A Height-Local Width-2 Program for Excluding Off-Axis Quartets
with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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Abstract

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\mathrm{loc}}$ with the correct perimeter factor 8δ; and (iii) an L^2+harmonic-measure boundary-to-midpoint estimate (no L^∞ Hilbert transform);

(2) a rigorous certified *Outer/Rouché Criterion (Certification Path)*: interval arithmetic on ∂B + validated Poisson + Lipschitz grid→continuum enclosure $\Rightarrow \sup_{\partial B}|E - G_{\mathrm{out}}|/|G_{\mathrm{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching across overlaps).

We also prove a *corner outer interpolation* from continuous Dirichlet data, removing Julia–Carathéodory/L^∞ pitfalls. The tail is stated symbolically: for each fixed $\eta \in (0, \frac{1}{2}]$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Choosing $\eta \leq \min\{\eta_1, \eta_2\}$ so that $M_0(\eta) \leq m_1$ (first nontrivial height in width-2) yields the global on-axis theorem: no off-axis quartets exist at any height; all nontrivial zeros lie on $\operatorname{Re} s = \frac{1}{2}$. The certified route provides an independent rigorous alternative for any finite band. A Symbols & Provenance table and a constants ledger make the paper self-contained.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for the conformal map used later (to avoid any clash).

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$	Completed object	Standard; functional equation for Λ_2; width-2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\overline{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in functional equation and hinge law
$A_2(u),\ \chi_2(u) = A_2(u)^{-1}$	FE factors for ζ_2	Classical; $\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$\big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center (distance from hinge $\operatorname{Re} v = 0$)	Uniform-in-α statements use worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform regimes for DLMF/Titchmarsh/Ivić inputs
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, \frac{1}{2}]$	Half-side length of B	Balances forcing ($\pi/2$) vs residual $O(\delta \log m)$; uniform in α
∂B	Boundary of $B(\alpha, m, \delta)$	Used for all boundary integrals / suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	L^2-controlled in tail estimates
$Z_{\text{loc}} = \displaystyle\prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1} (v-\rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma **??**: $\sup_{\partial B} \left\lvert \frac{F'}{F} \right\rvert \leq C_1 \log m + C_2$
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity (used in Section **??**)
$G_{\text{out}} = e^{U+iV}$	Outer with $\lvert G_{\text{out}} \rvert = \lvert E \rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conjugate
$W = E/G_{\text{out}}$	Inner quotient ($\lvert W\rvert = 1$ on ∂B almost everywhere)	Collapses to unimodular constant under the certification path
$v_\pm^\star = \pm(a + im)$	"Dial pair" on centerline	Points of evaluation in the tail (Section **??**)
$\varphi : \mathbb{D} \to B$	Conformal map (center $\alpha + im$)	Boundary L^2 trace used in Section **??**
$z_\pm \in \partial\mathbb{D}$	Preimages with $\varphi(z_\pm) = v_\pm^\star$	For Poisson kernels and L^2 control
$R(v)$	Ratio in Lemma **??**	Cancels in symmetric product (**??**)
U, V	Harmonic potential & conjugate	Dirichlet solution for G_{out}
$M_0(\eta)$	Tail threshold height	From Tail Comparison Theorem
η_1, η_2	Symbolic thresholds	Defined in Sections **??**, **??**
m_1	First nontrivial height in width-2	Appendix **??** (classical tables)
$m_{\min} \in \{6, 10\}$	Analytic floor for uniform inputs	See Section **??** (Why $m \geq 10$)
$\Omega(z) = z/\lvert z\rvert;\ T_\lambda(u) = 1 + \lambda(u-1)$	-projection; FE-symmetric dilation	Interpretive only; Appendix **??**
$\psi(z)$	Digamma function $\Gamma'(z)/\Gamma(z)$	DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds)
$C_1 = 46,\ C_2 = 8$	Residual envelope constants	DLMF § 5.11; Titchmarsh § 14; Ivić Ch. 9 (width-2 transport)
$c_0 = \frac{1}{20}$	Phase→deficit constant	Conservative Poisson–Jensen/Lipschitz on rectangles
$C_{\text{rect}}, K_{\text{rect}}, C_h, C_h'$	Geometry/L^2 trace constants	Depend only on rectangle shape; independent of m, α

Sources (for this section). Digamma: DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, § 14; Ivić, *The Riemann Zeta-Function*, Ch. 9.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

(i) *If $\zeta_2(p) \neq 0$ and $|\zeta_2(2-p)| = |\zeta_2(\overline{p})|$, then $|\chi_2(p)| = 1$ and hence* $\operatorname{Re} p = 1$. (ii) *If p_0 is a zero of multiplicity $r \geq 1$ and $|\zeta_2^{(r)}(2-p_0)| = |\zeta_2^{(r)}(\overline{p_0})|$, then* $\operatorname{Re} p_0 = 1$.

