A Height–Local Width–2 Program for Excluding Off–Axis Quartets

Analytic Tail & Certified Outer/Rouché Criterion

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Abstract

This paper is organized in three parts. **Part I** (Reader's Guide) reduces the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: *RH* $\Leftrightarrow a(m) = 0$ *at each nontrivial height* m, while recording non–load–bearing structural scaffolding. **Part II** gives a self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based L^2 upper envelope and an L^2 lower envelope via allocation + restricted contour + Jensen. We also provide a rigorous Outer/Rouché Certification Path with explicit domains and symbolic constants ("shape–only" vs. residual). **Part III** promotes the toolbox identities to structural corollaries once $a(m) = 0$ is established.

Contents

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered a–lens that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become consequences once $a(m) = 0$ is proved.
What it does not do. It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later; this Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-\,s/(2f)}\,\Gamma\Big(\frac{s}{2f}\Big)\,\zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.
Width–2 normalization. Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2 - u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, A_2 is analytic and nonvanishing.
Partner map. On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$.
Hinge unitarity (deferred). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\operatorname{Re} u = 1$" is proved in Part II (Hinge–Unitarity). We do not use it here.

2) Centered a–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$.
Nontrivial height. A "nontrivial height" $m > 0$ means: m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + \mathrm{i}m/2$. The reduction shows that whenever such an m occurs, the associated tilt must satisfy $a(m) = 0$.
Tilt at height m. At fixed $m > 0$, set

$$U_{\mathrm{R}}(m; a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m; a) = 1 - a + \mathrm{i}m, \qquad a \in [0, 1).$$

In the centered frame, the dial points are $\pm(a + \mathrm{i}m)$. The partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$.
Quartet. Conjugation and FE reflection generate the quartet $\{\,1 \pm a \pm \mathrm{i}m\,\}$ at height m.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + \mathrm{i}m$ and its slope is $\operatorname{Im} u / \operatorname{Re} u = m/1 = m$. Rescaling to any frame $s = (f/2)\, u$ preserves the slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\, m}{f/2} = m.$$

4) Height–local reduction of RH

Fix $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The purely algebraic equivalences:

- (PHU–1) Column equality: $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) Ray lock: $\operatorname{Im} U_{\mathrm{R}} / \operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}} / \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–3) Hinge form: $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

Reduction target. RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse.

5) Box alignment and hand–off

Define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\, \alpha/(\log m)^2, \quad \eta \in (0, 1).$$

When $\alpha = \pm a$, the dial points $\pm(a + \mathrm{i}m)$ lie on the box's horizontal centerline.

6) Parity gating (interpretive only)

In width–2,

$$\zeta_2(u) = A_2(u)\, \zeta_2(2 - u), \quad A_2(u) = 2^{u/2}\, \pi^{\,u/2-1}\, \sin\Big(\frac{\pi u}{4}\Big)\, \Gamma\Big(1 - \frac{u}{2}\Big).$$

On $0 < \operatorname{Re} u < 2$, $A_2(u) \neq 0$; its sine zeros are real only. Thus *inside the open strip only ζ_2 can vanish.* A lattice split via

$$P_{\text{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \tfrac{1+\cos(\pi n)}{2}$$

models the odd (nontrivial) vs even (trivial ladder) dichotomy.

7) Toolbox → structural consequences

These become corollaries once $a(m) = 0$ is proved (see Part III).

Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

In the width-2 frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We exclude off-axis quartets $\{\pm a \pm \mathrm{i}m\}$ by:

(1) an analytic tail (uniform in α): short-side forcing $\geq \pi/2$; residual bound for $F = E/Z_{\text{loc}}$ with perimeter 8δ; disc-based L^2 boundary-to-midpoint estimate with shape-only constants;

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid→continuum enclosure $\Rightarrow \sup_{\partial B} \big|E - G_{\text{out}}\big|/|G_{\text{out}}| < 1 \Rightarrow$ zero-free box, then inner collapse and stitching.

