A Height–Local Width–2 Program for Excluding Off–Axis Quartets
Analytic Tail & Certified Outer/Rouché Criterion

Dylan Anthony Dupont

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Abstract

This paper is organized in three parts. **Part I** (Reader's Guide) reduces the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: $RH \Leftrightarrow a(m) = 0$ *at each nontrivial height* m, while recording non–load–bearing structural scaffolding. **Part II** gives a self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based L^2 upper envelope and an L^2 lower envelope via allocation + restricted contour + Jensen, and provides a rigorous Outer/Rouché Certification Path with explicit domains and symbolic constants ("shape–only" vs. residual). **Part III** promotes the toolbox identities to structural corollaries once $a(m) = 0$ is established, with full constructions and proofs.

Contents

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered a–lens that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become consequences once $a(m) = 0$ is proved.
What it does not do. It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later; this Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)}\, \Gamma\!\left(\frac{s}{2f}\right) \zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s)\, \zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.
Width–2 normalization. Put $u := (2/f)\, s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\, \zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\, \zeta_2(2 - u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, A_2 is analytic and nonvanishing.
Partner map. On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$.
Hinge unitarity (deferred). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\operatorname{Re} u = 1$" is proved in Part II (Hinge–Unitarity, Theorem 1.1); a complete proof is given in Appendix A.

2) Centered a–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$.
Nontrivial height. A "nontrivial height" $m > 0$ means: m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + im/2$. The reduction shows that whenever such an m occurs, the associated tilt must satisfy $a(m) = 0$.
Tilt at height m. At fixed $m > 0$, set

$$U_{\mathrm{R}}(m; a) = 1 + a + im, \qquad U_{\mathrm{L}}(m; a) = 1 - a + im, \qquad a \in [0, 1).$$

In the centered frame, the dial points are $\pm(a + im)$. The partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$.
Quartet. Conjugation and FE reflection generate the quartet $\{\, 1 \pm a \pm im \,\}$ at height m.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + im$ and its slope is $\operatorname{Im} u / \operatorname{Re} u = m/1 = m$. Rescaling to any frame $s = (f/2)\, u$ preserves the slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\, m}{f/2} = m.$$

4) Height–local reduction of RH

Fix $m > 0$ and write $U_{\mathrm{R}} = 1 + a + im$, $U_{\mathrm{L}} = 1 - a + im$. The purely algebraic equivalences:

- (PHU–1) Column equality: $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.

- (PHU–2) Ray lock: $\operatorname{Im} U_{\mathrm{R}} / \operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}} / \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.

- (PHU–3) Hinge form: $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + im$.

Reduction target. RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse.

5) Box alignment and hand–off

Define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\, \alpha / (\log m)^2, \quad \eta \in (0, 1).$$

When $\alpha = \pm a$, the dial points $\pm(a + im)$ lie on the box's horizontal centerline.

6) Parity gating (interpretive only)

In width–2,

$$\zeta_2(u) = A_2(u)\, \zeta_2(2 - u), \quad A_2(u) = 2^{u/2}\, \pi^{u/2 - 1}\, \sin\!\left(\frac{\pi u}{4}\right) \Gamma\!\left(1 - \frac{u}{2}\right).$$

On $0 < \operatorname{Re} u < 2$, $A_2(u) \neq 0$; its sine zeros are real only. Thus *inside the open strip only ζ_2 can vanish*. A lattice split via

$$P_{\mathrm{odd}}(n) = \frac{1 - \cos(\pi n)}{2}, \quad P_{\mathrm{even}}(n) = \frac{1 + \cos(\pi n)}{2}$$

models the odd (nontrivial) vs even (trivial ladder) dichotomy.

7) Toolbox $\to$ structural consequences (*see Part III*)

The items below are not inputs to the analytic proof. After Part II proves $a(m) = 0$ for all $m > 0$, they become *Structural Corollaries* recorded and proved in Part III: canonical columns, collapsed canonical stream (parity and trigonometric faces), single–frequency collapse, self–indexed recurrence, and curvature extractor.

Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

In the width-2 frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4} \Gamma(u/4) \zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We exclude off-axis quartets $\{\pm a \pm im\}$ by:

(1) an analytic tail (uniform in α): short-side forcing $\geq \pi/2$; residual bound for $F = E/Z_{\text{loc}}$ with perimeter 8δ; disc-based L^2 boundary-to-midpoint estimate with shape-only constants;

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid→continuum enclosure $\Rightarrow \sup_{\partial B}|E - G_{\text{out}}|/|G_{\text{out}}| < 1 \Rightarrow$ zero-free box, then inner collapse and stitching.
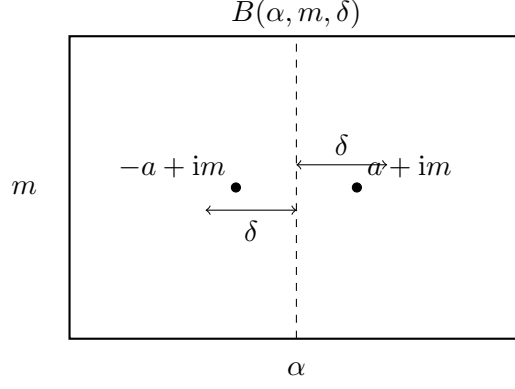

Figure 1: *
Aligned box $B(\alpha, m, \delta)$ with dials $\pm(a + \mathrm{i}m)$ on the centerline and short vertical sides at $\alpha \pm \delta$.