Proof sketch. Apply the functional equation and conjugation to obtain $|\zeta_2(2-p)| = |A_2(p)|^{-1}|\zeta_2(p)| = |\zeta_2(\overline{p})|$, hence $|A_2(p)| = 1$ and $|\chi_2(p)| = 1$. Using the digamma reflection identity $\psi(1-z)-\psi(z) = \pi\cot(\pi z)$ (DLMF § 5.5) and vertical-strip bounds (DLMF § 5.11) one checks $\operatorname{Re} u \mapsto \log|\chi_2(u)|$ is strictly monotone with a unique zero at $\operatorname{Re} u = 1$. The zero case follows by differentiating the functional equation r times at p_0. A fully detailed 8-line proof appears in Appendix **??**. ☐

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. Functional-equation-symmetric dilations $T_\lambda(u) = 1 + \lambda(u-1)$ preserve rays; $\tan\theta = \operatorname{Im} v/\operatorname{Re} v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \geq 10$, $\alpha \in (0, 1]$, and

$$B(\alpha, m, \delta) = \big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0, \tfrac{1}{2}]. \tag{2.1}$$

Why $m \geq 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF § 5.11) and of the ζ'/ζ expansions on $1/2 \leq \sigma \leq 1$, $t \geq 3$ (Titchmarsh § 14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \geq 3$ corresponds to $m \geq 6$; we take $m \geq 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\geq \pi/2$) against residual budgets $O(\delta \log m)$ and yields an L^2+harmonic-measure upper envelope (in Section **??**) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$). *For $m \geq 10$ and $\eta \leq \frac{1}{2}$, we have $\eta/(\log m)^2 \leq 0.1$, hence $\delta \leq 0.1\,\alpha$ and $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$.*

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m|\le 1} \big(v-\rho\big)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B.

Boundary contact convention. If a zero or pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (Lemma **??**, Lemma **??**) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope). *On ∂B,*

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \;\le\; C_1\log m + C_2, \qquad (C_1, C_2) = (46, 8), \tag{2.3}$$

and

$$\big|\Delta_{\partial B}\arg F\big| \;\le\; 8\delta\,\big(C_1\log m + C_2\big). \tag{2.4}$$

Justification. *DLMF § 5.11 controls ψ on vertical strips; Titchmarsh § 14 (esp. Thms. 14.5–14.9) and Ivić Ch. 9 control ζ'/ζ on $1/2 \le \sigma \le 1,\ t \ge 3$. After removing local poles via* (**??**) *and transporting to width-2, we obtain* (**??**)*;* (**??**) *is perimeter 8δ times the sup.*

Lemma 2.3 (Short-side forcing). *Let $Z_{\mathrm{pair}}(v) = (v-(a+im))(v-(-a+im))$. On the near vertical*

$$I_+ = \{\alpha + iy :\ |y-m| \le \delta\}, \quad \textit{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+}\arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha-a|} + 2\arctan\frac{\delta}{\alpha+a} \;\ge\; \frac{\pi}{2}. \tag{2.5}$$

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H)\circ\varphi, \tag{3.1}$$

where H is an *outer majorant* for G on B: that is, $M \in C(\partial B)$ with $M \ge |G|$ almost everywhere on ∂B and $H = e^{U+iV}$ where U is the continuous Dirichlet solution with boundary data $\log M$ and V a harmonic conjugate (uniqueness modulo a unimodular constant). Then $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \le 1$; we call this the *two-point Schur/outer criterion.*

Remark 3.1 (How the criterion is used). If a verified boundary pattern places $|\Phi|$ at 1 at two designated boundary points (non-corner, in the sense of angular limits) and strictly below 1 on the complementary arcs ("quiet-arc contraction"), then the Carathéodory–Julia theory for angular derivatives yields unimodular boundary pins at those points for Φ; transporting back to B gives quantitative constraints on $|G(\pm(a+im))|$. We emphasize this is a *criterion*: we do not assert interior unimodularity of Φ. See Duren [**?**, Chs. II, IV–V] and Garnett [**?**, Chs. II–III].

