A Height-Local Width-2 Program for Excluding Off-Axis Quartets
with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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Abstract

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\mathrm{loc}}$ with perimeter factor 8δ; and (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants (no strip/rectangle density comparison);

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow$ $\sup_{\partial B}\left|E - G_{\mathrm{out}}\right|/|G_{\mathrm{out}}| < 1$ $\Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed $\eta \in (0, 1)$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Combined with a certified base range below m_1 (first nontrivial height in width-2), this yields the global on-axis theorem. All constants appearing in the upper/lower envelope are *shape-only* (independent of m, α, a); residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps to boxes.

Symbol	**Definition / role**	**Provenance / why this form**
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$\big[\alpha - \delta, \alpha + \delta\big] \times \big[m - \delta, m + \delta\big]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Uniform-in-α uses worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, 1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
∂B	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	Controlled by L^2 trace & Hilbert
$Z_{\text{loc}}(v) = \prod_{\lvert \operatorname{Im} \rho - m\rvert \leq 1}(v - \rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma **??** (constants symbolic)
$G(v) = \dfrac{E(1 + v)}{E(1 - v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\text{out}} = e^{U+iV}$	Outer with $\lvert\, G_{\text{out}}\,\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\text{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v_\pm^\star = \pm(a + im)$	"Dial pair" on centerline	Points of evaluation in the tail
$Z_{\text{pair}}(v)$	$(v - (a + im))(v - (-a + im))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K_{\text{alloc}}^{(\star)}(\lambda)$	Allocation coefficient	Shape-only; Lemma **??**
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial (Lemma **??**)
C_{up}	Upper-envelope constant	Shape-only; disc-based bound (Lemma **??**)
C_h''	Horizontal budget constant	Shape-only; Lemma **??**

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$. These symmetries follow from $\Lambda_2(u) = \Lambda_2(2-u)$ and $\overline{\Lambda_2(\bar{z})} = \Lambda_2(z)$ on vertical strips, hence $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$ and conjugation invariance.

Theorem 1.1 (Hinge–Unitarity)**.** *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma+it)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\,\Re\psi\Big(\tfrac{\sigma+it}{4}\Big) - \tfrac{1}{4}\,\Re\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+it)\Big)\Big].$$

Moreover,

$$\big|\Re[\pi\cot(x+iy)]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (with m_1 defined in Appendix **??**) *the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,*

$$\Re\psi\Big(\frac{\sigma+it}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\operatorname{Re} u = 1$. For $|t| < m_1/2$ no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix **??***.*

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1+\lambda(u-1)$ preserve rays; $\tan\theta = \operatorname{Im} v/\operatorname{Re} v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{2.1}$$

Why $m \ge 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the ζ'/ζ expansions on $1/2 \le \sigma \le 1$, $t \ge 3$ (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \ge 3$ corresponds to $m \ge 6$; we take $m \ge 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\ge \pi/2$) against residual budgets $O(\delta \log m)$ and yields an L^2 + harmonic-measure upper envelope (in Section **??**) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$)**.** *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha,m,\delta) \subset \{\operatorname{Re} v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta \in (0,1)$ and $\log m \ge \log 10 > 0$, we have $\eta/(\log m)^2 < 1$, hence $\delta = \alpha\,\eta/(\log m)^2 < \alpha$. Therefore the left edge is at $\alpha - \delta > 0$, so the entire box lies strictly in $\{\operatorname{Re} v > 0\}$, uniformly for $\alpha \in (0,1]$. □

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\, |\operatorname{Im}\rho - m| \le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles within $|\operatorname{Im}\rho - m| \le 1$ have been removed).

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (*residual envelope*, *short-side forcing*) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope)**.** *On ∂B,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \;\le\; C_1 \log m + C_2, \tag{2.3}$$

and

$$\left| \Delta_{\partial B} \arg F \right| \;\le\; 8\delta \left(C_1 \log m + C_2 \right). \tag{2.4}$$

Justification. *DLMF §5.11 controls ψ on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control ζ'/ζ on $1/2 \le \sigma \le 1,\ t \ge 3$. After removing local poles via* (**??**) *and transporting to width-2, we obtain* (**??**)*. For* (**??**)*, write $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_s \arg F\, ds$ as the sum of side integrals (angular limits at the corners); then bound by $|\partial B| \sup_{\partial B} |F'/F| = 8\delta \sup |F'/F|$. The constants $C_1, C_2 > 0$ are absolute; we keep them symbolic (see Appendix S.2 for an optional instantiation).*

Lemma 2.3 (Logarithmic derivatives on ∂B)**.** *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}, \qquad \sup_{\partial B} \left| \frac{E'}{E} \right| \;\le\; \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho:\, |\operatorname{Im}\rho - m| \le 1} \; \sup_{v \in \partial B} \frac{m_\rho}{|v-\rho|}.$$

In particular, by the boundary-contact convention the right-hand side is finite.

