

A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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Abstract

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

- (1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\text{loc}}$ with perimeter factor 8δ ; and (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants (no strip/rectangle density comparison);
- (2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid→continuum enclosure $\Rightarrow \sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed $\eta \in (0, 1)$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α . Combined with a certified base range below m_1 (first nontrivial height in width-2), this yields the global on-axis theorem. All constants appearing in the upper/lower envelope are *shape-only* (independent of m , α , a); residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \rightarrow B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s, v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u)$	Completed object	Standard; FE for Λ_2 ; width-2 transport
$\pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right)$		
$E(v) = \Lambda_2(1 + v)$	Workhorse in v -plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2} \frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Uniform-in- α uses worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \frac{\eta \alpha}{(\log m)^2}, \eta \in (0, 1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
∂B	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
I_{\pm}	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	Controlled by L^2 trace & Hilbert
$Z_{\text{loc}}(v)$	Local zero/pole factors	De-singularizes E near ∂B
$\prod_{ \operatorname{Im} \rho - m \leq 1} (v - \rho)^{m_{\rho}}$		
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma ?? (constants symbolic)
$G(v) = \frac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\text{out}} = e^{U+iV}$	Outer with $ G_{\text{out}} = E $ on ∂B	$U = \log E \in C(\bar{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\text{out}}$	Inner quotient ($ W = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v_{\pm}^* = \pm(a + im)$	“Dial pair” on centerline	Points of evaluation in the tail
$Z_{\text{pair}}(v)$	$(v - (a + im))(v - (-a + im))$	Short-side forcing on I_+
Γ_{λ}	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_{λ}
$K_{\text{alloc}}^{(*)}(\lambda)$	Allocation coefficient	Shape-only; Lemma ??
$c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial (Lemma ??)
C_{up}	Upper-envelope constant	Shape-only; disc-based bound (Lemma ??)
C_h''	Horizontal budget constant	Shape-only; Lemma ??

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds). ζ'/ζ : Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right), \quad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$. These symmetries follow from $\Lambda_2(u) = \Lambda_2(2-u)$ and $\overline{\Lambda_2(\bar{z})} = \Lambda_2(z)$ on vertical strips, hence $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$ and conjugation invariance.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u) \zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2} \frac{\Gamma\left(\frac{2-u}{4}\right)}{\Gamma\left(\frac{u}{4}\right)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log |\chi_2(\sigma + it)|$. Then

$$f'(\sigma) = \frac{1}{2} \log \pi - \frac{1}{2} \Re \psi\left(\frac{\sigma + it}{4}\right) - \frac{1}{4} \Re \left[\pi \cot\left(\frac{\pi}{4}(\sigma + it)\right) \right].$$

Moreover,

$$|\Re[\pi \cot(x + iy)]| \leq \frac{\pi}{\cosh(2y) - 1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \geq m_1/2$ (with m_1 defined in Appendix ??) the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,

$$\Re \psi\left(\frac{\sigma + it}{4}\right) \geq \log\left(\frac{|t|}{4}\right) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on \mathbb{R} for all such t . Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\Re u = 1$. For $|t| < m_1/2$ no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix ??.

(Interpretive; non-load-bearing) Ω -continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1 + \lambda(u-1)$ preserve rays; $\tan \theta = \Im v / \Re v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \geq 10$, $\alpha \in (0, 1]$, and

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta \alpha}{(\log m)^2}, \quad \eta \in (0, 1). \quad (2.1)$$

Why $m \geq 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the ζ'/ζ expansions on $1/2 \leq \sigma \leq 1$, $t \geq 3$ (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \geq 3$ corresponds to $m \geq 6$; we take $m \geq 10$ for margin).

Why $\delta = \eta \alpha / (\log m)^2$. This balances the scale-free forcing ($\geq \pi/2$) against residual budgets $O(\delta \log m)$ and yields an L^2 + harmonic-measure upper envelope (in Section ??) that is uniformly small in α .

