A Height–Local Width–2 Program for Excluding Off–Axis Quartets
Analytic Tail & Certified Outer/Rouché Criterion

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Abstract

This paper is organized in three parts. **Part I** (Reader's Guide) reduces the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: *RH* $\Leftrightarrow a(m) = 0$ *at each nontrivial height* m, while recording non–load–bearing structural scaffolding. **Part II** gives a self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based L^2 upper envelope and an L^2 lower envelope via allocation + restricted contour + Jensen. We also provide a rigorous Outer/Rouché Certification Path with explicit domains and symbolic constants ("shape–only" vs. residual). **Part III** promotes the toolbox identities to structural corollaries once $a(m) = 0$ is established.

Contents

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered a–lens that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become consequences once $a(m) = 0$ is proved.
What it does not do. It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later; this Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)}\,\Gamma\Big(\frac{s}{2f}\Big)\,\zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.

Width–2 normalization. Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2 - u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, A_2 is analytic and nonvanishing.

*__Authorship and AI-use disclosure.__ The author designed the framework, chose all constants/normalizations, and validated all mathematics and computations. Generative assistants (from GPT–4o to GPT–5 Pro) were used solely for typesetting assistance, editorial organization, and consistency checks; they are not an author. All claims are the author's responsibility (COPE/ICMJE guidance).

Partner map. On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$, swapping the two column points at the same height.

Hinge unitarity (deferred). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$" iff $\operatorname{Re} u = 1$ is proved in Part II (Hinge–Unitarity). We do not use it here.

2) Centered a–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$.

Nontrivial height. A "nontrivial height" $m > 0$ means: m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + \mathrm{i}m/2$. The reduction shows that whenever such an m occurs, the associated tilt must satisfy $a(m) = 0$.

Tilt at height m. At fixed $m > 0$, set

$$U_{\mathrm{R}}(m; a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m; a) = 1 - a + \mathrm{i}m, \qquad a \in [0, 1).$$

In the centered frame, the dial points are $\pm(a + \mathrm{i}m)$. The partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$.

Quartet. Conjugation (top$\leftrightarrow$bottom) and FE reflection generate the quartet $\{\, 1 \pm a \pm \mathrm{i}m \,\}$ at height m.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + \mathrm{i}m$ and its slope is $\operatorname{Im} u / \operatorname{Re} u = m/1 = m$. Rescaling to any frame $s = (f/2)\, u$ preserves the slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\, m}{f/2} = m.$$

Thus $\{m_k\}$ simultaneously records the imaginary ordinates of the nontrivial zeros and their origin through slopes in every modulated frame—provided the per–height collapse holds.

4) Height–local reduction of RH

Fix a nontrivial height $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The following are purely algebraic and equivalent:

- (PHU–1) Column equality: $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) Ray (slope) lock: $\operatorname{Im} U_{\mathrm{R}} / \operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}} / \operatorname{Re} U_{\mathrm{L}}$, i.e. $m/(1+a) = m/(1-a) \iff a = 0$.
- (PHU–3) Hinge form: $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

Reduction target. RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse; nothing from this Guide is used there.

5) Box alignment and hand–off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\, \alpha / (\log m)^2, \quad \eta \in (0, 1).$$

When $\alpha = \pm a$, the dial points $\pm(a + \mathrm{i}m)$ lie on the box's horizontal centerline.

What Part II does. Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three–lines tools, Stirling–class envelopes, explicit control of ζ'/ζ away from zeros), Part II shows that any off–axis quartet forces a boundary lower bound larger than an explicit upper bound, hence $a(m) = 0$.

No circularity. The analytic proof is logically independent of this Guide.

6) Parity gating and selection devices (interpretive only)

Gating from the non–completed FE. In the width–2 frame the non–completed FE reads

$$\zeta_2(u) = A_2(u)\,\zeta_2(2-u), \quad A_2(u) = 2^{u/2}\,\pi^{\,u/2-1}\,\sin\Big(\frac{\pi u}{4}\Big)\,\Gamma\Big(1-\frac{u}{2}\Big).$$

On the open strip $0 < \operatorname{Re} u < 2$ with $\operatorname{Im} u \neq 0$, the prefactor $A_2(u)$ is nonzero and finite; its sine zeros (the "trivial ladder") lie on the real axis only. Thus *inside the open strip only ζ_2 can vanish* (nontrivial zeros), while the *trivial class is confined to the real axis.* This is the basic "odd/even lane" picture: the odd (upper) lane can host nontrivial zeros; the even (real) lane hosts the trivial ladder.

Orthogonal split on the integer lattice. To model this dichotomy as a clean input–space symmetry, decompose any lattice signal $X : \mathbb{Z} \to \mathbb{C}$ via the orthogonal projectors

$$P_{\text{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \qquad P_{\text{even}}(n) = \tfrac{1+\cos(\pi n)}{2},$$

so $X = P_{\text{odd}}X + P_{\text{even}}X$. We *assign the nontrivial stream to odd slots* (where $P_{\text{odd}} = 1$) and the *trivial ladder to even slots* (where $P_{\text{even}} = 1$). This mirrors the FE fact above without using it analytically.

