A Width–2 Boundary Program for Excluding Off–Axis Quartets with a Certified Tail Criterion and a Finite–Height Front–End (v35)

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Abstract

This document is a truth–latching consolidation (v35) of the width–2 boundary program. It records three proof–grade obstructions that prevent drift:

(i) *Exponent budget.* Under the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$ and the pointwise collar bound $\sup_{\partial B}\left|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}\right| \ll N_{\mathrm{loc}}(m)/(\kappa\,\delta)$, uniform η–shrinking absorption requires a residual δ–power $p - \theta \geq \frac{1}{2}$ (Theorem 10.12); with the pointwise endpoint one has $\theta = 1$.

(ii) *UE scaling NO–GO.* For pointwise/sup endpoints $\sup_{\partial B}|E'/E|$ with shape–only constants, the upper–envelope prefactor cannot have exponent $p > 1$ (Lemma 10.14); in particular the proved pointwise bound has $p = 1$.

(iii) *Forcing NO–GO.* In the single–box forcing architecture, the available forcing margin is $O(1)$ and cannot grow with m (Lemma 8.2).

Accordingly, the former η–absorption closure route based on the pointwise/sup upper–envelope plus collar is formally discarded (Appendix A). The manuscript's main unconditional output is the certified tail *criterion* at each height m (Theorem 11.1) together with a finite–height front–end. The remaining analytic frontier is reframed as **S5**: a non–pointwise upper–envelope redesign that controls the dial deviation $D_B(W)$ while avoiding the pointwise δ^{-1} collar blow–up.

For global hygiene, v35 also corrects completion/holomorphy: the working function is the entire width–2 completion $\Xi_2(u) := \xi(u/2)$, recentered as $E(v) := \Xi_2(1+v)$, so all uses of "E is entire" are literally true.

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Executive Proof Status

Status (v35): v35 is a *truth–latch* build: it codifies the decisive NO–GO constraints in the manuscript text so that the program cannot drift back to the (now invalid) v33 absorption narrative. The main unconditional output remains the *tail criterion family* (Theorem 11.1) together with a finite–height front–end (Definition 4.1). No claim of uniform tail closure is made in this version.

Proof–grade NO–GO constraints (now explicit):

1. *Exponent budget obstruction.* Under the nominal scaling $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$, a collar blow–up exponent θ and UE exponent p must satisfy $p - \theta \geq \frac{1}{2}$ for uniform η–shrinking closure (Theorem 10.12). In the present pointwise collar bound one has $\theta = 1$.

2. *UE scaling obstruction.* With the pointwise/sup endpoint $\sup_{\partial B} |E'/E|$ and shape–only constants, one cannot achieve any UE exponent $p > 1$ (Lemma 10.14). The proved pointwise bound has $p = 1$.

3. *Forcing is constant–limited.* In the current single–box forcing architecture, the available forcing margin is $O(1)$ and cannot grow with m (Lemma 8.2).

4. *Boundary modulus has no converse.* The condition $|W| = 1$ on ∂B does not exclude interior zeros; Bridge 1 is strictly one–directional (Remark 9.4).

Immediate consequence: the v33/v32 style η–absorption closure route based on the pointwise/-sup upper envelope and the collar is *formally discarded* in v35 (Appendix A). Any future closure must change the envelope endpoint and/or the local interface.

Completion / holomorphy hygiene (fixed): the working function is the entire width–2 completion $\Xi_2(u) := \xi(u/2)$ and $E(v) := \Xi_2(1+v)$ (Section 1). All uses of "E is entire" are now literally correct.

Open proof–grade blockers (v35):

1. **S5 UE redesign (primary frontier).** Replace the pointwise/sup UE endpoint by a non–pointwise functional that controls the same dial deviation $D_B(W)$ while avoiding the δ^{-1} collar blow–up (Section 12). (G–4/G–5 in the prior register.)

2. *Residual envelope ledger.* Lemma 7.2 still imports a standard RH–free bound for ζ'/ζ with local–zero subtraction; this must be proved in–text or cited in a referee–acceptable way with explicit constants. (G–1/G–12.)

3. *Front–end dependence.* The finite–height hypothesis remains an external input (Appendix C). (G–11.)

Reproducibility posture (v35): numerical artifacts remain an *audit harness* only (Appendix B). The v35 repro pack is hardened to record the endpoint functional and forcing architecture metadata so that future redesigns cannot silently mismatch the forcing chain.

Part I

Reader's Guide / Definitions and Reduction

1 Width–2 normalization

Let s denote the usual complex variable for $\zeta(s)$. We pass to the width–2 coordinate

$$u := 2s, \qquad \zeta_2(u) := \zeta(u/2).$$

Define the width–2 completed zeta

$$\Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\zeta(u/2).$$

Then Λ_2 is *meromorphic* (simple poles at $u = 0$ and $u = 2$) and satisfies the functional equation

$$\Lambda_2(u) = \Lambda_2(2 - u).$$

Define the entire completion

$$\Xi_2(u) := \xi(u/2) = \frac{u(u-2)}{8}\,\Lambda_2(u),$$

so that Ξ_2 is entire and obeys $\Xi_2(u) = \Xi_2(2-u)$.

We recenter at $u = 1$:

$$v := u - 1, \qquad E(v) := \Xi_2(1 + v).$$

Then E is entire and satisfies the evenness relation

$$E(v) = E(-v),$$

and complex conjugation gives $E(\overline{v}) = \overline{E(v)}$.

Remark 1.1 (Zeros). The zeros of $\Xi_2(u) = \xi(u/2)$ are exactly the *nontrivial* zeros of $\zeta(s)$ under the map $u = 2s$, with multiplicity. All boxes used in the tail program lie at heights $m \geq 10$, so the only zeros that can occur in the relevant local windows are nontrivial zeros.

2 Heights and horizontal displacement (RH–free)

Let $\rho = \beta + i\gamma$ be any nontrivial zero of $\zeta(s)$ (no assumption on β). In width–2 we write

$$u_\rho := 2\rho = (1 + a) + im, \qquad a := 2\beta - 1 \in (-1, 1), \qquad m := 2\gamma > 0. \tag{1}$$

Thus RH is equivalent to $a = 0$ for every nontrivial zero.

3 Quartet symmetry in width–2

The functional equation and conjugation imply that any off–axis zero with parameters (a, m) produces a quartet

$$\{\, 1 \pm a \pm im \,\} \subset \{u \in \mathbb{C} : \Xi_2(u) = 0\}. \tag{2}$$

In the centered v–coordinate this becomes $\{\pm a \pm im\} \subset \{v \in \mathbb{C} : E(v) = 0\}$.

4 Finite-height front-end after lowering the tail anchor

Once the tail anchor is lowered to $m_\star$, the analytic tail argument covers all $m \geq m_\star$. The remaining region corresponds to classical heights

$$0 < \mathrm{Im}(s) < H_0 := m_\star/2. \tag{3}$$

In v31 we take $m_\star = 10$, hence $H_0 = 5$.

Definition 4.1 (Front-end statement)**.** We say that *RH holds up to height* H_0 if every nontrivial zero $\rho = \beta + i\gamma$ with $0 < \gamma \leq H_0$ satisfies $\beta = 1/2$.

Remark 4.2 (How v31 discharges the front-end). The required statement for v31 is RH up to height $H_0 = 5$. This is a tiny special case of published rigorous verifications of RH to enormous heights. For example, Platt–Trudgian prove RH for all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ using interval arithmetic, which immediately implies RH up to $H_0 = 5$. Appendix C records this discharge in a pinned JSON file.

Part II

Self-Contained Boundary Program and Tail Closure

5 Aligned boxes and the $\delta(m)$ scale

Let $m > 0$ and $\alpha \in (0, 1]$. Fix a parameter $\eta \in (0, 1)$ and define the *nominal* box scale

$$\delta_0 = \delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}. \tag{4}$$

We will work with aligned boxes $B(\alpha, m, \delta)$ at scales $0 < \delta \le \delta_0$. By default one may take $\delta = \delta_0$, but later (Definition 10.5) we allow shrinking δ to enforce κ–admissibility; this is non-circular and monotone-safe (Lemmas 10.6 and 11.2).

Define the (width–2) box centered at $\alpha + im$ by

$$B(\alpha, m, \delta) := \{\, v \in \mathbb{C} : |\mathrm{Re}\, v - \alpha| \le \delta,\ |\mathrm{Im}\, v - m| \le \delta \,\}. \tag{5}$$

We will also use the symmetric dial centers $v_\pm := \pm\alpha + im$.

6 Local factor and finiteness

For a fixed $m > 0$, let

$$Z(m) := \{\, \rho : E(\rho) = 0,\ |\mathrm{Im}\, \rho - m| \le 1 \,\} \tag{6}$$

(zeros counted with multiplicity). Define the local zero factor and residual:

$$Z_{\mathrm{loc}}(v) := \prod_{\rho \in Z(m)} (v - \rho)^{m_\rho}. \tag{7}$$

$$F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{8}$$

Lemma 6.1 (Finiteness of Z_{loc})**.** *For each fixed $m > 0$ the set $Z(m)$ is finite; hence Z_{loc} is a finite product and F is meromorphic globally and analytic in any neighborhood of $\partial B(\alpha, m, \delta)$ that contains no zeros of E.*

Proof. Nontrivial zeros of ζ satisfy $0 < \beta < 1$, hence in the v–coordinate one has $\mathrm{Re}\, v \in (-1, 1)$ for all nontrivial zeros. Therefore the set $\{|\mathrm{Im}\, v - m| \le 1\} \cap \{|\mathrm{Re}\, v| \le 1\}$ is compact. Since E is entire and its zeros are discrete, only finitely many zeros can lie in this compact set. □

7 Residual envelope bound and the constants ledger

Remark 7.1 (Constant gate for the residual term (what is and is not assumed)). The tail criterion uses a bound of the form

$$\sup_{v \in \partial B(\alpha, m, \delta)} \left| \frac{F'(v)}{F(v)} \right| \ \le\ C_1 \log m + C_2,$$

with constants that must be (i) unconditional (not RH-equivalent) and (ii) uniform in $(\alpha, \delta, \eta, \kappa)$ once $m \ge 10$ and $0 < \alpha \le 1$. The proof below reduces this to standard RH-free bounds for ζ'/ζ in the critical strip with local-zero subtraction, plus a Stirling-type bound for Γ'/Γ.

