[Title from Part II manuscript]

A Height–Local, Boundary–Only Approach in the Width–2 Frame

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A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Criterion

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Authorship and AI-use disclosure. The author, Dylan Anthony Dupont, designed the framework, chose all constants/normalizations, and validated all mathematics and computations. Generative assistants (From GPT-4o to GPT-5 Pro) were used only for typesetting assistance, editorial organization, and consistency checks; and thus are not an author. All claims are the author's responsibility (based on COPE/ICMJE guidance).

Abstract

This paper is organized in three parts:

- **Part I** — Reader's Guide / Motivation, reducing the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: *RH* $\Leftrightarrow a(m) = 0$ *at each nontrivial height* m, while recording non–load–bearing structural scaffolding.
- **Part II** — A self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based L^2 upper envelope and an L^2 lower envelope via allocation + restricted contour + Jensen. A rigorous Outer/Rouché Certification Path, explicit domains and symbolic constants ("shape–only" vs. residual).
- **Part III** — Promotes the identities constructed in Part I to structural corollaries of the main theorem once $a(m) = 0$ is established.
- **Appendices** — Technical details supporting Part II (load–bearing for Part II; load–bearing for Part III if reference to appendices is required in Part III).

Dependency map (schematic): Part I $\rightarrow$ (no arrows into Part II); Part II $\rightarrow$ Part III; Appendices $\rightarrow$ Part II and Part III.

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered "a–lens" that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed$\rightarrow$rectifier) and explains how these become consequences once $a(m) = 0$ is proved. *What it does not do.* It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later. This Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-\,s/(2f)}\,\Gamma\!\left(\frac{s}{2f}\right)\zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f-s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f-s)$ with $A_f(s)A_f(f-s) \equiv 1$.

Width–2 normalization. Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2-u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$. In the open strip $0 < \Re u < 2$ and $\Im u \neq 0$, A_2 is analytic and nonvanishing.

Partner map. On $\Im u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$, swapping the two column points at the same height.

Hinge unitarity (deferred). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\Re u = 1$" is proved in Part II (Hinge–Unitarity). We do not use it here.

2) Centered a–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1+v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$.

Nontrivial height. A "nontrivial height" $m > 0$ means: m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + \mathrm{i}m/2$. The reduction shows that whenever such an m occurs, the associated tilt must satisfy $a(m) = 0$.

Tilt at height m. At fixed $m > 0$, set

$$U_{\mathrm{R}}(m;a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m;a) = 1 - a + \mathrm{i}m, \qquad a \in [0,1).$$

In the centered frame, the "dial points" are $\pm(a + \mathrm{i}m)$. The partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$.

Quartet. Conjugation (top$\leftrightarrow$bottom) and FE reflection generate the quartet $\{\,1 \pm a \pm \mathrm{i}m\,\}$ at height m.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + \mathrm{i}m$ and its slope is $\Im u/\Re u = m/1 = m$. Rescaling to any frame $s = (f/2)\,u$ preserves the slope:

$$\frac{\Im s}{\Re s} = \frac{(f/2)\,m}{f/2} = m.$$

Thus $\{m_k\}$ simultaneously records the imaginary ordinates of the nontrivial zeros and their origin–through slopes in every modulated frame—provided the per–height collapse holds.

4) Height–local reduction of RH

Fix a nontrivial height $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The following are purely algebraic and equivalent:

- (PHU–1) Column equality: $\Re U_{\mathrm{R}} = \Re U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) Ray (slope) lock: $\Im U_{\mathrm{R}}/\Re U_{\mathrm{R}} = \Im U_{\mathrm{L}}/\Re U_{\mathrm{L}}$, i.e. $m/(1+a) = m/(1-a) \iff a = 0$.
- (PHU–3) Hinge form: $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

Reduction target. RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse; nothing from this Guide is used there.

5) Box alignment and hand–off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\, \alpha/(\log m)^2, \ \ \eta \in (0, 1).$$

When $\alpha = \pm a$, the dial points $\pm(a + \mathrm{i}m)$ lie on the box's horizontal centerline.
What Part II does. Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three–lines tools, Stirling–class envelopes, explicit control of ζ'/ζ away from zeros), Part II shows that any off–axis quartet forces a boundary lower bound larger than an explicit upper bound, hence $a(m) = 0$.
No circularity. The analytic proof is logically independent of this Guide.

6) Parity gating and selection devices (interpretive only)

Gating from the non–completed FE. In the width–2 frame the non–completed FE reads

$$\zeta_2(u) = A_2(u)\, \zeta_2(2 - u), \quad A_2(u) = 2^{u/2}\, \pi^{\,u/2-1}\, \sin\Big(\frac{\pi u}{4}\Big)\, \Gamma\Big(1 - \frac{u}{2}\Big).$$

On the open strip $0 < \Re u < 2$ with $\Im u \neq 0$, the prefactor $A_2(u)$ is nonzero and finite; its sine zeros (the "trivial ladder") lie on the real axis only. Thus *inside the open strip only* ζ_2 *can vanish* (nontrivial zeros), while the *trivial class is confined to the real axis.* This is the basic "odd/even lane" picture: the odd (upper) lane can host nontrivial zeros; the even (real) lane hosts the trivial ladder.
Orthogonal split on the integer lattice. To model this dichotomy as a clean input–space symmetry, decompose any lattice signal $X : \mathbb{Z} \to \mathbb{C}$ via the orthogonal projectors

$$P_{\mathrm{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \qquad P_{\mathrm{even}}(n) = \tfrac{1+\cos(\pi n)}{2},$$

so $X = P_{\mathrm{odd}}X + P_{\mathrm{even}}X$. We *assign the nontrivial stream to odd slots* (where $P_{\mathrm{odd}} = 1$) and the *trivial ladder to even slots* (where $P_{\mathrm{even}} = 1$). This mirrors the FE fact above without using it analytically.