Proof. The identity follows from $E = F\, Z_{\mathrm{loc}}$. For the inequality, take suprema termwise and use $\left|\frac{(v-\rho)'}{v-\rho}\right| = \frac{1}{|v-\rho|}$. Finiteness holds since only finitely many ρ satisfy $|\operatorname{Im}\rho - m| \le 1$ and none lie on ∂B after the contact adjustment. □

Lemma 2.4 (Short-side forcing)**.** *Let $Z_{\mathrm{pair}}(v) = (v - (a+im))(v - (-a+im))$. On the near vertical*

$$I_+ = \{\alpha + iy :\ |y-m| \le \delta\}, \quad \text{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha - a|} + 2\arctan\frac{\delta}{\alpha + a} \;\ge\; \frac{\pi}{2}. \tag{2.5}$$

Proof. Along I_+, $\arg(v - (\pm a + im)) = \arctan\frac{y-m}{\alpha \mp a}$. As y runs from $m-\delta$ to $m+\delta$, the increment is $\arctan\frac{\delta}{|\alpha-a|} - \arctan\!\left(-\frac{\delta}{|\alpha-a|}\right) = 2\arctan\frac{\delta}{|\alpha-a|}$ for the near factor and $2\arctan\frac{\delta}{\alpha+a}$ for the far factor. Since $\alpha > 0$ and $a \ge 0$, $\alpha + a > 0$, and the sum is monotone in δ. When $|\alpha - a| \le \delta$, the first term contributes at least $\pi/2$ and the second is nonnegative, proving the bound. A symmetric formula holds on I_-, though not needed here. □

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H) \circ \varphi, \tag{3.1}$$

where H is an *outer majorant* for G on B: that is, $M \in C(\partial B)$ with $M \geq |G|$ a.e. on ∂B and $H = e^{U+iV}$ where U is the continuous Dirichlet solution with boundary data $\log M$ and V a harmonic conjugate (uniqueness modulo a unimodular constant). Then $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \leq 1$.

Remark 3.1 (How the criterion is used). If a verified boundary pattern places $|\Phi|$ at 1 at two designated boundary points (non-corner, in the sense of angular limits) and strictly below 1 on the complementary arcs ("quiet-arc contraction"), then Carathéodory–Julia theory yields unimodular boundary pins at those points for Φ; transporting back to B gives constraints on $|G(\pm(a+im))|$. Background: Duren [**?**, Chs. II, IV–V]; Garnett [**?**, Chs. II–III].

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$)**.** *For $v = a + im$ one has*

$$|G(v)| = \big|\chi_2(1+v)\big| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{3.2}$$

hence

$$|G(a+im)|\,|G(-a+im)| = \big|\chi_2(1+a+im)\big|\,\big|\chi_2(1-a+im)\big|. \tag{3.3}$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta(1+\frac{v}{2})}{\zeta(1-\frac{v}{2})} \right|, \qquad R(-v) = R(v)^{-1}.$$

Proof. *Expand Λ_2 at $1 \pm v$ and collect Γ and π factors; multiplying at $\pm v$ cancels R and yields* (**??**).

3.2 Outer/Rouché Certification Path

Let $U = \log|E| \in C(\overline{B})$ solve the Dirichlet problem on B and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (a.e. with respect to arclength). Existence/uniqueness follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

Proposition 3.3 (Outer/Rouché criterion)**.** *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \; < \; 1, \tag{3.4}$$

*then E is zero-free in B (Rouché's theorem; Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.*

Proposition 3.4 (Bridge 1: inner collapse)**.** *Under* (**??**)*,* $\log|W|$ *is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B.*

Proposition 3.5 (Bridge 2: stitching). *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

Remark 3.6 (Certification recipe and reproducibility). The verification of (**??**) is performed by a robust, rigorous pipeline detailed in Appendix G: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log |G_{\text{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid→continuum Lipschitz enclosure using $\sup_{\partial B} |E'/E|$ (Lemma **??**). Appendix G also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility.

