

# A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

Dylan [Surname]

**Authorship and AI-use disclosure.** The author, Dylan [Surname], designed the framework, chose all constants/normalizations, and validated all mathematics and computations. A generative assistant (GPT-5 Pro) was used only for typesetting assistance, editorial organization, and consistency checks; it is not an author. All claims are the author's responsibility (COPE/ICMJE guidance).

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## Abstract

In the width-2 centered frame  $u = 2s$ ,  $v = u - 1$ , let  $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$  and  $E(v) = \Lambda_2(1 + v)$ . We present a boundary-only, height-local program to exclude off-axis quartets  $\{\pm a \pm im\}$  via two complementary routes:

- (1) an analytic tail (uniform in  $\alpha \in (0, 1]$ ) using only: (i) explicit short-side forcing  $\geq \pi/2$ ; (ii) a residual bound for  $F = E/Z_{\text{loc}}$  with perimeter factor  $8\delta$ ; and (iii) a disc-based,  $L^2$  boundary-to-midpoint estimate with *shape-only* constants (no strip/rectangle density comparison);
- (2) a rigorous Outer/Rouché Certification Path: interval arithmetic on  $\partial B$  + validated Poisson + Lipschitz grid → continuum enclosure  $\Rightarrow \sup_{\partial B} |E - G_{\text{out}}| / |G_{\text{out}}| < 1 \Rightarrow$  zero-free box, followed by Bridge 1 (inner collapse  $W \equiv e^{i\theta}$ ) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed  $\eta \in (0, 1)$  there exists  $M_0(\eta)$  such that no off-axis quartet lies in any  $B(\alpha, m, \delta)$  with  $\delta = \eta\alpha/(\log m)^2$  for all  $m \geq M_0(\eta)$ , uniformly in  $\alpha$ . Combined with a certified base range below  $m_1$  (first nontrivial height in width-2), this yields the global on-axis theorem. All constants appearing in the upper/lower envelope are *shape-only* (independent of  $m$ ,  $\alpha$ ,  $a$ ); residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

## Symbols & Provenance (at a glance)

*Notation hygiene.* We reserve  $\psi$  for the digamma function and write  $\varphi : \mathbb{D} \rightarrow B$  for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s, v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right)$	Completed object	Standard; FE for $\Lambda_2$ ; width-2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in $v$ -plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$ Used in FE and hinge law
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	$\chi_2(u) = \pi^{u/2-1/2} \frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$\chi_2(u)$	FE factor inverse	Square (width & height $2\delta$ ) centered at $(\alpha, m)$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Uniform-in- $\alpha$ uses worst case $\alpha = 1$ (left dial handled by reflection $w = -v$ ; we keep $\alpha \geq 0$ )
$\alpha \in (0, 1]$	Horizontal center	Ensures uniform
$m \geq 10$	Height parameter	DLMF/Titchmarsh/Ivić regimes
$\delta = \frac{\eta \alpha}{(\log m)^2}, \eta \in (0, 1)$	Half-side length of $B$	Balances forcing vs residual $O(\delta \log m)$
$\partial B$	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
$I_{\pm}$	Short vertical sides of $\partial B$	Near/far verticals in forcing budgets
$Q$	Quiet arcs (horizontal sides of $\partial B$ )	Controlled by $L^2$ trace & Hilbert
$Z_{\text{loc}}(v)$	= Local zero/pole factors	De-singularizes $E$ near $\partial B$
$\prod_{ \operatorname{Im} \rho - m  \leq 1} \rho^{m_{\rho}}$	-	
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near $\partial B$ )	Lemma ?? (constants symbolic)
$G(v) = \frac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\text{out}} = e^{U+iV}$	Outer with $ G_{\text{out}}  =  E $ on $\partial B$	$U = \log  E  \in C(\overline{B})$ solves Dirichlet; $V$ harmonic conj.
$W = E/G_{\text{out}}$	Inner quotient ( $ W  = 1$ a.e. on $\partial B$ )	Collapses to unimodular constant upon certification
$v_{\pm}^* = \pm(a + im)$	“Dial pair” on centerline	Points of evaluation in the tail
$Z_{\text{pair}}(v)$	$(v - (a + im))(v - (-a + im))$	Short-side forcing on $I_+$
$\Gamma_{\lambda}$	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by $\Gamma_{\lambda}$
$K_{\text{alloc}}^{(*)}(\lambda)$	Allocation coefficient	Shape-only; Lemma ??
$c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$	Dial deficit constant ( $\lambda = \frac{1}{2}$ )	From Jensen at dial (Lemma ??)
$C_{\text{up}}$	Upper-envelope constant	Shape-only; disc-based bound (Lemma ??)
$C_h''$	Horizontal budget constant	Shape-only; Lemma ??