Lemma 7.2 (Residual envelope inequality (δ-uniform)). *Fix $m \geq 10$ and $\alpha \in (0,1]$. Let $\eta \in (0,1]$ and set the nominal width $\delta_0 := \eta\alpha/(\log m)^2$. Let $\delta \in (0, \delta_0]$ and set $B := B(\alpha, m, \delta)$.*

Define E, Z_{loc} and $F := E/Z_{\mathrm{loc}}$ as in §6 (equations (7)–(8)). Assume boundary-contact *on ∂B (i.e. $E \neq 0$ on ∂B; hence F is holomorphic on a neighborhood of ∂B). Then there exist absolute constants $C_1, C_2 > 0$ (independent of $m, \alpha, \delta, \eta, \kappa$ and of the zero configuration) such that*

$$\sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right| \leq C_1 \log m + C_2.$$

Proof sketch with explicit dependency control. Write $u := 1 + v$ and $s := u/2 = (1+v)/2 = \sigma + it$. For $v \in \partial B(\alpha, m, \delta)$ we have $\mathrm{Re}(s) \in [0,1]$ and

$$\mathrm{Im}(s) = \frac{\mathrm{Im}(v)}{2} \in \left[\frac{m}{2} - \frac{\delta}{2}, \frac{m}{2} + \frac{\delta}{2}\right].$$

Since $m \geq 10$ and $\delta \leq \delta_0 \leq 1/(\log 10)^2 < 1/5$, we have $\mathrm{Im}(s) \asymp m$ uniformly in δ.

1) Log-derivative identity in the s-frame. From $\Xi_2(u) = \frac{u(u-2)}{8}\Lambda_2(u)$ and $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ we obtain, for $u = 1 + v$,

$$\frac{E'(v)}{E(v)} = \frac{\Xi_2'(u)}{\Xi_2(u)} = \left(\frac{1}{u} + \frac{1}{u-2}\right) - \frac{1}{4}\log\pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{u}{4}\right) + \frac{1}{2}\frac{\zeta'}{\zeta}(s), \qquad (u = 1+v,\ s = u/2).$$

Since $u = 1 + v$ has $\mathrm{Im}(u) = m \geq 10$, the completion terms $(1/u + 1/(u-2))$ are $O(1/m)$ on ∂B and are absorbed into the absolute constants in the bound.

Moreover, since $v = 2s - 1$, the local factor derivative satisfies

$$\frac{Z_{\mathrm{loc}}'(v)}{Z_{\mathrm{loc}}(v)} = \sum_{\rho \in Z(m)} \frac{m_\rho}{v - \rho} = \frac{1}{2}\sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s},$$

where $Z_s(m)$ denotes the corresponding multiset of nontrivial zeros $\rho_s = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma - \frac{m}{2}| \leq \frac{1}{2}$.

Therefore

$$\frac{F'(v)}{F(v)} = \frac{E'(v)}{E(v)} - \frac{Z_{\mathrm{loc}}'(v)}{Z_{\mathrm{loc}}(v)} = -\frac{1}{4}\log\pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\left(\frac{1+v}{4}\right) + \frac{1}{2}\left(\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s}\right).$$

2) RH-free residual bound for ζ'/ζ with local-zero subtraction. A standard "local-zero decomposition" (unconditional) asserts that there exist absolute constants A_ζ, B_ζ such that for $0 \leq \sigma \leq 1$ and $t \geq 5$,

$$\left| \frac{\zeta'}{\zeta}(\sigma + it) - \sum_{|\gamma - t| \leq 1} \frac{1}{(\sigma + it) - \rho} \right| \leq A_\zeta \log(t+2) + B_\zeta. \tag{$\star$}$$

(For a self-contained route, $(\star)$ can be derived from the Hadamard product for $\xi(s)$ plus a Riemann–von Mangoldt bound for $N(T)$; otherwise cite a standard reference.)

For $v \in \partial B$ we have $|t - \frac{m}{2}| \leq \delta/2 < 1/10$, hence every zero in $Z_s(m)$ satisfies $|\gamma - t| \leq 1$ and is included in the sum in $(\star)$. Thus

$$\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s - \rho_s} = \left(\frac{\zeta'}{\zeta}(s) - \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho}\right) + \sum_{\substack{|\gamma - t| \leq 1 \\ |\gamma - \frac{m}{2}| > 1/2}} \frac{1}{s - \rho}.$$

In the remaining sum we have $|\gamma - t| \geq 1/2 - |t - \frac{m}{2}| \geq 2/5$, hence $|s - \rho| \geq 2/5$ and each term has modulus $\leq 5/2$. The number of zeros with $|\gamma - t| \leq 1$ is bounded by the manuscript's explicit local window majorant (Lemma 10.9) at height $\asymp m$, so this difference-of-windows sum is $\ll \log m$.

Combining these bounds yields absolute constants $A_{\text{res}}, B_{\text{res}}$ such that

$$\left|\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s - \rho_s}\right| \leq A_{\text{res}} \log m + B_{\text{res}},$$

uniformly for all $v \in \partial B$ and all $\delta \in (0, \delta_0]$.

3) Gamma factor bound (Stirling, uniform in δ). For $z = (1+v)/4$ we have $\mathrm{Re}(z) \in [1/4, 3/4]$ and $|\operatorname{Im}(z)| \asymp m$. A uniform Stirling-type bound gives

$$\left|\frac{\Gamma'}{\Gamma}(z)\right| \leq \log(|\operatorname{Im}(z)| + 2) + C_\Gamma \leq \log(m + 2) + C_\Gamma,$$

with an absolute constant C_Γ.

4) Conclusion. Insert the bounds from (2)–(3) into the identity in (1), and absorb harmless constants into (C_1, C_2). All constants are independent of $(\alpha, \delta, \eta, \kappa)$ because: (i) σ ranges over a fixed compact interval $[0, 1]$, (ii) $t \asymp m$ with $m \geq 10$ uniformly for $\delta \leq \delta_0$, and (iii) the difference-of-windows sum is controlled by Lemma 10.9, which is unconditional. $\square$

Remark 7.3 (Hard gate / certificates (v35)). The tail harness in Appendix B uses explicit numerical interval enclosures for the constant ledger (e.g. $C_1, C_2, C_{\text{up}}, C''_h, \kappa$) stored in `v35_repro_pack/v35_constants_m10.j`
It evaluates the tail inequality for a pinned parameter choice and records the UE exponent p explicitly. This is an *audit harness* only: it does not certify that the constants file is correct, and it does not, by itself, yield a uniform tail closure. An unconditional proof therefore still requires a referee-acceptable certification of the analytic constant ledger, and a resolution of the UE–Gate (Remark 10.11).

8 Short-side forcing

Assume an off-axis pair at height m with displacement $a > 0$ exists. On an aligned box with $\alpha = a$, the two upper zeros in the centered v–plane are at $v = \pm a + im$. The pair factor

$$Z_{\text{pair}}(v) := (v - (a + im))(v - (-a + im)) \tag{9}$$

produces a large phase rotation on the near vertical side.

Lemma 8.1 (Short-side forcing lower bound)**.** *Let $I_+ := \{\alpha + iy : |y - m| \leq \delta\}$ with $|\alpha - a| \leq \delta$. Then*

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan\left(\frac{\delta}{|\alpha - a|}\right) + 2 \arctan\left(\frac{\delta}{\alpha + a}\right) \geq \frac{\pi}{2}. \tag{10}$$

Lemma 8.2 (Single-box forcing is constant-limited)**.** *In the forcing setup of Lemma 8.1, the total phase variation of the pair factor along I_+ satisfies*

$$\Delta_{I_+} \arg Z_{\text{pair}} \leq 2\pi,$$

uniformly in the height m. Consequently the forcing constant c appearing in the tail inequality (Theorem 11.1) is an absolute constant, independent of m; in particular the forcing side cannot grow like $\log m$ *(or any unbounded function of m) as $m \to \infty$.*

Proof. On $I_+ = \{\alpha + iy : \ |y - m| \le \delta\}$ one has

$$Z_{\text{pair}}(\alpha + iy) = \big((\alpha - a) + i(y - m)\big)\big((\alpha + a) + i(y - m)\big).$$

Along $y \in [m - \delta, m + \delta]$ the argument of each linear factor varies by at most π (it is an arctan function whose range lies in an interval of length $\le \pi$), so the argument of the product varies by at most 2π, uniformly in m. The forcing chain converts a fixed positive portion of $\Delta_{I_+} \arg Z_{\text{pair}}$ into the constant c with fixed conversion scalars, so c is necessarily $O(1)$. $\square$

9 Outer factorization and the inner quotient (Bridge 1)

We work on a fixed box $B = B(\alpha, m, \delta)$ and write B° for its interior. Assume boundary-contact: $E \ne 0$ on ∂B (this will be enforced later by κ–admissibility; see Definition 10.5 and Lemma 10.6).

Lemma 9.1 (Dirichlet outer factor on a box)**.** *Let $B = B(\alpha, m, \delta)$ be the closed rectangle and B° its interior. Assume E is holomorphic on a neighborhood of $\overline{B}$ and $E \ne 0$ on ∂B. Then $\log|E| \in C(\partial B)$. Let $U \in C(\overline{B}) \cap \mathrm{Harm}(B^\circ)$ be the unique solution of the Dirichlet problem with boundary data $U|_{\partial B} = \log|E|$. Since B° is simply connected, there exists a harmonic conjugate V on B° (unique up to an additive constant) such that $U + iV$ is holomorphic on B°. Define*

$$G_{\text{out}}(v) := \exp(U(v) + iV(v)), \qquad v \in B^\circ.$$

Then G_{out} is holomorphic and zero-free on B°, satisfies $|G_{\text{out}}(v)| = e^{U(v)}$ for $v \in B^\circ$, and

$$\lim_{z \to \xi,\ z \in B^\circ} |G_{\text{out}}(z)| = |E(\xi)| \qquad (\xi \in \partial B).$$

Proof. Continuity of $\log|E|$ on ∂B follows from $E \ne 0$ on ∂B. Existence and uniqueness of U on a rectangle are standard. Since B° is simply connected, U admits a harmonic conjugate V on B°, unique up to an additive constant. The function $U + iV$ is holomorphic, hence so is $G_{\text{out}} = \exp(U + iV)$, and it is zero-free. Finally $|G_{\text{out}}| = e^U$ on B°, and by continuity of U on $\overline{B}$ we have $e^{U(\xi)} = |E(\xi)|$ on ∂B, yielding the boundary modulus identity in interior-limit form. $\square$

Define on B° the inner quotient

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

Then W is holomorphic on B° and $|W| = 1$ on ∂B in the sense of interior limits in modulus.