Lemma 2.1 (Short boxes stay in $\Re v > 0$). *For $m \geq 10$ and any $\eta \in (0, 1)$, one has $\delta < \alpha$ and $B(\alpha, m, \delta) \subset \{\Re v > 0\}$, uniformly in $\alpha \in (0, 1]$.*

Proof. Since $\eta \in (0, 1)$ and $\log m \geq \log 10 > 0$, we have $\eta / (\log m)^2 < 1$, hence $\delta = \alpha \eta / (\log m)^2 < \alpha$. Therefore the left edge is at $\alpha - \delta > 0$, so the entire box lies strictly in $\{\Re v > 0\}$, uniformly for $\alpha \in (0, 1]$. \square

De-singularization on ∂B . Let

$$Z_{\text{loc}}(v) = \prod_{\rho: |\operatorname{Im} \rho - m| \leq 1} (v - \rho)^{m_\rho}, \quad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (2.2)$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles within $|\operatorname{Im} \rho - m| \leq 1$ have been removed).

Boundary contact convention. If a zero/pole meets ∂B , shrink δ by a factor $1 - \varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (*residual envelope*, *short-side forcing*) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope). *On ∂B ,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2, \quad (2.3)$$

and

$$|\Delta_{\partial B} \arg F| \leq 8\delta (C_1 \log m + C_2). \quad (2.4)$$

Justification. *DLMF §5.11 controls ψ on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control ζ'/ζ on $1/2 \leq \sigma \leq 1$, $t \geq 3$. After removing local poles via (??) and transporting to width-2, we obtain (??). For (??), write $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F ds$ as the sum of side integrals (angular limits at the corners); then bound by $|\partial B| \sup_{\partial B} |F'/F| = 8\delta \sup |F'/F|$. The constants $C_1, C_2 > 0$ are absolute; we keep them symbolic (see Appendix ?? for an optional instantiation).*

Lemma 2.3 (Logarithmic derivatives on ∂B). *On ∂B ,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}, \quad \sup_{\partial B} \left| \frac{E'}{E} \right| \leq \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho: |\operatorname{Im} \rho - m| \leq 1} \sup_{v \in \partial B} \frac{m_\rho}{|v - \rho|}.$$

In particular, by the boundary-contact convention the right-hand side is finite.

Proof. The identity follows from $E = F Z_{\text{loc}}$. For the inequality, take suprema termwise and use $|\frac{(v-\rho)'}{v-\rho}| = \frac{1}{|v-\rho|}$. Finiteness holds since only finitely many ρ satisfy $|\operatorname{Im} \rho - m| \leq 1$ and none lie on ∂B after the contact adjustment. \square

Lemma 2.4 (Short-side forcing). *Let $Z_{\text{pair}}(v) = (v - (a + im))(v - (-a + im))$. On the near vertical*

$$I_+ = \{\alpha + iy : |y - m| \leq \delta\}, \quad \text{with } |\alpha - a| \leq \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan \frac{\delta}{|\alpha - a|} + 2 \arctan \frac{\delta}{\alpha + a} \geq \frac{\pi}{2}. \quad (2.5)$$

Proof. Along I_+ , $\arg(v - (\pm a + im)) = \arctan \frac{y-m}{\alpha \mp a}$. As y runs from $m - \delta$ to $m + \delta$, the increment is $\arctan \frac{\delta}{|\alpha - a|} - \arctan \left(-\frac{\delta}{|\alpha - a|} \right) = 2 \arctan \frac{\delta}{|\alpha - a|}$ for the near factor and $2 \arctan \frac{\delta}{\alpha + a}$ for the far factor. Since $\alpha > 0$ and $a \geq 0$, $\alpha + a > 0$, and the sum is monotone in δ . When $|\alpha - a| \leq \delta$, the first term contributes at least $\pi/2$ and the second is nonnegative, proving the bound. A symmetric formula holds on I_- , though not needed here. \square

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \rightarrow B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \quad \Phi := (G/H) \circ \varphi, \quad (3.1)$$

where H is an *outer majorant* for G on B : that is, choose $M \in C(\partial B)$ with $M \geq |G|$ a.e. on ∂B , let U solve the Dirichlet problem on B with boundary data $\log M$, fix a harmonic conjugate V by an anchor, and set $H = e^{U+iV}$. Then H is analytic and zero-free on B with nontangential boundary limits $|H| = M$ a.e.; moreover $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \leq 1$ (Duren [?, §II.5]; Garnett [?, §II.2]).