*Sources.* Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds).  $\zeta'/\zeta$ : Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

## 1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame  $u = 2s, v = u - 1$ , with

$$\Lambda_2(u) = \pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right), \quad E(v) := \Lambda_2(1+v).$$

Then  $E(v) = E(-v) = \overline{E(\bar{v})}$ ; off-axis zeros appear as quartets  $\{\pm a \pm im\}$ . These symmetries follow from  $\Lambda_2(u) = \Lambda_2(2-u)$  and  $\overline{\Lambda_2(\bar{z})} = \Lambda_2(z)$  on vertical strips, hence  $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$  and conjugation invariance.

**Theorem 1.1** (Hinge–Unitarity). *Let  $\zeta_2(u) = \zeta(u/2)$  and  $\zeta_2(u) = A_2(u) \zeta_2(2-u)$  with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2} \frac{\Gamma\left(\frac{2-u}{4}\right)}{\Gamma\left(\frac{u}{4}\right)}.$$

For each fixed  $t \neq 0$ , define  $f(\sigma) = \log |\chi_2(\sigma+it)|$ . Then

$$f'(\sigma) = \frac{1}{2} \log \pi - \frac{1}{2} \Re \psi\left(\frac{\sigma+it}{4}\right) - \frac{1}{4} \Re \left[ \pi \cot\left(\frac{\pi}{4}(\sigma+it)\right) \right].$$

Moreover,

$$|\Re[\pi \cot(x+iy)]| \leq \frac{\pi}{\cosh(2y)-1}.$$

Taking  $x = \frac{\pi}{4}\sigma$ ,  $y = \frac{\pi}{4}|t|$ , for  $|t| \geq m_1/2$  (with  $m_1$  defined in Appendix ??) the cotangent term is  $< 10^{-8}$ . Using vertical-strip bounds,

$$\Re \psi\left(\frac{\sigma+it}{4}\right) \geq \log\left(\frac{|t|}{4}\right) - \frac{2}{|t|},$$

hence  $f'(\sigma) < 0$  on  $\mathbb{R}$  for all such  $t$ . Since  $f(1) = 0$ , we have  $|\chi_2(u)| = 1$  iff  $\operatorname{Re} u = 1$ . For  $|t| < m_1/2$  no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix ??.

**(Interpretive; non-load-bearing)  $\Omega$ -continuum and ray invariance.** Let  $\Omega(z) = z/|z|$  forget scale. FE-symmetric dilations  $T_\lambda(u) = 1+\lambda(u-1)$  preserve rays;  $\tan \theta = \operatorname{Im} v / \operatorname{Re} v = m/a$ . At a nontrivial zero  $a = 0$ , the ray is vertical. This layer is contextual only; the proofs below do not use it.

## 2 Boxes, de-singularization, residual control, and forcing

Fix  $m \geq 10$ ,  $\alpha \in (0, 1]$ , and

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta \alpha}{(\log m)^2}, \quad \eta \in (0, 1). \quad (2.1)$$

**Why  $m \geq 10$ .** This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the  $\zeta'/\zeta$  expansions on  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$  (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since  $u = 2s$  doubles ordinates,  $t \geq 3$  corresponds to  $m \geq 6$ ; we take  $m \geq 10$  for margin).

**Why  $\delta = \eta\alpha/(\log m)^2$ .** This balances the scale-free forcing ( $\geq \pi/2$ ) against residual budgets  $O(\delta \log m)$  and yields an  $L^2 +$  harmonic-measure upper envelope (in Section ??) that is uniformly small in  $\alpha$ .

**Lemma 2.1** (Short boxes stay in  $\operatorname{Re} v > 0$ ). *For  $m \geq 10$  and any  $\eta \in (0, 1)$ , one has  $\delta < \alpha$  and  $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$ , uniformly in  $\alpha \in (0, 1]$ .*

*Proof.* Since  $\eta \in (0, 1)$  and  $\log m \geq \log 10 > 0$ , we have  $\eta/(\log m)^2 < 1$ , hence  $\delta = \alpha \eta/(\log m)^2 < \alpha$ . Therefore the left edge is at  $\alpha - \delta > 0$ , so the entire box lies strictly in  $\{\operatorname{Re} v > 0\}$ , uniformly for  $\alpha \in (0, 1]$ .  $\square$

**De-singularization on  $\partial B$ .** Let

$$Z_{\text{loc}}(v) = \prod_{\rho: |\text{Im } \rho - m| \leq 1} (v - \rho)^{m_\rho}, \quad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (2.2)$$

Then  $F$  is analytic and zero-free on a neighborhood of  $\partial B$  (all local zeros/poles within  $|\text{Im } \rho - m| \leq 1$  have been removed).

**Boundary contact convention.** If a zero/pole meets  $\partial B$ , shrink  $\delta$  by a factor  $1 - \varepsilon$  or shift  $\alpha$  by  $O(\delta)$ . All constants/inequalities below (*residual envelope, short-side forcing*) are stable under  $O(\delta)$  changes.

