

A Height–Local Width–2 Program for Excluding Off–Axis Quartets with an Analytic Tail and a Rigorous Certified Criterion

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Abstract

The manuscript is organized in three parts. **Part I** (Reader’s Guide) introduces the width–2 normalization and the centered a -parameter measuring horizontal displacement at a given height. **Part II** (Analytic Core) develops boundary-only tools on aligned boxes: (i) a hinge–unitarity law for the width–2 functional-equation factor; (ii) a de-singularized residual envelope for logarithmic derivatives; (iii) boundary criteria (including a certified Outer/Rouché route) designed for validated numerics. **Part III** records structural corollaries under on-axis collapse and presents a deterministic prime–locked *tick generator* together with a reproducible audit (supplementary).

Executive summary (what is proved vs. what is certified).

- **Unconditional analytic statements proved in Part II:** the width–2 setup and symmetries; Hinge–Unitarity (Thm. ??); residual/log-derivative envelopes (Lem. ??, ??); and the rigorous *Outer/Rouché certification criterion* (Prop. ??) together with the inner-collapse bridges (Prop. ??, ??).
- **Certified criterion (computational, but rigorous):** if the Rouché ratio (??) is validated on ∂B , then the box is zero-free (Prop. ??); the protocol is frozen in Appendix ??.
- **Analytic tail (programmatic):** Part II provides explicit symbolic inequalities that, once all shape-only constants are pinned (either from explicit operator bounds or from a certified computation on the normalized square), yield an explicit threshold $M_0(\eta)$ beyond which off-axis quartets are excluded.
- **Part III: supplementary.** All lattice corollaries follow from on-axis collapse; the tick generator is a deterministic equation audited against LMFDB data (Appendix ??) but is not used in Part II.

Contents

Part I — Reader’s Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces the width–2 normalization, defines the centered coordinate v and the horizontal displacement parameter a at a given height, and records structural (non-load-bearing) corollaries that become valid once on-axis collapse is established.

What it does not do. It contains no analytic estimates and no proofs. All analytic statements are proved in Part II.

1) Modulated frames and the width-2 pivot

For $f > 0$ define $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)} \Gamma\left(\frac{s}{2f}\right) \zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$.

Width-2 normalization. Put $u := (2/f)s$. Then

$$\zeta_2(u) := \zeta(u/2), \quad \Lambda_2(u) := \pi^{-u/4} \Gamma(u/4) \zeta(u/2), \quad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non-completed FE reads $\zeta_2(u) = A_2(u) \zeta_2(2 - u)$ where $A_2(u)A_2(2 - u) \equiv 1$.

Centered coordinate. Put $v := u - 1$ and define

$$E(v) := \Lambda_2(1 + v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$.

2) Heights and the width parameter a (RH-free definition)

Let $s = \beta + it$ be any nontrivial zero of $\zeta(s)$ with $0 < \beta < 1$, $t > 0$. In the width-2 frame $u = 2s$ and centered variable $v = u - 1$,

$$v = (2\beta - 1) + i(2t).$$

Define the *height* and *horizontal displacement* of this zero by

$$m := \operatorname{Im} v = 2t, \quad a := \operatorname{Re} v = 2\beta - 1.$$

Thus RH is equivalent to: every nontrivial zero has $a = 0$.

Quartet symmetry. If $E(a + im) = 0$ with $a \neq 0$, then by $E(v) = E(-v) = \overline{E(\bar{v})}$ the quartet $\{\pm a \pm im\}$ are zeros in the v -plane.

3) Height-local reduction

Define the height-local statement:

$$\text{PHU}(m) : \text{ every zero of } E \text{ with imaginary part } m \text{ has real part } 0.$$

Then RH is equivalent to $\text{PHU}(m)$ holding for every height m attained by a nontrivial zero.

4) Boxes and hand-off

For $x_0 \in (0, 1]$ and $m \geq 10$, define the aligned box

$$B(x_0, m, \delta) := [x_0 - \delta, x_0 + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta x_0}{(\log m)^2}, \quad \eta \in (0, 1).$$

Part II develops boundary criteria on ∂B designed to exclude off-axis quartets by contradiction and/or certification.

5) Toolbox → structural consequences (after collapse)

The parity-lattice constructions and structural corollaries in Part III become valid once collapse is established. They are not used as inputs in Part II.

Part II — Analytic Core and Certified Criterion on Aligned Boxes

Standing objects. In the width-2 centered frame:

$$\Lambda_2(u) = \pi^{-u/4} \Gamma(u/4) \zeta(u/2), \quad E(v) = \Lambda_2(1+v).$$

We work on boxes $B(x_0, m, \delta)$ with $m \geq 10$ and $\delta = \eta x_0 / (\log m)^2$.