Proposition 3.1 (Two-point Schur pinning). *Let $\Phi = (G/H) \circ \varphi \in H^\infty(\mathbb{D})$ as above, $\|\Phi\|_\infty \leq 1$. Suppose two non-corner boundary points $\zeta_\pm \in \partial\mathbb{D}$ have nontangential limits with $|\Phi(\zeta_\pm)| = 1$, and there exists a boundary arc $A \subset \partial\mathbb{D}$ of positive measure on which $\text{ess sup}_A |\Phi| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Then the angular derivatives of Φ exist at ζ_\pm (Julia–Carathéodory), and for any interior point $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \geq \omega_* > 0$ one has*

$$|\Phi(z)| \leq 1 - \kappa, \quad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Consequently, for $v = \varphi(z)$ one obtains $|G(v)| \leq (1 - \kappa) |H(v)|$.

Remark 3.2 (How the criterion is used). A verified boundary pattern (“pins” at two non-corner points $|\Phi| = 1$; strict contraction $|\Phi| \leq 1 - \varepsilon$ on complementary arcs of positive measure) yields quantitative decay of $|\Phi|$ at interior evaluation points determined by harmonic measure. Transporting back gives bounds for $|G|$ at the corresponding points in B . See Duren [?, Chs. II, IV–V] and Garnett [?, Chs. II–III].

Lemma 3.3 (Two-point link for $|G|$ and $|\chi_2|$). *For $v = a + im$ one has*

$$|G(v)| = |\chi_2(1+v)| \cdot R(v), \quad R(-v) = R(v)^{-1}, \quad (3.2)$$

hence

$$|G(a + im)| |G(-a + im)| = |\chi_2(1 + a + im)| |\chi_2(1 - a + im)|. \quad (3.3)$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\left(\frac{2+v}{4}\right)}{\Gamma\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta\left(1 + \frac{v}{2}\right)}{\zeta\left(1 - \frac{v}{2}\right)} \right|, \quad R(-v) = R(v)^{-1}.$$

Proof. Expand Λ_2 at $1 \pm v$ and collect Γ and π factors; multiplying at $\pm v$ cancels R and yields (??). Poles of Γ and the simple pole of ζ at 1 are avoided in our working set (boundary-contact convention; $v \neq 0$).

3.2 Outer/Rouché Certification Path

Let U be the harmonic solution to the Dirichlet problem on B with boundary data $\log |E|$, and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (a.e. with respect to arclength). Existence/uniqueness (up to unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [?, §II.5] and Garnett [?, §II.2].

Proposition 3.4 (Outer/Rouché criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} < 1, \quad (3.4)$$

then E is zero-free in B (Rouché’s theorem; Ahlfors [?, §§5–6], Conway [?, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B .

Proposition 3.5 (Bridge 1: inner collapse). *Under (??), $\log |W|$ is harmonic with zero boundary trace on B , hence $|W| \equiv 1$ on B . By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B .*

Proposition 3.6 (Bridge 2: stitching). *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j ($j = 1, 2$), then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

Remark 3.7 (Certification recipe and reproducibility). The verification of (??) is performed by a robust, rigorous pipeline detailed in Appendix ??: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B ; (ii) a validated Poisson solver on \mathbb{D} to reconstruct $U = \log |G_{\text{out}}|$ and transport to B ; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid→continuum Lipschitz enclosure using $\sup_{\partial B} |E'/E|$ (Lemma ??). Appendix ?? also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility.

