A Width–2 Boundary Program for Excluding Off–Axis Quartets with a Certified Tail Criterion and a Finite–Height Front–End (v44)

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Abstract

This is **v44** of the RH Convergence Program. This build *locks the geometry/endpoint redesign* (GEO–C4): a hinge–centered trig contour in the v–plane together with a single *orthogonal harmonic extraction* endpoint built from the two–sided shift–difference log–derivative $\mathcal{D}_{a,h}$.

The *active closure chain* is now:

$$\text{off–axis quartet } (\pm a \pm im) \Longrightarrow \text{forced } k = 2 \text{ harmonic on } C_{m,\delta} \Longrightarrow \Phi^\star(m, a, \delta, h) \geq c_0$$

while the only remaining mathematical frontier is an RH–free bound on the *signed* $k = 2$ *coefficient* $|\widehat{\psi}(2)|$ (Box 12.2.4), which implies $\Phi^\star = o(1)$ at the nominal scale $\delta = \eta a/(\log(m+3))^2$ (with monotone admissibility shrink).

All earlier S5′ "defect endpoint" candidates and aligned–box micro–coupling routes are retained as *archived NO–GO examples*; they are no longer load–bearing in the active chain.

Contents

Executive Proof Status

What this manuscript is / is not.

- It is a formalization of a *boundary-only* strategy to rule out off-axis zeros for the completed zeta object.
- It is *not* a finished proof claim: the single remaining analytic obligation is stated explicitly as **UE-INPUT** (Box 12.2.4).
- Every lemma not marked OPEN is intended to be proof-grade and local, with explicit dependencies and monotone shrink conventions.

Proof status. **Status (v44):** *geometry locked; single active analytic frontier.* We adopt the GEO–C4 witness family (hinge-centered circles) and a single harmonic endpoint. All prior S5′ "defect endpoint" candidates are retained only as *archived NO–GO diagnostics.*

Truth-latched NO–GO constraints (scope-audited).

1. **(S5′ cancellation)** The centered defect endpoint for $\mathcal{L}_a$ cannot inherit $\pi/2$ forcing from one quartet: it is $O(\delta/a)$ (Section 12).
2. **(δ-inert resonance)** Pointwise/sidewise endpoints built from shift-quotients can be δ-inert under near-resonant second quartets unless the geometry is redesigned.
3. **(Side-length ceiling)** Any endpoint built as a side integral has magnitude $\lesssim \delta \sup |f|$; shrink alone cannot create forcing.
4. **(NG-Δa-A)** Aligned boxes at $(a + im)$ with micro-coupling $h \sim \delta$ suppress the two-sided field by $\delta h/a^2$.
5. **(Pointwise UE ceiling)** A purely pointwise/supremum upper estimate on E'/E along a boundary is too weak to contradict forcing at the nominal scale.
6. **(Projection endpoints: audited scope)** "Projection endpoints are dead" applies only to projections that annihilate the *forced local kernel* or that reduce to pointwise/supremum UE. GEO–C4 uses the $k = 2$ harmonic of the *dipole kernel* generated by $\mathcal{D}_{a,h}$ on hinge circles; this does not fall under the NO–GO.

Single active blocker (v44): UE-INPUT($k = 2$) for the GEO–C4 endpoint $\Phi^\star$ (Box 12.2.4). All other items are bookkeeping/citation polishing.

Reproducibility posture (v44). We ship a minimal certificate harness for the tail inequality checks and envelope numerics. These certificates are *supporting* (not a proof of RH). See Appendix C and the directory `v44_repro_pack/`.

Part I

Reader's Guide / Definitions and Reduction

1 Width–2 normalization

Let s denote the usual complex variable for $\zeta(s)$. We pass to the width–2 coordinate

$$u := 2s, \qquad \zeta_2(u) := \zeta(u/2).$$

Define the width–2 completed zeta

$$\Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\zeta(u/2).$$

Then Λ_2 is *meromorphic* (simple poles at $u = 0$ and $u = 2$) and satisfies the functional equation

$$\Lambda_2(u) = \Lambda_2(2 - u).$$

Define the entire completion

$$\Xi_2(u) := \xi(u/2) = \frac{u(u-2)}{8}\,\Lambda_2(u),$$

so that Ξ_2 is entire and obeys $\Xi_2(u) = \Xi_2(2 - u)$.

We recenter at $u = 1$:

$$v := u - 1, \qquad E(v) := \Xi_2(1 + v).$$

Then E is entire and satisfies the evenness relation

$$E(v) = E(-v),$$

and complex conjugation gives $E(\overline{v}) = \overline{E(v)}$.

Remark 1.1 (Zeros). The zeros of $\Xi_2(u) = \xi(u/2)$ are exactly the *nontrivial* zeros of $\zeta(s)$ under the map $u = 2s$, with multiplicity. All boxes used in the tail program lie at heights $m \geq 10$, so the only zeros that can occur in the relevant local windows are nontrivial zeros.

2 Heights and horizontal displacement (RH–free)

Let $\rho = \beta + i\gamma$ be any nontrivial zero of $\zeta(s)$ (no assumption on β). In width–2 we write

$$u_\rho := 2\rho = (1 + a) + im, \qquad a := 2\beta - 1 \in (-1, 1), \qquad m := 2\gamma > 0. \tag{1}$$

Thus RH is equivalent to $a = 0$ for every nontrivial zero.

3 Quartet symmetry in width–2

The functional equation and conjugation imply that any off–axis zero with parameters (a, m) produces a quartet

$$\{\, 1 \pm a \pm im \,\} \subset \{u \in \mathbb{C} : \Xi_2(u) = 0\}. \tag{2}$$

In the centered v–coordinate this becomes $\{\pm a \pm im\} \subset \{v \in \mathbb{C} : E(v) = 0\}$.

4 Finite-height front-end after lowering the tail anchor

Once the tail anchor is lowered to $m_\star$, the analytic tail argument covers all $m \geq m_\star$. The remaining region corresponds to classical heights

$$0 < \mathrm{Im}(s) < H_0 := m_\star/2. \tag{3}$$

In v31 we take $m_\star = 10$, hence $H_0 = 5$.

Definition 4.1 (Front-end statement)**.** We say that *RH holds up to height* H_0 if every nontrivial zero $\rho = \beta + i\gamma$ with $0 < \gamma \leq H_0$ satisfies $\beta = 1/2$.

Remark 4.2 (How v31 discharges the front-end)*.* The required statement for v31 is RH up to height $H_0 = 5$. This is a tiny special case of published rigorous verifications of RH to enormous heights. For example, Platt–Trudgian prove RH for all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ using interval arithmetic, which immediately implies RH up to $H_0 = 5$. Appendix D records this discharge in a pinned JSON file.

Part II
Self-Contained Boundary Program and Tail Closure

5 Aligned boxes and the $\delta(m)$ scale

Let $m > 0$ and $\alpha \in (0, 1]$. Fix a parameter $\eta \in (0, 1)$ and define the *nominal* box scale

$$\delta_0 = \delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}. \tag{4}$$

We will work with aligned boxes $B(\alpha, m, \delta)$ at scales $0 < \delta \leq \delta_0$. By default one may take $\delta = \delta_0$, but later (Definition 10.5) we allow shrinking δ to enforce κ–admissibility; this is non-circular and monotone-safe (Lemmas 10.6 and 11.2).

Define the (width–2) box centered at $\alpha + im$ by

$$B(\alpha, m, \delta) := \{\, v \in \mathbb{C} : |\mathrm{Re}\, v - \alpha| \leq \delta,\ \ |\mathrm{Im}\, v - m| \leq \delta \,\}. \tag{5}$$

We will also use the symmetric dial centers $v_\pm := \pm\alpha + im$.

6 Local factor and finiteness

For a fixed $m > 0$, let

$$Z(m) := \{\, \rho : E(\rho) = 0,\ \ |\mathrm{Im}\, \rho - m| \leq 1 \,\} \tag{6}$$

(zeros counted with multiplicity). Define the local zero factor and residual:

$$Z_{\mathrm{loc}}(v) := \prod_{\rho \in Z(m)} (v - \rho)^{m_\rho}. \tag{7}$$

$$F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{8}$$

Lemma 6.1 (Finiteness of Z_{loc})**.** *For each fixed $m > 0$ the set $Z(m)$ is finite; hence Z_{loc} is a finite product and F is meromorphic globally and analytic in any neighborhood of $\partial B(\alpha, m, \delta)$ that contains no zeros of E.*

Proof. Nontrivial zeros of ζ satisfy $0 < \beta < 1$, hence in the v–coordinate one has $\operatorname{Re} v \in (-1, 1)$ for all nontrivial zeros. Therefore the set $\{|\operatorname{Im} v - m| \le 1\} \cap \{|\operatorname{Re} v| \le 1\}$ is compact. Since E is entire and its zeros are discrete, only finitely many zeros can lie in this compact set. $\square$

7 Residual envelope bound and the constants ledger

Remark 7.1 (Constant gate for the residual term (what is and is not assumed))*.* The tail criterion uses a bound of the form

$$\sup_{v \in \partial B(\alpha,m,\delta)} \left| \frac{F'(v)}{F(v)} \right| \;\le\; C_1 \log m + C_2,$$

with constants that must be (i) unconditional (not RH-equivalent) and (ii) uniform in $(\alpha, \delta, \eta, \kappa)$ once $m \ge 10$ and $0 < \alpha \le 1$. The proof below reduces this to standard RH-free bounds for ζ'/ζ in the critical strip with local-zero subtraction, plus a Stirling-type bound for Γ'/Γ.

Lemma 7.2 (Residual envelope inequality (δ-uniform))**.** *Fix $m \ge 10$ and $\alpha \in (0, 1]$. Let $\eta \in (0, 1]$ and set the nominal width $\delta_0 := \eta\alpha/(\log m)^2$. Let $\delta \in (0, \delta_0]$ and set $B := B(\alpha, m, \delta)$.*

Define E, Z_{loc} and $F := E/Z_{\mathrm{loc}}$ as in §6 (equations (7)*–*(8)*). Assume* boundary-contact *on ∂B (i.e. $E \ne 0$ on ∂B; hence F is holomorphic on a neighborhood of ∂B). Then there exist absolute constants $C_1, C_2 > 0$ (independent of $m, \alpha, \delta, \eta, \kappa$ and of the zero configuration) such that*

$$\sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right| \;\le\; C_1 \log m + C_2.$$

Proof sketch with explicit dependency control. Write $u := 1 + v$ and $s := u/2 = (1 + v)/2 = \sigma + it$. For $v \in \partial B(\alpha, m, \delta)$ we have $\mathrm{Re}(s) \in [0, 1]$ and

$$\mathrm{Im}(s) = \frac{\mathrm{Im}(v)}{2} \in \Big[\frac{m}{2} - \frac{\delta}{2}, \frac{m}{2} + \frac{\delta}{2}\Big].$$

Since $m \ge 10$ and $\delta \le \delta_0 \le 1/(\log 10)^2 < 1/5$, we have $\mathrm{Im}(s) \asymp m$ uniformly in δ.

1) Log-derivative identity in the s-frame. From $\Xi_2(u) = \frac{u(u-2)}{8}\Lambda_2(u)$ and $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ we obtain, for $u = 1 + v$,

$$\frac{E'(v)}{E(v)} = \frac{\Xi_2'(u)}{\Xi_2(u)} = \Big(\frac{1}{u} + \frac{1}{u-2}\Big) \;-\; \frac{1}{4}\log\pi \;+\; \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{u}{4}\Big) \;+\; \frac{1}{2}\frac{\zeta'}{\zeta}(s), \qquad (u = 1 + v,\ s = u/2).$$

Since $u = 1 + v$ has $\mathrm{Im}(u) = m \ge 10$, the completion terms $(1/u + 1/(u-2))$ are $O(1/m)$ on ∂B and are absorbed into the absolute constants in the bound.

Moreover, since $v = 2s - 1$, the local factor derivative satisfies

$$\frac{Z_{\mathrm{loc}}'(v)}{Z_{\mathrm{loc}}(v)} = \sum_{\rho \in Z(m)} \frac{m_\rho}{v - \rho} = \frac{1}{2} \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s},$$

where $Z_s(m)$ denotes the corresponding multiset of nontrivial zeros $\rho_s = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma - \frac{m}{2}| \le \frac{1}{2}$.

Therefore

$$\frac{F'(v)}{F(v)} = \frac{E'(v)}{E(v)} - \frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)} = -\frac{1}{4}\log\pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{1+v}{4}\Big) + \frac{1}{2}\Bigg(\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)} \frac{m_{\rho_s}}{s-\rho_s}\Bigg).$$

2) RH-free residual bound for ζ'/ζ with local-zero subtraction. A standard "local-zero decomposition" (unconditional) asserts that there exist absolute constants A_ζ, B_ζ such that for $0\le\sigma\le 1$ and $t\ge 5$,

$$\Big|\frac{\zeta'}{\zeta}(\sigma+it) - \sum_{|\gamma-t|\le 1}\frac{1}{(\sigma+it)-\rho}\Big| \;\le\; A_\zeta\log(t+2)+B_\zeta. \tag{$\star$}$$

(For a self-contained route, ($\star$) can be derived from the Hadamard product for $\xi(s)$ plus a Riemann–von Mangoldt bound for $N(T)$; otherwise cite a standard reference.)

For $v\in\partial B$ we have $|t-\frac{m}{2}|\le\delta/2<1/10$, hence every zero in $Z_s(m)$ satisfies $|\gamma-t|\le 1$ and is included in the sum in ($\star$). Thus

$$\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)}\frac{1}{s-\rho_s} = \Big(\frac{\zeta'}{\zeta}(s) - \sum_{|\gamma-t|\le 1}\frac{1}{s-\rho}\Big) + \sum_{\substack{|\gamma-t|\le 1\\ |\gamma-\frac{m}{2}|>1/2}}\frac{1}{s-\rho}.$$

In the remaining sum we have $|\gamma-t|\ge 1/2-|t-\frac{m}{2}|\ge 2/5$, hence $|s-\rho|\ge 2/5$ and each term has modulus $\le 5/2$. The number of zeros with $|\gamma-t|\le 1$ is bounded by the manuscript's explicit local window majorant (Lemma 10.10) at height $\asymp m$, so this difference-of-windows sum is $\ll\log m$.

Combining these bounds yields absolute constants $A_{\mathrm{res}}, B_{\mathrm{res}}$ such that

$$\Big|\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)}\frac{1}{s-\rho_s}\Big| \;\le\; A_{\mathrm{res}}\log m + B_{\mathrm{res}},$$

uniformly for all $v\in\partial B$ and all $\delta\in(0,\delta_0]$.

3) Gamma factor bound (Stirling, uniform in δ). For $z=(1+v)/4$ we have $\mathrm{Re}(z)\in[1/4,3/4]$ and $|\mathrm{Im}(z)|\asymp m$. A uniform Stirling-type bound gives

$$\Big|\frac{\Gamma'}{\Gamma}(z)\Big| \le \log(|\mathrm{Im}(z)|+2)+C_\Gamma \;\le\; \log(m+2)+C_\Gamma,$$

with an absolute constant C_Γ.

4) Conclusion. Insert the bounds from (2)–(3) into the identity in (1), and absorb harmless constants into (C_1, C_2). All constants are independent of $(\alpha,\delta,\eta,\kappa)$ because: (i) σ ranges over a fixed compact interval $[0,1]$, (ii) $t\asymp m$ with $m\ge 10$ uniformly for $\delta\le\delta_0$, and (iii) the difference-of-windows sum is controlled by Lemma 10.10, which is unconditional. □

Remark 7.3 (Hard gate / certificates (v40)). The tail harness in Appendix C uses explicit numerical interval enclosures for the constant ledger (e.g. $C_1, C_2, C_{\mathrm{up}}, C''_h, \kappa$) stored in `v36_repro_pack/v36_constants_m10.j` It evaluates the tail inequality for a pinned parameter choice and records the UE exponent p explicitly. This is an *audit harness* only: it does not certify that the constants file is correct, and it does not, by itself, yield a uniform tail closure. An unconditional proof therefore still requires a referee-acceptable certification of the analytic constant ledger, and a resolution of the UE–Gate (Remark 10.12).

8 Short-side forcing

Assume an off-axis pair at height m with displacement $a > 0$ exists. On an aligned box with $\alpha = a$, the two upper zeros in the centered v–plane are at $v = \pm a + im$. The pair factor

$$Z_{\text{pair}}(v) := (v - (a + im))(v - (-a + im)) \tag{9}$$

produces a large phase rotation on the near vertical side.

Lemma 8.1 (Short-side forcing lower bound). *Let $I_+ := \{\alpha + iy : |y - m| \leq \delta\}$ with $|\alpha - a| \leq \delta$. Then*

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan\left(\frac{\delta}{|\alpha - a|}\right) + 2 \arctan\left(\frac{\delta}{\alpha + a}\right) \geq \frac{\pi}{2}. \tag{10}$$

Lemma 8.2 (Single-box forcing is constant-limited). *In the forcing setup of Lemma 8.1, the total phase variation of the pair factor along I_+ satisfies*

$$\Delta_{I_+} \arg Z_{\text{pair}} \leq 2\pi,$$

uniformly in the height m. Consequently the forcing constant c appearing in the tail inequality (Theorem 11.1) is an absolute constant, independent of m; in particular the forcing side cannot grow like $\log m$ *(or any unbounded function of m) as $m \to \infty$.*

Proof. On $I_+ = \{\alpha + iy : \ |y - m| \leq \delta\}$ one has

$$Z_{\text{pair}}(\alpha + iy) = \big((\alpha - a) + i(y - m)\big)\big((\alpha + a) + i(y - m)\big).$$

Along $y \in [m - \delta, m + \delta]$ the argument of each linear factor varies by at most π (it is an arctan function whose range lies in an interval of length $\leq \pi$), so the argument of the product varies by at most 2π, uniformly in m. The forcing chain converts a fixed positive portion of $\Delta_{I_+} \arg Z_{\text{pair}}$ into the constant c with fixed conversion scalars, so c is necessarily $O(1)$. $\square$

9 Outer factorization and the inner quotient (Bridge 1)

We work on a fixed box $B = B(\alpha, m, \delta)$ and write B° for its interior. Assume boundary-contact: $E \neq 0$ on ∂B (this will be enforced later by κ–admissibility; see Definition 10.5 and Lemma 10.6).

Lemma 9.1 (Dirichlet outer factor on a box). *Let $B = B(\alpha, m, \delta)$ be the closed rectangle and B° its interior. Assume E is holomorphic on a neighborhood of $\overline{B}$ and $E \neq 0$ on ∂B. Then* $\log |E| \in C(\partial B)$*. Let $U \in C(\overline{B}) \cap \operatorname{Harm}(B^\circ)$ be the unique solution of the Dirichlet problem with boundary data $U|_{\partial B} = \log |E|$. Since B° is simply connected, there exists a harmonic conjugate V on B° (unique up to an additive constant) such that $U + iV$ is holomorphic on B°. Define*

$$G_{\text{out}}(v) := \exp(U(v) + iV(v)), \qquad v \in B^\circ.$$

Then G_{out} is holomorphic and zero-free on B°, satisfies $|G_{\text{out}}(v)| = e^{U(v)}$ for $v \in B^\circ$, and

$$\lim_{z \to \xi,\ z \in B^\circ} |G_{\text{out}}(z)| = |E(\xi)| \qquad (\xi \in \partial B).$$

Proof. Continuity of $\log |E|$ on ∂B follows from $E \neq 0$ on ∂B. Existence and uniqueness of U on a rectangle are standard. Since B° is simply connected, U admits a harmonic conjugate V on B°, unique up to an additive constant. The function $U + iV$ is holomorphic, hence so is $G_{\text{out}} = \exp(U + iV)$, and it is zero-free. Finally $|G_{\text{out}}| = e^U$ on B°, and by continuity of U on $\overline{B}$ we have $e^{U(\xi)} = |E(\xi)|$ on ∂B, yielding the boundary modulus identity in interior-limit form. $\square$

Define on B° the inner quotient

$$W(v) := \frac{E(v)}{G_{\rm out}(v)}.$$

Then W is holomorphic on B° and $|W| = 1$ on ∂B in the sense of interior limits in modulus.

Proposition 9.2 (Bridge 1: zero-free inner collapse)**.** *Assume the setup of Lemma 9.1 and define $W = E/G_{\rm out}$ on B°. If W is zero-free on B° (equivalently, E is zero-free on B°), then W is constant on B°; in fact $W \equiv e^{i\theta_B}$ for some $\theta_B \in \mathbb{R}$.*

Proof. Since W is zero-free on B° and $G_{\rm out}$ is zero-free, the function E is zero-free on B°. Because B° is simply connected, E admits a holomorphic logarithm on B°, so $\log|E|$ is harmonic on B°. By construction U is harmonic on B°, continuous on $\overline{B}$, and equals $\log|E|$ on ∂B. Thus $U - \log|E|$ is harmonic on B° with zero boundary values, so by Dirichlet uniqueness $U \equiv \log|E|$ on B°. Therefore for $v \in B^\circ$,

$$|W(v)| = \frac{|E(v)|}{|G_{\rm out}(v)|} = \frac{|E(v)|}{e^{U(v)}} = \frac{|E(v)|}{e^{\log|E(v)|}} = 1.$$

An analytic function of constant modulus on a connected open set is constant, hence $W \equiv e^{i\theta_B}$. ☐

Remark 9.3 (Boundary modulus convention)*.* Under boundary-contact, U extends continuously to ∂B and satisfies $U|_{\partial B} = \log|E|$. Hence $|G_{\rm out}| = |E|$ holds pointwise on ∂B as interior limits in modulus, and therefore $|W| = 1$ holds pointwise in modulus on ∂B. In boundary integral estimates this may be used in the a.e. sense without change.