Symbols & provenance (at a glance)

Symbol	Definition / role	Provenance / rationale
$u = 2s, v = u - 1$	width-2, centered frame	centers FE symmetry
$\Lambda_2(u), E(v)$	completed width-2 object; shifted workhorse	standard FE + conjugation
$\chi_2(u)$	inverse FE factor	$\chi_2(u) = \pi^{u/2-1/2} \frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(x_0, m, \delta)$	$[x_0 - \delta, x_0 + \delta] \times [m - \delta, m + \delta]$	aligned box around (x_0, m)
$\delta = \eta x_0 / (\log m)^2$	half-side length	balances forcing vs residual terms
I_L, I_R, H_{\pm}	left/right vertical sides; horizontals	boundary decomposition
Z_{loc}	product of local zeros near height band	de-singularizes logarithmic derivatives
$F = E/Z_{\text{loc}}$	residual analytic factor on ∂B	controlled by Lem. ??
$G(v) = E(1+v)/E(1-v)$	FE-symmetric quotient	used in Schur route
$G_{\text{out}} = e^{U+iV}$	outer function with $ G_{\text{out}} = E $ on ∂B	Dirichlet + harmonic conjugate
$W = E/G_{\text{out}}$	inner quotient ($ W = 1$ a.e. on ∂B)	collapses to constant under Rouché

Primary sources. Digamma bounds and reflection: DLMF §5.5, §5.11. Logarithmic derivative ζ'/ζ expansions: Titchmarsh [?, §14], Ivić [?, Ch. 9]. Lipschitz Cauchy/Hilbert boundedness: Coifman–McIntosh–Meyer [?]. Corner regularity and Dirichlet continuity for rectangles: Kellogg [?], Axler–Bourdon–Ramey [?].

1 Frames, symmetry, and the hinge law

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u) \zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2} \frac{\Gamma(\frac{2-u}{4})}{\Gamma(\frac{u}{4})}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log |\chi_2(\sigma+it)|$. Then

$$f'(\sigma) = \frac{1}{2} \log \pi - \frac{1}{2} \operatorname{Re} \psi\left(\frac{\sigma+it}{4}\right) - \frac{1}{4} \operatorname{Re} \left[\pi \cot\left(\frac{\pi}{4}(\sigma+it)\right) \right]. \quad (1.1)$$

Moreover,

$$|\operatorname{Re}[\pi \cot(x+iy)]| \leq \frac{\pi}{\cosh(2y) - 1}. \quad (1.2)$$

For $|t| \geq t_1$ (Appendix ??), $f'(\sigma) < 0$ for all real σ . Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\operatorname{Re} u = 1$ for all $|\operatorname{Im} u| \geq t_1$.

Proof. Differentiate $\log |\chi_2|$ using $\partial_\sigma \log |\Gamma(z)| = \operatorname{Re} \psi(z)$ and the reflection identity $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ (DLMF §5.5). This yields (??). For (??), write $\cot(x+iy) = \frac{\sin(2x) - i \sinh(2y)}{\cosh(2y) - \cos(2x)}$ so the real part is $\operatorname{Re} \cot(x+iy) = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)}$ and $|\sin(2x)| \leq 1$, $\cos(2x) \leq 1$ give (??). Finally, vertical-strip bounds for ψ (DLMF §5.11) imply

$$\operatorname{Re} \psi\left(\frac{\sigma+it}{4}\right) \geq \log\left(\frac{|t|}{4}\right) - \frac{2}{|t|} \quad (|t| \geq 10, \sigma \in \mathbb{R}),$$

so the dominating term in (??) is $-\frac{1}{2} \log(|t|/4)$ while the cotangent term is exponentially small in $|t|$. For $|t| \geq t_1$ this yields $f'(\sigma) < 0$ for all σ . \square

2 Boxes, de-singularization, residual control, and forcing

Fix $m \geq 10$, $x_0 \in (0, 1]$, and

$$B(x_0, m, \delta) = [x_0 - \delta, x_0 + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta x_0}{(\log m)^2}, \quad \eta \in (0, 1). \quad (2.1)$$

Write the boundary as:

$$\begin{aligned} I_L &= \{x_0 - \delta + iy : |y - m| \leq \delta\}, & I_R &= \{x_0 + \delta + iy : |y - m| \leq \delta\}, \\ H_- &= \{x + i(m - \delta) : |x - x_0| \leq \delta\}, & H_+ &= \{x + i(m + \delta) : |x - x_0| \leq \delta\}. \end{aligned}$$

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$). *For $m \geq 10$ and $\eta \in (0, 1)$, one has $\delta < x_0$ and $B(x_0, m, \delta) \subset \{\operatorname{Re} v > 0\}$.*

Proof. Since $\eta/(\log m)^2 < 1$, $\delta = x_0 \eta/(\log m)^2 < x_0$, hence $x_0 - \delta > 0$. \square

De-singularization. Let

$$Z_{\text{loc}}(v) := \prod_{\rho: |\operatorname{Im} \rho - m| \leq 1} (v - \rho)^{m_\rho}, \quad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (2.2)$$

By construction, F is analytic and zero-free on a neighborhood of ∂B (all zeros/poles of E in the band $|\operatorname{Im} \rho - m| \leq 1$ are removed).

