3]$ is $3^n - (3 + 3 \cdot (2^n - 2)) = 3^n - 3 \cdot 2^n + 3$.