Proposition 9.2 (Bridge 1: zero-free inner collapse)**.** *Assume the setup of Lemma 9.1 and define $W = E/G_{\text{out}}$ on B°. If W is zero-free on B° (equivalently, E is zero-free on B°), then W is constant on B°; in fact $W \equiv e^{i\theta_B}$ for some $\theta_B \in \mathbb{R}$.*

Proof. Since W is zero-free on B° and G_{out} is zero-free, the function E is zero-free on B°. Because B° is simply connected, E admits a holomorphic logarithm on B°, so $\log|E|$ is harmonic on B°. By construction U is harmonic on B°, continuous on $\overline{B}$, and equals $\log|E|$ on ∂B. Thus $U - \log|E|$ is harmonic on B° with zero boundary values, so by Dirichlet uniqueness $U \equiv \log|E|$ on B°. Therefore for $v \in B^\circ$,

$$|W(v)| = \frac{|E(v)|}{|G_{\text{out}}(v)|} = \frac{|E(v)|}{e^{U(v)}} = \frac{|E(v)|}{e^{\log|E(v)|}} = 1.$$

An analytic function of constant modulus on a connected open set is constant, hence $W \equiv e^{i\theta_B}$. $\square$

Remark 9.3 (Boundary modulus convention). Under boundary-contact, U extends continuously to ∂B and satisfies $U|_{\partial B} = \log|E|$. Hence $|G_{\text{out}}| = |E|$ holds pointwise on ∂B as interior limits in modulus, and therefore $|W| = 1$ holds pointwise in modulus on ∂B. In boundary integral estimates this may be used in the a.e. sense without change.

Remark 9.4 (No converse: boundary modulus does not exclude interior zeros). Lemma 9.1 implies that under boundary-contact the quotient $W := E/G_{\text{out}}$ satisfies $|W| = 1$ on ∂B (in the interior boundary-limit sense of Remark 9.3). This condition alone does *not* imply that W is zero-free or constant on B°: nonconstant holomorphic functions on B° can have $|W| = 1$ on ∂B and still possess prescribed interior zeros (e.g. via conformal transport of finite Blaschke products from the unit disc). Thus Proposition 9.2 is strictly one-directional: the additional hypothesis "W is zero-free on B°" is essential.

10 Shape-only invariance and the envelope constants

Let $T(v) := (v - (\alpha + im))/\delta$, mapping ∂B affinely onto the fixed square boundary ∂Q with $Q = [-1, 1]^2$.

Lemma 10.1 (Shape-only invariance)**.** *Any constant arising solely from geometric or boundary-operator estimates on ∂B that are invariant under affine rescaling depends only on ∂Q and is independent of (α, m, δ).*

Proof. Under T, arclength scales by δ and tangential derivatives by $1/\delta$. After normalization, all purely geometric quantities and operator norms reduce to fixed quantities on ∂Q. $\square$

Lemma 10.2 (Boundary-to-center evaluation in L^2 (sharp $\delta^{-1/2}$))**.** *Let $B = B(\alpha, m, \delta)$ be a box and let v_0 be its center. Let u be harmonic on B° and assume its boundary trace lies in $L^2(\partial B, ds)$. Then, writing $P_B(v_0, \xi) = d\omega^B_{v_0}/ds(\xi)$ for the Poisson kernel of B at v_0,*

$$|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} \, \|u\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q,

$$\|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} = \delta^{-1/2} \, \|P_Q(0, \cdot)\|_{L^2(\partial Q, ds)}.$$

In particular the exponent $\delta^{-1/2}$ is sharp (witnessed by the constant harmonic function $u \equiv 1$).

Proof. For harmonic u on B° with L^2 trace on ∂B, the Poisson representation gives

$$u(v_0) = \int_{\partial B} u(\xi) \, d\omega^B_{v_0}(\xi) = \int_{\partial B} u(\xi) \, P_B(v_0, \xi) \, ds(\xi).$$

Cauchy–Schwarz yields $|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2} \|u\|_{L^2}$.

For the scaling: under T, arclength scales by $ds = \delta \, ds_Q$ and Poisson kernels scale by $P_B(v_0, \xi) = \delta^{-1} P_Q(0, T(\xi))$. Hence

$$\int_{\partial B} P_B(v_0, \xi)^2 \, ds(\xi) = \int_{\partial Q} \delta^{-2} P_Q(0, \zeta)^2 \, \delta \, ds_Q(\zeta) = \delta^{-1} \int_{\partial Q} P_Q(0, \zeta)^2 \, ds_Q(\zeta),$$

giving $\|P_B(v_0, \cdot)\|_{L^2} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2}$.

Sharpness: for $u \equiv 1$ we have $|u(v_0)| = 1$ and $\|u\|_{L^2(\partial B)} = \sqrt{|\partial B|} \asymp \delta^{1/2}$, so the inequality forces $\|P_B(v_0, \cdot)\|_{L^2} \gtrsim \delta^{-1/2}$. $\square$

Lemma 10.3 (Upper envelope bound (outer-aligned form)). *Let $B = B(\pm a, m, \delta)$ be an aligned box and let $G_{\rm out}$ be the outer factor on B constructed from $\log|E|$ on ∂B (Section 9). Define the inner quotient*

$$W(v) := \frac{E(v)}{G_{\rm out}(v)}.$$

Assume the boundary-contact convention: E has no zeros on ∂B (hence W has unimodular boundary values a.e.). For each sign $\pm$ let $v_\pm := \pm a + im$ and let $e^{i\varphi_0^\pm} \in \mathbb{T}$ be an $L^2(\partial B, ds)$-best constant phase,

$$e^{i\varphi_0^\pm} \in \arg\min_{|c|=1} \int_{\partial B} |W(v) - c|^2\, ds(v).$$

Then there exists a shape-only *constant $C_{\rm up} > 0$ (depending only on the normalized square $Q = [-1,1]^2$) such that*

$$\sum_{\pm} \big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \;\le\; 2\, C_{\rm up}\, \delta \sup_{v\in\partial B} \Big|\frac{E'(v)}{E(v)}\Big|. \tag{11}$$

One admissible explicit definition is

$$C_{\rm up} := \Big(\sup_{\xi\in\partial Q} P_Q(0,\xi)\Big)^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2\to L^2}\big),$$

where $P_Q(0,\xi) = d\omega_0^Q/ds(\xi)$ is the Poisson kernel of Q at the center 0 with respect to arclength on ∂Q, and $H_{\partial Q}$ is the boundary conjugation (Hilbert/Cauchy) operator on ∂Q.

Remark 10.4 (No residual proxying in the upper envelope). Lemma 10.3 controls the inner quotient $W = E/G_{\rm out}$ and therefore depends on $\sup_{\partial B} |E'/E|$. Residual bounds for $F = E/Z_{\rm loc}$ control $\sup_{\partial B} |F'/F|$ and do *not* by themselves bound $\sup_{\partial B} |E'/E|$. Whenever the residual envelope is used to control dial deviation, it must be routed through the log-derivative split $E'/E = F'/F + Z'_{\rm loc}/Z_{\rm loc}$ (Lemma 10.7) together with the collar bound (Lemma 10.8), yielding Corollary 10.10.

Proof. Fix one sign and write $v_0 = v_\pm$ and $B = B(\pm a, m, \delta)$. We record the (RH-free) chain and indicate the scale factors explicitly.

1. **Evaluation from the boundary (harmonic measure; produces $\delta^{-1/2}$).** For any constant $c \in \mathbb{T}$, subharmonicity of $|W - c|^2$ implies

   $$|W(v_0) - c|^2 \le \int_{\partial B} |W(\xi) - c|^2\, d\omega_{v_0}^B(\xi) = \int_{\partial B} |W(\xi) - c|^2\, P_B(v_0,\xi)\, ds(\xi),$$

   so

   $$|W(v_0) - c| \le \|P_B(v_0,\cdot)\|^{1/2}_{L^\infty(\partial B)}\, \|W - c\|_{L^2(\partial B, ds)}.$$

   Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q, Poisson kernels scale by $\|P_B(v_0,\cdot)\|_\infty^{1/2} = \delta^{-1/2}\, \|P_Q(0,\cdot)\|_\infty^{1/2}$.

2. **Poincaré/Wirtinger on ∂B (produces δ).** For the L^2-best constant $c = e^{i\varphi_0^\pm}$ and $|\partial B| = 8\delta$, periodic Poincaré on a loop of length 8δ gives

   $$\|W - c\|_{L^2(\partial B)} \le \frac{|\partial B|}{2\pi}\, \|\partial_s W\|_{L^2(\partial B)} = \frac{4\delta}{\pi}\, \|\partial_s W\|_{L^2(\partial B)}.$$

3. **Outer factor control (no δ; uses bounded boundary conjugation).** Write $\log G_{\text{out}} = U + i\widetilde{U}$ with $U|_{\partial B} = \log|E|$ and $\widetilde{U} = H_{\partial B}U$. Differentiating tangentially, $\partial_s \log G_{\text{out}} = \partial_s U + i\,H_{\partial B}(\partial_s U)$. Since $\log W = \log E - \log G_{\text{out}}$,

$$\|\partial_s \log W\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2\to L^2}\big)\,\|\partial_s \log E\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2\to L^2}\big)\,\Big\|\frac{E'}{E}\Big\|_{L^2(\partial B)}.$$

On ∂B we have $|W| = 1$ a.e., hence $|\partial_s W| = |\partial_s \log W|$.

4. **L^2 to sup (produces $\delta^{1/2}$).** Using $|\partial B| = 8\delta$,

$$\Big\|\frac{E'}{E}\Big\|_{L^2(\partial B)} \le \sqrt{|\partial B|}\,\sup_{\partial B}\Big|\frac{E'}{E}\Big| = \sqrt{8\delta}\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|.$$

Combining the four steps yields

$$|W(v_0) - e^{i\varphi_0^{\pm}}| \le \|P_Q(0,\cdot)\|_\infty^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2\to L^2}\big) \cdot \delta \sup_{\partial B}\Big|\frac{E'}{E}\Big|,$$

where we used the similarity invariance $\|H_{\partial B}\|_{L^2\to L^2} = \|H_{\partial Q}\|_{L^2\to L^2}$. Summing over $\pm$ gives (11). $\square$

10.1 Local factor split and collar control

Definition 10.5 (Collar-admissible aligned boxes)**.** Fix once and for all a collar parameter $\kappa \in (0, 1/10)$. An aligned box $B = B(\alpha, m, \delta)$ is called *κ–admissible* if

$$\operatorname{dist}\big(\partial B,\ \mathcal{Z}(E)\big) \ge \kappa\delta.$$

Given any nominal scale $\delta_0 > 0$ and any center, there exists some $0 < \delta \le \delta_0$ for which κ–admissibility holds (Lemma 10.6). Whenever a chosen box is not κ–admissible, we shrink δ until κ–admissibility holds. Moreover the assembled tail inequality is monotone-safe under such δ–shrinking (Lemma 11.2).

