

# A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Criterion

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**Authorship and AI-use disclosure.** The author, Dylan Anthony Dupont, designed the framework, chose all constants/normalizations, and validated all mathematics and computations. Generative assistants (From GPT-4o to GPT-5 Pro) were used only for typesetting assistance, editorial organization, and consistency checks; and thus are not an author. All claims are the author’s responsibility (based on COPE/ICMJE guidance).

## Abstract

This paper is organized in three parts:

- **Part I** — Reader’s Guide / Motivation, reducing the Riemann Hypothesis (RH) to a height-local statement in the width-2 frame:  $RH \Leftrightarrow a(m) = 0$  at each nontrivial height  $m$ , while recording non-load-bearing structural scaffolding.
- **Part II** — A self-contained, boundary-only analytic proof that the per-height tilt satisfies  $a(m) = 0$  at every nontrivial height using a disc-based  $L^2$  upper envelope and an  $L^2$  lower envelope via allocation + restricted contour + Jensen. A rigorous Outer/Rouché Certification Path, explicit domains and symbolic constants (“shape-only” vs. residual).
- **Part III** — Promotes the identities constructed in Part I to structural corollaries of the main theorem once  $a(m) = 0$  is established.
- **Appendices** — Technical details supporting Part II (load-bearing for Part II; load-bearing for Part III if reference to appendices is required in Part III).

*Dependency map (schematic):* Part I → (no arrows into Part II); Part II → Part III; Appendices → Part II and Part III.

## Part I — Reader’s Guide / Motivation, Reduction & Implications

**What this section is (and is not).** *What it does.* It introduces modulated frames and the width-2 normalization, defines the centered “ $a$ -lens” that measures horizontal tilt at a fixed height, and reduces RH to the height-local target  $a(m) = 0$  for each nontrivial height  $m$ . It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become consequences once  $a(m) = 0$  is proved. *What it does not do.* It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later. This Guide is not used by the analytic part.

### 1) Modulated frames and the width-2 pivot

For  $f > 0$  define the modulated family  $\zeta_f(s) := \zeta(s/f)$  with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)} \Gamma\left(\frac{s}{2f}\right) \zeta_f(s),$$

so  $\Lambda_f$  is entire and satisfies  $\Lambda_f(s) = \Lambda_f(f - s)$ . Equivalently,  $\zeta_f(s) = A_f(s) \zeta_f(f - s)$  with  $A_f(s)A_f(f - s) \equiv 1$ .

**Width–2 normalization.** Put  $u := (2/f) s$ . Then

$$\zeta_2(u) := \zeta(u/2), \quad \Lambda_2(u) := \pi^{-u/4} \Gamma(u/4) \zeta(u/2), \quad \Lambda_2(u) = \Lambda_2(2-u).$$

The non–completed FE reads  $\zeta_2(u) = A_2(u) \zeta_2(2-u)$ . In the open strip  $0 < \Re u < 2$  and  $\Im u \neq 0$ ,  $A_2$  is analytic and nonvanishing.

**Partner map.** On  $\Im u > 0$ , FE + conjugation gives the involution  $J(u) = 2 - \bar{u}$ , swapping the two column points at the same height.

**Hinge unitarity (deferred).** The statement “ $|\chi_2(u)| = |A_2(u)|^{-1} = 1$  iff  $\Re u = 1$ ” is proved in Part II (Hinge–Unitarity). We do not use it here.

## 2) Centered $a$ –lens and the quartet

Let  $v := u - 1$  and  $E(v) := \Lambda_2(1+v)$ . Then  $E(v) = E(-v) = \overline{E(\bar{v})}$ .

**Nontrivial height.** A “nontrivial height”  $m > 0$  means:  $m$  occurs as the imaginary part of a nontrivial zero  $s = \frac{1}{2} + im/2$ . The reduction shows that whenever such an  $m$  occurs, the associated tilt must satisfy  $a(m) = 0$ .

**Tilt at height  $m$ .** At fixed  $m > 0$ , set

$$U_R(m; a) = 1 + a + im, \quad U_L(m; a) = 1 - a + im, \quad a \in [0, 1].$$

In the centered frame, the “dial points” are  $\pm(a + im)$ . The partner map  $J$  swaps  $U_R \leftrightarrow U_L$ .