Remark 9.4 (No converse: boundary modulus does not exclude interior zeros)*.* Lemma 9.1 implies that under boundary-contact the quotient $W := E/G_{\rm out}$ satisfies $|W| = 1$ on ∂B (in the interior boundary-limit sense of Remark 9.3). This condition alone does *not* imply that W is zero-free or constant on B°: nonconstant holomorphic functions on B° can have $|W| = 1$ on ∂B and still possess prescribed interior zeros (e.g. via conformal transport of finite Blaschke products from the unit disc). Thus Proposition 9.2 is strictly one-directional: the additional hypothesis "W is zero-free on B°" is essential.

10 Shape-only invariance and the envelope constants

Let $T(v) := (v - (\alpha + im))/\delta$, mapping ∂B affinely onto the fixed square boundary ∂Q with $Q = [-1, 1]^2$.

Lemma 10.1 (Shape-only invariance)**.** *Any constant arising solely from geometric or boundary-operator estimates on ∂B that are invariant under affine rescaling depends only on ∂Q and is independent of (α, m, δ).*

Proof. Under T, arclength scales by δ and tangential derivatives by $1/\delta$. After normalization, all purely geometric quantities and operator norms reduce to fixed quantities on ∂Q. ☐

Lemma 10.2 (Boundary-to-center evaluation in L^2 (sharp $\delta^{-1/2}$))**.** *Let $B = B(\alpha, m, \delta)$ be a box and let v_0 be its center. Let u be harmonic on B° and assume its boundary trace lies in $L^2(\partial B, ds)$. Then, writing $P_B(v_0, \xi) = d\omega^B_{v_0}/ds(\xi)$ for the Poisson kernel of B at v_0,*

$$|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} \, \|u\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q,

$$\|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2(\partial Q, ds)}.$$

In particular the exponent $\delta^{-1/2}$ is sharp (witnessed by the constant harmonic function $u \equiv 1$).

Proof. For harmonic u on B° with L^2 trace on ∂B, the Poisson representation gives

$$u(v_0) = \int_{\partial B} u(\xi)\, d\omega^B_{v_0}(\xi) = \int_{\partial B} u(\xi)\, P_B(v_0, \xi)\, ds(\xi).$$

Cauchy–Schwarz yields $|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2} \|u\|_{L^2}$.

For the scaling: under T, arclength scales by $ds = \delta\, ds_Q$ and Poisson kernels scale by $P_B(v_0, \xi) = \delta^{-1} P_Q(0, T(\xi))$. Hence

$$\int_{\partial B} P_B(v_0, \xi)^2\, ds(\xi) = \int_{\partial Q} \delta^{-2} P_Q(0, \zeta)^2\, \delta\, ds_Q(\zeta) = \delta^{-1} \int_{\partial Q} P_Q(0, \zeta)^2\, ds_Q(\zeta),$$

giving $\|P_B(v_0, \cdot)\|_{L^2} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2}$.

Sharpness: for $u \equiv 1$ we have $|u(v_0)| = 1$ and $\|u\|_{L^2(\partial B)} = \sqrt{|\partial B|} \asymp \delta^{1/2}$, so the inequality forces $\|P_B(v_0, \cdot)\|_{L^2} \gtrsim \delta^{-1/2}$. □

Lemma 10.3 (Upper envelope bound (outer-aligned form))**.** *Let $B = B(\pm a, m, \delta)$ be an aligned box and let G_{out} be the outer factor on B constructed from $\log |E|$ on ∂B (Section 9). Define the inner quotient*

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

Assume the boundary-contact convention: E has no zeros on ∂B (hence W has unimodular boundary values a.e.). For each sign $\pm$ let $v_\pm := \pm a + im$ and let $e^{i\varphi_0^\pm} \in \mathbb{T}$ be an $L^2(\partial B, ds)$-best constant phase,

$$e^{i\varphi_0^\pm} \in \arg\min_{|c|=1} \int_{\partial B} |W(v) - c|^2\, ds(v).$$

Then there exists a shape-only *constant $C_{\text{up}} > 0$ (depending only on the normalized square $Q = [-1, 1]^2$) such that*

$$\sum_{\pm} \left| W(v_\pm) - e^{i\varphi_0^\pm} \right| \;\le\; 2\, C_{\text{up}}\, \delta \sup_{v \in \partial B} \left| \frac{E'(v)}{E(v)} \right|. \tag{11}$$

One admissible explicit definition is

$$C_{\text{up}} := \Big(\sup_{\xi \in \partial Q} P_Q(0, \xi) \Big)^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2 \to L^2} \big),$$

where $P_Q(0, \xi) = d\omega_0^Q / ds(\xi)$ is the Poisson kernel of Q at the center 0 with respect to arclength on ∂Q, and $H_{\partial Q}$ is the boundary conjugation (Hilbert/Cauchy) operator on ∂Q.

Remark 10.4 (No residual proxying in the upper envelope). Lemma 10.3 controls the inner quotient $W = E/G_{\text{out}}$ and therefore depends on $\sup_{\partial B} |E'/E|$. Residual bounds for $F = E/Z_{\text{loc}}$ control $\sup_{\partial B} |F'/F|$ and do *not* by themselves bound $\sup_{\partial B} |E'/E|$. Whenever the residual envelope is used to control dial deviation, it must be routed through the log-derivative split $E'/E = F'/F + Z'_{\text{loc}}/Z_{\text{loc}}$ (Lemma 10.7) together with the collar bound (Lemma 10.8), yielding Corollary 10.11.

Proof. Fix one sign and write $v_0 = v_\pm$ and $B = B(\pm a, m, \delta)$. We record the (RH-free) chain and indicate the scale factors explicitly.

1. **Evaluation from the boundary (harmonic measure; produces $\delta^{-1/2}$).** For any constant $c \in \mathbb{T}$, subharmonicity of $|W - c|^2$ implies

$$|W(v_0) - c|^2 \le \int_{\partial B} |W(\xi) - c|^2 \, d\omega_{v_0}^B(\xi) = \int_{\partial B} |W(\xi) - c|^2 \, P_B(v_0, \xi) \, ds(\xi),$$

so

$$|W(v_0) - c| \le \|P_B(v_0, \cdot)\|_{L^\infty(\partial B)}^{1/2} \, \|W - c\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q, Poisson kernels scale by $\|P_B(v_0, \cdot)\|_\infty^{1/2} = \delta^{-1/2} \, \|P_Q(0, \cdot)\|_\infty^{1/2}$.

2. **Poincaré/Wirtinger on ∂B (produces δ).** For the L^2-best constant $c = e^{i\varphi_0^\pm}$ and $|\partial B| = 8\delta$, periodic Poincaré on a loop of length 8δ gives

$$\|W - c\|_{L^2(\partial B)} \le \frac{|\partial B|}{2\pi} \, \|\partial_s W\|_{L^2(\partial B)} = \frac{4\delta}{\pi} \, \|\partial_s W\|_{L^2(\partial B)}.$$

3. **Outer factor control (no δ; uses bounded boundary conjugation).** Write $\log G_{\text{out}} = U + i\widetilde{U}$ with $U|_{\partial B} = \log |E|$ and $\widetilde{U} = H_{\partial B} U$. Differentiating tangentially, $\partial_s \log G_{\text{out}} = \partial_s U + i \, H_{\partial B}(\partial_s U)$. Since $\log W = \log E - \log G_{\text{out}}$,

$$\|\partial_s \log W\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2 \to L^2}\big) \, \|\partial_s \log E\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2 \to L^2}\big) \left\| \frac{E'}{E} \right\|_{L^2(\partial B)}.$$

On ∂B we have $|W| = 1$ a.e., hence $|\partial_s W| = |\partial_s \log W|$.

4. **L^2 to sup (produces $\delta^{1/2}$).** Using $|\partial B| = 8\delta$,

$$\left\| \frac{E'}{E} \right\|_{L^2(\partial B)} \le \sqrt{|\partial B|} \, \sup_{\partial B} \left| \frac{E'}{E} \right| = \sqrt{8\delta} \, \sup_{\partial B} \left| \frac{E'}{E} \right|.$$

Combining the four steps yields

$$|W(v_0) - e^{i\varphi_0^\pm}| \le \|P_Q(0, \cdot)\|_\infty^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2 \to L^2}\big) \cdot \delta \sup_{\partial B} \left| \frac{E'}{E} \right|,$$

where we used the similarity invariance $\|H_{\partial B}\|_{L^2 \to L^2} = \|H_{\partial Q}\|_{L^2 \to L^2}$. Summing over $\pm$ gives (11). □

10.1 Local factor split and collar control

Definition 10.5 (Collar-admissible aligned boxes)**.** Fix once and for all a collar parameter $\kappa \in (0, 1/10)$. An aligned box $B = B(\alpha, m, \delta)$ is called *κ–admissible* if

$$\operatorname{dist}\big(\partial B, \, \mathcal{Z}(E)\big) \ge \kappa\delta.$$

Given any nominal scale $\delta_0 > 0$ and any center, there exists some $0 < \delta \le \delta_0$ for which κ–admissibility holds (Lemma 10.6). Whenever a chosen box is not κ–admissible, we shrink δ until κ–admissibility holds. Moreover the assembled tail inequality is monotone-safe under such δ–shrinking (Lemma 11.2).

Lemma 10.6 (Existence of a κ–admissible shrink)**.** *Fix $\kappa \in (0, 1/10)$ and a center $v_0 \in \mathbb{C}$. For every $\delta_0 > 0$ there exists $\delta' \in (0, \delta_0]$ such that the closed box*

$$B(v_0, \delta') := \{\, v \in \mathbb{C} : \ \|v - v_0\|_\infty \le \delta' \,\}$$

satisfies

$$\operatorname{dist}\big(\partial B(v_0, \delta'),\ \mathcal{Z}(E)\big) \ge \kappa\delta'.$$

In particular, given (α, m) and nominal $\delta_0 = \eta\alpha/(\log m)^2$, one may always choose a scale $0 < \delta \le \delta_0$ for which $B(\alpha, m, \delta)$ is κ–admissible.

Proof. Zeros of the entire function E are isolated. Choose $\varepsilon > 0$ such that $\mathcal{Z}(E) \cap \{\, 0 < \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) = 0$) or such that $\mathcal{Z}(E) \cap \{\, \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) \ne 0$). Set $\delta' := \min\{\delta_0, \varepsilon/(1+\kappa)\}$. Then every boundary point satisfies $\|v - v_0\|_\infty = \delta'$. Any zero $\rho \in \mathcal{Z}(E)$ is either $\rho = v_0$ (in which case $\operatorname{dist}(v, \rho) = \delta' \ge \kappa\delta'$) or satisfies $\|\rho - v_0\|_\infty \ge \varepsilon$ (in which case $\operatorname{dist}(v, \rho) \ge \varepsilon - \delta' \ge \kappa\delta'$). Therefore $\operatorname{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \ge \kappa\delta'$. $\square$

Lemma 10.7 (Log-derivative decomposition)**.** *With Z_{loc} and F as in* (7) *and* (8)*, one has on any region where E and Z_{loc} are holomorphic and nonvanishing (in particular on ∂B under the boundary-contact convention)*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}.$$

Lemma 10.8 (Buffered local factor bound on ∂B)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible in the sense of Definition 10.5. Then*

$$\sup_{v \in \partial B} \left| \frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)} \right| \le \frac{N_{\mathrm{loc}}(m)}{\kappa\, \delta},$$

where $N_{\mathrm{loc}}(m)$ counts zeros of E in the local window used to define Z_{loc}, with multiplicity.

Lemma 10.9 (Local log-derivative bound in $L^2(\partial B)$)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible (Definition 10.5), and let Z_{loc} be the local factor with local zero-count $N_{\mathrm{loc}}(m)$ as in Section 6. Then*

$$\left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^2(\partial B)} \le \frac{\sqrt{8}\, N_{\mathrm{loc}}(m)}{\kappa\, \delta^{1/2}}.$$

More generally, for any $1 \le r \le \infty$,

$$\left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^r(\partial B)} \le \frac{8^{1/r} N_{\mathrm{loc}}(m)}{\kappa\, \delta^{1-1/r}}.$$

Proof. Lemma 10.8 gives $\|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}\|_{L^\infty(\partial B)} \le N_{\mathrm{loc}}(m)/(\kappa\delta)$. Since $|\partial B| = 8\delta$, we have $\|f\|_{L^r(\partial B)} \le |\partial B|^{1/r} \|f\|_{L^\infty(\partial B)}$ for every $1 \le r \le \infty$, which yields the stated bounds. $\square$

Lemma 10.10 (Explicit local window zero count)**.** *Let $N(T)$ denote the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$, counted with multiplicity. Then for every $T \ge 5$,*

$$N(T+1) - N(T-1) \le 1.01 \log T + 17. \tag{12}$$

Consequently, for every $m \ge 10$,

$$N_{\mathrm{loc}}(m) \le 1.01 \log m + 17. \tag{13}$$

Proof. By [7, Theorem 1.1], for every $x \geq e$,

$$\Big| N(x) - \frac{x}{2\pi} \log\Big(\frac{x}{2\pi e}\Big)\Big| \leq 0.10076 \log x + 0.24460 \log\log x + 8.08344.$$

Let $M(x) := \frac{x}{2\pi} \log(\frac{x}{2\pi e})$, so $M'(x) = \frac{1}{2\pi} \log(\frac{x}{2\pi})$. For $T \geq 5$ we have $\log(T \pm 1) \leq \log(2T)$ and $\log\log x \leq \log x$ for $x \geq e$, hence

$$N(T+1) - N(T-1) \leq (M(T+1) - M(T-1)) + 2(0.10076 + 0.24460)\log(2T) + 2 \cdot 8.08344.$$

Moreover

$$M(T+1) - M(T-1) = \int_{T-1}^{T+1} M'(x)\, dx \leq \int_{T-1}^{T+1} \frac{1}{2\pi} \log x \, dx \leq \frac{1}{\pi} \log(2T).$$

Combining these bounds gives $N(T+1) - N(T-1) \leq 1.00903 \log T + 16.8663 \leq 1.01 \log T + 17$, establishing (12). Finally, in width–2 one has $m = 2T$. The local window $|\operatorname{Im} \rho - m| \leq 1$ corresponds to $|\gamma - T| \leq 1/2$ in the s–plane, so $N_{\mathrm{loc}}(m) = N(T + \frac{1}{2}) - N(T - \frac{1}{2}) \leq N(T+1) - N(T-1)$, yielding (13). □

Corollary 10.11 (Outer-aligned upper envelope in residual+local form)**.** *Let B be κ–admissible. Assume the residual envelope bound of Lemma 7.2, i.e.* $\sup_{\partial B} |F'/F| \leq L(m) := C_1 \log m + C_2$. *Then*

$$\sum_{\pm} |W(v_\pm) - e^{i\varphi_0^\pm}| \leq 2C_{\mathrm{up}} \left(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa}\right) \leq 2C_{\mathrm{up}} \left(\delta L(m) + \frac{1.01 \log m + 17}{\kappa}\right).$$

Remark 10.12 (UE gate = exponent budget at the local interface). Lemma 10.3 is the *only* step in the envelope chain that generates a positive power of δ in front of a boundary log-derivative endpoint. Abstractly, suppose an upper-envelope mechanism yields, for some $p > 0$,

$$\sum_{\pm} \big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \leq 2C_{\mathrm{up}}\, \delta^p \sup_{\partial B} \Big|\frac{E'}{E}\Big|,$$

and suppose the collar/local split yields, for some $\theta > 0$,

$$\sup_{\partial B} \Big|\frac{E'}{E}\Big| \leq L(m) \; + \; \frac{N_{\mathrm{loc}}(m)}{\kappa\, \delta^\theta}.$$

Then the local contribution in the envelope side scales as $\delta^{p-\theta} N_{\mathrm{loc}}(m)/\kappa$. Under the nominal choice $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ and the unconditional majorant $N_{\mathrm{loc}}(m) \ll \log m$, uniform η–shrinking tail closure is possible only if

$$p - \theta \;\geq\; \tfrac{1}{2}$$

(Theorem 10.13).

In the *proved* pointwise/sup architecture one has $p = 1$ (Lemma 10.3) and $\theta = 1$ (Lemma 10.8), so $p - \theta = 0$ and the local term is δ–inert; η–shrinking cannot suppress it (Lemma 10.14). Moreover, within this same endpoint class, a strengthened exponent $p > 1$ is impossible with shape–only constants (Lemma 10.15). Thus the former η–absorption closure route based on the pointwise/sup UE endpoint is a formal NO–GO and is recorded as discarded (Appendix A).

Theorem 10.13 (Exponent budget for η–shrinking under $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$)**.** *Let $m \ge 10$, $\alpha \in (0,1]$ and $\eta \in (0,1]$, and set the nominal scale*

$$\delta_0(m,\alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

Assume that for all $0 < \delta \le \delta_0(m,\alpha)$ one has:

(UE$_p$) (UE exponent) *for some $p > 0$,*

$$\mathrm{UE}(\delta) \;\le\; 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\Big|\frac{E'}{E}\Big|;$$

(COL$_\theta$) (Collar/local exponent) *for some $\theta > 0$,*

$$\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;\le\; L(m) \;+\; \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta^\theta},$$

with fixed $\kappa \in (0,1/10)$;

(GROW) (Majorants) *$L(m) \le A_L \log m + B_L$ and $N_{\mathrm{loc}}(m) \le A_N \log m + B_N$ for all $m \ge 10$;*

(FORCE) (Forcing side) *the forcing-vs-envelope tail inequality has a fixed positive forcing constant $c > 0$ and only δ–helpful subtractive terms on the RHS.*

Then at $\delta = \delta_0(m,\alpha)$ one has the explicit bound

$$\mathrm{UE}(\delta_0) \;\le\; 2C_{\mathrm{up}}\Big(\delta_0^p L(m) \;+\; \delta_0^{p-\theta}\frac{N_{\mathrm{loc}}(m)}{\kappa}\Big). \tag{BUDGET}$$

Moreover, uniform tail closure by η–shrinking *(i.e. there exists $\eta_\star > 0$ such that for every $\eta \le \eta_\star$ the tail inequality holds for all $m \ge 10$) is possible only if*

$$p - \theta \;\ge\; \tfrac{1}{2}. \tag{B1}$$

Proof. Insert (COL$_\theta$) into (UE$_p$) at $\delta = \delta_0$ to obtain (BUDGET). At $\alpha = 1$ one has $\delta_0(m,1) = \eta/(\log m)^2$, so the local term behaves as

$$\delta_0^{p-\theta} N_{\mathrm{loc}}(m) \;\ll\; \Big(\frac{\eta}{(\log m)^2}\Big)^{p-\theta}\log m = \eta^{p-\theta}(\log m)^{1-2(p-\theta)}.$$

If $p-\theta < 1/2$ then $1-2(p-\theta) > 0$, so the local contribution grows without bound as $m \to \infty$, while the forcing side tends to the fixed constant c because all RHS corrections are δ–helpful and vanish as $\delta_0 \to 0$. Hence uniform tail closure is impossible. If $p - \theta \ge 1/2$ then the local contribution is uniformly bounded by $O(\eta^{p-\theta})$ and tends to 0 as $\eta \downarrow 0$, enabling uniform absorption once all constants are δ–uniform. ☐

Lemma 10.14 (η–absorption obstruction under the pointwise UE exponent $p = 1$)**.** *Assume the hypotheses of Corollary 10.11. Then for every $\delta \le \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$,*

$$\sum_\pm \big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \le 2C_{\mathrm{up}}\Big(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa}\Big).$$

In particular, letting $\eta \downarrow 0$ (hence $\delta \downarrow 0$) only suppresses the residual term $\delta L(m)$; the local term $N_{\mathrm{loc}}(m)/\kappa$ does not *decay with η. Therefore any absorption-style closure that attempts to force the envelope side small by choosing η must additionally verify a* separate *inequality of the form*

$$\frac{2C_{\mathrm{up}}}{\kappa} N_{\mathrm{loc}}(m) \ < \ c$$

at the relevant anchor height(s), where c is the forcing constant in (14).