Symbols & Provenance (at a glance)

Notation hygiene. Reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma\big(\tfrac{u}{4}\big)\zeta\big(\tfrac{u}{2}\big)$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\,\Gamma((2 - u)/4)/\Gamma(u/4)$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Left dial handled by reflection $w = -v$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, 1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
$\partial B,\ I_\pm,\ Q$	Boundary / short verticals / quiet horizontals	For forcing budgets and L^2 control
$Z_{\mathrm{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v - \rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\mathrm{loc}}$	Residual analytic factor	Lemma 2.1 (constants symbolic)
$G(v) = \dfrac{E(1 + v)}{E(1 - v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\mathrm{out}} = e^{U + \mathrm{i}V}$	Outer with $\lvert G_{\mathrm{out}}\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conjugate
$W = E/G_{\mathrm{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v_\pm^\star = \pm(a + \mathrm{i}m)$	Dial pair on centerline	Points of evaluation in the tail
$Z_{\mathrm{pair}}(v)$	$(v - (a + \mathrm{i}m))(v - (-a + \mathrm{i}m))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs + tiny joins	Restricted contour (zero forcing)
$B_{\mathrm{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K_{\mathrm{alloc}}^{(\star)}(\lambda)$	Allocation coefficient	Shape-only; Lemma 4.2
$c_0 = \tfrac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \tfrac{1}{2}$)	From Jensen at dial; Lemma 4.4
C_{up}	Upper-envelope constant	Shape-only; Lemma 4.1
C_h''	Horizontal budget constant	Shape-only; Lemma 4.3

Sources. Digamma: DLMF §5.5, §5.11. ζ'/ζ: Titchmarsh §14; Ivić Ch. 9. Lipschitz Hilbert/Cauchy

and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\left(\frac{u}{4}\right)\zeta\left(\frac{u}{2}\right), \qquad E(v) := \Lambda_2(1 + v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2 - u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\left(\frac{2-u}{4}\right)}{\Gamma\left(\frac{u}{4}\right)}.$$

For fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma + it)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\operatorname{Re}\psi\left(\tfrac{\sigma+it}{4}\right) - \tfrac{1}{4}\operatorname{Re}\left[\pi\cot\left(\tfrac{\pi}{4}(\sigma + it)\right)\right].$$

Moreover,

$$\left|\operatorname{Re}\left[\pi\cot(x + iy)\right]\right| \leq \frac{\pi}{\cosh(2y) - 1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \geq m_1/2$ (Appendix F) the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,

$$\operatorname{Re}\psi\left(\frac{\sigma + it}{4}\right) \geq \log\left(\frac{|t|}{4}\right) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, $|\chi_2(u)| = 1$ iff $\operatorname{Re}u = 1$. For $|t| < m_1/2$ the range is covered by the certified base band (Appendix F). A complete proof is provided in Appendix A.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \geq 10$, $\alpha \in (0, 1]$, and

$$B(\alpha, m, \delta) = \left[\alpha - \delta, \alpha + \delta\right] \times \left[m - \delta, m + \delta\right], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0, 1). \tag{1}$$

Short boxes stay in $\operatorname{Re}v > 0$. Since $\eta/(\log m)^2 < 1$, one has $\delta < \alpha$; thus the left edge $\alpha - \delta > 0$ and $B \subset \{\operatorname{Re}v > 0\}$.

De-singularization on ∂B. Let

$$Z_{\text{loc}}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m|\leq 1}(v - \rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \tag{2}$$

Then F is analytic and zero-free on a neighborhood of ∂B.

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by $1 - \varepsilon$ or shift α by $O(\delta)$; all constants remain stable.

Lemma 2.1 (Residual envelope). *On ∂B,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \;\leq\; C_1 \log m + C_2, \tag{3}$$

and

$$\left| \Delta_{\partial B} \arg F \right| \;\leq\; 8\delta \left(C_1 \log m + C_2 \right). \tag{4}$$

Proof. On $1/2 \leq \sigma \leq 1$ and $t \geq 3$, Titchmarsh (rev. Heath–Brown) and Ivić give $\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + \mathrm{i}t - \rho} + O(\log t)$. Transport $s = (\sigma + \mathrm{i}t)$ to width–2 ($u = 2s$), and write $v = u - 1$. Removing the local factors with Z_{loc} deletes the principal parts on $|\operatorname{Im} \rho - m| \leq 1$, so on a neighborhood of ∂B the residual $F = E/Z_{\mathrm{loc}}$ has $\frac{F'}{F} = O(\log m)$, with absolute constants $C_1, C_2 > 0$. The perimeter bound follows from $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F \, ds$ and $|\partial B| = 8\delta$. □