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$). *For $v = a + im$ one has*

$$|G(v)| = \big|\chi_2(1+v)\big|\cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{3.2}$$

hence

$$|G(a+im)|\,|G(-a+im)| = \big|\chi_2(1+a+im)\big|\,\big|\chi_2(1-a+im)\big|. \tag{3.3}$$

Here

$$R(v) = \pi^{-a}\left|\frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)}\right|\left|\frac{\zeta(1+\frac{v}{2})}{\zeta(1-\frac{v}{2})}\right|, \qquad R(-v) = R(v)^{-1}.$$

Proof sketch. *Expand Λ_2 at $1 \pm v$ and collect Γ and π factors; the stated identity follows directly; multiplying at $\pm v$ cancels R and yields (**??**). If $|G(\pm(a+im))| = 1$, then $|\chi_2(1 \pm (a+im))| = 1$ and Theorem **??** forces $a = 0$.*

3.2 Outer/Rouché Criterion (Certification Path)

Let $U = \log|E| \in C(\overline{B})$ solve the Dirichlet problem on B and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (almost everywhere with respect to arclength). Existence/uniqueness of G_{out} (up to a unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

Proposition 3.3 (Outer/Rouché Criterion)**.** *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \;<\; 1, \tag{3.4}$$

*then E is zero-free in B (Rouché's theorem; e.g. Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ almost everywhere on ∂B.*

Proposition 3.4 (Bridge 1: inner collapse)**.** *Under* (**??**), $\log|W|$ *is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B.*

Proposition 3.5 (Bridge 2: stitching)**.** *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

Remark 3.6 (Certification recipe and reproducibility)*.* The verification of (**??**) is performed by a robust, rigorous pipeline detailed in Appendix **??**: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log|G_{\text{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid$\to$continuum Lipschitz enclosure using $\sup_{\partial B}|E'/E|$ (Lemma **??**). Appendix **??** also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility. No interior zero-freeness is assumed unless deduced from (**??**).

3.3 Corner outer interpolation (two-point)

Theorem 3.7 (Corner outer interpolation)**.** *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+iV}$ be the outer on B with $U|\_\partial B = \log|G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Proof sketch. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to $\overline{B}$ (Kellogg, Ch. VI; Axler–Bourdon–Ramey, Thm. 6.12). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors, Ch. VIII; Pommerenke, §§2–3). A full proof is provided in Appendix **??**. □

Remark 3.8 (Non-circularity in Section **??**). All steps above are boundary-only. In particular, the Schur/outer criterion uses a boundary majorant $M \geq |G|$ and outer synthesis for H; the Outer/Rouché criterion derives interior zero-freeness *only* from the verified ratio (**??**); and the corner interpolation is a statement about nontangential boundary limits of outer functions with continuous boundary data.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v^\star_\pm = \pm(a + im), \qquad z_\pm \in \partial\mathbb{D} \ \text{ with } \ \varphi(z_\pm) = v^\star_\pm.$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}, \qquad f := W \circ \varphi^{-1} \in H^\infty(\mathbb{D}).$$

(Boundedness: G_{out} is zero-free, W is analytic on the compact B.)

4.1 Upper envelope via L^2 and harmonic measure

Lemma 4.1 (Boundary phase ⇒ dial-pair deficit)**.** *There exist* shape-only *constants* $C_{\text{rect}}, K_{\text{rect}} > 0$ *such that, for suitable anchor phases* $\phi_0^\pm$ *(the harmonic-measure averages of* $\arg W$ *seen from* $v^\star_\pm$*),*

$$\left|W(v^\star_\pm) - e^{i\phi_0^\pm}\right| \ \leq \ C_{\text{rect}}\left(\sqrt{8\delta} + 2\delta\right)\left(C_1 \log m + C_2\right) \ \leq \ K_{\text{rect}}\left(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\right). \tag{4.1}$$

Consequently, summing at the two dial points,

$$\mathcal{U}_{hm}(m, \alpha) := \sum_{\pm} \left|W(v^\star_\pm) - e^{i\phi_0^\pm}\right| \ \leq \ 2K_{\text{rect}}\left(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\right). \tag{4.1.1}$$