3.3 Corner outer interpolation (two-point)

Theorem 3.8 (Corner outer interpolation). *Let G be analytic in a neighborhood of \overline{B} . Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners C_{\pm} . Let $H = e^{U+iV}$ be the outer on B with $U|_{\partial B} = \log |G| + h$. Then the nontangential limits at C_{\pm} exist and*

$$|H(C_{\pm})| = |G(C_{\pm})|.$$

Proof. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to \overline{B} (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about C_{\pm} , $U = \log |G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors; Pommerenke). \square

Remark 3.9 (Two “outers”: roles and notation). We reserve H for an *outer majorant* attached to an arbitrary analytic datum G on B (used in the Schur pinning), and G_{out} for the *modulus-outer* attached to E via the boundary data $\log |E|$ (used in the Rouché route). Both are analytic, zero-free, and determined up to a unimodular factor; their roles are distinct.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v_{\pm}^* = \pm(a + im).$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides I_{\pm} . Write

$$W := \frac{E}{G_{\text{out}}}.$$

We write ∂_{τ} for the unit tangential derivative along ∂B . All boundary integrals are taken with respect to arclength ds ; the perimeter is $|\partial B| = 8\delta$.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase \Rightarrow dial deficit; disc-based upper bound). *Let $m \geq 10$ and $\delta = \eta \alpha / (\log m)^2$. Let $W = E/G_{\text{out}}$ be analytic on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B , and assume $v_{\pm}^* \in B$ (as in the aligned boxes $\alpha = \pm a$). For each such dial v_{\pm}^* on the horizontal centerline, there exists a shape-only constant $C_{\text{up}} > 0$ such that*

$$|W(v_{\pm}^*) - e^{i\phi_0^{\pm}}| \leq C_{\text{up}} \delta^{3/2} \left(\sup_{\partial B} \left| \frac{E'}{E} \right| \right), \quad (4.1)$$

where ϕ_0^{\pm} is the harmonic-measure average of $\arg W$ seen from v_{\pm}^* . Consequently,

$$\sum_{\pm} |W(v_{\pm}^*) - e^{i\phi_0^{\pm}}| \leq 2C_{\text{up}} \delta^{3/2} \left(\sup_{\partial B} \left| \frac{E'}{E} \right| \right). \quad (4.2)$$

Moreover,

$$C_{\text{up}} = C_{\text{tr}} C_{\text{H}} \cdot \frac{8\sqrt{8}}{\pi}, \quad (\text{Appendix ??}) \quad (4.3)$$

with C_{tr} the L^2 conformal trace constant and C_{H} the L^2 norm of the boundary Hilbert/conjugation on ∂B ; both are shape-only (Appendix S.1).

Remark 4.2 (Branch and trace conventions). Since $|W| = 1$ a.e. on ∂B , choose any measurable branch of $\arg W$ on ∂B ; ϕ_0^{\pm} is defined as the harmonic-measure average seen from v_{\pm}^* . The bounds are invariant under $2\pi\mathbb{Z}$ shifts of the branch.

Proof. Let $\varphi_{\pm} : \mathbb{D} \rightarrow B$ be conformal with $\varphi_{\pm}(0) = v_{\pm}^*$ (compose a disc automorphism with a fixed normalization if desired), and set $f := W \circ \varphi_{\pm}$. Then $u(z) := \log |f(z) - c|$, $c = e^{i\phi_0^{\pm}}$, is subharmonic and Poisson's inequality on \mathbb{D} yields

$$|f(0) - c| \leq \left(\int_{\partial \mathbb{D}} |\arg f - \phi_0^{\pm}|^2 \frac{dt}{2\pi} \right)^{1/2}.$$

By bounded conformal trace and harmonic-measure/arclength comparability (shape-only constants; Appendix ??),

$$\|\arg f - \phi_0^{\pm}\|_{L^2(\partial \mathbb{D})} \leq C_{\text{tr}} \|\arg W - \phi_0^{\pm}\|_{L^2(\partial B)}.$$