**Lemma 2.2** (Residual envelope). *On  $\partial B$ ,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2, \quad (2.3)$$

and

$$|\Delta_{\partial B} \arg F| \leq 8\delta (C_1 \log m + C_2). \quad (2.4)$$

Justification. *DLMF §5.11 controls  $\psi$  on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control  $\zeta'/\zeta$  on  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$ . After removing local poles via (??) and transporting to width-2, we obtain (??). For (??), write  $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F ds$  as the sum of side integrals (angular limits at the corners); then bound by  $|\partial B| \sup_{\partial B} |F'/F| = 8\delta \sup |F'/F|$ . The constants  $C_1, C_2 > 0$  are absolute; we keep them symbolic (see Appendix ?? for an optional instantiation).*

**Lemma 2.3** (Logarithmic derivatives on  $\partial B$ ). *On  $\partial B$ ,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}, \quad \sup_{\partial B} \left| \frac{E'}{E} \right| \leq \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho: |\text{Im } \rho - m| \leq 1} \sup_{v \in \partial B} \frac{m_\rho}{|v - \rho|}.$$

In particular, by the boundary-contact convention the right-hand side is finite.

*Proof.* The identity follows from  $E = F Z_{\text{loc}}$ . For the inequality, take suprema termwise and use  $\left| \frac{(v-\rho)'}{v-\rho} \right| = \frac{1}{|v-\rho|}$ . Finiteness holds since only finitely many  $\rho$  satisfy  $|\text{Im } \rho - m| \leq 1$  and none lie on  $\partial B$  after the contact adjustment.  $\square$

**Lemma 2.4** (Short-side forcing). *Let  $Z_{\text{pair}}(v) = (v - (a + im))(v - (-a + im))$ . On the near vertical*

$$I_+ = \{\alpha + iy : |y - m| \leq \delta\}, \quad \text{with } |\alpha - a| \leq \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan \frac{\delta}{|\alpha - a|} + 2 \arctan \frac{\delta}{\alpha + a} \geq \frac{\pi}{2}. \quad (2.5)$$

*Proof.* Along  $I_+$ ,  $\arg(v - (\pm a + im)) = \arctan \frac{y-m}{\alpha \mp a}$ . As  $y$  runs from  $m - \delta$  to  $m + \delta$ , the increment is  $\arctan \frac{\delta}{|\alpha - a|} - \arctan \left( -\frac{\delta}{|\alpha - a|} \right) = 2 \arctan \frac{\delta}{|\alpha - a|}$  for the near factor and  $2 \arctan \frac{\delta}{\alpha + a}$  for the far factor. Since  $\alpha > 0$  and  $a \geq 0$ ,  $\alpha + a > 0$ , and the sum is monotone in  $\delta$ . When  $|\alpha - a| \leq \delta$ , the first term contributes at least  $\pi/2$  and the second is nonnegative, proving the bound. A symmetric formula holds on  $I_-$ , though not needed here.  $\square$

### 3 Boundary-only criteria, bridges, and corner interpolation

#### 3.1 Two-point Schur/outer criterion (boundary-only)

Let  $\varphi : \mathbb{D} \rightarrow B$  be a conformal bijection with  $\varphi(0)$  the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \quad \Phi := (G/H) \circ \varphi, \quad (3.1)$$

where  $H$  is an *outer majorant* for  $G$  on  $B$ : that is, choose  $M \in C(\partial B)$  with  $M \geq |G|$  a.e. on  $\partial B$ , let  $U$  solve the Dirichlet problem on  $B$  with boundary data  $\log M$ , fix a harmonic conjugate  $V$  by an anchor, and set  $H = e^{U+iV}$ . Then  $H$  is analytic and zero-free on  $B$  with nontangential boundary limits  $|H| = M$  a.e.; moreover  $\Phi \in H^\infty(\mathbb{D})$  with  $\|\Phi\|_\infty \leq 1$  (Duren [?, §II.5]; Garnett [?, §II.2]).

**Proposition 3.1** (Two-point Schur pinning). *Let  $\Phi = (G/H) \circ \varphi \in H^\infty(\mathbb{D})$  as above,  $\|\Phi\|_\infty \leq 1$ . Suppose two non-corner boundary points  $\zeta_\pm \in \partial\mathbb{D}$  have nontangential limits with  $|\Phi(\zeta_\pm)| = 1$ , and there exists a boundary arc  $A \subset \partial\mathbb{D}$  of positive measure on which  $\text{ess sup}_A |\Phi| \leq 1 - \varepsilon$  for some  $\varepsilon > 0$ . Then the angular derivatives of  $\Phi$  exist at  $\zeta_\pm$  (Julia–Carathéodory), and for any interior point  $z \in \mathbb{D}$  with harmonic measure  $\omega_z(A) \geq \omega_* > 0$  one has*

$$|\Phi(z)| \leq 1 - \kappa, \quad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Consequently, for  $v = \varphi(z)$  one obtains  $|G(v)| \leq (1 - \kappa) |H(v)|$ .

*Remark 3.2* (How the criterion is used). A verified boundary pattern (“pins” at two non-corner points  $|\Phi| = 1$ ; strict contraction  $|\Phi| \leq 1 - \varepsilon$  on complementary arcs of positive measure) yields quantitative decay of  $|\Phi|$  at interior evaluation points determined by harmonic measure. Transporting back gives bounds for  $|G|$  at the corresponding points in  $B$ . See Duren [?, Chs. II, IV–V] and Garnett [?, Chs. II–III].

