Homework 7 Solutions

(You must justify ALL your claims unless otherwise stated)

Problem 1

Let $f : \mathbb{N} \to \mathbb{N}$ and suppose that, for all $n \in \mathbb{N}$, we have

$$f(n) = \begin{cases} 3f(\frac{n}{3}) & \text{if } n \equiv 0 \bmod 3 \\ f(n-1)+1 & \text{if } n \equiv 1 \bmod 3 \\ f(n-1)+3 & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

(a) Prove that $f(n) \equiv n^2 \bmod 3$ for all $n \in \mathbb{N}$.

(b) Prove that $f(n) \geq n$ for all $n \in \mathbb{N}$.

Suggested solutions:

(a) We prove by strong induction. Base case: $f(0) = 3f(0)$ so $f(0) = 0 \equiv n^2$. Let $n \in \mathbb{N}$. Assume $f(k) \equiv k^2$ for $0 \leq k \leq n$. Consider $f(n+1)$. There are three cases:

 (i) $n \equiv 0$: then $n+1 \equiv 1$. $f(n+1) = f(n)+1 \equiv n^2+1$ by inductive assumption. On the other hand $(n+1)^2 = n^2+2n+1 \equiv n^2+0+1 = n^2+1$.

 (ii) $n \equiv 1$: then $n+1 \equiv 2$ and $f(n+1) = f(n)+3 \equiv n^2+3$ by inductive assumption. On the other hand $(n+1)^2 \equiv n^2+2\cdot 1+1 = n^2+3$.

 (iii) $n \equiv 2$: then $n+1 \equiv 0$ and $f(n+1) = 3f((n+1)/3) \equiv 3(n+1)^2/9$ by inductive assumption. But $n+1 \equiv 0$ so $f(n+1) \equiv 0 \equiv (n+1)^2$.

(b) We prove by strong induction. Base case: $f(0) = 0 \geq 0^2$. Let $n \in \mathbb{N}$. Assume $f(k) \geq k$ for $0 \leq k \leq n$. Consider $f(n+1)$. There are three cases:

 (i) $n \equiv 0$: then $n+1 \equiv 1$. $f(n+1) = f(n)+1 \geq n+1$ by inductive assumption.

 (ii) $n \equiv 1$: then $n+1 \equiv 2$ and $f(n+1) = f(n)+3 \geq n+3$ by inductive assumption which is greater than $n+1$.

 (iii) $n \equiv 2$: then $n+1 \equiv 0$ and $f(n+1) = 3f((n+1)/3) \geq 3(n+1)/3 = (n+1)$.

Problem 2

The *Tribonacci sequence* is the sequence $t_0, t_1, t_2, \ldots$ defined by

$$t_0 = 0, \quad t_1 = 1, \quad t_2 = 1 \quad \text{and} \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for all } n \geq 3$$

Prove that $t_n \equiv t_{n+8} \bmod 4$ for all $n \in \mathbb{N}$.

Suggested solution:

We prove by strong induction with three base cases. First we compute $t_3 = 0+1+1 = 2$, $t_4 = 1+1+2 = 4$, $t_5 = 1+2+4 = 7$, $t_6 = 2+4+7 = 13$, $t_7 = 4+7+13 = 24$, $t_8 = 7+13+24 = 44$, $t_9 = 13+24+44 = 81$, $t_{10} = 24+44+81 = 149$.

Base cases: $t_8 = 44 \equiv 0 = t_0$. $t_9 = 81 \equiv 1 = t_1$. $t_{10} = 149 \equiv 1 = t_2$.

Inductive assumption: Let $n \in \mathbb{N}$. Assume $t_n \equiv t_{n+8}$, $t_{n+1} \equiv t_{n+9}$ and $t_{n+2} \equiv t_{n+10}$.

Inductive step:

$$\begin{aligned} t_{n+11} &= t_{n+10} + t_{n+9} + t_{n+8} \text{ by recursive definition} \\ &= t_{n+2} + t_{n+1} + t_n \text{ by inductive assumption} \\ &= t_{n+3} \text{ by recursive definition} \end{aligned}$$

Problem 3

For each of the following relations, determine whether it is reflexive, whether it is symmetric, whether it is antisymmetric, whether it is transitive, whether it is connected, whether it is an equivalence relation, whether it is a partial order relation, and whether it is a total order relation.

(a) The relation $\updownarrow$ on $\mathbb{R}^2$ defined for all $(a,b),(c,d) \in \mathbb{R}^2$ by letting $(a,b) \updownarrow (c,d)$ if and only if either $a < c$, or $a = c$ and $b \leq d$.

(b) The relation $\pitchfork$ on $\mathcal{P}(\mathbb{R})$ defined for all $U, V \in \mathcal{P}(\mathbb{R})$ by letting $U \pitchfork V$ if and only if $U \subseteq V \cup \{0\}$.

(c) The relation $\bowtie$ on the set X of all functions $\mathbb{R} \to \mathbb{R}$ defined for all $f, g \in X$ by letting $f \bowtie g$ if and only if $f(x) - g(x) \in \mathbb{Q}$ for all $x \in \mathbb{R}$.