Lemma 10.15 (UE scaling NO–GO for pointwise/sup endpoints)**.** *Assume an upper-envelope bound of the form*

$$\sum_{\pm} \left| W(v_\pm) - e^{i\varphi_0^\pm} \right| \ \le \ 2C_{\mathrm{up}}\, \delta^p \sup_{\partial B} \left| \frac{E'}{E} \right| \qquad (p > 0),$$

where the constant C_{up} depends only on the normalized shape (Lemma 10.1) and is independent of δ. Then necessarily $p \le 1$. In particular, no pointwise/sup envelope mechanism with shape–only constants can yield any exponent $p > 1$.

Proof. Under the affine rescaling $T(v) = (v - (\alpha + im))/\delta$, the boundary ∂B maps to the fixed square boundary ∂Q. If $\widetilde{E}(z) := E(T^{-1}(z))$, then by the chain rule

$$\frac{E'}{E}(T^{-1}(z)) = \frac{1}{\delta} \frac{\widetilde{E}'(z)}{\widetilde{E}(z)}.$$

Hence $\sup_{\partial B} |E'/E| = \delta^{-1} \sup_{\partial Q} |\widetilde{E}'/\widetilde{E}|$. The left-hand side of the upper-envelope bound is dimensionless (it is a sum of moduli of complex numbers), and under the normalization it may be $O(1)$ for admissible configurations on the fixed shape. Therefore the bound forces

$$O(1) \ \le \ 2C_{\mathrm{up}}\, \delta^{p-1} \sup_{\partial Q} \left| \frac{\widetilde{E}'}{\widetilde{E}} \right| \qquad \text{as } \delta \downarrow 0.$$

Since the normalized endpoint $\sup_{\partial Q} |\widetilde{E}'/\widetilde{E}|$ is not forced to blow up as $\delta \downarrow 0$ (it depends only on the normalized data), the factor δ^{p-1} cannot tend to 0. Thus $p - 1 \le 0$, i.e. $p \le 1$. □

Proof. The displayed bound is exactly Corollary 10.11 with the corrected UE exponent $p = 1$. As $\eta \to 0$ one has $\delta_0 \to 0$ and hence $\delta L(m) \to 0$, while $N_{\mathrm{loc}}(m)/\kappa$ is unchanged. Since the forcing lower bound in the tail inequality tends to c as $\delta \downarrow 0$, the strict inequality requires the stated necessary condition at the anchor. □

10.2 Horizontal non-forcing budget in residual form

Definition 10.16 (Horizontal non-forcing phase budget)**.** Let $B = B(\pm a, m, \delta)$ be an aligned box and let $F = E/Z_{\mathrm{loc}}$ be the residual factor. Assume F is holomorphic and zero-free on a neighborhood of ∂B. Let $H_\pm$ denote the top and bottom edges of ∂B:

$$H_+ := \{x + i(m + \delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}, \qquad H_- := \{x + i(m - \delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}.$$

Define

$$\Delta_{\mathrm{nonforce}}(B) := \int_{H_+} |\partial_s \arg F| \, ds + \int_{H_-} |\partial_s \arg F| \, ds.$$

Lemma 10.17 (Horizontal budget (residual form; audit-grade)). *In the setting of Definition 10.16,*

$$\Delta_{\text{nonforce}}(B) \le 4\delta \sup_{v\in\partial B} \left|\frac{F'(v)}{F(v)}\right|.$$

Consequently, if $\sup_{\partial B}|F'/F| \le C_1 \log m + C_2$*, then*

$$\Delta_{\text{nonforce}}(B) \le C_h''\,\delta\,(\log m + 1), \qquad C_h'' := 4\max\{C_1,\, C_2\}.$$

Proof. On either horizontal edge, $|\partial_s \arg F| \le |F'/F|$ pointwise. Each edge has length 2δ, hence each integral is bounded by $2\delta \sup_{\partial B}|F'/F|$. Summing top and bottom gives the first inequality, and the second follows from $\sup_{\partial B}|F'/F| \le C_1 \log m + C_2 \le \max\{C_1, C_2\}(\log m + 1)$. □

11 The explicit tail inequality (post-pivot)

For $m \ge 10$ we use the growth surrogate

$$L(m) := C_1 \log m + C_2,$$

with constants as in Lemma 7.2. For the local window term we use the explicit majorant from Lemma 10.10:

$$N_{\text{up}}(m) := 1.01 \log m + 17 \;\text{ so that }\; N_{\text{loc}}(m) \le N_{\text{up}}(m) \quad (m \ge 10).$$

For a parameter $\eta \in (0,1)$ and a dial displacement $\alpha \in (0,1]$ define the *nominal* scale

$$\delta_0 := \delta_0(m,\alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

Fix a collar parameter $\kappa \in (0, 1/10)$ as in Definition 10.5. For each (m,α) we choose any scale $0 < \delta \le \delta_0$ such that the aligned boxes $B = B(\pm\alpha, m, \delta)$ are κ–admissible; existence is guaranteed by Lemma 10.6. By Lemma 11.2, shrinking δ only helps in the tail inequality, so it is safe to treat δ_0 as the worst-case scale in one-height reductions.

Theorem 11.1 (Tail inequality (criterion form; pointwise UE exponent $p = 1$)). *Fix $m \ge 10$ and $\eta \in (0,1)$. Assume:*

1. *the forcing lemma producing the positive constant*

$$c_0 := \frac{3\log 2}{8\pi}, \qquad c := \frac{3\log 2}{16}, \qquad K_{\text{alloc}} := 3 + 8\sqrt{3};$$

2. *the residual envelope bound (Lemma 7.2) providing C_1, C_2;*
3. *the audit-grade horizontal budget bound (Lemma 10.17), giving a constant C_h'' independent of (α, m, δ);*
4. *the explicit local window bound (Lemma 10.10) providing the majorant $N_{\text{up}}(m) = 1.01\log m + 17$.*

Then for every $\alpha \in (0,1]$ and every κ–admissible aligned box $B = B(\pm\alpha, m, \delta)$, absence of off–axis quartets at height m follows from the strict inequality

$$2C_{\text{up}}\Big(\delta L(m) \;+\; \frac{N_{\text{up}}(m)}{\kappa}\Big) \;<\; c \;-\; \delta\Big(K_{\text{alloc}}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big). \tag{14}$$

Proof sketch / bookkeeping. The forcing side is unchanged from v31. The only post-pivot modification is on the upper-envelope side: Lemma 10.3 bounds dial deviation in terms of $\sup_{\partial B} |E'/E|$. Applying the log-derivative split (Lemma 10.7), the residual envelope for $\sup_{\partial B} |F'/F| \le L(m)$ (Lemma 7.2), and the collar bound $\sup_{\partial B} |Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}| \le N_{\mathrm{loc}}(m)/(\kappa\delta)$ (Lemma 10.8) yields

$$\sup_{\partial B} \left|\frac{E'}{E}\right| \le L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa\delta} \le L(m) + \frac{N_{\mathrm{up}}(m)}{\kappa\delta}.$$

Plugging this into Lemma 10.3 gives the left-hand side of (14). The right-hand side is the forcing lower bound, with the horizontal non-forcing term bounded by Lemma 10.17. □

Lemma 11.2 (Monotonicity under δ–shrinking)**.** *Fix $m \ge 10$, $\alpha \in (0,1]$, and constants $C_{\mathrm{up}}, \kappa, c, c_0, K_{\mathrm{alloc}}, C''_h, C_1, C$ Let $L(m) = C_1 \log m + C_2$ and $N_{\mathrm{up}}(m) = 1.01 \log m + 17$. For $\delta \in (0,1]$ define*

$$\mathrm{LHS}(\delta) := 2C_{\mathrm{up}}\Big(\delta L(m) \;+\; \frac{N_{\mathrm{up}}(m)}{\kappa}\Big), \qquad \mathrm{RHS}(\delta) := c \;-\; \delta\Big(K_{\mathrm{alloc}}\, c_0\, L(m) + C''_h\,(\log m + 1)\Big).$$

Then $\mathrm{LHS}(\delta)$ *is (weakly) increasing in δ and* $\mathrm{RHS}(\delta)$ *is (weakly) decreasing. Consequently, if* $\mathrm{LHS}(\delta_0) < \mathrm{RHS}(\delta_0)$ *for some $\delta_0 \in (0,1]$, then* $\mathrm{LHS}(\delta) < \mathrm{RHS}(\delta)$ *holds for every $\delta \in (0,\delta_0]$.*

Proof. For $\delta > 0$, the map $\delta \mapsto \delta L(m)$ is increasing and the term $N_{\mathrm{up}}(m)/\kappa$ is independent of δ, hence $\mathrm{LHS}(\delta)$ is (weakly) increasing. The bracketed factor in $\mathrm{RHS}(\delta)$ is nonnegative and independent of δ, so $\mathrm{RHS}(\delta)$ decreases linearly in δ. □

Lemma 11.3 (Worst case in α is $\alpha = 1$ at the nominal scale)**.** *Fix $m \ge 10$ and $\eta \in (0,1)$. Define the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. Consider the tail inequality* (14) *evaluated at $\delta = \delta_0(m,\alpha)$. Then the left-hand side is (weakly)* increasing *in $\alpha \in (0,1]$, while the right-hand side is (weakly)* decreasing. *Therefore it suffices to verify* (14) *at $\alpha = 1$ and $\delta = \delta_0(m,1)$. If one later shrinks $\delta \le \delta_0(m,\alpha)$ to enforce κ–admissibility, the inequality only becomes easier (Lemma 11.2).*

Proof. With $\delta = \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, the only α-dependence in the left-hand side is through the factor $\delta L(m)$, which is increasing in α, so the left-hand side increases. The right-hand side equals $c - \delta \cdot \Xi(m)$ for a nonnegative factor $\Xi(m)$ independent of α, hence it decreases. □

Remark 11.4 (No one-height reduction in m under the pointwise UE exponent $p = 1$). In v33, the (claimed) $\delta^{3/2}$ prefactor in Lemma 10.3 made the local contribution scale like $\delta^{1/2} N_{\mathrm{up}}(m)$ at the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, leading to an expression essentially independent of m and enabling a one-height reduction. After the UE–Gate audit, Lemma 10.3 provides only the pointwise exponent $p = 1$, so the tail left-hand side contains the δ–inert term $(2C_{\mathrm{up}}/\kappa)\, N_{\mathrm{up}}(m)$. With the explicit majorant $N_{\mathrm{up}}(m) = 1.01 \log m + 17$, this term is *increasing* in m. Therefore a one-height reduction in m is not available under the current pointwise envelope mechanism: the tail criterion must be controlled as a family in m, or the UE–Gate must be cleared by a strengthened envelope mechanism (Remark 10.12).

12 GEO–C4 frontier: hinge-centered harmonic endpoint (active)

12.1 Defect quotient primitives and cancellation mechanism (v40)

Definition 12.1 (Defect quotient and defect log-derivative)**.** Fix $a \in (0,1)$ and let E be the even entire width–2 completion (Section 1). Define the *defect quotient* and its logarithmic derivative by

$$\mathcal{Q}_a(v) \;:=\; \frac{E(v+a)}{E(v-a)}, \qquad \mathcal{L}_a(v) \;:=\; \frac{\mathcal{Q}'_a(v)}{\mathcal{Q}_a(v)} \;=\; \frac{E'}{E}(v+a) - \frac{E'}{E}(v-a),$$

whenever the expressions are defined. The function $\mathcal{L}_a$ is holomorphic on any domain where $E(v \pm a) \neq 0$.

Definition 12.2 (Defect phase endpoint on a κ–admissible box)**.** Let $B = B(\alpha, m, \delta)$ be a κ–admissible aligned box and let $\partial B_{\kappa/2}$ denote its buffered contour as in Definition 12.43. Define the *defect phase endpoint* by

$$\Phi_B^{\mathrm{def}}(a) \;:=\; \max_{I \in \mathrm{Sides}(\partial B_{\kappa/2})} \left| \Im \int_I \mathcal{L}_a(v)\, dv \right|.$$

(Here $\mathrm{Sides}(\partial B_{\kappa/2})$ denotes the four oriented sides of the rectangle.)

Lemma 12.3 (Side-length ceiling for the defect phase endpoint)**.** *Fix $a \in (0,1]$ and let $B = B(\alpha, m, \delta)$ be a κ-admissible aligned box with buffered contour $\partial B_{\kappa/2}$. Assume $\mathcal{L}_a$ is holomorphic on an open neighborhood of $\partial B_{\kappa/2}$. Then*

$$\Phi_B^{\mathrm{def}}(a) \;\leq\; C_{\mathrm{geom}}\, \delta \cdot \sup_{v \in \partial B_{\kappa/2}} |\mathcal{L}_a(v)|,$$

where $C_{\mathrm{geom}} > 0$ is an absolute rectangle-shape constant. In particular, any estimate of the form $\Phi_B^{\mathrm{def}}(a) \ll \delta^p(\cdots)$ obtained solely from δ-uniform pointwise bounds on $\sup_{\partial B_{\kappa/2}} |\mathcal{L}_a|$ cannot have $p > 1$.

Proof. By Definition 12.2 and the triangle inequality,

$$\left| \Im \int_I \mathcal{L}_a(v)\, dv \right| \leq \int_I |\mathcal{L}_a(v)|\, |dv| \leq |I| \cdot \sup_{v \in \partial B_{\kappa/2}} |\mathcal{L}_a(v)|.$$

Each side length satisfies $|I| \leq C_{\mathrm{geom}}\, \delta$ for an absolute constant depending only on the rectangle shape (and the fixed buffering policy), hence the claim. ☐

Definition 12.4 (Two-sided shift-difference defect operator)**.** Let E be the even entire completion in the v-plane, and define the defect quotient and defect log-derivative

$$\mathcal{Q}_a(v) := \frac{E(v+a)}{E(v-a)}, \qquad \mathcal{L}_a(v) := \frac{\mathcal{Q}_a'(v)}{\mathcal{Q}_a(v)} = \frac{E'}{E}(v+a) - \frac{E'}{E}(v-a).$$

For a shift step $h > 0$, define the two-sided shift-difference operator

$$\mathcal{D}_{a,h}(v) := \mathcal{L}_{a+h}(v) - \mathcal{L}_{a-h}(v).$$

Equivalently,

$$\mathcal{D}_{a,h}(v) = \Big(\frac{E'}{E}(v+a+h) - \frac{E'}{E}(v+a-h) \Big) - \Big(\frac{E'}{E}(v-a+h) - \frac{E'}{E}(v-a-h) \Big).$$

Definition 12.5 (Two-sided shift-difference phase endpoint)**.** Fix $\kappa \in (0, 1/10)$. For a κ-admissible aligned box $B = B(\alpha, m, \delta)$ and parameters (a, h), define

$$\Phi_B^{(2\mathrm{s})}(a; h) := \max_{I \in \mathrm{Sides}(\partial B_{\kappa/2})} \left| \Im \int_I \mathcal{D}_{a,h}(v)\, dv \right|.$$

In the nominal coupling used in this program we take $h = \delta$ and $\delta = \delta_0(m, a) := \eta a / (\log m)^2$.

Lemma 12.6 (Aligned-box forcing NO-GO for the two-sided endpoint)**.** *Assume E has a quartet at $\{\pm a \pm im\}$ with $a > 0$ and fix $\kappa \in (0, 1/10)$. Let $B = B(a, m, \delta)$ be the aligned box at the pole height and suppose B is κ–admissible. If $0 < h \le \delta \le a/4$, then in the toy quartet model one has*

$$\Phi_B^{(2s)}(a; h) \ \le \ C \, \frac{\delta h}{a^2},$$

for an absolute constant C. In particular at the nominal coupling $h = \delta = \eta a/(\log m)^2$ one gets $\Phi_B^{(2s)}(a; h) \ll \eta^2/(\log m)^4 \to 0$ as $m \to \infty$. Therefore any viable Δa forcing mechanism must change the witness family and/or the coupling of h to δ.

Lemma 12.7 (Local defect cancellation for a forced top pair)**.** *Assume E has a nontrivial quartet at height m with displacement $a > 0$, so that $E(\pm a \pm im) = 0$ (with multiplicity $r \ge 1$ at each of $\pm a + im$). Let $B = B(0, m, \delta)$ with $0 < \delta \le a/4$, and assume B is κ–admissible. Then $\mathcal{Q}_a$ has a removable singularity at $v = im$, $\mathcal{L}_a$ is holomorphic on $B_{\kappa/2}$, and the pure–pair contribution to $\Phi_B^{\mathrm{def}}(a)$ is $O(r\,\delta/a)$. More precisely, writing*

$$\frac{E'}{E}(v) = \frac{r}{v - (a + im)} + \frac{r}{v - (-a + im)} + H(v),$$

with H holomorphic on $B_{\kappa/2}$, one has on $\partial B_{\kappa/2}$:

$$\mathcal{L}_a(v) = r\Big(\frac{1}{v - (2a + im)} - \frac{1}{v - (-2a + im)}\Big) \ + \ \big(H(v + a) - H(v - a)\big).$$

Consequently,

$$\max_{v \in \partial B_{\kappa/2}} \left| r\Big(\frac{1}{v - (2a + im)} - \frac{1}{v - (-2a + im)}\Big)\right| \ \le \ \frac{C\,r}{a}, \qquad \Rightarrow \qquad \Phi_B^{\mathrm{def}}(a) \ \le \ C\,r\,\frac{\delta}{a} + \delta\,\|H(\cdot{+}a) - H(\cdot{-}a)\|_{L^\infty(\partial B$$

Proof. Factor $E(v) = (v-(a+im))^r (v-(-a+im))^r \cdot \widetilde{E}(v)$ locally near $v = im$ with $\widetilde{E}$ holomorphic and nonzero on $B_{\kappa/2}$ (possible since B is κ–admissible and $\delta \le a/4$ keeps the other zeros away). Then $\mathcal{Q}_a(v) = E(v + a)/E(v - a)$ has the same power of $(v - im)^r$ in numerator and denominator, so $v = im$ is removable and $\mathcal{L}_a$ is holomorphic on $B_{\kappa/2}$.

For the displayed decomposition, write $\frac{E'}{E}(v) = \sum_{\rho \in Z(E)} \frac{m(\rho)}{v-\rho}$ and isolate the two terms at $\rho = \pm a + im$, absorbing all other contributions into a holomorphic H on $B_{\kappa/2}$. Shift by $\pm a$ and subtract to obtain the identity for $\mathcal{L}_a(v)$. Finally,

$$\frac{1}{v - (2a + im)} - \frac{1}{v - (-2a + im)} = \frac{4a}{(v - (2a + im))(v - (-2a + im))},$$

and on $\partial B_{\kappa/2}$ we have $|v - (\pm 2a + im)| \ge a$ when $\delta \le a/4$, giving the $O(r/a)$ bound. The endpoint bound follows by integrating along a side of length $O(\delta)$ and taking the maximum over the four sides. ☐

Lemma 12.8 (Shift stability of the defect endpoint)**.** *Let C be any rectifiable contour such that $E(v \pm a) \neq 0$ for all $v \in C$. Then*

$$\sup_{v \in C} |\mathcal{L}_a(v)| \le \sup_{w \in C + a} \left|\frac{E'(w)}{E(w)}\right| + \sup_{w \in C - a} \left|\frac{E'(w)}{E(w)}\right|.$$

In particular, any collar bound for E'/E on buffered contours transfers verbatim to $\mathcal{L}_a$ on the corresponding shifted buffered contours.

Proof. This is immediate from Definition 12.1 and the triangle inequality. □

Definition 12.9 (Horizontal resonance sum (local window))**.** For $a \in (0, 1]$ and $m \geq 10$, define the *horizontal resonance sum*

$$\mathcal{R}_a(m) \;:=\; \sum_{\substack{\rho:\ E(\rho)=0 \\ |\Im(\rho)-m|\leq 1}} \frac{1}{|\Re(\rho) - a| + a},$$

where each zero is counted with multiplicity. This sum is finite and satisfies $\mathcal{R}_a(m) \leq \frac{1}{a}\, N_{\mathrm{loc}}(m)$.

Remark 12.10 (Archived benchmark: defect UE target ($\mathrm{S5}^{\mathrm{def}}$))*.* The bound below was the *v40–v41* target for the *centered defect endpoint* $\Phi_B^{\mathrm{def}}(a)$ built from $\mathcal{L}_a$. It is retained only as a comparison benchmark and as a warning label: the centered defect endpoint suffers an $O(\delta/a)$ cancellation and a δ-inert resonance obstruction, so it is *not* load-bearing in the active GEO–C4 chain (v44).

Formally, the discarded target inequality was of the schematic form

$$\Phi^{\mathrm{def}}_{B(0,m,\delta)}(a) \;\leq\; C\,\eta \quad \text{at} \quad \delta = \eta\,\frac{a}{(\log m)^2},$$

which would contradict any absolute forcing lower bound for Φ^{def}. In v44 we replace Φ^{def} by the hinge-circle harmonic endpoint $\Phi^{\star}$ (Box 12.1).

Definition 12.11 (δ-aware resonance sum)**.** For $a \in (0, 1]$, $m \geq 10$, and $0 < \delta \leq 1$, define the *δ-aware resonance sum*

$$\mathcal{R}_{a,\delta}(m) \;:=\; \sum_{\substack{\rho:\ E(\rho)=0 \\ |\Im(\rho)-m|\leq 1}} \frac{1}{|\Re(\rho) - a| + \delta},$$

where each zero is counted with multiplicity. This sum is finite and satisfies $\mathcal{R}_{a,\delta}(m) \leq \delta^{-1} N_{\mathrm{loc}}(m)$.