Lemma 2.2 (Residual envelope). *There exist absolute constants $C_1, C_2 > 0$ such that on ∂B ,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2, \quad (2.3)$$

and hence

$$|\Delta_{\partial B} \arg F| \leq 8\delta (C_1 \log m + C_2). \quad (2.4)$$

Proof. On vertical strips, $\psi(z) = \log z + O(1)$ uniformly (DLMF §5.11), so the Gamma-factor contribution to E'/E is $O(\log m)$ on ∂B . For the zeta factor, one uses the classical local expansion (Titchmarsh [?, §14], Ivić [?, Ch. 9])

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t), \quad (1/2 \leq \sigma \leq 1, t \geq 3),$$

with the implied constant absolute. Transporting to width-2 and dividing by Z_{loc} cancels the local pole sum, leaving an $O(\log m)$ bound on ∂B . Collecting the Gamma and zeta contributions yields (??). Then $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F ds$ with $|\partial_\tau \arg F| \leq |F'/F|$ and $|\partial B| = 8\delta$ gives (??). \square

Lemma 2.3 (Bridge logs on ∂B). *On ∂B ,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\text{loc}}}{Z_{\text{loc}}}.$$

In particular,

$$\sup_{\partial B} \left| \frac{E'}{E} \right| \leq \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho: |\text{Im } \rho - m| \leq 1} \sup_{v \in \partial B} \frac{m_\rho}{|v - \rho|}.$$

Proof. Differentiate $E = FZ_{\text{loc}}$. The supremum bound is immediate from the triangle inequality. \square

Lemma 2.4 (Short-side forcing from a conjugate-symmetric pair). *Let $a \in (0, 1]$ and define the pair factor*

$$Z_{\text{pair}}(v) = (v - (a + im))(v - (-a + im)).$$

On the right side I_R of the aligned box $B(a, m, \delta)$, one has

$$\Delta_{I_R} \arg Z_{\text{pair}} = 2 \arctan\left(\frac{\delta}{\delta}\right) + 2 \arctan\left(\frac{\delta}{2a + \delta}\right) \geq \frac{\pi}{2}. \quad (2.5)$$

Proof. Parametrize $v(y) = a + \delta + iy$, $y \in [m - \delta, m + \delta]$. For $v(y) - (a + im) = \delta + i(y - m)$,

$$\Delta_{I_R} \arg(v - (a + im)) = \arctan\left(\frac{\delta}{\delta}\right) - \arctan\left(\frac{-\delta}{\delta}\right) = \frac{\pi}{2}.$$

For $v(y) - (-a + im) = (2a + \delta) + i(y - m)$,

$$\Delta_{I_R} \arg(v - (-a + im)) = 2 \arctan\left(\frac{\delta}{2a + \delta}\right) \geq 0.$$

Summing yields (??). \square

3 Boundary-only criteria: Schur route and certified Rouché route

3.1 Two-point Schur/outer criterion (boundary-only)

Define the quotient

$$G(v) := \frac{E(1+v)}{E(1-v)}. \quad (3.1)$$

This quotient is designed to compare symmetric points about the hinge. It is analytic on boxes avoiding the singular set of the denominator.

Proposition 3.1 (Two-point Schur pinning). *Let $\varphi : \mathbb{D} \rightarrow B$ be conformal with $\varphi(0)$ the center of B and nontangential boundary correspondence away from corners. Let $M \in C(\partial B)$ satisfy $M \geq |G|$ a.e. on ∂B , and let $H = e^{U+iV}$ be the outer majorant obtained by solving Dirichlet on B with boundary data $\log M$ and choosing a harmonic conjugate V . Then $\Phi := (G/H) \circ \varphi \in H^\infty(\mathbb{D})$ and $\|\Phi\|_\infty \leq 1$. If two non-corner boundary points $\zeta_\pm \in \partial \mathbb{D}$ satisfy $|\Phi(\zeta_\pm)| = 1$ and some boundary arc $A \subset \partial \mathbb{D}$ satisfies $\text{ess sup}_A |\Phi| \leq 1 - \varepsilon$, then for any $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \geq \omega_* > 0$,*

$$|\Phi(z)| \leq 1 - \kappa, \quad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Proof. This is a standard harmonic-measure/maximum-modulus argument in H^∞ : the subharmonic function $\log |\Phi| \leq 0$ has boundary values ≤ 0 a.e., equals 0 at two points, and is $\leq \log(1 - \varepsilon)$ on an arc. Poisson averaging at z yields a strict deficit κ once $\omega_z(A)$ is bounded below. See Duren [?, §II.5] and Garnett [?, §II.2]. \square

Lemma 3.2 (Two-point link between G and χ_2). *For $v = a + im$,*

$$|G(a + im)| |G(-a + im)| = |\chi_2(1 + a + im)| |\chi_2(1 - a + im)|. \quad (3.2)$$

Proof. By definition $E(v) = \Lambda_2(1 + v)$, so

$$G(v) = \frac{\Lambda_2(2 + v)}{\Lambda_2(2 - v)}.$$

Using $\Lambda_2(u) = \Lambda_2(2 - u)$ and the non-completed FE for ζ_2 in width-2 yields the identity $|G(v)| |G(-v)| = |\chi_2(1 + v)| |\chi_2(1 - v)|$. Evaluating at $v = a + im$ gives (??). \square

3.2 Outer/Rouché certification route (rigorous and checkable)

Let U solve the Dirichlet problem on B with boundary data $\log|E|$ (continuous if E has no zeros on ∂B), and let V be a harmonic conjugate. Define

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free in B , and $|G_{\text{out}}| = |E|$ on ∂B in the sense of boundary traces.

Proposition 3.3 (Outer/Rouché criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} < 1, \quad (3.3)$$

then E is zero-free in B . Consequently, the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing in B with $|W| = 1$ a.e. on ∂B .