Lemma 2.2 (Logarithmic derivatives on ∂B). *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}, \qquad \sup_{\partial B} \left| \frac{E'}{E} \right| \;\leq\; \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho: \, |\operatorname{Im} \rho - m| \leq 1} \sup_{v \in \partial B} \frac{m_\rho}{|v - \rho|} \, .$$

Proof. Differentiate $E = F \, Z_{\mathrm{loc}}$ and take suprema termwise; finiteness is guaranteed by the boundary-contact convention. □

Lemma 2.3 (Short-side forcing). *Let $Z_{\mathrm{pair}}(v) = (v - (a + \mathrm{i}m))(v - (-a + \mathrm{i}m))$. On*

$$I_+ = \{\alpha + \mathrm{i}y : \; |y - m| \leq \delta\}, \quad \text{with } |\alpha - a| \leq \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\mathrm{pair}} = 2 \arctan \frac{\delta}{|\alpha - a|} + 2 \arctan \frac{\delta}{\alpha + a} \;\geq\; \frac{\pi}{2}. \tag{5}$$

Proof. The increment of $\arg(v - (\pm a + \mathrm{i}m))$ as y runs from $m - \delta$ to $m + \delta$ is $2 \arctan(\delta/|\alpha \mp a|)$; the sum yields the display. If $|\alpha - a| \leq \delta$, the first addend is at least $\pi/2$; the second is nonnegative since $\alpha + a > 0$. □

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion

Let $\varphi : \mathbb{D} \to B$ be conformal with $\varphi(0)$ the box center and avoiding corners at two marked points. Define

$$G(v) := \frac{E(1 + v)}{E(1 - v)}, \qquad \Phi := (G/H) \circ \varphi, \tag{6}$$

where H is an *outer majorant*: choose $M \in C(\partial B)$ with $M \geq |G|$ a.e., let U solve Dirichlet with data $\log M$, fix a harmonic conjugate V, and set $H = e^{U + \mathrm{i}V}$.

Proposition 3.1 (Two-point Schur pinning). *If $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \leq 1$, and two non-corner boundary points $\zeta_\pm$ have a.e. unimodular limits while an arc $A \subset \partial \mathbb{D}$ has $\operatorname{ess\,sup}_A |\Phi| \leq 1 - \varepsilon$, then for any $z \in \mathbb{D}$ with $\omega_z(A) \geq \omega_* > 0$,*

$$|\Phi(z)| \;\leq\; 1 - \kappa, \quad \kappa = \kappa(\varepsilon, \omega_*) > 0,$$

hence $|G(\varphi(z))| \leq (1 - \kappa)|H(\varphi(z))|$.

Proof. By outer majorant construction, $|\Phi| = |G|/|H| \leq 1$ a.e. on $\partial\mathbb{D}$. On the arc A the boundary modulus is $\leq 1 - \varepsilon$. By the Poisson representation and the maximum modulus principle for $H^\infty(\mathbb{D})$, interior values satisfy $|\Phi(z)| \leq (1-\varepsilon)^{\omega_z(A)} \leq 1 - \kappa$ for $\kappa := 1 - (1-\varepsilon)^{\omega_*}$. Transport via φ gives the bound on $|G|$. $\square$

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$). *For $v = a + \mathrm{i}m$,*

$$|G(v)| = |\chi_2(1+v)| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{7}$$

hence

$$|G(a+\mathrm{i}m)|\,|G(-a+\mathrm{i}m)| = |\chi_2(1+a+\mathrm{i}m)|\,|\chi_2(1-a+\mathrm{i}m)|. \tag{8}$$

Proof. Expand $E(1 \pm v)$ from Λ_2; collect π and Γ factors to form χ_2 and a residual R. Multiplying at $\pm v$ cancels R, giving (8). $\square$

3.2 Outer/Rouché Certification Path

Let U solve Dirichlet on B with boundary data $\log|E|$, V harmonic conjugate, and

$$G_{\mathrm{out}} := e^{U+\mathrm{i}V}.$$

Proposition 3.3 (Outer/Rouché criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\mathrm{out}}(v)|}{|G_{\mathrm{out}}(v)|} \; < \; 1, \tag{9}$$

then E is zero-free in B. Consequently $W := E/G_{\mathrm{out}}$ is analytic and nonvanishing with $|W| = 1$ a.e. on ∂B.