**Lemma 3.3** (Two-point link for  $|G|$  and  $|\chi_2|$ ). *For  $v = a + im$  one has*

$$|G(v)| = |\chi_2(1+v)| \cdot R(v), \quad R(-v) = R(v)^{-1}, \quad (3.2)$$

hence

$$|G(a+im)| |G(-a+im)| = |\chi_2(1+a+im)| |\chi_2(1-a+im)|. \quad (3.3)$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\left(\frac{2+v}{4}\right)}{\Gamma\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta(1+\frac{v}{2})}{\zeta(1-\frac{v}{2})} \right|, \quad R(-v) = R(v)^{-1}.$$

Proof. Expand  $\Lambda_2$  at  $1 \pm v$  and collect  $\Gamma$  and  $\pi$  factors; multiplying at  $\pm v$  cancels  $R$  and yields (??). Poles of  $\Gamma$  and the simple pole of  $\zeta$  at 1 are avoided in our working set (boundary-contact convention;  $v \neq 0$ ).

#### 3.2 Outer/Rouché Certification Path

Let  $U$  be the harmonic solution to the Dirichlet problem on  $B$  with boundary data  $\log |E|$ , and let  $V$  be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then  $G_{\text{out}}$  is analytic and zero-free on  $B$  and satisfies  $|G_{\text{out}}| = |E|$  nontangentially on  $\partial B$  (a.e. with respect to arclength). Existence/uniqueness (up to unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [?, §II.5] and Garnett [?, §II.2].

**Proposition 3.4** (Outer/Rouché criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} < 1, \quad (3.4)$$

*then  $E$  is zero-free in  $B$  (Rouché’s theorem; Ahlfors [?, §§5–6], Conway [?, Ch. VI]). Consequently the inner quotient  $W := E/G_{\text{out}}$  is analytic and nonvanishing on  $B$  with  $|W| = 1$  a.e. on  $\partial B$ .*

**Proposition 3.5** (Bridge 1: inner collapse). *Under (??),  $\log |W|$  is harmonic with zero boundary trace on  $B$ , hence  $|W| \equiv 1$  on  $B$ . By the open mapping theorem,  $W \equiv e^{i\theta_B}$  on  $B$  for some real constant  $\theta_B$ .*

**Proposition 3.6** (Bridge 2: stitching). *If  $B_1, B_2$  overlap and  $W \equiv e^{i\theta_{B_j}}$  on  $B_j$  ( $j = 1, 2$ ), then  $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$  on  $B_1 \cap B_2$  by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

*Remark 3.7* (Certification recipe and reproducibility). The verification of (??) is performed by a robust, rigorous pipeline detailed in Appendix ??: (i) interval enclosures for  $|E|$  and  $\arg E$  on  $\partial B$ ; (ii) a validated Poisson solver on  $\mathbb{D}$  to reconstruct  $U = \log |G_{\text{out}}|$  and transport to  $B$ ; (iii) an interval reconstruction of  $\arg G_{\text{out}}$ ; and (iv) a grid→continuum Lipschitz enclosure using  $\sup_{\partial B} |E'/E|$  (Lemma ??). Appendix ?? also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility.

### 3.3 Corner outer interpolation (two-point)

**Theorem 3.8** (Corner outer interpolation). *Let  $G$  be analytic in a neighborhood of  $\overline{B}$ . Let  $h \in C(\partial B)$  satisfy  $h \geq 0$  and  $h \equiv 0$  on small boundary arcs containing the two top corners  $C_{\pm}$ . Let  $H = e^{U+iV}$  be the outer on  $B$  with  $U|_{\partial B} = \log |G| + h$ . Then the nontangential limits at  $C_{\pm}$  exist and*

$$|H(C_{\pm})| = |G(C_{\pm})|.$$

*Proof.* Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to  $\overline{B}$  (Kellogg; Axler–Bourdon–Ramey). Since  $h = 0$  on arcs about  $C_{\pm}$ ,  $U = \log |G|$  there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors; Pommerenke).  $\square$

*Remark 3.9* (Two “outers”: roles and notation). We reserve  $H$  for an *outer majorant* attached to an arbitrary analytic datum  $G$  on  $B$  (used in the Schur pinning), and  $G_{\text{out}}$  for the *modulus-outer* attached to  $E$  via the boundary data  $\log |E|$  (used in the Rouché route). Both are analytic, zero-free, and determined up to a unimodular factor; their roles are distinct.