Lemma 10.6 (Existence of a κ–admissible shrink)**.** *Fix $\kappa \in (0, 1/10)$ and a center $v_0 \in \mathbb{C}$. For every $\delta_0 > 0$ there exists $\delta' \in (0, \delta_0]$ such that the closed box*

$$B(v_0, \delta') := \{\, v \in \mathbb{C} :\ \|v - v_0\|_\infty \le \delta' \,\}$$

satisfies

$$\operatorname{dist}\big(\partial B(v_0, \delta'),\ \mathcal{Z}(E)\big) \ge \kappa\delta'.$$

In particular, given (α, m) and nominal $\delta_0 = \eta\alpha/(\log m)^2$, one may always choose a scale $0 < \delta \le \delta_0$ for which $B(\alpha, m, \delta)$ is κ–admissible.

Proof. Zeros of the entire function E are isolated. Choose $\varepsilon > 0$ such that $\mathcal{Z}(E) \cap \{\, 0 < \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) = 0$) or such that $\mathcal{Z}(E) \cap \{\, \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) \ne 0$). Set $\delta' := \min\{\delta_0, \varepsilon/(1+\kappa)\}$. Then every boundary point satisfies $\|v - v_0\|_\infty = \delta'$. Any zero $\rho \in \mathcal{Z}(E)$ is either $\rho = v_0$ (in which case $\operatorname{dist}(v, \rho) = \delta' \ge \kappa\delta'$) or satisfies $\|\rho - v_0\|_\infty \ge \varepsilon$ (in which case $\operatorname{dist}(v, \rho) \ge \varepsilon - \delta' \ge \kappa\delta'$). Therefore $\operatorname{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \ge \kappa\delta'$. $\square$

Lemma 10.7 (Log-derivative decomposition)**.** *With Z_{loc} and F as in* (7) *and* (8)*, one has on any region where E and Z_{loc} are holomorphic and nonvanishing (in particular on ∂B under the boundary-contact convention)*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}.$$

Lemma 10.8 (Buffered local factor bound on ∂B)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible in the sense of Definition 10.5. Then*

$$\sup_{v\in\partial B} \left|\frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)}\right| \le \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta},$$

where $N_{\mathrm{loc}}(m)$ counts zeros of E in the local window used to define Z_{loc}, with multiplicity.

Lemma 10.9 (Explicit local window zero count)**.** *Let $N(T)$ denote the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$, counted with multiplicity. Then for every $T \ge 5$,*

$$N(T+1) - N(T-1) \le 1.01 \log T + 17. \tag{12}$$

Consequently, for every $m \ge 10$,

$$N_{\mathrm{loc}}(m) \le 1.01 \log m + 17. \tag{13}$$

Proof. By [7, Theorem 1.1], for every $x \ge e$,

$$\left| N(x) - \frac{x}{2\pi}\log\Big(\frac{x}{2\pi e}\Big)\right| \le 0.10076 \log x + 0.24460 \log\log x + 8.08344.$$

Let $M(x) := \frac{x}{2\pi}\log(\frac{x}{2\pi e})$, so $M'(x) = \frac{1}{2\pi}\log(\frac{x}{2\pi})$. For $T \ge 5$ we have $\log(T \pm 1) \le \log(2T)$ and $\log\log x \le \log x$ for $x \ge e$, hence

$$N(T+1) - N(T-1) \le (M(T+1) - M(T-1)) + 2(0.10076 + 0.24460)\log(2T) + 2\cdot 8.08344.$$

Moreover

$$M(T+1) - M(T-1) = \int_{T-1}^{T+1} M'(x)\,dx \le \int_{T-1}^{T+1} \frac{1}{2\pi}\log x\,dx \le \frac{1}{\pi}\log(2T).$$

Combining these bounds gives $N(T+1) - N(T-1) \le 1.00903 \log T + 16.8663 \le 1.01 \log T + 17$, establishing (12). Finally, in width–2 one has $m = 2T$. The local window $|\operatorname{Im}\rho - m| \le 1$ corresponds to $|\gamma - T| \le 1/2$ in the s–plane, so $N_{\mathrm{loc}}(m) = N(T + \frac{1}{2}) - N(T - \frac{1}{2}) \le N(T+1) - N(T-1)$, yielding (13). □

Corollary 10.10 (Outer-aligned upper envelope in residual+local form)**.** *Let B be κ–admissible. Assume the residual envelope bound of Lemma 7.2, i.e. $\sup_{\partial B}|F'/F| \le L(m) := C_1 \log m + C_2$. Then*

$$\sum_{\pm} |W(v_\pm) - e^{i\varphi_0^\pm}| \le 2C_{\mathrm{up}}\left(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa}\right) \le 2C_{\mathrm{up}}\left(\delta L(m) + \frac{1.01\log m + 17}{\kappa}\right).$$

Remark 10.11 (UE gate = exponent budget at the local interface). Lemma 10.3 is the *only* step in the envelope chain that generates a positive power of δ in front of a boundary log-derivative endpoint. Abstractly, suppose an upper-envelope mechanism yields, for some $p > 0$,

$$\sum_{\pm}\left|W(v_\pm) - e^{i\varphi_0^\pm}\right| \le 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\left|\frac{E'}{E}\right|,$$

and suppose the collar/local split yields, for some $\theta > 0$,

$$\sup_{\partial B}\left|\frac{E'}{E}\right| \le L(m) \;+\; \frac{N_{\rm loc}(m)}{\kappa\,\delta^{\theta}}.$$

Then the local contribution in the envelope side scales as $\delta^{p-\theta}N_{\rm loc}(m)/\kappa$. Under the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$ and the unconditional majorant $N_{\rm loc}(m) \ll \log m$, uniform η–shrinking tail closure is possible only if

$$p-\theta \;\ge\; \tfrac{1}{2}$$

(Theorem 10.12).

In the *proved* pointwise/sup architecture one has $p = 1$ (Lemma 10.3) and $\theta = 1$ (Lemma 10.8), so $p-\theta = 0$ and the local term is δ–inert; η–shrinking cannot suppress it (Lemma 10.13). Moreover, within this same endpoint class, a strengthened exponent $p > 1$ is impossible with shape–only constants (Lemma 10.14). Thus the former η–absorption closure route based on the pointwise/sup UE endpoint is a formal NO–GO and is recorded as discarded (Appendix A).

Theorem 10.12 (Exponent budget for η–shrinking under $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$)**.** *Let $m \ge 10$, $\alpha \in (0,1]$ and $\eta \in (0,1]$, and set the nominal scale*

$$\delta_0(m,\alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

Assume that for all $0 < \delta \le \delta_0(m,\alpha)$ one has:

(UE_p) (UE exponent) *for some $p > 0$,*

$$\mathrm{UE}(\delta) \;\le\; 2C_{\rm up}\,\delta^{p}\,\sup_{\partial B}\left|\frac{E'}{E}\right|;$$

(COL_θ) (Collar/local exponent) *for some $\theta > 0$,*

$$\sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; L(m) \;+\; \frac{N_{\rm loc}(m)}{\kappa\,\delta^{\theta}},$$

with fixed $\kappa \in (0,1/10)$;

(GROW) (Majorants) *$L(m) \le A_L \log m + B_L$ and $N_{\rm loc}(m) \le A_N \log m + B_N$ for all $m \ge 10$;*

(FORCE) (Forcing side) *the forcing-vs-envelope tail inequality has a fixed positive forcing constant $c > 0$ and only δ–helpful subtractive terms on the RHS.*

Then at $\delta = \delta_0(m,\alpha)$ one has the explicit bound

$$\mathrm{UE}(\delta_0) \;\le\; 2C_{\rm up}\Big(\delta_0^{p}L(m) \;+\; \delta_0^{p-\theta}\frac{N_{\rm loc}(m)}{\kappa}\Big). \tag{BUDGET}$$

Moreover, uniform tail closure by η–shrinking *(i.e. there exists $\eta_\star > 0$ such that for every $\eta \le \eta_\star$ the tail inequality holds for all $m \ge 10$) is possible only if*

$$p-\theta \;\ge\; \tfrac{1}{2}. \tag{B1}$$

Proof. Insert (COL_θ) into (UE_p) at $\delta = \delta_0$ to obtain (BUDGET). At $\alpha = 1$ one has $\delta_0(m, 1) = \eta/(\log m)^2$, so the local term behaves as

$$\delta_0^{p-\theta} N_{\mathrm{loc}}(m) \ \ll \ \Big(\frac{\eta}{(\log m)^2}\Big)^{p-\theta} \log m = \eta^{p-\theta} (\log m)^{1-2(p-\theta)}.$$

If $p-\theta < 1/2$ then $1-2(p-\theta) > 0$, so the local contribution grows without bound as $m \to \infty$, while the forcing side tends to the fixed constant c because all RHS corrections are δ–helpful and vanish as $\delta_0 \to 0$. Hence uniform tail closure is impossible. If $p - \theta \geq 1/2$ then the local contribution is uniformly bounded by $O(\eta^{p-\theta})$ and tends to 0 as $\eta \downarrow 0$, enabling uniform absorption once all constants are δ–uniform. $\square$

Lemma 10.13 (η–absorption obstruction under the pointwise UE exponent $p = 1$)**.** *Assume the hypotheses of Corollary 10.10. Then for every $\delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$,*

$$\sum_{\pm} \big| W(v_\pm) - e^{i\varphi_0^\pm} \big| \leq 2C_{\mathrm{up}} \Big(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa} \Big).$$

In particular, letting $\eta \downarrow 0$ (hence $\delta \downarrow 0$) only suppresses the residual term $\delta L(m)$; the local term $N_{\mathrm{loc}}(m)/\kappa$ does not *decay with η. Therefore any absorption-style closure that attempts to force the envelope side small by choosing η must additionally verify a* separate *inequality of the form*

$$\frac{2C_{\mathrm{up}}}{\kappa} N_{\mathrm{loc}}(m) \ < \ c$$

at the relevant anchor height(s), where c is the forcing constant in (14)*.*

Lemma 10.14 (UE scaling NO–GO for pointwise/sup endpoints)**.** *Assume an upper-envelope bound of the form*

$$\sum_{\pm} \big| W(v_\pm) - e^{i\varphi_0^\pm} \big| \ \leq \ 2C_{\mathrm{up}}\, \delta^p \sup_{\partial B} \Big| \frac{E'}{E} \Big| \qquad (p > 0),$$

where the constant C_{up} depends only on the normalized shape (Lemma 10.1) and is independent of δ. Then necessarily $p \leq 1$. In particular, no pointwise/sup envelope mechanism with shape–only constants can yield any exponent $p > 1$.