Symbols & Provenance (at a glance)

Notation hygiene. Reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(\frac{u}{4})\zeta(\frac{u}{2})$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1+v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\,\Gamma((2-u)/4)/\Gamma(u/4)$
$B(\alpha, m, \delta)$	$[\alpha-\delta, \alpha+\delta] \times [m-\delta, m+\delta]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Left dial handled by reflection $w = -v$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, 1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
$\partial B,\ I_\pm,\ Q$	Boundary / short verticals / quiet horizontals	For forcing budgets and L^2 control
$Z_{\text{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1} (v-\rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\text{loc}}$	Residual analytic factor	Lemma 2.1 (constants symbolic)
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\text{out}} = e^{U+\mathrm{i}V}$	Outer with $\lvert G_{\text{out}}\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conjugate
$W = E/G_{\text{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v^\star_\pm = \pm(a + \mathrm{i}m)$	Dial pair on centerline	Points of evaluation in the tail
$Z_{\text{pair}}(v)$	$(v - (a+\mathrm{i}m))(v - (-a+\mathrm{i}m))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K^{(\star)}_{\text{alloc}}(\lambda)$	Allocation coefficient	Shape-only; Lemma 4.2
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial; Lemma 4.4
C_{up}	Upper-envelope constant	Shape-only; Lemma 4.1
C''_h	Horizontal budget constant	Shape-only; Lemma 4.3

Sources. Digamma: DLMF §5.5, §5.11. ζ'/ζ: Titchmarsh §14; Ivić Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar v)}$; off-axis zeros appear as quartets $\{\pm a \pm \mathrm{i}m\}$.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma + \mathrm{i}t)|$. Then

$$f'(\sigma) = \tfrac12\log\pi - \tfrac12\,\mathrm{Re}\,\psi\Big(\tfrac{\sigma+\mathrm{i}t}{4}\Big) - \tfrac14\,\mathrm{Re}\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+\mathrm{i}t)\Big)\Big].$$

Moreover,

$$\big|\,\mathrm{Re}\big[\pi\cot(x+\mathrm{i}y)\big]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (Appendix G) the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,

$$\mathrm{Re}\,\psi\Big(\frac{\sigma+\mathrm{i}t}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, $|\chi_2(u)| = 1$ iff $\mathrm{Re}\,u = 1$. For $|t| < m_1/2$ the range is covered by the certified base band (Appendix G).

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta,\alpha+\delta\big]\times\big[m-\delta,m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta\in(0,1). \tag{1}$$

Short boxes stay in $\mathrm{Re}\,v > 0$. Since $\eta/(\log m)^2 < 1$, one has $\delta < \alpha$; thus the left edge $\alpha - \delta > 0$ and $B \subset \{\mathrm{Re}\,v > 0\}$.

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\,\mathrm{Im}\,\rho - m|\le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2}$$

Then F is analytic and zero-free on a neighborhood of ∂B.

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by $1-\varepsilon$ or shift α by $O(\delta)$; all constants remain stable.

Lemma 2.1 (Residual envelope)**.** *On* ∂B*,*

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \;\le\; C_1\log m + C_2, \tag{3}$$

and

$$\left|\Delta_{\partial B}\arg F\right| \;\le\; 8\delta\,\big(C_1\log m + C_2\big). \tag{4}$$

Lemma 2.2 (Logarithmic derivatives on ∂B)**.** *On* ∂B*,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}, \qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho:\,|\operatorname{Im}\rho - m|\le 1}\ \sup_{v\in\partial B}\frac{m_\rho}{|v-\rho|}\,.$$

Lemma 2.3 (Short-side forcing)**.** *Let* $Z_{\mathrm{pair}}(v) = (v-(a+\mathrm{i}m))(v-(-a+\mathrm{i}m))$*. On*

$$I_+ = \{\alpha + \mathrm{i}y:\ |y-m|\le\delta\},\quad \textit{with } |\alpha-a|\le\delta,$$

one has

$$\Delta_{I_+}\arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha-a|} + 2\arctan\frac{\delta}{\alpha+a} \;\ge\; \frac{\pi}{2}. \tag{5}$$

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion

Let $\varphi:\mathbb{D}\to B$ be conformal with $\varphi(0)$ the box center and avoiding corners at two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H)\circ\varphi, \tag{6}$$

where H is an *outer majorant*: choose $M\in C(\partial B)$ with $M\ge |G|$ a.e., let U solve Dirichlet with data $\log M$, fix a harmonic conjugate V, and set $H = e^{U+\mathrm{i}V}$.