7) Toolbox $\to$ structural consequences (after the theorem)

The items below are not inputs to the analytic proof. After Part II proves $a(m) = 0$ for all nontrivial heights, they become *Structural Corollaries* describing the collapsed geometry and its lattice faces. (*Explicit formulas and brief proofs are recorded as corollaries in Part III; Part I intentionally omits them and they are not inputs to Part II.*)

- Pre–collapse columns (projector faces in the u–frame): right/left templates place odd–slot samples $x \pm \mathrm{i}m_k$ and the even ladder $-4(\cdot)$ via $P_{\mathrm{odd}}, P_{\mathrm{even}}$; scaffolding, not assumptions.
- Collapsed canonical stream $U(n)$: when per–height collapse holds ($x = 1$ on odd slots), the two columns coincide; parity form (via $P_{\mathrm{odd}}, P_{\mathrm{even}}$) and an equivalent trigonometric form (via $\sin^2(\pi n/2), \cos^2(\pi n/2)$).
- Single–frequency collapse (cosine face): a two–parameter cosine form $U(n) = (c + d) + (c - d)\cos(\pi n)$ recovers the same stream; c, d simple in the odd–indexer $k(n)$.
- Self–indexed recurrence (no explicit k): a short recurrence for $U(n)$ pulls the needed odd index from the previous even sample; encodes the collapsed geometry without an explicit $k(n)$.

- Curvature extractor & the $\zeta(2)$ disguise: the discrete second–difference of the imaginary part at even indices recovers m_j and admits an odd–square convolution form normalized by $\zeta(2)$.
- Seed $\to$ rectifier $\to$ physical streams: two–carrier seeds rectify under a mod–4 factor to yield the physical stream $S_f(n)$ proportional to $U(n)$; pre–collapse faces scale analogously.

8) Implications and one–sentence hand–off

The width–2 organization centralizes symmetry at $\Re u = 1$; the centered a–lens isolates the single per–height degree of freedom; parity–orthogonal scaffolding separates the nontrivial stream from the ladder without entering the proof. With these definitions, RH reduces to: for every nontrivial height $m > 0$, $a(m) = 0$.

Part II — Self-Contained Boundary–only contradiction on aligned boxes

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\mathrm{loc}}$ with perimeter factor 8δ; and (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants (no strip/rectangle density comparison);

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow \sup_{\partial B}\left|E - G_{\mathrm{out}}\right|/|G_{\mathrm{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed $\eta \in (0, 1)$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Combined with a certified base range below m_1 (first nontrivial height in width-2), this yields the global on-axis theorem. All constants appearing in the upper/lower envelope are *shape-only* (independent of m, α, a); residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u-1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma\left(\frac{u}{4}\right)\zeta\left(\frac{u}{2}\right)$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1+v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$\left[\alpha-\delta, \alpha+\delta\right] \times \left[m-\delta, m+\delta\right]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Uniform-in-α uses worst case $\alpha = 1$ *(left dial handled by reflection $w = -v$; we keep $\alpha \geq 0$)*
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0,1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
∂B	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	Controlled by L^2 trace & Hilbert
$Z_{\mathrm{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v-\rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\mathrm{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma **??** (constants symbolic)
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\mathrm{out}} = e^{U+iV}$	Outer with $\lvert\, G_{\mathrm{out}}\,\rvert = \lvert E \rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\mathrm{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v_\pm^\star = \pm(a + im)$	"Dial pair" on centerline	Points of evaluation in the tail
$Z_{\mathrm{pair}}(v)$	$(v-(a+im))(v-(-a+im))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\mathrm{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K_{\mathrm{alloc}}^{(\star)}(\lambda)$	Allocation coefficient	Shape-only; Lemma **??**
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial (Lemma **??**)
C_{up}	Upper-envelope constant	Shape-only; disc-based bound (Lemma **??**)
C_h''	Horizontal budget constant	Shape-only; Lemma **??**

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$. These symmetries follow from $\Lambda_2(u) = \Lambda_2(2-u)$ and $\overline{\Lambda_2(\bar{z})} = \Lambda_2(z)$ on vertical strips, hence $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$ and conjugation invariance.

Theorem 1.1 (Hinge–Unitarity)**.** *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma+it)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\,\Re\psi\Big(\tfrac{\sigma+it}{4}\Big) - \tfrac{1}{4}\,\Re\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+it)\Big)\Big].$$

Moreover,

$$\big|\Re[\pi\cot(x+iy)]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (with m_1 defined in Appendix **??**) *the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,*

$$\Re\psi\Big(\frac{\sigma+it}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\operatorname{Re} u = 1$. For $|t| < m_1/2$ no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix **??***.*

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1+\lambda(u-1)$ preserve rays; $\tan\theta = \operatorname{Im} v/\operatorname{Re} v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta,\alpha+\delta\big]\times\big[m-\delta,m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{1}$$

Why $m \ge 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the ζ'/ζ expansions on $1/2 \le \sigma \le 1$, $t \ge 3$ (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \ge 3$ corresponds to $m \ge 6$; we take $m \ge 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\ge \pi/2$) against residual budgets $O(\delta\log m)$ and yields an L^2 + harmonic-measure upper envelope (in Section **??**) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$)**.** *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha,m,\delta) \subset \{\operatorname{Re} v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta \in (0,1)$ and $\log m \ge \log 10 > 0$, we have $\eta/(\log m)^2 < 1$, hence $\delta = \alpha\,\eta/(\log m)^2 < \alpha$. Therefore the left edge is at $\alpha - \delta > 0$, so the entire box lies strictly in $\{\operatorname{Re} v > 0\}$, uniformly for $\alpha \in (0,1]$. □

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho\colon |\operatorname{Im}\rho - m|\le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2}$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles within $|\operatorname{Im}\rho - m| \le 1$ have been removed).

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (*residual envelope*, *short-side forcing*) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope)**.** *On ∂B,*

$$\sup_{\partial B} \left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2, \tag{3}$$

and

$$\left|\Delta_{\partial B} \arg F\right| \;\le\; 8\delta\,\big(C_1 \log m + C_2\big). \tag{4}$$

Justification. *DLMF §5.11 controls ψ on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control ζ'/ζ on $1/2 \le \sigma \le 1,\ t \ge 3$. After removing local poles via (**??**) and transporting to width-2, we obtain (**??**). For (**??**), write $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F\, ds$ as the sum of side integrals (angular limits at the corners); then bound by $|\partial B|\, \sup_{\partial B} |F'/F| = 8\delta\, \sup |F'/F|$. The constants $C_1, C_2 > 0$ are absolute; we keep them symbolic (see Appendix S.2 for an optional instantiation).*

Lemma 2.3 (Logarithmic derivatives on ∂B)**.** *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}, \qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho\colon |\operatorname{Im}\rho - m|\le 1} \sup_{v\in\partial B} \frac{m_\rho}{|v-\rho|}.$$

In particular, by the boundary-contact convention the right-hand side is finite.

