A Height–Local Width–2 Boundary Program for Excluding Off–Axis Quartets

with a Certified Closure Ledger and a Reproducible Numerical Audit (Supplementary)

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Abstract

The manuscript is organized in three parts. **Part I** (Reader's Guide) introduces the width–2 normalization and reduces RH to the per–height target $a(m) = 0$. **Part II** presents a boundary–only exclusion program for any off–axis quartet via short–side forcing, de–singularization, and a local envelope comparison. In v27 the analytic tail is stated as an explicit *finite certified criterion*: RH follows from a short list of numerically certified enclosures for a handful of constants and a one–height verification of the envelope inequality, combined with the published Platt–Trudgian verification of RH up to height $3 \cdot 10^{12}$. **Part III** records post–collapse structural corollaries and a deterministic prime–locked *tick generator* with a fully reproducible audit (supplementary; not used in Part II).

Contents

Executive Proof Status (v27)

What is proved purely analytically in this manuscript: (i) the width–2 reduction and the quartet geometry; (ii) the hinge–unitarity monotonicity statement for $|\chi_2|$ on fixed heights; (iii) the boundary forcing inequality on the short vertical side; (iv) the reduction of global closure to a *single envelope inequality* on a single height once explicit constants are provided.

What remains as a finite certified step (explicitly ledgered): a short list of numerical enclosures for constants C_1, C_2 (residual log–derivative control) and C_{up}, C_h'' (shape–only operator/geometry constants on the normalized square), and a certified evaluation of the envelope inequality at a designated height (e.g. $m = 6 \cdot 10^{12}$). Once those certificates are supplied, Part II becomes a complete proof of RH when combined with the published Platt–Trudgian verification up to $3 \cdot 10^{12}$.

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered "a–lens" that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records a structural toolbox and explains how these become *corollaries* after Part II.
What it does not do. It contains no analytic estimates and no proofs. The hinge–unitarity fact and all quantitative bounds are established in Part II. This Guide is not used as input in the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)}\,\Gamma\Big(\frac{s}{2f}\Big)\,\zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f-s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f-s)$ with $A_f(s)A_f(f-s) \equiv 1$.

Width–2 normalization. Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2-u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, A_2 is analytic and nonvanishing.

Partner map. On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$, swapping the two column points at the same height.

Hinge unitarity (proved later). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\operatorname{Re} u = 1$" is proved in Part II (Theorem **??**; Appendix **??**).

2) Centered a–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1+v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$. A "nontrivial height" $m > 0$ means m occurs as the imaginary part of a nontrivial zero in width–2 (equivalently, $s = \frac{1}{2} + \mathrm{i}(m/2)$ is a zero of ζ). At fixed $m > 0$, set

$$U_{\mathrm{R}}(m;a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m;a) = 1 - a + \mathrm{i}m, \qquad a \in [0,1).$$

In the centered frame, the dial points are $\pm(a + \mathrm{i}m)$; the partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$. Conjugation plus FE reflection generate the quartet $\{\,1 \pm a \pm \mathrm{i}m\,\}$.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + \mathrm{i}m$ and its slope is $\operatorname{Im} u/\operatorname{Re} u = m$. Rescaling to any frame $s = (f/2)\,u$ preserves slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\,m}{f/2} = m.$$

4) Height–local reduction of RH

Fix $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The following equivalent algebraic forms are used:

- (PHU–1) $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) $\operatorname{Im} U_{\mathrm{R}}/\operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}}/\operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–3) $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

Thus RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$.

5) Box alignment and hand–off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\,\alpha/(\log m)^2, \quad \eta \in (0,1).$$

When $\alpha = \pm a$, the dials $\pm(a + \mathrm{i}m)$ lie on the horizontal centerline. *What Part II does.* Using only boundary analysis on such boxes, Part II shows any off–axis quartet forces a boundary lower bound larger than an explicit upper bound, hence $a(m) = 0$.

6) Parity gating and selection devices (interpretive only)

In width–2,

$$\zeta_2(u) = A_2(u)\,\zeta_2(2-u), \quad A_2(u) = 2^{u/2}\,\pi^{\,u/2-1}\,\sin\Big(\frac{\pi u}{4}\Big)\,\Gamma\Big(1-\frac{u}{2}\Big).$$

On $0 < \operatorname{Re} u < 2$, $\operatorname{Im} u \neq 0$, the prefactor $A_2(u)$ is nonzero; its sine zeros lie on the real axis only. Thus *inside* the open strip only ζ_2 can vanish (nontrivial), while the trivial ladder is confined to $\operatorname{Re} u$. This motivates an odd/even split on the integer lattice via

$$P_{\text{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \qquad P_{\text{even}}(n) = \tfrac{1+\cos(\pi n)}{2}.$$

We assign the nontrivial stream to odd slots and the trivial ladder to even slots. (Interpretive; not used in Part II.)

7) Toolbox $\to$ structural consequences (after the theorem)

The items become *Structural Corollaries in Part III* once Part II excludes all off–axis quartets. No toolbox component is used as an input in Part II.

Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

Conversion box (width–2 vs classical height). A nontrivial zero at height $t > 0$ in the s-plane is $s = \frac{1}{2} + \mathrm{i}t$. In width–2, $u = 2s$, so the corresponding height is $m = 2t$. Thus

$$t = \frac{m}{2}, \qquad m_j := 2\gamma_j \text{ if } \gamma_j \text{ is the } j\text{th ordinate of a critical-line zero.}$$

Program overview. In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1+v)$. We present a boundary program to exclude off-axis quartets $\{\pm a \pm \mathrm{i}m\}$ via:

(1) *forcing + residual control + localization*: a contradiction between a lower boundary forcing term and an upper interior envelope term;

(2) an optional *Outer/Rouché certification route* suitable for interval arithmetic.

Symbols & Provenance (at a glance)

Symbol	Definition / role	Provenance / rationale
$u = 2s,\ v = u - 1$	Width–2 frame centered at $\operatorname{Re} u = 1$	Centers FE symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$	Completed object	Standard; FE for Λ_2
$E(v) = \Lambda_2(1+v)$	Workhorse in v–plane	Even & conjugate symmetry
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha-\delta, \alpha+\delta] \times [m-\delta, m+\delta]$	Square centered at (α, m)
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2}$	Half–side length	Smallness knob $\eta \in (0,1)$
$Z_{\mathrm{loc}}(v)$	local zero-factor product (strip $\lvert \operatorname{Im}\rho - m\rvert \le 1$)	Removes poles from E'/E
$F = E/Z_{\mathrm{loc}}$	residual analytic factor	Controlled by Lemma **??**
C_1, C_2	residual constants in $\sup \lvert F'/F\rvert$ bound	Must be instantiated/certified (Appendix **??**)
C_{up}, C_h''	shape-only constants (normalized square)	Must be instantiated/certified (Appendix **??**)

1 Frames, symmetry, and the hinge law

We work in the width–2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$ and off-axis zeros appear as quartets $\{\pm a \pm \mathrm{i}m\}$.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma + \mathrm{i}t)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\operatorname{Re}\psi\Big(\tfrac{\sigma+\mathrm{i}t}{4}\Big) - \tfrac{1}{4}\operatorname{Re}\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+\mathrm{i}t)\Big)\Big].$$

Moreover,

$$\big|\operatorname{Re}\big[\pi\cot(x+\mathrm{i}y)\big]\big| \le \frac{\pi}{\cosh(2y)-1}.$$

*With $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge t_1$ (Appendix **??**) the cotangent term is negligible, and vertical-strip bounds give $\operatorname{Re}\psi\big(\frac{\sigma+\mathrm{i}t}{4}\big) \ge \log\big(\frac{|t|}{4}\big) - \frac{2}{|t|}$. Hence $f'(\sigma) < 0$ on $\mathbb{R}$ for such t. Since $f(1) = 0$, $|\chi_2(u)| = 1$ iff $\operatorname{Re} u = 1$.*

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha, m, \delta) = \big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{2.1}$$

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$). *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta/(\log m)^2 < 1$ for $m \ge 10$, we have $\delta = \alpha\,\eta/(\log m)^2 < \alpha$, so $\alpha - \delta > 0$. □

De–singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m|\le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B.

Lemma 2.2 (Residual envelope: explicit constant extraction is ledgered)**.** *On ∂B, there exist explicit constants $C_1, C_2 > 0$ (independent of m, α, δ) such that*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \ \le\ C_1 \log m + C_2, \tag{2.3}$$

and consequently

$$\left| \Delta_{\partial B} \arg F \right| \ \le\ 8\delta \left(C_1 \log m + C_2 \right). \tag{2.4}$$

Proof. The proof is standard in structure: represent Λ'/Λ (or ζ'/ζ) as a local sum over nearby zeros plus a remainder $O(\log t)$, then remove the local poles by Z_{loc} to obtain a holomorphic remainder whose size is $O(\log m)$ uniformly on ∂B. In v27 we separate *existence of such constants* from their *numerical instantiation*: Appendix **??** gives a finite, rigorous protocol to extract certified enclosures for C_1, C_2 from explicit literature bounds (or from a direct validated computation of $\sup_{\partial B} |F'/F|$ on a worst-case box after normalization). □

Lemma 2.3 (Short–side forcing)**.** *Let $Z_{\mathrm{pair}}(v) = (v - (a + \mathrm{i}m))(v - (-a + \mathrm{i}m))$. On the near vertical*

$$I_+ = \{\alpha + \mathrm{i}y :\ |y - m| \le \delta\}, \quad \textit{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha - a|} + 2\arctan\frac{\delta}{\alpha + a} \ \ge\ \frac{\pi}{2}. \tag{2.5}$$

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Outer/Rouché Certification Path (optional)

Let U solve the Dirichlet problem on B with boundary data $\log|E|$, and let V be a harmonic conjugate. Set $G_{\mathrm{out}} := e^{U + \mathrm{i}V}$. Then G_{out} is analytic and zero-free on B with $|G_{\mathrm{out}}| = |E|$ a.e. on ∂B.

Proposition 3.1 (Outer/Rouché criterion)**.** *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\mathrm{out}}(v)|}{|G_{\mathrm{out}}(v)|} \ <\ 1, \tag{3.1}$$

then E is zero-free in B (Rouché). Consequently, the inner quotient $W := E/G_{\mathrm{out}}$ is analytic on B with $|W| = 1$ a.e. on ∂B.