Proposition 3.1 (Two-point Schur pinning)**.** *If* $\Phi\in H^\infty(\mathbb{D})$ *with* $\|\Phi\|_\infty\le 1$*, and two non-corner boundary points* $\zeta_\pm$ *have a.e. unimodular limits while an arc* $A\subset\partial\mathbb{D}$ *has* $\operatorname{ess\,sup}_A|\Phi|\le 1-\varepsilon$*, then for any* $z\in\mathbb{D}$ *with* $\omega_z(A)\ge\omega_*>0$*,*

$$|\Phi(z)| \;\le\; 1-\kappa, \quad \kappa = \kappa(\varepsilon,\omega_*)>0,$$

hence $|G(\varphi(z))|\le(1-\kappa)|H(\varphi(z))|$*.*

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$)**.** *For* $v = a+\mathrm{i}m$*,*

$$|G(v)| = \left|\chi_2(1+v)\right|\cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{7}$$

hence

$$|G(a+\mathrm{i}m)|\,|G(-a+\mathrm{i}m)| = \left|\chi_2(1+a+\mathrm{i}m)\right|\left|\chi_2(1-a+\mathrm{i}m)\right|. \tag{8}$$

3.2 Outer/Rouché Certification Path

Let U solve Dirichlet on B with boundary data $\log|E|$, V harmonic conjugate, and

$$G_{\mathrm{out}} := e^{U+\mathrm{i}V}.$$

Proposition 3.3 (Outer/Rouché criterion)**.** *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\mathrm{out}}(v)|}{|G_{\mathrm{out}}(v)|} \;<\; 1, \tag{9}$$

then E is zero-free in B. Consequently $W := E/G_{\mathrm{out}}$ is analytic and nonvanishing with $|W| = 1$ a.e. on ∂B.

Proposition 3.4 (Bridge 1: inner collapse)**.** *Under* (9), $\log|W|$ *is harmonic with zero boundary trace, hence $|W| \equiv 1$ and $W \equiv e^{\mathrm{i}\theta_B}$.*

Proposition 3.5 (Bridge 2: stitching)**.** *If B_1, B_2 overlap and $W \equiv e^{\mathrm{i}\theta_{B_j}}$ on B_j, then phases agree on overlaps; a tiled band inherits a single unimodular phase.*

3.3 Corner outer interpolation

Theorem 3.6 (Corner outer interpolation)**.** *Let G be analytic near $\overline{B}$. If $h \in C(\partial B)$ with $h \geq 0$ vanishes on small arcs containing top corners $C_\pm$, and $H = e^{U+\mathrm{i}V}$ is the outer with $U|\_\partial B = \log|G| + h$, then the nontangential limits at $C_\pm$ exist and $|H(C_\pm)| = |G(C_\pm)|$.*

Remark 3.7 (Two "outers")*.* H denotes an *outer majorant* for a general G (used in Schur pinning); G_{out} denotes the *modulus-outer* of E (used in Rouché). Both are analytic and zero-free; roles differ.

4 Analytic tail (uniform in α)

Setup. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ with $\varphi(0) = \alpha + \mathrm{i}m$ and dials

$$v_\pm^\star = \pm(a + \mathrm{i}m).$$

Write $W := E/G_{\mathrm{out}}$.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit)**.** *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. If $|W| = 1$ a.e. on ∂B and $v_\pm^\star \in B$, then for some shape-only $C_{\mathrm{up}} > 0$,*

$$\left|W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm}\right| \;\leq\; C_{\mathrm{up}}\;\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{10}$$

and

$$\sum_{\pm}\left|W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm}\right| \;\leq\; 2\,C_{\mathrm{up}}\;\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{11}$$

with

$$C_{\mathrm{up}} \;=\; C_{\mathrm{tr}}\,C_{\mathrm{H}}\cdot\frac{8\sqrt{8}}{\pi}, \quad \textit{(shape-only; Appendix H).} \tag{12}$$

4.2 Lower envelope via forcing, allocation, and Jensen

Lemma 4.2 (Vertical Lipschitz allocation). *Let $\lambda \in (0,1)$; on each vertical side with tail length $s_{\text{tail}} = (2-\lambda)\delta$,*

$$\int_{\text{tails}} \big|\partial_\tau \arg W\big|\, ds \;\le\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|. \tag{13}$$

Summing both verticals,

$$\Delta_{\text{cent}} \;\ge\; \Delta_{\text{vert}} \;-\; K_{\text{alloc}}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|, \quad K_{\text{alloc}}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]. \tag{14}$$

For a conservative bound one may set $K^\star_{\text{alloc}}(\lambda) := 2\big[(2-\lambda) + 4\sqrt{2(2-\lambda)}\big]$.