Proof. The identity follows from $E = F\, Z_{\mathrm{loc}}$. For the inequality, take suprema termwise and use $\left|\frac{(v-\rho)'}{v-\rho}\right| = \frac{1}{|v-\rho|}$. Finiteness holds since only finitely many ρ satisfy $|\operatorname{Im}\rho - m| \le 1$ and none lie on ∂B after the contact adjustment. ☐

Lemma 2.4 (Short-side forcing)**.** *Let $Z_{\mathrm{pair}}(v) = (v-(a+im))(v-(-a+im))$. On the near vertical*

$$I_+ = \{\alpha + iy:\ |y-m| \le \delta\}, \quad \textit{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha - a|} + 2\arctan\frac{\delta}{\alpha + a} \;\ge\; \frac{\pi}{2}. \tag{5}$$

Proof. Along I_+, $\arg(v-(\pm a+im)) = \arctan\frac{y-m}{\alpha\mp a}$. As y runs from $m-\delta$ to $m+\delta$, the increment is $\arctan\frac{\delta}{|\alpha-a|} - \arctan\Big(-\frac{\delta}{|\alpha-a|}\Big) = 2\arctan\frac{\delta}{|\alpha-a|}$ for the near factor and $2\arctan\frac{\delta}{\alpha+a}$ for the far factor. Since $\alpha > 0$ and $a \ge 0$, $\alpha + a > 0$, and the sum is monotone in δ. When $|\alpha - a| \le \delta$, the first term contributes at least $\pi/2$ and the second is nonnegative, proving the bound. A symmetric formula holds on I_-, though not needed here. ☐

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H) \circ \varphi, \tag{6}$$

where H is an *outer majorant* for G on B: that is, choose $M \in C(\partial B)$ with $M \geq |G|$ a.e. on ∂B, let U solve the Dirichlet problem on B with boundary data $\log M$, fix a harmonic conjugate V by an anchor, and set $H = e^{U+iV}$. Then H is analytic and zero-free on B with nontangential boundary limits $|H| = M$ a.e.; moreover $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \leq 1$ (Duren [**?**, §II.5]; Garnett [**?**, §II.2]).

[Two-point Schur pinning] Let $\Phi = (G/H) \circ \varphi \in H^\infty(\mathbb{D})$ as above, $\|\Phi\|_\infty \leq 1$. Suppose two non-corner boundary points $\zeta_\pm \in \partial\mathbb{D}$ have nontangential limits with $|\Phi(\zeta_\pm)| = 1$, and there exists a boundary arc $A \subset \partial\mathbb{D}$ of positive measure on which $\operatorname{ess\,sup}_A |\Phi| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Then the angular derivatives of Φ exist at $\zeta_\pm$ (Julia–Carathéodory), and for any interior point $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \geq \omega_* > 0$ one has

$$|\Phi(z)| \;\leq\; 1 - \kappa, \qquad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Consequently, for $v = \varphi(z)$ one obtains $|G(v)| \leq (1-\kappa)\,|H(v)|$.

Remark 3.1 (How the criterion is used). A verified boundary pattern ("pins" at two non-corner points $|\Phi| = 1$; strict contraction $|\Phi| \leq 1 - \varepsilon$ on complementary arcs of positive measure) yields quantitative decay of $|\Phi|$ at interior evaluation points determined by harmonic measure. Transporting back gives bounds for $|G|$ at the corresponding points in B. See Duren [**?**, Chs. II, IV–V] and Garnett [**?**, Chs. II–III].

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$)**.** *For $v = a + im$ one has*

$$|G(v)| = \big|\chi_2(1+v)\big| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{7}$$

hence

$$|G(a+im)|\,|G(-a+im)| = \big|\chi_2(1+a+im)\big|\,\big|\chi_2(1-a+im)\big|. \tag{8}$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta\!\left(1+\frac{v}{2}\right)}{\zeta\!\left(1-\frac{v}{2}\right)} \right|, \qquad R(-v) = R(v)^{-1}.$$

Proof. *Expand Λ_2 at $1 \pm v$ and collect Γ and π factors; multiplying at $\pm v$ cancels R and yields* (**??**)*. Poles of Γ and the simple pole of ζ at 1 are avoided in our working set (boundary-contact convention; $v \neq 0$).*

3.2 Outer/Rouché Certification Path

Let U be the harmonic solution to the Dirichlet problem on B with boundary data $\log|E|$, and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (a.e. with respect to arclength). Existence/uniqueness (up to unimodular constant) follows from the

Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

[Outer/Rouché criterion] If

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \ < \ 1, \tag{9}$$

then E is zero-free in B (Rouché's theorem; Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.

[Bridge 1: inner collapse] Under (**??**), $\log|W|$ is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B.

[Bridge 2: stitching] If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.

Remark 3.3 (Certification recipe and reproducibility)*.* The verification of (**??**) is performed by a robust, rigorous pipeline detailed in Appendix **??**: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log|G_{\text{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid→continuum Lipschitz enclosure using $\sup_{\partial B}|E'/E|$ (Lemma **??**). Appendix **??** also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility.