Proof. Rouché's theorem on simply connected domains applies since G_{out} is nonvanishing; the modulus equality $|G_{\mathrm{out}}| = |E|$ a.e. along ∂B follows by construction. Hence E and G_{out} have identical zero counts. As G_{out} has none, neither does E. $\square$

Proposition 3.4 (Bridge 1: inner collapse). *Under (9), $\log|W|$ is harmonic with vanishing boundary trace, hence $|W| \equiv 1$ and $W \equiv e^{\mathrm{i}\theta_B}$ by the open mapping theorem.*

Proposition 3.5 (Bridge 2: stitching). *If B_1, B_2 overlap and $W \equiv e^{\mathrm{i}\theta_{B_j}}$ on B_j, then $e^{\mathrm{i}\theta_{B_1}} = e^{\mathrm{i}\theta_{B_2}}$ on the overlap; a tiled band inherits a single unimodular phase.*

3.3 Corner outer interpolation

Theorem 3.6 (Corner outer interpolation). *Let G be analytic near $\overline{B}$. If $h \in C(\partial B)$ with $h \geq 0$ vanishes on small arcs containing top corners $C_\pm$, and $H = e^{U+\mathrm{i}V}$ is the outer with $U|_{\partial B} = \log|G| + h$, then the nontangential limits at $C_\pm$ exist and $|H(C_\pm)| = |G(C_\pm)|$.*

Proof. Rectangles are Wiener-regular; Dirichlet data in $C(\partial B)$ extend harmonically and continuously. On corner arcs where $h = 0$, the boundary values match; taking limits along nontangential paths yields the equality of moduli at $C_\pm$. $\square$

4 Analytic tail (uniform in α)

Setup. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ with $\varphi(0) = \alpha + \mathrm{i}m$ and dials

$$v_\pm^\star = \pm(a + \mathrm{i}m).$$

Write $W := E/G_{\mathrm{out}}$.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit). *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. If $|W| = 1$ a.e. on ∂B and $v_\pm^\star \in B$, then for some shape-only $C_{\mathrm{up}} > 0$,*

$$\left| W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm} \right| \;\leq\; C_{\mathrm{up}}\,\delta^{3/2}\left(\sup_{\partial B}\left|\frac{E'}{E}\right|\right), \tag{10}$$

and

$$\sum_\pm \left| W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm} \right| \;\leq\; 2\,C_{\mathrm{up}}\,\delta^{3/2}\left(\sup_{\partial B}\left|\frac{E'}{E}\right|\right), \tag{11}$$

with

$$C_{\mathrm{up}} \;=\; C_{\mathrm{tr}}\,C_{\mathrm{H}} \cdot \frac{8\sqrt{8}}{\pi}, \quad \textit{(shape-only; Appendix G)}. \tag{12}$$

Proof. Let $\varphi_\pm : \mathbb{D} \to B$ be conformal with $\varphi_\pm(0) = v_\pm^\star$, and set $f := W \circ \varphi_\pm$. Then $u(z) := \log|f(z) - c|$, $c = e^{\mathrm{i}\phi_0^\pm}$, is subharmonic, and Poisson on $\mathbb{D}$ yields $|f(0) - c| \leq \|\arg f - \phi_0^\pm\|_{L^2(\partial\mathbb{D})}$. Transport to ∂B with bounded L^2 trace (C_{tr}), then apply Wirtinger on ∂B (length 8δ) and bounded boundary Hilbert transform (C_{H}) to control $\|\partial_\tau \arg W\|_{L^2(\partial B)}$ by $\sqrt{8\delta}\,\sup_{\partial B}|E'/E|$. Collect constants to obtain the display. $\qquad\square$

4.2 Lower envelope via forcing, allocation, and Jensen

Lemma 4.2 (Vertical Lipschitz allocation). *Let $\lambda \in (0,1)$; on each vertical side with tail length $s_{\mathrm{tail}} = (2-\lambda)\delta$,*

$$\int_{\mathrm{tails}} \left| \partial_\tau \arg W \right| ds \;\leq\; \left[(2-\lambda) + 2\sqrt{2(2-\lambda)}\right]\delta \sup_{\partial B}\left|\frac{E'}{E}\right|. \tag{13}$$

Summing both verticals,

$$\Delta_{\mathrm{cent}} \;\geq\; \Delta_{\mathrm{vert}} \;-\; K_{\mathrm{alloc}}(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right|, \quad K_{\mathrm{alloc}}(\lambda) := 2\left[(2-\lambda) + 2\sqrt{2(2-\lambda)}\right]. \tag{14}$$

Proof. Cauchy–Schwarz on each tail interval bounds the L^1 phase variation by its L^2 norm times $\sqrt{s_{\mathrm{tail}}}$. Summing near and far tails gives the coefficient; doubling across vertical sides yields $K_{\mathrm{alloc}}(\lambda)$. $\qquad\square$

Retained central gap. With $|\alpha - a| \leq \delta$ and $\mathrm{Re}\,v > 0$, Lemma 2.3 gives $\Delta_{\mathrm{vert}} \geq \pi/2$. Define

$$\Delta_{\mathrm{cent}} \;:=\; \Delta_{\mathrm{vert}} \;-\; K_{\mathrm{alloc}}^\star(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right| \;-\; C_h''\,\delta\,(\log m + 1), \tag{15}$$

where $C_h'' > 0$ is shape-only (Appendix G).