**Quartet.** Conjugation (top↔bottom) and FE reflection generate the quartet  $\{1 \pm a \pm im\}$  at height  $m$ .

## 3) Why width–2: slope invariance

If the columns collapse at height  $m$  ( $a = 0$ ), the point is  $u = 1 + im$  and its slope is  $\Im u / \Re u = m/1 = m$ . Rescaling to any frame  $s = (f/2) u$  preserves the slope:

$$\frac{\Im s}{\Re s} = \frac{(f/2)m}{f/2} = m.$$

Thus  $\{m_k\}$  simultaneously records the imaginary ordinates of the nontrivial zeros and their origin–through slopes in every modulated frame—provided the per–height collapse holds.

## 4) Height–local reduction of RH

Fix a nontrivial height  $m > 0$  and write  $U_R = 1 + a + im$ ,  $U_L = 1 - a + im$ . The following are purely algebraic and equivalent:

- (PHU–1) Column equality:  $\Re U_R = \Re U_L \iff a = 0$ .
- (PHU–2) Ray (slope) lock:  $\Im U_R / \Re U_R = \Im U_L / \Re U_L$ , i.e.  $m/(1+a) = m/(1-a) \iff a = 0$ .
- (PHU–3) Hinge form:  $U_R = U_L = 1 + im$ .

*Reduction target.* RH  $\iff$  for every nontrivial height  $m > 0$ ,  $a(m) = 0$ . Part II proves this per–height collapse; nothing from this Guide is used there.

## 5) Box alignment and hand-off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta := \eta \alpha / (\log m)^2, \quad \eta \in (0, 1).$$

When  $\alpha = \pm a$ , the dial points  $\pm(a + im)$  lie on the box's horizontal centerline.

**What Part II does.** Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three–lines tools, Stirling–class envelopes, explicit control of  $\zeta'/\zeta$  away from zeros), Part II shows that any off–axis quartet forces a boundary lower bound larger than an explicit upper bound, hence  $a(m) = 0$ .

**No circularity.** The analytic proof is logically independent of this Guide.

## 6) Parity gating and selection devices (interpretive only)

**Gating from the non–completed FE.** In the width–2 frame the non–completed FE reads

$$\zeta_2(u) = A_2(u) \zeta_2(2 - u), \quad A_2(u) = 2^{u/2} \pi^{u/2-1} \sin\left(\frac{\pi u}{4}\right) \Gamma\left(1 - \frac{u}{2}\right).$$

On the open strip  $0 < \Re u < 2$  with  $\Im u \neq 0$ , the prefactor  $A_2(u)$  is nonzero and finite; its sine zeros (the “trivial ladder”) lie on the real axis only. Thus *inside the open strip only  $\zeta_2$  can vanish* (nontrivial zeros), while the *trivial class is confined to the real axis*. This is the basic “odd/even lane” picture: the odd (upper) lane can host nontrivial zeros; the even (real) lane hosts the trivial ladder.

**Orthogonal split on the integer lattice.** To model this dichotomy as a clean input–space symmetry, decompose any lattice signal  $X : \mathbb{Z} \rightarrow \mathbb{C}$  via the orthogonal projectors

$$P_{\text{odd}}(n) = \frac{1-\cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \frac{1+\cos(\pi n)}{2},$$

so  $X = P_{\text{odd}}X + P_{\text{even}}X$ . We assign the nontrivial stream to odd slots (where  $P_{\text{odd}} = 1$ ) and the trivial ladder to even slots (where  $P_{\text{even}} = 1$ ). This mirrors the FE fact above without using it analytically.

## 7) Toolbox → structural consequences (after the theorem)

The items below are not inputs to the analytic proof. After Part II proves  $a(m) = 0$  for all nontrivial heights, they become *Structural Corollaries* describing the collapsed geometry and its lattice faces. (*Explicit formulas and brief proofs are recorded as corollaries in Part III; Part I intentionally omits them and they are not inputs to Part II.*)