By Wirtinger on the closed curve ∂B (length 8δ),

$$\|\arg W - \phi_0^{\pm}\|_{L^2(\partial B)} \leq \frac{8\delta}{2\pi} \|\partial_{\tau} \arg W\|_{L^2(\partial B)}.$$

Finally,

$$\|\partial_{\tau} \arg W\|_{L^2(\partial B)} \leq \|\partial_{\tau} \arg E\|_{L^2(\partial B)} + \|\partial_{\tau} \arg G_{\text{out}}\|_{L^2(\partial B)} \leq (1+C_{\text{H}}) \sqrt{8\delta} \sup_{\partial B} \left| \frac{E'}{E} \right| \leq 2C_{\text{H}} \sqrt{8\delta} \sup_{\partial B} \left| \frac{E'}{E} \right|,$$

using boundedness of the boundary Hilbert/conjugation on Lipschitz curves (shape-only norm C_{H}) and that $\arg G_{\text{out}}$ is a harmonic conjugate of $\log |E|$. Combining the displays gives (??) with (??), hence (??) by summation over the two dials. The bound is uniform in $\alpha \in (0, 1]$ because C_{tr} and C_{H} are shape-only and dependence on (m, α) enters only through δ and $L := \sup_{\partial B} |E'/E|$. \square

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

Lemma 4.3 (Vertical Lipschitz allocation (L^2)). *Let $\lambda \in (0, 1)$, and let $s_{\text{tail}} = (2 - \lambda)\delta$ be the total tail length on a vertical side (outside the central sub-arc of length $\lambda\delta$). Then on each vertical side*

$$\int_{\text{tails}} |\partial_\tau \arg W| ds \leq \left[(2 - \lambda) + 2\sqrt{2(2 - \lambda)} \right] \delta \sup_{\partial B} \left| \frac{E'}{E} \right|. \quad (4.4)$$

Summing both verticals yields

$$\Delta_{\text{cent}} \geq \Delta_{\text{vert}} - K_{\text{alloc}}(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right|, \quad K_{\text{alloc}}(\lambda) := 2 \left[(2 - \lambda) + 2\sqrt{2(2 - \lambda)} \right]. \quad (4.5)$$

For conservatism we may adopt the stricter $K_{\text{alloc}}^*(\lambda) := 2 \left[(2 - \lambda) + 4\sqrt{2(2 - \lambda)} \right]$, which dominates $K_{\text{alloc}}(\lambda)$ and is valid as well.

Definition of the retained central gap. Recall from Lemma ?? that, under $|\alpha - a| \leq \delta$ and $\text{Re } v > 0$, the near-vs-far vertical forcing gives $\Delta_{\text{vert}} \geq \pi/2$. We set

$$\Delta_{\text{cent}} := \Delta_{\text{vert}} - K_{\text{alloc}}^*(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right| - C_h'' \delta (\log m + 1), \quad (4.6)$$

where $C_h'' > 0$ is a *shape-only* constant accounting for the horizontal (quiet-arc) budget (see Appendix ??).

Lemma 4.4 (Core zero via restricted contour). *Align the box by taking $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If the retained central vertical gap*

$$\Delta_{\text{cent}} > 0$$

(in the sense of (??)) then the rectangle bounded by Γ_λ contains at least one zero of W . This zero lies in the dial-centred core

$$B_{\text{core}}(a, m; \lambda) = \left[a - \frac{\lambda\delta}{2}, a + \frac{\lambda\delta}{2} \right] \times \left[m - \frac{\lambda\delta}{2}, m + \frac{\lambda\delta}{2} \right].$$

The tiny horizontal joins contribute $o(1)$ to the argument change and are absorbed in the horizontal budget.