7) Toolbox $\to$ structural consequences (after the theorem)

The items below are not inputs to the analytic proof. After Part II proves $a(m) = 0$ for all nontrivial heights, they become *Structural Corollaries* describing the collapsed geometry and its lattice faces (brief proofs appear in Part III).

- Pre–collapse columns (projector faces in the u–frame): right/left templates place odd–slot samples $x \pm \mathrm{i}m_k$ and the even ladder $-4(\cdot)$ via $P_{\text{odd}}, P_{\text{even}}$.
- Collapsed canonical stream $U(n)$: when per–height collapse holds ($x = 1$ on odd slots), the two columns coincide; parity face (via $P_{\text{odd}}, P_{\text{even}}$) and an equivalent trigonometric face.
- Single–frequency collapse (cosine face): a two–parameter cosine form $U(n) = (c+d) + (c-d)\cos(\pi n)$; c, d simple in the odd–indexer $k(n)$.
- Self–indexed recurrence (no explicit k): a short recurrence for $U(n)$ pulls the needed odd index from the previous even sample.
- Curvature extractor & the $\zeta(2)$ disguise: the discrete second difference of the imaginary part at even indices recovers m_j and admits an odd–square convolution normalized by $\zeta(2)$.
- Seed $\to$ rectifier $\to$ physical streams: two–carrier seeds rectify under a mod–4 factor to yield the physical stream $S_f(n) \propto U(n)$; pre–collapse faces scale analogously.

8) Implications and one–sentence hand–off

The width–2 organization centralizes symmetry at $\operatorname{Re} u = 1$; the centered a–lens isolates the single per–height degree of freedom; parity–orthogonal scaffolding separates the nontrivial stream from the ladder without entering the proof. With these definitions, RH reduces to: for every nontrivial height $m > 0$, $a(m) = 0$.

Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm \mathrm{i}m\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\mathrm{loc}}$ with perimeter factor 8δ; and (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants;

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow \sup_{\partial B}\big|E - G_{\mathrm{out}}\big|/|G_{\mathrm{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{\mathrm{i}\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed $\eta \in (0, 1)$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Combined with a certified base range below m_1 (first nontrivial height in width-2), this yields the global on-axis theorem. All constants in the upper/lower envelope are *shape-only*; residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(\frac{u}{4})\zeta(\frac{u}{2})$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1+v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha-\delta, \alpha+\delta] \times [m-\delta, m+\delta]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Left dial handled by reflection $w = -v$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, 1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
∂B	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	Controlled by L^2 trace & Hilbert
$Z_{\mathrm{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m \rvert \leq 1}(v - \rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\mathrm{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma **??** (constants symbolic)
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\mathrm{out}} = e^{U+iV}$	Outer with $\lvert G_{\mathrm{out}} \rvert = \lvert E \rvert$ on ∂B	$U = \log\lvert E \rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\mathrm{out}}$	Inner quotient ($\lvert W \rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v^\star_\pm = \pm(a + im)$	Dial pair on centerline	Points of evaluation in the tail
$Z_{\mathrm{pair}}(v)$	$(v - (a+im))(v - (-a+im))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\mathrm{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K^{(\star)}_{\mathrm{alloc}}(\lambda)$	Allocation coefficient	Shape-only; Lemma **??**
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial; Lemma **??**
C_{up}	Upper-envelope constant	Shape-only; Lemma **??**
C''_h	Horizontal budget constant	Shape-only; Lemma **??**

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm \mathrm{i}m\}$. These symmetries follow from $\Lambda_2(u) = \Lambda_2(2-u)$ and $\overline{\Lambda_2(\overline{z})} = \Lambda_2(z)$ on vertical strips, hence $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$ and conjugation invariance.

Theorem 1.1 (Hinge–Unitarity)**.** *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma + \mathrm{i}t)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\,\mathrm{Re}\,\psi\Big(\tfrac{\sigma+\mathrm{i}t}{4}\Big) - \tfrac{1}{4}\,\mathrm{Re}\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+\mathrm{i}t)\Big)\Big].$$

Moreover,

$$\big|\,\mathrm{Re}\big[\pi\cot(x+\mathrm{i}y)\big]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

*Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (with m_1 defined in Appendix **??**) the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,*

$$\mathrm{Re}\,\psi\Big(\frac{\sigma+\mathrm{i}t}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

*hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\mathrm{Re}\,u = 1$. For $|t| < m_1/2$ no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix **??**.*

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1+\lambda(u-1)$ preserve rays; $\tan\theta = \mathrm{Im}\,v/\,\mathrm{Re}\,v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta,\alpha+\delta\big] \times \big[m-\delta,m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{1}$$

Why $m \ge 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the ζ'/ζ expansions on $1/2 \le \sigma \le 1,\ t \ge 3$ (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \ge 3$ corresponds to $m \ge 6$; we take $m \ge 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\ge \pi/2$) against residual budgets $O(\delta\log m)$ and yields an L^2 + harmonic-measure upper envelope (in Section **??**) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\mathrm{Re}\,v > 0$)**.** *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha,m,\delta) \subset \{\mathrm{Re}\,v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta \in (0,1)$ and $\log m \ge \log 10 > 0$, we have $\eta/(\log m)^2 < 1$, hence $\delta = \alpha\,\eta/(\log m)^2 < \alpha$. Therefore the left edge is at $\alpha - \delta > 0$, so the entire box lies strictly in $\{\mathrm{Re}\,v > 0\}$. □

De-singularization on ∂B. Let

$$Z_{\rm loc}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m|\le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\rm loc}(v)}. \tag{2}$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles within $|\operatorname{Im}\rho - m| \le 1$ have been removed).