Proof. Under the affine rescaling $T(v) = (v - (\alpha + im))/\delta$, the boundary ∂B maps to the fixed square boundary ∂Q. If $\widetilde{E}(z) := E(T^{-1}(z))$, then by the chain rule

$$\frac{E'}{E}(T^{-1}(z)) = \frac{1}{\delta} \frac{\widetilde{E}'(z)}{\widetilde{E}(z)}.$$

Hence $\sup_{\partial B} |E'/E| = \delta^{-1} \sup_{\partial Q} |\widetilde{E}'/\widetilde{E}|$. The left-hand side of the upper-envelope bound is dimensionless (it is a sum of moduli of complex numbers), and under the normalization it may be $O(1)$ for admissible configurations on the fixed shape. Therefore the bound forces

$$O(1) \ \leq \ 2C_{\mathrm{up}}\, \delta^{p-1} \sup_{\partial Q} \Big| \frac{\widetilde{E}'}{\widetilde{E}} \Big| \qquad \text{as } \delta \downarrow 0.$$

Since the normalized endpoint $\sup_{\partial Q} |\widetilde{E}'/\widetilde{E}|$ is not forced to blow up as $\delta \downarrow 0$ (it depends only on the normalized data), the factor δ^{p-1} cannot tend to 0. Thus $p - 1 \leq 0$, i.e. $p \leq 1$. $\square$

Proof. The displayed bound is exactly Corollary 10.10 with the corrected UE exponent $p = 1$. As $\eta \to 0$ one has $\delta_0 \to 0$ and hence $\delta L(m) \to 0$, while $N_{\mathrm{loc}}(m)/\kappa$ is unchanged. Since the forcing lower bound in the tail inequality tends to c as $\delta \downarrow 0$, the strict inequality requires the stated necessary condition at the anchor. $\square$

10.2 Horizontal non-forcing budget in residual form

Definition 10.15 (Horizontal non-forcing phase budget)**.** Let $B = B(\pm a, m, \delta)$ be an aligned box and let $F = E/Z_{\mathrm{loc}}$ be the residual factor. Assume F is holomorphic and zero-free on a neighborhood of ∂B. Let $H_\pm$ denote the top and bottom edges of ∂B:

$$H_+ := \{x + i(m+\delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}, \qquad H_- := \{x + i(m-\delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}.$$

Define

$$\Delta_{\mathrm{nonforce}}(B) := \int_{H_+} |\partial_s \arg F|\, ds + \int_{H_-} |\partial_s \arg F|\, ds.$$

Lemma 10.16 (Horizontal budget (residual form; audit-grade))**.** *In the setting of Definition 10.15,*

$$\Delta_{\mathrm{nonforce}}(B) \le 4\delta \sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right|.$$

Consequently, if $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2$*, then*

$$\Delta_{\mathrm{nonforce}}(B) \le C_h''\, \delta\, (\log m + 1), \qquad C_h'' := 4 \max\{C_1,\, C_2\}.$$

Proof. On either horizontal edge, $|\partial_s \arg F| \le |F'/F|$ pointwise. Each edge has length 2δ, hence each integral is bounded by $2\delta \sup_{\partial B} |F'/F|$. Summing top and bottom gives the first inequality, and the second follows from $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2 \le \max\{C_1, C_2\}(\log m + 1)$. $\square$

11 The explicit tail inequality (post-pivot)

For $m \ge 10$ we use the growth surrogate

$$L(m) := C_1 \log m + C_2,$$

with constants as in Lemma 7.2. For the local window term we use the explicit majorant from Lemma 10.9:

$$N_{\mathrm{up}}(m) := 1.01 \log m + 17 \ \text{ so that } \ N_{\mathrm{loc}}(m) \le N_{\mathrm{up}}(m) \quad (m \ge 10).$$

For a parameter $\eta \in (0,1)$ and a dial displacement $\alpha \in (0,1]$ define the *nominal* scale

$$\delta_0 := \delta_0(m, \alpha) := \frac{\eta \alpha}{(\log m)^2}.$$

Fix a collar parameter $\kappa \in (0, 1/10)$ as in Definition 10.5. For each (m, α) we choose any scale $0 < \delta \le \delta_0$ such that the aligned boxes $B = B(\pm\alpha, m, \delta)$ are κ–admissible; existence is guaranteed by Lemma 10.6. By Lemma 11.2, shrinking δ only helps in the tail inequality, so it is safe to treat δ_0 as the worst-case scale in one-height reductions.

Theorem 11.1 (Tail inequality (criterion form; pointwise UE exponent $p = 1$))**.** *Fix* $m \ge 10$ *and* $\eta \in (0,1)$*. Assume:*

1. *the forcing lemma producing the positive constant*

$$c_0 := \frac{3 \log 2}{8\pi}, \qquad c := \frac{3 \log 2}{16}, \qquad K_{\mathrm{alloc}} := 3 + 8\sqrt{3};$$

2. the residual envelope bound (Lemma 7.2) providing C_1, C_2;

3. the audit-grade horizontal budget bound (Lemma 10.16), giving a constant C_h'' independent of (α, m, δ);

4. the explicit local window bound (Lemma 10.9) providing the majorant $N_{\rm up}(m) = 1.01 \log m + 17$.

Then for every $\alpha \in (0,1]$ and every κ–admissible aligned box $B = B(\pm\alpha, m, \delta)$, absence of off–axis quartets at height m follows from the strict inequality

$$2C_{\rm up}\Big(\delta L(m) \;+\; \frac{N_{\rm up}(m)}{\kappa}\Big) \;<\; c \;-\; \delta\Big(K_{\rm alloc}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big). \tag{14}$$

Proof sketch / bookkeeping. The forcing side is unchanged from v31. The only post-pivot modification is on the upper-envelope side: Lemma 10.3 bounds dial deviation in terms of $\sup_{\partial B} |E'/E|$. Applying the log-derivative split (Lemma 10.7), the residual envelope for $\sup_{\partial B} |F'/F| \le L(m)$ (Lemma 7.2), and the collar bound $\sup_{\partial B} |Z'_{\rm loc}/Z_{\rm loc}| \le N_{\rm loc}(m)/(\kappa\delta)$ (Lemma 10.8) yields

$$\sup_{\partial B}\Big|\frac{E'}{E}\Big| \le L(m) + \frac{N_{\rm loc}(m)}{\kappa\delta} \le L(m) + \frac{N_{\rm up}(m)}{\kappa\delta}.$$

Plugging this into Lemma 10.3 gives the left-hand side of (14). The right-hand side is the forcing lower bound, with the horizontal non-forcing term bounded by Lemma 10.16. □

Lemma 11.2 (Monotonicity under δ–shrinking)**.** *Fix $m \ge 10$, $\alpha \in (0,1]$, and constants $C_{\rm up}, \kappa, c, c_0, K_{\rm alloc}, C_h'', C_1,$ Let $L(m) = C_1 \log m + C_2$ and $N_{\rm up}(m) = 1.01 \log m + 17$. For $\delta \in (0,1]$ define*

$$\mathrm{LHS}(\delta) := 2C_{\rm up}\Big(\delta L(m) \;+\; \frac{N_{\rm up}(m)}{\kappa}\Big), \qquad \mathrm{RHS}(\delta) := c \;-\; \delta\Big(K_{\rm alloc}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big).$$

Then $\mathrm{LHS}(\delta)$ *is (weakly) increasing in δ and* $\mathrm{RHS}(\delta)$ *is (weakly) decreasing. Consequently, if* $\mathrm{LHS}(\delta_0) < \mathrm{RHS}(\delta_0)$ *for some $\delta_0 \in (0,1]$, then* $\mathrm{LHS}(\delta) < \mathrm{RHS}(\delta)$ *holds for every $\delta \in (0, \delta_0]$.*

Proof. For $\delta > 0$, the map $\delta \mapsto \delta L(m)$ is increasing and the term $N_{\rm up}(m)/\kappa$ is independent of δ, hence $\mathrm{LHS}(\delta)$ is (weakly) increasing. The bracketed factor in $\mathrm{RHS}(\delta)$ is nonnegative and independent of δ, so $\mathrm{RHS}(\delta)$ decreases linearly in δ. □

Lemma 11.3 (Worst case in α is $\alpha = 1$ at the nominal scale)**.** *Fix $m \ge 10$ and $\eta \in (0,1)$. Define the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. Consider the tail inequality* (14) *evaluated at $\delta = \delta_0(m,\alpha)$. Then the left-hand side is (weakly)* increasing *in $\alpha \in (0,1]$, while the right-hand side is (weakly)* decreasing. *Therefore it suffices to verify* (14) *at $\alpha = 1$ and $\delta = \delta_0(m,1)$. If one later shrinks $\delta \le \delta_0(m,\alpha)$ to enforce κ–admissibility, the inequality only becomes easier (Lemma 11.2).*

Proof. With $\delta = \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, the only α-dependence in the left-hand side is through the factor $\delta L(m)$, which is increasing in α, so the left-hand side increases. The right-hand side equals $c - \delta \cdot \Xi(m)$ for a nonnegative factor $\Xi(m)$ independent of α, hence it decreases. □

Remark 11.4 (No one-height reduction in m under the pointwise UE exponent $p = 1$). In v33, the (claimed) $\delta^{3/2}$ prefactor in Lemma 10.3 made the local contribution scale like $\delta^{1/2} N_{\rm up}(m)$ at the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, leading to an expression essentially independent of m and enabling a one-height reduction. After the UE–Gate audit, Lemma 10.3 provides only the pointwise exponent $p = 1$, so the tail left-hand side contains the δ–inert term $(2C_{\rm up}/\kappa)\, N_{\rm up}(m)$. With the explicit majorant $N_{\rm up}(m) = 1.01 \log m + 17$, this term is *increasing* in m. Therefore a one-height reduction in m is not available under the current pointwise envelope mechanism: the tail criterion must be controlled as a family in m, or the UE–Gate must be cleared by a strengthened envelope mechanism (Remark 10.11).