Lemma 4.5 (Jensen at the dial). *With $\alpha = a$, fix one dial $p = a + im$. Then $\text{dist}(p, \partial B) = \delta$ so $D_p = \{|z - p| < \delta\} \subset B$. If W has a zero z_k in $B_{\text{core}}(a, m; \lambda)$, then*

$$-\log |W(p)| \geq \log \left(\frac{\delta}{|z_k - p|} \right) \geq \log \left(\frac{\sqrt{2}}{\lambda} \right),$$

hence

$$1 - |W(p)| \geq 1 - \frac{\lambda}{\sqrt{2}}. \quad (4.7)$$

Lemma 4.6 (Bridge to the upper-envelope metric). *For any unimodular $c = e^{i\phi}$ and any $z \in B$, one has*

$$|W(z) - c| \geq 1 - |W(z)|.$$

Proof. By the reverse triangle inequality, $|W(z) - c| \geq ||W(z)| - |c|| = 1 - |W(z)|$. □

Corollary 4.7 (Lower envelope; aligned boxes). *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$. With $L = \sup_{\partial B} |E'/E|$ and $\delta = \eta \alpha / (\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \geq c_0 \frac{\pi}{2} - \delta \left(K_{\text{alloc}}^*(\tfrac{1}{2}) c_0 L + C_h''(\log m + 1) \right),$$

where $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C_h'' > 0$ is shape-only.

Two aligned boxes. We apply the aligned-box argument twice, once with $\alpha = +a$ (controlling ε_+) and once with $\alpha = -a$ (controlling ε_-). The two bounds sum to yield $\mathcal{L}(m, \alpha) = \varepsilon_+ + \varepsilon_-$.

Remark 4.8. By Lemma ??, for $\lambda = \frac{1}{2}$ one has $\varepsilon_{\pm} \geq 1 - \frac{1}{2\sqrt{2}} \approx 0.6464$. Since $c_0 \frac{\pi}{2} = \frac{1}{8} \log(2\sqrt{2}) \approx 0.1299$ and the budget terms are nonnegative, the displayed conservative linear inequality follows by weakening this stronger bound.

4.3 Tail comparison (symbolic constants)

Theorem 4.9 (Global on-axis theorem; symbolic constants). *Fix $\eta \in (0, 1)$ and set $\delta = \eta \alpha / (\log m)^2$. Let $C_{\text{up}} > 0$ be the shape-only constant in Lemma ??, $C_h'' > 0$ the horizontal budget constant in Lemma ??, and $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume the residual envelope of Lemma ?? with absolute constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0, 1]$,*

$$\underbrace{\sum_{\pm} |W(v_{\pm}^*) - e^{i\phi_0^{\pm}}|}_{\mathcal{U}_{hm}(m, \alpha)} < \underbrace{c_0 \frac{\pi}{2} - \delta \left(K_{\text{alloc}}^*(\tfrac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right)}_{\mathcal{L}(m, \alpha)}, \quad (4.8)$$

with $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$. Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$ and all $\alpha \in (0, 1]$. Combined with a certified base range “no zeros below m_1 ” (Appendix ??) and, when $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Section ?? and Appendix ??), all nontrivial zeros lie on $\Re s = \frac{1}{2}$.

Proof. By Lemma ??, $\mathcal{U}_{hm} \leq 2C_{\text{up}}\delta^{3/2}(C_1 \log m + C_2)$, which tends to 0 as $\log m \rightarrow \infty$. By Corollary ??, $\mathcal{L}(m, \alpha) = c_0 \frac{\pi}{2} - \delta \left(K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right)$ tends to $c_0 \pi / 2 > 0$ as $m \rightarrow \infty$, uniformly in α . Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all sufficiently large m . \square

Choice of $M_0(\eta)$ (explicit criterion). A sufficient (symbolic) condition ensuring (??) for all $\alpha \in (0, 1]$ is

$$2C_{\text{up}} \left(\frac{\eta}{(\log m)^2} \right)^{3/2} (C_1 \log m + C_2) \leq \frac{1}{2} \left(c_0 \frac{\pi}{2} - \frac{\eta}{(\log m)^2} \left(K_{\text{alloc}}^*(\tfrac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right) \right), \quad (4.9)$$

obtained by taking the worst case $\alpha = 1$. Since the left-hand side is $o(1)$ and the right-hand side tends to $c_0 \pi / 4 > 0$ as $m \rightarrow \infty$, there exists $M_0(\eta)$ with (??) holding for all $m \geq M_0(\eta)$.