Suggested solutions:

(a) Reflexive: Let $(a,b) \in \mathbb{R}^2$. Then $a = a$ and $b \leq b$ so $(a,b) \updownarrow (a,b)$.
Not symmetric (hence not an equivalence relation): $(1,3) \updownarrow (3,1)$ but not $(3,1) \updownarrow (1,3)$.
Transitive: Let $(a,b),(c,d),(e,f) \in \mathbb{R}^2$ such that $(a,b) \updownarrow (c,d)$ and $(c,d) \updownarrow (e,f)$. There are two cases:

  (i) $a < c$: Since $c < e$ or $c = e$, we have $a < e$, which implies $(a,b) \updownarrow (e,f)$.

  (ii) $a = c$ and $b \leq d$. Since $(c,d) \updownarrow (e,f)$, if $c < e$ then $a < e$ as in the previous case. If $c = e$ and $d \leq f$ then $a = f$ and $b \leq f$. In either scenario $(a,b) \updownarrow (e,f)$.

Antisymmetric: Let $(a,b),(c,d) \in \mathbb{R}^2$ such that $(a,b) \updownarrow (c,d)$ and $(c,d) \updownarrow (a,b)$. We cannot have $a < c$ because it contradicts $(c,d) \updownarrow (a,b)$. Therefore we have $a = c$ and $b \leq d$. Similarly we can conclude $c = a$ and $d \leq b$, which implies $b = d$. $(a,b) = (c,d)$.
Since it is reflexive, transitive and antisymmetric, it is a partial order.
Connected: Let $(a,b),(c,d) \in \mathbb{R}^2$ such that $(a,b) \not\updownarrow (c,d)$. We need to show $(c,d) \updownarrow (a,b)$. By assumption either $a \geq c$ and ($a \neq c$ or $b > d$). In the case that $a \neq c$, we have $a > c$, which implies $(c,d) \updownarrow (a,b)$. Otherwise, $a = c$ and $b > d$, which implies $(c,d) \updownarrow (a,b)$.
Since it is a partial order and is connected, it is a total order.

(b) Reflexive: For any $U \subseteq \mathbb{R}$, $U \subseteq U \cup \{0\}$.
Not symmetric (hence not an equivalence relation): $\{0\} \pitchfork \{1\}$ but not $\{1\} \pitchfork \{0\}$.
Transitive: Let $U, V, W \subseteq \mathbb{R}$ such that $U \subseteq V \cup \{0\}$ and $V \subseteq W \cup \{0\}$. Then $U \subseteq V \cup \{0\} \subseteq (W \cup \{0\}) \cup \{0\} = W \cup \{0\}$.
Not antisymmetric (hence not a partial/total order): $\{0\} \pitchfork \emptyset$ and $\emptyset \pitchfork \{0\}$ but $\emptyset \neq \{0\}$.
Not connected: $\{1\}$ and $\{2\}$ are not $\pitchfork$-related in either direction.

(c) Reflexive: For any $f \in X$ and $x \in \mathbb{R}$, $f(x) - f(x) = 0 \in \mathbb{Q}$.
Symmetric: For any $f, g \in X$ such that $f(x) - g(x) \in \mathbb{Q}$, we can write $f(x) - g(x) = m/n$ for some $m \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$. Then $g(x) - f(x) \in (-m)/n \in \mathbb{Q}$.
Transitive: For any $f, g, h \in X$ such that $f(x) - g(x) \in \mathbb{Q}$ and $g(x) - h(x) \in \mathbb{Q}$. We can write $f(x) - g(x) = m_1/n_1$ and $g(x) - h(x) = m_2/n_2$ for some $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $n_1, n_2 \neq 0$. Then $f(x) - h(x) = m_1/n_1 + m_2/n_2 = (m_1 n_2 + n_1 m_2)/n_1 n_2 \in \mathbb{Q}$.
Since it is reflexive, symmetric and transitive, it is an equivalence relation.
Not antisymmetric (hence not a partial/total order): The constant functions 1 and 2 satisfy $1 \bowtie 2$, $2 \bowtie 1$ but $1 \neq 0$ as functions.
Not connected: The constant functions 1 and $\sqrt{2}$ are not $\bowtie$-related in either direction.

Problem 4

Let X be a set and let $\sim$ be an equivalence relation on X.

(a) Prove that the function $q : X \to X/{\sim}$ defined by $q(a) = [a]_\sim$ for all $a \in X$ is a surjection.

(b) A function $f : X \to Y$ is said to *respect* $\sim$ if, for all $a, b \in X$, if $a \sim b$, then $f(a) = f(b)$. Prove that for all $f : X \to Y$, f respects $\sim$ if and only if $f = g \circ q$ for some function $g : X/{\sim} \to Y$.

(c) Prove that the function g from part (b) is unique, in the sense that if $h : X/{\sim} \to Y$ is a function such that $f = h \circ q$, then $g = h$.