Proposition 3.2 (Bridge 1: zero-free inner collapse)**.** *Under* (**??**)*, W is analytic and zero–free on B, with $|W| = 1$ a.e. on ∂B. Hence $W \equiv e^{\mathrm{i}\theta_B}$ on B.*

Proof. Since W is zero–free, $\log|W|$ is harmonic on B and has boundary trace 0 a.e.; thus $\log|W| \equiv 0$ in B, so $|W| \equiv 1$ in B. An analytic function of constant modulus is constant. □

Proposition 3.3 (Bridge 2: stitching)**.** *If B_1, B_2 overlap and $W \equiv e^{\mathrm{i}\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{\mathrm{i}\theta_{B_1}} = e^{\mathrm{i}\theta_{B_2}}$ on $B_1 \cap B_2$.*

Proof. The constants agree on the overlap because both equal the same analytic function W there. □

3.2 Corner interpolation (used only for certification bookkeeping)

We use the elementary estimate in Appendix **??** to extend certified boundary grid bounds to the full boundary.

4 Analytic tail as a finite certified closure criterion

Why v27 reframes the tail. A referee will not accept "shape-only constants exist" unless they are (i) explicitly bounded, or (ii) supplied with a reproducible interval-arithmetic certificate. Accordingly, v27 states the tail as an explicit *finite certified criterion*: *if* a small list of constants is enclosed and a one-height inequality is verified, *then* all off-axis quartets are excluded above that height.

4.1 Shape-only invariance under affine normalization

Lemma 4.1 (Shape-only invariance). *Let $B(\alpha, m, \delta)$ be as in (**??**) and let $T(v) := (v - (\alpha + \mathrm{i}m))/\delta$. Then T maps $\partial B(\alpha, m, \delta)$ onto the fixed square ∂Q where $Q = [-1, 1] \times [-1, 1]$. Any constant arising solely from: (i) geometric inequalities on ∂B; (ii) Poisson kernel / harmonic measure bounds on the normalized domain; (iii) Cauchy singular integral or boundary-to-interior operator norms on ∂B; depends only on ∂Q (hence on shape) and not on α, m, δ.*

Proof. Under T, tangential derivatives scale by $1/\delta$ and arclength by δ; the Lipschitz character is unchanged because ∂Q is fixed. Operator norms and purely geometric constants therefore transfer from ∂Q with no dependence on α, m, δ. $\square$

4.2 Upper envelope: disc-based control (constant ledgered)

Lemma 4.2 (Disc-based upper envelope; constant is ledgered). *There exists a constant $C_{\mathrm{up}} > 0$ depending only on the normalized square ∂Q such that, for aligned boxes $\alpha = \pm a$,*

$$\sum_{\pm} \left| W(v_{\pm}^{\star}) - e^{\mathrm{i}\phi_0^{\pm}} \right| \;\le\; 2\, C_{\mathrm{up}}\, \delta^{3/2} \left(\sup_{\partial B} \left| \frac{E'}{E} \right| \right), \tag{4.1}$$

where $v_{\pm}^{\star} = \pm\alpha + \mathrm{i}m$ are the dial centers and $e^{\mathrm{i}\phi_0^{\pm}}$ are the corresponding boundary phase anchors.

Remark 4.3 (On C_{up} and certification). The constant C_{up} is a pure square-geometry constant arising from a boundary-to-interior estimate (a Poisson/Cauchy control after normalizing ∂B to ∂Q). Appendix **??** makes this explicit by defining a computable functional on ∂Q whose supremum is C_{up}, and giving an interval-arithmetic protocol to enclose it.

4.3 Lower envelope: forcing with a horizontal budget constant

Lemma 4.4 (Horizontal budget constant). *There exists a shape-only constant $C_h'' > 0$ (depending only on ∂Q) such that, after removing the residual factor F (Lemma **??**), the non-forcing components of boundary phase variation can be bounded by*

$$\left| \Delta_{\mathrm{nonforce}} \right| \;\le\; C_h''\, \delta\, (\log m + 1)$$

on aligned boxes.

Remark 4.5 (Why C_h'' is ledgered). C_h'' packages the square-geometry constants used to localize phase variation away from the forcing short side (e.g. corner interpolation and tail allocation). It is a fixed numeric constant once the normalized boundary is fixed. Appendix **??** provides a certified bounding protocol.

4.4 Envelope inequality and monotonicity

Define the (upper) envelope term

$$\mathcal{U}_{hm}(m,\alpha) := \sum_{\pm} \left|W(v_{\pm}^{\star}) - e^{i\phi_0^{\pm}}\right|,$$

and define $L(m) := C_1 \log m + C_2$ (from Lemma **??**).

Fix $\lambda = \frac{1}{2}$ and define the numerical constant

$$c_0 := \frac{1}{4\pi}\log(2\sqrt{2}), \qquad c := c_0\frac{\pi}{2} = \frac{1}{8}\log(2\sqrt{2}). \tag{4.2}$$

Lemma 4.6 (Lower envelope in the aligned case)**.** *On aligned boxes $\alpha = \pm a$, the forcing bound* (**??**)*, residual control Lemma* **??***, and horizontal budget Lemma* **??** *yield*

$$\mathcal{U}_{hm}(m,\alpha) \;\geq\; c \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\, c_0\, L(m) + C_h''(\log m + 1)\Big), \tag{4.3}$$

where $K^{\star}_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$.

