A Width–2 Boundary Program for Excluding Off–Axis Quartets with a Certified Tail Criterion and a Finite–Height Front–End (v36)

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Abstract

This document is a truth–latching guardrail build (v36) of the width–2 boundary program. It records three proof–grade obstructions that prevent drift:

(i) *Exponent budget.* Under the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$ and the pointwise collar bound $\sup_{\partial B}\left|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}\right| \ll N_{\mathrm{loc}}(m)/(\kappa\,\delta)$, uniform η–shrinking absorption requires a residual δ–power $p-\theta \geq \frac{1}{2}$ (Theorem 10.13); with the pointwise endpoint one has $\theta = 1$.

(ii) *UE scaling NO–GO.* For pointwise/sup endpoints $\sup_{\partial B}|E'/E|$ with shape–only constants, the upper–envelope prefactor cannot have exponent $p > 1$ (Lemma 10.15); in particular the proved pointwise bound has $p = 1$.

(iii) *Forcing NO–GO.* In the single–box forcing architecture, the available forcing margin is $O(1)$ and cannot grow with m (Lemma 8.2).

Accordingly, the former η–absorption closure route based on the pointwise/sup upper–envelope plus collar is formally discarded (Appendix A). The manuscript's main unconditional output is the certified tail *criterion* at each height m (Theorem 11.1) together with a finite–height front–end. The remaining analytic frontier is reframed as **S5**: a non–pointwise upper–envelope redesign that controls the dial deviation $D_B(W)$ while avoiding the pointwise δ^{-1} collar blow–up.

For global hygiene, v35 corrected completion/holomorphy; v36 retains this convention: the working function is the entire width–2 completion $\Xi_2(u) := \xi(u/2)$, recentered as $E(v) := \Xi_2(1+v)$, so all uses of "E is entire" are literally true.

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Executive Proof Status

Status (v36): v36 is a *guardrail* build: it codifies the decisive NO–GO constraints and the S5 acceptance gates (Remark 12.1) so that the program cannot drift back to the (now invalid) v33 absorption narrative or into naive endpoint redesigns. The main unconditional output remains the *tail criterion family* (Theorem 11.1) together with a finite–height front–end (Definition 4.1). No claim of uniform tail closure is made in this version.

Proof–grade NO–GO constraints (now explicit):

1. *Exponent budget obstruction.* Under the nominal scaling $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$, a local/collar blow–up exponent θ and UE exponent p must satisfy $p - \theta \geq \frac{1}{2}$ for uniform η–shrinking closure (Theorem 10.13 and Theorem 12.2). In the present pointwise collar bound one has $\theta = 1$.

2. *UE scaling obstruction (pointwise/sup).* With the pointwise/sup endpoint $\sup_{\partial B} |E'/E|$ and shape–only constants, one cannot achieve any UE exponent $p > 1$ (Lemma 10.15). The proved pointwise bound has $p = 1$.

3. *Forceability gate + constant-limited forcing.* The current single–box forcing architecture forces only the dial deviation $D_B(W)$ and the available forcing margin is $O(1)$ (Lemma 8.2). Any S5 endpoint redesign is invalid unless it dominates $D_B(W)$ or supplies a new forcing lemma (Remark 12.5).

4. *Boundary modulus has no converse.* The condition $|W| = 1$ on ∂B does not exclude interior zeros; Bridge 1 is strictly one–directional (Remark 9.4).

5. *Absolute $L^r(\partial B)$ log-derivative endpoints are dead.* In any absolute L^r endpoint class, the best possible UE exponent collapses to $p(r) = 1 - 1/r$ (Lemma 12.6), while the collar/local split has the same exponent $\theta(r) = 1 - 1/r$ (Proposition 12.7), so $p(r) - \theta(r) = 0$ and the local term is δ–inert.

6. *Projection endpoints are dead under current forcing.* Endpoints that annihilate the local kernel span cannot control the forced dial deviation $D_B(W)$ without a new forcing link (Lemma 12.9).

Immediate consequence: the v33/v32 style η–absorption closure route based on the pointwise/sup upper envelope and the collar is *formally discarded* in v35 and remains discarded in v36 (Appendix A). Any future closure must change the envelope endpoint and/or the local interface.

Completion / holomorphy hygiene (fixed): the working function is the entire width–2 completion $\Xi_2(u) := \xi(u/2)$ and $E(v) := \Xi_2(1+v)$ (Section 1). All uses of "E is entire" are now literally correct.

Open proof–grade blockers (v36):

1. **S5 UE redesign (primary frontier).** Replace the pointwise/sup UE endpoint by a non–pointwise functional that controls the same dial deviation $D_B(W)$ while avoiding the δ^{-1} collar blow–up; see the S5 acceptance criterion (Remark 12.1) and the baseline NO–GO filters (Subsection 12.1) (Section 12). (G–4/G–5 in the prior register.)

2. *Residual envelope ledger.* Lemma 7.2 still imports a standard RH–free bound for ζ'/ζ with local–zero subtraction; this must be proved in–text or cited in a referee–acceptable way with explicit constants. (G–1/G–12.)

3. *Front–end dependence.* The finite–height hypothesis remains an external input (Appendix C). (G–11.)

Reproducibility posture (v36): numerical artifacts remain an *audit harness* only (Appendix B). The v36 repro pack is hardened to record endpoint metadata (functional class + exponent budget parameters + forceability mode). Missing fields render a certificate invalid (Appendix B), so future redesigns cannot silently mismatch the forcing chain.

Part I
Reader's Guide / Definitions and Reduction

1 Width–2 normalization

Let s denote the usual complex variable for $\zeta(s)$. We pass to the width–2 coordinate

$$u := 2s, \qquad \zeta_2(u) := \zeta(u/2).$$

Define the width–2 completed zeta

$$\Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\zeta(u/2).$$

Then Λ_2 is *meromorphic* (simple poles at $u = 0$ and $u = 2$) and satisfies the functional equation

$$\Lambda_2(u) = \Lambda_2(2 - u).$$

Define the entire completion

$$\Xi_2(u) := \xi(u/2) = \frac{u(u-2)}{8}\,\Lambda_2(u),$$

so that Ξ_2 is entire and obeys $\Xi_2(u) = \Xi_2(2 - u)$.

We recenter at $u = 1$:

$$v := u - 1, \qquad E(v) := \Xi_2(1 + v).$$

Then E is entire and satisfies the evenness relation

$$E(v) = E(-v),$$

and complex conjugation gives $E(\overline{v}) = \overline{E(v)}$.

Remark 1.1 (Zeros). The zeros of $\Xi_2(u) = \xi(u/2)$ are exactly the *nontrivial* zeros of $\zeta(s)$ under the map $u = 2s$, with multiplicity. All boxes used in the tail program lie at heights $m \geq 10$, so the only zeros that can occur in the relevant local windows are nontrivial zeros.

2 Heights and horizontal displacement (RH–free)

Let $\rho = \beta + i\gamma$ be any nontrivial zero of $\zeta(s)$ (no assumption on β). In width–2 we write

$$u_\rho := 2\rho = (1 + a) + im, \qquad a := 2\beta - 1 \in (-1, 1), \qquad m := 2\gamma > 0. \tag{1}$$

Thus RH is equivalent to $a = 0$ for every nontrivial zero.

3 Quartet symmetry in width–2

The functional equation and conjugation imply that any off–axis zero with parameters (a, m) produces a quartet

$$\{\, 1 \pm a \pm im \,\} \subset \{u \in \mathbb{C} : \Xi_2(u) = 0\}. \tag{2}$$

In the centered v–coordinate this becomes $\{\pm a \pm im\} \subset \{v \in \mathbb{C} : E(v) = 0\}$.

4 Finite-height front-end after lowering the tail anchor

Once the tail anchor is lowered to $m_\star$, the analytic tail argument covers all $m \geq m_\star$. The remaining region corresponds to classical heights

$$0 < \mathrm{Im}(s) < H_0 := m_\star/2. \tag{3}$$

In v31 we take $m_\star = 10$, hence $H_0 = 5$.

Definition 4.1 (Front-end statement)**.** We say that *RH holds up to height* H_0 if every nontrivial zero $\rho = \beta + i\gamma$ with $0 < \gamma \leq H_0$ satisfies $\beta = 1/2$.

Remark 4.2 (How v31 discharges the front-end)*.* The required statement for v31 is RH up to height $H_0 = 5$. This is a tiny special case of published rigorous verifications of RH to enormous heights. For example, Platt–Trudgian prove RH for all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ using interval arithmetic, which immediately implies RH up to $H_0 = 5$. Appendix C records this discharge in a pinned JSON file.

Part II
Self-Contained Boundary Program and Tail Closure

5 Aligned boxes and the $\delta(m)$ scale

Let $m > 0$ and $\alpha \in (0, 1]$. Fix a parameter $\eta \in (0, 1)$ and define the *nominal* box scale

$$\delta_0 = \delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}. \tag{4}$$

We will work with aligned boxes $B(\alpha, m, \delta)$ at scales $0 < \delta \leq \delta_0$. By default one may take $\delta = \delta_0$, but later (Definition 10.5) we allow shrinking δ to enforce κ–admissibility; this is non-circular and monotone-safe (Lemmas 10.6 and 11.2).

Define the (width–2) box centered at $\alpha + im$ by

$$B(\alpha, m, \delta) := \{\, v \in \mathbb{C} : |\mathrm{Re}\, v - \alpha| \leq \delta,\ |\mathrm{Im}\, v - m| \leq \delta \,\}. \tag{5}$$

We will also use the symmetric dial centers $v_\pm := \pm\alpha + im$.

6 Local factor and finiteness

For a fixed $m > 0$, let

$$Z(m) := \{\, \rho : E(\rho) = 0,\ |\mathrm{Im}\, \rho - m| \leq 1 \,\} \tag{6}$$

(zeros counted with multiplicity). Define the local zero factor and residual:

$$Z_{\mathrm{loc}}(v) := \prod_{\rho \in Z(m)} (v - \rho)^{m_\rho}. \tag{7}$$

$$F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{8}$$

Lemma 6.1 (Finiteness of Z_{loc})**.** *For each fixed $m > 0$ the set $Z(m)$ is finite; hence Z_{loc} is a finite product and F is meromorphic globally and analytic in any neighborhood of $\partial B(\alpha, m, \delta)$ that contains no zeros of E.*

Proof. Nontrivial zeros of ζ satisfy $0 < \beta < 1$, hence in the v–coordinate one has $\operatorname{Re} v \in (-1, 1)$ for all nontrivial zeros. Therefore the set $\{|\operatorname{Im} v - m| \le 1\} \cap \{|\operatorname{Re} v| \le 1\}$ is compact. Since E is entire and its zeros are discrete, only finitely many zeros can lie in this compact set. □

7 Residual envelope bound and the constants ledger

Remark 7.1 (Constant gate for the residual term (what is and is not assumed)). The tail criterion uses a bound of the form

$$\sup_{v \in \partial B(\alpha,m,\delta)} \left| \frac{F'(v)}{F(v)} \right| \ \le\ C_1 \log m + C_2,$$

with constants that must be (i) unconditional (not RH-equivalent) and (ii) uniform in $(\alpha, \delta, \eta, \kappa)$ once $m \ge 10$ and $0 < \alpha \le 1$. The proof below reduces this to standard RH-free bounds for ζ'/ζ in the critical strip with local-zero subtraction, plus a Stirling-type bound for Γ'/Γ.

Lemma 7.2 (Residual envelope inequality (δ-uniform))**.** *Fix $m \ge 10$ and $\alpha \in (0, 1]$. Let $\eta \in (0, 1]$ and set the nominal width $\delta_0 := \eta\alpha/(\log m)^2$. Let $\delta \in (0, \delta_0]$ and set $B := B(\alpha, m, \delta)$.*

Define E, Z_{loc} and $F := E/Z_{\mathrm{loc}}$ as in §6 (equations (7)–(8)). Assume boundary-contact *on ∂B (i.e. $E \ne 0$ on ∂B; hence F is holomorphic on a neighborhood of ∂B). Then there exist absolute constants $C_1, C_2 > 0$ (independent of $m, \alpha, \delta, \eta, \kappa$ and of the zero configuration) such that*

$$\sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right| \ \le\ C_1 \log m + C_2.$$

Proof sketch with explicit dependency control. Write $u := 1 + v$ and $s := u/2 = (1 + v)/2 = \sigma + it$. For $v \in \partial B(\alpha, m, \delta)$ we have $\operatorname{Re}(s) \in [0, 1]$ and

$$\operatorname{Im}(s) = \frac{\operatorname{Im}(v)}{2} \in \Big[\frac{m}{2} - \frac{\delta}{2}, \frac{m}{2} + \frac{\delta}{2}\Big].$$

Since $m \ge 10$ and $\delta \le \delta_0 \le 1/(\log 10)^2 < 1/5$, we have $\operatorname{Im}(s) \asymp m$ uniformly in δ.

