

A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Criterion

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Abstract

This paper is organized in three parts:

- **Part I** — Reader’s Guide / Motivation, reducing the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: $RH \Leftrightarrow a(m) = 0$ at each nontrivial height m , while recording non–load–bearing structural scaffolding.
- **Part II** — A self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based L^2 upper envelope and an L^2 lower envelope via allocation + restricted contour + Jensen. A rigorous Outer/Rouché Certification Path, explicit domains and symbolic constants (“shape–only” vs. residual).
- **Part III** — Promotes the identities constructed in Part I to structural corollaries of the main theorem once $a(m) = 0$ is established.
- **Appendices** — Technical details supporting Part II (load–bearing for Part II; load–bearing for Part III if reference to appendices is required in Part III).

Dependency map (schematic): Part I \rightarrow (no arrows into Part II); Part II \rightarrow Part III; Appendices \rightarrow Part II and Part III.

Part I — Reader’s Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered “ a –lens” that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m . It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed \rightarrow rectifier) and explains how these become consequences once $a(m) = 0$ is proved. *What it does not do.* It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later. This Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)} \Gamma\left(\frac{s}{2f}\right) \zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s) \zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.

Width–2 normalization. Put $u := (2/f)s$. Then

$$\zeta_2(u) := \zeta(u/2), \quad \Lambda_2(u) := \pi^{-u/4} \Gamma(u/4) \zeta(u/2), \quad \Lambda_2(u) = \Lambda_2(2-u).$$

The non-completed FE reads $\zeta_2(u) = A_2(u) \zeta_2(2-u)$. In the open strip $0 < \Re u < 2$ and $\Im u \neq 0$, A_2 is analytic and nonvanishing.

Partner map. On $\Im u > 0$, FE + conjugation gives the involution $J(u) = 2 - \bar{u}$, swapping the two column points at the same height.

Hinge unitarity (deferred). The statement “ $|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\Re u = 1$ ” is proved in Part II (Hinge–Unitarity). We do not use it here.

2) Centered a –lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\bar{v})}$.

Nontrivial height. A “nontrivial height” $m > 0$ means: m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + im/2$. The reduction shows that whenever such an m occurs, the associated tilt must satisfy $a(m) = 0$.

Tilt at height m . At fixed $m > 0$, set

$$U_R(m; a) = 1 + a + im, \quad U_L(m; a) = 1 - a + im, \quad a \in [0, 1).$$

In the centered frame, the “dial points” are $\pm(a + im)$. The partner map J swaps $U_R \leftrightarrow U_L$.

Quartet. Conjugation (top \leftrightarrow bottom) and FE reflection generate the quartet $\{1 \pm a \pm im\}$ at height m .

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1 + im$ and its slope is $\Im u / \Re u = m/1 = m$. Rescaling to any frame $s = (f/2)u$ preserves the slope:

$$\frac{\Im s}{\Re s} = \frac{(f/2)m}{f/2} = m.$$

Thus $\{m_k\}$ simultaneously records the imaginary ordinates of the nontrivial zeros and their origin–through slopes in every modulated frame—provided the per–height collapse holds.

4) Height–local reduction of RH

Fix a nontrivial height $m > 0$ and write $U_R = 1 + a + im$, $U_L = 1 - a + im$. The following are purely algebraic and equivalent:

- (PHU–1) Column equality: $\Re U_R = \Re U_L \iff a = 0$.
- (PHU–2) Ray (slope) lock: $\Im U_R / \Re U_R = \Im U_L / \Re U_L$, i.e. $m/(1+a) = m/(1-a) \iff a = 0$.
- (PHU–3) Hinge form: $U_R = U_L = 1 + im$.

Reduction target. RH \iff for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse; nothing from this Guide is used there.

5) Box alignment and hand-off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta := \eta \alpha / (\log m)^2, \quad \eta \in (0, 1).$$

When $\alpha = \pm a$, the dial points $\pm(a + im)$ lie on the box's horizontal centerline.

What Part II does. Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three-lines tools, Stirling-class envelopes, explicit control of ζ'/ζ away from zeros), Part II shows that any off-axis quartet forces a boundary lower bound larger than an explicit upper bound, hence $a(m) = 0$.

No circularity. The analytic proof is logically independent of this Guide.

6) Parity gating and selection devices (interpretive only)

Gating from the non-completed FE. In the width-2 frame the non-completed FE reads

$$\zeta_2(u) = A_2(u) \zeta_2(2 - u), \quad A_2(u) = 2^{u/2} \pi^{u/2-1} \sin\left(\frac{\pi u}{4}\right) \Gamma\left(1 - \frac{u}{2}\right).$$

On the open strip $0 < \Re u < 2$ with $\Im u \neq 0$, the prefactor $A_2(u)$ is nonzero and finite; its sine zeros (the “trivial ladder”) lie on the real axis only. Thus *inside the open strip only ζ_2 can vanish* (nontrivial zeros), while the *trivial class is confined to the real axis*. This is the basic “odd/even lane” picture: the odd (upper) lane can host nontrivial zeros; the even (real) lane hosts the trivial ladder.