## 4 Analytic tail (uniform in $\alpha$ )

**Setup and notation.** Let  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  be a conformal bijection with  $\varphi(0) = \alpha + im$ ; define the *dial pair* on the horizontal centerline by

$$v_{\pm}^* = \pm(a + im).$$

Split the boundary  $\partial B$  into the two *quiet arcs*  $Q$  (horizontal edges) and the two short vertical sides  $I_{\pm}$ . Write

$$W := \frac{E}{G_{\text{out}}}.$$

We write  $\partial_{\tau}$  for the unit tangential derivative along  $\partial B$ . All boundary integrals are taken with respect to arclength  $ds$ ; the perimeter is  $|\partial B| = 8\delta$ . *Left dial via reflection.* For the left dial  $-a + im$ , we either work in the reflected coordinate  $w = -v$  with a box centered at  $\alpha = a > 0$ , or equivalently use the reflected aligned box. All shape-only constants are unaffected.

## 4.1 Upper envelope via a disc-based $L^2$ route

**Lemma 4.1** (Boundary phase  $\Rightarrow$  dial deficit; disc-based upper bound). *Let  $m \geq 10$  and  $\delta = \eta \alpha / (\log m)^2$ . Let  $W = E/G_{\text{out}}$  be analytic on  $B(\alpha, m, \delta)$  with  $|W| = 1$  a.e. on  $\partial B$ , and assume  $v_\pm^* \in B$  (as in the aligned boxes  $\alpha = \pm a$ ). For each such dial  $v_\pm^*$  on the horizontal centerline, there exists a shape-only constant  $C_{\text{up}} > 0$  such that*

$$|W(v_\pm^*) - e^{i\phi_0^\pm}| \leq C_{\text{up}} \delta^{3/2} \left( \sup_{\partial B} \left| \frac{E'}{E} \right| \right), \quad (4.1)$$

where  $\phi_0^\pm$  is the harmonic-measure average of  $\arg W$  seen from  $v_\pm^*$ . Consequently,

$$\sum_{\pm} |W(v_\pm^*) - e^{i\phi_0^\pm}| \leq 2C_{\text{up}} \delta^{3/2} \left( \sup_{\partial B} \left| \frac{E'}{E} \right| \right), \quad (4.2)$$

where the sum is obtained by applying (??) separately on the two aligned boxes (right and left; or, equivalently, in  $v$  and in  $w = -v$  with the same  $\alpha = a$ ) and adding the two bounds; the two half-side lengths coincide as  $\delta = \eta a / (\log m)^2$  in both applications. Moreover,

$$C_{\text{up}} = C_{\text{tr}} C_{\text{H}} \cdot \frac{8\sqrt{8}}{\pi}, \quad (\text{Appendix ??}) \quad (4.3)$$

with  $C_{\text{tr}}$  the  $L^2$  conformal trace constant and  $C_{\text{H}}$  the  $L^2$  norm of the boundary Hilbert/conjugation on  $\partial B$ ; both are shape-only (Appendix S.1).

*Remark 4.2* (Branch and trace conventions). Since  $|W| = 1$  a.e. on  $\partial B$ , choose any measurable branch of  $\arg W$  on  $\partial B$ ;  $\phi_0^\pm$  is defined as the harmonic-measure average seen from  $v_\pm^*$ . The bounds are invariant under  $2\pi\mathbb{Z}$  shifts of the branch.

*Proof.* Let  $\varphi_\pm : \mathbb{D} \rightarrow B$  be conformal with  $\varphi_\pm(0) = v_\pm^*$  (compose a disc automorphism with a fixed normalization if desired), and set  $f := W \circ \varphi_\pm$ . Then  $u(z) := \log |f(z) - c|$ ,  $c = e^{i\phi_0^\pm}$ , is subharmonic and Poisson's inequality on  $\mathbb{D}$  yields

$$|f(0) - c| \leq \left( \int_{\partial\mathbb{D}} |\arg f - \phi_0^\pm|^2 \frac{dt}{2\pi} \right)^{1/2}.$$

By bounded conformal trace and harmonic-measure/arclength comparability (shape-only constants; Appendix ??),

$$\|\arg f - \phi_0^\pm\|_{L^2(\partial\mathbb{D})} \leq C_{\text{tr}} \|\arg W - \phi_0^\pm\|_{L^2(\partial B)}.$$

By Wirtinger on the closed curve  $\partial B$  (length  $8\delta$ ),

$$\|\arg W - \phi_0^\pm\|_{L^2(\partial B)} \leq \frac{8\delta}{2\pi} \|\partial_\tau \arg W\|_{L^2(\partial B)}.$$

Finally,

$$\|\partial_\tau \arg W\|_{L^2(\partial B)} \leq \|\partial_\tau \arg E\|_{L^2(\partial B)} + \|\partial_\tau \arg G_{\text{out}}\|_{L^2(\partial B)} \leq (1+C_{\text{H}}) \sqrt{8\delta} \sup_{\partial B} \left| \frac{E'}{E} \right| \leq 2C_{\text{H}} \sqrt{8\delta} \sup_{\partial B} \left| \frac{E'}{E} \right|,$$

using boundedness of the boundary Hilbert/conjugation on Lipschitz curves (shape-only norm  $C_{\text{H}}$ ) and that  $\arg G_{\text{out}}$  is a harmonic conjugate of  $\log |E|$ . Combining the displays gives (??) with (??), hence (??) by summation. The bound is uniform in  $\alpha \in (0, 1]$  because  $C_{\text{tr}}$  and  $C_{\text{H}}$  are shape-only and dependence on  $(m, \alpha)$  enters only through  $\delta$  and  $L := \sup_{\partial B} |E'/E|$ .  $\square$