1) Log-derivative identity in the s-frame. From $\Xi_2(u) = \frac{u(u-2)}{8}\Lambda_2(u)$ and $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ we obtain, for $u = 1 + v$,

$$\frac{E'(v)}{E(v)} = \frac{\Xi_2'(u)}{\Xi_2(u)} = \Big(\frac{1}{u} + \frac{1}{u-2}\Big) \ -\ \frac{1}{4}\log\pi \ +\ \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{u}{4}\Big) \ +\ \frac{1}{2}\frac{\zeta'}{\zeta}(s), \qquad (u = 1 + v,\ s = u/2).$$

Since $u = 1 + v$ has $\operatorname{Im}(u) = m \ge 10$, the completion terms $(1/u + 1/(u - 2))$ are $O(1/m)$ on ∂B and are absorbed into the absolute constants in the bound.

Moreover, since $v = 2s - 1$, the local factor derivative satisfies

$$\frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)} = \sum_{\rho \in Z(m)} \frac{m_\rho}{v - \rho} = \frac{1}{2} \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s},$$

where $Z_s(m)$ denotes the corresponding multiset of nontrivial zeros $\rho_s = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma - \frac{m}{2}| \le \frac{1}{2}$.

Therefore

$$\frac{F'(v)}{F(v)} = \frac{E'(v)}{E(v)} - \frac{Z'_{\rm loc}(v)}{Z_{\rm loc}(v)} = -\frac{1}{4}\log\pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{1+v}{4}\Big) + \frac{1}{2}\Bigg(\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)} \frac{m_{\rho_s}}{s-\rho_s}\Bigg).$$

2) RH-free residual bound for ζ'/ζ with local-zero subtraction. A standard "local-zero decomposition" (unconditional) asserts that there exist absolute constants A_ζ, B_ζ such that for $0 \le \sigma \le 1$ and $t \ge 5$,

$$\Big|\frac{\zeta'}{\zeta}(\sigma+it) - \sum_{|\gamma-t|\le 1} \frac{1}{(\sigma+it)-\rho}\Big| \;\le\; A_\zeta \log(t+2) + B_\zeta. \tag{$\star$}$$

(For a self-contained route, ($\star$) can be derived from the Hadamard product for $\xi(s)$ plus a Riemann–von Mangoldt bound for $N(T)$; otherwise cite a standard reference.)

For $v \in \partial B$ we have $|t - \frac{m}{2}| \le \delta/2 < 1/10$, hence every zero in $Z_s(m)$ satisfies $|\gamma - t| \le 1$ and is included in the sum in ($\star$). Thus

$$\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)} \frac{1}{s-\rho_s} = \Big(\frac{\zeta'}{\zeta}(s) - \sum_{|\gamma-t|\le 1} \frac{1}{s-\rho}\Big) + \sum_{\substack{|\gamma-t|\le 1\\ |\gamma-\frac{m}{2}|>1/2}} \frac{1}{s-\rho}.$$

In the remaining sum we have $|\gamma - t| \ge 1/2 - |t - \frac{m}{2}| \ge 2/5$, hence $|s-\rho| \ge 2/5$ and each term has modulus $\le 5/2$. The number of zeros with $|\gamma - t| \le 1$ is bounded by the manuscript's explicit local window majorant (Lemma 10.10) at height $\asymp m$, so this difference-of-windows sum is $\ll \log m$.

Combining these bounds yields absolute constants $A_{\rm res}, B_{\rm res}$ such that

$$\Big|\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s\in Z_s(m)} \frac{1}{s-\rho_s}\Big| \;\le\; A_{\rm res}\log m + B_{\rm res},$$

uniformly for all $v \in \partial B$ and all $\delta \in (0, \delta_0]$.

3) Gamma factor bound (Stirling, uniform in δ). For $z = (1+v)/4$ we have $\mathrm{Re}(z) \in [1/4, 3/4]$ and $|\,\mathrm{Im}(z)| \asymp m$. A uniform Stirling-type bound gives

$$\Big|\frac{\Gamma'}{\Gamma}(z)\Big| \le \log(|\,\mathrm{Im}(z)| + 2) + C_\Gamma \;\le\; \log(m+2) + C_\Gamma,$$

with an absolute constant C_Γ.

4) Conclusion. Insert the bounds from (2)–(3) into the identity in (1), and absorb harmless constants into (C_1, C_2). All constants are independent of $(\alpha, \delta, \eta, \kappa)$ because: (i) σ ranges over a fixed compact interval $[0,1]$, (ii) $t \asymp m$ with $m \ge 10$ uniformly for $\delta \le \delta_0$, and (iii) the difference-of-windows sum is controlled by Lemma 10.10, which is unconditional. □

Remark 7.3 (Hard gate / certificates (v36)). The tail harness in Appendix B uses explicit numerical interval enclosures for the constant ledger (e.g. $C_1, C_2, C_{\rm up}, C''_h, \kappa$) stored in `v36_repro_pack/v36_constants_m10.j`
It evaluates the tail inequality for a pinned parameter choice and records the UE exponent p explicitly. This is an *audit harness* only: it does not certify that the constants file is correct, and it does not, by itself, yield a uniform tail closure. An unconditional proof therefore still requires a referee-acceptable certification of the analytic constant ledger, and a resolution of the UE–Gate (Remark 10.12).

8 Short-side forcing

Assume an off-axis pair at height m with displacement $a > 0$ exists. On an aligned box with $\alpha = a$, the two upper zeros in the centered v–plane are at $v = \pm a + im$. The pair factor

$$Z_{\text{pair}}(v) := (v - (a + im))(v - (-a + im)) \tag{9}$$

produces a large phase rotation on the near vertical side.

Lemma 8.1 (Short-side forcing lower bound). *Let $I_+ := \{\alpha + iy : |y - m| \le \delta\}$ with $|\alpha - a| \le \delta$. Then*

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2\arctan\left(\frac{\delta}{|\alpha - a|}\right) + 2\arctan\left(\frac{\delta}{\alpha + a}\right) \ge \frac{\pi}{2}. \tag{10}$$

Lemma 8.2 (Single-box forcing is constant-limited). *In the forcing setup of Lemma 8.1, the total phase variation of the pair factor along I_+ satisfies*

$$\Delta_{I_+} \arg Z_{\text{pair}} \le 2\pi,$$

uniformly in the height m. Consequently the forcing constant c appearing in the tail inequality (Theorem 11.1) is an absolute constant, independent of m; in particular the forcing side cannot grow like $\log m$ (or any unbounded function of m) as $m \to \infty$.

Proof. On $I_+ = \{\alpha + iy : \ |y - m| \le \delta\}$ one has

$$Z_{\text{pair}}(\alpha + iy) = \big((\alpha - a) + i(y - m)\big)\big((\alpha + a) + i(y - m)\big).$$

Along $y \in [m - \delta, m + \delta]$ the argument of each linear factor varies by at most π (it is an arctan function whose range lies in an interval of length $\le \pi$), so the argument of the product varies by at most 2π, uniformly in m. The forcing chain converts a fixed positive portion of $\Delta_{I_+} \arg Z_{\text{pair}}$ into the constant c with fixed conversion scalars, so c is necessarily $O(1)$. $\square$

9 Outer factorization and the inner quotient (Bridge 1)

We work on a fixed box $B = B(\alpha, m, \delta)$ and write B° for its interior. Assume boundary-contact: $E \ne 0$ on ∂B (this will be enforced later by κ–admissibility; see Definition 10.5 and Lemma 10.6).

Lemma 9.1 (Dirichlet outer factor on a box). *Let $B = B(\alpha, m, \delta)$ be the closed rectangle and B° its interior. Assume E is holomorphic on a neighborhood of $\overline{B}$ and $E \ne 0$ on ∂B. Then $\log|E| \in C(\partial B)$. Let $U \in C(\overline{B}) \cap \operatorname{Harm}(B^\circ)$ be the unique solution of the Dirichlet problem with boundary data $U|_{\partial B} = \log|E|$. Since B° is simply connected, there exists a harmonic conjugate V on B° (unique up to an additive constant) such that $U + iV$ is holomorphic on B°. Define*

$$G_{\text{out}}(v) := \exp(U(v) + iV(v)), \qquad v \in B^\circ.$$

Then G_{out} is holomorphic and zero-free on B°, satisfies $|G_{\text{out}}(v)| = e^{U(v)}$ for $v \in B^\circ$, and

$$\lim_{z \to \xi,\ z \in B^\circ} |G_{\text{out}}(z)| = |E(\xi)| \qquad (\xi \in \partial B).$$

Proof. Continuity of $\log|E|$ on ∂B follows from $E \ne 0$ on ∂B. Existence and uniqueness of U on a rectangle are standard. Since B° is simply connected, U admits a harmonic conjugate V on B°, unique up to an additive constant. The function $U + iV$ is holomorphic, hence so is $G_{\text{out}} = \exp(U + iV)$, and it is zero-free. Finally $|G_{\text{out}}| = e^U$ on B°, and by continuity of U on $\overline{B}$ we have $e^{U(\xi)} = |E(\xi)|$ on ∂B, yielding the boundary modulus identity in interior-limit form. $\square$

Define on B° the inner quotient

$$W(v) := \frac{E(v)}{G_{\mathrm{out}}(v)}.$$

Then W is holomorphic on B° and $|W| = 1$ on ∂B in the sense of interior limits in modulus.

Proposition 9.2 (Bridge 1: zero-free inner collapse)**.** *Assume the setup of Lemma 9.1 and define $W = E/G_{\mathrm{out}}$ on B°. If W is zero-free on B° (equivalently, E is zero-free on B°), then W is constant on B°; in fact $W \equiv e^{i\theta_B}$ for some $\theta_B \in \mathbb{R}$.*

Proof. Since W is zero-free on B° and G_{out} is zero-free, the function E is zero-free on B°. Because B° is simply connected, E admits a holomorphic logarithm on B°, so $\log|E|$ is harmonic on B°. By construction U is harmonic on B°, continuous on $\overline{B}$, and equals $\log|E|$ on ∂B. Thus $U - \log|E|$ is harmonic on B° with zero boundary values, so by Dirichlet uniqueness $U \equiv \log|E|$ on B°. Therefore for $v \in B^\circ$,

$$|W(v)| = \frac{|E(v)|}{|G_{\mathrm{out}}(v)|} = \frac{|E(v)|}{e^{U(v)}} = \frac{|E(v)|}{e^{\log|E(v)|}} = 1.$$

An analytic function of constant modulus on a connected open set is constant, hence $W \equiv e^{i\theta_B}$. □

Remark 9.3 (Boundary modulus convention)*.* Under boundary-contact, U extends continuously to ∂B and satisfies $U|_{\partial B} = \log|E|$. Hence $|G_{\mathrm{out}}| = |E|$ holds pointwise on ∂B as interior limits in modulus, and therefore $|W| = 1$ holds pointwise in modulus on ∂B. In boundary integral estimates this may be used in the a.e. sense without change.

Remark 9.4 (No converse: boundary modulus does not exclude interior zeros)*.* Lemma 9.1 implies that under boundary-contact the quotient $W := E/G_{\mathrm{out}}$ satisfies $|W| = 1$ on ∂B (in the interior boundary-limit sense of Remark 9.3). This condition alone does *not* imply that W is zero-free or constant on B°: nonconstant holomorphic functions on B° can have $|W| = 1$ on ∂B and still possess prescribed interior zeros (e.g. via conformal transport of finite Blaschke products from the unit disc). Thus Proposition 9.2 is strictly one-directional: the additional hypothesis "W is zero-free on B°" is essential.

10 Shape-only invariance and the envelope constants

Let $T(v) := (v - (\alpha + im))/\delta$, mapping ∂B affinely onto the fixed square boundary ∂Q with $Q = [-1, 1]^2$.

Lemma 10.1 (Shape-only invariance)**.** *Any constant arising solely from geometric or boundary-operator estimates on ∂B that are invariant under affine rescaling depends only on ∂Q and is independent of (α, m, δ).*

Proof. Under T, arclength scales by δ and tangential derivatives by $1/\delta$. After normalization, all purely geometric quantities and operator norms reduce to fixed quantities on ∂Q. □

Lemma 10.2 (Boundary-to-center evaluation in L^2 (sharp $\delta^{-1/2}$))**.** *Let $B = B(\alpha, m, \delta)$ be a box and let v_0 be its center. Let u be harmonic on B° and assume its boundary trace lies in $L^2(\partial B, ds)$. Then, writing $P_B(v_0, \xi) = d\omega^B_{v_0}/ds(\xi)$ for the Poisson kernel of B at v_0,*

$$|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)}\, \|u\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q,

$$\|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} = \delta^{-1/2} \, \|P_Q(0, \cdot)\|_{L^2(\partial Q, ds)}.$$

In particular the exponent $\delta^{-1/2}$ is sharp (witnessed by the constant harmonic function $u \equiv 1$).