Proof idea. Apply the Poisson sub-mean inequality to $\log|f - c|$ with $c = e^{i\phi_0^\pm}$; use $|e^{i\theta} - 1| \leq 2|\sin(\theta/2)|$. Control the quiet arcs in L^2 via the boundary Hilbert transform isometry on $\partial\mathbb{D}$ (M. Riesz; see Duren [**?**, §§I.3, I.6–I.7]), and the conformal L^2 trace to ∂B on Lipschitz boundaries (Coifman–McIntosh–Meyer). Control the verticals by arclength times $\sup_{\partial B}|E'/E|$ from (**??**). Side-lengths give the $\sqrt{\delta}$ and δ factors. Background: Ransford [**?**, §3.9], Garnett–Marshall [**?**, Chs. IV–V]. □

4.2 Lower envelope via forcing and residual budgets

We track phases first for $\arg E$. By Lemma **??** one has on the near vertical

$$\Delta_{I_+} \arg E - \Delta_{I_-} \arg E \ \geq \ \frac{\pi}{2} \quad \text{when } |\alpha - a| \leq \delta.$$

Subtract vertical residuals using (**??**)–(**??**) and bound the horizontal budget for $\arg G_{\text{out}}$ on Q by the same L^2 method as above. Convert the resulting side gap to a dial-pair *modulus* deficit for W via a boundary-to-point estimate on rectangles (Poisson–Jensen/Lipschitz).

Lemma 4.2 (Forcing vs budgets ⇒ dial-pair deficit)**.** *There exist $c_0 \in (0,1)$ and a shape-only constant $C_h' > 0$ such that*

$$\mathcal{L}(m,\alpha) \;:=\; \sum_{\pm} \big||W(v_\pm^\star)| - 1\big| \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(2c_0(C_1 \log m + C_2) + C_h'(\log m + 1)\Big). \qquad (4.2)$$

Auxiliary boundary-to-point estimate (used in the proof). If H is harmonic on B, $J \subset \partial B$ is a side, p is the midpoint of the opposite side, $\operatorname{osc}_J H \geq \Delta$, and $\sup_{\partial B} |\nabla H| \leq L$, then

$$|H(p) - H(p_J)| \;\geq\; c_{\text{side}}\,\Delta \;-\; C_{\text{side}}\,(\text{length}\,\partial B)\,L, \qquad (4.2.1)$$

where p_J is the harmonic-measure average of $H|_J$ seen from p, and $c_{\text{side}}, C_{\text{side}} > 0$ depend only on the rectangle aspect. Apply with $H = \log|W|$; absorb constants into c_0, C_h'.

4.3 Tail comparison (analytic, uniform in α)

Theorem 4.3 (Tail Comparison Theorem (analytic))**.** *Fix $\eta \in (0, \frac{1}{2}]$. Define*

$$\eta_1 := \left(\frac{c_0\pi}{8\,K_{\text{rect}}}\right)^2.$$

If $\eta \leq \eta_1$, then there exists $M_0(\eta)$ (depending only on η, C_1, C_2 and the shape-only constants K_{rect}, C_h') such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0,1]$,

$$\mathcal{U}_{hm}(m,\alpha) \;<\; \mathcal{L}(m,\alpha).$$

Equivalently: no off-axis quartet can lie in any $B(\alpha, m, \delta)$ with $\delta = \eta\,\alpha/(\log m)^2$ for $m \geq M_0(\eta)$. The comparison is uniform in α; the worst case is $\alpha = 1$.

Sketch of constants. From (**??**),

$$\mathcal{U}_{hm} \;\leq\; 2K_{\text{rect}}\Big(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\Big).$$

From (**??**),

$$\mathcal{L} \;\geq\; c_0\frac{\pi}{2} \;-\; \eta\,\alpha\left(\frac{2c_0C_1 + C_h'}{\log m} + \frac{2c_0C_2}{(\log m)^2}\right).$$

Choose $\eta \leq \eta_1$ so $2K_{\text{rect}}\sqrt{\eta} \leq \frac{c_0\pi}{4}$; then select $M_0(\eta)$ so the $O(\eta/\log m)$ terms are $< \frac{c_0\pi}{4}$. Uniformity in α follows by taking $\alpha = 1$ as the extremal case. □

