Homework 1 Solutions

Problem 1

Prove the following proposition:
Let $a, b, c \in \mathbb{Z}$. If a divides b and a divides c, then a divides $b - c$.

Suggested solutions:
Since a divides b, there is $p \in \mathbb{Z}$ such that $b = ap$. Since a divides c, there is $q \in \mathbb{Z}$ such that $c = aq$. Then $b - c = ap - aq = a(p - q)$. Since $p - q$ is in $\mathbb{Z}$, a divides $b - c$.

Problem 2

(a) Let A be the set of all ordered pairs of integers whose sum is negative. Write A in set-builder notation.

(b) Let B be the set of all ordered pairs of real numbers which lie on the graph of the function $f(x) = \sin(x^2 + 1)$. Write B in set-builder notation.

(c) Let $C = \{\dots, -14, -10, -6, -2, 2, 6, 10, 14, \dots\}$. Write C in set-builder notation.

Suggested solutions:

(a) $A = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b < 0\}$

(b) $B = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid b = \sin(a^2 + 1)\}$

(c) $C = \{x \in \mathbb{Z} \mid 4 \text{ divides } (x - 2)\}$

Problem 3

Determine with proof whether the following propositions are true or false.

(a) If n is an even integer, then n^2 is an even integer.

(b) All prime numbers are odd integers.

(c) All odd integers are prime numbers.

Suggested solutions:

(a) True: assume n is even. There is $k \in \mathbb{Z}$ such that $n = 2k$. Then $n^2 = (2k)(2k) = 2(2k^2)$. Since $2k^2 \in \mathbb{Z}$, n^2 is even.

(b) False: 2 is an even prime number.

(c) False: 9 is an odd composite number.

Problem 4

Write each of the following sets in list notation.

(a) $D = \{n \in \mathbb{N} : 2 \text{ divides } n \text{ or } 3 \text{ divides } n\}$

(b) $E = \{n \in \mathbb{N} : 2 \text{ divides } n \text{ and } 3 \text{ divides } n\}$

(c) $F = \{n \in \mathbb{Z} : 3 \text{ is the only prime that divides } n\}$

Suggested solutions:

(a) $D = \{0, 2, 3, 4, 6, 8, 9, \dots\}$

(b) $E = \{0, 6, 12, 18, \dots\}$

(c) $F = \{3, -3, 9, -9, 27, \dots\}$

Problem 5

For fixed $n \in \mathbb{N}$, let p represent the proposition 'n is even', let q represent the proposition 'n is prime' and let r represent the proposition '$n = 2$'. Translate each of the following propositional formulae into a plain English sentence—note that your sentence will include the variable n.

(a) $(p \wedge q) \Rightarrow r$;

(b) $p \Rightarrow ((\neg q) \vee r)$;

(c) $(q \wedge (\neg r)) \Rightarrow (\neg p)$;

(d) $(p \wedge (\neg q)) \vee (\neg r)$.

Suggested solutions:

(a) If n is even and n is prime, then $n = 2$. (This is true.)

(b) If n is even, then either n is not prime, or $n = 2$. (This is true.)

(c) If n is prime and n is not 2, then n is not even. (This is true.)

(d) Either n is even but not prime, or n is not 2. (This is false: for example when $n = 2$.)

Problem 6

Prove that there is an irrational real number x such that $x^{\sqrt{2}} \in \mathbb{Q}$.
[Hint: we know that $\sqrt{2} \notin \mathbb{Q}$ is true, but we don't know whether ${\sqrt{2}}^{\sqrt{2}} \in \mathbb{Q}$ is true or false. However we *do* know that ${\sqrt{2}}^{\sqrt{2}} \in \mathbb{Q} \vee {\sqrt{2}}^{\sqrt{2}} \notin \mathbb{Q}$ is true. Consider doing a proof by cases.]

Suggested solutions:
Suppose ${\sqrt{2}}^{\sqrt{2}} \in \mathbb{Q}$, then $x = \sqrt{2}$ is an example of an irrational real number such that $x^{\sqrt{2}} \in \mathbb{Q}$.
Otherwise, ${\sqrt{2}}^{\sqrt{2}} \notin \mathbb{Q}$. Take $x = {\sqrt{2}}^{\sqrt{2}}$ which is irrational. We check that $x^{\sqrt{2}} = \left({\sqrt{2}}^{\sqrt{2}}\right)^{\sqrt{2}} = {\sqrt{2}}^{\sqrt{2}\cdot\sqrt{2}} = {\sqrt{2}}^{2} = 2 \in \mathbb{Q}$.