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (residual envelope, short-side forcing) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope)**.** *On ∂B,*

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \;\le\; C_1\log m + C_2, \tag{3}$$

and

$$\left|\Delta_{\partial B}\arg F\right| \;\le\; 8\delta\,\big(C_1\log m + C_2\big). \tag{4}$$

Justification. *DLMF §5.11 controls ψ on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control ζ'/ζ on $1/2 \le \sigma \le 1,\ t\ge 3$. After removing local poles via (**??**) and transporting to width-2, we obtain (**??**). For (**??**), write $\Delta_{\partial B}\arg F = \int_{\partial B}\partial_\tau \arg F\,ds$ as the sum of side integrals (angular limits at the corners); then bound by $|\partial B|\,\sup_{\partial B}|F'/F| = 8\delta\,\sup|F'/F|$. The constants $C_1, C_2 > 0$ are absolute; we keep them symbolic (see Appendix **??** for an optional instantiation).*

Lemma 2.3 (Logarithmic derivatives on ∂B)**.** *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\rm loc})'}{Z_{\rm loc}}, \qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho:\,|\operatorname{Im}\rho-m|\le 1}\ \sup_{v\in\partial B}\frac{m_\rho}{|v-\rho|}\,.$$

In particular, by the boundary-contact convention the right-hand side is finite.

Lemma 2.4 (Short-side forcing)**.** *Let $Z_{\rm pair}(v) = (v-(a+\mathrm{i}m))(v-(-a+\mathrm{i}m))$. On the near vertical*

$$I_+ = \{\alpha + \mathrm{i}y:\ |y-m|\le\delta\}, \quad \text{with } |\alpha - a|\le\delta,$$

one has

$$\Delta_{I_+}\arg Z_{\rm pair} = 2\arctan\frac{\delta}{|\alpha-a|} + 2\arctan\frac{\delta}{\alpha+a} \;\ge\; \frac{\pi}{2}. \tag{5}$$

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi:\mathbb{D}\to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H)\circ\varphi, \tag{6}$$

where H is an *outer majorant* for G on B: choose $M\in C(\partial B)$ with $M\ge |G|$ a.e. on ∂B, let U solve the Dirichlet problem on B with boundary data $\log M$, fix a harmonic conjugate V by an anchor, and set $H = e^{U+\mathrm{i}V}$. Then H is analytic and zero-free on B with nontangential boundary limits $|H| = M$ a.e.; moreover $\Phi\in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty\le 1$ (Duren [**?**, §II.5]; Garnett [**?**, §II.2]).

[Two-point Schur pinning] Let $\Phi = (G/H) \circ \varphi \in H^\infty(\mathbb{D})$ as above, $\|\Phi\|_\infty \le 1$. Suppose two non-corner boundary points $\zeta_\pm \in \partial\mathbb{D}$ have nontangential limits with $|\Phi(\zeta_\pm)| = 1$, and there exists a boundary arc $A \subset \partial\mathbb{D}$ of positive measure on which $\operatorname{ess\,sup}_A |\Phi| \le 1 - \varepsilon$ for some $\varepsilon > 0$. Then the angular derivatives of Φ exist at $\zeta_\pm$ (Julia–Carathéodory), and for any interior point $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \ge \omega_* > 0$ one has

$$|\Phi(z)| \;\le\; 1 - \kappa, \qquad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Consequently, for $v = \varphi(z)$ one obtains $|G(v)| \le (1-\kappa)\,|H(v)|$.

Lemma 3.1 (Two-point link for $|G|$ and $|\chi_2|$)**.** *For $v = a + \mathrm{i}m$ one has*

$$|G(v)| = \big|\chi_2(1+v)\big| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{7}$$

hence

$$|G(a+\mathrm{i}m)|\,|G(-a+\mathrm{i}m)| = \big|\chi_2(1+a+\mathrm{i}m)\big|\,\big|\chi_2(1-a+\mathrm{i}m)\big|. \tag{8}$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta(1+\frac{v}{2})}{\zeta(1-\frac{v}{2})} \right|, \qquad R(-v) = R(v)^{-1}.$$

3.2 Outer/Rouché Certification Path

Let U be the harmonic solution to the Dirichlet problem on B with boundary data $\log|E|$, and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\mathrm{out}} := e^{U + \mathrm{i}V}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\mathrm{out}}| = |E|$ nontangentially on ∂B (a.e.). Existence/uniqueness (up to unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

[Outer/Rouché criterion] If

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\mathrm{out}}(v)|}{|G_{\mathrm{out}}(v)|} \;<\; 1, \tag{9}$$

then E is zero-free in B (Rouché's theorem; Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\mathrm{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.