Proof. For harmonic u on B° with L^2 trace on ∂B, the Poisson representation gives

$$u(v_0) = \int_{\partial B} u(\xi) \, d\omega_{v_0}^B(\xi) = \int_{\partial B} u(\xi) \, P_B(v_0, \xi) \, ds(\xi).$$

Cauchy–Schwarz yields $|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2} \|u\|_{L^2}$.

For the scaling: under T, arclength scales by $ds = \delta \, ds_Q$ and Poisson kernels scale by $P_B(v_0, \xi) = \delta^{-1} P_Q(0, T(\xi))$. Hence

$$\int_{\partial B} P_B(v_0, \xi)^2 \, ds(\xi) = \int_{\partial Q} \delta^{-2} P_Q(0, \zeta)^2 \, \delta \, ds_Q(\zeta) = \delta^{-1} \int_{\partial Q} P_Q(0, \zeta)^2 \, ds_Q(\zeta),$$

giving $\|P_B(v_0, \cdot)\|_{L^2} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2}$.

Sharpness: for $u \equiv 1$ we have $|u(v_0)| = 1$ and $\|u\|_{L^2(\partial B)} = \sqrt{|\partial B|} \asymp \delta^{1/2}$, so the inequality forces $\|P_B(v_0, \cdot)\|_{L^2} \gtrsim \delta^{-1/2}$. $\square$

Lemma 10.3 (Upper envelope bound (outer-aligned form))**.** *Let $B = B(\pm a, m, \delta)$ be an aligned box and let G_{out} be the outer factor on B constructed from $\log |E|$ on ∂B (Section 9). Define the inner quotient*

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

Assume the boundary-contact convention: E has no zeros on ∂B (hence W has unimodular boundary values a.e.). For each sign $\pm$ let $v_\pm := \pm a + im$ and let $e^{i\varphi_0^\pm} \in \mathbb{T}$ be an $L^2(\partial B, ds)$-best constant phase,

$$e^{i\varphi_0^\pm} \in \arg\min_{|c|=1} \int_{\partial B} |W(v) - c|^2 \, ds(v).$$

Then there exists a shape-only *constant $C_{\text{up}} > 0$ (depending only on the normalized square $Q = [-1, 1]^2$) such that*

$$\sum_{\pm} \left| W(v_\pm) - e^{i\varphi_0^\pm} \right| \;\le\; 2 \, C_{\text{up}} \, \delta \sup_{v \in \partial B} \left| \frac{E'(v)}{E(v)} \right|. \tag{11}$$

One admissible explicit definition is

$$C_{\text{up}} := \Big(\sup_{\xi \in \partial Q} P_Q(0, \xi) \Big)^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2 \to L^2} \big),$$

where $P_Q(0, \xi) = d\omega_0^Q / ds(\xi)$ is the Poisson kernel of Q at the center 0 with respect to arclength on ∂Q, and $H_{\partial Q}$ is the boundary conjugation (Hilbert/Cauchy) operator on ∂Q.

Remark 10.4 (No residual proxying in the upper envelope)*.* Lemma 10.3 controls the inner quotient $W = E/G_{\text{out}}$ and therefore depends on $\sup_{\partial B} |E'/E|$. Residual bounds for $F = E/Z_{\text{loc}}$ control $\sup_{\partial B} |F'/F|$ and do *not* by themselves bound $\sup_{\partial B} |E'/E|$. Whenever the residual envelope is used to control dial deviation, it must be routed through the log-derivative split $E'/E = F'/F + Z_{\text{loc}}'/Z_{\text{loc}}$ (Lemma 10.7) together with the collar bound (Lemma 10.8), yielding Corollary 10.11.

Proof. Fix one sign and write $v_0 = v_\pm$ and $B = B(\pm a, m, \delta)$. We record the (RH-free) chain and indicate the scale factors explicitly.

1. **Evaluation from the boundary (harmonic measure; produces $\delta^{-1/2}$).** For any constant $c \in \mathbb{T}$, subharmonicity of $|W-c|^2$ implies

$$|W(v_0) - c|^2 \le \int_{\partial B} |W(\xi) - c|^2 \, d\omega_{v_0}^B(\xi) = \int_{\partial B} |W(\xi) - c|^2 \, P_B(v_0, \xi) \, ds(\xi),$$

so

$$|W(v_0) - c| \le \|P_B(v_0, \cdot)\|_{L^\infty(\partial B)}^{1/2} \, \|W - c\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q, Poisson kernels scale by $\|P_B(v_0, \cdot)\|_\infty^{1/2} = \delta^{-1/2} \, \|P_Q(0, \cdot)\|_\infty^{1/2}$.

2. **Poincaré/Wirtinger on ∂B (produces δ).** For the L^2-best constant $c = e^{i\varphi_0^\pm}$ and $|\partial B| = 8\delta$, periodic Poincaré on a loop of length 8δ gives

$$\|W - c\|_{L^2(\partial B)} \le \frac{|\partial B|}{2\pi} \, \|\partial_s W\|_{L^2(\partial B)} = \frac{4\delta}{\pi} \, \|\partial_s W\|_{L^2(\partial B)}.$$

3. **Outer factor control (no δ; uses bounded boundary conjugation).** Write $\log G_{\text{out}} = U + i\widetilde{U}$ with $U|_{\partial B} = \log|E|$ and $\widetilde{U} = H_{\partial B} U$. Differentiating tangentially, $\partial_s \log G_{\text{out}} = \partial_s U + i \, H_{\partial B}(\partial_s U)$. Since $\log W = \log E - \log G_{\text{out}}$,

$$\|\partial_s \log W\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2 \to L^2}\big) \, \|\partial_s \log E\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2 \to L^2}\big) \, \Big\| \frac{E'}{E} \Big\|_{L^2(\partial B)}.$$

On ∂B we have $|W| = 1$ a.e., hence $|\partial_s W| = |\partial_s \log W|$.

4. **L^2 to sup (produces $\delta^{1/2}$).** Using $|\partial B| = 8\delta$,

$$\Big\| \frac{E'}{E} \Big\|_{L^2(\partial B)} \le \sqrt{|\partial B|} \, \sup_{\partial B} \Big| \frac{E'}{E} \Big| = \sqrt{8\delta} \, \sup_{\partial B} \Big| \frac{E'}{E} \Big|.$$

Combining the four steps yields

$$|W(v_0) - e^{i\varphi_0^\pm}| \le \|P_Q(0, \cdot)\|_\infty^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2 \to L^2}\big) \cdot \delta \sup_{\partial B} \Big| \frac{E'}{E} \Big|,$$

where we used the similarity invariance $\|H_{\partial B}\|_{L^2 \to L^2} = \|H_{\partial Q}\|_{L^2 \to L^2}$. Summing over $\pm$ gives (11). $\square$

10.1 Local factor split and collar control

Definition 10.5 (Collar-admissible aligned boxes)**.** Fix once and for all a collar parameter $\kappa \in (0, 1/10)$. An aligned box $B = B(\alpha, m, \delta)$ is called *κ–admissible* if

$$\operatorname{dist}\big(\partial B, \, \mathcal{Z}(E)\big) \ge \kappa\delta.$$

Given any nominal scale $\delta_0 > 0$ and any center, there exists some $0 < \delta \le \delta_0$ for which κ–admissibility holds (Lemma 10.6). Whenever a chosen box is not κ–admissible, we shrink δ until κ–admissibility holds. Moreover the assembled tail inequality is monotone-safe under such δ–shrinking (Lemma 11.2).

Lemma 10.6 (Existence of a κ–admissible shrink)**.** *Fix $\kappa \in (0, 1/10)$ and a center $v_0 \in \mathbb{C}$. For every $\delta_0 > 0$ there exists $\delta' \in (0, \delta_0]$ such that the closed box*

$$B(v_0, \delta') := \{\, v \in \mathbb{C} : \ \|v - v_0\|_\infty \le \delta' \,\}$$

satisfies

$$\operatorname{dist}\big(\partial B(v_0, \delta'),\ \mathcal{Z}(E)\big) \ge \kappa\delta'.$$

In particular, given (α, m) and nominal $\delta_0 = \eta\alpha/(\log m)^2$, one may always choose a scale $0 < \delta \le \delta_0$ for which $B(\alpha, m, \delta)$ is κ–admissible.

Proof. Zeros of the entire function E are isolated. Choose $\varepsilon > 0$ such that $\mathcal{Z}(E) \cap \{\, 0 < \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) = 0$) or such that $\mathcal{Z}(E) \cap \{\, \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) \ne 0$). Set $\delta' := \min\{\delta_0, \varepsilon/(1 + \kappa)\}$. Then every boundary point satisfies $\|v - v_0\|_\infty = \delta'$. Any zero $\rho \in \mathcal{Z}(E)$ is either $\rho = v_0$ (in which case $\operatorname{dist}(v, \rho) = \delta' \ge \kappa\delta'$) or satisfies $\|\rho - v_0\|_\infty \ge \varepsilon$ (in which case $\operatorname{dist}(v, \rho) \ge \varepsilon - \delta' \ge \kappa\delta'$). Therefore $\operatorname{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \ge \kappa\delta'$. □

Lemma 10.7 (Log-derivative decomposition)**.** *With Z_{loc} and F as in* (7) *and* (8)*, one has on any region where E and Z_{loc} are holomorphic and nonvanishing (in particular on ∂B under the boundary-contact convention)*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}.$$

Lemma 10.8 (Buffered local factor bound on ∂B)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible in the sense of Definition 10.5. Then*

$$\sup_{v \in \partial B} \left| \frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)} \right| \le \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta},$$

where $N_{\mathrm{loc}}(m)$ counts zeros of E in the local window used to define Z_{loc}, with multiplicity.

Lemma 10.9 (Local log-derivative bound in $L^2(\partial B)$)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible (Definition 10.5), and let Z_{loc} be the local factor with local zero-count $N_{\mathrm{loc}}(m)$ as in Section 6. Then*

$$\left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^2(\partial B)} \le \frac{\sqrt{8}\, N_{\mathrm{loc}}(m)}{\kappa\, \delta^{1/2}}.$$

More generally, for any $1 \le r \le \infty$,

$$\left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^r(\partial B)} \le \frac{8^{1/r} N_{\mathrm{loc}}(m)}{\kappa\, \delta^{1-1/r}}.$$

Proof. Lemma 10.8 gives $\|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}\|_{L^\infty(\partial B)} \le N_{\mathrm{loc}}(m)/(\kappa\delta)$. Since $|\partial B| = 8\delta$, we have $\|f\|_{L^r(\partial B)} \le |\partial B|^{1/r} \|f\|_{L^\infty(\partial B)}$ for every $1 \le r \le \infty$, which yields the stated bounds. □

Lemma 10.10 (Explicit local window zero count)**.** *Let $N(T)$ denote the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$, counted with multiplicity. Then for every $T \ge 5$,*

$$N(T + 1) - N(T - 1) \le 1.01 \log T + 17. \tag{12}$$

Consequently, for every $m \ge 10$,

$$N_{\mathrm{loc}}(m) \le 1.01 \log m + 17. \tag{13}$$

Proof. By [7, Theorem 1.1], for every $x \ge e$,

$$\Big| N(x) - \frac{x}{2\pi}\log\Big(\frac{x}{2\pi e}\Big)\Big| \le 0.10076\log x + 0.24460\log\log x + 8.08344.$$

Let $M(x) := \frac{x}{2\pi}\log(\frac{x}{2\pi e})$, so $M'(x) = \frac{1}{2\pi}\log(\frac{x}{2\pi})$. For $T \ge 5$ we have $\log(T \pm 1) \le \log(2T)$ and $\log\log x \le \log x$ for $x \ge e$, hence

$$N(T+1) - N(T-1) \le (M(T+1) - M(T-1)) + 2(0.10076 + 0.24460)\log(2T) + 2 \cdot 8.08344.$$

Moreover

$$M(T+1) - M(T-1) = \int_{T-1}^{T+1} M'(x)\,dx \le \int_{T-1}^{T+1} \frac{1}{2\pi}\log x\,dx \le \frac{1}{\pi}\log(2T).$$

Combining these bounds gives $N(T+1) - N(T-1) \le 1.00903\log T + 16.8663 \le 1.01\log T + 17$, establishing (12). Finally, in width–2 one has $m = 2T$. The local window $|\operatorname{Im}\rho - m| \le 1$ corresponds to $|\gamma - T| \le 1/2$ in the s–plane, so $N_{\mathrm{loc}}(m) = N(T + \frac{1}{2}) - N(T - \frac{1}{2}) \le N(T+1) - N(T-1)$, yielding (13). $\square$

Corollary 10.11 (Outer-aligned upper envelope in residual+local form)**.** *Let B be κ–admissible. Assume the residual envelope bound of Lemma 7.2, i.e.* $\sup_{\partial B}|F'/F| \le L(m) := C_1\log m + C_2$*. Then*

$$\sum_{\pm} |W(v_\pm) - e^{i\varphi_0^\pm}| \le 2C_{\mathrm{up}}\left(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa}\right) \le 2C_{\mathrm{up}}\left(\delta L(m) + \frac{1.01\log m + 17}{\kappa}\right).$$

Remark 10.12 (UE gate = exponent budget at the local interface). Lemma 10.3 is the *only* step in the envelope chain that generates a positive power of δ in front of a boundary log-derivative endpoint. Abstractly, suppose an upper-envelope mechanism yields, for some $p > 0$,

$$\sum_{\pm}\big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \le 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\Big|\frac{E'}{E}\Big|,$$

and suppose the collar/local split yields, for some $\theta > 0$,

$$\sup_{\partial B}\Big|\frac{E'}{E}\Big| \le L(m) \;+\; \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta^\theta}.$$

Then the local contribution in the envelope side scales as $\delta^{p-\theta}N_{\mathrm{loc}}(m)/\kappa$. Under the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$ and the unconditional majorant $N_{\mathrm{loc}}(m) \ll \log m$, uniform η–shrinking tail closure is possible only if

$$p - \theta \;\ge\; \tfrac{1}{2}$$

(Theorem 10.13).