Proof. By (??), $|E - G_{\text{out}}| < |G_{\text{out}}|$ on ∂B . Since G_{out} is zero-free in B , Rouché's theorem implies that E and G_{out} have the same number of zeros in B , hence none. The stated properties of W follow. \square

Proposition 3.4 (Bridge 1: inner collapse). *Under (??), $W \equiv e^{i\theta_B}$ on B for some constant phase $\theta_B \in \mathbb{R}$.*

Proof. Since W is analytic and nonvanishing, $\log|W|$ is harmonic. The boundary trace satisfies $\log|W| = 0$ a.e. because $|W| = |E|/|G_{\text{out}}| = 1$ on ∂B . By uniqueness for the Dirichlet problem, $\log|W| \equiv 0$ on B , hence $|W| \equiv 1$. The open mapping theorem forces an analytic map of a domain into the unit circle to be constant, so $W \equiv e^{i\theta_B}$. \square

Proposition 3.5 (Bridge 2: stitching). *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j for $j = 1, 2$, then $\theta_{B_1} \equiv \theta_{B_2} \pmod{2\pi}$.*

Proof. On $B_1 \cap B_2$, both analytic constants agree pointwise, hence globally. \square

3.3 Corner interpolation (regularity reminder)

Rectangles are Wiener-regular, so continuous boundary data admit harmonic extensions continuous up to \overline{B} . This justifies using continuous boundary moduli in the outer construction and evaluating corner limits; see Kellogg [?] and Axler–Bourdon–Ramey [?]. A detailed note is recorded in Appendix ??.

4 Analytic tail: symbolic inequalities and pinned constants (programmatic)

Purpose of this section. This section records the symbolic inequalities that connect: (i) short-side forcing (Lemma ??), (ii) residual envelopes (Lemma ??), and (iii) an upper–lower comparison on a covering family of boxes. Once all shape-only constants are pinned (either from explicit operator-norm bounds or from a certified computation on the normalized square), these inequalities yield an explicit threshold $M_0(\eta)$ for the tail. The pinned-constants closure is recorded in Appendix ??.

Remark 4.1 (Why constants are “shape-only”). Under the affine normalization $T(v) = (v - (x_0 + im))/\delta$, the boundary $\partial B(x_0, m, \delta)$ maps to the fixed square boundary ∂Q , $Q = [-1, 1]^2$. Any constant arising solely from boundary singular-integral operator norms (Cauchy/Hilbert), Poisson/trace operators, or geometric decompositions of the square boundary depends only on ∂Q , hence is independent of m, x_0, δ .

Theorem 4.2 (Tail closure criterion: symbolic form). *Fix $\eta \in (0, 1)$ and set $\delta = \eta x_0 / (\log m)^2$. Assume:*

- *residual envelope constants C_1, C_2 as in Lemma ??;*
- *pinned shape-only constants C_{up}, C_h'' controlling the relevant boundary-to-interior and allocation estimates on the normalized square (Appendix ?? and Appendix ??).*

Then there exists an explicit computable function $M_0(\eta)$ (depending only on these constants) such that: for all $m \geq M_0(\eta)$ and all $x_0 \in (0, 1]$, any off-axis quartet at height m would violate at least one of the pinned inequalities, hence is excluded.

Remark 4.3 (How $M_0(\eta)$ is used in practice). In an implementation, one combines:

- (a) a finite verified band $0 < \text{Im } s \leq T_{\text{ver}}$ from Platt/Platt–Trudgian (Appendix ??);
- (b) a tail threshold $M_0(\eta)$ for $\text{Im } u = \text{Im}(2s) = 2 \text{Im } s$.

If $M_0(\eta) \leq 2T_{\text{ver}}$, then the finite band plus tail imply global RH. Appendix ?? records the pinned-constants logic needed to drive $M_0(\eta)$ down (potentially to m_1).

Part III — Structural Corollaries (supplementary)

Standing basis. Part III assumes the on-axis collapse conclusion (i.e. every nontrivial zero has $a = 0$) and records structural consequences. Nothing in Part III is used in Part II.

Corollary 4.4 (Canonical columns). *Define $P_{\text{odd}}(n) = (1 - \cos \pi n)/2$ and $P_{\text{even}}(n) = (1 + \cos \pi n)/2$. Let $k(2j - 1) = j$, $k(2j) = j + 1$. For any $x \in (0, 2)$,*

$$U_R(x, n) = P_{\text{odd}}(n) (x + i m_{k(n)}) - 4(n+1-k(n)) P_{\text{even}}(n), \quad U_L(x, n) = P_{\text{odd}}(n) (2 - x + i m_{k(n)}) - 4(n+1-k(n)) P_{\text{even}}(n),$$

Under collapse, the canonical choice $x = 1$ gives $U_R(1, n) = U_L(1, n)$ for all n .

Corollary 4.5 (Collapsed canonical stream: mod-4 face).

$$U(n) := P_{\text{odd}}(n) (1 + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n),$$

so $U(2j - 1) = 1 + im_j$ and $U(2j) = -4(j + 1)$.

