Homework 3 Solutions

Problem 1

Prove that:

$$\forall A, B[A \subseteq B \text{ if and only if } A \setminus B = \varnothing]$$

Suggested solution:

Assume $A \subseteq B$. Suppose $A \setminus B \neq \emptyset$. Then there is $x \in A \setminus B$. $x \in A$ and $x \notin B$. Since $x \in A$ and $A \subseteq B$, $x \in B$. This is a contradiction.

Assume $A \setminus B = \emptyset$. Let $x \in A$ (if A is empty then the following is vacuously true). Since $x \notin \emptyset = A \setminus B$, it is false that $x \in A \wedge x \notin B$. We know $x \in A$ is true, so it must be the case that $x \in B$.

Problem 2

For each of the following statements, prove it is true for all sets X, Y, otherwise, give a counterexample showing that the statement is false.

(a) $\mathscr{P}(X \cap Y) = \mathscr{P}(X) \cap \mathscr{P}(Y)$

(b) $\mathscr{P}(X \setminus Y) = \mathscr{P}(X) \setminus \mathscr{P}(Y)$

(c) $\mathscr{P}(X \times Y) = \mathscr{P}(X) \times \mathscr{P}(Y)$

(d) $\mathscr{P}(X \cup Y) = \mathscr{P}(X) \cup \mathscr{P}(Y)$

Suggested solutions:

(a) True: let $A \in \mathscr{P}(X \cap Y)$. Then $A \subseteq X \cap Y$. For any element $x \in A$, $x \in X \cap Y$, which implies $x \in X$. Therefore $A \subseteq X$ and $A \in \mathscr{P}(X)$. Similarly, for any element $y \in A$, $y \in X \cap Y$, which implies $y \in Y$. Therefore $A \subseteq Y$ and $A \in \mathscr{P}(Y)$. Therefore, $A \in \mathscr{P}(X) \cap \mathscr{P}(Y)$. We have shown $\mathscr{P}(X \cap Y) \subseteq \mathscr{P}(X) \cap \mathscr{P}(Y)$.

Conversely, let $A \in \mathscr{P}(X) \cap \mathscr{P}(Y)$. $A \in \mathscr{P}(X)$ and $A \in \mathscr{P}(Y)$. $A \subseteq X$ and $A \subseteq Y$. Let $x \in A$. By the subset relations, we have both $x \in X$ and $x \in Y$. Therefore $x \in X \cap Y$. Since x is arbitrary, $A \subseteq X \cap Y$. $A \in \mathscr{P}(X \cap Y)$.

(b) False: consider the intervals $X = [0, 2]$ and $Y = [0, 1]$. Then $X \setminus Y = (1, 2]$. The set $A = \{0, 2\}$ is in $\mathscr{P}(X) \setminus \mathscr{P}(Y)$ but not in $\mathscr{P}(X \setminus Y)$.

(c) False: consider the intervals $X = Y = [0, 1]$. The set $A = \{(0, 0), (0, 1), (1, 1)\}$ (three corners of the square) is in $\mathscr{P}(X \times Y)$ but not in $\mathscr{P}(X) \times \mathscr{P}(Y)$.

(d) False: consider the intervals $X = [0, 2]$ and $Y = [2, 4]$. The set $A = \{1, 3\}$ is in $\mathscr{P}(X \cup Y)$ but not $\mathscr{P}(X) \cup \mathscr{P}(Y)$.

Problem 3

Consider the following subsets of $\mathbb{R} \times \mathbb{R}$.

- $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}$
- $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1 \vee |y| \leq 1\}$
- $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1 \wedge |y| \leq 1\}$
- $D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \max\{|x|, |y|\} \leq 1\}$

(a) Describe each of the above sets geometrically in $\mathbb{R} \times \mathbb{R}$ (You may include a graph to support your answer)

(b) For each of the following statements, prove it is true, or give a counterexample showing that the statement is false.

$$A \subseteq B \qquad C \subseteq A \qquad D \subseteq C \qquad C \subseteq D \qquad B \subseteq C$$

Suggested solutions:

(a) A is the unit circle. B is the union of $[-1, 1] \times \mathbb{R}$ and $\mathbb{R} \times [-1, 1]$ (like the cross in England flag). $C = D$ are the square $[-1, 1] \times [-1, 1]$.