In the *proved* pointwise/sup architecture one has $p = 1$ (Lemma 10.3) and $\theta = 1$ (Lemma 10.8), so $p - \theta = 0$ and the local term is δ–inert; η–shrinking cannot suppress it (Lemma 10.14). Moreover, within this same endpoint class, a strengthened exponent $p > 1$ is impossible with shape–only constants (Lemma 10.15). Thus the former η–absorption closure route based on the pointwise/sup UE endpoint is a formal NO–GO and is recorded as discarded (Appendix A).

Theorem 10.13 (Exponent budget for η–shrinking under $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$)**.** *Let $m \geq 10$, $\alpha \in (0,1]$ and $\eta \in (0,1]$, and set the nominal scale*

$$\delta_0(m,\alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

Assume that for all $0 < \delta \leq \delta_0(m,\alpha)$ one has:

(UE$_p$) (UE exponent) *for some $p > 0$,*

$$\mathrm{UE}(\delta) \;\leq\; 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\Big|\frac{E'}{E}\Big|;$$

(COL$_\theta$) (Collar/local exponent) *for some $\theta > 0$,*

$$\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;\leq\; L(m) \;+\; \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta^\theta},$$

with fixed $\kappa \in (0, 1/10)$;

(GROW) (Majorants) *$L(m) \leq A_L \log m + B_L$ and $N_{\mathrm{loc}}(m) \leq A_N \log m + B_N$ for all $m \geq 10$;*

(FORCE) (Forcing side) *the forcing-vs-envelope tail inequality has a fixed positive forcing constant $c > 0$ and only δ–helpful subtractive terms on the RHS.*

Then at $\delta = \delta_0(m,\alpha)$ one has the explicit bound

$$\mathrm{UE}(\delta_0) \;\leq\; 2C_{\mathrm{up}}\Big(\delta_0^p L(m) \;+\; \delta_0^{p-\theta}\frac{N_{\mathrm{loc}}(m)}{\kappa}\Big). \tag{BUDGET}$$

Moreover, uniform tail closure by η–shrinking *(i.e. there exists $\eta_\star > 0$ such that for every $\eta \leq \eta_\star$ the tail inequality holds for all $m \geq 10$) is possible only if*

$$p - \theta \;\geq\; \tfrac{1}{2}. \tag{B1}$$

Proof. Insert (COL$_\theta$) into (UE$_p$) at $\delta = \delta_0$ to obtain (BUDGET). At $\alpha = 1$ one has $\delta_0(m,1) = \eta/(\log m)^2$, so the local term behaves as

$$\delta_0^{p-\theta} N_{\mathrm{loc}}(m) \;\ll\; \Big(\frac{\eta}{(\log m)^2}\Big)^{p-\theta}\log m = \eta^{p-\theta}(\log m)^{1-2(p-\theta)}.$$

If $p-\theta < 1/2$ then $1-2(p-\theta) > 0$, so the local contribution grows without bound as $m \to \infty$, while the forcing side tends to the fixed constant c because all RHS corrections are δ–helpful and vanish as $\delta_0 \to 0$. Hence uniform tail closure is impossible. If $p - \theta \geq 1/2$ then the local contribution is uniformly bounded by $O(\eta^{p-\theta})$ and tends to 0 as $\eta \downarrow 0$, enabling uniform absorption once all constants are δ–uniform. $\square$

Lemma 10.14 (η–absorption obstruction under the pointwise UE exponent $p = 1$)**.** *Assume the hypotheses of Corollary 10.11. Then for every $\delta \leq \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$,*

$$\sum_{\pm}\big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \leq 2C_{\mathrm{up}}\Big(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa}\Big).$$

In particular, letting $\eta \downarrow 0$ (hence $\delta \downarrow 0$) only suppresses the residual term $\delta L(m)$; the local term $N_{\rm loc}(m)/\kappa$ does not *decay with η. Therefore any absorption-style closure that attempts to force the envelope side small by choosing η must additionally verify a* separate *inequality of the form*

$$\frac{2C_{\rm up}}{\kappa}\,N_{\rm loc}(m) \;<\; c$$

at the relevant anchor height(s), where c is the forcing constant in (14)*.*

Lemma 10.15 (UE scaling NO–GO for pointwise/sup endpoints)**.** *Assume an upper-envelope bound of the form*

$$\sum_{\pm}\bigl|W(v_\pm)-e^{i\varphi_0^\pm}\bigr| \;\le\; 2C_{\rm up}\,\delta^p\,\sup_{\partial B}\Bigl|\frac{E'}{E}\Bigr| \qquad (p>0),$$

where the constant $C_{\rm up}$ depends only on the normalized shape (Lemma 10.1) and is independent of δ. Then necessarily $p \le 1$. In particular, no pointwise/sup envelope mechanism with shape–only constants can yield any exponent $p>1$.

Proof. Under the affine rescaling $T(v)=(v-(\alpha+im))/\delta$, the boundary ∂B maps to the fixed square boundary ∂Q. If $\widetilde E(z):=E(T^{-1}(z))$, then by the chain rule

$$\frac{E'}{E}(T^{-1}(z)) = \frac{1}{\delta}\,\frac{\widetilde E'(z)}{\widetilde E(z)}.$$

Hence $\sup_{\partial B}|E'/E| = \delta^{-1}\sup_{\partial Q}|\widetilde E'/\widetilde E|$. The left-hand side of the upper-envelope bound is dimensionless (it is a sum of moduli of complex numbers), and under the normalization it may be $O(1)$ for admissible configurations on the fixed shape. Therefore the bound forces

$$O(1) \;\le\; 2C_{\rm up}\,\delta^{p-1}\,\sup_{\partial Q}\Bigl|\frac{\widetilde E'}{\widetilde E}\Bigr| \qquad \text{as } \delta\downarrow 0.$$

Since the normalized endpoint $\sup_{\partial Q}|\widetilde E'/\widetilde E|$ is not forced to blow up as $\delta \downarrow 0$ (it depends only on the normalized data), the factor δ^{p-1} cannot tend to 0. Thus $p-1\le 0$, i.e. $p \le 1$. ☐

Proof. The displayed bound is exactly Corollary 10.11 with the corrected UE exponent $p=1$. As $\eta \to 0$ one has $\delta_0 \to 0$ and hence $\delta L(m)\to 0$, while $N_{\rm loc}(m)/\kappa$ is unchanged. Since the forcing lower bound in the tail inequality tends to c as $\delta \downarrow 0$, the strict inequality requires the stated necessary condition at the anchor. ☐

10.2 Horizontal non-forcing budget in residual form

Definition 10.16 (Horizontal non-forcing phase budget)**.** Let $B=B(\pm a,m,\delta)$ be an aligned box and let $F=E/Z_{\rm loc}$ be the residual factor. Assume F is holomorphic and zero-free on a neighborhood of ∂B. Let $H_\pm$ denote the top and bottom edges of ∂B:

$$H_+ := \{x+i(m+\delta):\ x\in[\pm a-\delta,\pm a+\delta]\}, \qquad H_- := \{x+i(m-\delta):\ x\in[\pm a-\delta,\pm a+\delta]\}.$$

Define

$$\Delta_{\rm nonforce}(B) := \int_{H_+}|\partial_s \arg F|\,ds + \int_{H_-}|\partial_s \arg F|\,ds.$$

Lemma 10.17 (Horizontal budget (residual form; audit-grade)). *In the setting of Definition 10.16,*

$$\Delta_{\text{nonforce}}(B) \le 4\delta \sup_{v\in\partial B} \left|\frac{F'(v)}{F(v)}\right|.$$

Consequently, if $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2$*, then*

$$\Delta_{\text{nonforce}}(B) \le C_h'' \,\delta\, (\log m + 1), \qquad C_h'' := 4\max\{C_1,\, C_2\}.$$

Proof. On either horizontal edge, $|\partial_s \arg F| \le |F'/F|$ pointwise. Each edge has length 2δ, hence each integral is bounded by $2\delta \sup_{\partial B} |F'/F|$. Summing top and bottom gives the first inequality, and the second follows from $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2 \le \max\{C_1, C_2\}(\log m + 1)$. □

11 The explicit tail inequality (post-pivot)

For $m \ge 10$ we use the growth surrogate

$$L(m) := C_1 \log m + C_2,$$

with constants as in Lemma 7.2. For the local window term we use the explicit majorant from Lemma 10.10:

$$N_{\text{up}}(m) := 1.01 \log m + 17 \;\text{ so that }\; N_{\text{loc}}(m) \le N_{\text{up}}(m) \quad (m \ge 10).$$

For a parameter $\eta \in (0,1)$ and a dial displacement $\alpha \in (0,1]$ define the *nominal* scale

$$\delta_0 := \delta_0(m,\alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

Fix a collar parameter $\kappa \in (0, 1/10)$ as in Definition 10.5. For each (m,α) we choose any scale $0 < \delta \le \delta_0$ such that the aligned boxes $B = B(\pm\alpha, m, \delta)$ are κ–admissible; existence is guaranteed by Lemma 10.6. By Lemma 11.2, shrinking δ only helps in the tail inequality, so it is safe to treat δ_0 as the worst-case scale in one-height reductions.

Theorem 11.1 (Tail inequality (criterion form; pointwise UE exponent $p = 1$)). *Fix* $m \ge 10$ *and* $\eta \in (0,1)$*. Assume:*

1. *the forcing lemma producing the positive constant*

$$c_0 := \frac{3\log 2}{8\pi}, \qquad c := \frac{3\log 2}{16}, \qquad K_{\text{alloc}} := 3 + 8\sqrt{3};$$

2. *the residual envelope bound (Lemma 7.2) providing* C_1, C_2*;*
3. *the audit-grade horizontal budget bound (Lemma 10.17), giving a constant* C_h'' *independent of* (α, m, δ)*;*
4. *the explicit local window bound (Lemma 10.10) providing the majorant* $N_{\text{up}}(m) = 1.01 \log m + 17$*.*

Then for every $\alpha \in (0,1]$ *and every* κ*–admissible aligned box* $B = B(\pm\alpha, m, \delta)$*, absence of off–axis quartets at height* m *follows from the strict inequality*

$$2C_{\text{up}}\Big(\delta L(m) \;+\; \frac{N_{\text{up}}(m)}{\kappa}\Big) \;<\; c \;-\; \delta\Big(K_{\text{alloc}}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big). \tag{14}$$

Proof sketch / bookkeeping. The forcing side is unchanged from v31. The only post-pivot modification is on the upper-envelope side: Lemma 10.3 bounds dial deviation in terms of $\sup_{\partial B}|E'/E|$. Applying the log-derivative split (Lemma 10.7), the residual envelope for $\sup_{\partial B}|F'/F| \le L(m)$ (Lemma 7.2), and the collar bound $\sup_{\partial B}|Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}| \le N_{\mathrm{loc}}(m)/(\kappa\delta)$ (Lemma 10.8) yields

$$\sup_{\partial B}\left|\frac{E'}{E}\right| \le L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa\delta} \le L(m) + \frac{N_{\mathrm{up}}(m)}{\kappa\delta}.$$

Plugging this into Lemma 10.3 gives the left-hand side of (14). The right-hand side is the forcing lower bound, with the horizontal non-forcing term bounded by Lemma 10.17. □