Corollary 4.6 (Collapsed canonical stream: mod-2 face). *Using $\sin^2(\pi n/2) = P_{\text{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\text{even}}(n)$,*

$$U(n) = \sin^2\left(\frac{\pi n}{2}\right)(1 + i m_{k(n)}) - 4(n+1-k(n)) \cos^2\left(\frac{\pi n}{2}\right).$$

Corollary 4.7 (Single-frequency collapse). *There exist functions $c(n), d(n)$ with*

$$U(n) = (c+d) + (c-d) \cos(\pi n), \quad c = 2(k(n) - n - 1), \quad d = \frac{1 + i m_{k(n)}}{2}.$$

Corollary 4.8 (Self-indexed recurrence). *With $U(0) = -4$ and $U(1) = 1 + im_1$, for all $n \geq 2$,*

$$U(n) = P_{\text{odd}}(n) \left(1 + i m_{-U(n-1)/4}\right) - P_{\text{even}}(n) \left(U(n-2) + 4(n+1)\right).$$

Corollary 4.9 (Seed \rightarrow rectifier \rightarrow physical streams). *Let $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$. For $f > 0$ and gain $\lambda \in \mathbb{R}$,*

$$s_{f,k}(n) = f\lambda \left[\sin\left(\frac{\pi n}{2}\right)(1 + i m_k) - 4n \cos\left(\frac{\pi n}{2}\right) \right],$$

then $\chi_4(n) s_{f,k}(n) = f\lambda [P_{\text{odd}}(n)(1 + im_k) - 4n P_{\text{even}}(n)]$. With $\lambda = \frac{1}{2}$ and $k = k(n)$ we get $S_f(n) = \frac{f}{2} U(n)$.

Corollary 4.10 (Curvature extractor & $\zeta(2)$ disguise). *Let $F(n) := \text{Im } U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2} \text{Im}(U''(2j)) = \frac{1}{3\zeta(2)} \text{Im}(U''(2j)) = \frac{2}{3\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{(2(j-\ell)+1)^2}.$$

For $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, $\text{Im } \Delta^2 U(2j) = m_{j+1} + m_j$.

RH-dependency ledger (Part III).

- *RH-free*: pure parity-lattice algebra and identities in Cor. ??–??.
- *Uses collapse*: interpreting the odd-lane imaginary parts as the complete set of nontrivial ordinates.

Part III (continued) — Prime-Locked Tick Generator (audited; supplementary)

Notation (true zeros vs. generated ticks). Let $\gamma_1 < \gamma_2 < \dots$ denote ordinates of zeros on $\text{Re } s = \frac{1}{2}$ and set $m_j := 2\gamma_j$. Define a deterministic *tick sequence* $\tilde{t}_1, \tilde{t}_2, \dots$ by the generator equation below and set $\tilde{m}_j := 2\tilde{t}_j$. The numerical audit compares \tilde{m}_j against the true m_j . Part II does not use this section.

Let $\theta(t)$ be the Riemann-Siegel theta function.

Fix

$$A := \frac{3}{2}, \quad X(t) := C(\log t)^A \quad (C \geq 1), \tag{4.1}$$

and the smooth cutoff weight W of Appendix ??.

Define for $t > 0$ and $\Delta > 0$:

$$\mathcal{P}_{X(t)}(t, \Delta) := - \sum_{p^k \geq 1} \frac{1}{k p^{k/2}} W\left(\frac{p^k}{X(t)}\right) \left[\sin((t+\Delta)k \log p) - \sin(tk \log p) \right].$$

Theorem 4.11 (Deterministic prime-locked tick generator). Fix $C \geq 1$, $A = \frac{3}{2}$, and W as above. Seed $\tilde{t}_1 := \gamma_1$ (Appendix ??). Given \tilde{t}_j , define \tilde{t}_{j+1} as the unique solution of

$$\theta(\tilde{t}_{j+1}) - \theta(\tilde{t}_j) + \mathcal{P}_{X(\tilde{t}_j)}(\tilde{t}_j, \tilde{t}_{j+1} - \tilde{t}_j) = \pi. \quad (4.2)$$

For all sufficiently large j , (??) has a unique solution $\tilde{t}_{j+1} > \tilde{t}_j$, and a deterministic bracketed bisection converges.

Proof. Define $F_j(\Delta) := \theta(\tilde{t}_j + \Delta) - \theta(\tilde{t}_j) + \mathcal{P}_{X(\tilde{t}_j)}(\tilde{t}_j, \Delta) - \pi$. Then $F_j(0) = -\pi < 0$ and $\theta(\tilde{t}_j + \Delta) - \theta(\tilde{t}_j) \rightarrow \infty$ as $\Delta \rightarrow \infty$, while \mathcal{P} is bounded for fixed cutoff $X(\tilde{t}_j)$. Hence a root exists. Moreover

$$F'_j(\Delta) = \theta'(\tilde{t}_j + \Delta) - \sum_{p^k \leq X(\tilde{t}_j)} \frac{\log p}{p^{k/2}} W\left(\frac{p^k}{X(\tilde{t}_j)}\right) \cos((\tilde{t}_j + \Delta)k \log p).$$

As $t \rightarrow \infty$, $\theta'(t) = \frac{1}{2} \log(t/2\pi) + O(1/t)$. The prime sum is $O(\sum_{p^k \leq X} \log p / p^{k/2}) = O(\sqrt{X})$. With $X(\tilde{t}_j) = C(\log \tilde{t}_j)^{3/2}$, we have $\sqrt{X} = O((\log \tilde{t}_j)^{3/4}) = o(\log \tilde{t}_j)$, hence $F'_j(\Delta) > 0$ for all large j , so F_j is strictly increasing and the root is unique. Bisection converges by monotonicity. \square

Numerical audit to $j = 50$: error-vs-cutoff (fixed $A = \frac{3}{2}$)

We compare $\tilde{m}_j = 2\tilde{t}_j$ to true $m_j = 2\gamma_j$ from LMFDB (Ref. [?]; Appendix ??). Errors exclude the seeded $j = 1$.