3.3 Corner outer interpolation (two-point)

Theorem 3.7 (Corner outer interpolation). *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+iV}$ be the outer on B with $U|\_\partial B = \log |G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Proof. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log |G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors; Pommerenke). □

Normalization note (shape-only constants). For each box $B(\alpha, m, \delta)$ define the affine normalization

$$\mathcal{N}_{\alpha,m,\delta}(v) = \frac{v - (\alpha + im)}{\delta},$$

which sends $B(\alpha, m, \delta)$ onto the unit square $[-1, 1] \times [-1, 1]$. All operator-norm constants used below (trace to/from $\partial\mathbb{D}$, boundary Hilbert/conjugation in L^2, the harmonic-measure floor for fixed subarcs) are computed on this normalized unit square and are therefore *shape-only* (independent of m, α, and a). The only places m and α enter are through $\delta = \eta\, \alpha/(\log m)^2$ and the residual factor $L = \sup_{\partial B} |E'/E|$.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v_\pm^\star = \pm(a + im), \qquad z_\pm \in \partial\mathbb{D} \;\text{ with }\; \varphi(z_\pm) = v_\pm^\star.$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}.$$

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase ⇒ dial deficit; disc-based upper bound). *Let $m \geq 10$ and $\delta = \eta\, \alpha/(\log m)^2$. Let $W = E/G_{\text{out}}$ be analytic and nonvanishing on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B. For each dial $v_\pm^\star$ on the horizontal centerline, there exists a shape-only constant $C_{\text{up}} > 0$ such that*

$$\left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \;\leq\; C_{\text{up}}\, \delta^{3/2} \left(\sup_{\partial B} \left|\frac{E'}{E}\right|\right),$$

where $\phi_0^{\pm}$ is the harmonic-measure average of $\arg W$ *as seen from $v_{\pm}^{\star}$. Consequently,*

$$\sum_{\pm} \big| W(v_{\pm}^{\star}) - e^{i\phi_0^{\pm}} \big| \;\le\; 2\, C_{\mathrm{up}}\; \delta^{3/2}\; \Big(\sup_{\partial B} \Big| \frac{E'}{E} \Big| \Big).$$

Proof. Let $\psi : \mathbb{D} \to B$ be conformal with $\psi(0) = v_{\pm}^{\star}$, and set $f := W \circ \psi$. Then $u(z) := \log|f(z) - c|$ with $c = e^{i\phi_0^{\pm}}$ is subharmonic and Poisson's inequality on $\mathbb{D}$ yields

$$|f(0) - c| \;\le\; \Big(\int_{\partial\mathbb{D}} |\arg f - \phi_0^{\pm}|^2 \,\frac{dt}{2\pi} \Big)^{1/2}.$$

By bounded conformal trace from ∂B to $\partial\mathbb{D}$ on Lipschitz domains (shape-only constant C_{tr}),

$$\|\arg f - \phi_0^{\pm}\|_{L^2(\partial\mathbb{D})} \;\le\; C_{\mathrm{tr}}\, \|\arg W - \phi_0^{\pm}\|_{L^2(\partial B)}.$$

By Wirtinger on the closed curve ∂B (length 8δ),

$$\|\arg W - \phi_0^{\pm}\|_{L^2(\partial B)} \;\le\; \frac{8\delta}{2\pi}\, \|\partial_s \arg W\|_{L^2(\partial B)}.$$

Finally,

$$\|\partial_s \arg W\|_{L^2(\partial B)} \;\le\; \|\partial_s \arg E\|_{L^2(\partial B)} + \|\partial_s \arg G_{\mathrm{out}}\|_{L^2(\partial B)} \;\le\; 2\sqrt{8\delta}\; \sup_{\partial B} \Big| \frac{E'}{E} \Big|,$$

using the L^2 boundary Hilbert/conjugation isometry on Lipschitz curves (constant 1). Combine to obtain the claim with $C_{\mathrm{up}} = C_{\mathrm{tr}} \cdot \frac{8\sqrt{8}}{\pi}$ shape-only. □