12 S5 frontier: non-pointwise UE redesign (open)

At fixed (m, α) the tail inequality (14) is a strict forcing–vs–envelope condition. In v35 the combination of Theorem 10.12, Lemma 10.14, and Lemma 8.2 formally rules out the former "η–absorption" closure route based on the pointwise/sup endpoint $\sup_{\partial B} |E'/E|$ together with the pointwise collar bound.

What must change. The forcing chain produces a lower bound for the *dial deviation*

$$D_B(W) \ := \ \sum_{\pm} \left| W(v_\pm) - e^{i\varphi_0^\pm} \right|$$

appearing in Lemma 10.3. In the current architecture this deviation is upper-bounded by a pointwise endpoint $\delta \sup_{\partial B} |E'/E|$, which (via the collar) introduces the sharp δ^{-1} blow–up. To obtain a tail closure mechanism one must redesign the envelope endpoint and/or the local interface so that the exponent budget $p - \theta \geq \frac{1}{2}$ is met *uniformly in m*.

Remark 12.1 (Forcing compatibility for redesigned endpoints). The existing forcing chain lower-bounds $D_B(W)$ (via the pair-factor phase rotation) by a fixed constant c up to δ–small corrections. If one proposes a redesigned envelope endpoint Φ_B (non-pointwise, e.g. an L^2 or energy functional), then the current forcing lower bound is useful only if it implies a corresponding lower bound for Φ_B. A sufficient (and simplest) compatibility condition is:

$$\Phi_B \ \geq \ D_B(W) \qquad \text{for all admissible boxes and quotients } W,$$

so that the forcing lower bound propagates unchanged. If this domination fails, then a *new forcing lemma* must be proved that lower-bounds Φ_B directly.

S5 design targets (open). A future closure route (S5) should provide a non–pointwise endpoint Φ_B and a UE–type inequality of the schematic form

$$D_B(W) \ \leq \ C_{\mathrm{up}}\, \delta^p \, \Phi_B(E) \qquad (p > 0),$$

together with a local/residual split of $\Phi_B(E)$ whose local contribution scales as $\delta^{-\theta}$ with $\theta < p - \frac{1}{2}$, or more generally satisfies the exponent budget of Theorem 10.12. The point is *not* to recover the specific exponent $\frac{3}{2}$ from older drafts, but to obtain any effective gain $p - \theta > \frac{1}{2}$ with proof–grade uniformity.

Remark 12.2 (Recorded open lemmas (S5 checklist)). A proof–grade S5 implementation would minimally require:

1. **(S5–UE)** a redesigned upper-envelope inequality with a forceable endpoint Φ_B;
2. **(S5–RES)** a δ–uniform residual envelope bound in the same endpoint class;
3. **(S5–LOC)** a collar/local bound in the same endpoint class that avoids the pointwise δ^{-1} blow–up;
4. **(S5–FORCE)** either $\Phi_B \geq D_B(W)$ or a new forcing lemma as in Remark 12.1.

13 Global RH from a finite front-end + the tail criterion family

Theorem 13.1 (Global closure (criterion-first logical form))**.** *Assume:*

1. *(Front-end) All nontrivial zeros with* $0 < \mathrm{Im}(s) \le 5$ *lie on the critical line.*
2. *(Tail criterion) Fix some* $\eta \in (0,1)$ *and* $\kappa \in (0, 1/10)$*, and assume the analytic inputs Lemmas 10.3–10.9 and Lemma 10.16 with finite constants. Assume moreover that for every* $m \ge 10$ *and every* $\alpha \in (0,1]$ *there exists a* κ*–admissible scale* $0 < \delta \le \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$ *such that the strict tail inequality* (14) *holds.*

Then all nontrivial zeros of $\zeta(s)$ *lie on the critical line.*

Proof. For each $m \ge 10$, Theorem 11.1 turns the strict inequality (14) into exclusion of off-axis quartets at height m. By the tail criterion hypothesis, no off-axis quartets exist at any height $m \ge 10$. By the front-end hypothesis, there are no off-axis zeros below height 5. Hence there are no off-axis zeros at any height, so every nontrivial zero lies on the critical line. $\square$

Remark 13.2 (Role of computations and the repro pack (v35)). Appendix B provides a small interval-arithmetic harness that evaluates the tail inequality for pinned parameters and a pinned constant ledger. In v35 this is used only for audit purposes (e.g. exponent tracking), not as a proof substitute.

A Discarded closure routes (as of v35)

This appendix records closure routes that were explored in earlier iterations (v32–v34) but are now ruled out *under the currently proved inputs.* The purpose is to prevent future drift: these routes should not be re-opened unless a genuinely new analytic input (e.g. an S5 endpoint redesign) is supplied.

A.1 D1: Pointwise UE endpoint $\sup_{\partial B} |E'/E|$ + collar + η–absorption (S1/S1′)

The former absorption narrative attempted to close the tail family by shrinking η in the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. In the pointwise/sup architecture the UE step has exponent $p = 1$ (Lemma 10.3) and the collar/local split has exponent $\theta = 1$ (Lemma 10.8), so the local contribution is δ–inert and cannot be suppressed by η (Lemma 10.13). More strongly, the exponent budget (Theorem 10.12) shows that uniform η–shrinking requires $p - \theta \ge \frac{1}{2}$, while the scaling NO–GO (Lemma 10.14) forbids any $p > 1$ within this endpoint class. Finally, the forcing margin is constant–limited in the single–box architecture (Lemma 8.2), so one cannot compensate by "making forcing grow with m".

Proposition A.1 (Historical record: formal anchor absorption under a hypothetical strengthened UE exponent)**.** *This proposition is* not used *in v35. It is recorded only to document the logical shape of the discarded absorption idea.*

Assume that, for some $p > 1$*, an upper-envelope step admits the strengthened form*

$$D_B(W) \;\le\; 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\left|\frac{E'}{E}\right|$$

with the same constant ledger, and that all other constants in (14) *are finite. Fix an anchor height* $m_\star \ge 10$ *and evaluate* (14) *at* $(m,\alpha) = (m_\star, 1)$ *with the nominal scale* $\delta_0(m_\star, 1) = \eta/(\log m_\star)^2$*. Then there exists* $\eta_\star(m_\star, p) > 0$ *such that* (14) *holds at* $(m_\star, 1)$ *for every* $\eta \in (0, \eta_\star]$*.*

Warning: *within the pointwise/sup endpoint class, Lemma 10.14 forbids any $p > 1$, so this proposition cannot be invoked without an S5 redesign.*

Proof. Under a strengthened exponent $p > 1$, the envelope side becomes $A\,\eta^p + B\,\eta^{p-1}$ for finite constants A, B depending on $(m_\star, p)$ and the constant ledger, while the forcing side equals $c - D\,\eta$ for a finite D. Since $p > 1$, one has $\eta^p \to 0$, $\eta^{p-1} \to 0$, and $\eta \to 0$ as $\eta \downarrow 0$, so the strict inequality holds for all sufficiently small η. □

A.2 D2: Shrinking the local window / short-interval zero counts

A tempting workaround is to replace the fixed local window $|\gamma - t| \le 1$ in the residual/collar interface by a shrinking window $|\gamma - t| \le \delta^\beta$ to reduce the local term. However, without additional analytic input, available RH–free methods control $N(t+1) - N(t-1)$ at unit scale and do *not* provide a proof–grade bound for $N(t+\delta^\beta) - N(t-\delta^\beta)$ as $\delta \downarrow 0$. Thus v35 does not pursue window-shrinking as a substitute for the missing UE gain.

A.3 D3: "Make forcing grow with m" within single-box forcing

Because $\Delta_{I_+} \arg Z_{\mathrm{pair}} \le 2\pi$ uniformly (Lemma 8.2), the forcing constant c in the tail inequality is $O(1)$. Any attempt to obtain a forcing side that grows like $\log m$ (or any unbounded function of m) would require a different forcing architecture (not the v35 single-box forcing chain).

A.4 D4: "Boundary modulus implies interior zero-freeness" converse

Under boundary-contact, the quotient $W = E/G_{\mathrm{out}}$ satisfies $|W| = 1$ on ∂B (Remark 9.3), but this has no converse implication toward zero-freeness or constancy (Remark 9.4). Therefore, any closure route that implicitly treats $|W| = 1$ as "almost zero-free" is invalid.

B Tail harness bundle and reproducibility (v35)

B.1 What the tail checks prove (and what they do not)

Each tail check records the statement:

> Given a constants file that provides interval enclosures for $(C_1, C_2, C_{\mathrm{up}}, C_h'', \kappa)$, the chosen parameters (m, η, α), and the recorded UE exponent p, the harness computes interval bounds for the left-hand side LHS and right-hand side RHS in (14) and reports whether the strict separation $\mathrm{LHS}_{\mathrm{hi}} < \mathrm{RHS}_{\mathrm{lo}}$ holds.

It does *not* certify that the constants file is correct.

B.2 SHA–256 table (exact artifacts)

The file `v35_repro_pack/SHA256SUMS.txt` is the canonical hash list.

```
98e39f3c557d0bdcbcb79fde73159daad381b38740e570646fae5da9e6c7fea4  README.md
572bc9f01b1a7b3b38eb3c1c740b50ca01dc4d407bc6482b667cebe29f76455f  SHA256SUMS.txt
1a63129c5af2576cf62aefcda2358b541f7a10b5004fc570a9176e5bf5621793  v35_constants_m10.json
c35112fe7ba00aab82a1332e8af278e4656b55e4195b03ba32f78e4e60aeb692  v35_frontend_certificate.json
c1debbda3583dbaf0dc7120684ba89c457fef1227f4aa13504b21cf11e029acb  v35_frontend_verifier_output.txt
cb9f61fd9a605ba1c2df478eb3ee304e3186367bd1078b246ee3137fa8d21e1d  v35_generate_frontend_certificate
    .py
```

```
12c700ac72c7dc3fee1f66263f8e91e68aa77f427e630ea2ca81aa26ad159174  v35_generate_tail_check.py
5f75f9d692727786c289e3c2cbe3c2fcb0b89784fffaf01e280093a9083ef5fc  v35_tail_check_m10.json
3b2c15c47e9eaadc4edbac24296091f41e9c04062de51536469584fc1783a307
    v35_tail_check_verifier_output_m10.txt
76f736fa4e6722a8c929f625968c046ad68befda20186e1ba17f6082c8f0c30a  v35_verify_frontend_certificate.
    py
60227835eecbdef40d8ccca1745456068a525f25dc8cc6d96c89acb7fb7c383f  v35_verify_tail_check.py
```