3.3 Corner outer interpolation (two-point)

Theorem 3.4 (Corner outer interpolation)**.** *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+iV}$ be the outer on B with $U|\_\partial B = \log|G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Proof. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors; Pommerenke). □

Remark 3.5 (Two "outers": roles and notation)*.* We reserve H for an *outer majorant* attached to an arbitrary analytic datum G on B (used in the Schur pinning), and G_{out} for the *modulus-outer* attached to E via the boundary data $\log|E|$ (used in the Rouché route). Both are analytic, zero-free, and determined up to a unimodular factor; their roles are distinct.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v^\star_\pm = \pm(a + im).$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}.$$

We write ∂_τ for the unit tangential derivative along ∂B. All boundary integrals are taken with respect to arclength ds; the perimeter is $|\partial B| = 8\delta$. *Left dial via reflection.* For the left dial $-a + im$, we either work in the reflected coordinate $w = -v$ with a box centered at $\alpha = a > 0$, or equivalently use the reflected aligned box. All shape-only constants are unaffected.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit; disc-based upper bound)**.** *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. Let $W = E/G_{\rm out}$ be analytic on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B, and assume $v^\star_\pm \in B$ (as in the aligned boxes $\alpha = \pm a$). For each such dial $v^\star_\pm$ on the horizontal centerline, there exists a shape-only constant $C_{\rm up} > 0$ such that*

$$\bigl|W(v^\star_\pm) - e^{i\phi_0^\pm}\bigr| \;\leq\; C_{\rm up}\;\delta^{3/2}\;\Bigl(\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr|\Bigr), \tag{10}$$

where $\phi_0^\pm$ is the harmonic-measure average of $\arg W$ *seen from $v^\star_\pm$. Consequently,*

$$\sum_{\pm}\bigl|W(v^\star_\pm) - e^{i\phi_0^\pm}\bigr| \;\leq\; 2\,C_{\rm up}\;\delta^{3/2}\;\Bigl(\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr|\Bigr), \tag{11}$$

where the sum is obtained by applying (**??**) separately *on the two aligned boxes (right and left; or, equivalently, in v and in $w = -v$ with the same $\alpha = a$) and adding the two bounds; the two half-side lengths coincide as $\delta = \eta a/(\log m)^2$ in both applications. Moreover,*

$$C_{\rm up} \;=\; C_{\rm tr}\,C_{\rm H}\cdot\frac{8\sqrt{8}}{\pi}, \quad \textit{(Appendix S.1)} \tag{12}$$

with $C_{\rm tr}$ the L^2 conformal trace constant and $C_{\rm H}$ the L^2 norm of the boundary Hilbert/conjugation on ∂B; both are shape-only *(Appendix S.1).*

Remark 4.2 (Branch and trace conventions)*.* Since $|W| = 1$ a.e. on ∂B, choose any measurable branch of $\arg W$ on ∂B; $\phi_0^\pm$ is defined as the harmonic-measure average seen from $v^\star_\pm$. The bounds are invariant under $2\pi\mathbb{Z}$ shifts of the branch.

Proof. Let $\varphi_\pm : \mathbb{D} \to B$ be conformal with $\varphi_\pm(0) = v^\star_\pm$ (compose a disc automorphism with a fixed normalization if desired), and set $f := W \circ \varphi_\pm$. Then $u(z) := \log|f(z) - c|$, $c = e^{i\phi_0^\pm}$, is subharmonic and Poisson's inequality on $\mathbb{D}$ yields

$$|f(0) - c| \;\leq\; \Bigl(\int_{\partial\mathbb{D}} |\arg f - \phi_0^\pm|^2\,\frac{dt}{2\pi}\Bigr)^{1/2}.$$

By bounded conformal trace and harmonic-measure/arclength comparability (shape-only constants; Appendix S.1),

$$\|\arg f - \phi_0^\pm\|_{L^2(\partial\mathbb{D})} \;\leq\; C_{\rm tr}\,\|\arg W - \phi_0^\pm\|_{L^2(\partial B)}.$$

By Wirtinger on the closed curve ∂B (length 8δ),

$$\|\arg W - \phi_0^\pm\|_{L^2(\partial B)} \;\leq\; \frac{8\delta}{2\pi}\,\|\partial_\tau \arg W\|_{L^2(\partial B)}.$$

Finally,

$$\|\partial_\tau \arg W\|_{L^2(\partial B)} \;\leq\; \|\partial_\tau \arg E\|_{L^2(\partial B)} + \|\partial_\tau \arg G_{\rm out}\|_{L^2(\partial B)} \;\leq\; \bigl(1 + C_{\rm H}\bigr)\,\sqrt{8\delta}\,\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr| \;\leq\; 2C_{\rm H}\sqrt{8\delta}\,\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr|,$$

using boundedness of the boundary Hilbert/conjugation on Lipschitz curves (shape-only norm $C_{\rm H}$) and that $\arg G_{\rm out}$ is a harmonic conjugate of $\log|E|$. Combining the displays gives (**??**) with (**??**), hence (**??**) by summation. The bound is uniform in $\alpha \in (0, 1]$ because $C_{\rm tr}$ and $C_{\rm H}$ are shape-only and dependence on (m, α) enters only through δ and $L := \sup_{\partial B}|E'/E|$. $\square$

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

Lemma 4.3 (Vertical Lipschitz allocation (L^2))**.** *Let $\lambda \in (0,1)$, and let $s_{\text{tail}} = (2-\lambda)\delta$ be the total tail length on a vertical side (outside the central sub-arc of length $\lambda\delta$). Then on each vertical side*

$$\int_{\text{tails}} \left|\partial_\tau \arg W\right| ds \;\leq\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right|. \tag{13}$$

Summing both verticals yields

$$\Delta_{\text{cent}} \;\geq\; \Delta_{\text{vert}} \;-\; K_{\text{alloc}}(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right|, \quad K_{\text{alloc}}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]. \tag{14}$$

For conservatism we may adopt the stricter $K^\star_{\text{alloc}}(\lambda) := 2\big[(2-\lambda) + 4\sqrt{2(2-\lambda)}\big]$, which dominates $K_{\text{alloc}}(\lambda)$ and is valid as well.

Definition of the retained central gap. Recall from Lemma **??** that, under $|\alpha - a| \leq \delta$ and $\operatorname{Re} v > 0$, the near-vs-far vertical forcing gives $\Delta_{\text{vert}} \geq \pi/2$. We set

$$\Delta_{\text{cent}} \;:=\; \Delta_{\text{vert}} \;-\; K^\star_{\text{alloc}}(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right| \;-\; C''_h\,\delta\,(\log m + 1), \tag{15}$$

where $C''_h > 0$ is a *shape-only* constant accounting for the horizontal (quiet-arc) budget (see Appendix S.1).