- Pre-collapse columns (projector faces in the  $u$ –frame): right/left templates place odd–slot samples  $x \pm im_k$  and the even ladder  $-4(\cdot)$  via  $P_{\text{odd}}, P_{\text{even}}$ ; scaffolding, not assumptions.
- Collapsed canonical stream  $U(n)$ : when per–height collapse holds ( $x = 1$  on odd slots), the two columns coincide; parity form (via  $P_{\text{odd}}, P_{\text{even}}$ ) and an equivalent trigonometric form (via  $\sin^2(\pi n/2), \cos^2(\pi n/2)$ ).
- Single–frequency collapse (cosine face): a two–parameter cosine form  $U(n) = (c + d) + (c - d) \cos(\pi n)$  recovers the same stream;  $c, d$  simple in the odd–indexer  $k(n)$ .
- Self–indexed recurrence (no explicit  $k$ ): a short recurrence for  $U(n)$  pulls the needed odd index from the previous even sample; encodes the collapsed geometry without an explicit  $k(n)$ .

- Curvature extractor & the  $\zeta(2)$  disguise: the discrete second-difference of the imaginary part at even indices recovers  $m_j$  and admits an odd-square convolution form normalized by  $\zeta(2)$ .
- Seed  $\rightarrow$  rectifier  $\rightarrow$  physical streams: two-carrier seeds rectify under a mod-4 factor to yield the physical stream  $S_f(n)$  proportional to  $U(n)$ ; pre-collapse faces scale analogously.

## 8) Implications and one-sentence hand-off

The width-2 organization centralizes symmetry at  $\Re u = 1$ ; the centered  $a$ -lens isolates the single per-height degree of freedom; parity-orthogonal scaffolding separates the nontrivial stream from the ladder without entering the proof. With these definitions, RH reduces to: for every nontrivial height  $m > 0$ ,  $a(m) = 0$ .

**Hand-off.** **Part II** now proves this per-height collapse by a boundary-only contradiction on aligned boxes; nothing from this Guide is used in that proof.

## Part II — Analytic Core (self-contained; boundary-only)

## Part III — Structural Corollaries (after the main theorem)

**Standing assumption for this part.** Assume the *Main Theorem (Part II)*: for every nontrivial height  $m > 0$ , the per-height tilt satisfies  $a(m) = 0$ . Equivalently, in the universal strip  $0 < \Re u < 2$  each upper partner has real part 1.

**Parity projectors and indexer.** On  $\mathbb{Z}$  set

$$P_{\text{odd}}(n) = \frac{1 - \cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \frac{1 + \cos(\pi n)}{2}, \quad k(n) = \frac{n}{2} + \frac{1 - \cos(\pi n)}{4}.$$

Then  $k(2j - 1) = j$  and  $k(2j) = j$  for  $j \in \mathbb{Z}$ .

**Corollary 0.1** (Canonical columns). *For  $x \in (0, 2)$  define*

$$U_R(x, n) = P_{\text{odd}}(n) (x + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n),$$

$$U_L(x, n) = P_{\text{odd}}(n) (2 - x + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n).$$

*Let  $x = 1 \pm a$  with  $a \in (-1, 1)$ . Under the standing assumption  $a(m) = 0$  at each nontrivial height, the canonical choice  $x = 1$  yields  $U_R(1, n) = U_L(1, n)$  for all  $n \in \mathbb{Z}$ .*

*Proof.* On odd  $n = 2j - 1$ , both columns give  $1 + i m_j$ . On even  $n = 2j$ , both give  $-4((2j) + 1 - k(2j)) = -4(j + 1)$ .  $\square$

**Corollary 0.2** (Collapsed canonical stream:  $U(n)$ ). *Define*

$$U(n) := U_R(1, n) = U_L(1, n) = P_{\text{odd}}(n) (1 + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n).$$

*Then  $U(2j - 1) = 1 + i m_j$  and  $U(2j) = -4(j + 1)$  for all  $j \in \mathbb{Z}$ .*

*Proof.* Immediate from  $P_{\text{odd}}(2j - 1) = 1$ ,  $P_{\text{even}}(2j - 1) = 0$  and  $P_{\text{odd}}(2j) = 0$ ,  $P_{\text{even}}(2j) = 1$ , with  $k(2j - 1) = k(2j) = j$ .  $\square$

**Corollary 0.3** (Squared-trig form). *Using  $\sin^2(\pi n/2) = P_{\text{odd}}(n)$  and  $\cos^2(\pi n/2) = P_{\text{even}}(n)$ ,*

$$U(n) = \sin^2\left(\frac{\pi n}{2}\right) (1 + i m_{k(n)}) - 4(n + 1 - k(n)) \cos^2\left(\frac{\pi n}{2}\right).$$