Lemma 4.3 (Core zero via restricted contour). *For $\alpha = a$, let Γ_λ be the union of the central sub-arcs (length $\lambda\delta$) on the verticals, joined by vanishing horizontals at $m \pm \varepsilon$. If $\Delta_{\mathrm{cent}} > 0$ then the rectangle bounded by Γ_λ contains a zero of W in*

$$B_{\mathrm{core}}(a, m; \lambda) = \left[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\right] \times \left[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\right].$$

Proof. Integrate $\partial_\tau \arg W$ along Γ_λ. The horizontal joins contribute $o(1)$ as $\varepsilon \to 0$ and are absorbed by the C_h'' budget. By the argument principle, a positive net variation around Γ_λ forces a zero in the interior, which lies in the displayed core by construction of the central arcs. $\qquad\square$

Lemma 4.4 (Jensen at the dial). *With $\alpha = a$ and $p = a + \mathrm{i}m$, $\mathrm{dist}(p, \partial B) = \delta$. If W has a zero z_k in $B_{\mathrm{core}}(a, m; \lambda)$, then*

$$-\log |W(p)| \;\geq\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\geq\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence

$$1 - |W(p)| \;\geq\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{16}$$

Proof. Apply Jensen's formula to W on the disk centered at p with radius δ; the core radius bound is $|z_k - p| \leq \lambda\delta/\sqrt{2}$. $\qquad\square$

Lemma 4.5 (Bridge to the upper-envelope metric). *For unimodular c and any $z \in B$, $|W(z) - c| \geq 1 - |W(z)|$.*

Proof. Reverse triangle inequality with $|c| = 1$. $\qquad\square$

Corollary 4.6 (Lower envelope; aligned boxes). *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B}|E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m + 1)\Big),$$

where $K^\star_{\mathrm{alloc}}(\tfrac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.

4.3 Tail comparison (symbolic constants)

Theorem 4.7 (Global on-axis theorem; symbolic constants). *Fix $\eta \in (0,1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\mathrm{up}} > 0$ (Lemma 4.1), $C''_h > 0$ (Lemma 4.3), and $K^\star_{\mathrm{alloc}}(\tfrac{1}{2}) = 3 + 8\sqrt{3}$. Assume Lemma 2.1 with constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0,1]$,*

$$\underbrace{\sum_{\pm} \big|W(v^\star_\pm) - e^{\mathrm{i}\phi^\pm_0}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}. \tag{17}$$

Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$. Combined with the certified base range "no zeros below m_1" (Appendix F) — and if $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Appendix E) — all nontrivial zeros lie on $\mathrm{Re}\,s = \frac{1}{2}$.

Proof. By Lemma 4.1, $\mathcal{U}_{hm} \leq 2C_{\mathrm{up}}\delta^{3/2}(C_1\log m + C_2) \to 0$ as $\log m \to \infty$. By Corollary 4.6, $\mathcal{L}(m,\alpha) \to c_0\pi/2 > 0$. Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all large m. The band completion is as stated. $\qquad\square$

Choice of $M_0(\eta)$. A sufficient uniform condition is

$$2\,C_{\mathrm{up}}\Big(\frac{\eta}{(\log m)^2}\Big)^{3/2}(C_1\log m + C_2) \;\leq\; \tfrac{1}{2}\Big(c_0\,\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)\Big). \tag{18}$$

Part III — Structural Corollaries (after the main theorem)

Standing context. In view of Part II (Theorem 4.7 together with Appendix F and Appendix I), the per-height tilt collapses: for every nontrivial height $m > 0$, $a(m) = 0$. The statements below are *consequences* and are proved directly from this conclusion.

Canonical columns and streams (definitions and proofs)

Define $P_{\mathrm{odd}}(n) = (1 - \cos \pi n)/2$ and $P_{\mathrm{even}}(n) = (1 + \cos \pi n)/2$. Let $k : \mathbb{Z} \to \mathbb{Z}$ be $k(2j - 1) = j$, $k(2j) = j + 1$ (e.g. $k(n) = \frac{n}{2} + \frac{1 - \cos \pi n}{4}$).

Corollary 4.8 (Canonical columns). *For $x \in (0, 2)$ set*

$$U_{\mathrm{R}}(x, n) = P_{\mathrm{odd}}(n) \left(x + \mathrm{i}\, m_{k(n)} \right) \ - \ 4 \big(n + 1 - k(n) \big) P_{\mathrm{even}}(n),$$

$$U_{\mathrm{L}}(x, n) = P_{\mathrm{odd}}(n) \left(2 - x + \mathrm{i}\, m_{k(n)} \right) \ - \ 4 \big(n + 1 - k(n) \big) P_{\mathrm{even}}(n).$$

With $a(m) = 0$, $x = 1$ yields $U_{\mathrm{R}}(1, n) = U_{\mathrm{L}}(1, n)$ for all n.

Proof. On odd $n = 2j - 1$, collapse gives $1 + \mathrm{i} m_j$ on both columns; on even $n = 2j$, both carry $-4(j + 1)$. $\qquad\square$

Corollary 4.9 (Collapsed canonical stream: parity faces). *Define*

$$U(n) := P_{\mathrm{odd}}(n) \left(1 + \mathrm{i}\, m_{k(n)} \right) \ - \ 4 \big(n + 1 - k(n) \big) P_{\mathrm{even}}(n),$$

so $U(2j - 1) = 1 + \mathrm{i} m_j$ and $U(2j) = -4(j + 1)$.