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

Lemma 4.2 (Vertical Lipschitz allocation (L^2))**.** *Let $\lambda \in (0,1)$, and let $s_{\mathrm{tail}} = (2-\lambda)\delta$ be the total tail length on a vertical side (outside the central sub-arc of length $\lambda\delta$). Then on each vertical side*

$$\int_{\mathrm{tails}} \big|\partial_\tau \arg W\big|\, ds \;\le\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\, \delta\, \sup_{\partial B} \Big| \frac{E'}{E} \Big|.$$

Summing both verticals yields

$$\Delta_{\mathrm{cent}} \;\ge\; \Delta_{\mathrm{vert}} \;-\; K_{\mathrm{alloc}}(\lambda)\, \delta\, \sup_{\partial B} \Big| \frac{E'}{E} \Big|, \quad K_{\mathrm{alloc}}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big].$$

For conservatism we may adopt the stricter $K_{\mathrm{alloc}}^{\star}(\lambda) := 2\big[(2-\lambda) + 4\sqrt{2(2-\lambda)}\big]$, which dominates $K_{\mathrm{alloc}}(\lambda)$ and is valid as well.

Proof. Split $\partial_\tau \arg W = \partial_\tau \arg E - \partial_\tau \arg G_{\mathrm{out}}$. The first term integrates $\le s_{\mathrm{tail}} \sup_{\partial B} |E'/E|$. For the second, by the L^2 conjugation isometry on ∂B,

$$\|\partial_\tau \arg G_{\mathrm{out}}\|_{L^2(\partial B)} = \|\partial_\tau \log|E|\|_{L^2(\partial B)} \le \sqrt{|\partial B|}\, \sup_{\partial B} \Big| \frac{E'}{E} \Big| = \sqrt{8\delta}\; \sup_{\partial B} \Big| \frac{E'}{E} \Big|.$$

Cauchy–Schwarz on the tails gives $\int_{\mathrm{tails}} |\partial_\tau \arg G_{\mathrm{out}}| \le \sqrt{s_{\mathrm{tail}}}\, \sqrt{8\delta}\, \sup|E'/E|$. Summing the two contributions on one side gives $\big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\big]\delta \sup|E'/E|$. Doubling yields the first display; the stricter $K_{\mathrm{alloc}}^{\star}$ trivially dominates it. □

Lemma 4.3 (Core zero via restricted contour). *Align the box by taking $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If the retained central vertical gap*

$$\Delta_{\text{cent}} \;:=\; \Delta_{\text{vert}} \;-\; K^{\star}_{\text{alloc}}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;-\; C''_h\,\delta\,(\log m + 1) \;>\; 0$$

(with a shape-only constant $C''_h > 0$ for the horizontal budget) then the rectangle bounded by Γ_λ contains at least one zero of W. This zero lies in the dial-centred core

$$B_{\text{core}}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

Proof. Along Γ_λ the net change in $\arg W$ is $\Delta_{\text{cent}} > 0$ by hypothesis. The tiny horizontals carry vanishing contribution in the $\varepsilon \downarrow 0$ limit (already absorbed in the horizontal budget). The argument principle then forces at least one interior zero. The geometry of Γ_λ confines the zero to $B_{\text{core}}(a, m; \lambda)$. □

Lemma 4.4 (Jensen at the dial). *With $\alpha = a$, fix one dial $p = a + im$. Then $\operatorname{dist}(p, \partial B) = \delta$ so $D_p = \{|z - p| < \delta\} \subset B$. If W has a zero z_k in $B_{\text{core}}(a, m; \lambda)$, then*

$$-\log|W(p)| \;\geq\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\geq\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence, since $1 - e^{-u} \geq u/2$ for $u \in [0, 1]$ and $u = \log(\sqrt{2}/\lambda) \leq \log(2\sqrt{2}) < 1$,

$$1 - |W(p)| \;\geq\; \frac{1}{2}\,\log\Big(\frac{\sqrt{2}}{\lambda}\Big).$$

Corollary 4.5 (Lower envelope; aligned boxes). *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B}|E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m + 1)\Big),$$

where $K^{\star}_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.

Two aligned boxes. We apply the aligned-box argument twice, once with $\alpha = +a$ (controlling ε_+) and once with $\alpha = -a$ (controlling ε_-). The two bounds sum to yield $\mathcal{L}(m, \alpha) = \varepsilon_+ + \varepsilon_-$.