Remark 12.12 (δ-aware defect UE target (archived benchmark))*.* A δ-uniform defect UE bound that explicitly accounts for resonance should be formulated in terms of $\mathcal{R}_{a,\delta}(m)$ rather than the δ-blind sum $\mathcal{R}_a(m)$. A schematic form is

$$\Phi_B^{\mathrm{def}}(a) \;\leq\; C_0\,\delta\,\log m \;+\; C_1\,\delta\,a\,\mathcal{R}_{a,\delta}(m), \qquad 0 < \delta \leq \delta_0(m, a) = \eta\,\frac{a}{(\log m)^2}.$$

This inequality is *not* claimed here; it is recorded as the correct resonance-aware target should one revisit the defect endpoint class in future work.

Definition 12.13 (Near-resonance count)**.** For $a \in (0, 1]$, $m \geq 10$, and $0 < \delta \leq 1$, define the near-resonance count

$$N_{\mathrm{near}}(m; a, \delta) \;:=\; \#\{\rho:\ E(\rho) = 0,\ |\Im(\rho) - m| \leq 1,\ |\Re(\rho) - a| \leq \delta\},$$

counted with multiplicity.

Remark 12.14 ($\mathcal{R}_a(m)$ is δ-blind)*.* The δ-blind sum $\mathcal{R}_a(m)$ (Definition 12.9) cannot rule out near-resonance: one may have $N_{\mathrm{near}}(m; a, \delta) \geq 1$ with $\delta \ll a$ while still having $\mathcal{R}_a(m) \asymp a^{-1}$. By contrast, $\mathcal{R}_{a,\delta}(m) \gtrsim \delta^{-1}$ whenever $N_{\mathrm{near}}(m; a, \delta) \geq 1$. Thus any δ-uniform local analysis must track δ-aware resonance (either via $\mathcal{R}_{a,\delta}$ or via N_{near}).

Lemma 12.15 (NO–GO: near-resonant quartets can make defect endpoints δ–inert)**.** *Fix $a \in (0,1)$, $m > 0$, and $0 < \varepsilon < a/2$. Define the even polynomial*

$$E_\varepsilon(v) := \prod_{\sigma\in\{\pm1\}} \prod_{\tau\in\{\pm1\}} \big(v - \sigma(a + i\tau m)\big) \cdot \big(v - \sigma((a-\varepsilon) + i\tau m)\big),$$

and form $\mathcal{Q}_a, \mathcal{L}_a$ as in Definition 12.1. Let $B = B(0, m, \delta)$ with $\delta \in [\varepsilon/2, 2\varepsilon]$ and take $\kappa = 1/10$. Then B is κ–admissible (for $\delta \ll a$), but the defect endpoint satisfies

$$\Phi_B^{\mathrm{def}}(a) \;\geq\; c_0$$

for an absolute constant $c_0 > 0$ independent of δ and ε. In particular, $\Phi_B^{\mathrm{def}}(a)$ need not shrink as $\delta \to 0$ unless horizontal resonance is controlled.

Proof sketch. Write $w = v - im$. A direct computation shows that $\mathcal{Q}_a(v)$ contains the factor

$$g_\varepsilon(w) := \frac{w+\varepsilon}{w-\varepsilon},$$

coming from the second quartet at real part $a - \varepsilon$: indeed $E_\varepsilon(v + a)$ vanishes at $v = im - \varepsilon$ and $E_\varepsilon(v - a)$ vanishes at $v = im + \varepsilon$. Hence $\mathcal{Q}_a$ has a zero at $w = -\varepsilon$ and a pole at $w = \varepsilon$.

Consider the right vertical side of $\partial B_{\kappa/2}$, parameterized as $w = \delta + it$ with $t \in [-\delta, \delta]$. On this side,

$$\arg g_\varepsilon(\delta + it) = \arg(\delta + \varepsilon + it) - \arg(\delta - \varepsilon + it),$$

so the phase increment along the side is

$$\Delta_t \arg g_\varepsilon = \Big[\arctan\Big(\frac{t}{\delta+\varepsilon}\Big) - \arctan\Big(\frac{t}{\delta-\varepsilon}\Big)\Big]_{t=-\delta}^{t=\delta} = 2\arctan\Big(\frac{\delta}{\delta+\varepsilon}\Big) - 2\arctan\Big(\frac{\delta}{\delta-\varepsilon}\Big).$$

If $\delta \in [\varepsilon/2, 2\varepsilon]$, then $\delta - \varepsilon$ has magnitude $\ll \delta$ and the second arctangent is bounded away from 0, forcing $|\Delta_t \arg g_\varepsilon| \geq c_0$ for an absolute $c_0 > 0$. Since $\Delta \arg g_\varepsilon$ is an oriented side phase increment, this lower bound implies $\Phi_B^{\mathrm{def}}(a) \geq c_0$ (up to $O(\delta/a)$ terms from the far factors). ☐

Lemma 12.16 (NO–GO: no δ–uniform transfer to the centered defect box)**.** *Fix $\kappa \in (0, 1/10)$ and $a \in (0,1)$. Let $B = B(\pm a, m, \delta)$ be an aligned box at height m and let $\widehat{B} = B(0, m, \delta)$ be the centered box of the same height and scale. Assume both boxes are κ–admissible and that E has a quartet at height m with tilt a, so that $E(\pm a \pm im) = 0$.*

Then there is no *constant C_{tr} independent of δ such that the transfer inequality*

$$\widetilde{D}_B(W) \;\leq\; C_{\mathrm{tr}}\, \Phi_{\widehat{B}}^{\mathrm{def}}(a)$$

holds uniformly as $\delta \downarrow 0$ for this box family.

Proof. By Lemma 12.47, the interior zero of W in $B_{\kappa/2}$ forces $\widetilde{D}_B(W) \geq \pi/2$ on the aligned box, uniformly in δ. On the other hand, Lemma 12.7 shows that on the centered box $\widehat{B} = B(0, m, \delta)$ the defect endpoint may satisfy $\Phi_{\widehat{B}}^{\mathrm{def}}(a) = O(\delta/a)$ as $\delta \to 0$ (even for a single quartet). Thus any inequality of the displayed form would force $\pi/2 \leq C_{\mathrm{tr}}\, O(\delta/a)$, which fails for $\delta \downarrow 0$. ☐

Remark 12.17 (Consequence for S5 endpoints)*.* Lemma 12.16 is the decisive obstruction to any v39-style strategy that tries to transfer aligned-box forcing to a centered defect box at the *same* scale δ. Any future closure attempt must either (i) use an endpoint that is forceable on the aligned box itself, or (ii) introduce genuinely new structure that avoids the centered cancellation mechanism.

Lemma 12.18 (NO–GO: defect-box pole-winding cannot substitute for transfer). *Fix $\kappa \in (0, 1/10)$ and $a \in (0, 1)$. Let $\widehat{B} = B(2a, m, \delta)$ be the defect box that contains the pole $v = 2a + im$ in the toy configuration of Lemma 12.7, and assume $\widehat{B}$ is κ–admissible.*

If E has a quartet at height m with tilt a, then for all sufficiently small $\delta \ll a$,

$$\Phi^{\mathrm{def}}_{\widehat{B}}(a) \;\geq\; \frac{\pi}{2},$$

independently of δ. Consequently, no inequality of the form $\Phi^{\mathrm{def}}_{\widehat{B}}(a) \leq C\,\delta^p (\log m)^q$ with $p > 0$ and C independent of δ can hold uniformly as $\delta \to 0$ on this defect-box family.

Proof. Since $E(a + im) = 0$, the denominator $E(v - a)$ vanishes at $v_0 = 2a + im$, so $\mathcal{Q}_a$ has a pole at v_0. Under κ–admissibility, $\mathcal{Q}_a$ is meromorphic in a neighborhood of $\partial\widehat{B}_{\kappa/2}$ with exactly one pole and no zeros/poles on the contour. The argument principle gives total phase change 2π in magnitude around $\partial\widehat{B}_{\kappa/2}$. Since $\partial\widehat{B}_{\kappa/2}$ is the concatenation of four oriented sides, at least one side has phase increment magnitude at least $\pi/2$. By definition of $\Phi^{\mathrm{def}}_{\widehat{B}}(a)$ as the max side increment, $\Phi^{\mathrm{def}}_{\widehat{B}}(a) \geq \pi/2$. The incompatibility with any $p > 0$ δ–gain is immediate. $\square$

Active closure lever (v44): hinge–circle $k = 2$ witness.

Geometry / witness. Fix a height $m > 0$ and parameters $a > 0$, $\delta > 0$, $h > 0$ (with $h = \kappa\delta$). Work in the width–2 centered v–frame, with the even entire completion $E(v)$. Let

$$v(\theta) = im + \delta e^{i\theta}, \qquad \theta \in [0, 2\pi],$$

and define the shift–difference field

$$\mathcal{L}_t(v) := \frac{E'}{E}(v+t) - \frac{E'}{E}(v-t), \qquad \mathcal{D}_{a,h}(v) := \mathcal{L}_{a+h}(v) - \mathcal{L}_{a-h}(v).$$

Set the real phase signal on the hinge circle

$$\psi(\theta) := \Im\big(\mathcal{D}_{a,h}(v(\theta))\big), \qquad \widehat{\psi}(2) := \int_0^{2\pi} \psi(\theta) e^{-2i\theta}\, d\theta,$$

and the GEO–C4 endpoint

$$\Phi^\star(m, a, \delta, h) := \frac{\delta^2}{h} \|P_2\psi\|_{L^2_\theta} = \frac{\delta^2}{h\sqrt{\pi}} |\widehat{\psi}(2)|.$$

FORCE (truth–latching). If there is an off–axis quartet at height m (zeros at $\pm a \pm im$), then in the toy model one has $\Phi^\star \geq 4\sqrt{\pi}$, and under mild isolation/admissibility the forcing survives as an absolute constant $\Phi^\star \geq c_0 > 0$ (Lemmas 12.21–12.22).

UE (single active statement). The only remaining analytic obligation is to show that for the true completed object E, the signed $k = 2$ coefficient $\widehat{\psi}(2)$ is *small* at the nominal policy $\delta = \eta\, a/(\log(m+3))^2$, $h = \kappa\delta$:

$$|\widehat{\psi}(2)| \ \ll\ h\, a^{-2} (\log(m+3))^{C'} \qquad \text{(Box 12.2.4)}.$$

By Lemma 12.24, this yields an $o(1)$ upper bound on $\Phi^\star$ and contradicts the forcing constant. Hence no off–axis quartet exists, and RH follows.

Why this is a geometry change. The hinge–centered trig contour and the orthogonal $k = 2$ extraction replace "sidewise maxima" by a *single signed harmonic channel.* This is designed to (i) evade the $O(\delta/a)$ cancellation that killed earlier defect endpoints, (ii) remain δ–aware under near–resonant quartets, and (iii) preserve phase information needed for potential Weil/Li identification (Appendix F).

12.2 GEO–C4 endpoint: toy forcing, stability, and UE reduction

The previous S5$'$ defect endpoints are now *archived*; the active v44 endpoint is the hinge–circle harmonic functional $\Phi^\star$ from Box 12.1. We record here (i) a proof–grade toy forcing computation, (ii) a stability lemma showing the forcing persists for the true completed object after monotone admissibility shrink, and (iii) an explicit reduction of the needed UE bound to a single derivative–field estimate (UE-INPUT).

12.2.1 Shift-admissibility on hinge circles

Definition 12.19 (Shift-admissible hinge circle)**.** Fix (m, a, δ, h) with $0 < h < \delta$. We say the hinge circle $C_{m,\delta}$ is *shift-admissible at* (a, h) if

$$E\big(v(\theta) \pm (a \pm h)\big) \neq 0 \qquad \text{for all } \theta \in [0, 2\pi],$$

so that $\mathcal{D}_{a,h}(v(\theta))$ is well-defined and continuous on $[0, 2\pi]$. We say it is buf–*admissible* if, additionally,

$$\min_{\theta} \min_{\pm,\pm} \ \mathrm{dist}\Big(v(\theta) \pm (a \pm h),\, Z(E)\Big) \ \geq \ \mathrm{buf} \cdot \delta,$$

for some fixed buf $\in (0, 1)$.

Remark 12.20 (Monotone admissibility shrink)*.* Zeros of an entire function are isolated. Hence, if a fixed quartet $\{\pm a \pm im\} \subset Z(E)$ exists, then by shrinking δ one can always arrange buf–admissibility for a hinge circle centered at im with coupling $h = \kappa\delta$ ($\kappa \in (0, 1)$ fixed). In the active closure chain we always allow this shrink; UE bounds are required only in a form monotone in δ ("smaller δ makes UE easier").

12.2.2 Toy forcing computation (one quartet)

Lemma 12.21 (Toy forcing for the $k = 2$ harmonic endpoint)**.** *Let*

$$E_{\mathrm{toy}}(v) := \prod_{\sigma=\pm1} \prod_{\tau=\pm1} (v - \sigma a - i\tau m) = (v^2 - (a + im)^2)(v^2 - (a - im)^2),$$

and define $\mathcal{L}_t, \mathcal{D}_{a,h}$ and $\Phi^\star$ as in Box 12.1. Assume $0 < h < \delta$ and set $v(\theta) = im + \delta e^{i\theta}$. Then the induced signal $\psi_{a,h}(\theta) = \Im(\mathcal{D}_{a,h}(v(\theta)))$ has

$$A_c := \int_0^{2\pi} \psi_{a,h}(\theta) \cos(2\theta)\, d\theta = 0, \qquad A_s := \int_0^{2\pi} \psi_{a,h}(\theta) \sin(2\theta)\, d\theta = \frac{4\pi h}{\delta^2},$$

and therefore

$$\big\|P_2 \psi_{a,h}\big\|_{L^2([0,2\pi])} = \frac{|A_s|}{\sqrt{\pi}} = \frac{4\sqrt{\pi}\, h}{\delta^2}, \qquad \boxed{\Phi^\star(m, a, \delta, h) = 4\sqrt{\pi}}.$$

Proof. Write $F(v) := E'_{\mathrm{toy}}(v)/E_{\mathrm{toy}}(v) = \sum_{\rho \in \{\pm a \pm im\}} \frac{1}{v-\rho}$. Fix $v = im + u$. The two *top* zeros $\rho = \pm a + im$ contribute

$$F(v + a \pm h) \supset \frac{1}{u \pm h}, \qquad F(v - a \pm h) \supset \frac{1}{u \mp h},$$

hence (by the definition of $\mathcal{D}_{a,h}$)

$$\mathcal{D}_{a,h}(im+u) = \Big(\frac{1}{u+h} - \frac{1}{u-h}\Big) - \Big(\frac{1}{u-h} - \frac{1}{u+h}\Big) + \text{(terms analytic in a neighborhood of } u = 0),$$

so the singular part is exactly

$$\mathcal{D}^{\mathrm{sing}}_{a,h}(im + u) = \frac{-4h}{u^2 - h^2}.$$

On the hinge circle $u = \delta e^{i\theta}$, set $r := h/\delta \in (0, 1)$ and rewrite

$$\mathcal{D}^{\mathrm{sing}}_{a,h}(v(\theta)) = \frac{-4h}{\delta^2 e^{2i\theta} - h^2} = \frac{-4h}{\delta^2} \cdot \frac{1}{e^{2i\theta} - r^2} = \frac{-4h}{\delta^2}\, e^{-2i\theta} \sum_{n \geq 0} r^{2n} e^{-2in\theta},$$

where we used the geometric series $(1 - r^2 e^{-2i\theta})^{-1} = \sum_{n\ge 0} r^{2n} e^{-2in\theta}$ (valid since $r < 1$). Taking imaginary parts gives the exact Fourier series

$$\psi^{\mathrm{sing}}_{a,h}(\theta) = \Im\big(\mathcal{D}^{\mathrm{sing}}_{a,h}(v(\theta))\big) = \frac{4h}{\delta^2} \sum_{k\ge 1} r^{2(k-1)} \sin(2k\theta).$$

In particular, the $k = 1$ term shows that the $\sin(2\theta)$ coefficient equals $4h/\delta^2$, and there is no $\cos(2\theta)$ term. Hence

$$A_s = \int_0^{2\pi} \psi^{\mathrm{sing}}_{a,h}(\theta) \sin(2\theta)\, d\theta = \frac{4h}{\delta^2} \int_0^{2\pi} \sin^2(2\theta)\, d\theta = \frac{4h}{\delta^2} \cdot \pi = \frac{4\pi h}{\delta^2},$$

and $A_c = 0$. Since P_2 only depends on (A_c, A_s), this yields the claimed $\|P_2\psi\|_{L^2}$ and therefore $\Phi^\star = 4\sqrt{\pi}$. ☐

12.2.3 Forcing for the true completed object: stability under analytic remainders

Lemma 12.22 (Local forcing is stable after admissibility shrink)**.** *Let E be the completed even entire object in the v–plane. Fix a height $m > 0$ and suppose $E(\pm a \pm im) = 0$ for some $a > 0$. Fix $\kappa \in (0, 1)$ and set $h = \kappa\delta$ with $0 < \delta \le \delta_0$. Assume $C_{m,\delta}$ is* buf*–admissible at (a, h) in the sense of Definition 12.19. Then there exists $\delta_* = \delta_*(E, m, a, \kappa, \mathrm{buf}) > 0$ such that for all $0 < \delta \le \delta_*$,*

$$\boxed{\Phi^\star(m, a, \delta, h) \ \ge\ 2\sqrt{\pi}}.$$

In particular, after shrinking δ if needed (Remark 12.20), an off–axis quartet forces a uniform positive lower bound for $\Phi^\star$.

Proof. Write $F := E'/E$. Near each simple zero v_0 of E one has $F(v) = \frac{1}{v - v_0} + H(v)$ with H holomorphic near v_0. Under buf–admissibility, the four shifted traces $v(\theta) \pm (a \pm h)$ avoid all zeros, so $\mathcal{D}_{a,h}(v(\theta))$ is continuous in θ.

Fix δ and decompose on the circle:

$$\mathcal{D}_{a,h}(v(\theta)) = \mathcal{D}^{\mathrm{sing}}_{a,h}(v(\theta)) + \mathcal{R}_{a,h}(v(\theta)),$$

where $\mathcal{D}^{\mathrm{sing}}_{a,h}$ denotes the contribution of the two top zeros $\rho = \pm a + im$ to $\mathcal{D}_{a,h}$ (hence it has the same local pole structure as in Lemma 12.21), and $\mathcal{R}_{a,h}$ is continuous on $C_{m,\delta}$ (it collects all remaining analytic terms).

By Lemma 12.21 applied to the singular part,

$$\frac{\delta^2}{h} \left\| P_2 \Im(\mathcal{D}^{\mathrm{sing}}_{a,h} \circ v) \right\|_{L^2} = 4\sqrt{\pi}.$$

For the remainder, the projection is bounded by its L^2 norm:

$$\left\| P_2 \Im(\mathcal{R}_{a,h} \circ v) \right\|_{L^2} \le \left\| \Im(\mathcal{R}_{a,h} \circ v) \right\|_{L^2} \le \sqrt{2\pi} \sup_\theta |\mathcal{R}_{a,h}(v(\theta))|.$$

Since $\mathcal{R}_{a,h}$ is continuous in a neighborhood of the center as $\delta \to 0$ (all singularities are carried by $\mathcal{D}^{\mathrm{sing}}_{a,h}$), we have $\sup_\theta |\mathcal{R}_{a,h}(v(\theta))| = O_{E,m,a,\kappa,\mathrm{buf}}(1)$ as $\delta \to 0$. Multiplying by $\delta^2/h = \delta/(\kappa)$ shows the normalized remainder contribution is $O(\delta)$. Thus for sufficiently small δ,

$$\Phi^\star = \frac{\delta^2}{h} \left\| P_2(\psi^{\mathrm{sing}} + \psi^{\mathrm{rem}}) \right\|_{L^2} \ge 4\sqrt{\pi} - O(\delta) \ \ge\ 2\sqrt{\pi},$$

as claimed. ☐

Lemma 12.23 (Quartet-induced L^1 blow-up on the hinge circle (toy model))**.** *In the single-quartet toy model of Lemma 12.21, with $v(\theta) = im + \delta e^{i\theta}$ and $h = \kappa\delta$ ($\kappa \in (0,1)$), the singular contribution*

$$\mathcal{D}_{a,h}^{\mathrm{sing}}\big(v(\theta)\big) = -\frac{4h}{\delta^2 e^{2i\theta} - h^2}$$

satisfies the lower bound

$$\int_0^{2\pi} \Big|\mathcal{D}_{a,h}^{\mathrm{sing}}\big(v(\theta)\big)\Big| \, d\theta \ \geq \ \frac{8\pi}{1+\kappa^2}\,\frac{h}{\delta^2}.$$

In particular, any UE-input of the form $\int_0^{2\pi} |\mathcal{D}_{a,h}(v(\theta))|\, d\theta \ll h\,(\log m)^C/a^2$ is incompatible with the presence of an off-axis quartet at that height.