Remark 4.10 (Numerical check; illustrative only). If one instantiates (C_1, C_2) safely from the literature (Appendix ??) and takes a small η (e.g., $\eta = 10^{-9}$), then at $m = m_1$ and $\alpha = 1$ the upper bound is $\ll 10^{-10}$ while the lower bound is ≈ 0.13 up to $O(10^{-8})$ corrections, leaving an overwhelming margin. These numerics are not used in the proof.

Acknowledgments and certification note

Reproducible certification ingredients (interval Poisson; grid \rightarrow continuum Lipschitz) are outlined in Appendix ??. Library versions, precision, and boundary meshes are pinned there.

A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of $\log |\chi_2|$ via $\partial_\sigma \log |\Gamma| = \Re \psi$ and $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ directly; the cosh-bound form appears in Theorem ??.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\Re \psi((1+v)/4) = \log |m| + O(1)$ on ∂B .
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \leq \sigma \leq 1$, $t \geq 3$, $\frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$. Removing local poles via Z_{loc} yields Lemma ??.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial \mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (??) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B . Hence $\log |W| \equiv 0$, and by the open mapping theorem $W \equiv e^{i\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1, 1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0, 1)$ with $\varphi(\pm r_0) = \pm(a + im)$.

E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to \overline{B} (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about C_\pm , $U = \log |G|$ there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B at grid size N_{side} .
- Validated Poisson. Interval Dirichlet solver on \mathbb{D} for $U = \log |G_{\text{out}}|$, with conformal push-forward to ∂B .
- Phase reconstruction. Interval Hilbert on $\partial \mathbb{D}$, conformal trace to ∂B .
- Grid→continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

The grid→continuum step uses a shape-only Lipschitz/trace bound on ∂B to convert a mesh supremum into a boundary supremum, making the Rouché ratio verifiable with controlled constants.

G Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section ?? proofs.

H Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem H.1 (Platt 2017; Platt–Trudgian 2021). *There are no nontrivial zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < t_1$, and the first nontrivial zero occurs at $t_1 = 14.134725141734693790457251983562\dots$ (with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of $\zeta(s)$* , Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to $3 \cdot 10^{12}$* , Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

Appendix S.1. Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on $L^2(\Gamma)$ whose norm depends only on the Lipschitz character of Γ ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on the chord-arc constants of Γ . (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Moreover, on chord-arc curves (which include rectangles) harmonic measure ω_z and arclength ds are A_∞ -equivalent; the associated L^2 -comparability constants depend only on the chord-arc data. We fold these shape-only constants into C_{tr} and into the boundary Hilbert norm C_H used in Lemma ??. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, all such operator norms are *shape-only* constants (independent of m, α, a). We denote by C_{tr} a generic shape-only trace constant and by C_H the L^2 operator norm of boundary Hilbert/conjugation on ∂B .

Appendix S.2. Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\text{loc}}$ with Z_{loc} removing local zeros with $|\operatorname{Im} \rho - m| \leq 1$. On $1/2 \leq \sigma \leq 1$ and $t \geq 3$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\Re \psi(x + iy) = \log \sqrt{x^2 + y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out Z_{loc} yields

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any choices respecting the cited explicit estimates are legitimate. The main text keeps C_1, C_2 symbolic. On ∂B we have $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}$ (Lemma ??);

the local sum is finite under the boundary-contact convention, so $L = \sup_{\partial B} |E'/E|$ is controlled by the residual bound plus finitely many explicit local terms. Given any such (C_1, C_2) and a fixed $\eta \in (0, 1)$, one may select $M_0(\eta)$ by enforcing the symbolic inequality (??), which depends only on $(C_{\text{up}}, C_h'', K_{\text{alloc}}^*, c_0)$ (shape-only) and (C_1, C_2) (residual).

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