4.3 Tail comparison (symbolic constants)

Theorem 4.6 (Global on-axis theorem; symbolic constants). *Fix $\eta \in (0, 1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\text{up}} > 0$ be the shape-only constant in Lemma* **??**, *$C''_h > 0$ the horizontal budget constant in Lemma* **??**, *and $K^{\star}_{\text{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume the residual envelope of Lemma* **??** *with absolute constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0, 1]$,*

$$\underbrace{\sum_{\pm}\big|W(v^{\star}_{\pm}) - e^{i\phi_0^{\pm}}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^{\star}_{\text{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)},$$

with $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$ and all $\alpha \in (0, 1]$. Combined with a certified base range "no zeros below m_1" (Appendix I), all nontrivial zeros lie on $\Re s = \frac{1}{2}$.

Proof. By Lemma ??, $\mathcal{U}_{hm} \le 2C_{\rm up}\delta^{3/2}(C_1 \log m + C_2)$, which tends to 0 as $\log m \to \infty$. By Corollary ??, $\mathcal{L}(m,\alpha) = c_0\frac{\pi}{2} - \delta\big(K^\star_{\rm alloc}(\tfrac12)\, c_0\,(C_1\log m + C_2) + C_h''(\log m+1)\big)$ tends to $c_0\pi/2 > 0$ as $m \to \infty$, uniformly in α. Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all sufficiently large m. □

Remark 4.7 (Numerical check; illustrative only). If one instantiates (C_1, C_2) safely from the literature (Appendix S.2) and takes a small η (e.g., $\eta = 10^{-9}$), then at $m = m_1$ and $\alpha = 1$ the upper bound is $\ll 10^{-10}$ while the lower bound is ≈ 0.13 up to $O(10^{-8})$ corrections, leaving an overwhelming margin. These numerics are not used in the proof.

Acknowledgments and certification note

Reproducible certification ingredients (interval Poisson; grid→continuum Lipschitz) are outlined in Appendix G. Library versions, precision, and boundary meshes are pinned there.

A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of $\log|\chi_2|$ via $\partial_\sigma \log|\Gamma| = \Re\psi$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$ directly; the cosh-bound form appears in Theorem ??.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\Re\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1,\ t \ge 3$, $\dfrac{\zeta'}{\zeta}(\sigma+it) = \displaystyle\sum_{|\operatorname{Im}\rho - t|\le 1} \frac{1}{\sigma+it-\rho} +$ $O(\log t)$. Removing local poles via $Z_{\rm loc}$ yields Lemma ??.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (??) holds, then E and $G_{\rm out}$ have the same zero count, $G_{\rm out}$ is zero-free, $|W| = 1$ on ∂B. Hence $\log|W| \equiv 0$, and by the open mapping theorem $W \equiv e^{i\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\psi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\psi(0) = \alpha + im$ and $\psi(\pm 1)$ the top corners. By symmetry, $\psi((-1,1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0,1)$ with $\psi(\pm r_0) = \pm(a + im)$.

E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B at grid size N_{side}.
- Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\text{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid→continuum. Lipschitz enclosure via $\sup_{\partial B}|E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B}|E - G_{\text{out}}|/|G_{\text{out}}| < 1$.

G Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section **??** proofs.

H Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem H.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < t_1$, and the first nontrivial zero occurs at $t_1 = 14.134725141734693790457251983562\ldots$ (with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of $\zeta(s)$*, Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to $3 \cdot 10^{12}$*, Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

Appendix S.1. Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on $L^2(\Gamma)$ whose norm depends only on the Lipschitz character of Γ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on the chord-arc constants of Γ. (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, all such operator norms are *shape-only* constants (independent of m, α, a). We denote by C_{tr} a generic shape-only trace constant and by "Hilbert isometry" the L^2 identity on $\partial\mathbb{D}$ transported to ∂B with shape-only dependence.

Appendix S.2. Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\rm loc}$ with $Z_{\rm loc}$ removing local zeros with $|\operatorname{Im}\rho - m| \le 1$. On $1/2 \le \sigma \le 1$ and $t \ge 3$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\Re\psi(x+iy) = \log\sqrt{x^2+y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out $Z_{\rm loc}$ yields

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any choices respecting the cited explicit estimates are legitimate. The main text keeps C_1, C_2 symbolic. On ∂B we have $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\rm loc})'}{Z_{\rm loc}}$ (Lemma **??**); the local sum is finite under the boundary-contact convention, so $L = \sup_{\partial B} |E'/E|$ is controlled by the residual bound plus finitely many explicit local terms.

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