[Bridge 1: inner collapse] Under (**??**), $\log|W|$ is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{\mathrm{i}\theta_B}$ on B for some real constant θ_B.

[Bridge 2: stitching] If B_1, B_2 overlap and $W \equiv e^{\mathrm{i}\theta_{B_j}}$ on B_j $(j = 1,2)$, then $e^{\mathrm{i}\theta_{B_1}} = e^{\mathrm{i}\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.

Remark 3.2 (Certification recipe and reproducibility)*.* The verification of (**??**) is performed by a rigorous pipeline (Appendix **??**): (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log|G_{\mathrm{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\mathrm{out}}$; and (iv) a grid$\to$continuum Lipschitz enclosure using $\sup_{\partial B} |E'/E|$ (Lemma **??**). Appendix **??** also pins libraries (e.g., Arb), precisions, and boundary meshes to ensure reproducibility.

3.3 Corner outer interpolation (two-point)

Theorem 3.3 (Corner outer interpolation)**.** *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+\mathrm{i}V}$ be the outer on B with $U|\_\partial B = \log|G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Remark 3.4 (Two "outers": roles and notation)*.* We reserve H for an *outer majorant* attached to an arbitrary analytic datum G on B (used in the Schur pinning), and G_{out} for the *modulus-outer* attached to E via the boundary data $\log|E|$ (used in the Rouché route). Both are analytic, zero-free, and determined up to a unimodular factor; their roles are distinct.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + \mathrm{i}m$; define the *dial pair* on the horizontal centerline by

$$v_\pm^\star = \pm(a + \mathrm{i}m).$$

Split the boundary ∂B into the two *quiet arcs Q* (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\mathrm{out}}}.$$

We write ∂_τ for the unit tangential derivative along ∂B. All boundary integrals are taken with respect to arclength ds; the perimeter is $|\partial B| = 8\delta$. For the left dial $-a + \mathrm{i}m$, we either work in the reflected coordinate $w = -v$ with a box centered at $\alpha = a > 0$, or equivalently use the reflected aligned box (shape-only constants are unaffected).

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit; disc-based upper bound)**.** *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. Let $W = E/G_{\mathrm{out}}$ be analytic on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B, and assume $v_\pm^\star \in B$ (as in the aligned boxes $\alpha = \pm a$). For each such dial $v_\pm^\star$ on the horizontal centerline, there exists a shape-only constant $C_{\mathrm{up}} > 0$ such that*

$$\bigl|W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm}\bigr| \;\leq\; C_{\mathrm{up}}\; \delta^{3/2} \;\Bigl(\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr|\Bigr), \tag{10}$$

where $\phi_0^\pm$ is the harmonic-measure average of $\arg W$ *seen from $v_\pm^\star$. Consequently,*

$$\sum_\pm \bigl|W(v_\pm^\star) - e^{\mathrm{i}\phi_0^\pm}\bigr| \;\leq\; 2\,C_{\mathrm{up}}\; \delta^{3/2} \;\Bigl(\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr|\Bigr), \tag{11}$$

where the sum is obtained by applying (**??**) *separately on the two aligned boxes (right and left; or in v and $w = -v$ with the same $\alpha = a$) and adding the bounds. Moreover,*

$$C_{\mathrm{up}} \;=\; C_{\mathrm{tr}}\, C_{\mathrm{H}} \cdot \frac{8\sqrt{8}}{\pi}, \tag{12}$$

with C_{tr} the L^2 conformal trace constant and C_{H} the L^2 norm of the boundary Hilbert/conjugation on ∂B (both shape-only; see Appendix **??***).*

Remark 4.2 (Branch and trace conventions)*.* Since $|W| = 1$ a.e. on ∂B, choose any measurable branch of $\arg W$ on ∂B; $\phi_0^\pm$ is defined as the harmonic-measure average seen from $v_\pm^\star$. The bounds are invariant under $2\pi\mathbb{Z}$ shifts of the branch.

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

Lemma 4.3 (Vertical Lipschitz allocation (L^2))**.** *Let $\lambda \in (0,1)$, and let $s_{\text{tail}} = (2-\lambda)\delta$ be the total tail length on a vertical side (outside the central sub-arc of length $\lambda\delta$). Then on each vertical side*

$$\int_{\text{tails}} \left|\partial_\tau \arg W\right| ds \;\le\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right|. \tag{13}$$

Summing both verticals yields

$$\Delta_{\text{cent}} \;\ge\; \Delta_{\text{vert}} \;-\; K_{\text{alloc}}(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right|, \quad K_{\text{alloc}}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]. \tag{14}$$

For conservatism we may adopt $K^\star_{\text{alloc}}(\lambda) := 2\big[(2-\lambda) + 4\sqrt{2(2-\lambda)}\big]$.