C	max $ \tilde{m} - m $	mean $ \tilde{m} - m $	max rel. err	mean rel. err
16	0.106406	0.028070	0.000476	0.000165
32	0.087644	0.022884	0.000395	0.000133
48	0.057151	0.017504	0.000323	0.000109

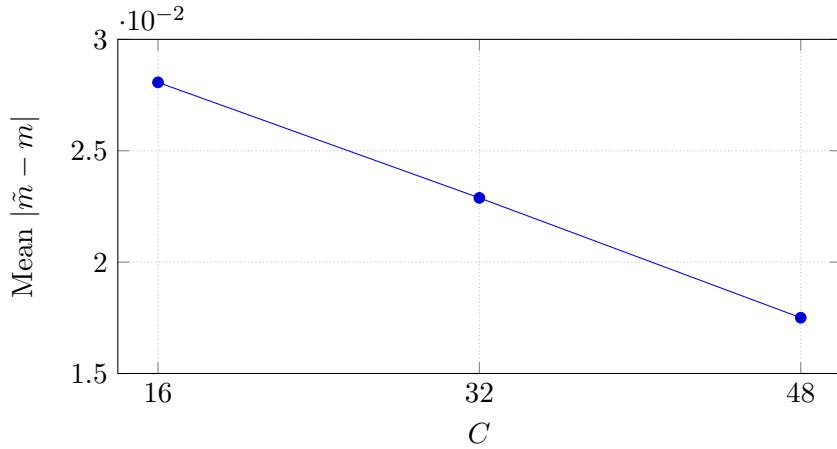


Figure 1: Mean absolute tick error decreases as C grows (fixed $A = 3/2$; $j = 2, \dots, 50$).

A Corner interpolation (detail)

Rectangles are Wiener-regular domains; if boundary data are continuous on ∂B , the Dirichlet solution is continuous on \overline{B} . This allows one to treat $\log |E|$ as boundary data for the outer construction away from boundary zeros and to interpret corner limits in the usual way. See Kellogg [?] and Axler–Bourdon–Ramey [?, Ch. 8].

B Outer/Rouché certification protocol (rigorous outline)

- **Boundary sampling.** Partition each edge of ∂B into intervals; on each, compute interval enclosures for $E(v)$ and (optionally) $E'(v)$.
- **Validated Dirichlet.** Solve Dirichlet for U with boundary data $\log|E|$ using a validated Poisson solver (map to \mathbb{D} if desired).
- **Phase reconstruction.** Obtain V as a harmonic conjugate (validated Hilbert transform on the circle after conformal mapping, or a direct validated Cauchy integral on ∂B).
- **Outer function.** Form $G_{\text{out}} = e^{U+iV}$ with interval arithmetic.
- **Rouché check.** Verify $\sup_{\partial B} |E - G_{\text{out}}| / |G_{\text{out}}| < 1$ with grid-to-continuum Lipschitz enclosure (using E'/E bounds).

C Certified first nontrivial zero and verified band

We cite rigorously verified computations:

Theorem C.1 (Platt 2017; Platt–Trudgian 2021). *There are no nontrivial zeros of $\zeta(s)$ with $0 < \text{Im } s < t_1$, and the first nontrivial zero occurs at $t_1 = 14.134725141734693790457251983562\dots$ with rigorous interval bounds. Moreover, RH has been verified for all zeros with $0 < \text{Im } s \leq 3 \cdot 10^{12}$.*

Set $m_1 := 2t_1$.

Appendix S.1. Shape-only operator norms (dependency statement)

On a Lipschitz Jordan curve (in particular, the square boundary), the Cauchy singular integral and boundary Hilbert transform are bounded on L^2 with operator norms depending only on the Lipschitz character (Coifman–McIntosh–Meyer [?]). Under affine normalization of ∂B to the fixed square ∂Q , these norms become *shape-only*. A fully explicit numerical enclosure of these norms is possible via certified quadrature on ∂Q .

Appendix S.3. Pinned constants and tail threshold logic (program)

Theorem ?? reduces the tail threshold $M_0(\eta)$ to a finite family of constants: residual constants C_1, C_2 and shape-only constants (trace/Hilbert/geometric allocation). To turn the tail into a complete proof, one must:

- (1) instantiate C_1, C_2 from explicit literature bounds for ζ'/ζ on vertical strips;
- (2) certify the shape-only operator norms on the normalized square boundary (Appendix ??) or bound them explicitly;
- (3) compute $M_0(\eta)$ and verify $M_0(\eta) \leq 2 \cdot 3 \cdot 10^{12}$ so that the Platt–Trudgian band closes the remainder.

The v24-style numerical pinning example can be carried out here, but for submission it should be presented as a *certified* enclosure rather than floating-point approximations.

Appendix PW. Smooth cutoff weight

Define $W : [0, 1] \rightarrow [0, 1]$ by

$$W(y) := \begin{cases} \exp\left(1 - \frac{1}{1-y}\right), & 0 \leq y < 1, \\ 0, & y = 1. \end{cases}$$

Interpret $W(y) = 0$ for $y > 1$.

Appendix NA. Deterministic audit protocol and full reference script

Purpose. This appendix freezes the data source, theta function, weight, cutoff rule, solver, and error metrics used in the Part III audit.