## 4.2 Lower envelope via forcing, $L^2$ allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

**Lemma 4.3** (Vertical Lipschitz allocation ( $L^2$ )). *Let  $\lambda \in (0, 1)$ , and let  $s_{\text{tail}} = (2 - \lambda)\delta$  be the total tail length on a vertical side (outside the central sub-arc of length  $\lambda\delta$ ). Then on each vertical side*

$$\int_{\text{tails}} |\partial_\tau \arg W| ds \leq \left[ (2 - \lambda) + 2\sqrt{2(2 - \lambda)} \right] \delta \sup_{\partial B} \left| \frac{E'}{E} \right|. \quad (4.4)$$

Summing both verticals yields

$$\Delta_{\text{cent}} \geq \Delta_{\text{vert}} - K_{\text{alloc}}(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right|, \quad K_{\text{alloc}}(\lambda) := 2 \left[ (2 - \lambda) + 2\sqrt{2(2 - \lambda)} \right]. \quad (4.5)$$

For conservatism we may adopt the stricter  $K_{\text{alloc}}^*(\lambda) := 2[(2 - \lambda) + 4\sqrt{2(2 - \lambda)}]$ , which dominates  $K_{\text{alloc}}(\lambda)$  and is valid as well.

*Definition of the retained central gap.* Recall from Lemma ?? that, under  $|\alpha - a| \leq \delta$  and  $\operatorname{Re} v > 0$ , the near-vs-far vertical forcing gives  $\Delta_{\text{vert}} \geq \pi/2$ . We set

$$\Delta_{\text{cent}} := \Delta_{\text{vert}} - K_{\text{alloc}}^*(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right| - C_h'' \delta (\log m + 1), \quad (4.6)$$

where  $C_h'' > 0$  is a *shape-only* constant accounting for the horizontal (quiet-arc) budget (see Appendix ??).

**Lemma 4.4** (Core zero via restricted contour). *Align the box by taking  $\alpha = a$ . Let  $\Gamma_\lambda$  be the union of the two central sub-arcs (length  $\lambda\delta$ ) on the vertical sides, joined by vanishing horizontals at heights  $m \pm \varepsilon$  as  $\varepsilon \downarrow 0$ . If the retained central vertical gap*

$$\Delta_{\text{cent}} > 0$$

(in the sense of ??) then the rectangle bounded by  $\Gamma_\lambda$  contains at least one zero of  $W$ . This zero lies in the dial-centred core

$$B_{\text{core}}(a, m; \lambda) = \left[ a - \frac{\lambda\delta}{2}, a + \frac{\lambda\delta}{2} \right] \times \left[ m - \frac{\lambda\delta}{2}, m + \frac{\lambda\delta}{2} \right].$$

The tiny horizontal joins contribute  $o(1)$  to the argument change and are absorbed in the horizontal budget.

**Lemma 4.5** (Jensen at the dial). *With  $\alpha = a$ , fix one dial  $p = a + im$ . Then  $\operatorname{dist}(p, \partial B) = \delta$  so  $D_p = \{|z - p| < \delta\} \subset B$ . If  $W$  has a zero  $z_k$  in  $B_{\text{core}}(a, m; \lambda)$ , then*

$$-\log |W(p)| \geq \log \left( \frac{\delta}{|z_k - p|} \right) \geq \log \left( \frac{\sqrt{2}}{\lambda} \right),$$

hence

$$1 - |W(p)| \geq 1 - \frac{\lambda}{\sqrt{2}}. \quad (4.7)$$

**Lemma 4.6** (Bridge to the upper-envelope metric). *For any unimodular  $c = e^{i\phi}$  and any  $z \in B$ , one has*

$$|W(z) - c| \geq 1 - |W(z)|.$$

*Proof.* By the reverse triangle inequality,  $|W(z) - c| \geq ||W(z)| - |c|| = 1 - |W(z)|$ .  $\square$

**Corollary 4.7** (Lower envelope; aligned boxes). *Pick  $\lambda = \frac{1}{2}$  and denote  $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$ . With  $L = \sup_{\partial B} |E'/E|$  and  $\delta = \eta \alpha / (\log m)^2$ ,*

$$\varepsilon_+ + \varepsilon_- \geq c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 L + C_h''(\log m + 1) \right),$$

where  $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$  and  $C_h'' > 0$  is shape-only.

*Two aligned boxes.* We apply the aligned-box argument twice, once with  $\alpha = +a$  (controlling  $\varepsilon_+$ ) and once with  $\alpha = -a$  (controlling  $\varepsilon_-$ ). The two bounds sum to yield  $\mathcal{L}(m, \alpha) = \varepsilon_+ + \varepsilon_-$ .