Proof. For $h = \kappa\delta$ one has

$$\Big|\mathcal{D}_{a,h}^{\mathrm{sing}}\big(v(\theta)\big)\Big| = \frac{4h}{|\delta^2 e^{2i\theta} - h^2|} = \frac{4h}{\delta^2\,|e^{2i\theta} - \kappa^2|} \geq \frac{4h}{\delta^2(1+\kappa^2)},$$

since $|e^{2i\theta} - \kappa^2| \leq |e^{2i\theta}| + |\kappa^2| = 1 + \kappa^2$. Integrating over $\theta \in [0, 2\pi]$ gives the stated bound. $\square$

12.2.4 UE reduction via the signed $k = 2$ coefficient

The point of GEO–C4 is that the endpoint is a *single Fourier channel*. Rather than bounding ψ pointwise, or bounding $\int |\mathcal{D}_{a,h}|$ (which discards phase), we reduce UE directly to controlling the *signed* $k = 2$ coefficient $\widehat{\psi}(2)$. This is the minimal interface compatible with (i) harmonic extraction and (ii) potential Weil/Li bridges.

Lemma 12.24 (UE from a signed $k = 2$ coefficient bound)**.** *Let $\psi(\theta) = \Im(\mathcal{D}_{a,h}(v(\theta)))$ on $v(\theta) = im + \delta e^{i\theta}$ and set*

$$\widehat{\psi}(2) := \int_0^{2\pi} \psi(\theta) e^{-2i\theta}\, d\theta.$$

Then

$$\|P_2\psi\|_{L^2_\theta} = \frac{|\widehat{\psi}(2)|}{\sqrt{\pi}}, \qquad \textit{and hence} \qquad \Phi^\star(m,a,\delta,h) = \frac{\delta^2}{h\sqrt{\pi}}\,|\widehat{\psi}(2)|.$$

In particular, any bound of the form $|\widehat{\psi}(2)| \leq B(m,a)\,h$ implies $\Phi^\star \leq (\delta^2/\sqrt{\pi})\,B(m,a)$.

Proof. Write the real Fourier expansion $\psi(\theta) = a_0 + \sum_{k\geq 1}\big(a_k\cos(k\theta) + b_k\sin(k\theta)\big)$. Then the $k = 2$ projection is $P_2\psi = a_2\cos(2\theta) + b_2\sin(2\theta)$ and

$$\|P_2\psi\|_{L^2_\theta}^2 = \int_0^{2\pi} \big(a_2\cos(2\theta) + b_2\sin(2\theta)\big)^2\, d\theta = \pi(a_2^2 + b_2^2).$$

On the other hand,

$$\widehat{\psi}(2) = \int_0^{2\pi} \psi(\theta) e^{-2i\theta}\, d\theta = \pi a_2 - i\pi b_2,$$

so $|\widehat{\psi}(2)| = \pi\sqrt{a_2^2 + b_2^2}$ and therefore $\|P_2\psi\|_{L^2_\theta} = |\widehat{\psi}(2)|/\sqrt{\pi}$. Substituting into the definition of $\Phi^\star$ gives the identity. $\square$

UE-INPUT($k = 2$) (v44, single active statement).
Fix $\kappa \in (0,1)$ and let E be the completed even entire object in the v–plane. For all sufficiently large m and all $a \in (0,1)$, set

$$\delta = \eta \, \frac{a}{(\log(m+3))^2}, \qquad h := \kappa\delta,$$

and allow smaller δ if needed to enforce shift/buffer admissibility of the hinge circle $v(\theta) = im + \delta e^{i\theta}$.
With $\psi(\theta) = \Im(\mathcal{D}_{a,h}(v(\theta)))$ and $\widehat{\psi}(2) = \int_0^{2\pi} \psi(\theta) e^{-2i\theta}\, d\theta$, prove that there exist absolute constants $C, C' > 0$ such that

$$\boxed{|\widehat{\psi}(2)| \;\le\; C \, \frac{(\log(m+3))^{C'}}{a^2} \, h.}$$

With this in place, the GEO–C4 chain closes: FORCE gives a constant lower bound on $\Phi^\star$, while Lemma 12.24 converts UE–INPUT($k = 2$) into an $o(1)$ upper bound on $\Phi^\star$ under the nominal δ–policy.

12.2.5 Dependency map and NO–GO cross-check

The active dependency chain is now:

$$\boxed{(\pm a \pm im) \in Z(E) \;\Rightarrow\; \Phi^\star \ge c_0 \quad \text{and} \quad \text{UE-INPUT}(k=2) \Rightarrow \Phi^\star = o(1) \;\Rightarrow\; a = 0 \;\Rightarrow\; \text{RH.}}$$

Archived NO–GO / obstruction (v40–v41)	**How GEO–C4 avoids it (v44)**
Centered defect endpoint for $\mathcal{L}_a$ cancels to $O(\delta/a)$.	GEO–C4 uses the *double difference* $\mathcal{D}_{a,h}$ and reads a $k = 2$ harmonic channel forced by the dipole kernel $-4h/(u^2 - h^2)$.
δ–inert resonance for pointwise/sidewise shift endpoints (near-resonant second quartet).	We allow monotone admissibility shrink in δ and use an orthogonal projection endpoint; the forcing survives shrink (Lemma 12.22).
Aligned-box micro-coupling (NG-Δa-A) suppresses forcing by $\delta h/a^2$.	GEO–C4 is hinge-centered: the singularities of $\mathcal{D}_{a,h}$ occur at $v = im \pm h$ inside the witness disk, so there is no a^{-2} suppression.
Pointwise UE ceiling: $\sup |E'/E|$ on a boundary cannot beat forcing.	Harmonic extraction reduces UE to a *signed* $k = 2$ coefficient bound (Lemma 12.24), avoiding absolute values and preserving phase (Weil/Li–compatible).

Lemma 12.25 (Endpoint-only NO–GO: $\theta = 0$ per pole forbids any $p > 0$ UE gain)**.** *Fix $\kappa \in (0, 1/10)$. Let $B = B(\alpha, m, \delta)$ be an aligned box and assume κ–admissibility so that $\widetilde{D}_B(W)$ is defined (Definition 12.43). Let E be holomorphic on a neighborhood of $\overline{B}$ with outer factorization $E = G_{\text{out}}\, W$ on B.*

Assume Φ_B is a boundary functional acting on the trace of E'/E and that there exist constants $C_{\text{loc}} = C_{\text{loc}}(\kappa)$ and $u \ge 0$ such that, for every $\rho \in B_{\kappa/2}$ and the test function $E_\rho(v) := v - \rho$, one has the per-pole bound

$$\Phi_B\!\left(\frac{E'_\rho}{E_\rho}\right) = \Phi_B\!\left(\frac{1}{v-\rho}\right) \le C_{\text{loc}}\, \kappa^{-u},$$

uniformly in δ (this is the strong form of "$\theta = 0$ per pole").

Then there do not exist constants $C_{\rm up} > 0$ and $p > 0$ (independent of δ) such that the phase-class UE inequality

$$\widetilde{D}_B(W) \;\le\; C_{\rm up}\, \delta^p\, \Phi_B\!\left(\frac{E'}{E}\right)$$

holds for all such boxes B and all such holomorphic E. In fact, for $E = E_\rho(v) = v - \rho$ with $\rho \in B_{\kappa/2}$, the inequality forces

$$\frac{\pi}{2} \;\le\; C_{\rm up}\, C_{\rm loc}\, \kappa^{-u}\, \delta^p,$$

which fails for sufficiently small δ.

Proof. Fix $\rho \in B_{\kappa/2}$ and take $E(v) = E_\rho(v) = v - \rho$. Then E has a zero in $B^\circ_{\kappa/2}$, hence its quotient $W = E/G_{\rm out}$ also has a zero there (since $G_{\rm out}$ is holomorphic and zero-free on B°). By Lemma 12.47 this implies $\widetilde{D}_B(W) \ge \pi/2$. On the other hand $E'/E = 1/(v-\rho)$, so by hypothesis $\Phi_B(E'/E) \le C_{\rm loc}\kappa^{-u}$ uniformly in δ. Substituting into the claimed UE bound gives $\pi/2 \le C_{\rm up}C_{\rm loc}\kappa^{-u}\delta^p$, contradicting $p > 0$ as $\delta \to 0$. □

Remark 12.26 (Consequence: endpoint-only $\theta = 0$ cannot yield any $p > 0$ gain). Any attempt to close the S5′ envelope obligation by choosing an endpoint class Φ_B that is δ–inert on each local Cauchy kernel $(v-\rho)^{-1}$ ("$\theta = 0$ per pole") cannot produce a δ^p UE gain with $p > 0$ via a purely local analytic argument: the test input $E(v) = v-\rho$ contradicts such a bound (Lemma 12.25). Therefore any successful phase-class UE inequality with $p \ge 1/2$ must incorporate additional structure that defeats the one-pole model (e.g. forcing redesign or pair-isolation/cancellation in the local factor).

Lemma 12.27 (LOCAL isolation needed to beat the one-pole obstruction)**.** *Fix $\kappa \in (0, 1/10)$ and let Φ_B be the endpoint class targeted in S5′. To obtain any UE gain $p > 0$ with a local exponent $\theta < p$ in the S5′ budget theorem, it is necessary to prove a structural statement of the following form:*

Whenever $B = B(\alpha, m, \delta)$ is κ-admissible at the nominal scale $\delta \le \delta_0(m, \alpha)$ and $Z_{\rm loc}$ is the local factor of E on B, there exists a factorization

$$Z_{\rm loc} \;=\; Z_{\rm forced} \cdot Z_{\rm rest}$$

such that (i) $Z_{\rm forced}$ contains only $O(1)$ zeros (the "forced pair"), and (ii) in the chosen endpoint class *one has a δ-small bound*

$$\Phi_B\!\left(\frac{Z'_{\rm rest}}{Z_{\rm rest}}\right) \;\le\; C\,\kappa^{-u}\,(\log m)^{q_{\rm eff}}\,\delta^{-(\theta_{\rm eff})} \qquad \textit{with} \qquad \theta_{\rm eff} < p$$

(and ideally $\theta_{\rm eff} = 0$ with $q_{\rm eff} = 0$ or with an extra δ factor).

Remark 12.28 (S5 acceptance criterion (budget calculus; no drift)). Any proposed S5 redesign must specify a boundary functional Φ_B (shape–invariant; δ–normalized) and prove two explicit inequalities uniformly for all $m \ge 10$, all $\alpha \in (0, 1]$, and all κ–admissible $0 < \delta \le \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$:

1. **(S5–UE)** a forceable upper-envelope bound

$$D_B(W) \le C_{\rm up}\, \delta^p\, \Phi_B(E'/E)$$

with an explicit exponent $p > 0$ and δ–uniform constant $C_{\rm up}$;

2. **(S5–LOC)** a local/collar bound in the same endpoint class

$$\Phi_B(Z'_{\rm loc}/Z_{\rm loc}) \le C_{\rm loc}\,\delta^{-\theta}\,G(N_{\rm loc}(m),\kappa)$$

with explicit $\theta \ge 0$ and an explicit growth model for G (e.g. $G(n,\kappa) \ll \kappa^{-u}n^q$).

The redesign is budget–viable for uniform η–shrinking closure under δ_0 only if the S5 Budget Theorem yields $2(p-\theta) \ge q$ (and $p-\theta > 0$ for shrinkability). If $p-\theta < 0$, the standard κ–admissible shrink policy is unsafe (shrinking δ can increase the envelope term) and must be redesigned.

Finally, the forcing chain remains phrased in terms of $D_B(W)$; therefore S5 must include either $\Phi_B \ge D_B(W)$ on all admissible boxes or a new forcing lemma that lower-bounds Φ_B directly (Remark 12.34).

Theorem 12.29 (S5 Budget Theorem (general endpoint functional))**.** *Fix $\eta \in (0,1]$ and $\kappa \in (0,1/10)$ and define the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. Let Φ_B be a boundary functional and assume that for every $m \ge 10$, $\alpha \in (0,1]$, and every κ–admissible $0 < \delta \le \delta_0(m,\alpha)$ one has:*

(i) ***(S5–UE)*** *$D_B(W) \le C_{\rm up}\,\delta^p\,\Phi_B(E'/E)$ for some $p > 0$ and δ–uniform constant $C_{\rm up}$;*

(ii) ***(S5–SPLIT)*** *$\Phi_B(E'/E) \le \mathrm{Res}(m) + \Phi_B(Z'_{\rm loc}/Z_{\rm loc})$;*

(iii) ***(S5–LOC)*** *$\Phi_B(Z'_{\rm loc}/Z_{\rm loc}) \le C_{\rm loc}\,\delta^{-\theta}\,G(N_{\rm loc}(m),\kappa)$ for some $\theta \ge 0$ and δ–uniform $C_{\rm loc}$.*

Assume moreover that $N_{\rm loc}(m) \le A_N \log m + B_N$ and $\mathrm{Res}(m) \le A_L(\log m)^{r_L} + B_L$ for absolute constants, and that for some $q, u \ge 0$ one has the growth model

$$G(n,\kappa) \le C_G\,\kappa^{-u}\,n^q \qquad (n \ge 1),$$

with C_G independent of (m,α,δ).

Then at the nominal choice $\delta = \delta_0(m,\alpha)$,

$$D_B(W) \;\le\; C_{\rm up}\Big(\delta_0^p\,\mathrm{Res}(m) \;+\; C_{\rm loc}\,\delta_0^{p-\theta}\,G(N_{\rm loc}(m),\kappa)\Big). \tag{15}$$

Furthermore:

1. ***(Uniformity in** m**)*** *The local contribution in* (15) *is uniformly bounded in $m \ge 10$ only if*

$$2(p-\theta) \;\ge\; q. \tag{16}$$

2. ***(η–shrinkability)*** *If* (16) *holds and $p-\theta > 0$, then*

$$\sup_{m\ge 10}\; \delta_0(m,1)^{p-\theta}\,G(N_{\rm loc}(m),\kappa) \;=\; O\big(\eta^{p-\theta}\big),$$

so the local penalty can be made arbitrarily small by choosing η sufficiently small.

3. ***(δ–shrink monotonicity)*** *If $p \ge 0$ and $p-\theta \ge 0$, then the right-hand side of* (15) *is non-decreasing in δ (for fixed m,α); hence replacing δ_0 by a smaller κ–admissible $\delta \le \delta_0$ can only improve the inequality. If $p-\theta < 0$, κ–shrinking can* worsen *the envelope term.*

Proof. Combine (S5–UE) with (S5–SPLIT) and (S5–LOC) to obtain

$$D_B(W) \le C_{\mathrm{up}}\,\delta^p\,\mathrm{Res}(m) + C_{\mathrm{up}}\,C_{\mathrm{loc}}\,\delta^{p-\theta}\,G(N_{\mathrm{loc}}(m),\kappa).$$

Set $\delta = \delta_0(m,\alpha)$ to obtain (15).

For the local contribution at $\alpha = 1$ use $\delta_0 = \eta/(\log m)^2$, the growth model $G(n,\kappa) \le C_G\kappa^{-u}n^q$, and $N_{\mathrm{loc}}(m) \ll \log m$ to get

$$\delta_0^{p-\theta}\,G(N_{\mathrm{loc}}(m),\kappa) \;\ll\; \kappa^{-u}\,\eta^{p-\theta}\,(\log m)^{-2(p-\theta)}\,(\log m)^q = \kappa^{-u}\,\eta^{p-\theta}\,(\log m)^{q-2(p-\theta)}.$$

This is uniformly bounded in m only if $q - 2(p-\theta) \le 0$, i.e. (16). If additionally $p - \theta > 0$, the factor $\eta^{p-\theta}$ yields η–shrinkability.

Finally, the monotonicity claim follows because $\delta \mapsto \delta^p$ and $\delta \mapsto \delta^{p-\theta}$ are nondecreasing on $(0,\infty)$ exactly when $p \ge 0$ and $p - \theta \ge 0$. □

Theorem 12.30 (S5′ closure from a forceable phase endpoint)**.** *Fix $\kappa \in (0, 1/10)$ and $\eta \in (0,1)$ and define $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. Let $\widetilde{D}_B$ be a boundary phase endpoint functional assigned to each κ–admissible aligned box $B = B(\pm a, m, \delta)$ and its boundary quotient W. Assume:*

(S5′–FORCE) Under an off-axis quartet at height m with displacement $a > 0$, there exists an aligned κ–admissible box B (with $\alpha \approx a$) such that $\widetilde{D}_B(W) \ge c_{\mathrm{force}} - \delta\,\Xi(m)$ with $c_{\mathrm{force}} > 0$ absolute and $\Xi(m) \ge 0$ explicit.

(S5′–UE+SPLIT) For every κ–admissible aligned box,

$$\widetilde{D}_B(W) \le C_{\mathrm{up}}\,\delta^p\Big(\mathrm{Res}(m) + C_{\mathrm{loc}}\,\delta^{-\theta}\,G(N_{\mathrm{loc}}(m),\kappa)\Big),$$

where $p > 0$, $\theta \ge 0$, and $C_{\mathrm{up}}, C_{\mathrm{loc}}$ are δ–uniform, and $G(n,\kappa) \le C_G\,\kappa^{-u}n^q$ for fixed $u, q \ge 0$.

Suppose additionally that $2p \ge 1$, $2(p-\theta) \ge q$, and $p - \theta > 0$. Then there exists $\eta_\star \in (0,1)$ (depending on the displayed constants and κ) such that for every $\eta \in (0, \eta_\star]$ the S5′ tail inequality holds at $\delta = \delta_0(m,\alpha)$ for all $m \ge 10$ and all $\alpha \in (0,1]$, and hence no off-axis quartet exists at any height $m \ge 10$. Combined with any finite-height front-end, this implies RH.

Remark 12.31 (S5′ acceptance gate for phase endpoints (no drift)). Any proposed S5′ endpoint built from boundary phase data (e.g. $\Delta \arg$ or an integral of $\Im(\log\text{-derivative})$) must declare its exponent budget data (p,θ,q) in the schematic bound

$$\widetilde{D}_B(W) \le C_{\mathrm{up}}\,\delta^p\Big(\mathrm{Res}(m) + C_{\mathrm{loc}}\,\delta^{-\theta}\,G(N_{\mathrm{loc}}(m),\kappa)\Big),$$

and must satisfy the uniformity/shrink conditions of Theorem 12.30: $2p \ge 1$, $2(p-\theta) \ge q$, and $p - \theta > 0$. Pure $\Delta \arg$ endpoints have $p = 0$ and are rejected. Any phase endpoint whose proof reduces to an absolute $L^r(\partial B)$ estimate for E'/E is also rejected by Lemma 12.50 and Proposition 12.51.

At fixed (m,α) the tail inequality (14) is a strict forcing–vs–envelope condition. In v39 (inherited from v36) the combination of Theorem 10.13, Lemma 10.15, and Lemma 8.2 formally rules out the former "η–absorption" closure route based on the pointwise/sup endpoint $\sup_{\partial B}|E'/E|$ together with the pointwise collar bound.

What must change. The forcing chain produces a lower bound for the *dial deviation*

$$D_B(W) \;:=\; \sum_{\pm}\big|W(v_\pm) - e^{i\varphi_0^{\pm}}\big|$$

appearing in Lemma 10.3. In the current architecture this deviation is upper-bounded by a pointwise endpoint $\delta \sup_{\partial B} |E'/E|$, which (via the collar) introduces the sharp δ^{-1} blow–up. To obtain a tail closure mechanism one must redesign the envelope endpoint and/or the local interface so that the exponent budget $p - \theta \geq \frac{1}{2}$ is met *uniformly in* m.

Remark 12.32 (Forcing compatibility for redesigned endpoints)*.* The existing forcing chain lower-bounds $D_B(W)$ (via the pair-factor phase rotation) by a fixed constant c up to δ–small corrections. If one proposes a redesigned envelope endpoint Φ_B (non-pointwise, e.g. an L^2 or energy functional), then the current forcing lower bound is useful only if it implies a corresponding lower bound for Φ_B. A sufficient (and simplest) compatibility condition is:

$$\Phi_B \;\geq\; D_B(W) \qquad \text{for all admissible boxes and quotients } W,$$

so that the forcing lower bound propagates unchanged. If this domination fails, then a *new forcing lemma* must be proved that lower-bounds Φ_B directly.

Lemma 12.33 (Forceability transfer by domination)**.** *Let B be a κ–admissible aligned box and W the associated boundary quotient. Suppose a boundary endpoint functional Φ_B satisfies*

$$\Phi_B \geq D_B(W) \quad \textit{for all admissible } (B, W).$$

Then the existing single–box forcing lower bound for $D_B(W)$ implies the same forcing lower bound for Φ_B with no change in the forcing constants.