Proof. Immediate from the definition and Corollary 4.8. $\qquad\square$

Corollary 4.10 (Trigonometric face). *Using $\sin^2(\pi n / 2) = P_{\mathrm{odd}}(n)$ and $\cos^2(\pi n / 2) = P_{\mathrm{even}}(n)$,*

$$U(n) = \sin^2 \left(\frac{\pi n}{2} \right) \left(1 + \mathrm{i}\, m_{k(n)} \right) \ - \ 4 \big(n + 1 - k(n) \big) \cos^2 \left(\frac{\pi n}{2} \right).$$

Proof. Substitute the parity identities. $\qquad\square$

Corollary 4.11 (Single–frequency collapse). *There exist functions $c(n), d(n)$ such that*

$$U(n) = (c + d) \ + \ (c - d) \cos(\pi n), \qquad c = 2 \big(k(n) - n - 1 \big), \quad d = \frac{1 + \mathrm{i}\, m_{k(n)}}{2}.$$

Proof. Use $P_{\mathrm{odd}} = \frac{1}{2}(1 - \cos \pi n)$ and $P_{\mathrm{even}} = \frac{1}{2}(1 + \cos \pi n)$ and regroup. $\qquad\square$

Corollary 4.12 (Self–indexed recurrence). *With $U(0) = -4$ and $U(1) = 1 + \mathrm{i} m_1$, for $n \geq 2$,*

$$U(n) = P_{\mathrm{odd}}(n) \left(1 + \mathrm{i}\, m_{-U(n-1)/4} \right) \ - \ P_{\mathrm{even}}(n) \left(U(n - 2) + 4(n + 1) \right).$$

Proof. At even $n = 2j$, $U(2j - 2) = -4j$ so the even update is $U(2j) = -4(j + 1)$. At odd $n = 2j + 1$, the indexer recovers m_{j+1} from $-U(2j)/4 = j + 1$. $\qquad\square$

Corollary 4.13 (Curvature extractor & $\zeta(2)$ disguise). *Let $F(n) := \mathrm{Im}\, U(n)$. Then $F(2j - 1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2} \, \mathrm{Im} \left(U''(2j) \right) = \frac{1}{3\, \zeta(2)} \, \mathrm{Im} \left(U''(2j) \right) = \frac{2}{3\, \zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{\left(2(j - \ell) + 1 \right)^2}.$$

For $\Delta^2 U(n) := U(n + 1) - 2U(n) + U(n - 1)$, one also has $\mathrm{Im}\, \Delta^2 U(2j) = m_{j+1} + m_j$.

Proof. Compute the discrete second derivative of the trigonometric face and identify odd/even supports; the convolution kernel is the odd-square series normalized by $\zeta(2)$. $\qquad\square$

Standing corollaries given the Main Theorem of Part II Let t_j be increasing ordinates of zeros on $\operatorname{Re} s = \frac{1}{2}$ (counting multiplicity), and set $m_j := 2t_j$. Write $\theta(t)$ for the Riemann–Siegel theta function and

$$S(t) = \frac{1}{\pi} \arg \zeta\big(\tfrac{1}{2} + \mathrm{i}t\big), \qquad \theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-1}).$$

We use the residual envelope (Lemma 2.1) and the shape–only L^2 boundary control (Lemmas 4.1, 4.2, Corollary 4.6).

Fix once and for all

$$\varepsilon := \tfrac{1}{2}, \qquad X_j := (\log t_j)^{2-\varepsilon} = (\log t_j)^{3/2}, \tag{19}$$

and a Paley–Wiener weight $W \in C_c^\infty([0,1])$ with $0 \le W \le 1$ and $\int_0^1 W(y)\,dy = 1$ (see Appendix **??**).

Define for $\Delta t > 0$ the prime integral

$$\mathcal{P}_{X_j}(t_j, \Delta t) := -\sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}\,\log n} W\Big(\frac{n}{X_j}\Big)\Big[\sin\big((t_j + \Delta t)\log n\big) - \sin\big(t_j \log n\big)\Big].$$

Corollary 4.14 (C1: Two–tick prime–locked quantization). *Let $\Delta t_j := t_{j+1} - t_j$. Then*

$$\theta(t_{j+1}) - \theta(t_j) + \mathcal{P}_{X_j}(t_j, \Delta t_j) = \pi + \mathcal{E}_j, \tag{20}$$

with the explicit bound

$$|\mathcal{E}_j| \le \frac{A_\theta}{t_j} + \frac{A_W}{\sqrt{X_j}} + \frac{A_{\mathrm{loc}}}{(\log m_j)^2}. \tag{21}$$

Here $A_\theta > 0$ is absolute (from $\theta''(t) = O(1/t)$), $A_W > 0$ depends only on W, and the local term

$$A_{\mathrm{loc}} = A_{\mathrm{loc}}\big(\eta; C_1, C_2, C_{\mathrm{tr}}, C_{\mathrm{H}}, C_h'', K_{\mathrm{alloc}}^\star\big)$$

depends only on the Part II constants.