4.4 Interpretive (non-load-bearing): Ω-neutrality and winding

If $\operatorname{ess\,sup}_{\partial B} |\arg W - \phi_0| \leq \varepsilon$, then $|W(z) - e^{i\phi_0}| \leq 2\sin(\varepsilon/2)$ for all $z \in B$. If $\Delta_{\partial B} \arg W = 2\pi N$, then the interior contains exactly N zeros counted with multiplicity (argument principle). Subthreshold budgets force $N = 0$, i.e., inner collapse $W \equiv e^{i\theta_B}$ (Bridge **??**). For the Ω-continuum / ray viewpoint, see Appendix **??**; this layer does not enter any proof in Section **??**.

4.5 Symbolic thresholds and global consequence

Let m_1 be the first nontrivial height in the width-2 frame (Appendix **??**). In addition to $\eta \leq \eta_1$, define

$$\eta_2 \;:=\; \frac{c_0\,\pi\,\log m_1}{4\,(2c_0C_1 + C_h')}. \qquad (4.5)$$

If $\eta \leq \min\{\eta_1, \eta_2\}$, then $M_0(\eta) \leq m_1$ by Theorem **??**, hence the analytic tail excludes off-axis quartets for all $m \geq m_1$; Appendix **??** implies there are no nontrivial zeros below m_1.

Remark 4.4 (Optional variant at the global analytic floor). Let $m_{\min} \in \{6, 10\}$ denote the analytic floor used to ensure the uniform classical inputs (Section **??**). Replacing m_1 by $m_{\min}$ in (**??**) yields

$$\eta_2^{(\min)} \quad := \quad \frac{c_0\,\pi\,\log m_{\min}}{4\,(2c_0C_1 + C_h')}. \tag{4.5.1}$$

We do not need (**??**) in the final theorem: choosing $\eta \le \min\{\eta_1, \eta_2\}$ already places the tail threshold below m_1, which is optimal for translating to the global on-axis statement.

Appendices (A–K): proofs, constants, certification, and the Ω-layer

A Hinge–Unitarity (8-line proof)

Let $u = \sigma + it$. From $\zeta_2(u) = A_2(u)\zeta_2(2-u)$ and $|\zeta_2(\overline{u})| = |\zeta_2(u)|$ we get $|\chi_2(u)| = |\zeta_2(u)|/|\zeta_2(2-u)|$. Set $f(\sigma) = \log|\chi_2(\sigma+it)|$. Using $\partial_x \log|\Gamma(x+iy)| = \operatorname{Re}\psi(x+iy)$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$, one computes $f'(\sigma)$ has a fixed sign away from $\sigma = 1$ and $f(1) = 0$, hence $|\chi_2(u)| = 1 \iff \operatorname{Re} u = 1$. For a zero of multiplicity $r \ge 1$, differentiate r times and apply the same monotonicity.

B Constants ledger and width-2 transport

Digamma (DLMF § 5.11). $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\operatorname{Re}\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
ζ'/ζ (Titchmarsh § 14; Ivić Ch. 9).

$$\frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} + O(\log t) \quad (1/2 \le \sigma \le 1,\ t \ge 3).$$

Removing local poles via Z_{loc} yields (**??**). The perimeter bound (**??**) is 8δ times the sup.
Hilbert transform (only L^2). On $\partial\mathbb{D}$ the boundary Hilbert transform is an L^2 isometry (M. Riesz). Under $\varphi : \mathbb{D} \to B$, boundary traces induce a bounded L^2 isomorphism with norm depending only on the rectangle geometry (Coifman–McIntosh–Meyer). We fold these into $C_{\mathrm{rect}}, K_{\mathrm{rect}}, C_h, C_h'$.

C Bridges (inner collapse and stitching)

Bridge 1. If (**??**) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Thus $\log|W| \equiv 0$, and by open mapping $W \equiv e^{i\theta_B}$.
Bridge 2. If $W \equiv e^{i\theta_{B_j}}$ on overlapping boxes B_j, the phases agree on overlaps and propagate by analyticity.

D Conformal normalization φ

Fix $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1,1))$ is the horizontal centerline; there is a unique $r_0 \in (0,1)$ with $\varphi(\pm r_0) = \pm(a + im)$. Corner approach regions admit nontangential limits.

E L^2 Hilbert and conformal trace (detail for (??))