Retained central gap. Under $|\alpha - a| \le \delta$ and $\operatorname{Re} v > 0$, the near/far vertical forcing gives $\Delta_{\text{vert}} \ge \pi/2$ (Lemma **??**). We set

$$\Delta_{\text{cent}} \;:=\; \Delta_{\text{vert}} \;-\; K^\star_{\text{alloc}}(\lambda)\,\delta \sup_{\partial B}\left|\frac{E'}{E}\right| \;-\; C''_h\,\delta\,(\log m + 1), \tag{15}$$

where $C''_h > 0$ is a shape-only constant accounting for the horizontal (quiet-arc) budget (Appendix **??**).

Lemma 4.4 (Core zero via restricted contour)**.** *Align the box by taking $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If $\Delta_{\text{cent}} > 0$ in the sense of* (**??**)*, then the rectangle bounded by Γ_λ contains at least one zero of W. This zero lies in the dial-centred core*

$$B_{\text{core}}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

The tiny horizontal joins contribute $o(1)$ to the argument change and are absorbed in the horizontal budget.

Lemma 4.5 (Jensen at the dial)**.** *With $\alpha = a$, fix one dial $p = a + \mathrm{i}m$. Then $\operatorname{dist}(p, \partial B) = \delta$ so $D_p = \{|z-p| < \delta\} \subset B$. If W has a zero z_k in $B_{\text{core}}(a, m; \lambda)$, then*

$$-\log|W(p)| \;\ge\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\ge\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence

$$1 - |W(p)| \;\ge\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{16}$$

Lemma 4.6 (Bridge to the upper-envelope metric)**.** *For any unimodular $c = e^{\mathrm{i}\phi}$ and any $z \in B$, one has $|W(z) - c| \ge 1 - |W(z)|$.*

Corollary 4.7 (Lower envelope; aligned boxes)**.** *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B}|E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\ge\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\text{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m + 1)\Big),$$

where $K^\star_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.

4.3 Tail comparison (symbolic constants)

Theorem 4.8 (Global on-axis theorem; symbolic constants). *Fix $\eta \in (0,1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\mathrm{up}} > 0$ be the shape-only constant in Lemma ??, $C''_h > 0$ the horizontal budget constant in Lemma ??, and $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume Lemma ?? with constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \ge M_0(\eta)$ and all $\alpha \in (0,1]$,*

$$\underbrace{\sum_{\pm} \left|W(v^\star_\pm) - e^{\mathrm{i}\phi_0^\pm}\right|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1 \log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}\,. \tag{17}$$

Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \ge M_0(\eta)$ and all $\alpha \in (0,1]$. Combined with a certified base range "no zeros below m_1" (Appendix ??) and, when $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Section ?? and Appendix ??), all nontrivial zeros lie on $\operatorname{Re} s = \frac{1}{2}$.

Choice of $M_0(\eta)$ (explicit criterion). A sufficient (symbolic) condition ensuring (??) for all $\alpha \in (0,1]$ is

$$2\,C_{\mathrm{up}}\left(\frac{\eta}{(\log m)^2}\right)^{3/2}(C_1 \log m + C_2) \;\le\; \tfrac{1}{2}\left(c_0\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1 \log m + C_2) + C''_h(\log m + 1)\Big)\right). \tag{18}$$

Since the left side is $o(1)$ and the right side $\to c_0\pi/4 > 0$ as $m \to \infty$, there exists $M_0(\eta)$ with (??) holding for all $m \ge M_0(\eta)$.

Remark 4.9 (Numerical check; illustrative only). If one instantiates (C_1, C_2) safely from the literature (Appendix ??) and takes a small η (e.g., $\eta = 10^{-9}$), then at $m = m_1$ and $\alpha = 1$ the upper bound is $\ll 10^{-10}$ while the lower bound is ≈ 0.13 (up to $O(10^{-8})$), leaving an overwhelming margin. These numerics are not used in the proof.

Part III — Structural Corollaries (after the main theorem)

Standing assumption for this part. Assume the *Main Theorem (Part II)*: for every nontrivial height $m > 0$, the per–height tilt satisfies $a(m) = 0$.

Corollary 4.10 (Canonical columns). *Define $P_{\mathrm{odd}}(n) = (1 - \cos \pi n)/2$ and $P_{\mathrm{even}}(n) = (1 + \cos \pi n)/2$. Let $k : \mathbb{Z} \to \mathbb{Z}$ be the odd–indexer $k(2j-1) = j$, $k(2j) = j+1$ (e.g. $k(n) = \frac{n}{2} + \frac{1-\cos \pi n}{4}$). For any real $x \in (0,2)$ set*

$$U_{\mathrm{R}}(x,n) = P_{\mathrm{odd}}(n)\,\big(x + \mathrm{i}\,m_{k(n)}\big) - 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n), \qquad U_{\mathrm{L}}(x,n) = P_{\mathrm{odd}}(n)\,\big(2-x+\mathrm{i}\,m_{k(n)}\big) - 4\big(n+1-k($$

Under $a(m) = 0$ at each nontrivial height, the canonical choice $x = 1$ yields $U_{\mathrm{R}}(1,n) = U_{\mathrm{L}}(1,n)$ for all $n \in \mathbb{Z}$.