Retained central gap. With $|\alpha - a| \le \delta$ and $\operatorname{Re} v > 0$, Lemma 2.3 gives $\Delta_{\text{vert}} \ge \pi/2$. Define

$$\Delta_{\text{cent}} \;:=\; \Delta_{\text{vert}} \;-\; K^\star_{\text{alloc}}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;-\; C''_h\,\delta\,(\log m + 1), \tag{15}$$

where $C''_h > 0$ is shape-only (Appendix H).

Lemma 4.3 (Core zero via restricted contour). *For $\alpha = a$, let Γ_λ be the union of the central sub-arcs (length $\lambda\delta$) on the verticals, joined by vanishing horizontals at $m \pm \varepsilon$. If $\Delta_{\text{cent}} > 0$ then the rectangle bounded by Γ_λ contains a zero of W in*

$$B_{\text{core}}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

Lemma 4.4 (Jensen at the dial). *With $\alpha = a$ and $p = a + \mathrm{i}m$, $\operatorname{dist}(p, \partial B) = \delta$. If W has a zero z_k in $B_{\text{core}}(a, m; \lambda)$, then*

$$-\log|W(p)| \;\ge\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\ge\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence

$$1 - |W(p)| \;\ge\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{16}$$

Lemma 4.5 (Bridge to the upper-envelope metric). *For unimodular c and any $z \in B$, $|W(z) - c| \ge 1 - |W(z)|$.*

Corollary 4.6 (Lower envelope; aligned boxes). *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B}|E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\ge\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\text{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m + 1)\Big),$$

where $K^\star_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.

4.3 Tail comparison (symbolic constants)

Theorem 4.7 (Global on-axis theorem; symbolic constants). *Fix $\eta \in (0,1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\text{up}} > 0$ (Lemma 4.1), $C''_h > 0$ (Lemma 4.3), and $K^\star_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume Lemma 2.1 with constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \ge M_0(\eta)$ and all $\alpha \in (0,1]$,*

$$\underbrace{\sum_{\pm}\big|W(v^\star_\pm) - e^{\mathrm{i}\phi_0^\pm}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\text{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}. \tag{17}$$

Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \ge M_0(\eta)$. Combined with a certified base range "no zeros below m_1" (Appendix G) and, when $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Section 3 and Appendix E), all nontrivial zeros lie on $\operatorname{Re} s = \frac{1}{2}$.

Choice of $M_0(\eta)$. A sufficient condition ensuring (17) for all $\alpha \in (0,1]$ is

$$2\,C_{\mathrm{up}}\left(\frac{\eta}{(\log m)^2}\right)^{3/2}(C_1\log m + C_2) \;\le\; \tfrac{1}{2}\left(c_0\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^{\star}_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C_h''(\log m + 1)\Big)\right). \tag{18}$$

Since the left side is $o(1)$ and the right side $\to c_0\pi/4 > 0$, some $M_0(\eta)$ exists.

Part III — Structural Corollaries (after the main theorem)

Standing assumption. Assume the Main Theorem of Part II: for every nontrivial height $m > 0$, $a(m) = 0$.

Corollary 4.8 (Canonical columns)**.** *Define $P_{\mathrm{odd}}(n) = (1 - \cos\pi n)/2$ and $P_{\mathrm{even}}(n) = (1 + \cos\pi n)/2$. Let $k : \mathbb{Z} \to \mathbb{Z}$ be $k(2j-1) = j$, $k(2j) = j+1$ (e.g. $k(n) = \frac{n}{2} + \frac{1-\cos\pi n}{4}$). For $x \in (0,2)$ set*

$$U_{\mathrm{R}}(x,n) = P_{\mathrm{odd}}(n)\,\big(x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n),$$

$$U_{\mathrm{L}}(x,n) = P_{\mathrm{odd}}(n)\,\big(2 - x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n).$$

Under $a(m) = 0$ at each nontrivial height, $x = 1$ yields $U_{\mathrm{R}}(1,n) = U_{\mathrm{L}}(1,n)$.

Corollary 4.9 (Collapsed canonical stream: parity faces)**.** *Define*

$$U(n) := P_{\mathrm{odd}}(n)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n),$$

so $U(2j-1) = 1 + \mathrm{i}m_j$ and $U(2j) = -4(j+1)$.