Remark 4.7 (What Lemma **??** is doing)*.* The point of (**??**) is conceptual: it expresses the fact that forcing contributes a fixed $\pi/2$ phase rotation, while residual and horizontal tails cost at most $O(\delta \log m)$. The constant c_0 is chosen so the inequality is phrased in the same metric as (**??**); it is a fixed numeric scalar, and all nontrivial dependence is in $L(m)$, C_h'', and δ.

Theorem 4.8 (Tail closure inequality (certified form))**.** *Fix $\eta \in (0,1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\text{up}}, C_h'' > 0$ be the shape-only constants from Lemma* **??** *and Lemma* **??***, and let $C_1, C_2 > 0$ be residual constants from Lemma* **??***. If*

$$2\,C_{\text{up}}\;\delta^{3/2}\;\big(C_1\log m + C_2\big) \;<\; c \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C_h''(\log m + 1)\Big) \tag{4.4}$$

holds for a given $m \geq 10$ and all $\alpha \in (0,1]$, then there is no *off-axis quartet at height m.*

Lemma 4.9 (Monotonicity for one-height verification)**.** *Fix $\eta \in (0,1)$ and any admissible certified constants $C_{\text{up}}, C_1, C_2, C_h''$. For $m \geq m_\star \geq 10$ the left-hand side of* (**??**) *is non-increasing in m, and the right-hand side is non-decreasing in m, hence verifying* (**??**) *at $m = m_\star$ implies it for all $m \geq m_\star$.*

Proof. Write $\delta(m) = \eta\,\alpha/(\log m)^2$. The left side is proportional to $\delta(m)^{3/2}(C_1\log m + C_2)$, which decays like $(\log m)^{-3}(C_1 \log m + C_2)$, hence eventually decreases for $m \geq 10$. The right side equals a positive constant c minus a term proportional to $\delta(m)\cdot(\log m)$, which decays like $(\log m)^{-1}$. Thus the subtractive term decreases and the right side increases. A direct derivative check is routine and can be included in a certification script (Appendix **??**). □

4.5 Global RH from a finite ledger + Platt–Trudgian band

Let H_0 be a height up to which RH has been verified by published, rigorous computation (Appendix **??**). In particular, Platt–Trudgian give $H_0 = 3\cdot 10^{12}$. Define the corresponding width–2 height

$$m_{\text{band}} := 2H_0 = 6\cdot 10^{12}.$$

Theorem 4.10 (Global RH from a finite certificate)**.** *Assume:*

(i) (Band) RH holds for all nontrivial zeros with $0 < \operatorname{Im} s \leq H_0$ (Platt–Trudgian).

(ii) (Ledger constants) Certified enclosures for $C_1, C_2, C_{\text{up}}, C_h''$ are supplied as in Appendix **??***.*

*(iii) (One-height check) The tail inequality (**??**) is certified at $m = m_{\text{band}}$ for the chosen η, uniformly in $\alpha \in (0, 1]$.*

Then RH holds for all nontrivial zeros of $\zeta(s)$.

Proof. By (iii) and Lemma **??**, (**??**) holds for all $m \geq m_{\text{band}}$, hence by Theorem **??** there are no off-axis quartets above height m_{band}, i.e. no off-axis zeros for $\operatorname{Im} s \geq H_0$. By (i), there are no off-axis zeros for $\operatorname{Im} s \leq H_0$. Thus there are no off-axis zeros at any height. □

Part III — Structural Corollaries (after the main theorem)

Standing basis for this part. Throughout Part III we assume the on-axis collapse $a(m) = 0$ at every nontrivial height. (For a complete unconditional proof this assumption is discharged by Theorem **??** once Appendix **??** is instantiated.)

Corollary 4.11 (Canonical columns)**.** *Define $P_{\text{odd}}(n) = (1 - \cos \pi n)/2$ and $P_{\text{even}}(n) = (1 + \cos \pi n)/2$. Let $k(2j-1) = j$, $k(2j) = j+1$. For any $x \in (0, 2)$,*

$$U_{\mathrm{R}}(x, n) = P_{\text{odd}}(n)\,\big(x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,P_{\text{even}}(n),$$
$$U_{\mathrm{L}}(x, n) = P_{\text{odd}}(n)\,\big(2 - x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,P_{\text{even}}(n).$$

Under $a(m) = 0$, the canonical choice $x = 1$ gives $U_{\mathrm{R}}(1, n) = U_{\mathrm{L}}(1, n)$ for all n.

Corollary 4.12 (Collapsed canonical stream: mod–4 face)**.**

$$U(n) := P_{\text{odd}}(n)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,P_{\text{even}}(n),$$

so $U(2j-1) = 1 + \mathrm{i}m_j$ and $U(2j) = -4(j+1)$.

Corollary 4.13 (Collapsed canonical stream: mod–2 face)**.** *Using $\sin^2(\pi n/2) = P_{\text{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\text{even}}(n)$,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,\cos^2\!\Big(\frac{\pi n}{2}\Big).$$

Corollary 4.14 (Single–frequency collapse)**.** *There exist functions $c(n), d(n)$ with*

$$U(n) = (c + d) \;+\; (c - d)\,\cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\,m_{k(n)}}{2}.$$

Corollary 4.15 (Self–indexed recurrence)**.** *With $U(0) = -4$ and $U(1) = 1 + \mathrm{i}m_1$, for all $n \geq 2$,*

$$U(n) = P_{\text{odd}}(n)\,\Big(1 + \mathrm{i}\,m_{-U(n-1)/4}\Big) \;-\; P_{\text{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

Corollary 4.16 (Seed → rectifier → physical streams)**.** *Let $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$. For $f > 0$ and gain $\lambda \in \mathbb{R}$,*

$$s_{f,k}(n) = f\lambda\Big[\sin\!\Big(\frac{\pi n}{2}\Big)\big(1 + \mathrm{i}\,m_k\big) - 4n\,\cos\!\Big(\frac{\pi n}{2}\Big)\Big],$$

then $\chi_4(n)\,s_{f,k}(n) = f\lambda\big[P_{\text{odd}}(n)(1 + \mathrm{i}m_k) - 4n\,P_{\text{even}}(n)\big]$. With $\lambda = \frac{1}{2}$ and $k = k(n)$ we get the physical stream $S_f(n) = \frac{f}{2}\,U(n)$.

