Homework 10

(You must justify ALL your claims unless otherwise stated)

Problem 1

An organization consists of 13 men and 12 women.

(a) The organization wishes to form a 10 person committee consisting of 5 men and 5 women. However, there are 2 men, call them Adam and Bob, who refuse to work together and thus cannot be on the committee together. How many such committees can be formed?

(b) The entire organization wishes to form a line. How many ways can this be done in which at least one woman is standing next to another?

Problem 2

3. Let a, b, k be three positive integers with $a + b \geq k$. Prove that

$$\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i} \binom{b}{k-i}$$

by counting in two ways argument. Use the exact form given. Do not simplify algebraically.

Problem 3

If I flip a coin 20 times, I get a sequence of Heads H and tails T.

(a) How many different sequences of heads and tails are possible?

(b) How may different sequences of heads and tails have exactly five heads?

(c) How many different sequences have at most 2 heads?

(d) How many different sequences have at least 3 heads?

Problem 4

Let X be a finite set, let $|X| = n$. Define $E, O \subseteq \mathcal{P}(X)$ as follows:

$$E = \{U \subseteq X : |U| \text{ is even}\} \text{ and } O = \{U \subseteq X : |U| \text{ is odd}\}$$

(a) Use the addition principle to prove that

$$|E| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$$

(b) Use the addition principle to prove that

$$|O| = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$$

(c) (c) Use parts (a) and (b) to conclude that, for all $n > 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Problem 5

Use the multiplication principle to prove the following:

(a) The number of ordered pairs (A, a) such that $A \subseteq [n]$ and $a \in A$ is equal to $n \cdot 2^{n-1}$ for all $n > 0$.

(b) The number of injections $[k] \to [n]$ is equal to $\binom{n}{k} \cdot k!$, for all $n, k \in \mathbb{N}$.

Problem 6

Use the addition and multiplication principles to prove that, for all $n > 0$, the number of surjections $[n] \to [3]$ is equal to $3^n - 3 \cdot 2^n + 3$.