Orthogonal split on the integer lattice. To model this dichotomy as a clean input-space symmetry, decompose any lattice signal $X : \mathbb{Z} \rightarrow \mathbb{C}$ via the orthogonal projectors

$$P_{\text{odd}}(n) = \frac{1 - \cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \frac{1 + \cos(\pi n)}{2},$$

so $X = P_{\text{odd}}X + P_{\text{even}}X$. We assign the nontrivial stream to odd slots (where $P_{\text{odd}} = 1$) and the trivial ladder to even slots (where $P_{\text{even}} = 1$). This mirrors the FE fact above without using it analytically.

7) Toolbox \rightarrow structural consequences (after the theorem)

The items below are not inputs to the analytic proof. After Part II proves $a(m) = 0$ for all nontrivial heights, they become *Structural Corollaries* describing the collapsed geometry and its lattice faces. (*Explicit formulas and brief proofs are recorded as corollaries in Part III; Part I intentionally omits them and they are not inputs to Part II.*)

- Pre-collapse columns (projector faces in the u -frame): right/left templates place odd-slot samples $x \pm im_k$ and the even ladder $-4(\cdot)$ via $P_{\text{odd}}, P_{\text{even}}$; scaffolding, not assumptions.
- Collapsed canonical stream $U(n)$: when per-height collapse holds ($x = 1$ on odd slots), the two columns coincide; parity form (via $P_{\text{odd}}, P_{\text{even}}$) and an equivalent trigonometric form (via $\sin^2(\pi n/2), \cos^2(\pi n/2)$).
- Single-frequency collapse (cosine face): a two-parameter cosine form $U(n) = (c + d) + (c - d) \cos(\pi n)$ recovers the same stream; c, d simple in the odd-indexer $k(n)$.
- Self-indexed recurrence (no explicit k): a short recurrence for $U(n)$ pulls the needed odd index from the previous even sample; encodes the collapsed geometry without an explicit $k(n)$.

- Curvature extractor & the $\zeta(2)$ disguise: the discrete second-difference of the imaginary part at even indices recovers m_j and admits an odd-square convolution form normalized by $\zeta(2)$.
- Seed \rightarrow rectifier \rightarrow physical streams: two-carrier seeds rectify under a mod-4 factor to yield the physical stream $S_f(n)$ proportional to $U(n)$; pre-collapse faces scale analogously.

8) Implications and one-sentence hand-off

The width-2 organization centralizes symmetry at $\Re u = 1$; the centered a -lens isolates the single per-height degree of freedom; parity-orthogonal scaffolding separates the nontrivial stream from the ladder without entering the proof. With these definitions, RH reduces to: for every nontrivial height $m > 0$, $a(m) = 0$.

Hand-off. Part II now proves this per-height collapse by a boundary-only contradiction on aligned boxes; nothing from this Guide is used in that proof.

Part II — Analytic Core (self-contained; boundary-only)

Part III — Structural Corollaries (after the main theorem)

Standing assumption for this part. Assume the *Main Theorem (Part II)*: for every nontrivial height $m > 0$, the per-height tilt satisfies $a(m) = 0$. Equivalently, in the universal strip $0 < \Re u < 2$ each upper partner has real part 1.

Parity projectors and indexer. On \mathbb{Z} set

$$P_{\text{odd}}(n) = \frac{1 - \cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \frac{1 + \cos(\pi n)}{2}, \quad k(n) = \frac{n}{2} + \frac{1 - \cos(\pi n)}{4}.$$

Then $k(2j - 1) = j$ and $k(2j) = j$ for $j \in \mathbb{Z}$.

Corollary 0.1 (Canonical columns). *For $x \in (0, 2)$ define*

$$U_{\text{R}}(x, n) = P_{\text{odd}}(n) (x + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n),$$

$$U_{\text{L}}(x, n) = P_{\text{odd}}(n) (2 - x + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n).$$

Let $x = 1 \pm a$ with $a \in (-1, 1)$. Under the standing assumption $a(m) = 0$ at each nontrivial height, the canonical choice $x = 1$ yields $U_{\text{R}}(1, n) = U_{\text{L}}(1, n)$ for all $n \in \mathbb{Z}$.