(b)
- $A \subseteq B$ is true: we prove by contrapositive. If $(x, y) \notin B$, then both $|x| > 1$ and $|y| > 1$. $x^2 + y^2 = |x|^2 + |y|^2 > 1$. Hence $(x, y) \notin A$.
- $C \subseteq A$ is false: the point $(1, 1)$ is in C but not A.
- $D \subseteq C$ is true: if $(x, y) \in D$, then $\max\{|x|, |y|\} \leq 1$. We have $|x| \leq \max\{|x|, |y|\} \leq 1$ and $|y| \leq \max\{|x|, |y|\} \leq 1$. Hence $(x, y) \in C$.
- $C \subseteq D$ is true: if $(x, y) \in C$, then $|x| \leq 1$ and $|y| \leq 1$. $\max\{|x|, |y|\} \leq \max\{1, 1\} = 1$. Hence $(x, y) \in D$.
- $B \subseteq C$ is false: the point $(2, 0)$ is in B but not C.

Problem 4

A subset $U \subseteq \mathbb{R}$ is said to be **open** if

$$\forall x \in U, \exists \delta \in (0, \infty), \forall y \in \mathbb{R}, (x - \delta < y < x + \delta \Rightarrow y \in U)$$

(a) Find a maximally negated logical formula that is equivalent to the assertion that a subset $U \subseteq \mathbb{R}$ is *not* open. [By the way, 'closed' does not mean the same thing as 'not open'.]

(b) Prove that for all $a, b \in \mathbb{R}$ with $a < b$, the interval (a, b) is open.

(c) Prove that for all $a, b \in \mathbb{R}$ with $a < b$, the interval $[a, b)$ is not open.

(d) (Extra practice, not to be submitted) Determine whether $\mathbb{R} \setminus \mathbb{Z}$ is open and whether $\mathbb{Q}$ is open.

Suggested solutions:

(a)
$$\exists x \in U, \forall \delta \in (0, \infty), \exists y \in \mathbb{R}, (x - \delta < y < x + \delta \wedge y \notin U)$$

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Let $x \in (a, b)$. Pick $\delta = \min\{b - x, x - a\} \in (0, \infty)$. For any $y \in \mathbb{R}$ with $x - \delta < y < x + \delta$, we have

$$\begin{aligned} y &< x + \delta \\ &\leq x + (b - x) \\ &= b \end{aligned}$$

and

$$\begin{aligned} y &> x - \delta \\ &\geq x + (a - x) \\ &= a \end{aligned}$$

Therefore $y \in (a, b)$.

(c) Let $a, b \in \mathbb{R}$ with $a < b$. Choose $x = a \in [a, b)$. For any $\delta \in (0, \infty)$, we can pick $y = x - \delta/2$. Then $x - \delta < x - \delta/2 = y < x < x + \delta$ and $y \notin [a, b)$.

(d) $\mathbb{R} \setminus \mathbb{Z}$ is open but $\mathbb{Q}$ is not.

Problem 5

Let I_{op} be the set of all bounded open intervals in $\mathbb{R}$, and I_{cl} be the set of all closed intervals in $\mathbb{R}$. Specifically:

$$\begin{aligned} I_{\text{op}} &= \{U \subseteq \mathbb{R} \mid U = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ with } a < b\} \\ I_{\text{cl}} &= \{U \subseteq \mathbb{R} \mid U = [a, b] \text{ for some } a, b \in \mathbb{R} \text{ with } a < b\} \end{aligned}$$

For each of the following statements, determine whether it is true or false.