*Remark 4.8.* By Lemma ??, for  $\lambda = \frac{1}{2}$  one has  $\varepsilon_\pm \geq 1 - \frac{1}{2\sqrt{2}} \approx 0.6464$ . Since  $c_0 \frac{\pi}{2} = \frac{1}{8} \log(2\sqrt{2}) \approx 0.1299$  and the budget terms are nonnegative, the displayed conservative linear inequality follows by weakening this stronger bound.

### 4.3 Tail comparison (symbolic constants)

**Theorem 4.9** (Global on-axis theorem; symbolic constants). *Fix  $\eta \in (0, 1)$  and set  $\delta = \eta \alpha / (\log m)^2$ . Let  $C_{\text{up}} > 0$  be the shape-only constant in Lemma ??,  $C_h'' > 0$  the horizontal budget constant in Lemma ??, and  $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$ . Assume the residual envelope of Lemma ?? with absolute constants  $C_1, C_2 > 0$ . Then there exists  $M_0(\eta)$  such that, for all  $m \geq M_0(\eta)$  and all  $\alpha \in (0, 1]$ ,*

$$\underbrace{\sum_{\pm} |W(v_{\pm}^*) - e^{i\phi_0^{\pm}}|}_{\mathcal{U}_{hm}(m, \alpha)} < \underbrace{c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right)}_{\mathcal{L}(m, \alpha)}, \quad (4.8)$$

with  $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$ . Consequently, no off-axis quartet lies in any  $B(\alpha, m, \delta)$  for  $m \geq M_0(\eta)$  and all  $\alpha \in (0, 1]$ . Combined with a certified base range “no zeros below  $m_1$ ” (Appendix ??) and, when  $M_0(\eta) > m_1$ , certification of the finite band  $[m_1, M_0(\eta)]$  via the Outer/Rouché pipeline (Section ?? and Appendix ??), all nontrivial zeros lie on  $\Re s = \frac{1}{2}$ .

*Clarifying note.* For exclusion at a given abscissa  $a \in (0, 1]$ , we apply (??) on the two aligned boxes with  $\alpha = a$  (right) and, by reflection  $w = -v$ , also cover the left dial; the right-hand side is monotone in  $\alpha$  via  $\delta \propto \alpha$ , so the worst case  $\alpha = 1$  provides a uniform  $M_0(\eta)$ .

*Proof.* By Lemma ??,  $\mathcal{U}_{hm} \leq 2C_{\text{up}}\delta^{3/2}(C_1 \log m + C_2)$ , which tends to 0 as  $\log m \rightarrow \infty$ . By Corollary ??,  $\mathcal{L}(m, \alpha) = c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right)$  tends to  $c_0\pi/2 > 0$  as  $m \rightarrow \infty$ , uniformly in  $\alpha$ . Hence  $\mathcal{U}_{hm} < \mathcal{L}$  for all sufficiently large  $m$ .  $\square$

**Choice of  $M_0(\eta)$  (explicit criterion).** A sufficient (symbolic) condition ensuring (??) for all  $\alpha \in (0, 1]$  is

$$2C_{\text{up}} \left( \frac{\eta}{(\log m)^2} \right)^{3/2} (C_1 \log m + C_2) \leq \frac{1}{2} \left( c_0 \frac{\pi}{2} - \frac{\eta}{(\log m)^2} \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right) \right), \quad (4.9)$$

obtained by taking the worst case  $\alpha = 1$ . Since the left-hand side is  $o(1)$  and the right-hand side tends to  $c_0\pi/4 > 0$  as  $m \rightarrow \infty$ , there exists  $M_0(\eta)$  with (??) holding for all  $m \geq M_0(\eta)$ .

*Remark 4.10* (Numerical check; illustrative only). If one instantiates  $(C_1, C_2)$  safely from the literature (Appendix ??) and takes a small  $\eta$  (e.g.,  $\eta = 10^{-9}$ ), then at  $m = m_1$  and  $\alpha = 1$  the upper bound is  $\ll 10^{-10}$  while the lower bound is  $\approx 0.13$  up to  $O(10^{-8})$  corrections, leaving an overwhelming margin. These numerics are not used in the proof.

## Acknowledgments and certification note

Reproducible certification ingredients (interval Poisson; grid→continuum Lipschitz) are outlined in Appendix ???. Library versions, precision, and boundary meshes are pinned there.

## A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of  $\log |\chi_2|$  via  $\partial_\sigma \log |\Gamma| = \Re \psi$  and  $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$  directly; the cosh-bound form appears in Theorem ??.