Remark 12.34 (Forceability gate for S5 endpoints (NO–GO unless met))*.* The current forcing architecture (Section 8) forces only the dial deviation $D_B(W)$ by an $O(1)$ constant up to δ–small deductions (Lemma 8.2). Consequently, any S5 redesign that replaces $D_B(W)$ by a different endpoint $\widetilde{D}_B$ (or Φ_B) is *invalid* unless it proves either:

(i) $\widetilde{D}_B \geq D_B(W)$ for all admissible boxes/quotients (domination transfer), or

(ii) a new forcing lemma that lower–bounds $\widetilde{D}_B$ directly under an off–axis quartet.

Without (i) or (ii), the forcing half of the tail inequality becomes logically disconnected from the envelope half.

12.3 S5′ phase endpoints: winding / argument-increment toolkit

Definition 12.35 (Phase increment on a boundary arc)**.** Let $\Gamma \subset \mathbb{C}$ be a piecewise C^1 oriented curve and let f be holomorphic on an open neighborhood of Γ with $f(v) \neq 0$ for all $v \in \Gamma$. Define the phase increment of f along Γ by

$$\Delta_\Gamma \arg f \;:=\; \Im \int_\Gamma \frac{f'(v)}{f(v)}\, dv.$$

(Equivalently, $\Delta_\Gamma \arg f$ is the total change of a continuous branch of $\arg f$ along Γ.)

Lemma 12.36 (Phase increment identity and vertical specialization)**.** *Under the hypotheses of Definition 12.35, the phase increment is additive under concatenation of curves and satisfies:*

1. *If $\Gamma = \Gamma_1 \cup \Gamma_2$ (oriented concatenation), then $\Delta_\Gamma \arg f = \Delta_{\Gamma_1} \arg f + \Delta_{\Gamma_2} \arg f$.*

2. If Γ is the vertical segment $I_+ := \{\alpha + iy : \ |y - m| \le \delta\}$ oriented upward, then

$$\Delta_{I_+} \arg f \ = \ \Im \int_{m-\delta}^{m+\delta} \frac{f'(\alpha + iy)}{f(\alpha + iy)}\, i\, dy \ = \ \int_{m-\delta}^{m+\delta} \Re\Big(\frac{f'(\alpha + iy)}{f(\alpha + iy)} \Big)\, dy.$$

Remark 12.37 (Parentheses hygiene for phase endpoints)*.* For non-horizontal arcs, one must distinguish

$$\Im \int_\Gamma \frac{f'}{f}\, dv \qquad \text{from} \qquad \int_\Gamma \Im\Big(\frac{f'}{f}\Big)\, dv.$$

Only the former is a phase increment. This distinction is essential on vertical segments where $dv = i\, dy$.

Definition 12.38 (Shifted near-vertical segment)**.** Let $B = B(\alpha, m, \delta)$ be an aligned box and let $\lambda \in (0,1)$. Define the shifted segment

$$I_{+,\lambda} := \{\alpha + \lambda\delta + iy : \ |y - m| \le \delta\},$$

oriented upward. (*This lies strictly inside B. It is separated from the unshifted vertical line $I_+ = \{\alpha + iy : |y - m| \le \delta\}$ by horizontal distance $\lambda\delta$, and its distance to ∂B is at least $(1 - \lambda)\delta$.*)

Lemma 12.39 (Phase split on $I_{+,\lambda}$)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible and aligned, and let $I_{+,\lambda}$ be as in Definition 12.38. Assume E, Z_{loc} and $F := E/Z_{\mathrm{loc}}$ are holomorphic and nonvanishing on an open neighborhood of $I_{+,\lambda}$. Then*

$$\Delta_{I_{+,\lambda}} \arg E \ = \ \Delta_{I_{+,\lambda}} \arg F \ + \ \Delta_{I_{+,\lambda}} \arg Z_{\mathrm{loc}}.$$

Moreover,

$$\big|\Delta_{I_{+,\lambda}} \arg F\big| \ \le \ 2\delta \sup_{v \in I_{+,\lambda}} \left|\frac{F'(v)}{F(v)}\right| \ \le \ 2\delta \sup_{v \in \partial B} \left|\frac{F'(v)}{F(v)}\right|.$$

Corollary 12.40 (Residual phase budget (δ–uniform))**.** *Assume the residual envelope bound of Lemma 7.2, i.e. $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2$ on every κ–admissible aligned box. Then, for every $\lambda \in (0,1)$,*

$$\big|\Delta_{I_{+,\lambda}} \arg F\big| \ \le \ 2\delta\, (C_1 \log m + C_2).$$

Lemma 12.41 (Local phase is δ–inert on line segments (per-zero contribution is $O(1)$))**.** *Let $S \subset \mathbb{C}$ be any oriented line segment and let $\rho \notin S$. Choose a continuous branch of $\arg(v - \rho)$ along S. Then*

$$\left|\Im \int_S \frac{dv}{v - \rho}\right| = |\arg(v_1 - \rho) - \arg(v_0 - \rho)| \le \pi,$$

where v_0, v_1 are the endpoints of S. Consequently, writing $Z_{\mathrm{loc}}(v) = \prod_{\rho \in Z_{\mathrm{loc}}(m)} (v - \rho)^{m_\rho}$, for any segment S avoiding $Z_{\mathrm{loc}}(m)$ one has

$$\left|\Im \int_S \frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)}\, dv\right| = |\Delta_S \arg Z_{\mathrm{loc}}| \le \pi\, N_{\mathrm{loc}}(m).$$

Corollary 12.42 (Prototype phase upper bound (residual + local))**.** *Under the hypotheses of Lemma 12.39 and Corollary 12.40,*

$$\big|\Delta_{I_{+,\lambda}} \arg E\big| \le 2\delta\, (C_1 \log m + C_2) \ + \ \big|\Delta_{I_{+,\lambda}} \arg Z_{\mathrm{loc}}\big|.$$

In particular, the residual contribution is $O(\delta \log m)$ while the local contribution is δ–inert in the phase class.

Definition 12.43 (Buffered boundary phase endpoint)**.** Let $B = B(\alpha, m, \delta)$ be an aligned box and assume κ–admissibility: $\operatorname{dist}(\partial B, Z(E)) \geq \kappa\delta$. Let G_{out} be the Dirichlet outer factor on B° and $W := E/G_{\text{out}}$ the inner quotient. Define the buffered inner rectangle

$$B_{\kappa/2} := \{v \in B : \operatorname{dist}(v, \partial B) \geq \tfrac{\kappa\delta}{2}\},$$

and write its oriented boundary as $\partial B_{\kappa/2} = \bigcup_{j=1}^{4} S_j$ (counterclockwise). Define the sidewise phase increment

$$\Delta_{S_j} \arg W \;\; := \;\; \Im \int_{S_j} \frac{W'(v)}{W(v)}\, dv,$$

and the phase endpoint

$$\widetilde{D}_B(W) := \max_{1 \leq j \leq 4} \Big| \Delta_{S_j} \arg W \Big|.$$

Lemma 12.44 (Collar nonvanishing for buffered phase endpoints)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible: $\operatorname{dist}(\partial B, \mathcal{Z}(E)) \geq \kappa\delta$. Let $B_{\kappa/2}$ be the buffered inner rectangle from Definition 12.43. Then*

$$\operatorname{dist}(\partial B_{\kappa/2}, \mathcal{Z}(E)) \;\geq\; \frac{\kappa\delta}{2}.$$

In particular, if G_{out} is the Dirichlet outer factor on B° and $W = E/G_{\text{out}}$, then both G_{out} and W are holomorphic and nonvanishing on an open neighborhood of $\partial B_{\kappa/2}$, so the phase increments $\Delta_{S_j} \arg W$ are well-defined (no branch crossing).

Corollary 12.45 (Local term on the buffered boundary phase endpoint class)**.** *Assume the hypotheses of Definition 12.43 and write $\partial B_{\kappa/2} = \bigcup_{j=1}^{4} S_j$. Then the local factor satisfies*

$$\max_{1 \leq j \leq 4} \left| \Im \int_{S_j} \frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)}\, dv \right| \leq \pi\, N_{\text{loc}}(m).$$

Equivalently,

$$\max_{1 \leq j \leq 4} \left| \Delta_{S_j} \arg Z_{\text{loc}} \right| \leq \pi\, N_{\text{loc}}(m).$$

Lemma 12.46 (Refined per-zero phase bound by horizontal separation)**.** *Let $S \subset \mathbb{C}$ be a line segment of length $|S|$ and let $\rho \notin S$. Write $d := \operatorname{dist}(\rho, S)$. Then*

$$\left| \Im \int_S \frac{dv}{v - \rho} \right| \leq \min\left\{ \pi,\; \frac{|S|}{d} \right\}.$$

Lemma 12.47 (Phase forcing from an interior zero)**.** *Assume the setup of Definition 12.43. If W has at least one zero in $B^\circ_{\kappa/2}$ (equivalently E has at least one zero there), then*

$$\widetilde{D}_B(W) \geq \frac{\pi}{2}.$$

Proof. Since W is holomorphic and nonvanishing on a neighborhood of $\partial B_{\kappa/2}$, the argument principle gives

$$\oint_{\partial B_{\kappa/2}} \frac{W'(v)}{W(v)}\, dv = 2\pi i\, N,$$

where $N \ge 1$ is the number of zeros of W in $B^{\circ}_{\kappa/2}$, counted with multiplicity. Taking imaginary parts and decomposing $\partial B_{\kappa/2}$ into four sides yields

$$\sum_{j=1}^{4} \Delta_{S_j} \arg W = 2\pi N.$$

Hence

$$\widetilde{D}_B(W) \ \ge\ \frac{1}{4}\left|\sum_{j=1}^{4} \Delta_{S_j} \arg W\right| \ =\ \frac{\pi N}{2} \ \ge\ \frac{\pi}{2}.$$

$\square$

Corollary 12.48 (Forcing hypothesis is automatic for the buffered phase endpoint)**.** *Let $\widetilde{D}_B$ be the buffered boundary phase endpoint of Definition 12.43. Then whenever W has a zero in $B^{\circ}_{\kappa/2}$ one has $\widetilde{D}_B(W) \ge \pi/2$. In particular, in Theorem 12.30 the forcing condition* (S5′–FORCE) *may be taken with $c_{\mathrm{force}} = \pi/2$ and $\Xi(m) \equiv 0$ whenever the contradiction endpoint is $\widetilde{D}_B$.*

Proof. This is exactly Lemma 12.47. $\square$

Remark 12.49 (Forceability gate for phase endpoints)*.* The single-box forcing chain in this manuscript supplies a lower bound only for a *forced endpoint*. For the buffered phase endpoint $\widetilde{D}_B$ this lower bound is provided by Lemma 12.47 (and recorded as a hypothesis-discharge in Corollary 12.48). Consequently, any proposed S5′ contradiction endpoint Φ_B is admissible only if either:

1. $\Phi_B(W) \ge \widetilde{D}_B(W)$ on every κ–admissible aligned box (forcing transfers), or
2. a new forcing lemma is proved that lower-bounds $\Phi_B(W)$ directly under an interior zero/off-axis quartet.

Without such a link, forcing and envelope are logically disconnected.

12.4 Baseline NO–GO results for naive non-pointwise endpoints

The S5 goal is to replace the pointwise/sup endpoint in Lemma 10.3 by a non–pointwise functional that still controls the same dial deviation $D_B(W)$ appearing in the forcing chain. The next two results prevent drift into two large endpoint classes that cannot work under the present $D_B(W)$ target and the v36 local split/collar interface (unchanged from v35).

Lemma 12.50 (Absolute L^r endpoint scaling collapse)**.** *Let $B = B(\pm a, m, \delta)$ be an aligned box and let G_{out} and $W = E/G_{\mathrm{out}}$ be as in Lemma 10.3. Assume boundary contact so that W has unimodular boundary values a.e. Fix $r \in [1, \infty]$ and write $L^r(\partial B)$ for $L^r(\partial B, ds)$. Then there exists a shape-only constant $C_r > 0$ (depending only on the normalized square $Q = [-1, 1]^2$) such that for each sign $\pm$,*

$$\left|W(v_\pm) - e^{i\varphi_0^\pm}\right| \ \le\ C_r\, \delta^{\,1-1/r} \left\|\frac{E'}{E}\right\|_{L^r(\partial B)}. \tag{17}$$

In particular, any upper-envelope mechanism whose endpoint is an absolute $L^r(\partial B)$ *norm of E'/E cannot have a δ–prefactor exponent exceeding $p(r) = 1 - 1/r$ within this endpoint class.*

Proof. Repeat the proof of Lemma 10.3 with L^2 replaced by L^r throughout. Evaluation from the boundary gives $|W(v_\pm)-c| \le \|P_B(v_\pm,\cdot)\|_{L^q}\,\|W-c\|_{L^r}$ for $1/r+1/q=1$, and under affine rescaling $\|P_B\|_{L^q} \asymp \delta^{-1/r}$. Boundary Poincaré in L^r yields $\|W-c\|_{L^r} \le C'_r\,\delta\,\|\partial_s W\|_{L^r}$ with a shape-only constant C'_r, and outer factor control bounds $\|\partial_s W\|_{L^r}$ by a shape-only constant times $\|E'/E\|_{L^r}$. Choosing $c = e^{i\varphi_0^\pm}$ gives (17), with overall factor $\delta^{-1/r}\cdot\delta = \delta^{1-1/r}$. □

Proposition 12.51 (NO–GO: absolute L^r log-derivative endpoints cannot clear the UE–Gate)**.** *Assume in addition that B is κ–admissible and hence the pointwise collar bound holds:* $\sup_{\partial B}|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}| \le N_{\mathrm{loc}}(m)/(\kappa\delta)$ *(Lemma 10.8). Then for every $r\in[1,\infty]$,*

$$\delta^{\,1-1/r}\left\|\frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}\right\|_{L^r(\partial B)} \;\le\; 8^{1/r}\,\frac{N_{\mathrm{loc}}(m)}{\kappa},$$

independent of δ. In particular, under the nominal scale $\delta_0(m,\alpha)=\eta\alpha/(\log m)^2$ and the unconditional majorant $N_{\mathrm{loc}}(m)\ll\log m$, uniform η–shrinking cannot suppress the local term within any envelope mechanism whose endpoint is an absolute $L^r(\partial B)$ norm of E'/E.

Proof. Use $|\partial B| = 8\delta$ and $\|f\|_{L^r}\le|\partial B|^{1/r}\|f\|_{L^\infty}$ to get

$$\left\|\frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}\right\|_{L^r(\partial B)} \le (8\delta)^{1/r}\cdot\frac{N_{\mathrm{loc}}(m)}{\kappa\delta} = 8^{1/r}\,\frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta^{\,1-1/r}}.$$

Multiply by $\delta^{\,1-1/r}$. □

Remark 12.52 (Implication for S5 endpoint design). Lemmas 12.50–12.51 rule out the entire family of S5 proposals that attempt to replace $\sup_{\partial B}|E'/E|$ by an *absolute* $L^r(\partial B)$ norm of E'/E while keeping the same κ–collar local control. Any viable S5 redesign must instead (i) exploit cancellation (argument-principle style *signed* endpoints) and/or (ii) move to a less singular boundary object (e.g. endpoints built from $\log|E|$ / BMO-type control).

Lemma 12.53 (NO–GO: local-kernel projection endpoints cannot control $D_B(W)$ without a new forcing link)**.** *Fix an aligned box B and consider an endpoint functional of the form*

$$\Phi_B(E) \;:=\; \|(I-\Pi_B)(E'/E)\|_{X(\partial B)}$$

for some normed boundary space $X(\partial B)$ and a bounded projection Π_B satisfying $\Pi_B(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$ whenever Z_{loc} is the local factor associated to B (so that the local term is annihilated under the split $E'/E = F'/F + Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$). Then there is no universal inequality of the form

$$D_B(W) \;\le\; C\,\delta^p\,\Phi_B(E)$$

(valid for all forcing-aligned boxes under the boundary-contact convention), for any fixed $p>0$ and constant C, unless one supplies a new forcing link that lower-bounds Φ_B directly under an off-axis quartet.

Proof. This is a structural counterexample: in the class of holomorphic functions E obeying the boundary-contact convention, take $E = Z_{\mathrm{loc}}$ on a box for which Z_{loc} has a zero at one of the dial points $v_\pm$. Then $F\equiv 1$ and $E'/E = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$, so by assumption $(I-\Pi_B)(E'/E)=0$ and hence $\Phi_B(E)=0$. However G_{out} is zero-free, so $W = E/G_{\mathrm{out}}$ shares the same interior zeros as E and $W(v_\pm)=0$ for at least one sign, giving $D_B(W)\ge 1$. Thus no inequality $D_B(W)\le C\delta^p\Phi_B(E)$ can hold from these hypotheses alone; any attempt to use such an endpoint must replace $D_B(W)$ as the forced object and provide a forcing-transfer lemma (Remark 12.34). □

Remark 12.54 (Consequence for S5 searches). Lemmas 12.51 and 12.53 close two broad endpoint classes: (i) absolute $L^r(\partial B)$ norms of E'/E (including L^2) under the current collar interface, and (ii) endpoints that annihilate the local kernel span while still targeting the forced dial deviation $D_B(W)$. Any viable S5 redesign must introduce a genuinely new local-interface input and/or a new forcing-compatible endpoint.

S5 design targets (open). A future closure route (S5) should provide a non–pointwise endpoint Φ_B and a UE–type inequality of the schematic form

$$D_B(W) \;\leq\; C_{\mathrm{up}}\,\delta^p\,\Phi_B(E) \qquad (p > 0),$$

together with a local/residual split of $\Phi_B(E)$ whose local contribution scales as $\delta^{-\theta}$ with $\theta < p - \frac{1}{2}$, or more generally satisfies the exponent budget of Theorem 10.13. The point is *not* to recover the specific exponent $\frac{3}{2}$ from older drafts, but to obtain any effective gain $p - \theta > \frac{1}{2}$ with proof–grade uniformity.

Remark 12.55 (Recorded open lemmas (S5 checklist)). A proof–grade S5 implementation would minimally require:

1. **(S5–UE)** a redesigned upper-envelope inequality with a forceable endpoint Φ_B;
2. **(S5–RES)** a δ–uniform residual envelope bound in the same endpoint class;
3. **(S5–LOC)** a collar/local bound in the same endpoint class that avoids the pointwise δ^{-1} blow–up;
4. **(S5–FORCE)** either $\Phi_B \geq D_B(W)$ or a new forcing lemma as in Remark 12.32.

13 Global RH from a finite front-end + the tail criterion family

Theorem 13.1 (Global closure (criterion-first logical form)). *Assume:*

1. *(Front-end) All nontrivial zeros with* $0 < \mathrm{Im}(s) \leq 5$ *lie on the critical line.*
2. *(Tail criterion) Fix some* $\eta \in (0, 1)$ *and* $\kappa \in (0, 1/10)$*, and assume the analytic inputs Lemmas 10.3–10.10 and Lemma 10.17 with finite constants. Assume moreover that for every* $m \geq 10$ *and every* $\alpha \in (0, 1]$ *there exists a* κ*–admissible scale* $0 < \delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ *such that the strict tail inequality* (14) *holds.*

Then all nontrivial zeros of $\zeta(s)$ *lie on the critical line.*

Proof. For each $m \geq 10$, Theorem 11.1 turns the strict inequality (14) into exclusion of off-axis quartets at height m. By the tail criterion hypothesis, no off-axis quartets exist at any height $m \geq 10$. By the front-end hypothesis, there are no off-axis zeros below height 5. Hence there are no off-axis zeros at any height, so every nontrivial zero lies on the critical line. □

Remark 13.2 (Role of computations and the repro pack (v40)). Appendix C provides a small interval-arithmetic harness that evaluates the tail inequality for pinned parameters and a pinned constant ledger. In v36 this is used only for audit purposes (e.g. exponent tracking), not as a proof substitute.

A Discarded closure routes (as of v44)

This appendix records closure routes that were explored in earlier iterations (v32–v34) but are now ruled out *under the currently proved inputs*. The purpose is to prevent future drift: these routes should not be re-opened unless a genuinely new analytic input (e.g. an S5 endpoint redesign) is supplied.

D0: Centered defect endpoint closure ($\mathrm{S5}^{\mathrm{def}}$) is retired

The v39 "defect endpoint" family Φ^{def} on centered boxes cannot serve as a load-bearing closure route: transfer from aligned-box forcing to a centered defect box at the same δ is impossible (Lemma 12.16); the defect endpoint has a side-length ceiling preventing any $p > 1$ gain from pointwise bounds (Lemma 12.3); and near-resonant quartets can make Φ^{def} δ-inert (Lemma 12.15). The defect endpoint is therefore retained only as a cautionary NO–GO example.

A.1 D1: Pointwise UE endpoint $\sup_{\partial B} |E'/E|$ + collar + η–absorption (S1/S1′)

The former absorption narrative attempted to close the tail family by shrinking η in the nominal scale $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$. In the pointwise/sup architecture the UE step has exponent $p = 1$ (Lemma 10.3) and the collar/local split has exponent $\theta = 1$ (Lemma 10.8), so the local contribution is δ–inert and cannot be suppressed by η (Lemma 10.14). More strongly, the exponent budget (Theorem 10.13) shows that uniform η–shrinking requires $p - \theta \geq \frac{1}{2}$, while the scaling NO–GO (Lemma 10.15) forbids any $p > 1$ within this endpoint class. Finally, the forcing margin is constant–limited in the single–box architecture (Lemma 8.2), so one cannot compensate by "making forcing grow with m".

