Full name	Andrew ID

21-127 Test 1 (Practice)

Wednesday, 15 February 2023

Please read the following instructions carefully before the test begins.

Before the test

- Do not open the test until instructed to do so.
- Write your full name and Andrew ID in the boxes at the top of this page.
- Place your Carnegie Mellon University ID card face-up in front of you.
- Turn off your electronic devices (e.g. phone, tablet, laptop, calculator), and store any devices, notes or books out of sight (e.g. in a closed bag).

During the test

- Write clearly and legibly with a pen or pencil that is dark enough to be readable when scanned.
- You must justify all answers and claims with mathematical proof, unless otherwise specified.
- If you continue a solution on one of the extra pages (pages 13–14), you should clearly indicate in your solution the page number where it is continued.
- You may not use notes, books, other reference materials, calculators or electronic devices on this test.
- You may not communicate with others or attempt to look at other students' work during the test.
- If you require assistance, please raise your hand and wait for a proctor to come to you.
- If you need to leave the classroom (e.g. to use the bathroom), please raise your hand, show your CMU ID card to a proctor, and leave your belongings in the classroom.
- If you finish the test with 5 minutes or more remaining, you may turn in your test and leave the classroom discreetly; otherwise, please remain seated until the test ends.

After the test

- Stop working immediately when you are instructed to do so.
- Turn in all 15 pages of this test; if you tore out any pages, put them back in their correct positions.

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1. (a) Write the definition of the empty set $\varnothing$ using a logical formula **[5]**

(b) For each of the following, name one element from the set. If it is not possible, **prove** that it is the empty set. **[10]**

(i) $\mathscr{P}(\mathbb{Z} \setminus \mathbb{Q})$

(ii) $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$

(iii) $\{x \in \mathbb{R} : \exists n \in \mathbb{Z}, (x = \sqrt{n} \wedge \exists k \in \mathbb{Z}, n = 4k - 1)\}$

(iv) $\{x \in \mathbb{Q} : x^2 + 4x + 4 = 0\}$

Suggested solutions:

(a) $\varnothing = \{x \mid x \neq x\}$

(b) (i) An element is $\varnothing$ (it is a subset of any power set).

(ii) Impossible: an element $x \in \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$ satisfies $(0, \frac{1}{n})$ for each n. Therefore x is positive. However, there is no positive number strictly less than all $1/n$.

(iii) $\sqrt{3}$ (when $n = 3$ and $k = 1$).

(iii) -2.

More space for (Q1)

2. (a) State the extensionality axiom **[5]**

(b) Let A,B be two arbitrary sets. Prove that $A \setminus B = A \setminus (A \cap B)$ **[10]**

Suggested solutions:

(a) $\forall A \forall B\, (\forall x\, (x \in A \Leftrightarrow x \in B) \implies A = B)$

(b) Let $x \in A \setminus B$. We know $x \in A$ and $x \notin B$. Since we have $x \in A$ already, it suffices to prove $x \notin A \cap B$ to conclude $x \in A \setminus (A \cap B)$. Observe the following equivalences:

$$\begin{aligned} & x \notin A \cap B \\ & \Leftrightarrow \neg(x \in A \cap B) \\ & \Leftrightarrow \neg(x \in A \land x \in B) \\ & \Leftrightarrow x \notin A \lor x \notin B \end{aligned}$$

The last disjunction is true because we already know that $x \notin B$.
Conversely, let $x \in A \setminus (A \cap B)$. We know $x \in A$ and $x \notin A \cap B$. The latter implies $x \notin A \lor x \notin B$. Since we had $x \in A$, we must have $x \notin B$. Therefore $x \in A$ and $x \notin B$. $x \in A \setminus B$.

More space for (Q2)

3. (a) Define the set of all rational numbers in set-builder notation **[5]**

(b) Let a, b, c be integers. Prove that **[10]**

$$a|b \wedge a|c \Rightarrow \forall s, t \in \mathbb{Z},\ a|(sb+tc)$$

Suggested solutions:

(a) $\mathbb{Q} = \{q \in \mathbb{R} \mid \exists a, b \in \mathbb{Z}, (b \neq 0 \wedge q = a/b)\}$

(b) Let $s, t \in \mathbb{Z}$. We need to find $r \in \mathbb{Z}$ such that $ar = sb + tc$. Since $a|b$, we can find $r_1 \in \mathbb{Z}$ such that $b = ar_1$. Similarly, since $a|c$, we can find $r_2 \in \mathbb{Z}$ such that $c = ar_2$. Thus

$$sb + tc = s(ar_1) + t(ar_2) = a(sr_1 + tr_2).$$

Since $s_r 1 + tr_2 \in \mathbb{Z}$, we can conclude that $a|(sb+tc)$.

More space for (Q3)

4. Give an example of logical formulae $p(x), q(x)$ and a set S such that the following are not logically equivalent. Justify your answer with a proof. **[10]**

- $\exists x \in S, (p(x) \wedge q(x))$
- $(\exists x \in S, p(x)) \wedge (\exists x \in S, q(x))$

Suggested solution: Let $S = \mathbb{R}$, $p(x)$ be "$x \in \mathbb{Q}$" and $q(x)$ be "$x \in \mathbb{R} \setminus \mathbb{Q}$".

- $\exists x \in S, (p(x) \wedge q(x))$ is not true because $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are disjoint.
- $(\exists x \in S, p(x)) \wedge (\exists x \in S, q(x))$ is true because we can witness $\exists x \in S, p(x)$ by $x = 1$ and $\exists x \in S, q(x)$ by $x = \sqrt{2}$.

Since the propositions have different truth values, they cannot be logically equivalent.

More space for (Q4)

5. Consider the proposition $\varphi : \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}(y = x \vee y^2 > x^2)$

(a) Determine whether φ is True or False. Justify your answer with a proof. **[8]**

(b) Write $\neg\varphi$ in maximally negated form. **[7]**

Suggested solutions:

(a) φ is true. Choose $x = 0 \in \mathbb{Z}$. For any $y = \mathbb{Z}$, there are two cases: if $y = 0$, then $y = x$ so $y = x \vee y^2 > x^2$. Otherwise, $y \neq 0$ and thus $y^2 > 0 = x^2$.

(b) $\neg\varphi : \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y \neq x \wedge y^2 \leq x^2)$

More space for (Q5)

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 13).

If you use this page to continue a solution to a question, please clearly indicate on the first page of your solution where it is continued (this is page 14).

Initials

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