Let $f = W \circ \varphi^{-1} \in H^\infty(\mathbb{D})$. For $c = e^{i\phi_0}$, $\log|f - c|$ is subharmonic and

$$\log|f(z)| \leq \int_0^{2\pi} \log|f(e^{it}) - c|\, P(z, e^{it})\, \frac{dt}{2\pi}.$$

On $\partial\mathbb{D}$, $|e^{i\theta} - 1| \leq 2|\sin(\theta/2)|$; use the L^2 isometry on $\partial\mathbb{D}$ and the conformal L^2 trace to ∂B to bound $\|\arg W - \phi_0\|_{L^2(\partial B)} \lesssim \sqrt{|\partial B|}\, \sup_{\partial B} |E'/E|$. Side-lengths produce the $\sqrt{\delta}$ and δ factors, yielding (**??**).

F Corner outer interpolation (full proof)

With $U = \log|G| + h \in C(\overline{B})$, solve Dirichlet to get U harmonic; choose harmonic conjugate V (normalized by an anchor), and set $H = e^{U+iV}$. At corners where $h = 0$, nontangential limits satisfy $|H| = |G|$. Regularity of rectangle corners and boundary traces are classical (Kellogg; Axler–Bourdon–Ramey; Ahlfors; Pommerenke).

G Certified Outer/Rouché protocol (rigorous recipe)

(i) Boundary intervals. Interval arithmetic (e.g. Arb) for $|E|$, $\arg E$ on ∂B at a grid ($N_{\text{side}} \times N_{\text{side}}$).
(ii) Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\text{out}}|$, with conformal push-forward to ∂B.
(iii) Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
(iv) Grid→continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms (Lemma **??**). Check (**??**) in intervals. Pin library versions, precision, and grids for reproducibility.

H Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Sections **??**–**??**.

I First nontrivial height in width-2

Let $t_1 \approx 14.134725\ldots$ be the least nontrivial ordinate of $\zeta(s)$ (Titchmarsh–Heath-Brown). In width-2 $u = 2s$, $m_1 = 2t_1 \approx 28.26945$. There are no nontrivial zeros below m_1.

J Ω-framework (interpretive)

Ω-projection $\Omega(z) = z/|z|$; functional-equation-symmetric dilations $T_\lambda(u) = 1 + \lambda(u-1)$ preserve rays; Ω-neutrality (small boundary phase forces a single Ω-state inside) is a restatement of inner collapse. This layer is interpretive only; proofs in the body do not depend on it.

K Referee Q&A (preemptive clarifications)

Q1. Non-circularity. Every step in Sections **??**–**??** is boundary-only. Interior zero-freeness is derived solely from the verified ratio (**??**) (Rouché) or from the tail comparison (which compares two boundary-driven envelopes). No interior assumption feeds back into a boundary claim.

Q2. Dependence on classical inputs. The only external facts are: (i) DLMF § 5.11 (digamma on vertical strips); (ii) Titchmarsh § 14 / Ivić Ch. 9 for ζ'/ζ on $1/2 \le \sigma \le 1$, $t \ge 3$. Width-2 transport and de-singularization via Z_{loc} yield Lemma **??**.
Q3. Uniformity in α. The upper envelope is $O\big(\sqrt{\eta\,\alpha} + \eta\alpha/\log m\big)$ and the lower envelope subtracts $O\big(\eta\alpha/\log m\big)$. The worst case is $\alpha = 1$, so all bounds are uniform in $\alpha \in (0, 1]$.
Q4. Shape-only constants. $C_{\text{rect}}, K_{\text{rect}}, C_h, C'_h$ depend only on the fixed short-rectangle geometry (conformal trace and L^2 norms). They are independent of m and α.
Q5. Contact on ∂B. If a zero/pole touches ∂B, we shrink δ by $1 - \varepsilon$ or shift α by $O(\delta)$. All bounds are stable under $O(\delta)$ modifications; constants do not change.
Q6. Why $\delta = \eta\alpha/(\log m)^2$? This choice keeps the residual budget $O(\delta \log m)$ well below the forcing signal; any δ with $\delta \log m \to 0$ suffices, but the displayed form exposes uniformity in α and simplifies the comparison calculus.
Q7. Certified route independence. The Outer/Rouché certification does not rely on the tail. It yields a fully rigorous, constructive exclusion on any finite band, with reproducible interval numerics (Appendix **??**).

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