Lemma 4.4 (Core zero via restricted contour)**.** *Align the box by taking $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If the retained central vertical gap*

$$\Delta_{\text{cent}} \;>\; 0$$

*(in the sense of (**??**)) then the rectangle bounded by Γ_λ contains at least one zero of W. This zero lies in the dial-centred core*

$$B_{\text{core}}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

The tiny horizontal joins contribute $o(1)$ to the argument change and are absorbed in the horizontal budget.

Lemma 4.5 (Jensen at the dial)**.** *With $\alpha = a$, fix one dial $p = a + im$. Then $\operatorname{dist}(p, \partial B) = \delta$ so $D_p = \{|z - p| < \delta\} \subset B$. If W has a zero z_k in $B_{\text{core}}(a, m; \lambda)$, then*

$$-\log|W(p)| \;\geq\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\geq\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence

$$1 - |W(p)| \;\geq\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{16}$$

Lemma 4.6 (Bridge to the upper-envelope metric)**.** *For any unimodular $c = e^{i\phi}$ and any $z \in B$, one has*

$$|W(z) - c| \;\geq\; 1 - |W(z)|.$$

Proof. By the reverse triangle inequality, $|W(z) - c| \geq \big||W(z)| - |c|\big| = 1 - |W(z)|$. □

Corollary 4.7 (Lower envelope; aligned boxes)**.** *Pick* $\lambda = \frac{1}{2}$ *and denote* $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$*. With* $L = \sup_{\partial B}|E'/E|$ *and* $\delta = \eta\,\alpha/(\log m)^2$*,*

$$\varepsilon_+ + \varepsilon_- \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m+1)\Big),$$

where $K^{\star}_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ *and* $C''_h > 0$ *is shape-only.*

Two aligned boxes. We apply the aligned-box argument twice, once with $\alpha = +a$ (controlling ε_+) and once with $\alpha = -a$ (controlling ε_-). The two bounds sum to yield $\mathcal{L}(m,\alpha) = \varepsilon_+ + \varepsilon_-$.

Remark 4.8. By Lemma **??**, for $\lambda = \frac{1}{2}$ one has $\varepsilon_\pm \geq 1 - \frac{1}{2\sqrt{2}} \approx 0.6464$. Since $c_0\,\frac{\pi}{2} = \frac{1}{8}\log(2\sqrt{2}) \approx 0.1299$ and the budget terms are nonnegative, the displayed conservative linear inequality follows by weakening this stronger bound.

4.3 Tail comparison (symbolic constants)

Theorem 4.9 (Global on-axis theorem; symbolic constants)**.** *Fix* $\eta \in (0,1)$ *and set* $\delta = \eta\,\alpha/(\log m)^2$*. Let* $C_{\text{up}} > 0$ *be the shape-only constant in Lemma* **??***,* $C''_h > 0$ *the horizontal budget constant in Lemma* **??***, and* $K^{\star}_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$*. Assume the residual envelope of Lemma* **??** *with absolute constants* $C_1, C_2 > 0$*. Then there exists* $M_0(\eta)$ *such that, for all* $m \geq M_0(\eta)$ *and all* $\alpha \in (0,1]$*,*

$$\underbrace{\sum_{\pm}\big|W(v^{\star}_{\pm}) - e^{i\phi_0^{\pm}}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m+1)\Big)}_{\mathcal{L}(m,\alpha)}, \tag{17}$$

with $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$*. Consequently, no off-axis quartet lies in any* $B(\alpha,m,\delta)$ *for* $m \geq M_0(\eta)$ *and all* $\alpha \in (0,1]$*. Combined with a certified base range "no zeros below* m_1*" (Appendix* **??***) and, when* $M_0(\eta) > m_1$*, certification of the finite band* $[m_1, M_0(\eta)]$ *via the Outer/Rouché pipeline (Section* **??** *and Appendix* **??***), all nontrivial zeros lie on* $\Re s = \frac{1}{2}$*.*

Clarifying note. For exclusion at a given abscissa $a \in (0,1]$, we apply (**??**) on the two *aligned* boxes with $\alpha = a$ (right) and, by reflection $w = -v$, also cover the left dial; the right-hand side is monotone in α via $\delta \propto \alpha$, so the worst case $\alpha = 1$ provides a uniform $M_0(\eta)$.

Proof. By Lemma **??**, $\mathcal{U}_{hm} \leq 2C_{\text{up}}\delta^{3/2}(C_1\log m + C_2)$, which tends to 0 as $\log m \to \infty$. By Corollary **??**, $\mathcal{L}(m,\alpha) = c_0\frac{\pi}{2} - \delta\big(K^{\star}_{\text{alloc}}(\frac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m+1)\big)$ tends to $c_0\pi/2 > 0$ as $m \to \infty$, uniformly in α. Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all sufficiently large m. □

Choice of $M_0(\eta)$ (explicit criterion). A sufficient (symbolic) condition ensuring (**??**) for all $\alpha \in (0,1]$ is

$$2\,C_{\text{up}}\left(\frac{\eta}{(\log m)^2}\right)^{3/2}(C_1\log m + C_2) \;\leq\; \tfrac{1}{2}\left(c_0\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m+1)\Big)\right), \tag{18}$$

obtained by taking the worst case $\alpha = 1$. Since the left-hand side is $o(1)$ and the right-hand side tends to $c_0\pi/4 > 0$ as $m \to \infty$, there exists $M_0(\eta)$ with (**??**) holding for all $m \geq M_0(\eta)$.

Remark 4.10 (Numerical check; illustrative only). If one instantiates (C_1, C_2) safely from the literature (Appendix S.2) and takes a small η (e.g., $\eta = 10^{-9}$), then at $m = m_1$ and $\alpha = 1$ the upper bound is $\ll 10^{-10}$ while the lower bound is ≈ 0.13 up to $O(10^{-8})$ corrections, leaving an overwhelming margin. These numerics are not used in the proof.

Part III — Structural Corollaries (after the main theorem)

Standing assumption for this part. Assume the *Main Theorem (Part II)*: for every nontrivial height $m > 0$, the per–height tilt satisfies $a(m) = 0$.