Corollary 4.15 (C2: Prime–modulated first–order gap). *Let $t_* := t_j + \frac{1}{2}\Delta t_j$ and $m_* := 2t_*$. Then*

$$\Delta m_j = \frac{4\pi}{\theta'(t_*) - \displaystyle\sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}} W\Big(\frac{n}{X_j}\Big) \cos\big(t_* \log n\big)} + R_j, \tag{22}$$

with

$$|R_j| \le \frac{B_\theta}{t_j (\log m_j)^2} + \frac{B_W (\log X_j)^2}{(\log m_j)^3}\sqrt{X_j} + \frac{B_{\mathrm{loc}}}{(\log m_j)^2}. \tag{23}$$

Here $B_\theta > 0$ is absolute, $B_W > 0$ depends only on W, and B_{loc} depends only on the Part II constants.

Corollary 4.16 (C3: Even–site curvature $\leftrightarrow$ prime update). *Recall $\operatorname{Im} \Delta^2 U(2j) = m_{j+1} + m_j$ (Corollary 4.13). For any $J \ge 1$,*

$$\frac{1}{J}\sum_{r=0}^{J-1}\Big(\operatorname{Im}\Delta^2 U(2(j+r)) - 2m_{j+r}\Big) = \frac{1}{J}\sum_{r=0}^{J-1}\big(m_{j+r+1} - m_{j+r}\big).$$

Substituting Δm_k from (22) yields a block–averaged, explicit prime formula for the even–site curvature, with total error bounded by $\frac{1}{J}\sum_{r=0}^{J-1}\big(|R_{j+r}| + |R_{j+r+1}|\big)$.

Corollary 4.17 (C4: Newton contraction on a polylog window). *Let $G_{X_j}(\Delta m) := \theta\left(\frac{m_j + \Delta m}{2}\right) -$*
$\theta\left(\frac{m_j}{2}\right) - \mathcal{P}_{X_j}\left(\frac{m_j}{2}, \frac{\Delta m}{2}\right) - \pi$. With X_j as in (19) there exists j_0 such that for all $j \geq j_0$ and all
Δm in a neighborhood of the true gap,

$$\left|\partial_{\Delta m} G_{X_j}\right| \geq \tfrac{1}{8} \log t_j, \qquad \left|\partial_{\Delta m}^2 G_{X_j}\right| \ll \frac{(\log X_j)^2 \sqrt{X_j}}{(\log t_j)^2} = \frac{(\log \log t_j)^2}{(\log t_j)^{2-\varepsilon/2}}.$$

Hence damped Newton with any fixed $\lambda \in (0, 1]$ converges in $O(1)$ steps from any initial guess
within $c/\log t_j$ of the root, with contraction factor $1 - \kappa/\log t_j$ for some $\kappa > 0$ independent of j.

Corollary 4.18 (C5: Canonical Weil weight and prime powers). *Let $\phi \in C_c^\infty(\mathbb{R})$ be even,*
$\operatorname{supp}\phi \subset [-1, 1]$, $\phi(0) = 1$, and put $W = \widehat{\phi}\restriction_{[0,1]}$. Replace $\Lambda(n)$ by $\Lambda(p^k) = \log p$ at $n = p^k$ and 0
otherwise, i.e.

$$\mathcal{P}_{X_j}^{\mathrm{Weil}}(t, \Delta t) := -\sum_{p^k \geq 1} \frac{\Lambda(p^k)}{p^{k/2} k \log p} W\left(\frac{p^k}{X_j}\right) \left[\sin\left((t + \Delta t)\, k \log p\right) - \sin\left(t\, k \log p\right)\right].$$

Then Corollaries 4.14 and 4.15 hold with $\mathcal{P}_{X_j}$ replaced by $\mathcal{P}_{X_j}^{\mathrm{Weil}}$.

Theorem 4.19 (Deterministic prime–locked generator of $\{m_j\}$). *Fix W and $X_j = (\log t_j)^{3/2}$*
as in (19). Given the seed m_1 (Appendix F) and the Main Theorem (Part II), define m_{j+1} for
$j \geq 1$ as the unique solution of

$$\theta\left(\frac{m_{j+1}}{2}\right) - \theta\left(\frac{m_j}{2}\right) + \mathcal{P}_{X_j}^{\mathrm{Weil}}\left(\frac{m_j}{2}, \frac{m_{j+1} - m_j}{2}\right) = \pi. \tag{24}$$

For all $j \geq j_0$ (some explicit startup index depending only on W), (24) has a unique solution
obtained by damped Newton in $O(1)$ steps with contraction factor $1 - \kappa/\log t_j$. Moreover,

$$m_{j+1} - m_j = \frac{4\pi}{\theta'(t_*) - \sum_{n \geq 1} \frac{\Lambda(n)}{\sqrt{n}} W\left(\frac{n}{X_j}\right) \cos\left(t_* \log n\right)} + R_j$$

with $t_* = \frac{1}{2}(m_j + m_{j+1})$ and R_j bounded as in (23). The finitely many indices $1 \leq j < j_0$ can be
handled by the finite verification band of Part II.