## B Constants ledger (sources & transport)

- Digamma (DLMF §5.11):  $\psi(z) = \log z + O(1)$  uniformly on vertical strips; transported to width-2 gives  $\Re \psi((1+v)/4) = \log |m| + O(1)$  on  $\partial B$ .
- $\zeta'/\zeta$  (Titchmarsh §14; Ivić Ch. 9): for  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$ ,  $\frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im} \rho-t| \leq 1} \frac{1}{\sigma+it-\rho} + O(\log t)$ . Removing local poles via  $Z_{\text{loc}}$  yields Lemma ??.
- Lipschitz Hilbert/Cauchy: bounded on  $L^2(\Gamma)$  for Lipschitz curves; boundary traces between  $\partial\mathbb{D}$  and  $\Gamma$  are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

## C Bridges (one-liners)

- Bridge 1. If (??) holds, then  $E$  and  $G_{\text{out}}$  have the same zero count,  $G_{\text{out}}$  is zero-free,  $|W| = 1$  on  $\partial B$ . Hence  $\log |W| \equiv 0$ , and by the open mapping theorem  $W \equiv e^{i\theta_B}$ .
- Bridge 2. If  $W_1, W_2$  are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

## D Conformal normalization

Take  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  conformal with  $\varphi(0) = \alpha + im$  and  $\varphi(\pm 1)$  the top corners. By symmetry,  $\varphi((-1, 1))$  is the horizontal centerline; thus there exists a unique  $r_0 \in (0, 1)$  with  $\varphi(\pm r_0) = \pm(a + im)$ .

## E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to  $\overline{B}$  (Kellogg; Axler–Bourdon–Ramey). Since  $h = 0$  on arcs about  $C_\pm$ ,  $U = \log |G|$  there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

## F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for  $|E|$ ,  $\arg E$  on  $\partial B$  at grid size  $N_{\text{side}}$ .
- Validated Poisson. Interval Dirichlet solver on  $\mathbb{D}$  for  $U = \log |G_{\text{out}}|$ , with conformal push-forward to  $\partial B$ .

- Phase reconstruction. Interval Hilbert on  $\partial\mathbb{D}$ , conformal trace to  $\partial B$ .
- Grid→continuum. Lipschitz enclosure via  $\sup_{\partial B} |E'/E|$  and explicit pair terms.
- Certificate. Check  $\sup_{\partial B} |E - G_{\text{out}}| / |G_{\text{out}}| < 1$ .

The grid→continuum step uses a shape-only Lipschitz/trace bound on  $\partial B$  to convert a mesh supremum into a boundary supremum, making the Rouché ratio verifiable with controlled constants.

## G Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section ?? proofs.

## H Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

**Theorem H.1** (Platt 2017; Platt–Trudgian 2021). *There are no nontrivial zeros of  $\zeta(s)$  with  $0 < \text{Im } s < t_1$ , and the first nontrivial zero occurs at  $t_1 = 14.134725141734693790457251983562\dots$  (with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of  $\zeta(s)$* , Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to  $3 \cdot 10^{12}$* , Bull. Lond. Math. Soc. 53 (2021), 792–797. Set  $m_1 := 2t_1$ .

## Appendix S.1. Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve  $\Gamma$  (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on  $L^2(\Gamma)$  whose norm depends only on the Lipschitz character of  $\Gamma$ ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between  $\partial\mathbb{D}$  and  $\Gamma$  are bounded in  $L^2$  with operator norms depending only on the chord–arc constants of  $\Gamma$ . (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Moreover, on chord–arc curves (which include rectangles) harmonic measure  $\omega_z$  and arclength  $ds$  are  $A_\infty$ -equivalent; the associated  $L^2$ -comparability constants depend only on the chord–arc data. We fold these shape-only constants into  $C_{\text{tr}}$  and into the boundary Hilbert norm  $C_H$  used in Lemma ???. Since  $B(\alpha, m, \delta)$  normalizes to the unit square via an affine map, all such operator norms are *shape-only* constants (independent of  $m, \alpha, a$ ). We denote by  $C_{\text{tr}}$  a generic shape-only trace constant and by  $C_H$  the  $L^2$  operator norm of boundary Hilbert/conjugation on  $\partial B$ .

## Appendix S.2. Instantiating $(C_1, C_2)$ from explicit literature bounds (optional)

Let  $F = E/Z_{\text{loc}}$  with  $Z_{\text{loc}}$  removing local zeros with  $|\text{Im } \rho - m| \leq 1$ . On  $1/2 \leq \sigma \leq 1$  and  $t \geq 3$ ,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\text{Im } \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips  $\psi$  satisfies  $\Re\psi(x+iy) = \log\sqrt{x^2+y^2} + O(1)$  (DLMF §5.11). Transporting to width 2 and dividing out  $Z_{\text{loc}}$  yields

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2,$$

with absolute constants  $C_1, C_2 > 0$ ; any choices respecting the cited explicit estimates are legitimate. The main text keeps  $C_1, C_2$  symbolic. On  $\partial B$  we have  $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}$  (Lemma ??); the local sum is finite under the boundary-contact convention, so  $L = \sup_{\partial B} |E'/E|$  is controlled by the residual bound plus finitely many explicit local terms. Given any such  $(C_1, C_2)$  and a fixed  $\eta \in (0, 1)$ , one may select  $M_0(\eta)$  by enforcing the symbolic inequality (??), which depends only on  $(C_{\text{up}}, C_h'', K_{\text{alloc}}^*, c_0)$  (shape-only) and  $(C_1, C_2)$  (residual).

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