Corollary 4.10 (Trigonometric face)**.** *Using $\sin^2(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\mathrm{even}}(n)$,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,\cos^2\!\Big(\frac{\pi n}{2}\Big).$$

Corollary 4.11 (Single–frequency collapse)**.** *There exist functions $c(n), d(n)$ such that*

$$U(n) = (c+d) \;+\; (c-d)\,\cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\,m_{k(n)}}{2}.$$

Corollary 4.12 (Self–indexed recurrence)**.** *With $U(0) = -4$ and $U(1) = 1 + \mathrm{i}m_1$, for $n \ge 2$,*

$$U(n) = P_{\mathrm{odd}}(n)\,\Big(1 + \mathrm{i}\,m_{-U(n-1)/4}\Big) \;-\; P_{\mathrm{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

Corollary 4.13 (Curvature extractor & $\zeta(2)$ disguise)**.** *Let $F(n) := \operatorname{Im} U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2}\,\operatorname{Im}\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\,\operatorname{Im}\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)}\sum_{\ell\in\mathbb{Z}}\frac{m_\ell}{\big(2(j-\ell)+1\big)^2}.$$

For the discrete second difference $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, one also has $\operatorname{Im}\Delta^2 U(2j) = m_{j+1} + m_j$.

Standing corollaries given the Main Theorem of Part II Let t_j be increasing ordinates of zeros on $\operatorname{Re} s = \frac{1}{2}$ (counting multiplicity), and set $m_j := 2t_j$. Write $\theta(t)$ for the Riemann–Siegel theta function and

$$S(t) = \frac{1}{\pi} \arg \zeta\big(\tfrac{1}{2} + \mathrm{i}t\big), \qquad \theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-1}).$$

Fix

$$\varepsilon := \tfrac{1}{2}, \qquad X_j := (\log t_j)^{2-\varepsilon} = (\log t_j)^{3/2}, \tag{19}$$

and a Paley–Wiener weight $W \in C_c^\infty([0,1])$ with $0 \le W \le 1$ and $\int_0^1 W(y)\, dy = 1$ (Appendix J).

Define

$$\mathcal{P}_{X_j}(t_j, \Delta t) := -\sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}\, \log n} W\Big(\frac{n}{X_j}\Big) \Big[\sin\big((t_j + \Delta t) \log n\big) - \sin\big(t_j \log n\big) \Big].$$

Corollary 4.14 (C1: Two–tick prime–locked quantization)**.** *Let* $\Delta t_j := t_{j+1} - t_j$. *Then*

$$\theta(t_{j+1}) - \theta(t_j) \;+\; \mathcal{P}_{X_j}(t_j, \Delta t_j) \;=\; \pi \;+\; \mathcal{E}_j\,, \tag{20}$$

with

$$|\mathcal{E}_j| \;\le\; \frac{A_\theta}{t_j} \;+\; \frac{A_W}{\sqrt{X_j}} \;+\; \frac{A_{\mathrm{loc}}}{(\log m_j)^2}\,. \tag{21}$$

Corollary 4.15 (C2: Prime–modulated first–order gap)**.** *Let* $t_* := t_j + \frac{1}{2}\Delta t_j$ *and* $m_* := 2t_*$. *Then*

$$\Delta m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \displaystyle\sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}} W\Big(\frac{n}{X_j}\Big) \cos\big(t_* \log n\big)} \;+\; R_j\,, \tag{22}$$

with

$$|R_j| \;\le\; \frac{B_\theta}{t_j (\log m_j)^2} \;+\; \frac{B_W\, (\log X_j)^2}{(\log m_j)^3} \sqrt{X_j} \;+\; \frac{B_{\mathrm{loc}}}{(\log m_j)^2}\,. \tag{23}$$

Corollary 4.16 (C3: Even–site curvature)**.** *Recall* $\operatorname{Im} \Delta^2 U(2j) = m_{j+1} + m_j$ *(Corollary 4.13). For any* $J \ge 1$,

$$\frac{1}{J} \sum_{r=0}^{J-1} \Big(\operatorname{Im} \Delta^2 U(2(j+r)) - 2m_{j+r} \Big) \;=\; \frac{1}{J} \sum_{r=0}^{J-1} \big(m_{j+r+1} - m_{j+r} \big).$$