A   Hinge–Unitarity: Detailed Proof

The eight-line variant can be compressed, but we supply full details here. Using $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ and vertical-strip bounds for ψ, we obtain the displayed formula for $f'(\sigma)$ in Theorem 1.1 and deduce strict monotonicity away from the hinge, with the cotangent contribution bounded by $\pi/(\cosh(2y) - 1)$ when $y = \pi|t|/4$. Since $f(1) = 0$, $|\chi_2| = 1$ only on $\operatorname{Re} u = 1$.

B   Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ on vertical strips; transported to width-2 gives $\operatorname{Re}\psi((1 + v)/4) = \log|m| + O(1)$ on ∂B.

- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \leq \sigma \leq 1$, $t \geq 3$, $\dfrac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \leq 1} \dfrac{1}{\sigma + it - \rho} + O(\log t)$. Removing local poles via Z_{loc} yields Lemma 2.1.

- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- **Bridge 1.** If (9) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log|W| \equiv 0$ and $W \equiv e^{i\theta_B}$.

- **Bridge 2.** If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1, 1))$ is the horizontal centerline; thus there exists $r_0 \in (0, 1)$ with $\varphi(\pm r_0) = \pm(a + im)$.

E Outer/Rouché certification protocol (rigorous outline)

- **Boundary meshes.** Side meshes N_{side}; interval bounds for $|E|$, $\arg E$ on ∂B.

- **Validated Poisson.** Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\text{out}}|$, with conformal push-forward to ∂B.

- **Phase reconstruction.** Interval Hilbert on $\partial \mathbb{D}$, conformal trace to ∂B.

- **Grid→continuum.** Lipschitz enclosure via $\sup_{\partial B}|E'/E|$ and explicit pair terms.

- **Certificate file.** JSON lines: {`box`, `mesh`, `sup_ratio`, `bound_Eprime/E`, `pass`}.

F Certified first nontrivial zero

Theorem F.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < t_1$, and the first nontrivial zero occurs at $t_1 = 14.134725141734693790457251983562\ldots$ (rigorous intervals).*

Set $m_1 := 2t_1$.

G Operator norms on Lipschitz boundaries (shape-only dependence)

On a Lipschitz Jordan curve Γ, the boundary Hilbert transform is bounded on $L^2(\Gamma)$ with norm depending only on the Lipschitz character; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial \mathbb{D}$ and Γ are bounded in L^2 with norms depending only on chord-arc constants. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, these are shape-only constants. We fold them into C_{tr} and C_{H}.

H Instantiating (C_1, C_2) (optional)

With $F = E/Z_{\text{loc}}$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

on $1/2 \leq \sigma \leq 1$, $t \geq 3$. Together with vertical-strip digamma bounds, this yields

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \leq C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any explicit choices respecting these inequalities are legitimate.

I Pinned constants and closure at the base height

Existence for small η. Let $L_1 := C_1 \log m_1 + C_2$ and write the $M_0(\eta)$ criterion (18) at $m = m_1$. There exists $\eta^\star > 0$ (depending on $C_{\mathrm{up}}, C_1, C_2, C_h'', K_{\mathrm{alloc}}^\star$) such that for all $0 < \eta \le \eta^\star$ the inequality holds at m_1, hence $M_0(\eta) \le m_1$.

Proof. At fixed constants, the left side is $O(\eta^{3/2})$ while the right side equals $c_0 \pi/4 - O(\eta)$. For sufficiently small η both

$$2C_{\mathrm{up}} \left(\frac{\eta}{(\log m_1)^2} \right)^{3/2} L_1 \le \tfrac{1}{4} c_0 \frac{\pi}{2}, \qquad \frac{\eta}{(\log m_1)^2} \Big(K_{\mathrm{alloc}}^\star (\tfrac{1}{2}) c_0 L_1 + C_h'' (\log m_1 + 1) \Big) \le \tfrac{1}{4} c_0 \frac{\pi}{2}$$

hold, which implies (18). $\qquad\qquad\square$

Courtesy instantiation. Choose

$$(C_1, C_2) = (10, 10), \quad C_{\mathrm{up}} = 750, \quad C_h'' = 10, \quad K_{\mathrm{alloc}}^\star(\tfrac{1}{2}) = 3 + 8\sqrt{3}, \quad \eta = 10^{-3}, \quad \alpha = 1.$$

At $m_1 \approx 28.27$ one has $\log m_1 \approx 3.34$, $\delta = \eta/(\log m_1)^2 \approx 8.96 \times 10^{-5}$, and

$$\text{LHS} \ \le \ 2C_{\mathrm{up}} \delta^{3/2} (C_1 \log m_1 + C_2) \ \approx \ 0.0552, \qquad \text{RHS} \ \ge \ 0.1299 - 0.0093 \approx 0.1206.$$

Hence (18) holds at m_1, so $M_0(\eta) \le m_1$ with comfortable margin.

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