(a) $I_{\text{op}} \in \mathscr{P}(\mathscr{P}(\mathbb{R}))$

(b) $I_{\text{op}} \subseteq I_{\text{cl}}$

(c) $I_{\text{op}} \cap I_{\text{cl}} \neq \varnothing$

(d) $\forall U \in I_{\text{op}}, \exists V \in I_{\text{cl}}, (U \subseteq V \wedge V \nsubseteq U)$

Suggested solutions:

(a) True: we show that $I_{\text{op}} \subseteq \mathscr{P}(\mathbb{R})$. Let $U \in I_{\text{op}}$. By definition $U \subseteq \mathbb{R}$. Therefore $U \in \mathscr{P}(\mathbb{R})$.

(b) False: $(0,1)$ is in I_{op} but not I_{cl}. Otherwise, $(0,1)$ is in $I_{\text{op}} \cap I_{\text{cl}}$. That will contradict the proof in (c).

(c) False: suppose $A \in I_{\text{op}} \cap I_{\text{cl}}$. Then there are $a < b$ and $c < d$ such that $A = (a,b) = [c,d]$. Since $c \in A$, we have $a < c < b$. Also $a < (a+c)/2 < b$. While $(a+c)/2 \in (a,b)$, $(a+c)/2 < (c+c)/2 = c$ so it is not in $[c,d]$, contradicting $(a+c)/2 \in A = [c,d]$.

(d) True: let $U \in I_{\text{op}}$. We can write $U = (a,b)$ for some $a < b$. Define $V = [a-1, b+1] \in I_{\text{cl}}$. We have $U \subseteq V$ but $V \nsubseteq U$ (the latter is witnessed by $a-1$).

Problem 6

Given sets A and B, let $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(a) Prove that for all sets A and B, we have $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(b) Prove that for all sets A and B, we have $A \triangle B = \varnothing$ if and only if $A = B$.

(c) Prove that for all sets A, B, C, we have $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

(d) (Extra practice, not to be submitted) Prove that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$ for all sets A, B and C.

Suggested solutions:

(a) Let $x \in A \triangle B$. Then $x \in A \cup B$ and $x \notin A \cap B$. Suppose $x \notin A \setminus B$. We need to show that $x \in B \setminus A$. Since $x \notin A \setminus B$, it is false that ($x \in A$ and $x \notin B$). Either $x \notin A$ or $x \in B$. In the first case $x \notin A$, since $x \in A \cup B$, we must have $x \in B$. Therefore $x \in B \setminus A$. In the second case $x \in B$, since $x \notin A \cap B$, x cannot be in A at the same time. Therefore $x \in B \setminus A$.

Conversely, let $x \in (A \setminus B) \cup (B \setminus A)$. By symmetry we can assume $x \in (A \setminus B)$. $x \in A$ and $x \notin B$. The former $x \in A$ implies $A \cup B$. The latter $x \notin B$ implies $x \notin A \cap B$. Therefore, $x \in (A \cup B) \setminus (A \cap B) = A \triangle B$.

(b) If $A \triangle B = \emptyset$, by Q1 $A \cup B \subseteq A \cap B$. But $A \cap B \subseteq A \subseteq A \cup B$. Hence $A \cup B = A = A \cap B$. Similarly $A \cup B = B = A \cap B$. We have $A = B$.

Conversely, if $A = B$, then $A \triangle B = (A \setminus B) \cup (B \setminus A) = \emptyset \cup \emptyset = \emptyset$. (The second last equality is by Q1.)

(c) Let $x \in A \cap (B \triangle C)$. $x \in A$ and $x \in B \triangle C$. The latter implies $x \in B \cup C$ but $x \notin B \cap C$. Either $x \notin B$ or $x \notin C$. Consider the case $x \notin B$ (the other case $x \notin C$ is symmetric). Since $x \notin B$, $x \notin A \cap B$. In particular, $x \notin (A \cap B) \cap (A \cap C)$. Since $x \notin B$ and $x \in B \cup C$, we know $x \in C$. Together with $x \in A$, we have $x \in A \cap C$. In particular $x \in (A \cap B) \cup (A \cap C)$. Therefore $x \in (A \cap B) \triangle (A \cap C)$.