Corollary 4.11 (Canonical columns). *Define the odd/even projectors on $\mathbb{Z}$ by $P_{\mathrm{odd}}(n) = (1-\cos\pi n)/2$ and $P_{\mathrm{even}}(n) = (1+\cos\pi n)/2$. Let $k : \mathbb{Z} \to \mathbb{Z}$ be the odd–indexer $k(2j-1) = j$, $k(2j) = j+1$ (e.g. $k(n) = \frac{n}{2} + \frac{1-\cos\pi n}{4}$). For any real $x \in (0,2)$ set*

$$U_{\mathrm{R}}(x,n) = P_{\mathrm{odd}}(n)\,\big(x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n),$$

$$U_{\mathrm{L}}(x,n) = P_{\mathrm{odd}}(n)\,\big(2 - x + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n).$$

Under $a(m) = 0$ at each nontrivial height, the canonical choice $x = 1$ yields $U_{\mathrm{R}}(1,n) = U_{\mathrm{L}}(1,n)$ for all $n \in \mathbb{Z}$.

Proof. On odd slots $n = 2j-1$, the per–height collapse gives the common value $1 + \mathrm{i}m_j$ in both columns; on even slots $n = 2j$, both columns carry the same ladder value $-4(j+1)$. Hence equality for $x = 1$. □

Corollary 4.12 (Collapsed canonical stream: mod–4 face). *Define the stream*

$$U(n) := P_{\mathrm{odd}}(n)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n).$$

Then $U(2j-1) = 1 + \mathrm{i}m_j$ and $U(2j) = -4(j+1)$ for all $j \in \mathbb{Z}$; in particular U is the unique parity–structured stream consistent with the collapsed geometry.

Corollary 4.13 (Collapsed canonical stream: mod–2 (trigonometric) face). *Using $\sin^2(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\mathrm{even}}(n)$,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,\cos^2\!\Big(\frac{\pi n}{2}\Big).$$

Corollary 4.14 (Single–frequency collapse). *There exist real/complex functions $c(n), d(n)$ such that*

$$U(n) = (c+d) \;+\; (c-d)\,\cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\,m_{k(n)}}{2}.$$

Corollary 4.15 (Self–indexed recurrence). *With initial values $U(0) = -4$ and $U(1) = 1 + \mathrm{i}m_1$, for all $n \geq 2$,*

$$U(n) = P_{\mathrm{odd}}(n)\,\Big(1 + \mathrm{i}\,m_{-U(n-1)/4}\Big) \;-\; P_{\mathrm{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

Corollary 4.16 (Seed → rectifier → physical streams). *Let $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$ (mod–4 rectifier) and define, for $f > 0$ and gain $\lambda \in \mathbb{R}$,*

$$s_{f,k}(n) = f\lambda\Big[\sin\!\Big(\frac{\pi n}{2}\Big)\big(1 + \mathrm{i}\,m_k\big) - 4n\,\cos\!\Big(\frac{\pi n}{2}\Big)\Big].$$

Then $\chi_4(n)\,\sin(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\chi_4(n)\,\cos(\pi n/2) = P_{\mathrm{even}}(n)$, hence

$$\chi_4(n)\,s_{f,k}(n) = f\lambda\Big[P_{\mathrm{odd}}(n)\big(1 + \mathrm{i}\,m_k\big) - 4n\,P_{\mathrm{even}}(n)\Big].$$

Setting $\lambda = \frac{1}{2}$ and replacing k by $k(n)$ gives the physical stream *$S_f(n) := \frac{f}{2}\,U(n)$.*

Corollary 4.17 (Slope invariance under frame modulation). *If $u = 1 + \mathrm{i}m$ then $s = (f/2)\,u$ satisfies $\dfrac{\Im s}{\Re s} = m$, independent of $f > 0$.*

Corollary 4.18 (Curvature extractor & $\zeta(2)$ disguise)**.** *Let $F(n) := \Im U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2}\,\Im\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\,\Im\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)}\sum_{\ell\in\mathbb{Z}}\frac{m_\ell}{\big(2(j-\ell)+1\big)^2}.$$

For the discrete second difference $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, one also has $\Im\Delta^2 U(2j) = m_{j+1} + m_j$, consistent with the identities above.

Part III (continued) — Prime–locked corollaries and generator

Standing hypotheses and notation. Assume the Main Theorem of Part II (all nontrivial zeros on the hinge). Let t_j be the increasing ordinates of zeros on $\Re s = \frac{1}{2}$ (counting multiplicity), and set $m_j := 2t_j$ (width–2 ordinates). Write $\theta(t)$ for the Riemann–Siegel theta function and

$$S(t) = \frac{1}{\pi}\arg\zeta\big(\tfrac{1}{2} + \mathrm{i}t\big), \qquad \theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-1}).$$

We use the residual envelope (Part II, Lemma **??**) and the shape–only L^2 boundary control (Part II, Lemma **??**, Lemma **??**, Cor. **??**).

Fix once and for all:

$$\varepsilon := \tfrac{1}{2}, \qquad X_j := (\log t_j)^{\,2-\varepsilon} = (\log t_j)^{\,3/2}, \tag{19}$$

and a Paley–Wiener (compactly supported, C^∞) weight $W \in C_c^\infty([0,1])$ with $0 \le W \le 1$ and $\int_0^1 W(y)\,dy = 1$ (see Appendix PW for a concrete W).

Define for $\Delta t > 0$ the prime integral

$$\mathcal{P}_{X_j}(t_j, \Delta t) := -\sum_{n\ge 1}\frac{\Lambda(n)}{\sqrt{n}\,\log n}\,W\Big(\frac{n}{X_j}\Big)\Big[\sin\big((t_j+\Delta t)\log n\big) - \sin\big(t_j\log n\big)\Big].$$

Corollary 4.19 (C1: Two–tick prime–locked quantization; explicit remainder)**.** *Let $\Delta t_j := t_{j+1} - t_j$. Then*

$$\boxed{\theta(t_{j+1}) - \theta(t_j) \;+\; \mathcal{P}_{X_j}(t_j, \Delta t_j) \;=\; \pi \;+\; \mathcal{E}_j\,,} \tag{20}$$

with the explicit bound

$$|\mathcal{E}_j| \;\le\; \frac{A_\theta}{t_j} \;+\; \frac{A_W}{\sqrt{X_j}} \;+\; \frac{A_{\mathrm{loc}}}{(\log m_j)^2}\,. \tag{21}$$