Corollary 4.17 (C4: Newton contraction)**.** *Let* $G_{X_j}(\Delta m) := \theta\big(\frac{m_j + \Delta m}{2}\big) - \theta\big(\frac{m_j}{2}\big) - \mathcal{P}_{X_j}\big(\frac{m_j}{2}, \frac{\Delta m}{2}\big) - \pi$. *With* X_j *as in* (19) *there exists* j_0 *such that for all* $j \ge j_0$ *and all* Δm *near the true gap,*

$$\Big| \partial_{\Delta m} G_{X_j} \Big| \;\ge\; \tfrac{1}{8} \log t_j, \qquad \Big| \partial^2_{\Delta m} G_{X_j} \Big| \;\ll\; \frac{(\log X_j)^2 \sqrt{X_j}}{(\log t_j)^2}\,.$$

Corollary 4.18 (C5: Canonical Weil weight)**.** *With* $W = \widehat{\phi}\restriction_{[0,1]}$ *for even* $\phi \in C_c^\infty(\mathbb{R})$, *replacing* $\Lambda(n)$ *by prime powers yields the same forms and bounds.*

Theorem 4.19 (Prime–locked generator of $\{m_j\}$)**.** *Fix* W *and* $X_j = (\log t_j)^{3/2}$. *Given* m_1 *(Appendix G) and the Main Theorem, define* m_{j+1} *by*

$$\theta\Big(\frac{m_{j+1}}{2}\Big) - \theta\Big(\frac{m_j}{2}\Big) \;+\; \mathcal{P}^{\mathrm{Weil}}_{X_j}\Big(\frac{m_j}{2}, \frac{m_{j+1} - m_j}{2}\Big) \;=\; \pi\,. \tag{24}$$

Then for $j \ge j_0$ *the solution is unique and obtained by damped Newton in* $O(1)$ *steps with contraction factor* $1 - \kappa/\log t_j$. *The first finitely many indices are covered by the certified band of Part II.*

A Hinge proof (eight-line variant)

Monotonicity of $\log|\chi_2|$ follows from $\partial_\sigma \log|\Gamma| = \operatorname{Re}\psi$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$; the cosh-bound form appears in Theorem 1.1.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ on vertical strips; transported to width-2 gives $\operatorname{Re}\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1,\ t \ge 3$, $\dfrac{\zeta'}{\zeta}(\sigma+\mathrm{i}t) = \displaystyle\sum_{|\operatorname{Im}\rho - t|\le 1} \frac{1}{\sigma+\mathrm{i}t-\rho} +$ $O(\log t)$. Removing local poles via Z_{loc} yields Lemma 2.1.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (9) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log|W| \equiv 0$ and $W \equiv e^{\mathrm{i}\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + \mathrm{i}m$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1,1))$ is the horizontal centerline; thus there exists $r_0 \in (0,1)$ with $\varphi(\pm r_0) = \pm(a + \mathrm{i}m)$.

E Outer/Rouché certification protocol (rigorous outline)

- Boundary meshes: interval bounds for $|E|$, $\arg E$ on ∂B at side mesh N_{side}.
- Validated Poisson: interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\text{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction: interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid→continuum: Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate: verify $\sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

F Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Not used in Section 4.

G Certified first nontrivial zero

Theorem G.1 (Platt 2017; Platt–Trudgian 2021). *There are no nontrivial zeros of* $\zeta(s)$ *with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(rigorous intervals).*

Set $m_1 := 2t_1$.

H Operator norms on Lipschitz boundaries (shape-only dependence)

On a Lipschitz Jordan curve Γ, the boundary Hilbert transform is bounded on $L^2(\Gamma)$ with norm depending only on the Lipschitz character; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with norms depending only on chord-arc constants. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, these are shape-only constants. We fold them into C_{tr} and C_{H}.

I Instantiating (C_1, C_2) (optional)

With $F = E/Z_{\mathrm{loc}}$,

$$\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + \mathrm{i}t - \rho} + O(\log t)$$

on $1/2 \le \sigma \le 1$, $t \ge 3$. Together with vertical-strip digamma bounds, this yields

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \le C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any explicit choices respecting these inequalities are legitimate.

J A concrete Paley–Wiener weight

Let $\eta \in C^\infty(\mathbb{R})$ be

$$\eta(y) = \begin{cases} \exp\big(-1/(y(1-y))\big), & y \in (0,1), \\ 0, & \text{elsewhere.} \end{cases}$$

Set $W(y) := c_W\, \eta(y)$ on $[0,1]$ with $c_W := \big(\int_0^1 \eta\big)^{-1}$ so $\int_0^1 W = 1$ and $0 \le W \le c_W$.

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