Lemma 11.2 (Monotonicity under δ–shrinking)**.** *Fix $m \ge 10$, $\alpha \in (0,1]$, and constants $C_{\mathrm{up}}, \kappa, c, c_0, K_{\mathrm{alloc}}, C''_h, C_1, C$ Let $L(m) = C_1 \log m + C_2$ and $N_{\mathrm{up}}(m) = 1.01\log m + 17$. For $\delta \in (0,1]$ define*

$$\mathrm{LHS}(\delta) := 2C_{\mathrm{up}}\Big(\delta L(m) \;+\; \frac{N_{\mathrm{up}}(m)}{\kappa}\Big), \qquad \mathrm{RHS}(\delta) := c \;-\; \delta\Big(K_{\mathrm{alloc}}\, c_0\, L(m) + C''_h\,(\log m + 1)\Big).$$

Then $\mathrm{LHS}(\delta)$ *is (weakly) increasing in δ and* $\mathrm{RHS}(\delta)$ *is (weakly) decreasing. Consequently, if* $\mathrm{LHS}(\delta_0) < \mathrm{RHS}(\delta_0)$ *for some $\delta_0 \in (0,1]$, then* $\mathrm{LHS}(\delta) < \mathrm{RHS}(\delta)$ *holds for every $\delta \in (0,\delta_0]$.*

Proof. For $\delta > 0$, the map $\delta \mapsto \delta L(m)$ is increasing and the term $N_{\mathrm{up}}(m)/\kappa$ is independent of δ, hence $\mathrm{LHS}(\delta)$ is (weakly) increasing. The bracketed factor in $\mathrm{RHS}(\delta)$ is nonnegative and independent of δ, so $\mathrm{RHS}(\delta)$ decreases linearly in δ. □

Lemma 11.3 (Worst case in α is $\alpha = 1$ at the nominal scale)**.** *Fix $m \ge 10$ and $\eta \in (0,1)$. Define the nominal scale $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$. Consider the tail inequality* (14) *evaluated at $\delta = \delta_0(m,\alpha)$. Then the left-hand side is (weakly)* increasing *in $\alpha \in (0,1]$, while the right-hand side is (weakly)* decreasing*. Therefore it suffices to verify* (14) *at $\alpha = 1$ and $\delta = \delta_0(m,1)$. If one later shrinks $\delta \le \delta_0(m,\alpha)$ to enforce κ–admissibility, the inequality only becomes easier (Lemma 11.2).*

Proof. With $\delta = \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, the only α-dependence in the left-hand side is through the factor $\delta L(m)$, which is increasing in α, so the left-hand side increases. The right-hand side equals $c - \delta\cdot\Xi(m)$ for a nonnegative factor $\Xi(m)$ independent of α, hence it decreases. □

Remark 11.4 (No one-height reduction in m under the pointwise UE exponent $p = 1$). In v33, the (claimed) $\delta^{3/2}$ prefactor in Lemma 10.3 made the local contribution scale like $\delta^{1/2}N_{\mathrm{up}}(m)$ at the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, leading to an expression essentially independent of m and enabling a one-height reduction. After the UE–Gate audit, Lemma 10.3 provides only the pointwise exponent $p = 1$, so the tail left-hand side contains the δ–inert term $(2C_{\mathrm{up}}/\kappa)\,N_{\mathrm{up}}(m)$. With the explicit majorant $N_{\mathrm{up}}(m) = 1.01\log m + 17$, this term is *increasing* in m. Therefore a one-height reduction in m is not available under the current pointwise envelope mechanism: the tail criterion must be controlled as a family in m, or the UE–Gate must be cleared by a strengthened envelope mechanism (Remark 10.12).

12 S5 frontier: non-pointwise UE redesign (open)

Remark 12.1 (S5 acceptance criterion (budget calculus; no drift)). Any proposed S5 redesign must specify a boundary functional Φ_B (shape–invariant; δ–normalized) and prove two explicit inequalities uniformly for all $m \ge 10$, all $\alpha \in (0,1]$, and all κ–admissible $0 < \delta \le \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$:

1. **(S5–UE)** a forceable upper-envelope bound

$$D_B(W) \le C_{\rm up}\, \delta^p\, \Phi_B(E'/E)$$

   with an explicit exponent $p > 0$ and δ–uniform constant $C_{\rm up}$;

2. **(S5–LOC)** a local/collar bound in the same endpoint class

$$\Phi_B(Z'_{\rm loc}/Z_{\rm loc}) \le C_{\rm loc}\, \delta^{-\theta}\, G(N_{\rm loc}(m), \kappa)$$

   with explicit $\theta \ge 0$ and an explicit growth model for G (e.g. $G(n, \kappa) \ll \kappa^{-u} n^q$).

The redesign is budget–viable for uniform η–shrinking closure under δ_0 only if the S5 Budget Theorem yields $2(p-\theta) \ge q$ (and $p-\theta > 0$ for shrinkability). If $p-\theta < 0$, the standard κ–admissible shrink policy is unsafe (shrinking δ can increase the envelope term) and must be redesigned.

Finally, the forcing chain remains phrased in terms of $D_B(W)$; therefore S5 must include either $\Phi_B \ge D_B(W)$ on all admissible boxes or a new forcing lemma that lower-bounds Φ_B directly (Remark 12.5).

Theorem 12.2 (S5 Budget Theorem (general endpoint functional))**.** *Fix $\eta \in (0, 1]$ and $\kappa \in (0, 1/10)$ and define the nominal scale $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$. Let Φ_B be a boundary functional and assume that for every $m \ge 10$, $\alpha \in (0, 1]$, and every κ–admissible $0 < \delta \le \delta_0(m, \alpha)$ one has:*

(i) ***(S5–UE)*** *$D_B(W) \le C_{\rm up}\, \delta^p\, \Phi_B(E'/E)$ for some $p > 0$ and δ–uniform constant $C_{\rm up}$;*

(ii) ***(S5–SPLIT)*** *$\Phi_B(E'/E) \le \mathrm{Res}(m) + \Phi_B(Z'_{\rm loc}/Z_{\rm loc})$;*

(iii) ***(S5–LOC)*** *$\Phi_B(Z'_{\rm loc}/Z_{\rm loc}) \le C_{\rm loc}\, \delta^{-\theta}\, G(N_{\rm loc}(m), \kappa)$ for some $\theta \ge 0$ and δ–uniform $C_{\rm loc}$.*

Assume moreover that $N_{\rm loc}(m) \le A_N \log m + B_N$ and $\mathrm{Res}(m) \le A_L(\log m)^{r_L} + B_L$ for absolute constants, and that for some $q, u \ge 0$ one has the growth model

$$G(n, \kappa) \le C_G\, \kappa^{-u}\, n^q \qquad (n \ge 1),$$

with C_G independent of (m, α, δ).

Then at the nominal choice $\delta = \delta_0(m, \alpha)$,

$$D_B(W) \;\le\; C_{\rm up}\Big(\delta_0^p\, \mathrm{Res}(m) \;+\; C_{\rm loc}\, \delta_0^{p-\theta}\, G(N_{\rm loc}(m), \kappa)\Big). \tag{15}$$

Furthermore:

1. ***(Uniformity in** m**)*** *The local contribution in (15) is uniformly bounded in $m \ge 10$ only if*

$$2(p-\theta) \;\ge\; q. \tag{16}$$

2. ***(η–shrinkability)*** *If (16) holds and $p - \theta > 0$, then*

$$\sup_{m \ge 10}\; \delta_0(m, 1)^{p-\theta}\, G(N_{\rm loc}(m), \kappa) \;=\; O\big(\eta^{p-\theta}\big),$$

   so the local penalty can be made arbitrarily small by choosing η sufficiently small.

3. ***(δ–shrink monotonicity)*** *If $p \ge 0$ and $p - \theta \ge 0$, then the right-hand side of (15) is non-decreasing in δ (for fixed m, α); hence replacing δ_0 by a smaller κ–admissible $\delta \le \delta_0$ can only improve the inequality. If $p - \theta < 0$, κ–shrinking can* worsen *the envelope term.*

Proof. Combine (S5–UE) with (S5–SPLIT) and (S5–LOC) to obtain

$$D_B(W) \le C_{\rm up}\, \delta^p\, \mathrm{Res}(m) + C_{\rm up}\, C_{\rm loc}\, \delta^{p-\theta}\, G(N_{\rm loc}(m), \kappa).$$

Set $\delta = \delta_0(m, \alpha)$ to obtain (15).

For the local contribution at $\alpha = 1$ use $\delta_0 = \eta/(\log m)^2$, the growth model $G(n, \kappa) \le C_G \kappa^{-u} n^q$, and $N_{\rm loc}(m) \ll \log m$ to get

$$\delta_0^{p-\theta}\, G(N_{\rm loc}(m), \kappa) \;\ll\; \kappa^{-u}\, \eta^{p-\theta}\, (\log m)^{-2(p-\theta)}\, (\log m)^q = \kappa^{-u}\, \eta^{p-\theta}\, (\log m)^{q-2(p-\theta)}.$$

This is uniformly bounded in m only if $q - 2(p-\theta) \le 0$, i.e. (16). If additionally $p - \theta > 0$, the factor $\eta^{p-\theta}$ yields η–shrinkability.

Finally, the monotonicity claim follows because $\delta \mapsto \delta^p$ and $\delta \mapsto \delta^{p-\theta}$ are nondecreasing on $(0, \infty)$ exactly when $p \ge 0$ and $p - \theta \ge 0$. □

At fixed (m, α) the tail inequality (14) is a strict forcing–vs–envelope condition. In v36 (inherited from v35) the combination of Theorem 10.13, Lemma 10.15, and Lemma 8.2 formally rules out the former "η–absorption" closure route based on the pointwise/sup endpoint $\sup_{\partial B} |E'/E|$ together with the pointwise collar bound.

What must change. The forcing chain produces a lower bound for the *dial deviation*

$$D_B(W) \;:=\; \sum_{\pm} \bigl| W(v_\pm) - e^{i\varphi_0^\pm} \bigr|$$

appearing in Lemma 10.3. In the current architecture this deviation is upper-bounded by a pointwise endpoint $\delta \sup_{\partial B} |E'/E|$, which (via the collar) introduces the sharp δ^{-1} blow–up. To obtain a tail closure mechanism one must redesign the envelope endpoint and/or the local interface so that the exponent budget $p - \theta \ge \frac{1}{2}$ is met *uniformly in* m.

Remark 12.3 (Forcing compatibility for redesigned endpoints)*.* The existing forcing chain lower-bounds $D_B(W)$ (via the pair-factor phase rotation) by a fixed constant c up to δ–small corrections. If one proposes a redesigned envelope endpoint Φ_B (non-pointwise, e.g. an L^2 or energy functional), then the current forcing lower bound is useful only if it implies a corresponding lower bound for Φ_B. A sufficient (and simplest) compatibility condition is:

$$\Phi_B \;\ge\; D_B(W) \qquad \text{for all admissible boxes and quotients } W,$$

so that the forcing lower bound propagates unchanged. If this domination fails, then a *new forcing lemma* must be proved that lower-bounds Φ_B directly.

Lemma 12.4 (Forceability transfer by domination)**.** *Let B be a κ–admissible aligned box and W the associated boundary quotient. Suppose a boundary endpoint functional Φ_B satisfies*

$$\Phi_B \ge D_B(W) \quad \textit{for all admissible } (B, W).$$

Then the existing single–box forcing lower bound for $D_B(W)$ implies the same forcing lower bound for Φ_B with no change in the forcing constants.

Remark 12.5 (Forceability gate for S5 endpoints (NO–GO unless met))*.* The current forcing architecture (Section 8) forces only the dial deviation $D_B(W)$ by an $O(1)$ constant up to δ–small deductions (Lemma 8.2). Consequently, any S5 redesign that replaces $D_B(W)$ by a different endpoint $\widetilde{D}_B$ (or Φ_B) is *invalid* unless it proves either:

(i) $\widetilde{D}_B \geq D_B(W)$ for all admissible boxes/quotients (domination transfer), or

(ii) a new forcing lemma that lower–bounds $\widetilde{D}_B$ directly under an off–axis quartet.

Without (i) or (ii), the forcing half of the tail inequality becomes logically disconnected from the envelope half.

12.1 Baseline NO–GO results for naive non-pointwise endpoints

The S5 goal is to replace the pointwise/sup endpoint in Lemma 10.3 by a non–pointwise functional that still controls the same dial deviation $D_B(W)$ appearing in the forcing chain. The next two results prevent drift into two large endpoint classes that cannot work under the present $D_B(W)$ target and the v36 local split/collar interface (unchanged from v35).