Proof. On odd $n = 2j - 1$, both columns give $1 + i m_j$. On even $n = 2j$, both give $-4((2j) + 1 - k(2j)) = -4(j + 1)$. \square

Corollary 0.2 (Collapsed canonical stream: $U(n)$). *Define*

$$U(n) := U_{\text{R}}(1, n) = U_{\text{L}}(1, n) = P_{\text{odd}}(n) (1 + i m_{k(n)}) - 4(n + 1 - k(n)) P_{\text{even}}(n).$$

Then $U(2j - 1) = 1 + i m_j$ and $U(2j) = -4(j + 1)$ for all $j \in \mathbb{Z}$.

Proof. Immediate from $P_{\text{odd}}(2j - 1) = 1$, $P_{\text{even}}(2j - 1) = 0$ and $P_{\text{odd}}(2j) = 0$, $P_{\text{even}}(2j) = 1$, with $k(2j - 1) = k(2j) = j$. \square

Corollary 0.3 (Squared-trig form). *Using $\sin^2(\pi n/2) = P_{\text{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\text{even}}(n)$,*

$$U(n) = \sin^2\left(\frac{\pi n}{2}\right) (1 + i m_{k(n)}) - 4(n + 1 - k(n)) \cos^2\left(\frac{\pi n}{2}\right).$$

Proof. Substitute the trigonometric projector identities into Cor. 0.2. \square

Corollary 0.4 (Single-frequency collapse). *There exist $c(n) \in \mathbb{R}$ and $d(n) \in \mathbb{C}$ such that*

$$U(n) = (c + d) + (c - d) \cos(\pi n), \quad c = 2(k(n) - n - 1), \quad d = \frac{1 + i m_{k(n)}}{2}.$$

Proof. From Cor. 0.3, use $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ with $\theta = \pi n/2$, and collect constant and $\cos(\pi n)$ parts. \square

Corollary 0.5 (Self-indexed recurrence). *With initial values $U(0) = -4$ and $U(1) = 1 + i m_1$, for all $n \geq 2$,*

$$U(n) = P_{\text{odd}}(n) \left(1 + i m_{-U(n-1)/4}\right) - P_{\text{even}}(n) \left(U(n-2) + 4(n+1)\right).$$

Proof. On odd $n = 2j - 1$, we have $-U(n-1)/4 = -(U(2j-2))/4 = j$, so the odd sample is $1 + i m_j$. On even $n = 2j$, the update gives $-4(j+1)$. This matches Cor. 0.2. \square

Corollary 0.6 (Curvature extractor & $\zeta(2)$ disguise). *Let $F(n) := \Im U(n)$ and let Δ^2 denote the discrete second difference in n . Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$F(2j-1) = \frac{2}{\pi^2} \Im(\Delta^2 U(2j)) = \frac{1}{3\zeta(2)} \Im(\Delta^2 U(2j)) = \frac{2}{3\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{(2(j-\ell)+1)^2} = m_j.$$

Proof. Apply Δ^2 to the trigonometric form in Cor. 0.3; at even sites only the squared factors contribute. Use $\zeta(2) = \pi^2/6$ to rewrite $2/\pi^2 = 1/(3\zeta(2))$. The odd-square kernel arises from the sampled second difference of $\cos^2(\pi n/2)$. \square

Corollary 0.7 (Seed \rightarrow rectifier \rightarrow physical streams). *Let $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$ and define the even-ladder duplicator*

$$E(n) := 2(n+1-k(n)) \quad \text{so that} \quad E(2j) = 2(j+1), \quad E(2j-1) = 0.$$

For $f > 0$ and gain $\lambda \in \mathbb{R}$, set the seed

$$s_f(n) = f\lambda \left[\sin\left(\frac{\pi n}{2}\right) (1 + i m_{k(n)}) - E(n) \cos\left(\frac{\pi n}{2}\right) \right].$$

On \mathbb{Z} we have $\chi_4(n) \sin(\pi n/2) = P_{\text{odd}}(n)$ and $\chi_4(n) \cos(\pi n/2) = P_{\text{even}}(n)$, hence

$$\chi_4(n) s_f(n) = f\lambda \left[P_{\text{odd}}(n) (1 + i m_{k(n)}) - E(n) P_{\text{even}}(n) \right].$$

Choosing $\lambda = \frac{1}{2}$ and recalling $E(n) = 2(n+1-k(n))$ yields the physical stream

$$S_f(n) := \frac{f}{2} U(n).$$

Proof. The mod-4 identities map sine/cosine to parity projectors on \mathbb{Z} ; the even ladder $E(n)$ then gives the correct $-4(j+1)$ at $n = 2j$. Setting $\lambda = \frac{1}{2}$ collapses to $S_f = (f/2) U$. \square

Corollary 0.8 (Slope invariance under frame modulation). *If $u = 1 + i m$ then $s = (f/2) u$ satisfies $\frac{\Im s}{\Re s} = m$, independent of $f > 0$.*

Proof. $(\Im s)/(\Re s) = ((f/2)m)/(f/2) = m$. \square