Homework 5

Problem 1

For each of the following functions, find its inverse if it has one, or prove that it does not have an inverse. [If the function does not have an inverse by showing that it is not bijective.

(a) $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = \dfrac{n + |n|}{2}$ for all $n \in \mathbb{Z}$.

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, $h(A) = \mathbb{N} \setminus A$ for all $A \in \mathcal{P}(\mathbb{N})$.

Problem 2

Let $f : X \to Y$ be a function. Let A and B be subsets of X. If f is injective, can we conclude that $f[A \setminus B] = f[A] \setminus f[B]$? Prove your answer.

Problem 3

The *Fibonacci Numbers* is a sequence of natural numbers defined by:

(i) $f_0 = 0$

(ii) $f_1 = 1$

(ii) $\forall n \in \mathbb{N}, f_{n+2} = f_{n+1} + f_n$

Use induction to prove that 3 divides f_{4n}, for all $n \in \mathbb{N}$.

Problem 4

Prove that for all $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Problem 5

A sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = 5 \quad \text{and} \quad a_{n+1} = 3a_n - 8 \text{ for all } n \in \mathbb{N}$$

Find an expression for a general term a_n in terms of $n \in \mathbb{N}$ and prove your formula by weak induction. [Hint: Try to find a formula for $a_{n+1} - 4$ in terms of a_n]

Problem 6

The operators of *indexed conjunction* $\bigwedge_{i=1}^{n}$ and *indexed disjunction* $\bigvee_{i=1}^{n}$ are defined by recursion on $n \in \mathbb{N}$ as follows:

- $\bigwedge_{i=1}^{0} p_i = \top$ and $\bigwedge_{i=1}^{n+1} p_i = \left(\bigwedge_{i=1}^{n} p_i\right) \wedge p_{n+1}$, for all $n \in \mathbb{N}$;
- $\bigvee_{i=1}^{0} p_i = \bot$ and $\bigvee_{i=1}^{n+1} p_i = \left(\bigvee_{i=1}^{n} p_i\right) \vee p_{n+1}$, for all $n \in \mathbb{N}$.

where $\top$ represents the true proposition '$0 = 0$', and $\bot$ represents the false proposition '$0 = 1$'.

Prove by induction that $\left(\bigvee_{i=1}^{n} p_i\right) \Rightarrow q \equiv \bigwedge_{i=1}^{n}(p_i \Rightarrow q)$ for all $n \in \mathbb{N}$, where $p_1, p_2, \ldots$ and q are propositional variables.

Problem 7 (Extra practice, not to be submitted)

Consider a chessboard of size $2^n \times 2^n$ for some arbitrary positive integer n. Remove any square from the board. Is it possible to tile the remaining squares with L-shaped triominoes (showed below)? If your answer is Yes, prove it. If your answer is No, provide a counterexample.