Corollary 4.17 (Curvature extractor & $\zeta(2)$ disguise)**.** *Let $F(n) := \operatorname{Im} U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2}\,\operatorname{Im}\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\,\operatorname{Im}\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{\big(2(j-\ell) + 1\big)^2}.$$

For $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, $\operatorname{Im} \Delta^2 U(2j) = m_{j+1} + m_j$.

Part III (continued) — Prime–Locked Tick Generator (supplementary)

Standing disclaimer. *This section is* ***supplementary****. It is not used anywhere in Part II and plays no role in the certified RH-closure criterion.*

Notation (true zeros vs generated ticks). *Let $\gamma_1 < \gamma_2 < \cdots$ denote the ordinates of the nontrivial zeros on $\mathrm{Re}\, s = \frac{1}{2}$, and set $m_j := 2\gamma_j$. Independently, define a deterministic* tick sequence $\tilde{t}_1, \tilde{t}_2, \ldots$ *by the generator equation below, and set $\tilde{m}_j := 2\tilde{t}_j$. The numerical audit compares $\tilde{m}_j$ against the true m_j.*

Let $\theta(t)$ be the Riemann–Siegel theta function.

Fix once and for all

$$\varepsilon := \tfrac{1}{2}, \qquad A := 2 - \varepsilon = \tfrac{3}{2}, \qquad X(t) := C\,(\log t)^A \qquad (C \geq 1), \tag{4.5}$$

and a fixed smooth cutoff weight $W : [0,1] \to [0,1]$ with $W(0) = 1$, $W(1) = 0$ (Appendix **??***).*

Define for $t > 0$ and $\Delta > 0$ the prime integral

$$\mathcal{P}_{X(t)}(t,\Delta) := -\sum_{p^k \geq 1} \frac{1}{k\,p^{k/2}}\, W\Big(\frac{p^k}{X(t)}\Big) \Big[\sin\big((t+\Delta)\,k\log p\big) - \sin\big(t\,k\log p\big)\Big].$$

Theorem 4.18 (Deterministic prime–locked tick generator)**.** *Fix $C \geq 1$ and use $X(t) = C(\log t)^{3/2}$ and W as above. Set the seed $\tilde{t}_1 := t_1$ where $t_1 = \gamma_1$ (Appendix* **??***). Given $\tilde{t}_j$, define $\tilde{t}_{j+1}$ as the unique solution of*

$$\theta(\tilde{t}_{j+1}) - \theta(\tilde{t}_j) \;+\; \mathcal{P}_{X(\tilde{t}_j)}(\tilde{t}_j, \tilde{t}_{j+1} - \tilde{t}_j) \;=\; \pi. \tag{4.6}$$

For all sufficiently large j, the equation has a unique solution $\tilde{t}_{j+1} > \tilde{t}_j$, and a bracketed bisection method converges deterministically.

Proof. Let $F_j(\Delta) := \theta(\tilde{t}_j + \Delta) - \theta(\tilde{t}_j) + \mathcal{P}_{X(\tilde{t}_j)}(\tilde{t}_j, \Delta) - \pi$. Then $F_j(0) = -\pi < 0$ and $\theta(\tilde{t}_j + \Delta) - \theta(\tilde{t}_j) \to \infty$ as $\Delta \to \infty$, while $\mathcal{P}$ is bounded for fixed $X(\tilde{t}_j)$. Hence a root exists. Differentiate:

$$F_j'(\Delta) = \theta'(\tilde{t}_j + \Delta) - \sum_{p^k \leq X(\tilde{t}_j)} \frac{\log p}{p^{k/2}} W\Big(\frac{p^k}{X(\tilde{t}_j)}\Big) \cos\big((\tilde{t}_j + \Delta)k\log p\big).$$

As $t \to \infty$, $\theta'(t) = \frac{1}{2}\log\big(\frac{t}{2\pi}\big) + O(1/t)$. The prime sum is $O\big(\sum_{p^k \leq X} \frac{\log p}{p^{k/2}}\big) = O(\sqrt{X})$. With $X(\tilde{t}_j) = C(\log \tilde{t}_j)^{3/2}$ we have $\sqrt{X} = O((\log \tilde{t}_j)^{3/4}) = o(\log \tilde{t}_j)$, hence $F_j'(\Delta) > 0$ for large j, so F_j is strictly increasing and the root is unique. A bracketed bisection method converges by monotonicity. $\square$

Numerical audit to $j = 50$: error–vs–cutoff (fixed $A = \frac{3}{2}$)