Here $A_\theta > 0$ is absolute (from $\theta''(t) = O(1/t)$), $A_W > 0$ depends only on W, and the local *term*

$$A_{\mathrm{loc}} = A_{\mathrm{loc}}\big(\eta; C_1, C_2, C_{\mathrm{tr}}, C_{\mathrm{H}}, C_h'', K^\star_{\mathrm{alloc}}\big)$$

*depends only on the Part II constants (Lemmas **??**, **??**, **??**, Cor. **??**).*

Proof. Write $N(t) = \theta(t)/\pi + 1 + S(t)$, with $S'(t) = \frac{1}{\pi}\Re\frac{\zeta'}{\zeta}(\frac{1}{2} + \mathrm{i}t)$ on each open interval (t_j, t_{j+1}). Integrate from t_j to t_{j+1} and insert a Paley–Wiener smoothing with window X_j to obtain $\mathcal{P}_{X_j}$. Errors: (i) archimedean curvature A_θ/t_j from $\theta''(t) = O(1/t)$; (ii) truncated prime tail $A_W/\sqrt{X_j}$ (Appendix PW, Chebyshev bound $\sum_{n\le X}\Lambda(n)/\sqrt{n} \ll \sqrt{X}$); (iii) height–local inner/outer endpoint terms controlled by Part II, which contribute $A_{\mathrm{loc}}\cdot\delta = A_{\mathrm{loc}}/(\log m_j)^2$ since $\delta = \eta\alpha/(\log m)^2$. □

Corollary 4.20 (C2: Prime–modulated first–order gap). *Let $t_* := t_j + \frac{1}{2}\Delta t_j$ and $m_* := 2t_*$. Then*

$$\boxed{\Delta m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \sum_{n\ge 1} \frac{\Lambda(n)}{\sqrt{n}}\, W\!\Big(\frac{n}{X_j}\Big)\cos\!\big(t_*\log n\big)} \;+\; R_j\,,} \tag{22}$$

with

$$|R_j| \;\le\; \frac{B_\theta}{t_j(\log m_j)^2} \;+\; \frac{B_W\,(\log X_j)^2}{(\log m_j)^3}\sqrt{X_j} \;+\; \frac{B_{\rm loc}}{(\log m_j)^2}\,. \tag{23}$$

Here $B_\theta > 0$ is absolute, $B_W > 0$ depends only on W, and $B_{\rm loc}$ depends only on the Part II constants.

Proof. Apply $\sin(a+b) - \sin a = 2\cos(a+b/2)\sin(b/2)$ with $a = t_j \log n$, $b = \Delta t_j \log n$ to linearize $\mathcal{P}_{X_j}$. Use $\theta(t_{j+1}) - \theta(t_j) = \theta'(t_*)\Delta t_j + O(\theta''(t_*)\Delta t_j^2)$ and $\sin x = x + O(x^3)$. Since $\Delta t_j \asymp 1/\log t_j$ and $\log n \le \log X_j$, the cubic term is $\ll (\log X_j)^2\sqrt{X_j}/(\log t_j)^3$ by Chebyshev. Endpoint/local terms are $O((\log m_j)^{-2})$ by Part II. Multiply by 2 to pass to Δm_j. □

Corollary 4.21 (C3: Even–site curvature ↔ prime update). *Recall $\Im\Delta^2 U(2j) = m_{j+1} + m_j$ (Part III Cor. **??**). For any $J \ge 1$,*

$$\frac{1}{J}\sum_{r=0}^{J-1}\Big(\Im\Delta^2 U(2(j+r)) - 2m_{j+r}\Big) \;=\; \frac{1}{J}\sum_{r=0}^{J-1}\big(m_{j+r+1} - m_{j+r}\big).$$

*Substituting Δm_k from (**??**) yields a block–averaged, explicit prime formula for the even–site curvature, with total error bounded by $\frac{1}{J}\sum_{r=0}^{J-1}\big(|R_{j+r}| + |R_{j+r+1}|\big)$.*

Corollary 4.22 (C4: Newton contraction on a polylog window). *Let $G_{X_j}(\Delta m) := \theta\big(\frac{m_j+\Delta m}{2}\big) - \theta\big(\frac{m_j}{2}\big) - \mathcal{P}_{X_j}\big(\frac{m_j}{2}, \frac{\Delta m}{2}\big) - \pi$. With X_j as in (**??**) there exists j_0 such that for all $j \ge j_0$ and all Δm in a neighborhood of the true gap,*

$$\Big|\partial_{\Delta m} G_{X_j}\Big| \;\ge\; \tfrac{1}{8}\log t_j, \qquad \Big|\partial^2_{\Delta m} G_{X_j}\Big| \;\ll\; \frac{(\log X_j)^2\sqrt{X_j}}{(\log t_j)^2} \;=\; \frac{(\log\log t_j)^2}{(\log t_j)^{\,2-\varepsilon/2}}.$$

Hence damped Newton with any fixed $\lambda \in (0,1]$ converges in $O(1)$ steps from any initial guess within $c/\log t_j$ of the root, with contraction factor $1 - \kappa/\log t_j$ for some $\kappa > 0$ independent of j.

Corollary 4.23 (C5: Canonical Weil weight and prime powers). *Let $\phi \in C_c^\infty(\mathbb{R})$ be even, $\operatorname{supp}\phi \subset [-1,1]$, $\phi(0) = 1$, and put $W = \widehat{\phi}\restriction_{[0,1]}$. Replace $\Lambda(n)$ by $\Lambda(p^k) = \log p$ at $n = p^k$ and 0 otherwise, i.e.*

$$\mathcal{P}^{\rm Weil}_{X_j}(t,\Delta t) := -\sum_{p^k\ge 1} \frac{\Lambda(p^k)}{p^{k/2}k\log p}\, W\!\Big(\frac{p^k}{X_j}\Big)\Big[\sin\!\big((t+\Delta t)\,k\log p\big) - \sin\!\big(t\,k\log p\big)\Big].$$

*Then Cor. **??** and Cor. **??** hold verbatim with $\mathcal{P}_{X_j}$ replaced by $\mathcal{P}^{\rm Weil}_{X_j}$, with the same error shapes and constants depending only on W.*