Suggested solutions:

(a) Let $A \in X/{\sim}$. By definition there is $a \in X$ such that $A = [a]_\sim$. Thus $q(a) = [a]_\sim = A$.

(b) Let $f : X \to Y$. Assume f respects $\sim$. We need to define $g : X/{\sim} \to Y$ such that $f = g \circ q$. Let $[a]_\sim \in X/{\sim}$. We define $g([a]_\sim) = f(a)$. Then $f = g \circ q$. It remains to check g is well-defined: if $[a]_\sim = [b]_\sim$, then $g([a]_\sim) = f(a) = f(b) = g([b]_\sim)$ where the middle equality $f(a) = f(b)$ uses the assumption.
Conversely, assume $f = g \circ q$ for some $g : X/{\sim} \to Y$. We show f respects $\sim$. Let $a, b \in X$ such that $a \sim b$. This implies $[a]_\sim = [b]_\sim$. Then $f(a) = g(q(a)) = g([a]_\sim) = g([b]_\sim) = f(b)$.

(c) Let g, h as in the question. Let $[a]_\sim \in X/{\sim}$. Then $g([a]_\sim) = h(q(a)) = f(a) = h(q(a)) = h([a]_\sim)$.

Problem 5

For each of the following sets X, partial orders $\preccurlyeq$ on X and subsets $A \subseteq X$, find $\sup_\preccurlyeq(A)$ if it exists (or prove that it doesn't), and find $\inf_\preccurlyeq(A)$ if it exists (or prove that it doesn't).

(a) $X = \mathbb{R}$, $\preccurlyeq \,=\, \leq$ and $A = [0, 1) \cap (\mathbb{R} \setminus \mathbb{Q})$;

(b) $X = \mathbb{N}$, $\preccurlyeq \,=\, \mid$ and $A = \{p \in \mathbb{N} \mid p \text{ is prime}\}$;

(c) $X = \{U \subseteq \{1, 2, 3, 4\} \mid |U| \text{ is even}\}$, $\preccurlyeq \,=\, \subseteq$, and $A = \{\{1, 2\}, \{2, 3\}\}$;

Suggested solutions:

(a) $\sup_\preccurlyeq(A) = 1$: 1 is greater than any element in A. If $\delta < 1$, then there is some irrational number $x \in A$ such that $\delta < x < 1$. $\inf_\preccurlyeq(A) = 0$: 0 is less than any element in A. If $\delta > 0$, then there is some irrational $y \in A$ such that $0 < y < \delta$.

(b) $\sup_\preccurlyeq(A) = 0$ because all prime numbers divide 0, and 0 is already the smallest natural number. $\inf_\preccurlyeq(A) = 1$ because 1 is the only number that divides all prime numbers (no other lower bounds to compare).

(c) $\sup_\preccurlyeq(A) = \{1, 2, 3, 4\}$: any upper bound of A must contain 1, 2 and 3, so it must be $\{1, 2, 3, 4\}$ (no other upper bounds to compare). $\inf_\preccurlyeq(A) = \emptyset$: any lower bound is a subset of $\{2\}$, which must be $\emptyset$ (no other lower bounds to compare).

Problem 6

Assume A is a nonempty set and $(B, \preccurlyeq_B)$ is a nonempty poset. Let $\mathcal{F}$ be the set of all functions with domain A and codomain B. Define an $\preccurlyeq_\mathcal{F}$ on $\mathcal{F}$ to be the pointwise ordering. That is,

$$f \preccurlyeq_\mathcal{F} g \text{ iff } f(t) \preccurlyeq_B g(t) \text{ for all } t \in A.$$

Prove that $\preccurlyeq_\mathcal{F}$ is a partial order on $\mathcal{F}$.
Suggested solution:
Reflexive: Let $f \in \mathcal{F}$ and $t \in A$. Then $f(t) \preccurlyeq_B f(t)$ because $\preccurlyeq_B$ is a partial order. Therefore $f \preccurlyeq_\mathcal{F} f$.

Antisymmetric: Let $f, g \in \mathcal{F}$ such that $f \preccurlyeq_{\mathcal{F}} g$ and $g \preccurlyeq_{\mathcal{F}} f$. This implies $f(t) \preccurlyeq_B g(t)$ and $g(t) \preccurlyeq_B f(t)$ for all $t \in A$. Since $\preccurlyeq_B$ is antisymmetric, $f(t) = g(t)$ for all $t \in A$. This implies $f = g$.
Transitive: Let $f, g, h \in \mathcal{F}$ such that $f \preccurlyeq_{\mathcal{F}} g$ and $g \preccurlyeq_{\mathcal{F}} h$. Then for all $t \in A$, $f(t) \preccurlyeq_B g(t)$ and $g(t) \preccurlyeq_B h(t)$. Since $\preccurlyeq_B$ is transitive, $f(t) \preccurlyeq_B h(t)$. As $t \in A$ is arbitrary, we can conclude that $f \preccurlyeq_{\mathcal{F}} h$.