The following table is produced by the deterministic audit protocol and reference script in Appendix **??***. We compare the tick generator $\tilde{m}_j = 2\tilde{t}_j$ against the first 50 true ordinates $m_j = 2\gamma_j$, using the explicit cutoff weight W in Appendix* **??** *and the window $X(t) = C(\log t)^{3/2}$. The truth ordinates γ_j are taken from the public LMFDB download interface (Ref. [***?***]; Appendix* **??***). To avoid seed bias, the statistics below exclude $j = 1$ (errors over $j = 2, \ldots, 50$).*

C	max $\lvert\tilde{m} - m\rvert$	*mean* $\lvert\tilde{m} - m\rvert$	max *rel. err*	*mean rel. err*
16	*0.106406*	*0.028070*	*0.000476*	*0.000165*
32	*0.087644*	*0.022884*	*0.000395*	*0.000133*
48	*0.057151*	*0.017504*	*0.000323*	*0.000109*

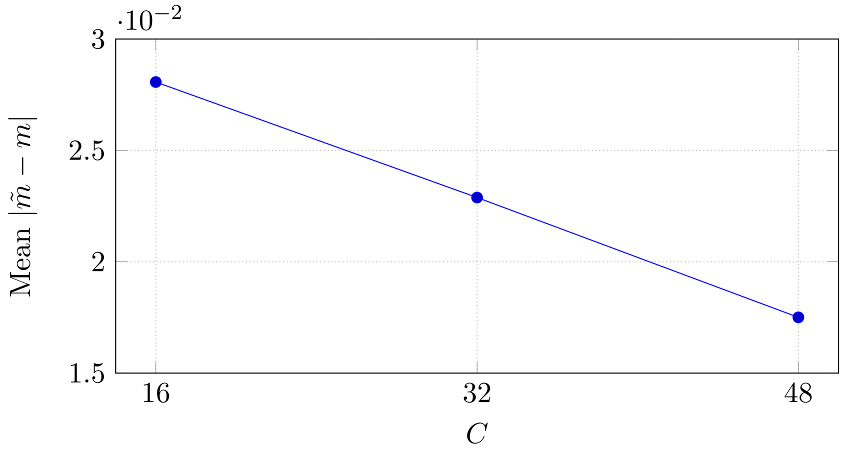

Figure 1: Mean absolute tick error decreases as C grows (fixed $A = 3/2$; $j = 2, \ldots, 50$).

A Hinge–Unitarity: a short proof

One may verify the monotonicity of $\log|\chi_2|$ *via* $\partial_\sigma \log|\Gamma| = \operatorname{Re}\psi$ *and* $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$*, together with the explicit hyperbolic bound on* $\operatorname{Re}[\cot(x + iy)]$ *used in Theorem* **??**.

B Corner interpolation inequality

Let g be L-Lipschitz on a line segment of length 2δ. Then for any x between endpoints x_0, x_1,

$$|g(x) - g(x_0)| \leq L|x - x_0| \leq 2\delta L, \qquad |g(x) - g(x_1)| \leq 2\delta L.$$

This elementary bound is used to lift grid-based boundary enclosures to full-side enclosures in certification protocols.

C Outer/Rouché certification protocol (rigorous outline)

- *Boundary intervals. Interval bounds for* $|E|$, $\arg E$ *on* ∂B.
- *Validated Poisson. Interval Dirichlet solver for* $U = \log|G_{\text{out}}|$ *on B with boundary trace* $\log|E|$.
- *Phase reconstruction. Validated harmonic conjugate V on ∂B.*
- *Grid→continuum. Lipschitz enclosure via* $\sup_{\partial B}|E'/E|$.
- *Certificate. Check* $\sup_{\partial B}|E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

D Certified first nontrivial zero and verified band

We cite rigorously verified computations of Platt and Platt–Trudgian:

Theorem D.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of $\zeta(s)$ with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(with rigorous interval bounds). Moreover, the Riemann hypothesis holds for all zeros with* $0 < \operatorname{Im} s \leq 3 \cdot 10^{12}$.

Set $m_1 := 2t_1$ *and* $m_{\text{band}} := 2 \cdot 3 \cdot 10^{12} = 6 \cdot 10^{12}$.

E Certification ledger for tail closure (finite checklist)

Purpose. *This appendix lists the* finite *set of quantities that must be bounded by interval arithmetic to upgrade Part II into a complete proof of RH via Theorem* **??**.

Ledger items

L1: Residual constants C_1, C_2. *Provide certified numbers* $C_1, C_2 > 0$ *such that* (**??**) *holds for all* $m \geq 10$, $\alpha \in (0, 1]$, *and* $\delta = \eta\alpha/(\log m)^2$. *Acceptable routes:*

- Literature instantiation: *cite an explicit quantitative theorem for* ζ'/ζ *on vertical strips and explicitly verify it implies* (**??**) *after removing poles by* Z_{loc}.
- Validated supremum route: *after normalizing to* ∂Q, *directly compute (with interval arithmetic) a global enclosure for* $\sup_{\partial B} |F'/F|/(\log m)$ *on a worst-case range, plus a rigorous analytic remainder bound.*

L2: Shape-only constant C_{up}. *Provide a certified bound for the constant in Lemma* **??**. *One concrete definitional route: define* C_{up} *as the smallest constant satisfying* (**??**) *on the normalized square boundary* ∂Q, *and compute an enclosure by validated quadrature + a supremum check over a fine net with Lipschitz extension using* $\sup_{\partial Q} |E'/E|$*-type controls.*