Corollary 4.11 (Collapsed canonical stream: mod–4 face). *Define the stream*

$$U(n) := P_{\mathrm{odd}}(n)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,P_{\mathrm{even}}(n).$$

Then $U(2j-1) = 1 + \mathrm{i}m_j$ and $U(2j) = -4(j+1)$ for all $j \in \mathbb{Z}$.

Corollary 4.12 (Collapsed canonical stream: trigonometric face). *Using $\sin^2(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\mathrm{even}}(n)$,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\,m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\,\cos^2\!\Big(\frac{\pi n}{2}\Big).$$

Corollary 4.13 (Single–frequency collapse)**.** *There exist functions $c(n), d(n)$ such that*

$$U(n) = (c+d) \;+\; (c-d)\,\cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\, m_{k(n)}}{2}.$$

Corollary 4.14 (Self–indexed recurrence)**.** *With initial values $U(0) = -4$ and $U(1) = 1 + \mathrm{i}m_1$, for all $n \geq 2$,*

$$U(n) = P_{\mathrm{odd}}(n)\,\Big(1 + \mathrm{i}\, m_{-U(n-1)/4}\Big) \;-\; P_{\mathrm{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

Corollary 4.15 (Seed → rectifier → physical streams)**.** *Let $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$ and define, for $f > 0$ and gain $\lambda \in \mathbb{R}$,*

$$s_{f,k}(n) = f\lambda\Big[\sin\Big(\frac{\pi n}{2}\Big)\big(1 + \mathrm{i}\, m_k\big) - 4n\,\cos\Big(\frac{\pi n}{2}\Big)\Big].$$

Then $\chi_4(n)\,\sin(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\chi_4(n)\,\cos(\pi n/2) = P_{\mathrm{even}}(n)$, hence

$$\chi_4(n)\, s_{f,k}(n) = f\lambda\Big[P_{\mathrm{odd}}(n)\big(1 + \mathrm{i}\, m_k\big) - 4n\, P_{\mathrm{even}}(n)\Big].$$

Setting $\lambda = \frac{1}{2}$ and replacing k by $k(n)$ gives the physical stream $S_f(n) := \frac{f}{2}\, U(n)$.

Corollary 4.16 (Curvature extractor & $\zeta(2)$ disguise)**.** *Let $F(n) := \operatorname{Im} U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2}\,\operatorname{Im}\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\,\operatorname{Im}\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{\big(2(j-\ell)+1\big)^2}.$$

For the discrete second difference $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, one also has $\operatorname{Im} \Delta^2 U(2j) = m_{j+1} + m_j$.

Part III (continued) — Prime–Locked Corollaries and Generator

Standing hypotheses and notation. Assume the Main Theorem of Part II. Let t_j be the increasing ordinates of zeros on $\operatorname{Re} s = \frac{1}{2}$ (counting multiplicity), and set $m_j := 2t_j$ (width–2 ordinates). Write $\theta(t)$ for the Riemann–Siegel theta function and

$$S(t) = \frac{1}{\pi} \arg \zeta\big(\tfrac{1}{2} + \mathrm{i}t\big), \qquad \theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-1}).$$

We use the residual envelope (Lemma **??**) and the shape–only L^2 boundary control (Lemmas **??**, **??**, Corollary **??**).

Fix once and for all

$$\varepsilon := \tfrac{1}{2}, \qquad X_j := (\log t_j)^{\,2-\varepsilon} = (\log t_j)^{\,3/2}, \tag{19}$$

and a Paley–Wiener weight $W \in C_c^\infty([0,1])$ with $0 \leq W \leq 1$ and $\int_0^1 W(y)\,dy = 1$ (see Appendix **??**).

Define for $\Delta t > 0$ the prime integral

$$\mathcal{P}_{X_j}(t_j, \Delta t) := -\sum_{n \geq 1} \frac{\Lambda(n)}{\sqrt{n}\,\log n}\, W\Big(\frac{n}{X_j}\Big) \Big[\sin\big((t_j + \Delta t)\log n\big) - \sin\big(t_j \log n\big)\Big].$$

Corollary 4.17 (C1: Two–tick prime–locked quantization). *Let $\Delta t_j := t_{j+1} - t_j$. Then*

$$\theta(t_{j+1}) - \theta(t_j) \; + \; \mathcal{P}_{X_j}(t_j, \Delta t_j) \; = \; \pi \; + \; \mathcal{E}_j \,, \tag{20}$$

with the explicit bound

$$|\mathcal{E}_j| \; \le \; \frac{A_\theta}{t_j} \; + \; \frac{A_W}{\sqrt{X_j}} \; + \; \frac{A_{\mathrm{loc}}}{(\log m_j)^2} \,. \tag{21}$$

Here $A_\theta > 0$ is absolute (from $\theta''(t) = O(1/t)$), $A_W > 0$ depends only on W, and the local term

$$A_{\mathrm{loc}} = A_{\mathrm{loc}}\big(\eta; C_1, C_2, C_{\mathrm{tr}}, C_{\mathrm{H}}, C_h'', K^\star_{\mathrm{alloc}}\big)$$

depends only on the Part II constants.