Lemma 12.6 (Absolute L^r endpoint scaling collapse)**.** *Let $B = B(\pm a, m, \delta)$ be an aligned box and let G_{out} and $W = E/G_{\mathrm{out}}$ be as in Lemma 10.3. Assume boundary contact so that W has unimodular boundary values a.e. Fix $r \in [1, \infty]$ and write $L^r(\partial B)$ for $L^r(\partial B, ds)$. Then there exists a shape-only constant $C_r > 0$ (depending only on the normalized square $Q = [-1, 1]^2$) such that for each sign $\pm$,*

$$\left|W(v_\pm) - e^{i\varphi_0^\pm}\right| \;\leq\; C_r\, \delta^{\,1-1/r} \left\| \frac{E'}{E} \right\|_{L^r(\partial B)}. \tag{17}$$

In particular, any upper-envelope mechanism whose endpoint is an absolute $L^r(\partial B)$ *norm of E'/E cannot have a δ–prefactor exponent exceeding $p(r) = 1 - 1/r$ within this endpoint class.*

Proof. Repeat the proof of Lemma 10.3 with L^2 replaced by L^r throughout. Evaluation from the boundary gives $|W(v_\pm) - c| \leq \|P_B(v_\pm, \cdot)\|_{L^q} \|W - c\|_{L^r}$ for $1/r + 1/q = 1$, and under affine rescaling $\|P_B\|_{L^q} \asymp \delta^{-1/r}$. Boundary Poincaré in L^r yields $\|W - c\|_{L^r} \leq C'_r\, \delta\, \|\partial_s W\|_{L^r}$ with a shape-only constant C'_r, and outer factor control bounds $\|\partial_s W\|_{L^r}$ by a shape-only constant times $\|E'/E\|_{L^r}$. Choosing $c = e^{i\varphi_0^\pm}$ gives (17), with overall factor $\delta^{-1/r} \cdot \delta = \delta^{1-1/r}$. □

Proposition 12.7 (NO–GO: absolute L^r log-derivative endpoints cannot clear the UE–Gate)**.** *Assume in addition that B is κ–admissible and hence the pointwise collar bound holds:* $\sup_{\partial B} |Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}| \leq N_{\mathrm{loc}}(m)/(\kappa\delta)$ *(Lemma 10.8). Then for every $r \in [1, \infty]$,*

$$\delta^{\,1-1/r} \left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^r(\partial B)} \;\leq\; 8^{1/r}\, \frac{N_{\mathrm{loc}}(m)}{\kappa},$$

independent of δ. In particular, under the nominal scale $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ and the unconditional majorant $N_{\mathrm{loc}}(m) \ll \log m$, uniform η–shrinking cannot suppress the local term within any envelope mechanism whose endpoint is an absolute $L^r(\partial B)$ norm of E'/E.

Proof. Use $|\partial B| = 8\delta$ and $\|f\|_{L^r} \leq |\partial B|^{1/r} \|f\|_{L^\infty}$ to get

$$\left\| \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}} \right\|_{L^r(\partial B)} \leq (8\delta)^{1/r} \cdot \frac{N_{\mathrm{loc}}(m)}{\kappa\delta} = 8^{1/r}\, \frac{N_{\mathrm{loc}}(m)}{\kappa\, \delta^{\,1-1/r}}.$$

Multiply by $\delta^{\,1-1/r}$. □

Remark 12.8 (Implication for S5 endpoint design)*.* Lemmas 12.6–12.7 rule out the entire family of S5 proposals that attempt to replace $\sup_{\partial B} |E'/E|$ by an *absolute* $L^r(\partial B)$ norm of E'/E while keeping the same κ–collar local control. Any viable S5 redesign must instead (i) exploit cancellation (argument-principle style *signed* endpoints) and/or (ii) move to a less singular boundary object (e.g. endpoints built from $\log |E|$ / BMO-type control).

Lemma 12.9 (NO–GO: local-kernel projection endpoints cannot control $D_B(W)$ without a new forcing link)**.** *Fix an aligned box B and consider an endpoint functional of the form*

$$\Phi_B(E) \; := \; \|(I - \Pi_B)(E'/E)\|_{X(\partial B)}$$

for some normed boundary space $X(\partial B)$ and a bounded projection Π_B satisfying $\Pi_B(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$ whenever Z_{loc} is the local factor associated to B (so that the local term is annihilated under the split $E'/E = F'/F + Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$). Then there is no universal inequality of the form

$$D_B(W) \; \leq \; C\, \delta^p\, \Phi_B(E)$$

(valid for all forcing-aligned boxes under the boundary-contact convention), for any fixed $p > 0$ and constant C, unless one supplies a new forcing link that lower-bounds Φ_B directly under an off-axis quartet.

Proof. This is a structural counterexample: in the class of holomorphic functions E obeying the boundary-contact convention, take $E = Z_{\mathrm{loc}}$ on a box for which Z_{loc} has a zero at one of the dial points $v_\pm$. Then $F \equiv 1$ and $E'/E = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$, so by assumption $(I - \Pi_B)(E'/E) = 0$ and hence $\Phi_B(E) = 0$. However G_{out} is zero-free, so $W = E/G_{\mathrm{out}}$ shares the same interior zeros as E and $W(v_\pm) = 0$ for at least one sign, giving $D_B(W) \geq 1$. Thus no inequality $D_B(W) \leq C\delta^p \Phi_B(E)$ can hold from these hypotheses alone; any attempt to use such an endpoint must replace $D_B(W)$ as the forced object and provide a forcing-transfer lemma (Remark 12.5). $\square$

Remark 12.10 (Consequence for S5 searches). Lemmas 12.7 and 12.9 close two broad endpoint classes: (i) absolute $L^r(\partial B)$ norms of E'/E (including L^2) under the current collar interface, and (ii) endpoints that annihilate the local kernel span while still targeting the forced dial deviation $D_B(W)$. Any viable S5 redesign must introduce a genuinely new local-interface input and/or a new forcing-compatible endpoint.

S5 design targets (open). A future closure route (S5) should provide a non–pointwise endpoint Φ_B and a UE–type inequality of the schematic form

$$D_B(W) \; \leq \; C_{\mathrm{up}}\, \delta^p\, \Phi_B(E) \qquad (p > 0),$$

together with a local/residual split of $\Phi_B(E)$ whose local contribution scales as $\delta^{-\theta}$ with $\theta < p - \frac{1}{2}$, or more generally satisfies the exponent budget of Theorem 10.13. The point is *not* to recover the specific exponent $\frac{3}{2}$ from older drafts, but to obtain any effective gain $p - \theta > \frac{1}{2}$ with proof–grade uniformity.

Remark 12.11 (Recorded open lemmas (S5 checklist)). A proof–grade S5 implementation would minimally require:

1. **(S5–UE)** a redesigned upper-envelope inequality with a forceable endpoint Φ_B;
2. **(S5–RES)** a δ–uniform residual envelope bound in the same endpoint class;
3. **(S5–LOC)** a collar/local bound in the same endpoint class that avoids the pointwise δ^{-1} blow–up;
4. **(S5–FORCE)** either $\Phi_B \geq D_B(W)$ or a new forcing lemma as in Remark 12.3.

13 Global RH from a finite front-end + the tail criterion family

Theorem 13.1 (Global closure (criterion-first logical form))**.** *Assume:*

1. *(Front-end) All nontrivial zeros with* $0 < \mathrm{Im}(s) \le 5$ *lie on the critical line.*
2. *(Tail criterion) Fix some* $\eta \in (0, 1)$ *and* $\kappa \in (0, 1/10)$*, and assume the analytic inputs Lemmas 10.3–10.10 and Lemma 10.17 with finite constants. Assume moreover that for every* $m \ge 10$ *and every* $\alpha \in (0, 1]$ *there exists a* κ*–admissible scale* $0 < \delta \le \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ *such that the strict tail inequality* (14) *holds.*

Then all nontrivial zeros of $\zeta(s)$ *lie on the critical line.*

Proof. For each $m \ge 10$, Theorem 11.1 turns the strict inequality (14) into exclusion of off-axis quartets at height m. By the tail criterion hypothesis, no off-axis quartets exist at any height $m \ge 10$. By the front-end hypothesis, there are no off-axis zeros below height 5. Hence there are no off-axis zeros at any height, so every nontrivial zero lies on the critical line. □

Remark 13.2 (Role of computations and the repro pack (v36)). Appendix B provides a small interval-arithmetic harness that evaluates the tail inequality for pinned parameters and a pinned constant ledger. In v36 this is used only for audit purposes (e.g. exponent tracking), not as a proof substitute.

A Discarded closure routes (as of v36)

This appendix records closure routes that were explored in earlier iterations (v32–v34) but are now ruled out *under the currently proved inputs.* The purpose is to prevent future drift: these routes should not be re-opened unless a genuinely new analytic input (e.g. an S5 endpoint redesign) is supplied.

A.1 D1: Pointwise UE endpoint $\sup_{\partial B} |E'/E|$ + collar + η–absorption (S1/S1′)

The former absorption narrative attempted to close the tail family by shrinking η in the nominal scale $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$. In the pointwise/sup architecture the UE step has exponent $p = 1$ (Lemma 10.3) and the collar/local split has exponent $\theta = 1$ (Lemma 10.8), so the local contribution is δ–inert and cannot be suppressed by η (Lemma 10.14). More strongly, the exponent budget (Theorem 10.13) shows that uniform η–shrinking requires $p - \theta \ge \frac{1}{2}$, while the scaling NO–GO (Lemma 10.15) forbids any $p > 1$ within this endpoint class. Finally, the forcing margin is constant–limited in the single–box architecture (Lemma 8.2), so one cannot compensate by "making forcing grow with m".

Proposition A.1 (Historical record: formal anchor absorption under a hypothetical strengthened UE exponent)**.** *This proposition is* not used *in v36. It is recorded only to document the logical shape of the discarded absorption idea.*

Assume that, for some $p > 1$*, an upper-envelope step admits the strengthened form*

$$D_B(W) \;\le\; 2C_{\mathrm{up}}\,\delta^p \sup_{\partial B}\Big|\frac{E'}{E}\Big|$$

with the same constant ledger, and that all other constants in (14) *are finite. Fix an anchor height* $m_\star \ge 10$ *and evaluate* (14) *at* $(m, \alpha) = (m_\star, 1)$ *with the nominal scale* $\delta_0(m_\star, 1) = \eta/(\log m_\star)^2$*. Then there exists* $\eta_\star(m_\star, p) > 0$ *such that* (14) *holds at* $(m_\star, 1)$ *for every* $\eta \in (0, \eta_\star]$*.*

Warning: *within the pointwise/sup endpoint class, Lemma 10.15 forbids any $p > 1$, so this proposition cannot be invoked without an S5 redesign.*

Proof. Under a strengthened exponent $p > 1$, the envelope side becomes $A\,\eta^p + B\,\eta^{p-1}$ for finite constants A, B depending on $(m_\star, p)$ and the constant ledger, while the forcing side equals $c - D\,\eta$ for a finite D. Since $p > 1$, one has $\eta^p \to 0$, $\eta^{p-1} \to 0$, and $\eta \to 0$ as $\eta \downarrow 0$, so the strict inequality holds for all sufficiently small η. □

A.2 D2: Shrinking the local window / short-interval zero counts

A tempting workaround is to replace the fixed local window $|\gamma - t| \le 1$ in the residual/collar interface by a shrinking window $|\gamma - t| \le \delta^\beta$ to reduce the local term. However, without additional analytic input, available RH–free methods control $N(t+1) - N(t-1)$ at unit scale and do *not* provide a proof–grade bound for $N(t+\delta^\beta) - N(t-\delta^\beta)$ as $\delta \downarrow 0$. Thus v36 does not pursue window-shrinking as a substitute for the missing UE gain.

A.3 D3: "Make forcing grow with m" within single-box forcing

Because $\Delta_{I_+} \arg Z_{\text{pair}} \le 2\pi$ uniformly (Lemma 8.2), the forcing constant c in the tail inequality is $O(1)$. Any attempt to obtain a forcing side that grows like $\log m$ (or any unbounded function of m) would require a different forcing architecture (not the v36 single-box forcing chain).

A.4 D4: "Boundary modulus implies interior zero-freeness" converse

Under boundary-contact, the quotient $W = E/G_{\text{out}}$ satisfies $|W| = 1$ on ∂B (Remark 9.3), but this has no converse implication toward zero-freeness or constancy (Remark 9.4). Therefore, any closure route that implicitly treats $|W| = 1$ as "almost zero-free" is invalid.

A.5 D5: Absolute L^r log-derivative endpoints (NO–GO)

Replacing the pointwise endpoint $\sup_{\partial B} |E'/E|$ by an *absolute* boundary $L^r(\partial B)$ norm of E'/E does not improve the exponent budget: Lemma 12.6 forces the UE prefactor exponent to be $p(r) = 1 - 1/r$, while Proposition 12.7 shows the local/collar contribution has the same exponent $\theta(r) = 1 - 1/r$, hence $p(r) - \theta(r) = 0$ and the local leakage is δ–inert.

A.6 D6: Projecting out the local kernel span (NO–GO)

A tempting idea is to define an endpoint by projecting E'/E off the span of local Cauchy kernels so that the local term vanishes. Lemma 12.9 shows this cannot control the forced dial deviation $D_B(W)$ without changing the contradiction endpoint or supplying a new forcing link.