*Proof.* Substitute the trigonometric projector identities into Cor. 0.2.  $\square$

**Corollary 0.4** (Single-frequency collapse). *There exist  $c(n) \in \mathbb{R}$  and  $d(n) \in \mathbb{C}$  such that*

$$U(n) = (c + d) + (c - d) \cos(\pi n), \quad c = 2(k(n) - n - 1), \quad d = \frac{1 + i m_{k(n)}}{2}.$$

*Proof.* From Cor. 0.3, use  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$  and  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$  with  $\theta = \pi n/2$ , and collect constant and  $\cos(\pi n)$  parts.  $\square$

**Corollary 0.5** (Self-indexed recurrence). *With initial values  $U(0) = -4$  and  $U(1) = 1 + im_1$ , for all  $n \geq 2$ ,*

$$U(n) = P_{\text{odd}}(n) \left(1 + i m_{-U(n-1)/4}\right) - P_{\text{even}}(n) \left(U(n-2) + 4(n+1)\right).$$

*Proof.* On odd  $n = 2j - 1$ , we have  $-U(n-1)/4 = -(U(2j-2))/4 = j$ , so the odd sample is  $1 + im_j$ . On even  $n = 2j$ , the update gives  $-4(j+1)$ . This matches Cor. 0.2.  $\square$

**Corollary 0.6** (Curvature extractor &  $\zeta(2)$  disguise). *Let  $F(n) := \Im U(n)$  and let  $\Delta^2$  denote the discrete second difference in  $n$ . Then  $F(2j-1) = m_j$ ,  $F(2j) = 0$ , and*

$$F(2j-1) = \frac{2}{\pi^2} \Im(\Delta^2 U(2j)) = \frac{1}{3\zeta(2)} \Im(\Delta^2 U(2j)) = \frac{2}{3\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{(2(j-\ell)+1)^2} = m_j.$$

*Proof.* Apply  $\Delta^2$  to the trigonometric form in Cor. 0.3; at even sites only the squared factors contribute. Use  $\zeta(2) = \pi^2/6$  to rewrite  $2/\pi^2 = 1/(3\zeta(2))$ . The odd-square kernel arises from the sampled second difference of  $\cos^2(\pi n/2)$ .  $\square$

**Corollary 0.7** (Seed  $\rightarrow$  rectifier  $\rightarrow$  physical streams). *Let  $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$  and define the even-ladder duplicator*

$$E(n) := 2(n+1-k(n)) \quad \text{so that} \quad E(2j) = 2(j+1), \quad E(2j-1) = 0.$$

For  $f > 0$  and gain  $\lambda \in \mathbb{R}$ , set the seed

$$s_f(n) = f\lambda \left[ \sin\left(\frac{\pi n}{2}\right) (1 + i m_{k(n)}) - E(n) \cos\left(\frac{\pi n}{2}\right) \right].$$

On  $\mathbb{Z}$  we have  $\chi_4(n) \sin(\pi n/2) = P_{\text{odd}}(n)$  and  $\chi_4(n) \cos(\pi n/2) = P_{\text{even}}(n)$ , hence

$$\chi_4(n) s_f(n) = f\lambda \left[ P_{\text{odd}}(n) (1 + i m_{k(n)}) - E(n) P_{\text{even}}(n) \right].$$

Choosing  $\lambda = \frac{1}{2}$  and recalling  $E(n) = 2(n+1-k(n))$  yields the physical stream

$$S_f(n) := \frac{f}{2} U(n).$$

*Proof.* The mod-4 identities map sine/cosine to parity projectors on  $\mathbb{Z}$ ; the even ladder  $E(n)$  then gives the correct  $-4(j+1)$  at  $n = 2j$ . Setting  $\lambda = \frac{1}{2}$  collapses to  $S_f = (f/2) U$ .  $\square$

**Corollary 0.8** (Slope invariance under frame modulation). *If  $u = 1 + im$  then  $s = (f/2)u$  satisfies  $\frac{\Im s}{\Re s} = m$ , independent of  $f > 0$ .*

*Proof.*  $(\Im s)/(\Re s) = ((f/2)m)/(f/2) = m$ .  $\square$