Truth ordinates. Download $\gamma_1, \dots, \gamma_{50}$ from the LMFDB plain-text endpoint:

```
https://www.lmfdb.org/zeros/zeta/list?download=yes&limit=100.
```

Parse index/value pairs $j \ \gamma_j$.

Audit protocol.

- Seed: $\tilde{t}_1 := \gamma_1$.
- Theta: $\theta(t) = \operatorname{Im} \log \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$ (principal branch).
- Weight: W from Appendix ??.
- Cutoff: $X(t) = C(\log t)^{3/2}$.
- Root solve: bisection on $F_j(\Delta) = 0$ where F_j is defined in Thm. ??.
- Statistics: compare $\tilde{m}_j = 2\tilde{t}_j$ to $m_j = 2\gamma_j$, exclude $j = 1$ in summary stats.

Reference script (Python 3).

```
#!/usr/bin/env python3
"""
Deterministic audit script for the Part III prime-locked tick generator.

```

Reproduces the v26 Part III table ($A = 3/2$; C in $\{16, 32, 48\}$; $J = 50$) with deterministic bisection and a fixed smooth cutoff weight W .

Truth ordinates are fetched from LMFDB's plain-text endpoint; if the fetch fails, the script falls back to an embedded list for $j=1..50$.

Notation:

- gamma_j : true ordinates from LMFDB
- ttilde_j : generated tick ordinates
- $\text{m}_j = 2*\text{gamma}_j$, $\text{mtilde}_j = 2*\text{ttilde}_j$

No circularity:

```

- X(t) uses the predicted tick ttilde_j at each step.
- Truth list is used only for reporting errors.
"""

import argparse
import math
import urllib.request

import mpmath as mp
mp.mp.dps = 60 # fixed precision for theta

LMFDB_URL = "https://www.lmfdb.org/zeros/zeta/list?download=yes&limit=100"

FALLBACK_GAMMA_50 = [
"14.1347251417346937904572519835625",
"21.0220396387715549926284795938969",
"25.0108575801456887632137909925628",
"30.4248761258595132103118975305840",
"32.9350615877391896906623689640747",
"37.5861781588256712572177634807053",
"40.9187190121474951873981269146334",
"43.3270732809149995194961221654068",
"48.0051508811671597279424727494277",
"49.7738324776723021819167846785638",
"52.9703214777144606441472966088808",
"56.4462476970633948043677594767060",
"59.3470440026023530796536486749922",
"60.8317785246098098442599018245241",
"65.1125440480816066608750542531836",
"67.0798105294941737144788288965221",
"69.5464017111739792529268575265547",
"72.0671576744819075825221079698261",
"75.7046906990839331683269167620305",
"77.1448400688748053726826648563047",
"79.3373750202493679227635928771161",
"82.9103808540860301831648374947706",
"84.7354929805170501057353112068276",
"87.4252746131252294065316678509191",
"88.8091112076344654236823480795095",
"92.4918992705584842962597252418105",
"94.6513440405198869665979258152080",
"95.8706342282453097587410292192467",
"98.8311942181936922333244201386224",
"101.3178510057313912287854479402924",
"103.7255380404783394163984081086952",
"105.4466230523260944936708324141119",
"107.1686111842764075151233519630860",
"111.0295355431696745246564503099445",
"111.8746591769926370856120787167707",
"114.3202209154527127658909372761910",
"116.2266803208575543821608043120647",

```

```

"118.7907828659762173229791397026999",
"121.3701250024206459189455329704998",
"122.9468292935525882008174603307700",
"124.2568185543457671847320079661301",
"127.5166838795964951242793237669060",
"129.5787041999560509857680339061800",
"131.0876885309326567235663724615015",
"133.4977372029975864501304920426407",
"134.7565097533738713313260641571699",
"138.1160420545334432001915551902824",
"139.7362089521213889504500465233824",
"141.1237074040211237619403538184753",
"143.1118458076206327394051238689139",
]

def fetch_lmfdb_gammas(limit: int = 50, url: str = LMFDB_URL, timeout: int = 20):
    """
    Returns a list of decimal strings gamma_1..gamma_limit.
    Endpoint returns plain text: "1 gamma1 2 gamma2 ..."
    """
    try:
        with urllib.request.urlopen(url, timeout=timeout) as f:
            txt = f.read().decode("utf-8", errors="replace").strip()
            parts = txt.split()
            gammas = parts[1::2]
            if len(gammas) < limit:
                raise ValueError("LMFDB response too short")
            return gammas[:limit], "LMFDB"
    except Exception:
        return FALLBACK_GAMMA_50[:limit], "FALLBACK"

def theta_float(t: float) -> float:
    # mpmath's siegeltheta implements the standard Riemann-Siegel theta.
    return float(mp.siegeltheta(t))

def W_cutoff(y: float) -> float:
    # Smooth one-sided cutoff on (0,1): W(0+)=1, W(y)->0 rapidly as y->1-.
    if y <= 0.0 or y >= 1.0:
        return 0.0
    return math.exp(1.0 - 1.0/(1.0 - y))

def primes_up_to(n: int):
    if n < 2:
        return []
    sieve = bytearray(b"\x01") * (n + 1)
    sieve[:2] = b"\x00\x00"
    r = int(n ** 0.5)
    for p in range(2, r + 1):
        if sieve[p]:
            start = p * p
            step = p

