Homework 8

(You must justify ALL your claims unless otherwise stated)

Problem 1

Let S be the set of all functions from $\mathbb{N}$ to $\{0,1\}$. That is,

$$S = \{f : \mathbb{N} \to \{0,1\} : f \text{ is a function}\}$$

Define a bijection $\psi : \mathcal{P}(\mathbb{N}) \to S$ and prove that it is a bijection.

Problem 2

Suppose that A and B are sets with the same cardinality (that is, there is a bijection $f : A \to B$). Prove that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality by finding a function $F : \mathcal{P}(A) \to \mathcal{P}(B)$ and proving that it is bijective.

Problem 3

Use the previous problem to prove that: For any set A, $\mathcal{P}(A)$ is either finite or uncountable.

Problem 4

Prove that the unit circle $\mathcal{C}$ is uncountable. Recall

$$\mathcal{C} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

(Hint: Can you find an interval (a,b) and a bijective function between that interval and a part of $\mathcal{C}$? Can you find a bijection between (a,b) and $(0,1)$?)

Problem 5

Show that the set of all polynomials $\mathbb{Z}[x]$ with integer coefficients is countable by proving the following statements:

(a) For $n \in \mathbb{N}$, let $P_n[\mathbb{Z}]$ be the set of all polynomials of degree n with integer coefficients. Prove that $P_n[\mathbb{Z}]$ is countable.

(b) Prove that

$$\mathbb{Z}[x] = \bigcup_{n \in \mathbb{N}} P_n[\mathbb{Z}]$$

(c) Prove that $\mathbb{Z}[x]$ is countable.

Problem 6

Find a set S of subsets of $\mathbb{R} \times \mathbb{R}$ (That is, $S \subseteq \mathcal{P}(\mathbb{R} \times \mathbb{R})$) that satisfies that following properties:

(a) S is countable.

(b) For every $(x, y) \in \mathbb{R} \times \mathbb{R}$, there exists a set $A \in S$ and there exists $(a, b) \in A$ such that the distance between (x, y) and (a, b) is less than $\frac{1}{2}$.
(You can use the fact that for every real number $r \in \mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$, with $|r - q| < \frac{1}{2}$. This is still true if $\frac{1}{2}$ is replaced by any other positive number).

You must prove that the set S you define does in fact satisfy conditions (a) and (b).