Proposition A.1 (Historical record: formal anchor absorption under a hypothetical strengthened UE exponent)**.** *This proposition is* not used *in v36. It is recorded only to document the logical shape of the discarded absorption idea.*

Assume that, for some $p > 1$, an upper-envelope step admits the strengthened form

$$D_B(W) \;\leq\; 2C_{\mathrm{up}}\, \delta^p \sup_{\partial B} \Big| \frac{E'}{E} \Big|$$

with the same constant ledger, and that all other constants in (14) *are finite. Fix an anchor height $m_\star \geq 10$ and evaluate* (14) *at $(m, \alpha) = (m_\star, 1)$ with the nominal scale $\delta_0(m_\star, 1) = \eta/(\log m_\star)^2$. Then there exists $\eta_\star(m_\star, p) > 0$ such that* (14) *holds at $(m_\star, 1)$ for every $\eta \in (0, \eta_\star]$.*

Warning: *within the pointwise/sup endpoint class, Lemma 10.15 forbids any $p > 1$, so this proposition cannot be invoked without an S5 redesign.*

Proof. Under a strengthened exponent $p > 1$, the envelope side becomes $A\,\eta^p + B\,\eta^{p-1}$ for finite constants A, B depending on $(m_\star, p)$ and the constant ledger, while the forcing side equals $c - D\,\eta$ for a finite D. Since $p > 1$, one has $\eta^p \to 0$, $\eta^{p-1} \to 0$, and $\eta \to 0$ as $\eta \downarrow 0$, so the strict inequality holds for all sufficiently small η. □

A.2 D2: Shrinking the local window / short-interval zero counts

A tempting workaround is to replace the fixed local window $|\gamma - t| \leq 1$ in the residual/collar interface by a shrinking window $|\gamma - t| \leq \delta^\beta$ to reduce the local term. However, without additional analytic input, available RH–free methods control $N(t+1) - N(t-1)$ at unit scale and do *not* provide a proof–grade bound for $N(t+\delta^\beta) - N(t-\delta^\beta)$ as $\delta \downarrow 0$. Thus v36 does not pursue window-shrinking as a substitute for the missing UE gain.

A.3 D3: "Make forcing grow with m" within single-box forcing

Because $\Delta_{I_+} \arg Z_{\mathrm{pair}} \le 2\pi$ uniformly (Lemma 8.2), the forcing constant c in the tail inequality is $O(1)$. Any attempt to obtain a forcing side that grows like $\log m$ (or any unbounded function of m) would require a different forcing architecture (not the v36 single-box forcing chain).

A.4 D4: "Boundary modulus implies interior zero-freeness" converse

Under boundary-contact, the quotient $W = E/G_{\mathrm{out}}$ satisfies $|W| = 1$ on ∂B (Remark 9.3), but this has no converse implication toward zero-freeness or constancy (Remark 9.4). Therefore, any closure route that implicitly treats $|W| = 1$ as "almost zero-free" is invalid.

A.5 D5: Absolute L^r log-derivative endpoints (NO–GO)

Replacing the pointwise endpoint $\sup_{\partial B} |E'/E|$ by an *absolute* boundary $L^r(\partial B)$ norm of E'/E does not improve the exponent budget: Lemma 12.50 forces the UE prefactor exponent to be $p(r) = 1 - 1/r$, while Proposition 12.51 shows the local/collar contribution has the same exponent $\theta(r) = 1 - 1/r$, hence $p(r) - \theta(r) = 0$ and the local leakage is δ–inert.

A.6 D6: Projecting out the local kernel span (NO–GO)

A tempting idea is to define an endpoint by projecting E'/E off the span of local Cauchy kernels so that the local term vanishes. Lemma 12.53 shows this cannot control the forced dial deviation $D_B(W)$ without changing the contradiction endpoint or supplying a new forcing link.

Supporting documentation for D6 (not a viable endpoint under current forcing). The next definition and lemmas formalize the projection setup and the exact cancellation of the local term. They are recorded only to document the mechanism behind the NO–GO.

Definition A.2 (Local Cauchy subspace and L^2 projection (supporting documentation))**.** Let $B = B(\alpha, m, \delta)$ be κ–admissible and let $Z_{\mathrm{loc}}(m)$ denote the multiset of zeros of E used to define Z_{loc} (counted with multiplicity). Define the finite-dimensional subspace

$$K_B := \mathrm{span}\{\, k_\rho : \partial B \to \mathbb{C},\ k_\rho(v) = (v-\rho)^{-1} \ :\ \rho \in Z_{\mathrm{loc}}(m) \,\} \subset L^2(\partial B, ds),$$

and let $\Pi_B : L^2(\partial B) \to K_B$ be the orthogonal projection.

Lemma A.3 (Projection kills the local log-derivative (supporting documentation))**.** *With notation as in Definition A.2,*

$$\frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}(v) = \sum_{\rho \in Z_{\mathrm{loc}}(m)} \frac{m_\rho}{v-\rho} \in K_B \quad (v \in \partial B).$$

Hence $\Pi_B(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$ *and* $(I - \Pi_B)(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = 0$ *in* $L^2(\partial B)$ *(and thus pointwise on* ∂B*). Consequently, using Lemma 10.7,*

$$(I - \Pi_B)\left(\frac{E'}{E}\right) = (I - \Pi_B)\left(\frac{F'}{F}\right) \quad \text{on } \partial B.$$

Moreover $\|\Pi_B\|_{L^2 \to L^2} = 1$.

Remark A.4 (Conditioning caveat for coefficient representations (supporting documentation)). Lemma A.3 uses only the abstract orthogonal projection Π_B (a contraction). No uniform bound on the inverse Gram matrix of the spanning kernels k_ρ is available without a lower bound on pairwise zero separations in $Z_{\mathrm{loc}}(m)$. Therefore any coefficient-level formula for Π_B must be treated as non-uniform unless additional spacing structure is proved.

B S6 harness: explicit-formula interpretation (non-closure)

This appendix is an *interpretation harness only.* It is not used in any implication in the manuscript. Its purpose is to connect the v–frame "off-axis" language to the classical explicit formula for prime-counting functions.

B.1 D7: ML-Δa on aligned boxes is ruled out (v43)

Lemma 12.6 shows that the two–sided shift–difference endpoint $\Phi_B^{(2s)}(a;h)$ cannot be made forceable on the aligned witness family $B(a,m,\delta)$ at the nominal micro–step coupling $h \asymp \delta \asymp \eta a/(\log m)^2$: in the toy quartet model one has $\Phi_B^{(2s)}(a;h) \ll \delta h/a^2 \to 0$ as $m \to \infty$. Therefore, any v40–style "force $\Phi^{(2s)}$ on aligned boxes" closure route is invalid. Future work must pivot to the v44 GEO–C4 closure lever in Box 12.1: invent a different witness family and/or a different endpoint coupling that is simultaneously forceable, budget eligible, and robust under near–resonance.

B.2 Frame mapping: v–displacement and the real part β

A nontrivial zero $\rho = \beta + i\gamma$ in the s–frame corresponds to

$$u_\rho = 2\rho = 2\beta + i\,2\gamma, \qquad v_\rho = u_\rho - 1 = (2\beta - 1) + i\,2\gamma.$$

Thus an off-critical-line zero ($\beta \neq \frac{1}{2}$) is exactly an off-axis v–zero ($\Re v_\rho \neq 0$), with displacement $a := \Re v_\rho = 2\beta - 1$.

B.3 Explicit formula: off-axis zeros as amplitude leaks

In a standard explicit formula (e.g. for $\psi(x) = \sum_{n \le x} \Lambda(n)$), nontrivial zeros enter through terms of the form x^ρ/ρ (or $\mathrm{Li}(x^\rho)$). If $\rho = \beta + i\gamma$, then

$$x^\rho = x^\beta e^{i\gamma \log x},$$

so the amplitude is governed by x^β. In v–variables, $\beta = \frac{1}{2} + \frac{a}{2}$, so any $a > 0$ corresponds to an $x^{1/2+a/2}$-scale contribution (an "amplitude leak" beyond the square-root scale).

B.4 What S5′ would mean in this language

A successful S5′ closure would exclude all off-axis v–zeros, hence prove RH and thereby eliminate all amplitude leaks with $\beta > 1/2$. However, the present manuscript does *not* claim any new prime-error bounds directly: the S6 harness is only a translation layer for interpreting off-axis zeros in the classical explicit-formula setting.

S6 cross-check for GEO–C4. The GEO–C4 endpoint $\Phi^\star$ is a *tilt-sensitivity functional*, but now read through a hinge-centered *harmonic channel.* It acts on the double-difference field $\mathcal{D}_{a,h}$, i.e. a second finite difference in the shift parameter a of the log-derivative E'/E, sampled on the hinge circle $C_{m,\delta}$ and then orthogonally projected onto the $k = 2$ Fourier carrier.

In explicit-formula language, varying a corresponds to varying $\beta = \frac{1}{2} + \frac{a}{2}$, i.e. changing the exponent in the zero-term $x^\rho = x^{\beta+i\gamma}$. An off-axis quartet at tilt $a > 0$ produces an amplitude leak factor $x^{a/2}$ relative to the critical-line baseline. GEO–C4 is engineered so that such a leak yields a nonzero local dipole kernel for $\mathcal{D}_{a,h}$ near $v = im$, which forces a $k = 2$ boundary harmonic (Lemma 12.21), while UE control can be pursued via derivative field bounds rather than a pointwise supremum (Box 12.2.4).

C Tail harness bundle and reproducibility (v44)

C.1 What the tail checks prove (and what they do not)

Each tail check records the statement:

> Given a constants file that provides interval enclosures for $(C_1, C_2, C_{\mathrm{up}}, C_h'', \kappa)$, the chosen parameters (m, η, α), and the recorded UE exponent p, the harness computes interval bounds for the left-hand side LHS and right-hand side RHS in (14) and reports whether the strict separation $\mathrm{LHS}_{\mathrm{hi}} < \mathrm{RHS}_{\mathrm{lo}}$ holds.

It does *not* certify that the constants file is correct.

C.2 SHA–256 table (exact artifacts)

The file `v40_repro_pack/SHA256SUMS.txt` is the canonical hash list.

```
71213ecdda4c8c7dfb63c917bb336cb48b1e865bcebf70289f61a3306dfc4910  README.md
102373187bf21b6770a0e148a5b9c53a48538553c9c556fe15adda07a357119e  v41_constants_m10.json
fca5f82eb2bf2f3ade3988daaee7687a50b9c07118507b351400070ff9504204  v41_frontend_certificate.json
c1debbda3583dbaf0dc7120684ba89c457fef1227f4aa13504b21cf11e029acb  v41_frontend_verifier_output.txt
7e62ddb4e0aa8993fe6fb2e09b93cfa4a54bcd5ed541c17e762590e2069c9745  v41_generate_frontend_certificate
    .py
c75f51c8d62cb702c4707397e954959d0699f05152f127711166c8ae86fd17c9  v41_generate_tail_check.py
9c9162526c238831c02d05e6e970f0118c535aba4179dcf15aafb1d72ba99edc  v41_tail_check_m10.json
3b2c15c47e9eaadc4edbac24296091f41e9c04062de51536469584fc1783a307
    v41_tail_check_verifier_output_m10.txt
911529a08273a7fc4925a266779b5da17cfd7af572092551b1c83e7297f1c640  v41_verify_frontend_certificate.
    py
fa1f6eb05727581a23beb1bad4cbfeeeb5beaa6999f0a89468a51dccf53ecbe3  v41_verify_tail_check.py
```

C.3 Commands

From the directory `v40_repro_pack/`:

1. `sha256sum -c SHA256SUMS.txt`
2. `python3 v36_verify_tail_check.py --constants v36_constants_m10.json --certificate v36_tail_check_m10.json`
3. `python3 v36_verify_frontend_certificate.py --certificate v36_frontend_certificate.json`

C.4 Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false)

```
LHS_hi =
    850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076
RHS_lo =
    0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351
STRICT (LHS_hi < RHS_lo) = False
REGEN_MATCH = True
INEQUALITY_STRICT = False
CERT_REPORTED_PASS = False
OK
```

C.5 Bundle files (verbatim)


```
{
  "UE_endpoint_class": "pointwise/sup",
  "UE_exponent_p": "1",
  "alpha_worst": "1",
  "budget_tuple": {
    "notes": "Budget tuple recorded for the legacy D1 tail harness; v40 active frontier is ML-\
    u0394a (two-sided shift-diff), whose UE budget remains OPEN in-text (Box box:missing-lever-v40)
    .",
    "p": "1",
    "q": "1",
    "theta": "1"
  },
  "certificate_version": "v40_repro_pack",
  "created_utc": "2026-01-28T02:06:02Z",
  "endpoint_family": "v40: legacy D1 tail harness + active frontier ML-\u0394a (two-sided shift-
    diff)",
  "endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
  "eta": "1e-14",
  "forceability_mode": "identity: forced object is D_B(W)",
  "forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma
    force-constant-limited)",
  "forcing_constants": {
    "Kalloc_expression": "3 + 8*sqrt(3)",
    "c0_expression": "(3*ln(2))/(8*pi)",
    "c_expression": "(3*ln(2))/16"
  },
  "forcing_target": "Tail harness targets D_B(W); v40 active Missing Lever targets \u03a6^{(2s)}_B(
    a;\u03b4) (ML-\u0394a)",
  "growth_model": {
    "form": "G(n,kappa) <= C_G * kappa^{-u} * n^q",
    "notes": "Growth model parameters are not certified by this harness; recorded for audit
    bookkeeping.",
    "q": "1",
    "u": "OPEN"
  },
  "intervals": {
    "C1": {
      "hi": "15.2",
      "lo": "15.1"
    },
    "C2": {
      "hi": "37.4",
      "lo": "37.3"
    },
    "C_hpp": {
      "hi": "1100.5",
      "lo": "1100"
    },
    "C_up": {
      "hi": "1100.5",
      "lo": "1100"
    }
  },
  "kappa": "0.05",
  "local_exponent_theta": "1",
  "local_growth_q": "1",
  "m_band": "10",
```

"manuscript_version": "v40",
"missing_lever_open": true,
"ml_deltaa_frontier_installed": true,
"notes": [
"Demo-only intervals carried forward from v31-style scaffolding; replace with audit-proven enclosures when G-1/G-12 are closed.",
"The verifier/generator implement directed-rounding interval arithmetic with Python's decimal module.",
"The local-window majorant N_up(m)=1.01*log(m)+17 is hard-coded from Lemma Nloc-logm in manuscript_v36.",
"UE_exponent_p is recorded explicitly to prevent exponent drift across versions.",
"v36 adds explicit metadata for endpoint_functional and forcing_architecture to prevent silent mismatch under S5 redesign.",
"UE_endpoint_class='pointwise/sup' is the class for which Lemma UE-scaling-nogo forbids any exponent p>1 with shape-only constants.",
"v36 schema latch: required fields (endpoint_functional, UE_exponent_p, local_exponent_theta, local_growth_q, forceability_mode, forcing_architecture) must be present; verifier fails closed if missing.",
"v40 schema latch: phase_endpoint_only_nogo_installed=true records that Lemma phase-UE-theta0-nogo is installed in-text (endpoint-only NO-GO)."
],
"phase_endpoint_only_nogo_installed": true
}

{
"certificate_version": "v40_repro_pack",
"m_band": "10",
"eta": "1e-14",
"alpha": "1",
"kappa": "0.05",
"UE_exponent_p": "1",
"UE_endpoint_class": "pointwise/sup",
"endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
"forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma force-constant-limited)",
"forceability_mode": "identity: forced object is D_B(W)",
"endpoint_family": "v40: legacy D1 tail harness + active frontier ML-\u0394a (two-sided shift-diff)",
"forcing_target": "Tail harness targets D_B(W); v40 active Missing Lever targets \u03a6^{(2s)}_B(a;\u03b4) (ML-\u0394a)",
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"p": "1",
"q": "1",
"theta": "1"
},
"growth_model": {
"form": "G(n,kappa) <= C_G * kappa^{-u} * n^q",
"notes": "Growth model parameters are not certified by this harness; recorded for audit bookkeeping.",
"q": "1",
"u": "OPEN"
},
"missing_lever_open": true,

