Homework 3

Problem 1

Prove that:

$$\forall A, B[A \subseteq B \text{ if and only if } A \setminus B = \varnothing]$$

Problem 2

For each of the following statements, prove it is true for all sets X, Y, otherwise, give a counterexample showing that the statement is false.

(a) $\mathscr{P}(X \cap Y) = \mathscr{P}(X) \cap \mathscr{P}(Y)$

(b) $\mathscr{P}(X \setminus Y) = \mathscr{P}(X) \setminus \mathscr{P}(Y)$

(c) $\mathscr{P}(X \times Y) = \mathscr{P}(X) \times \mathscr{P}(Y)$

(d) $\mathscr{P}(X \cup Y) = \mathscr{P}(X) \cup \mathscr{P}(Y)$

Problem 3

Consider the following subsets of $\mathbb{R} \times \mathbb{R}$.

- $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}$
- $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1 \vee |y| \leq 1\}$
- $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1 \wedge |y| \leq 1\}$
- $D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \max\{|x|, |y|\} \leq 1\}$

(a) Describe each of the above sets geometrically in $\mathbb{R} \times \mathbb{R}$ (You may include a graph to support your answer)

(b) For each of the following statements, prove it is true, or give a counterexample showing that the statement is false.

$$A \subseteq B \qquad C \subseteq A \qquad D \subseteq C \qquad C \subseteq D \qquad B \subseteq C$$

Problem 4

A subset $U \subseteq \mathbb{R}$ is said to be **open** if

$$\forall x \in U,\ \exists \delta \in (0, \infty),\ \forall y \in \mathbb{R},\ (x - \delta < y < x + \delta \Rightarrow y \in U)$$

(a) Find a maximally negated logical formula that is equivalent to the assertion that a subset $U \subseteq \mathbb{R}$ is *not* open. [By the way, 'closed' does not mean the same thing as 'not open'.]

(b) Prove that for all $a, b \in \mathbb{R}$ with $a < b$, the interval (a, b) is open.

(c) Prove that for all $a, b \in \mathbb{R}$ with $a < b$, the interval $[a, b)$ is not open.

(d) (Extra practice, not to be submitted) Determine whether $\mathbb{R} \setminus \mathbb{Z}$ is open and whether $\mathbb{Q}$ is open.

Problem 5

Let I_{op} be the set of all bounded open intervals in $\mathbb{R}$, and I_{cl} be the set of all closed intervals in $\mathbb{R}$. Specifically:

$$I_{\text{op}} = \{U \subseteq \mathbb{R} \mid U = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ with } a < b\}$$
$$I_{\text{cl}} = \{U \subseteq \mathbb{R} \mid U = [a, b] \text{ for some } a, b \in \mathbb{R} \text{ with } a < b\}$$

For each of the following statements, determine whether it is true or false.

(a) $I_{\text{op}} \in \mathscr{P}(\mathscr{P}(\mathbb{R}))$

(b) $I_{\text{op}} \subseteq I_{\text{cl}}$

(c) $I_{\text{op}} \cap I_{\text{cl}} \neq \varnothing$

(d) $\forall U \in I_{\text{op}},\ \exists V \in I_{\text{cl}},\ (U \subseteq V \wedge V \nsubseteq U)$

Problem 6

Given sets A and B, let $A \triangle B = (A \cup B) \setminus (A \cap B)$.

(a) Prove that for all sets A and B, we have $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(b) Prove that for all sets A and B, we have $A \triangle B = \varnothing$ if and only if $A = B$.

(c) Prove that for all sets A, B, C, we have $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

(d) (Extra practice, not to be submitted) Prove that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$ for all sets A, B and C.