B.3 Commands

From the directory `v35_repro_pack/`:

1. `sha256sum -c SHA256SUMS.txt`
2. `python3 v35_verify_tail_check.py --constants v35_constants_m10.json --certificate v35_tail_check_m10.json`
3. `python3 v35_verify_frontend_certificate.py --certificate v35_frontend_certificate.json`

B.4 Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false)

```
LHS_hi =
    850713.3937515341704749092892085477855953048194518160830781594256609802627304191409602400076
RHS_lo =
    0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351
STRICT (LHS_hi < RHS_lo) = False
REGEN_MATCH = True
INEQUALITY_STRICT = False
CERT_REPORTED_PASS = False
OK
```

B.5 Bundle files (verbatim)


```
{
  "certificate_version": "v35",
  "created_utc": "2026-01-23T00:00:00Z",
  "m_band": "10",
  "eta": "1e-14",
  "alpha_worst": "1",
  "kappa": "0.05",
  "intervals": {
    "C1": {
      "lo": "15.1",
      "hi": "15.2"
    },
    "C2": {
      "lo": "37.3",
      "hi": "37.4"
    },
    "C_up": {
      "lo": "1100",
      "hi": "1100.5"
    },
    "C_hpp": {
      "lo": "1100",
```

```
      "hi": "1100.5"
    }
  },
  "notes": [
    "Demo-only intervals carried forward from v31-style scaffolding; replace with audit-proven
    enclosures when G-1/G-12 are closed.",
    "The verifier/generator implement directed-rounding interval arithmetic with Python's decimal
    module.",
    "The local-window majorant N_up(m)=1.01*log(m)+17 is hard-coded from Lemma Nloc-logm in
    manuscript_v35.",
    "UE_exponent_p is recorded explicitly to prevent exponent drift across versions.",
    "v35 adds explicit metadata for endpoint_functional and forcing_architecture to prevent silent
    mismatch under S5 redesign.",
    "UE_endpoint_class='pointwise/sup' is the class for which Lemma UE-scaling-nogo forbids any
    exponent p>1 with shape-only constants."
  ],
  "UE_exponent_p": "1",
  "manuscript_version": "v35",
  "UE_endpoint_class": "pointwise/sup",
  "endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
  "forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma
    force-constant-limited)",
  "forcing_constants": {
    "c_expression": "(3*ln(2))/16",
    "c0_expression": "(3*ln(2))/(8*pi)",
    "Kalloc_expression": "3 + 8*sqrt(3)"
  }
}
```

```
{
  "certificate_version": "v35",
  "m_band": "10",
  "eta": "1e-14",
  "alpha": "1",
  "kappa": "0.05",
  "UE_exponent_p": "1",
  "UE_endpoint_class": "pointwise/sup",
  "endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
  "forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma
    force-constant-limited)",
  "forcing_constants": {
    "c_expression": "(3*ln(2))/16",
    "c0_expression": "(3*ln(2))/(8*pi)",
    "Kalloc_expression": "3 + 8*sqrt(3)"
  },
  "manuscript_version": "v35",
  "prec": 90,
  "pi_interval": {
    "lo": "3.14159265358979323846264338327950288419716939937510",
    "hi": "3.14159265358979323846264338327950288419716939937511"
  },
  "logm_interval": {
    "lo":
    "2.302585092994045684017991454684364207601101488628772976033327900967572609677352480235997 21",
    "hi":
    "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721"
```

```
},
"delta_interval": {
  "lo":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197468E
  -15",
  "hi":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197469E
  -15"
},
"L_interval": {
  "lo":
  "72.069034904210089828671670965733899534776632478294471938103251304610346406128022451563578",
  "hi":
  "72.399293413509494397073470111202335955536742627157349235706584094707103667095757699587157 6"
},
"Nup_interval": {
  "lo":
  "19.325610943923986140858171369231207849677112503515060705793661179977248335774126005038 3571",
  "hi":
  "19.325610943923986140858171369231207849677112503515060705793661179977248335774126005038 3572"
},
"kappa_interval": {
  "lo": "0.05",
  "hi": "0.05"
},
"lhs_interval": {
  "lo":
  "850326.88153265568924514499623682060899067688357982393949159379150356541572603490436657 1925",
  "hi":
  "850713.39375153417047490928920854778559530481945181608307815942566098026273041914096024 0076"
},
"rhs_interval": {
  "lo":
  "0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351",
  "hi":
  "0.129965096347948199005209691457697222838229716361348668790498959759558243588339370271250251"
},
"derived_constants": {
  "ln2_interval": {
    "lo":
  "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327",
    "hi":
  "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327"
  },
  "c_interval": {
    "lo":
  "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099373",
    "hi":
  "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099375"
  },
  "c0_interval": {
    "lo":
  "0.0827383500572443475236711620442491341185086557736206913728528561387020242248387512851407512",

    "hi":
  "0.0827383500572443475236711620442491341185086557736209547372007536994885577445868650239268751"
  },
  "Kalloc_interval": {
    "lo":
```

```
    "16.856406460551018348219570732046978935542442030483045024446455835615464135270400296649169 5",
      "hi":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491696"
    }
  },
  "pass": false
}
```


```python
#!/usr/bin/env python3
"""
v35_generate_tail_check.py

Deterministically generates v35_tail_check_m10.json from v35_constants_m10.json using
directed-rounding interval arithmetic implemented with Python's decimal module.

This generator is intended to be auditable: no network access, no randomness, and no external
    libraries.

Tail inequality evaluation (for given inputs):
  LHS(delta) < RHS(delta), where
  LHS(delta) = 2*C_up*( delta^p*L(m) + delta^(p-1)*N_up(m)/kappa )
  RHS(delta) = c - delta*( Kalloc*c0*L(m) + C_hpp*(log(m)+1) )

with
  L(m)    = C1*log(m) + C2,
  N_up(m) = 1.01*log(m) + 17,
  c  = (3 ln 2)/16,
  c0 = (3 ln 2)/(8 pi),
  Kalloc = 3 + 8*sqrt(3).

Usage:
  python3 v35_generate_tail_check.py v35_constants_m10.json v35_tail_check_m10.json
"""

import json
import sys
from dataclasses import dataclass
from decimal import Decimal, getcontext, localcontext, ROUND_FLOOR, ROUND_CEILING


# ---- Fixed enclosure for pi (50 decimal places) ----
# pi = 3.14159265358979323846264338327950288419716939937510...
PI_LO = Decimal("3.14159265358979323846264338327950288419716939937510")
PI_HI = Decimal("3.14159265358979323846264338327950288419716939937511")


@dataclass
class Interval:
    lo: Decimal
    hi: Decimal

    def __post_init__(self) -> None:
        if self.lo > self.hi:
            raise ValueError(f"Bad interval: {self.lo} > {self.hi}")
```

```python
def ctx(prec: int, rounding):
    c = getcontext().copy()
    c.prec = prec
    c.rounding = rounding
    return c


def iv(lo: str, hi: str | None = None) -> Interval:
    if hi is None:
        hi = lo
    return Interval(Decimal(lo), Decimal(hi))


def add(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo + b.lo
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi + b.hi
    return Interval(lo, hi)


def sub(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo - b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi - b.lo
    return Interval(lo, hi)


def mul(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        cands_lo = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        lo = min(cands_lo)
    with localcontext(ctx(prec, ROUND_CEILING)):
        cands_hi = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        hi = max(cands_hi)
    return Interval(lo, hi)


def div(a: Interval, b: Interval, prec: int) -> Interval:
    if b.lo <= 0 <= b.hi:
        raise ZeroDivisionError("Interval division by an interval containing 0.")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        rlo = Decimal(1) / b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        rhi = Decimal(1) / b.lo
    return mul(a, Interval(rlo, rhi), prec)


def sqrt(a: Interval, prec: int) -> Interval:
    if a.lo < 0:
        raise ValueError("sqrt of negative interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.sqrt()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.sqrt()
    return Interval(lo, hi)
```

```
def ln(a: Interval, prec: int) -> Interval:
    if a.lo <= 0:
        raise ValueError("ln of nonpositive interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.ln()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.ln()
    return Interval(lo, hi)


def pow_3_2(a: Interval, prec: int) -> Interval:
    return mul(a, sqrt(a, prec), prec)


def compute(constants: dict, prec: int = 90) -> dict:
    m = iv(constants["m_band"])
    eta = iv(constants["eta"])
    alpha = iv(constants["alpha_worst"])
    kappa = iv(constants["kappa"])

    p = str(constants.get("UE_exponent_p", "1"))

    C1 = iv(constants["intervals"]["C1"]["lo"], constants["intervals"]["C1"]["hi"])
    C2 = iv(constants["intervals"]["C2"]["lo"], constants["intervals"]["C2"]["hi"])
    Cup = iv(constants["intervals"]["C_up"]["lo"], constants["intervals"]["C_up"]["hi"])
    Chpp = iv(constants["intervals"]["C_hpp"]["lo"], constants["intervals"]["C_hpp"]["hi"])

    logm = ln(m, prec)
    delta = div(mul(eta, alpha, prec), mul(logm, logm, prec), prec)

    # L(m) = C1*logm + C2
    L = add(mul(C1, logm, prec), C2, prec)

    # N_up(m) = 1.01*logm + 17
    Nup = add(mul(iv("1.01"), logm, prec), iv("17"), prec)

    # ln 2
    ln2 = ln(iv("2"), prec)

    # c = (3 ln 2)/16
    c = div(mul(iv("3"), ln2, prec), iv("16"), prec)

    # c0 = (3 ln 2)/(8 pi), pi enclosed
    pi = Interval(PI_LO, PI_HI)
    c0 = div(mul(iv("3"), ln2, prec), mul(iv("8"), pi, prec), prec)

    # Kalloc = 3 + 8 sqrt(3)
    sqrt3 = sqrt(iv("3"), prec)
    Kalloc = add(iv("3"), mul(iv("8"), sqrt3, prec), prec)

    logm_plus1 = add(logm, iv("1"), prec)

    # UE exponent p: LHS = 2*Cup*(delta^p * L + delta^(p-1) * Nup/kappa).
    # We support p="1" (pointwise UE proved in v35) and p="3/2" (hypothetical strengthened gate).
    if p in ("1", "1.0", "1.00"):
        local_term = div(Nup, kappa, prec)              # delta^(p-1)=1
        residual_term = mul(delta, L, prec)             # delta^p = delta
    elif p in ("3/2", "1.5", "1.50"):
        sqrt_delta = sqrt(delta, prec)
```

```
        local_term = mul(sqrt_delta, div(Nup, kappa, prec), prec)     # delta^(1/2)
        residual_term = mul(mul(delta, sqrt_delta, prec), L, prec)    # delta^(3/2)
    else:
        raise ValueError(f"Unsupported UE_exponent_p={p!r}; use '1' or '3/2'.")

    lhs = mul(mul(iv("2"), Cup, prec), add(residual_term, local_term, prec), prec)

    # RHS = c - delta*(Kalloc*c0*L + Chpp*(logm+1))
    term1 = mul(mul(Kalloc, c0, prec), L, prec)
    term2 = mul(Chpp, logm_plus1, prec)
    rhs = sub(c, mul(delta, add(term1, term2, prec), prec), prec)

    passed = (lhs.hi < rhs.lo)

    return {
        "prec": prec,
        "UE_exponent_p": p,
        "pi_interval": {"lo": str(PI_LO), "hi": str(PI_HI)},
        "logm_interval": {"lo": str(logm.lo), "hi": str(logm.hi)},
        "delta_interval": {"lo": str(delta.lo), "hi": str(delta.hi)},
        "L_interval": {"lo": str(L.lo), "hi": str(L.hi)},
        "Nup_interval": {"lo": str(Nup.lo), "hi": str(Nup.hi)},
        "kappa_interval": {"lo": str(kappa.lo), "hi": str(kappa.hi)},
        "lhs_interval": {"lo": str(lhs.lo), "hi": str(lhs.hi)},
        "rhs_interval": {"lo": str(rhs.lo), "hi": str(rhs.hi)},
        "derived_constants": {
            "ln2_interval": {"lo": str(ln2.lo), "hi": str(ln2.hi)},
            "c_interval": {"lo": str(c.lo), "hi": str(c.hi)},
            "c0_interval": {"lo": str(c0.lo), "hi": str(c0.hi)},
            "Kalloc_interval": {"lo": str(Kalloc.lo), "hi": str(Kalloc.hi)},
        },
        "pass": bool(passed),
    }


def main() -> int:
    if len(sys.argv) != 3:
        print("Usage: v35_generate_tail_check.py constants.json tail_check.json", file=sys.stderr)
        return 2

    with open(sys.argv[1], "r", encoding="utf-8") as f:
        constants = json.load(f)

    out = {
        "certificate_version": "v35",
        "m_band": constants["m_band"],
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        "UE_exponent_p": constants.get("UE_exponent_p", "1"),
        "UE_endpoint_class": constants.get("UE_endpoint_class", "pointwise/sup"),
        "endpoint_functional": constants.get("endpoint_functional", "sup_{\\partial B} |E\'/E|"),
        "forcing_architecture": constants.get("forcing_architecture", "single-box forcing"),
        "forcing_constants": constants.get("forcing_constants", {}),
        "manuscript_version": constants.get("manuscript_version", "v35"),
    }
    out.update(compute(constants, prec=90))

    with open(sys.argv[2], "w", encoding="utf-8") as f:
```

```
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[2])
    print("[generate] PASS =", out["pass"])
    print("[generate] lhs_interval.hi =", out["lhs_interval"]["hi"])
    print("[generate] rhs_interval.lo =", out["rhs_interval"]["lo"])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