"phase_endpoint_only_nogo_installed": true,
"local_exponent_theta": "1",
"local_growth_q": "1",
"forcing_constants": {
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  "c0_expression": "(3*ln(2))/(8*pi)",
  "c_expression": "(3*ln(2))/16"
},
"manuscript_version": "v40",
"prec": 90,
"pi_interval": {
  "lo": "3.14159265358979323846264338327950288419716939937510",
  "hi": "3.14159265358979323846264338327950288419716939937511"
},
"logm_interval": {
  "lo":
  "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721",
  "hi":
  "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721"
},
"delta_interval": {
  "lo":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197468E
  -15",
  "hi":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197469E
  -15"
},
"L_interval": {
  "lo":
  "72.0690349042100898286716709657338995347766324782944719381032513046103464061280224515635578",
  "hi":
  "72.3992934135094943970734701112023359555367426271573492357065840947071036670957576995871576"
},
"Nup_interval": {
  "lo":
  "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383571",
  "hi":
  "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383572"
},
"kappa_interval": {
  "lo": "0.05",
  "hi": "0.05"
},
"lhs_interval": {
  "lo":
  "850326.881532655689245144996236820608990676883579823939491593791503565415726034904366571925",
  "hi":
  "850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076"
},
"rhs_interval": {
  "lo":
  "0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351",
  "hi":
  "0.129965096347948199005209691457697222838229716361348668790498959759558243588339370271250251"
},
"derived_constants": {
  "ln2_interval": {
    "lo":

```
    "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327",
      "hi":
    "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327"
    },
    "c_interval": {
      "lo":
    "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099373",
      "hi":
    "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099375"
    },
    "c0_interval": {
      "lo":
    "0.0827383500572443475236711620442491341185086557736206913728528561387020242248387512851407512",

      "hi":
    "0.0827383500572443475236711620442491341185086557736209547372007536994885577445868650239268751"
    },
    "Kalloc_interval": {
      "lo":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491695",
      "hi":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491696"
    }
  },
  "pass": false
}
```


```
#!/usr/bin/env python3
"""
v40_generate_tail_check.py

Deterministically generates v40_tail_check_m10.json from v40_constants_m10.json using
directed-rounding interval arithmetic implemented with Python's decimal module.

This generator is intended to be auditable: no network access, no randomness, and no external
    libraries.

Tail inequality evaluation (for given inputs):
  LHS(delta) < RHS(delta), where
  LHS(delta) = 2*C_up*( delta^p*L(m) + delta^(p-1)*N_up(m)/kappa )
  RHS(delta) = c - delta*( Kalloc*c0*L(m) + C_hpp*(log(m)+1) )

with
  L(m)    = C1*log(m) + C2,
  N_up(m) = 1.01*log(m) + 17,
  c  = (3 ln 2)/16,
  c0 = (3 ln 2)/(8 pi),
  Kalloc = 3 + 8*sqrt(3).

Usage:
  python3 v40_generate_tail_check.py v40_constants_m10.json v40_tail_check_m10.json
"""

import json
import sys
from dataclasses import dataclass
```

```
from decimal import Decimal, getcontext, localcontext, ROUND_FLOOR, ROUND_CEILING


# ---- Fixed enclosure for pi (50 decimal places) ----
# pi = 3.14159265358979323846264338327950288419716939937510...
PI_LO = Decimal("3.14159265358979323846264338327950288419716939937510")
PI_HI = Decimal("3.14159265358979323846264338327950288419716939937511")


@dataclass
class Interval:
    lo: Decimal
    hi: Decimal

    def __post_init__(self) -> None:
        if self.lo > self.hi:
            raise ValueError(f"Bad interval: {self.lo} > {self.hi}")


def ctx(prec: int, rounding):
    c = getcontext().copy()
    c.prec = prec
    c.rounding = rounding
    return c


def iv(lo: str, hi: str | None = None) -> Interval:
    if hi is None:
        hi = lo
    return Interval(Decimal(lo), Decimal(hi))


def add(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo + b.lo
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi + b.hi
    return Interval(lo, hi)


def sub(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo - b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi - b.lo
    return Interval(lo, hi)


def mul(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        cands_lo = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        lo = min(cands_lo)
    with localcontext(ctx(prec, ROUND_CEILING)):
        cands_hi = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        hi = max(cands_hi)
    return Interval(lo, hi)


def div(a: Interval, b: Interval, prec: int) -> Interval:
```

```
    if b.lo <= 0 <= b.hi:
        raise ZeroDivisionError("Interval division by an interval containing 0.")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        rlo = Decimal(1) / b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        rhi = Decimal(1) / b.lo
    return mul(a, Interval(rlo, rhi), prec)


def sqrt(a: Interval, prec: int) -> Interval:
    if a.lo < 0:
        raise ValueError("sqrt of negative interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.sqrt()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.sqrt()
    return Interval(lo, hi)


def ln(a: Interval, prec: int) -> Interval:
    if a.lo <= 0:
        raise ValueError("ln of nonpositive interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.ln()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.ln()
    return Interval(lo, hi)


def pow_3_2(a: Interval, prec: int) -> Interval:
    return mul(a, sqrt(a, prec), prec)


def compute(constants: dict, prec: int = 90) -> dict:
    m = iv(constants["m_band"])
    eta = iv(constants["eta"])
    alpha = iv(constants["alpha_worst"])
    kappa = iv(constants["kappa"])

    p = str(constants.get("UE_exponent_p", "1"))

    C1 = iv(constants["intervals"]["C1"]["lo"], constants["intervals"]["C1"]["hi"])
    C2 = iv(constants["intervals"]["C2"]["lo"], constants["intervals"]["C2"]["hi"])
    Cup = iv(constants["intervals"]["C_up"]["lo"], constants["intervals"]["C_up"]["hi"])
    Chpp = iv(constants["intervals"]["C_hpp"]["lo"], constants["intervals"]["C_hpp"]["hi"])

    logm = ln(m, prec)
    delta = div(mul(eta, alpha, prec), mul(logm, logm, prec), prec)

    # L(m) = C1*logm + C2
    L = add(mul(C1, logm, prec), C2, prec)

    # N_up(m) = 1.01*logm + 17
    Nup = add(mul(iv("1.01"), logm, prec), iv("17"), prec)

    # ln 2
    ln2 = ln(iv("2"), prec)

    # c = (3 ln 2)/16
```

```
    c = div(mul(iv("3"), ln2, prec), iv("16"), prec)

    # c0 = (3 ln 2)/(8 pi), pi enclosed
    pi = Interval(PI_LO, PI_HI)
    c0 = div(mul(iv("3"), ln2, prec), mul(iv("8"), pi, prec), prec)

    # Kalloc = 3 + 8 sqrt(3)
    sqrt3 = sqrt(iv("3"), prec)
    Kalloc = add(iv("3"), mul(iv("8"), sqrt3, prec), prec)

    logm_plus1 = add(logm, iv("1"), prec)

    # UE exponent p: LHS = 2*Cup*(delta^p * L + delta^(p-1) * Nup/kappa).
    # We support p="1" (pointwise UE proved in v36) and p="3/2" (hypothetical strengthened gate).
    if p in ("1", "1.0", "1.00"):
        local_term = div(Nup, kappa, prec)              # delta^(p-1)=1
        residual_term = mul(delta, L, prec)             # delta^p = delta
    elif p in ("3/2", "1.5", "1.50"):
        sqrt_delta = sqrt(delta, prec)
        local_term = mul(sqrt_delta, div(Nup, kappa, prec), prec)     # delta^(1/2)
        residual_term = mul(mul(delta, sqrt_delta, prec), L, prec)    # delta^(3/2)
    else:
        raise ValueError(f"Unsupported UE_exponent_p={p!r}; use '1' or '3/2'.")

    lhs = mul(mul(iv("2"), Cup, prec), add(residual_term, local_term, prec), prec)

    # RHS = c - delta*(Kalloc*c0*L + Chpp*(logm+1))
    term1 = mul(mul(Kalloc, c0, prec), L, prec)
    term2 = mul(Chpp, logm_plus1, prec)
    rhs = sub(c, mul(delta, add(term1, term2, prec), prec), prec)

    passed = (lhs.hi < rhs.lo)

    return {
        "prec": prec,
        "UE_exponent_p": p,
        "pi_interval": {"lo": str(PI_LO), "hi": str(PI_HI)},
        "logm_interval": {"lo": str(logm.lo), "hi": str(logm.hi)},
        "delta_interval": {"lo": str(delta.lo), "hi": str(delta.hi)},
        "L_interval": {"lo": str(L.lo), "hi": str(L.hi)},
        "Nup_interval": {"lo": str(Nup.lo), "hi": str(Nup.hi)},
        "kappa_interval": {"lo": str(kappa.lo), "hi": str(kappa.hi)},
        "lhs_interval": {"lo": str(lhs.lo), "hi": str(lhs.hi)},
        "rhs_interval": {"lo": str(rhs.lo), "hi": str(rhs.hi)},
        "derived_constants": {
            "ln2_interval": {"lo": str(ln2.lo), "hi": str(ln2.hi)},
            "c_interval": {"lo": str(c.lo), "hi": str(c.hi)},
            "c0_interval": {"lo": str(c0.lo), "hi": str(c0.hi)},
            "Kalloc_interval": {"lo": str(Kalloc.lo), "hi": str(Kalloc.hi)},
        },
        "pass": bool(passed),
    }


def main() -> int:
    if len(sys.argv) != 3:
        print("Usage: v40_generate_tail_check.py constants.json tail_check.json", file=sys.stderr)
        return 2
```

```
with open(sys.argv[1], "r", encoding="utf-8") as f:
    constants = json.load(f)

# ---- fail-closed metadata latch (v36) ----
REQUIRED_META = [
    "UE_exponent_p",
    "UE_endpoint_class",
    "endpoint_functional",
    "forcing_architecture",
    "forceability_mode",
    "local_exponent_theta",
    "local_growth_q",
    # v40 schema extensions (fail-closed)
    "endpoint_family",
    "forcing_target",
    "budget_tuple",
    "growth_model",
    "missing_lever_open",
    "phase_endpoint_only_nogo_installed",
]
for k in REQUIRED_META:
    if k not in constants or constants[k] in (None, ""):
        raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")

out = {
    "certificate_version": constants.get("certificate_version", "v40_repro_pack"),
    "m_band": constants["m_band"],
    "eta": constants["eta"],
    "alpha": constants["alpha_worst"],
    "kappa": constants["kappa"],
    # required metadata latch (fail-closed)
    "UE_exponent_p": constants["UE_exponent_p"],
    "UE_endpoint_class": constants["UE_endpoint_class"],
    "endpoint_functional": constants["endpoint_functional"],
    "forcing_architecture": constants["forcing_architecture"],
    "forceability_mode": constants["forceability_mode"],

    # v40 schema extensions (fail-closed metadata latch)
    "endpoint_family": constants["endpoint_family"],
    "forcing_target": constants["forcing_target"],
    "budget_tuple": constants["budget_tuple"],
    "growth_model": constants["growth_model"],
    "missing_lever_open": constants["missing_lever_open"],
    "phase_endpoint_only_nogo_installed": constants["phase_endpoint_only_nogo_installed"],
    "local_exponent_theta": constants["local_exponent_theta"],
    "local_growth_q": constants["local_growth_q"],
    "forcing_constants": constants.get("forcing_constants", {}),
    "manuscript_version": constants.get("manuscript_version", "v40"),
}

out.update(compute(constants, prec=90))

with open(sys.argv[2], "w", encoding="utf-8") as f:
    json.dump(out, f, indent=2)

print("[generate] wrote", sys.argv[2])
print("[generate] PASS =", out["pass"])
print("[generate] lhs_interval.hi =", out["lhs_interval"]["hi"])
print("[generate] rhs_interval.lo =", out["rhs_interval"]["lo"])
```

```
    return 0


if __name__ == "__main__":
    raise SystemExit(main())




#!/usr/bin/env python3
"""
v40_verify_tail_check.py

Verifier for v40_tail_check_m10.json. This script:
- loads the constants JSON and the pinned certificate JSON
- regenerates the certificate from constants
- checks exact JSON equality on the computed fields
- reports PASS/FAIL and prints the strict-separation check LHS_hi < RHS_lo.

Usage:
  python3 v40_verify_tail_check.py --constants v40_constants_m10.json --certificate
    v40_tail_check_m10.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json
import sys

from v40_generate_tail_check import compute


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v40 tail check (m=10).")
    ap.add_argument("--constants", required=True, help="Path to v40_constants_m10.json")
    ap.add_argument("--certificate", required=True, help="Path to v40_tail_check_m10.json")
    args = ap.parse_args()

    with open(args.constants, "r", encoding="utf-8") as f:
        constants = json.load(f)

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    # ---- fail-closed metadata latch (v40) ----
    REQUIRED_META = [
        "UE_exponent_p",
        "UE_endpoint_class",
        "endpoint_functional",
        "forcing_architecture",
        "forceability_mode",
        "local_exponent_theta",
        "local_growth_q",
        # v40 schema extensions (fail-closed)
```

```
    "endpoint_family",
    "forcing_target",
    "budget_tuple",
    "growth_model",
    "missing_lever_open",
    "phase_endpoint_only_nogo_installed",
]
for k in REQUIRED_META:
    if k not in constants or constants[k] in (None, ""):
        raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")
    if k not in cert or cert[k] in (None, ""):
        raise KeyError(f"Missing required metadata field {k!r} in certificate (fail-closed).")

regen = {
    "certificate_version": constants.get("certificate_version", "v40_repro_pack"),
    "m_band": constants["m_band"],
    "eta": constants["eta"],
    "alpha": constants["alpha_worst"],
    "kappa": constants["kappa"],
    # required metadata latch (fail-closed)
    "UE_exponent_p": constants["UE_exponent_p"],
    "UE_endpoint_class": constants["UE_endpoint_class"],
    "endpoint_functional": constants["endpoint_functional"],
    "forcing_architecture": constants["forcing_architecture"],
    "forceability_mode": constants["forceability_mode"],
    "local_exponent_theta": constants["local_exponent_theta"],
    "local_growth_q": constants["local_growth_q"],
    # v40 schema extensions (fail-closed)
    "endpoint_family": constants["endpoint_family"],
    "forcing_target": constants["forcing_target"],
    "budget_tuple": constants["budget_tuple"],
    "growth_model": constants["growth_model"],
    "missing_lever_open": constants["missing_lever_open"],
    "phase_endpoint_only_nogo_installed": constants["phase_endpoint_only_nogo_installed"],
    "forcing_constants": constants.get("forcing_constants", {}),
    "manuscript_version": constants.get("manuscript_version", "v40"),
}

# v40 fail-closed checks: missing_lever_open must be explicitly True and budget_tuple structure
 must be explicit
if constants.get("missing_lever_open") is not True:
    raise SystemExit("FAIL: constants missing_lever_open must be true in v40.")
if cert.get("missing_lever_open") is not True:
    raise SystemExit("FAIL: certificate missing_lever_open must be true in v40.")

if constants.get("phase_endpoint_only_nogo_installed") is not True:
    raise SystemExit("FAIL: constants phase_endpoint_only_nogo_installed must be true in v40.")
if cert.get("phase_endpoint_only_nogo_installed") is not True:
    raise SystemExit("FAIL: certificate phase_endpoint_only_nogo_installed must be true in v40
.")
for obj, name in [(constants, "constants"), (cert, "certificate")]:
    bt = obj.get("budget_tuple")
    if not isinstance(bt, dict):
        raise SystemExit(f"FAIL: {name}.budget_tuple must be an object.")
    for k in ["p", "theta", "q"]:
        if k not in bt:
            raise SystemExit(f"FAIL: {name}.budget_tuple missing key '{k}'.")
regen.update(compute(constants, prec=90))
```

```
    # Compare all keys that regen produces (ignore any extra keys in cert)
    ok = True
    for k, v in regen.items():
        if cert.get(k) != v:
            ok = False
            print(f"MISMATCH key={k}")
            print("  cert :", cert.get(k))
            print("  regen:", v)

    lhs_hi = regen["lhs_interval"]["hi"]
    rhs_lo = regen["rhs_interval"]["lo"]
    strict = (float(lhs_hi) < float(rhs_lo))

    print("LHS_hi =", lhs_hi)
    print("RHS_lo =", rhs_lo)
    print("STRICT (LHS_hi < RHS_lo) =", strict)

    print("REGEN_MATCH =", ok)
    print("INEQUALITY_STRICT =", strict)
    print("CERT_REPORTED_PASS =", regen.get("pass"))

    if not ok:
        print("FAIL (mismatch)")
        return 1

    print("OK")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

D Finite-height front-end certificate (literature-based)

The required front-end is RH up to height $H_0 = 5$. We record a discharge using Platt–Trudgian's published verification of RH up to $3 \cdot 10^{12}$.


```
{
  "certificate_version": "v40_repro_pack",
  "created_utc": "2026-01-28T02:06:02Z",
  "discharged_by": {
    "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=H0.",
    "reference": {
      "arxiv": "2004.09765",
      "authors": "D. J. Platt and T. S. Trudgian",
      "doi": "10.1112/blms.12460",
      "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (rigorous
    interval arithmetic).",
      "title": "The Riemann hypothesis is true up to 3*10^12",
      "venue": "Bulletin of the London Mathematical Society",
      "year": 2021
    },
    "type": "literature_citation",
```

```
    "verification_height": 3000000000000.0
  },
  "manuscript_version": "v40",
  "needed_frontend_statement": {
    "H0": 5.0,
    "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2.",
    "type": "RH_to_height"
  },
  "notes": [
    "This JSON is not itself a computation of zeros; it is a pinned statement+reference used by v40
    .",
    "For a fully self-contained proof without external computational input, one would need to
    implement and certify an argument-principle zero count in this region using ball arithmetic (
    not provided here)."
  ]
}
```


```
H0 (needed) = 5.0
H_cited     = 3000000000000.0
CHECK: H0 <= H_cited : True
PASS
```

```python
#!/usr/bin/env python3
"""v40_generate_frontend_certificate.py

Creates a pinned JSON certificate for the finite-height front-end assumption used by v40.

This script does NOT compute zeta zeros. It encodes a minimal (H0, citation) logic statement:
if RH has been verified up to H_cited and H0 <= H_cited, then RH holds up to height H0.

Usage:
  python3 v40_generate_frontend_certificate.py v40_frontend_certificate.json
"""

from __future__ import annotations

import json
from datetime import datetime, timezone
import sys


def main() -> int:
    if len(sys.argv) != 2:
        print("Usage: v40_generate_frontend_certificate.py output.json", file=sys.stderr)
        return 2

    out = {
        "certificate_version": "v40_repro_pack",
        "created_utc": datetime.now(timezone.utc).strftime("%Y-%m-%dT%H:%M:%SZ"),
        "needed_frontend_statement": {
            "type": "RH_to_height",
            "H0": 5.0,
            "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
        },
```

```
        "discharged_by": {
            "type": "literature_citation",
            "verification_height": 3e12,
            "reference": {
                "authors": "D. J. Platt and T. S. Trudgian",
                "title": "The Riemann hypothesis is true up to 3*10^12",
                "venue": "Bulletin of the London Mathematical Society",
                "year": 2021,
                "doi": "10.1112/blms.12460",
                "arxiv": "2004.09765",
                "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (
    rigorous interval arithmetic)."
            },
            "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=
    H0."
        },
        "notes": [
            "This JSON is not itself a computation of zeros; it is a pinned statement+reference
    used by v40.",
            "For a fully self-contained proof without external computational input, one would need
    to implement and certify an argument-principle zero count in this region using ball arithmetic
    (not provided here)."
        ]
    }

    with open(sys.argv[1], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[1])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())

#!/usr/bin/env python3
"""v40_verify_frontend_certificate.py

Verifier for the front-end certificate JSON produced by v40_generate_frontend_certificate.py.

This verifier checks the internal logic only:
- parses the JSON
- confirms that the required finite-height H0 is <= the cited verification height

It does NOT re-run the cited large-scale computation (Platt--Trudgian); that result is treated as
    an
external, peer-reviewed input in the manuscript.

Usage:
  python3 v40_verify_frontend_certificate.py --certificate v40_frontend_certificate.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
```

```
import json


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v40 front-end certificate JSON (internal logic
     only).")
    ap.add_argument("--certificate", required=True, help="Path to v40_frontend_certificate.json")
    args = ap.parse_args()

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    needed = cert.get("needed_frontend_statement", {})
    discharged = cert.get("discharged_by", {})

    H0 = float(needed.get("H0"))
    Hc = float(discharged.get("verification_height"))

    ok = H0 <= Hc

    print("H0 (needed) =", H0)
    print("H_cited     =", Hc)
    print(f"CHECK: H0 <= H_cited : {ok}")

    if not ok:
        print("FAIL")
        return 1

    print("PASS")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

References

E UE playbook (attack surface)

This one-page appendix isolates the **exact UE attack surface** for the current GEO–C4 closure chain. Everything outside this appendix is treated as locked for the v44 build.

Definitions (v–frame)

- $E(v)$: the even entire completion in the width–2 centered v–plane.
- $f(v) := E'(v)/E(v)$.
- $\mathcal{L}_t(v) := f(v+t) - f(v-t)$ (odd real–shift difference).
- $\mathcal{D}_{a,h}(v) := \mathcal{L}_{a+h}(v) - \mathcal{L}_{a-h}(v)$ (second difference in t).

- Hinge circle at height m: $v(\theta) = im + \delta e^{i\theta}$, $\theta \in [0, 2\pi]$.
- Phase signal: $\psi(\theta) := \Im(\mathcal{D}_{a,h}(v(\theta)))$ and the signed $k = 2$ coefficient

$$\widehat{\psi}(2) := \int_0^{2\pi} \psi(\theta) e^{-2i\theta}\, d\theta.$$

- Endpoint: $\Phi^\star = \dfrac{\delta^2}{h} \|P_2\psi\|_{L^2} = \dfrac{\delta^2}{h\sqrt{\pi}} |\widehat{\psi}(2)|$ (Lemma 12.24).

Target inequality (single active statement)

At the nominal policy $\delta = \eta a / (\log(m+3))^2$ and $h = \kappa\delta$, prove

$$|\widehat{\psi}(2)| \;\ll\; h\, a^{-2} (\log(m+3))^{C'} \qquad \text{(Box 12.2.4)}.$$

This immediately yields $\Phi^\star = o(1)$ (or $\Phi^\star \ll \eta^2$ at the exponent boundary), contradicting the forcing constant from any off–axis quartet.

Known decompositions / reductions

- **Archimedean absorption.** For the completed E, the Γ–factor contribution to $f = E'/E$ is smooth in t and its second difference contributes $O(h)$ uniformly on hinge circles (cf. Batch 15 ABSORB). Thus the UE burden is concentrated in the ζ'/ζ component.
- **Residue kernel viewpoint.** On any contour buffered away from zeros, $f(v) = \sum_\rho \dfrac{1}{v - \rho} +$ (analytic) and $\mathcal{D}_{a,h}$ becomes a sum of explicit rational kernels. The $k = 2$ coefficient extracts a *single signed moment* of this field.

Candidate analytic routes

1. **Kernel sum over zeros (height–local).** Express $\widehat{\psi}(2)$ as $\sum_\rho K_{m,a,h,\delta}(\rho)$, isolate the local quartet contribution, and bound the remainder by tail/multiplicity controls. The certified tail harness can handle the far zeros once the kernel decay is explicit.
2. **Hardy/Poisson control on the circle.** Interpret the $k = 2$ coefficient as a boundary pairing and use harmonic analysis on the disk to bound it by interior norms of $\mathcal{D}_{a,h}$ (or of $\partial_t \mathcal{L}_t$) without taking absolute values on the boundary.
3. **Explicit formula / Weil functional identification (optional).** Try to match $K_{m,a,h,\delta}(\rho)$ to a Mellin kernel $\widehat{g}(\rho)\widehat{g}(1-\rho)$ so that $\widehat{\psi}(2)$ becomes a Weil-type functional. This is phase-sensitive and is only plausible because UE–INPUT($k = 2$) is a signed channel (Appendix F).

F Weil/Li bridge scoping (optional)

This appendix records a *possible* bridge between the hinge–circle $k = 2$ channel and classical explicit-formula positivity criteria (Weil, and hence Li). It is **not** used in the v44 closure chain, but it may make future builds strictly easier.

Weil positivity in one line

One formulation (see Burnol's exposition [9]) asserts that RH is equivalent to the nonnegativity of a quadratic functional $W(g)$ over all smooth compactly supported g on $(0, \infty)$, with $W(g)$ expressed as a signed sum over nontrivial zeros via Mellin transforms. Moreover, the contrapositive is constructive: an off–axis zero implies the existence of a negativity witness g.

Why our UE channel is compatible

Our endpoint is already a *signed* harmonic channel:

$$\widehat{\psi}(2) = \int_0^{2\pi} \Im(\mathcal{D}_{a,h}(v(\theta)))e^{-2i\theta}\, d\theta.$$

If one can rewrite $\widehat{\psi}(2)$ as a sum over zeros $\sum_\rho K(\rho)$ and then factor or dominate $K(\rho)$ by a Weil kernel $\widehat{g}(\rho)\widehat{g}(1-\rho)$, the negativity-witness logic could be imported.

A sharp warning

Absolute-value UE interfaces (e.g. $\int |\mathcal{D}_{a,h}|$) discard the phase information that drives Weil/Li sign changes. The present UE–INPUT($k = 2$) interface is designed to avoid this loss, but an actual bridge would still require a precise kernel identification and a regularization argument to match the compact-support hypotheses on g.

References

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