```

```

        sieve[start:n+1:step] = b"\x00" * (((n - start) // step) + 1)
    return [i for i in range(2, n + 1) if sieve[i]]}

def prime_power_terms(X: float, t: float):
    """
    Precompute omega=log(p^k), coeff=W(p^k/X)/(k*sqrt(p^k)),
    and sin(t*omega), cos(t*omega) for fast evaluation of P(t,Delta).
    """
    N = int(X)
    ps = primes_up_to(N)
    omegas, coeffs, sin0, cos0 = [], [], [], []
    for p in ps:
        n = p
        k = 1
        while n <= N:
            y = n / X
            w = W_cutoff(y)
            if w != 0.0:
                coeff = w / (k * math.sqrt(n))
                omega = math.log(n)
                omegas.append(omega)
                coeffs.append(coeff)
                ang = t * omega
                sin0.append(math.sin(ang))
                cos0.append(math.cos(ang))
            k += 1
            n *= p
    return omegas, coeffs, sin0, cos0

def P_prime_increment(terms, Delta: float) -> float:
    """
    Compute P_{X}(t,Delta) using the precomputed terms at t.
    """
    omegas, coeffs, sin0, cos0 = terms
    s = 0.0
    for omega, coeff, s0, c0 in zip(omegas, coeffs, sin0, cos0):
        # sin((t+Delta)omega)-sin(t omega) = s0*(cos(Delta omega)-1) + c0*sin(Delta omega)
        d = s0 * (math.cos(Delta * omega) - 1.0) + c0 * math.sin(Delta * omega)
        s += coeff * d
    return -s

def next_tick(tj: float, C: float, A: float, tol: float = 1e-12, max_iter: int = 100):
    """
    Solve for the next tick t_{j+1} from t_j by bisection on F_j(Delta)=0.
    """
    X = C * (math.log(tj) ** A)
    terms = prime_power_terms(X, tj)
    theta_tj = theta_float(tj)

    def F(Delta: float) -> float:
        return (theta_float(tj + Delta) - theta_tj) + P_prime_increment(terms, Delta) - math

```

```

# Initial heuristic gap near 2pi/log(t/2pi)
denom = math.log(max(tj / (2.0 * math.pi), 1.0000001))
gap0 = 2.0 * math.pi / denom if denom > 0 else 10.0

lo = 0.0
hi = max(1.0, 2.0 * gap0)
f_hi = F(hi)
it = 0
while f_hi <= 0.0 and it < 60:
    hi *= 2.0
    f_hi = F(hi)
    it += 1
if f_hi <= 0.0:
    raise RuntimeError(f"Failed to bracket root at t={tj} (hi={hi}, F(hi)={f_hi})")

for _ in range(max_iter):
    mid = 0.5 * (lo + hi)
    if F(mid) <= 0.0:
        lo = mid
    else:
        hi = mid
    if hi - lo < tol:
        break

return tj + 0.5 * (lo + hi)

def generate_ticks(t1: float, J: int, C: float, A: float):
    ts = [t1]
    t = t1
    for _ in range(1, J):
        t = next_tick(t, C=C, A=A)
        ts.append(t)
    return ts

def error_stats(ts_tick, gammas_true, exclude_seed: bool = True):
    start = 1 if exclude_seed else 0
    m_tick = [2.0 * t for t in ts_tick[start:]]
    m_true = [2.0 * float(g) for g in gammas_true[start:len(ts_tick)]]

    abs_err = [abs(a - b) for a, b in zip(m_tick, m_true)]
    rel_err = [ae / abs(mt) for ae, mt in zip(abs_err, m_true)]

    return {
        "max_abs": max(abs_err),
        "mean_abs": sum(abs_err) / len(abs_err),
        "max_rel": max(rel_err),
        "mean_rel": sum(rel_err) / len(rel_err),
    }

def main():

```

```

ap = argparse.ArgumentParser()
ap.add_argument("--J", type=int, default=50)
ap.add_argument("--A", type=float, default=1.5)
ap.add_argument("--Cs", type=str, default="16,32,48")
ap.add_argument("--no_fetch", action="store_true")
ap.add_argument("--include_seed_in_stats", action="store_true")
args = ap.parse_args()

Cs = [float(x.strip()) for x in args.Cs.split(",") if x.strip()]

if args.no_fetch:
    gammas, source = (FALLBACK_GAMMA_50[:args.J], "FALLBACK(forced)")
else:
    gammas, source = fetch_lmfdb_gammas(limit=max(args.J, 50))

# Seed uses gamma_1 to match the frozen protocol
t1 = float(gammas[0])

exclude_seed = not args.include_seed_in_stats

print(f"[audit] source={source} J={args.J} A={args.A} Cs={Cs}")
print("[audit] computing...")

# LaTeX-ready rows
for C in Cs:
    ts_tick = generate_ticks(t1, args.J, C=C, A=args.A)
    st = error_stats(ts_tick, gammas, exclude_seed=exclude_seed)
    print(
        f"{int(C)} &
{st['max_abs']:.6f} & {st['mean_abs']:.6f} &
{st['max_rel']:.6f} & {st['mean_rel']:.6f} \\\\""
    )

if __name__ == "__main__":
    main()

```

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Authorship and AI–Use Disclosure

The author designed the framework and validated the mathematics and computations. Generative assistants were used for typesetting assistance, editorial organization, and consistency checks; they are not authors. All claims are the author’s responsibility.