Supporting documentation for D6 (not a viable endpoint under current forcing). The next definition and lemmas formalize the projection setup and the exact cancellation of the local term. They are recorded only to document the mechanism behind the NO–GO.

Definition A.2 (Local Cauchy subspace and L^2 projection (supporting documentation))**.** Let $B = B(\alpha, m, \delta)$ be κ–admissible and let $Z_{\text{loc}}(m)$ denote the multiset of zeros of E used to define Z_{loc} (counted with multiplicity). Define the finite-dimensional subspace

$$K_B := \operatorname{span}\{\, k_\rho : \partial B \to \mathbb{C},\ k_\rho(v) = (v - \rho)^{-1} \ :\ \rho \in Z_{\text{loc}}(m) \,\} \subset L^2(\partial B, ds),$$

and let $\Pi_B : L^2(\partial B) \to K_B$ be the orthogonal projection.

Lemma A.3 (Projection kills the local log-derivative (supporting documentation))**.** *With notation as in Definition A.2,*

$$\frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}(v) = \sum_{\rho \in Z_{\mathrm{loc}}(m)} \frac{m_\rho}{v-\rho} \in K_B \quad (v \in \partial B).$$

Hence $\Pi_B(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$ *and* $(I-\Pi_B)(Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}) = 0$ *in* $L^2(\partial B)$ *(and thus pointwise on* ∂B*). Consequently, using Lemma 10.7,*

$$(I-\Pi_B)\left(\frac{E'}{E}\right) = (I-\Pi_B)\left(\frac{F'}{F}\right) \quad \text{on } \partial B.$$

Moreover $\|\Pi_B\|_{L^2\to L^2} = 1$.

Remark A.4 (Conditioning caveat for coefficient representations (supporting documentation)). Lemma A.3 uses only the abstract orthogonal projection Π_B (a contraction). No uniform bound on the inverse Gram matrix of the spanning kernels k_ρ is available without a lower bound on pairwise zero separations in $Z_{\mathrm{loc}}(m)$. Therefore any coefficient-level formula for Π_B must be treated as non-uniform unless additional spacing structure is proved.

B Tail harness bundle and reproducibility (v36)

B.1 What the tail checks prove (and what they do not)

Each tail check records the statement:

> Given a constants file that provides interval enclosures for $(C_1, C_2, C_{\mathrm{up}}, C''_h, \kappa)$, the chosen parameters (m, η, α), and the recorded UE exponent p, the harness computes interval bounds for the left-hand side LHS and right-hand side RHS in (14) and reports whether the strict separation $\mathrm{LHS}_{\mathrm{hi}} < \mathrm{RHS}_{\mathrm{lo}}$ holds.

It does *not* certify that the constants file is correct.

B.2 SHA–256 table (exact artifacts)

The file `v36_repro_pack/SHA256SUMS.txt` is the canonical hash list.

```
55bfca839dd319f41fac71be7306dc3d3df2774ad9726175da92334a8852b1b4  README.md
413ac771650f09e6175c4c915cd2ac190d42436ff09121cb2e6cb56310cb28c7  v36_constants_m10.json
60a4d0c46841664c1d1ddc03744b53e224d1e42be7d6e30085c0c86e9edbcaf2  v36_frontend_certificate.json
c1debbda3583dbaf0dc7120684ba89c457fef1227f4aa13504b21cf11e029acb  v36_frontend_verifier_output.txt
cb9f61fd9a605ba1c2df478eb3ee304e3186367bd1078b246ee3137fa8d21e1d  v36_generate_frontend_certificate
    .py
8daa19ad6e7fbb3feda159b4a42f6fd451c7066c1ba78e6fd01195823a5ddf3d  v36_generate_tail_check.py
31e37cf69fd26ae41d36d734ce7a6aa2047dc4b547545ab632c612d25da4001b  v36_tail_check_m10.json
3b2c15c47e9eaadc4edbac24296091f41e9c04062de51536469584fc1783a307
    v36_tail_check_verifier_output_m10.txt
76f736fa4e6722a8c929f625968c046ad68befda20186e1ba17f6082c8f0c30a  v36_verify_frontend_certificate.
    py
0f36e0a23047f0ffcd413ecc024adadb11ffbde7924b43892728ec1416245b85  v36_verify_tail_check.py
```

B.3 Commands

From the directory `v36_repro_pack/`:

1. `sha256sum -c SHA256SUMS.txt`
2. `python3 v36_verify_tail_check.py --constants v36_constants_m10.json --certificate v36_tail_check_m10.json`
3. `python3 v36_verify_frontend_certificate.py --certificate v36_frontend_certificate.json`

B.4 Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false)

```
LHS_hi =
    850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076
RHS_lo =
    0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351
STRICT (LHS_hi < RHS_lo) = False
REGEN_MATCH = True
INEQUALITY_STRICT = False
CERT_REPORTED_PASS = False
OK
```

B.5 Bundle files (verbatim)


```
{
  "certificate_version": "v36",
  "created_utc": "2026-01-24T00:00:00Z",
  "m_band": "10",
  "eta": "1e-14",
  "alpha_worst": "1",
  "kappa": "0.05",
  "intervals": {
    "C1": {
      "lo": "15.1",
      "hi": "15.2"
    },
    "C2": {
      "lo": "37.3",
      "hi": "37.4"
    },
    "C_up": {
      "lo": "1100",
      "hi": "1100.5"
    },
    "C_hpp": {
      "lo": "1100",
      "hi": "1100.5"
    }
  },
  "notes": [
    "Demo-only intervals carried forward from v31-style scaffolding; replace with audit-proven
    enclosures when G-1/G-12 are closed.",
    "The verifier/generator implement directed-rounding interval arithmetic with Python's decimal
    module.",
    "The local-window majorant N_up(m)=1.01*log(m)+17 is hard-coded from Lemma Nloc-logm in
    manuscript_v36.",
```

"UE_exponent_p is recorded explicitly to prevent exponent drift across versions.",
"v36 adds explicit metadata for endpoint_functional and forcing_architecture to prevent silent mismatch under S5 redesign.",
"UE_endpoint_class='pointwise/sup' is the class for which Lemma UE-scaling-nogo forbids any exponent p>1 with shape-only constants.",
"v36 schema latch: required fields (endpoint_functional, UE_exponent_p, local_exponent_theta, local_growth_q, forceability_mode, forcing_architecture) must be present; verifier fails closed if missing."
],
"UE_exponent_p": "1",
"manuscript_version": "v36",
"UE_endpoint_class": "pointwise/sup",
"endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
"forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma force-constant-limited)",
"forcing_constants": {
"c_expression": "(3*ln(2))/16",
"c0_expression": "(3*ln(2))/(8*pi)",
"Kalloc_expression": "3 + 8*sqrt(3)"
},
"local_exponent_theta": "1",
"local_growth_q": "1",
"forceability_mode": "identity: forced object is D_B(W)"
}

{
"certificate_version": "v36",
"m_band": "10",
"eta": "1e-14",
"alpha": "1",
"kappa": "0.05",
"UE_exponent_p": "1",
"UE_endpoint_class": "pointwise/sup",
"endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
"forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma force-constant-limited)",
"forceability_mode": "identity: forced object is D_B(W)",
"local_exponent_theta": "1",
"local_growth_q": "1",
"forcing_constants": {
"c_expression": "(3*ln(2))/16",
"c0_expression": "(3*ln(2))/(8*pi)",
"Kalloc_expression": "3 + 8*sqrt(3)"
},
"manuscript_version": "v36",
"prec": 90,
"pi_interval": {
"lo": "3.14159265358979323846264338327950288419716939937510",
"hi": "3.14159265358979323846264338327950288419716939937511"
},
"logm_interval": {
"lo":
"2.302585092994045684017991454684364207601101488628772976033327900967572609677352480235997 21",
"hi":
"2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721"
},

```
"delta_interval": {
  "lo":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197468E
  -15",
  "hi":
  "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197469E
  -15"
},
"L_interval": {
  "lo":
  "72.0690349042100898286716709657338995347766324782944719381032513046103464061280224515635578",
  "hi":
  "72.3992934135094943970734701112023359555367426271573492357065840947071036670957576995871576"
},
"Nup_interval": {
  "lo":
  "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383571",
  "hi":
  "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383572"
},
"kappa_interval": {
  "lo": "0.05",
  "hi": "0.05"
},
"lhs_interval": {
  "lo":
  "850326.881532655689245144996236820608990676883579823939491593791503565415726034904366571925",
  "hi":
  "850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076"
},
"rhs_interval": {
  "lo":
  "0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351",
  "hi":
  "0.129965096347948199005209691457697222838229716361348668790498959759558243588339370271250251"
},
"derived_constants": {
  "ln2_interval": {
    "lo":
  "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327",
    "hi":
  "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327"
  },
  "c_interval": {
    "lo":
  "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099373",
    "hi":
  "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099375"
  },
  "c0_interval": {
    "lo":
  "0.0827383500572443475236711620442491341185086557736206913728528561387020242248387512851407512",

    "hi":
  "0.0827383500572443475236711620442491341185086557736209547372007536994885577445868650239268751"
  },
  "Kalloc_interval": {
    "lo":
  "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491695",
```

```
      "hi":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491696"
    }
  },
  "pass": false
}
```


```
#!/usr/bin/env python3
"""
v36_generate_tail_check.py

Deterministically generates v36_tail_check_m10.json from v36_constants_m10.json using
directed-rounding interval arithmetic implemented with Python's decimal module.

This generator is intended to be auditable: no network access, no randomness, and no external
    libraries.

Tail inequality evaluation (for given inputs):
  LHS(delta) < RHS(delta), where
  LHS(delta) = 2*C_up*( delta^p*L(m) + delta^(p-1)*N_up(m)/kappa )
  RHS(delta) = c - delta*( Kalloc*c0*L(m) + C_hpp*(log(m)+1) )

with
  L(m)    = C1*log(m) + C2,
  N_up(m) = 1.01*log(m) + 17,
  c  = (3 ln 2)/16,
  c0 = (3 ln 2)/(8 pi),
  Kalloc = 3 + 8*sqrt(3).

Usage:
  python3 v36_generate_tail_check.py v36_constants_m10.json v36_tail_check_m10.json
"""

import json
import sys
from dataclasses import dataclass
from decimal import Decimal, getcontext, localcontext, ROUND_FLOOR, ROUND_CEILING


# ---- Fixed enclosure for pi (50 decimal places) ----
# pi = 3.14159265358979323846264338327950288419716939937510...
PI_LO = Decimal("3.14159265358979323846264338327950288419716939937510")
PI_HI = Decimal("3.14159265358979323846264338327950288419716939937511")


@dataclass
class Interval:
    lo: Decimal
    hi: Decimal

    def __post_init__(self) -> None:
        if self.lo > self.hi:
            raise ValueError(f"Bad interval: {self.lo} > {self.hi}")


def ctx(prec: int, rounding):
```

```
    c = getcontext().copy()
    c.prec = prec
    c.rounding = rounding
    return c


def iv(lo: str, hi: str | None = None) -> Interval:
    if hi is None:
        hi = lo
    return Interval(Decimal(lo), Decimal(hi))


def add(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo + b.lo
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi + b.hi
    return Interval(lo, hi)


def sub(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo - b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi - b.lo
    return Interval(lo, hi)


def mul(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        cands_lo = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        lo = min(cands_lo)
    with localcontext(ctx(prec, ROUND_CEILING)):
        cands_hi = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        hi = max(cands_hi)
    return Interval(lo, hi)


def div(a: Interval, b: Interval, prec: int) -> Interval:
    if b.lo <= 0 <= b.hi:
        raise ZeroDivisionError("Interval division by an interval containing 0.")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        rlo = Decimal(1) / b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        rhi = Decimal(1) / b.lo
    return mul(a, Interval(rlo, rhi), prec)


def sqrt(a: Interval, prec: int) -> Interval:
    if a.lo < 0:
        raise ValueError("sqrt of negative interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.sqrt()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.sqrt()
    return Interval(lo, hi)


def ln(a: Interval, prec: int) -> Interval:
```

```
    if a.lo <= 0:
        raise ValueError("ln of nonpositive interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.ln()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.ln()
    return Interval(lo, hi)


def pow_3_2(a: Interval, prec: int) -> Interval:
    return mul(a, sqrt(a, prec), prec)


def compute(constants: dict, prec: int = 90) -> dict:
    m = iv(constants["m_band"])
    eta = iv(constants["eta"])
    alpha = iv(constants["alpha_worst"])
    kappa = iv(constants["kappa"])

    p = str(constants.get("UE_exponent_p", "1"))

    C1 = iv(constants["intervals"]["C1"]["lo"], constants["intervals"]["C1"]["hi"])
    C2 = iv(constants["intervals"]["C2"]["lo"], constants["intervals"]["C2"]["hi"])
    Cup = iv(constants["intervals"]["C_up"]["lo"], constants["intervals"]["C_up"]["hi"])
    Chpp = iv(constants["intervals"]["C_hpp"]["lo"], constants["intervals"]["C_hpp"]["hi"])

    logm = ln(m, prec)
    delta = div(mul(eta, alpha, prec), mul(logm, logm, prec), prec)

    # L(m) = C1*logm + C2
    L = add(mul(C1, logm, prec), C2, prec)

    # N_up(m) = 1.01*logm + 17
    Nup = add(mul(iv("1.01"), logm, prec), iv("17"), prec)

    # ln 2
    ln2 = ln(iv("2"), prec)

    # c = (3 ln 2)/16
    c = div(mul(iv("3"), ln2, prec), iv("16"), prec)

    # c0 = (3 ln 2)/(8 pi), pi enclosed
    pi = Interval(PI_LO, PI_HI)
    c0 = div(mul(iv("3"), ln2, prec), mul(iv("8"), pi, prec), prec)

    # Kalloc = 3 + 8 sqrt(3)
    sqrt3 = sqrt(iv("3"), prec)
    Kalloc = add(iv("3"), mul(iv("8"), sqrt3, prec), prec)

    logm_plus1 = add(logm, iv("1"), prec)

    # UE exponent p: LHS = 2*Cup*(delta^p * L + delta^(p-1) * Nup/kappa).
    # We support p="1" (pointwise UE proved in v36) and p="3/2" (hypothetical strengthened gate).
    if p in ("1", "1.0", "1.00"):
        local_term = div(Nup, kappa, prec)              # delta^(p-1)=1
        residual_term = mul(delta, L, prec)             # delta^p = delta
    elif p in ("3/2", "1.5", "1.50"):
        sqrt_delta = sqrt(delta, prec)
        local_term = mul(sqrt_delta, div(Nup, kappa, prec), prec)     # delta^(1/2)
```

```
        residual_term = mul(mul(delta, sqrt_delta, prec), L, prec)    # delta^(3/2)
    else:
        raise ValueError(f"Unsupported UE_exponent_p={p!r}; use '1' or '3/2'.")

    lhs = mul(mul(iv("2"), Cup, prec), add(residual_term, local_term, prec), prec)

    # RHS = c - delta*(Kalloc*c0*L + Chpp*(logm+1))
    term1 = mul(mul(Kalloc, c0, prec), L, prec)
    term2 = mul(Chpp, logm_plus1, prec)
    rhs = sub(c, mul(delta, add(term1, term2, prec), prec), prec)

    passed = (lhs.hi < rhs.lo)

    return {
        "prec": prec,
        "UE_exponent_p": p,
        "pi_interval": {"lo": str(PI_LO), "hi": str(PI_HI)},
        "logm_interval": {"lo": str(logm.lo), "hi": str(logm.hi)},
        "delta_interval": {"lo": str(delta.lo), "hi": str(delta.hi)},
        "L_interval": {"lo": str(L.lo), "hi": str(L.hi)},
        "Nup_interval": {"lo": str(Nup.lo), "hi": str(Nup.hi)},
        "kappa_interval": {"lo": str(kappa.lo), "hi": str(kappa.hi)},
        "lhs_interval": {"lo": str(lhs.lo), "hi": str(lhs.hi)},
        "rhs_interval": {"lo": str(rhs.lo), "hi": str(rhs.hi)},
        "derived_constants": {
            "ln2_interval": {"lo": str(ln2.lo), "hi": str(ln2.hi)},
            "c_interval": {"lo": str(c.lo), "hi": str(c.hi)},
            "c0_interval": {"lo": str(c0.lo), "hi": str(c0.hi)},
            "Kalloc_interval": {"lo": str(Kalloc.lo), "hi": str(Kalloc.hi)},
        },
        "pass": bool(passed),
    }


def main() -> int:
    if len(sys.argv) != 3:
        print("Usage: v36_generate_tail_check.py constants.json tail_check.json", file=sys.stderr)
        return 2

    with open(sys.argv[1], "r", encoding="utf-8") as f:
        constants = json.load(f)

    # ---- fail-closed metadata latch (v36) ----
    REQUIRED_META = [
        "UE_exponent_p",
        "UE_endpoint_class",
        "endpoint_functional",
        "forcing_architecture",
        "forceability_mode",
        "local_exponent_theta",
        "local_growth_q",
    ]
    for k in REQUIRED_META:
        if k not in constants or constants[k] in (None, ""):
            raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")

    out = {
        "certificate_version": "v36",
        "m_band": constants["m_band"],
```

```
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        # required metadata latch (fail-closed)
        "UE_exponent_p": constants["UE_exponent_p"],
        "UE_endpoint_class": constants["UE_endpoint_class"],
        "endpoint_functional": constants["endpoint_functional"],
        "forcing_architecture": constants["forcing_architecture"],
        "forceability_mode": constants["forceability_mode"],
        "local_exponent_theta": constants["local_exponent_theta"],
        "local_growth_q": constants["local_growth_q"],
        "forcing_constants": constants.get("forcing_constants", {}),
        "manuscript_version": constants.get("manuscript_version", "v36"),
    }

    out.update(compute(constants, prec=90))

    with open(sys.argv[2], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[2])
    print("[generate] PASS =", out["pass"])
    print("[generate] lhs_interval.hi =", out["lhs_interval"]["hi"])
    print("[generate] rhs_interval.lo =", out["rhs_interval"]["lo"])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