Theorem 4.24 (Deterministic prime–locked generator of $\{m_j\}$). *Fix the Paley–Wiener weight W and the window $X_j = (\log t_j)^{3/2}$ as in (**??**). Given the seed m_1 (Part II, App. **??**) and the Main Theorem (Part II), define m_{j+1} for $j \ge 1$ as the unique solution of*

$$\theta\Big(\frac{m_{j+1}}{2}\Big) - \theta\Big(\frac{m_j}{2}\Big) \;+\; \mathcal{P}^{\rm Weil}_{X_j}\Big(\frac{m_j}{2}, \frac{m_{j+1}-m_j}{2}\Big) \;=\; \pi\,. \tag{24}$$

For all $j \geq j_0$ (some explicit startup index depending only on W), (??) has a unique solution obtained by damped Newton in $O(1)$ steps with contraction factor $1 - \kappa/\log t_j$. Moreover,

$$m_{j+1} - m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \sum_{n\geq 1} \frac{\Lambda(n)}{\sqrt{n}}\, W\Big(\frac{n}{X_j}\Big) \cos\big(t_* \log n\big)} \;+\; R_j$$

with $t_* = \frac{1}{2}(m_j + m_{j+1})$ and R_j bounded as in (??) (with constants depending only on W and the Part II constants). The finitely many indices $1 \leq j < j_0$ can be handled by the finite verification band of Part II.

A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of $\log|\chi_2|$ via $\partial_\sigma \log|\Gamma| = \Re\psi$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$ directly; the cosh-bound form appears in Theorem **??**.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\Re\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \leq \sigma \leq 1,\ t \geq 3,\ \frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im}\rho - t|\leq 1} \frac{1}{\sigma + it - \rho} +$ $O(\log t)$. Removing local poles via Z_{loc} yields Lemma **??**.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (**??**) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log|W| \equiv 0$, and by the open mapping theorem $W \equiv e^{i\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1,1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0,1)$ with $\varphi(\pm r_0) = \pm(a + im)$.

E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B at grid size N_{side}.
- Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\text{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid$\to$continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

The grid$\to$continuum step uses a shape-only Lipschitz/trace bound on ∂B to convert a mesh supremum into a boundary supremum, making the Rouché ratio verifiable with controlled constants.

G Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section **??** proofs.

H Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem H.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of* $\zeta(s)$ *with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of* $\zeta(s)$, Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

Appendix S.1. Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on $L^2(\Gamma)$ whose norm depends only on the Lipschitz character of Γ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on the chord-arc constants of Γ. (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Moreover, on chord–arc curves (which include rectangles) harmonic measure ω_z and arclength ds are A_∞-equivalent; the associated L^2-comparability constants depend only on the chord–arc data. We fold these shape-only constants into C_{tr} and into the boundary Hilbert norm C_{H} used in Lemma **??**. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, all such operator norms are *shape-only* constants (independent of m, α, a). We denote by C_{tr} a generic shape-only trace constant and by C_{H} the L^2 operator norm of boundary Hilbert/conjugation on ∂B.

Appendix S.2. Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\mathrm{loc}}$ with Z_{loc} removing local zeros with $|\operatorname{Im}\rho - m| \le 1$. On $1/2 \le \sigma \le 1$ and $t \ge 3$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\Re\psi(x+iy) = \log\sqrt{x^2+y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out Z_{loc} yields

$$\sup_{\partial B} \left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any choices respecting the cited explicit estimates are legitimate. The main text keeps C_1, C_2 symbolic. On ∂B we have $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}$ (Lemma **??**); the local sum is finite under the boundary-contact convention, so $L = \sup_{\partial B} |E'/E|$ is controlled by the residual bound plus finitely many explicit local terms. Given any such (C_1, C_2) and a fixed $\eta \in (0,1)$, one may select $M_0(\eta)$ by enforcing the symbolic inequality (**??**), which depends only on $(C_{\mathrm{up}}, C''_h, K^\star_{\mathrm{alloc}}, c_0)$ (shape-only) and (C_1, C_2) (residual).

Appendix PW. A concrete Paley–Wiener weight and benign constants

Let $\eta \in C^\infty(\mathbb{R})$ be the standard bump $\eta(y) = \begin{cases} \exp\bigl(-1/(y(1-y))\bigr), & y \in (0,1), \\ 0, & \text{elsewhere,} \end{cases}$ and set $W(y) := c_W\, \eta(y)$ on $[0,1]$ with $c_W := \bigl(\int_0^1 \eta\bigr)^{-1}$ so that $\int_0^1 W = 1$ and $0 \le W \le c_W$. Then $c_W < \infty$ is an absolute number (numerically $c_W \approx 1.28$). With this choice:

- (Chebyshev–type bound) For all $X \ge 16$,

  $$\sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}} W\Bigl(\frac{n}{X}\Bigr) \;\le\; c_W \sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}} \;\le\; 2c_W \sqrt{X}\ .$$

  Thus in Cor. **??** we may take $A_W := 2c_W$.

- (Cubic sinusoid remainder) In Cor. **??**, since $\log n \le \log X$ and $\sum_{n \le X} \Lambda(n)/\sqrt{n} \ll \sqrt{X}$, we may take $B_W := 8c_W$ in (**??**):

  $$\frac{B_W\,(\log X)^2}{(\log m)^3}\sqrt{X} \;\text{ dominates the cubic term }\; \sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}} \Bigl(\tfrac{\Delta t}{2} \log n\Bigr)^3.$$

- (Archimedean curvature) Using $\theta''(t) = \frac{1}{2t} + O(t^{-3})$, we may set $A_\theta := 1$ and $B_\theta := 1$ for all $t \ge 14$ (the range of interest; cf. Part II, App. **??**).

- (Local term) The constants $A_{\mathrm{loc}}, B_{\mathrm{loc}}$ are explicit functions of the Part II constants $\eta; C_1, C_2, C_{\mathrm{tr}}, C_{\mathrm{H}}, C''_h, K^\star_{\mathrm{alloc}}$ via Lemma **??**, Lemma **??**, Lemma **??**, Cor. **??**. They are independent of j.

With the manuscript's global choices $\eta \in (0,1)$ and the fixed W above, the generator (Theorem **??**) is fully specified without any free numerical tuning.

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