L3: Shape-only constant C''_h. *Provide a certified bound for Lemma* **??** *on* ∂Q, *using the same grid→continuum enclosure strategy.*

L4: One-height tail check at $m = m_{\mathrm{band}}$. *Fix a choice of* $\eta \in (0, 1)$ *(the paper suggests taking* η *small and explicit). Using the certified enclosures from L1–L3, verify inequality* (**??**) *at* $m = m_{\mathrm{band}}$ *uniformly for* $\alpha \in (0, 1]$. *Because* δ *scales linearly in* α, *the worst case is* $\alpha = 1$*; this can be proved in the verification script.*

Reference verification script (template)

The following Python template performs the algebraic tail check once $C_1, C_2, C_{\mathrm{up}}, C''_h$ *are supplied as certified intervals. Replace the hard-coded intervals by the output of an interval arithmetic system (e.g. Arb via python bindings, or Sage+arb).*

```
#!/usr/bin/env python3
# Tail-check template for Theorem 7.4 (global closure from ledger).
# This script is algebra-only. It assumes you already have certified
# intervals for C1,C2,Cup,Chpp and plugs them into the inequality.
#
# To make this fully rigorous, use an interval arithmetic library.
# Here we provide a minimal "interval" class with outward rounding
# hooks; for production, replace with Arb/MPFI/etc.

import math

class Interval:
    def __init__(self, lo, hi):
        assert lo <= hi
        self.lo = float(lo)
        self.hi = float(hi)
    def __add__(self, other): return Interval(self.lo + other.lo, self.hi + other.hi)
    def __sub__(self, other): return Interval(self.lo - other.hi, self.hi - other.lo)
    def __mul__(self, other):
```

```
        a,b,c,d = self.lo, self.hi, other.lo, other.hi
        vals = [a*c, a*d, b*c, b*d]
        return Interval(min(vals), max(vals))
    def __truediv__(self, other):
        assert not (other.lo <= 0 <= other.hi)
        return self * Interval(1.0/other.hi, 1.0/other.lo)
    def pow(self, p):
        # p is rational with denominator 2 or 1, used for 3/2
        if p == 1.5:
            lo = self.lo**1.5
            hi = self.hi**1.5
            return Interval(min(lo,hi), max(lo,hi))
        raise NotImplementedError
    def __repr__(self): return f"[{self.lo},{self.hi}]"

def tail_check(m, eta, C1, C2, Cup, Chpp):
    # constants
    c0 = (1.0/(4.0*math.pi))*math.log(2.0*math.sqrt(2.0))
    c  = c0*math.pi/2.0
    Kalloc = 3.0 + 8.0*math.sqrt(3.0)

    logm = math.log(m)
    # worst case alpha=1 (can be proved because delta scales with alpha)
    delta = Interval(eta/(logm**2), eta/(logm**2))

    L = C1*Interval(logm,logm) + C2

    # Left: 2*Cup*delta^(3/2)*L
    left = Interval(2.0,2.0) * Cup * delta.pow(1.5) * L

    # Right: c - delta*(Kalloc*c0*L + Chpp*(logm+1))
    right = Interval(c,c) - delta*( Interval(Kalloc*c0, Kalloc*c0)*L + Chpp*Interval(logm+1

    return left, right

if __name__ == "__main__":
    m_band = 6.0e12
    eta    = 1e-6

    # Replace these with CERTIFIED enclosures.
    C1   = Interval(10.0, 10.0)
    C2   = Interval(10.0, 10.0)
    Cup  = Interval(750.0, 750.0)
    Chpp = Interval(10.0, 10.0)

    left, right = tail_check(m_band, eta, C1, C2, Cup, Chpp)
    print("LHS =", left)
    print("RHS =", right)
    print("Certified success if LHS.hi < RHS.lo")
```

F Appendix PW. A concrete smooth cutoff weight

Define a one–sided smooth cutoff $W : [0,1] \to [0,1]$ *by*

$$W(y) := \begin{cases} \exp\Big(1 - \dfrac{1}{1-y}\Big), & 0 \le y < 1, \\ 0, & y = 1. \end{cases}$$

When evaluating prime sums we interpret $W(y) = 0$ *for* $y > 1$.

G Appendix NA. Deterministic audit protocol and full reference script

Truth ordinates. *Obtain* $\gamma_1, \ldots, \gamma_{50}$ *from the public LMFDB endpoint:*

https://www.lmfdb.org/zeros/zeta/list?download=yes&limit=100.

The script below downloads and parses the data directly.