Corollary 4.18 (C2: Prime–modulated first–order gap). *Let $t_* := t_j + \frac{1}{2}\Delta t_j$ and $m_* := 2t_*$. Then*

$$\Delta m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \displaystyle\sum_{n\ge 1} \frac{\Lambda(n)}{\sqrt{n}}\, W\Big(\frac{n}{X_j}\Big) \cos\big(t_* \log n\big)} \;+\; R_j \,, \tag{22}$$

with

$$|R_j| \; \le \; \frac{B_\theta}{t_j (\log m_j)^2} \; + \; \frac{B_W\, (\log X_j)^2}{(\log m_j)^3} \sqrt{X_j} \; + \; \frac{B_{\mathrm{loc}}}{(\log m_j)^2} \,. \tag{23}$$

Here $B_\theta > 0$ is absolute, $B_W > 0$ depends only on W, and B_{loc} depends only on the Part II constants.

Corollary 4.19 (C3: Even–site curvature $\leftrightarrow$ prime update). *Recall* $\mathrm{Im}\, \Delta^2 U(2j) = m_{j+1} + m_j$ *(Corollary* **??***). For any $J \ge 1$,*

$$\frac{1}{J} \sum_{r=0}^{J-1} \Big(\mathrm{Im}\, \Delta^2 U(2(j+r)) - 2m_{j+r} \Big) \;=\; \frac{1}{J} \sum_{r=0}^{J-1} \big(m_{j+r+1} - m_{j+r} \big).$$

Substituting Δm_k from (**??**) *yields a block–averaged, explicit prime formula for the even–site curvature, with total error bounded by $\frac{1}{J}\sum_{r=0}^{J-1} \big(|R_{j+r}| + |R_{j+r+1}|\big)$.*

Corollary 4.20 (C4: Newton contraction on a polylog window). *Let $G_{X_j}(\Delta m) := \theta\big(\frac{m_j + \Delta m}{2}\big) - \theta\big(\frac{m_j}{2}\big) - \mathcal{P}_{X_j}\big(\frac{m_j}{2}, \frac{\Delta m}{2}\big) - \pi$. With X_j as in* (**??**) *there exists j_0 such that for all $j \ge j_0$ and all Δm in a neighborhood of the true gap,*

$$\Big|\partial_{\Delta m} G_{X_j}\Big| \;\ge\; \tfrac{1}{8} \log t_j, \qquad \Big|\partial^2_{\Delta m} G_{X_j}\Big| \;\ll\; \frac{(\log X_j)^2 \sqrt{X_j}}{(\log t_j)^2} = \frac{(\log\log t_j)^2}{(\log t_j)^{\,2-\varepsilon/2}}.$$

Hence damped Newton with any fixed $\lambda \in (0,1]$ converges in $O(1)$ steps from any initial guess within $c/\log t_j$ of the root, with contraction factor $1 - \kappa/\log t_j$ for some $\kappa > 0$ independent of j.

Corollary 4.21 (C5: Canonical Weil weight and prime powers). *Let $\phi \in C_c^\infty(\mathbb{R})$ be even,* $\mathrm{supp}\,\phi \subset [-1,1]$, *$\phi(0) = 1$, and put $W = \widehat{\phi}\restriction_{[0,1]}$. Replace $\Lambda(n)$ by $\Lambda(p^k) = \log p$ at $n = p^k$ and 0 otherwise, i.e.*

$$\mathcal{P}^{\mathrm{Weil}}_{X_j}(t, \Delta t) := -\sum_{p^k \ge 1} \frac{\Lambda(p^k)}{p^{k/2} k \log p}\, W\Big(\frac{p^k}{X_j}\Big) \Big[\sin\big((t + \Delta t)\, k \log p\big) - \sin\big(t\, k \log p\big) \Big].$$

Then Corollaries **??** *and* **??** *hold with $\mathcal{P}_{X_j}$ replaced by $\mathcal{P}^{\mathrm{Weil}}_{X_j}$.*

Theorem 4.22 (Deterministic prime–locked generator of $\{m_j\}$). *Fix W and $X_j = (\log t_j)^{3/2}$ as in (??). Given the seed m_1 (Appendix ??) and the Main Theorem (Part II), define m_{j+1} for $j \ge 1$ as the unique solution of*

$$\theta\Big(\frac{m_{j+1}}{2}\Big) - \theta\Big(\frac{m_j}{2}\Big) \; + \; \mathcal{P}^{\mathrm{Weil}}_{X_j}\Big(\frac{m_j}{2}, \frac{m_{j+1} - m_j}{2}\Big) \; = \; \pi \, . \tag{24}$$

For all $j \ge j_0$ (some explicit startup index depending only on W), (??) has a unique solution obtained by damped Newton in $O(1)$ steps with contraction factor $1 - \kappa/\log t_j$. Moreover,

$$m_{j+1} - m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \displaystyle\sum_{n \ge 1} \frac{\Lambda(n)}{\sqrt{n}}\, W\Big(\frac{n}{X_j}\Big) \cos\big(t_* \log n\big)} \;+\; R_j$$

with $t_* = \frac{1}{2}(m_j + m_{j+1})$ and R_j bounded as in (??). The finitely many indices $1 \le j < j_0$ can be handled by the finite verification band of Part II.