```
#!/usr/bin/env python3
"""
v35_verify_tail_check.py

Verifier for v35_tail_check_m10.json. This script:
- loads the constants JSON and the pinned certificate JSON
- regenerates the certificate from constants
- checks exact JSON equality on the computed fields
- reports PASS/FAIL and prints the strict-separation check LHS_hi < RHS_lo.

Usage:
  python3 v35_verify_tail_check.py --constants v35_constants_m10.json --certificate
    v35_tail_check_m10.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json
import sys

from v35_generate_tail_check import compute


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v35 tail check (m=10).")
    ap.add_argument("--constants", required=True, help="Path to v35_constants_m10.json")
    ap.add_argument("--certificate", required=True, help="Path to v35_tail_check_m10.json")
    args = ap.parse_args()

    with open(args.constants, "r", encoding="utf-8") as f:
        constants = json.load(f)

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    regen = {
        "certificate_version": "v35",
        "m_band": constants["m_band"],
        "eta": constants["eta"],
```

```
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        "UE_exponent_p": constants.get("UE_exponent_p", "1"),
        "UE_endpoint_class": constants.get("UE_endpoint_class", "pointwise/sup"),
        "endpoint_functional": constants.get("endpoint_functional"),
        "forcing_architecture": constants.get("forcing_architecture"),
        "forcing_constants": constants.get("forcing_constants"),
        "manuscript_version": constants.get("manuscript_version", "v35"),
    }
    regen.update(compute(constants, prec=90))

    # Compare all keys that regen produces (ignore any extra keys in cert)
    ok = True
    for k, v in regen.items():
        if cert.get(k) != v:
            ok = False
            print(f"MISMATCH key={k}")
            print("  cert :", cert.get(k))
            print("  regen:", v)

    lhs_hi = regen["lhs_interval"]["hi"]
    rhs_lo = regen["rhs_interval"]["lo"]
    strict = (float(lhs_hi) < float(rhs_lo))

    print("LHS_hi =", lhs_hi)
    print("RHS_lo =", rhs_lo)
    print("STRICT (LHS_hi < RHS_lo) =", strict)

    print("REGEN_MATCH =", ok)
    print("INEQUALITY_STRICT =", strict)
    print("CERT_REPORTED_PASS =", regen.get("pass"))

    if not ok:
        print("FAIL (mismatch)")
        return 1

    print("OK")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

C Finite-height front-end certificate (literature-based)

The required front-end is RH up to height $H_0 = 5$. We record a discharge using Platt–Trudgian's published verification of RH up to $3 \cdot 10^{12}$.


```
{
  "certificate_version": "v35",
  "created_utc": "2026-01-23T00:00:00Z",
  "needed_frontend_statement": {
    "type": "RH_to_height",
```

"H0": 5.0,
    "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
  },
  "discharged_by": {
    "type": "literature_citation",
    "verification_height": 3000000000000.0,
    "reference": {
      "authors": "D. J. Platt and T. S. Trudgian",
      "title": "The Riemann hypothesis is true up to 3*10^12",
      "venue": "Bulletin of the London Mathematical Society",
      "year": 2021,
      "doi": "10.1112/blms.12460",
      "arxiv": "2004.09765",
      "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (rigorous
    interval arithmetic)."
    },
    "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=H0."
  },
  "notes": [
    "This JSON is not itself a computation of zeros; it is a pinned statement+reference used by v31
    .",
    "For a fully self-contained proof without external computational input, one would need to
    implement and certify an argument-principle zero count in this region using ball arithmetic (
    not provided here)."
  ],
  "manuscript_version": "v35"
}

```
H0 (needed) = 5.0
H_cited     = 3000000000000.0
CHECK: H0 <= H_cited : True
PASS
```

```
#!/usr/bin/env python3
"""v34_generate_frontend_certificate.py

Creates a pinned JSON certificate for the finite-height front-end assumption used by v34.

This script does NOT compute zeta zeros. It encodes a minimal (H0, citation) logic statement:
if RH has been verified up to H_cited and H0 <= H_cited, then RH holds up to height H0.

Usage:
  python3 v34_generate_frontend_certificate.py v34_frontend_certificate.json
"""

from __future__ import annotations

import json
from datetime import datetime, timezone
import sys


def main() -> int:
    if len(sys.argv) != 2:
```

```
        print("Usage: v34_generate_frontend_certificate.py output.json", file=sys.stderr)
        return 2

    out = {
        "certificate_version": "v34",
        "created_utc": datetime.now(timezone.utc).strftime("%Y-%m-%dT%H:%M:%SZ"),
        "needed_frontend_statement": {
            "type": "RH_to_height",
            "H0": 5.0,
            "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
        },
        "discharged_by": {
            "type": "literature_citation",
            "verification_height": 3e12,
            "reference": {
                "authors": "D. J. Platt and T. S. Trudgian",
                "title": "The Riemann hypothesis is true up to 3*10^12",
                "venue": "Bulletin of the London Mathematical Society",
                "year": 2021,
                "doi": "10.1112/blms.12460",
                "arxiv": "2004.09765",
                "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (
    rigorous interval arithmetic)."
            },
            "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=
    H0."
        },
        "notes": [
            "This JSON is not itself a computation of zeros; it is a pinned statement+reference
    used by v34.",
            "For a fully self-contained proof without external computational input, one would need
    to implement and certify an argument-principle zero count in this region using ball arithmetic
    (not provided here)."
        ]
    }

    with open(sys.argv[1], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[1])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())

#!/usr/bin/env python3
"""v34_verify_frontend_certificate.py

Verifier for the front-end certificate JSON produced by v34_generate_frontend_certificate.py.

This verifier checks the internal logic only:
- parses the JSON
- confirms that the required finite-height H0 is <= the cited verification height

It does NOT re-run the cited large-scale computation (Platt--Trudgian); that result is treated as
    an
external, peer-reviewed input in the manuscript.
```

```
Usage:
  python3 v34_verify_frontend_certificate.py --certificate v34_frontend_certificate.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v34 front-end certificate JSON (internal logic only).")
    ap.add_argument("--certificate", required=True, help="Path to v34_frontend_certificate.json")
    args = ap.parse_args()

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    needed = cert.get("needed_frontend_statement", {})
    discharged = cert.get("discharged_by", {})

    H0 = float(needed.get("H0"))
    Hc = float(discharged.get("verification_height"))

    ok = H0 <= Hc

    print("H0 (needed) =", H0)
    print("H_cited     =", Hc)
    print(f"CHECK: H0 <= H_cited : {ok}")

    if not ok:
        print("FAIL")
        return 1

    print("PASS")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

References

References

[1] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Mathematics (2) **116** (1982), no. 2, 361–387.

[2] T. A. Driscoll and L. N. Trefethen, *Schwarz–Christoffel Mapping*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2002.

[3] P. L. Duren, *Theory of H^p Spaces*, Academic Press, 1970.

[4] J. B. Garnett, *Bounded Analytic Functions*, Graduate Texts in Mathematics, Springer, 2007.

[5] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Wiley-Interscience, 1985.

[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.

[7] A. Bellotti and T. Wong, *An improved explicit bound on the argument of the Riemann zeta function on the critical line*, arXiv:2412.15470v2 (2024).

[8] D. Platt and T. Trudgian, *The Riemann hypothesis is true up to* $3\cdot 10^{12}$, Bulletin of the London Mathematical Society **53** (2021), no. 3, 792–797.