Conversely, let $x \in (A \cap B) \triangle (A \cap C)$. $x \in (A \cap B) \cup (A \cap C)$ but $x \notin (A \cap B) \cap (A \cap C)$. The former has two cases: $x \in A \cap B$ or $x \in A \cap C$. By symmetry assume $x \in A \cap B$. $x \in A$ and $x \in B$. On the other hand, we unwind $x \notin (A \cap B) \cap (A \cap C)$ to give $x \notin A \cap B$ or $x \notin A \cap C$. Since x is in A, we have $x \notin B$ or $x \notin C$. In our case $x \in B$ so it narrows down to $x \notin C$. We have shown that $x \in A$, $x \in B$ and $x \notin C$. The latter two implies $x \in B \cup C$ but $x \notin B \cap C$. Hence $x \in B \triangle C$. Therefore $x \in A \cap (B \triangle C)$.

(d) Let x be an element. We need to show that $x \in A \triangle (B \triangle C)$ if and only if $x \in (A \triangle B) \triangle C$. Write p to be the proposition $x \in A$, q be $x \in B$ and r be $x \in C$. We can rewrite $x \in A \triangle (B \triangle C)$ as

$$
\begin{gathered}
x \in [A \cup (B \triangle C)] \setminus [A \cap (B \triangle C)] \\
x \in A \cup (B \triangle C)] \wedge x \notin A \cap (B \triangle C) \\
(x \in A \vee x \in B \triangle C) \wedge (x \notin A \vee x \notin B \triangle C) \\
(x \in A \vee (x \in B \cup C \wedge x \notin B \cap C)) \wedge (x \notin A \vee (x \notin B \cup C \vee x \in B \cap C)) \\
(x \in A \vee ((x \in B \vee x \in C) \wedge (x \notin B \vee x \notin C))) \wedge (x \notin A \vee (x \notin B \wedge x \notin C) \vee (x \in B \wedge x \in C)) \\
(p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee (\neg q \wedge \neg r) \vee (q \wedge r)) \quad (i)
\end{gathered}
$$

We check the truth tables of (i). Let ϕ_1 be $\neg p \vee (\neg q \wedge \neg r) \vee (q \wedge r)$ and ϕ_2 be $p \vee ((q \vee r) \wedge (\neg q \vee \neg r))$.

p	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$q \wedge r$	ϕ_1	$q \vee r$	$\neg q \vee \neg r$	ϕ_2	(i)
T	T	T	F	F	F	F	T	T	T	F	T	T
T	T	F	F	F	T	F	F	F	T	T	T	F
T	F	T	F	T	F	F	F	F	T	T	T	F
T	F	F	F	T	T	T	F	T	F	T	T	T
F	T	T	T	F	F	F	T	T	T	F	F	F
F	T	F	T	F	T	F	F	T	T	T	T	T
F	F	T	T	T	F	F	F	T	T	T	T	T
F	F	F	T	T	T	T	F	T	F	T	F	F

By the symmetry of $\cup$ and $\cap$, we know that $\triangle$ is symmetric ($X \triangle Y = Y \triangle X$). Hence the formula $x \in (A \triangle B) \triangle C$ amounts to swapping p, r in (i), namely

$$(r \vee ((q \vee p) \wedge (\neg q \vee \neg p))) \wedge (\neg r \vee (\neg p \wedge \neg q) \vee (p \wedge q)) \quad (ii)$$

Observe that (i) is true when p, q, r are all true or exactly one of p, q, r is true. Swapping p, r gives that (ii) is true when r, q, p are all true or exactly one of r, q, p is true, which is the same scenario. Therefore the column for (ii) in a truth table would be the same as (i). We can conclude that $x \in A \triangle (B \triangle C)$ if and only if $x \in (A \triangle B) \triangle C$.