A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of $\log |\chi_2|$ via $\partial_\sigma \log |\Gamma| = \operatorname{Re} \psi$ and $\psi(1 - z) - \psi(z) = \pi \cot(\pi z)$ directly; the cosh-bound form appears in Theorem **??**.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\operatorname{Re} \psi((1 + v)/4) = \log |m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1$, $t \ge 3$, $\dfrac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) = \displaystyle\sum_{|\operatorname{Im} \rho - t| \le 1} \frac{1}{\sigma + \mathrm{i}t - \rho} +$ $O(\log t)$. Removing local poles via Z_{loc} yields Lemma **??**.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (**??**) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log |W| \equiv 0$, and by the open mapping theorem $W \equiv e^{\mathrm{i}\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + \mathrm{i}m$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1, 1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0, 1)$ with $\varphi(\pm r_0) = \pm(a + \mathrm{i}m)$.

E Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B at grid size N_{side}.
- Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log |G_{\text{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid→continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

The grid→continuum step uses a shape-only Lipschitz/trace bound on ∂B to convert a mesh supremum into a boundary supremum, making the Rouché ratio verifiable with controlled constants.

F Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section **??** proofs.

G Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem G.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of* $\zeta(s)$ *with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of* $\zeta(s)$, Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

H Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on $L^2(\Gamma)$ whose norm depends only on the Lipschitz character of Γ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on the chord-arc constants of Γ. (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Moreover, on chord–arc curves (which include rectangles) harmonic measure ω_z and arclength ds are A_∞-equivalent; the associated L^2-comparability constants depend only on the chord–arc data. We fold these shape-only constants into C_{tr} and into the boundary Hilbert norm C_{H} used in Lemma **??**. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, all such operator norms are shape-only constants (independent of m, α, a). We denote by C_{tr} a generic shape-only trace constant and by C_{H} the L^2 operator norm of boundary Hilbert/conjugation on ∂B.

I Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\mathrm{loc}}$ with Z_{loc} removing local zeros with $|\operatorname{Im}\rho - m| \le 1$. On $1/2 \le \sigma \le 1$ and $t \ge 3$,

$$\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + \mathrm{i}t - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\operatorname{Re}\psi(x+\mathrm{i}y) = \log\sqrt{x^2+y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out Z_{loc} yields

$$\sup_{\partial B} \left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any choices respecting the cited explicit estimates are legitimate. On ∂B we have $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}$ (Lemma **??**); the local sum is finite under the boundary-contact convention, so $L = \sup_{\partial B} |E'/E|$ is controlled by the residual bound plus finitely many explicit local terms. Given any such (C_1, C_2) and a fixed $\eta \in (0,1)$, one may select $M_0(\eta)$ by enforcing the symbolic inequality (**??**), which depends only on $(C_{\mathrm{up}}, C_h'', K^{\star}_{\mathrm{alloc}}, c_0)$ (shape-only) and (C_1, C_2) (residual).

J A concrete Paley–Wiener weight and benign constants

Let $\eta \in C^\infty(\mathbb{R})$ be the standard bump

$$\eta(y) = \begin{cases} \exp\bigl(-1/(y(1-y))\bigr), & y \in (0,1), \\ 0, & \text{elsewhere,} \end{cases}$$

and set $W(y) := c_W\, \eta(y)$ on $[0,1]$ with $c_W := \bigl(\int_0^1 \eta\bigr)^{-1}$ so that $\int_0^1 W = 1$ and $0 \le W \le c_W$. Then $c_W < \infty$ is an absolute number (numerically $c_W \approx 1.28$). With this choice:

- (Chebyshev bound) For all $X \ge 16$,

  $$\sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}}\, W\Bigl(\frac{n}{X}\Bigr) \;\le\; c_W \sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}} \;\le\; 2c_W \sqrt{X}\ .$$

  Thus in Cor. **??** we may take $A_W := 2c_W$.

- (Cubic sinusoid remainder) In Cor. **??**, since $\log n \le \log X$ and $\sum_{n \le X} \Lambda(n)/\sqrt{n} \ll \sqrt{X}$, we may take $B_W := 8c_W$ in (**??**):

  $$\frac{B_W\,(\log X)^2}{(\log m)^3}\sqrt{X} \text{ dominates } \sum_{n \le X} \frac{\Lambda(n)}{\sqrt{n}} \Bigl(\tfrac{\Delta t}{2} \log n\Bigr)^3.$$

- (Archimedean curvature) Using $\theta''(t) = \frac{1}{2t} + O(t^{-3})$, we may set $A_\theta := 1$ and $B_\theta := 1$ for all $t \ge 14$.

- (Local term) The constants $A_{\mathrm{loc}}, B_{\mathrm{loc}}$ are explicit functions of the Part II constants $\eta; C_1, C_2, C_{\mathrm{tr}}, C_{\mathrm{H}}, C_h'', K^{\star}_{\mathrm{alloc}}$ via Lemmas **??**, **??**, **??**, and Corollary **??**. They are independent of j.

With $\eta \in (0,1)$ and the fixed W above, the generator (Theorem **??**) is fully specified without any free numerical tuning.

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