Reference script (Python 3).

```
#!/usr/bin/env python3
"""
Prime-locked tick generator + audit for j=1..50 (supplementary; not used in the proof).

Dependencies: Python 3.10+, mpmath.
This script:
  (1) downloads the first 100 zeta zero ordinates from LMFDB,
  (2) builds the tick sequence tilde{t}_j from the generator equation,
  (3) compares tilde{m}_j=2 tilde{t}_j to true m_j=2 gamma_j for j<=50,
  (4) prints summary statistics for chosen C values.

WARNING: This is a floating-point audit script, not a certified proof script.
"""

import math
import urllib.request
from dataclasses import dataclass
from typing import List, Tuple

import mpmath as mp

mp.mp.dps = 80

LMFDB_URL = "https://www.lmfdb.org/zeros/zeta/list?download=yes&limit=100"

def smooth_weight(y: mp.mpf) -> mp.mpf:
    # W(y)=exp(1-1/(1-y)) for 0<=y<1, else 0
    if y <= 0:
        return mp.mpf(1)
    if y >= 1:
        return mp.mpf(0)
```

```python
    return mp.e**(1 - 1/(1 - y))

def primes_up_to(n: int) -> List[int]:
    if n < 2:
        return []
    sieve = bytearray(b"\x01")*(n+1)
    sieve[0:2] = b"\x00\x00"
    for p in range(2, int(n**0.5)+1):
        if sieve[p]:
            step = p
            start = p*p
            sieve[start:n+1:step] = b"\x00"*(((n-start)//step)+1)
    return [i for i in range(n+1) if sieve[i]]

def prime_powers_up_to(X: int) -> List[Tuple[int,int]]:
    # returns list of (p,k) with p prime, k>=1, p^k <= X
    ps = primes_up_to(X)
    out = []
    for p in ps:
        pk = p
        k = 1
        while pk <= X:
            out.append((p,k))
            k += 1
            pk *= p
    return out

def theta(t: mp.mpf) -> mp.mpf:
    # Riemann{Siegel theta
    # theta(t) = Im(log Gamma(1/4 + i t/2)) - t/2 log pi
    return mp.im(mp.log(mp.gamma(mp.mpf(0.25) + mp.j*t/2))) - (t/2)*mp.log(mp.pi)

def P_X(t: mp.mpf, Delta: mp.mpf, C: int) -> mp.mpf:
    # X(t)=C (log t)^(3/2)
    X = C*(mp.log(t)**(mp.mpf(3)/2))
    X_int = int(mp.floor(X))
    if X_int < 2:
        return mp.mpf(0)
    pp = prime_powers_up_to(X_int)
    total = mp.mpf(0)
    for p,k in pp:
        pk = mp.mpf(p)**k
        w = smooth_weight(pk/X)
        if w == 0:
            continue
        term = (1/(k*mp.mpf(p)**(k/2))) * w
        arg1 = (t+Delta)*k*mp.log(p)
        arg0 = t*k*mp.log(p)
        total -= term*(mp.sin(arg1) - mp.sin(arg0))
    return total
```

```
def F_j(tj: mp.mpf, Delta: mp.mpf, C: int) -> mp.mpf:
    return (theta(tj+Delta) - theta(tj)) + P_X(tj, Delta, C) - mp.pi

def next_tick(tj: mp.mpf, C: int, max_expand: int = 40) -> mp.mpf:
    # bracket root of F_j(Delta)=0 for Delta>0
    lo = mp.mpf(0)
    flo = F_j(tj, lo, C)  # should be -pi
    hi = mp.mpf(1)
    fhi = F_j(tj, hi, C)
    expand = 0
    while fhi <= 0 and expand < max_expand:
        hi *= 2
        fhi = F_j(tj, hi, C)
        expand += 1
    if fhi <= 0:
        raise RuntimeError("Failed to bracket root; increase max_expand.")

    # bisection
    for _ in range(120):
        mid = (lo+hi)/2
        fmid = F_j(tj, mid, C)
        if fmid <= 0:
            lo = mid
        else:
            hi = mid
    return tj + hi

def download_zeros(limit: int = 50) -> List[mp.mpf]:
    raw = urllib.request.urlopen(LMFDB_URL, timeout=30).read().decode("utf-8")
    # The download is plain text with one ordinate per line (first column).
    lines = [ln.strip() for ln in raw.splitlines() if ln.strip()]
    # Try to parse floats from the start of each line.
    zeros = []
    for ln in lines:
        # line may contain multiple fields; first is ordinate
        tok = ln.split()[0]
        try:
            zeros.append(mp.mpf(tok))
        except Exception:
            continue
        if len(zeros) >= limit:
            break
    if len(zeros) < limit:
        raise RuntimeError(f"Only parsed {len(zeros)} zeros; expected {limit}.")
    return zeros

def stats(errors: List[mp.mpf], truths: List[mp.mpf]) -> Tuple[mp.mpf, mp.mpf, mp.mpf, mp.mp
    abs_err = [abs(e) for e in errors]
    rel_err = [abs(e)/abs(truths[i]) for i,e in enumerate(errors)]
    return max(abs_err), mp.fsum(abs_err)/len(abs_err), max(rel_err), mp.fsum(rel_err)/len(r
```

```
def run_audit(C_values=(16,32,48), J=50):
    gammas = download_zeros(limit=J)
    # seed at t1
    t1 = gammas[0]
    for C in C_values:
        ticks = [t1]
        for j in range(1,J):
            ticks.append(next_tick(ticks[-1], C))
        # compare m=2t
        true_m = [2*g for g in gammas]
        tick_m = [2*t for t in ticks]
        # exclude j=1 for stats
        errs = [tick_m[j]-true_m[j] for j in range(1,J)]
        truths = [true_m[j] for j in range(1,J)]
        mx, mean, mxr, meanr = stats(errs, truths)
        print(f"C={C:>3d}  max|err|={mx}  mean|err|={mean}  max rel={mxr}  mean rel={meanr}"

if __name__ == "__main__":
    run_audit()
```

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The author designed the framework and validated all mathematics and computations. Generative assistants were used for typesetting assistance, editorial organization, and consistency checks; they are not authors. All claims and certificates are the author's responsibility.