Homework 2

Problem 1

Determine (with proof) if the following statements are true or false.

(a) $\exists x \in \mathbb{R}, e^x \leq 0$.

(b) $\forall n \in \mathbb{N}, \exists y \in \mathbb{R}, y = \sqrt{n}$.

(c) $\exists y \in \mathbb{R}, \forall n \in \mathbb{N}, y = \sqrt{n}$.

(d) $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, mn = 1$.

(e) $\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, mn = 1$.

(f) Let $F(\mathbb{R})$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

$$\forall f \in F(\mathbb{R}), [[(\forall a, b \in \mathbb{R}, f(a) = f(b) \Rightarrow a = b) \wedge (\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y)] \\ \Rightarrow [\exists g \in F(\mathbb{R}), \forall x, y \in \mathbb{R}, y = f(x) \Leftrightarrow x = g(y)]]$$

Problem 2

For each of the following statements, write the logical negation in maximally negated form. Then decide which proposition (the original or the negation) is true, and why.

- Let $P(x)$ be the variable proposition "$1 \leq x \leq 3$"
- Let $R(x)$ be the variable proposition "$x^2 = 2$"
- Let $S(x)$ be the variable proposition "$x = 1$"

(a) $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, (m \neq n \wedge P(m) \wedge R(n))$.

(b) $\forall n \in \mathbb{Z}, (R(n) \wedge S(n) \Rightarrow P(n) \wedge \neg P(n))$.

Problem 3

Prove the following proposition:

prove that for all $m \in \mathbb{Z}$, m is odd if and only if 8 divides $m^2 - 1$.

Problem 4

Use truth tables to determine whether or not the following statements are tautologies. p and q are arbitrary propositional variables.

(a) $((p \Rightarrow q) \wedge p) \Rightarrow q$

(b) $((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p$

(c) $(p \Rightarrow q) \vee (q \Rightarrow p)$

(d) $((p \Leftrightarrow q) \wedge p) \Leftrightarrow \neg q$

Problem 5

(a) Let $a, b \in \mathbb{R}$ with $a < b$, and let $C[a, b]$ be the set of all continuous functions on the interval $[a, b]$.

Write a logical formula describing the Extreme Value Theorem. You are not allowed to use any English words in your logical formula.

(b) **Euclid's lemma**— If a prime p divides the product ab of two integers a and b, then p must divide at least one of those integers a or b.

Write a logical formula describing Euclid's lemma. You are not allowed to use any English words in your logical formula.

Problem 6

Find a propositional formula φ whose truth table is as follows.

p	q	r	...	φ
T	T	T	...	F
T	T	F	...	T
T	F	T	...	F
T	F	F	...	F
F	T	T	...	T
F	T	F	...	F
F	F	T	...	F
F	F	F	...	F

The columns for the subformulae of φ should be included in your solution. Completing the truth table correctly (without extra justification) will be considered sufficient proof that your formula is correct, but you should indicate how you came up with your formula.

Problem 7 (Extra practice, not to be submitted)

(a) Use Euclid's Lemma to prove that:
For any prime p and for any integer a, p divides a^2 if and only if p divides a.

(b) Use the previous part to prove that: For any prime p, $\sqrt{p}$ is irrational.