```
#!/usr/bin/env python3
"""
v36_verify_tail_check.py

Verifier for v36_tail_check_m10.json. This script:
- loads the constants JSON and the pinned certificate JSON
- regenerates the certificate from constants
- checks exact JSON equality on the computed fields
- reports PASS/FAIL and prints the strict-separation check LHS_hi < RHS_lo.

Usage:
  python3 v36_verify_tail_check.py --constants v36_constants_m10.json --certificate
    v36_tail_check_m10.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json
import sys

from v36_generate_tail_check import compute
```

```python
def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v36 tail check (m=10).")
    ap.add_argument("--constants", required=True, help="Path to v36_constants_m10.json")
    ap.add_argument("--certificate", required=True, help="Path to v36_tail_check_m10.json")
    args = ap.parse_args()

    with open(args.constants, "r", encoding="utf-8") as f:
        constants = json.load(f)

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    # ---- fail-closed metadata latch (v36) ----
    REQUIRED_META = [
        "UE_exponent_p",
        "UE_endpoint_class",
        "endpoint_functional",
        "forcing_architecture",
        "forceability_mode",
        "local_exponent_theta",
        "local_growth_q",
    ]
    for k in REQUIRED_META:
        if k not in constants or constants[k] in (None, ""):
            raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")
        if k not in cert or cert[k] in (None, ""):
            raise KeyError(f"Missing required metadata field {k!r} in certificate (fail-closed).")

    regen = {
        "certificate_version": "v36",
        "m_band": constants["m_band"],
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        # required metadata latch (fail-closed)
        "UE_exponent_p": constants["UE_exponent_p"],
        "UE_endpoint_class": constants["UE_endpoint_class"],
        "endpoint_functional": constants["endpoint_functional"],
        "forcing_architecture": constants["forcing_architecture"],
        "forceability_mode": constants["forceability_mode"],
        "local_exponent_theta": constants["local_exponent_theta"],
        "local_growth_q": constants["local_growth_q"],
        "forcing_constants": constants.get("forcing_constants", {}),
        "manuscript_version": constants.get("manuscript_version", "v36"),
    }

    regen.update(compute(constants, prec=90))

    # Compare all keys that regen produces (ignore any extra keys in cert)
    ok = True
    for k, v in regen.items():
        if cert.get(k) != v:
            ok = False
            print(f"MISMATCH key={k}")
            print("  cert :", cert.get(k))
            print("  regen:", v)
```

```
    lhs_hi = regen["lhs_interval"]["hi"]
    rhs_lo = regen["rhs_interval"]["lo"]
    strict = (float(lhs_hi) < float(rhs_lo))

    print("LHS_hi =", lhs_hi)
    print("RHS_lo =", rhs_lo)
    print("STRICT (LHS_hi < RHS_lo) =", strict)

    print("REGEN_MATCH =", ok)
    print("INEQUALITY_STRICT =", strict)
    print("CERT_REPORTED_PASS =", regen.get("pass"))

    if not ok:
        print("FAIL (mismatch)")
        return 1

    print("OK")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

C Finite-height front-end certificate (literature-based)

The required front-end is RH up to height $H_0 = 5$. We record a discharge using Platt–Trudgian's published verification of RH up to $3 \cdot 10^{12}$.


```
{
  "certificate_version": "v36",
  "created_utc": "2026-01-24T00:00:00Z",
  "needed_frontend_statement": {
    "type": "RH_to_height",
    "H0": 5.0,
    "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
  },
  "discharged_by": {
    "type": "literature_citation",
    "verification_height": 3000000000000.0,
    "reference": {
      "authors": "D. J. Platt and T. S. Trudgian",
      "title": "The Riemann hypothesis is true up to 3*10^12",
      "venue": "Bulletin of the London Mathematical Society",
      "year": 2021,
      "doi": "10.1112/blms.12460",
      "arxiv": "2004.09765",
      "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (rigorous
    interval arithmetic)."
    },
    "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=H0."
  },
  "notes": [
    "This JSON is not itself a computation of zeros; it is a pinned statement+reference used by v36
    .",
```

```
    "For a fully self-contained proof without external computational input, one would need to
    implement and certify an argument-principle zero count in this region using ball arithmetic (
    not provided here)."
  ],
  "manuscript_version": "v36"
}
```

```
H0 (needed) = 5.0
H_cited     = 3000000000000.0
CHECK: H0 <= H_cited : True
PASS
```

```
#!/usr/bin/env python3
"""v34_generate_frontend_certificate.py

Creates a pinned JSON certificate for the finite-height front-end assumption used by v34.

This script does NOT compute zeta zeros. It encodes a minimal (H0, citation) logic statement:
if RH has been verified up to H_cited and H0 <= H_cited, then RH holds up to height H0.

Usage:
  python3 v34_generate_frontend_certificate.py v34_frontend_certificate.json
"""

from __future__ import annotations

import json
from datetime import datetime, timezone
import sys


def main() -> int:
    if len(sys.argv) != 2:
        print("Usage: v34_generate_frontend_certificate.py output.json", file=sys.stderr)
        return 2

    out = {
        "certificate_version": "v34",
        "created_utc": datetime.now(timezone.utc).strftime("%Y-%m-%dT%H:%M:%SZ"),
        "needed_frontend_statement": {
            "type": "RH_to_height",
            "H0": 5.0,
            "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
        },
        "discharged_by": {
            "type": "literature_citation",
            "verification_height": 3e12,
            "reference": {
                "authors": "D. J. Platt and T. S. Trudgian",
                "title": "The Riemann hypothesis is true up to 3*10^12",
                "venue": "Bulletin of the London Mathematical Society",
                "year": 2021,
                "doi": "10.1112/blms.12460",
                "arxiv": "2004.09765",
```

```
            "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (
    rigorous interval arithmetic)."
        },
        "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=
    H0."
      },
      "notes": [
        "This JSON is not itself a computation of zeros; it is a pinned statement+reference
    used by v34.",
        "For a fully self-contained proof without external computational input, one would need
    to implement and certify an argument-principle zero count in this region using ball arithmetic
    (not provided here)."
      ]
    }

    with open(sys.argv[1], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[1])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())

#!/usr/bin/env python3
"""v34_verify_frontend_certificate.py

Verifier for the front-end certificate JSON produced by v34_generate_frontend_certificate.py.

This verifier checks the internal logic only:
- parses the JSON
- confirms that the required finite-height H0 is <= the cited verification height

It does NOT re-run the cited large-scale computation (Platt--Trudgian); that result is treated as
     an
external, peer-reviewed input in the manuscript.

Usage:
  python3 v34_verify_frontend_certificate.py --certificate v34_frontend_certificate.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v34 front-end certificate JSON (internal logic
     only).")
    ap.add_argument("--certificate", required=True, help="Path to v34_frontend_certificate.json")
    args = ap.parse_args()

    with open(args.certificate, "r", encoding="utf-8") as f:
```

```
        cert = json.load(f)

    needed = cert.get("needed_frontend_statement", {})
    discharged = cert.get("discharged_by", {})

    H0 = float(needed.get("H0"))
    Hc = float(discharged.get("verification_height"))

    ok = H0 <= Hc

    print("H0 (needed) =", H0)
    print("H_cited     =", Hc)
    print(f"CHECK: H0 <= H_cited : {ok}")

    if not ok:
        print("FAIL")
        return 1

    print("PASS")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

References

References

[1] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Mathematics (2) **116** (1982), no. 2, 361–387.

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