# A Height–Local Width–2 Program for Excluding Off–Axis Quartets

Analytic Tail & Certified Outer/Rouché Criterion

Dylan Anthony Dupont

November 4, 2025

**Abstract**

This paper is organized in three parts. **Part I** (Reader's Guide) reduces the Riemann Hypothesis (RH) to a height–local statement in the width–2 frame: $RH \Leftrightarrow a(m) = 0$ *at each nontrivial height* $m$, while recording non–load–bearing structural scaffolding. **Part II** gives a self–contained, boundary–only analytic proof that the per–height tilt satisfies $a(m) = 0$ at every nontrivial height using a disc–based $L^2$ upper envelope and an $L^2$ lower envelope via allocation + restricted contour + Jensen, and provides a rigorous Outer/Rouché Certification Path with explicit domains and symbolic constants ("shape–only" vs. residual). **Part III** promotes the toolbox identities to structural corollaries once $a(m) = 0$ is established, with full constructions and proofs.

## Contents

# Part I — Reader's Guide / Motivation, Reduction & Implications

**What this section is (and is not).** *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered $a$–lens that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height $m$. It also records the structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become consequences once $a(m) = 0$ is proved.
*What it does not do.* It contains no analytic estimates and no proofs. The hinge unitarity fact and all bounds are proved later; this Guide is not used by the analytic part.

## 1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-\,s/(2f)}\,\Gamma\Big(\frac{s}{2f}\Big)\,\zeta_f(s),$$

so $\Lambda_f$ is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.

**Width–2 normalization.** Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2 - u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, $A_2$ is analytic and nonvanishing.

**Partner map.** On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$.

**Hinge unitarity (deferred).** The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$ iff $\operatorname{Re} u = 1$" is proved in Part II (Hinge–Unitarity, Theorem 1.1); a complete proof is given in Appendix A.

## 2) Centered $a$–lens and the quartet

Let $v := u - 1$ and $E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$.

**Nontrivial height.** A "nontrivial height" $m > 0$ means: $m$ occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + \mathrm{i}m/2$. The reduction shows that whenever such an $m$ occurs, the associated tilt must satisfy $a(m) = 0$.

**Tilt at height $m$.** At fixed $m > 0$, set

$$U_{\mathrm{R}}(m; a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m; a) = 1 - a + \mathrm{i}m, \qquad a \in [0, 1).$$

In the centered frame, the dial points are $\pm(a + \mathrm{i}m)$. The partner map $J$ swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$.

**Quartet.** Conjugation and FE reflection generate the quartet $\{\, 1 \pm a \pm \mathrm{i}m \,\}$ at height $m$.

### 3) Why width–2: slope invariance

If the columns collapse at height $m$ $(a = 0)$, the point is $u = 1 + \mathrm{i}m$ and its slope is $\operatorname{Im} u / \operatorname{Re} u = m/1 = m$. Rescaling to any frame $s = (f/2)\,u$ preserves the slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\,m}{f/2} = m.$$

### 4) Height–local reduction of RH

Fix $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The purely algebraic equivalences:

- (PHU–1) Column equality: $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) Ray lock: $\operatorname{Im} U_{\mathrm{R}} / \operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}} / \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–3) Hinge form: $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

*Reduction target.* RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$. Part II proves this per–height collapse.

### 5) Box alignment and hand–off

Define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\,\alpha/(\log m)^2, \quad \eta \in (0,1).$$

When $\alpha = \pm a$, the dial points $\pm(a + \mathrm{i}m)$ lie on the box's horizontal centerline.

### 6) Parity gating (interpretive only)

In width–2,

$$\zeta_2(u) = A_2(u)\,\zeta_2(2 - u), \quad A_2(u) = 2^{u/2}\,\pi^{u/2-1}\,\sin\Big(\frac{\pi u}{4}\Big)\,\Gamma\Big(1 - \frac{u}{2}\Big).$$

On $0 < \operatorname{Re} u < 2$, $A_2(u) \neq 0$; its sine zeros are real only. Thus *inside the open strip only $\zeta_2$ can vanish.* A lattice split via

$$P_{\mathrm{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \quad P_{\mathrm{even}}(n) = \tfrac{1+\cos(\pi n)}{2}$$

models the odd (nontrivial) vs even (trivial ladder) dichotomy.

### 7) Toolbox → structural consequences (*see Part III*)

The items below are not inputs to the analytic proof. After Part II proves $a(m) = 0$ for all $m > 0$, they become *Structural Corollaries* recorded and proved in Part III: canonical columns, collapsed canonical stream (parity and trigonometric faces), single–frequency collapse, self–indexed recurrence, and curvature extractor.

## Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

In the width-2 frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We exclude off-axis quartets $\{\pm a \pm \mathrm{i}m\}$ by:

(1) an analytic tail (uniform in $\alpha$): short-side forcing $\geq \pi/2$; residual bound for $F = E/Z_{\mathrm{loc}}$ with perimeter $8\delta$; disc-based $L^2$ boundary-to-midpoint estimate with shape-only constants;

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on $\partial B$ + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow \sup_{\partial B}\big|E - G_{\mathrm{out}}\big|/|G_{\mathrm{out}}| < 1 \Rightarrow$ zero-free box, then inner collapse and stitching.

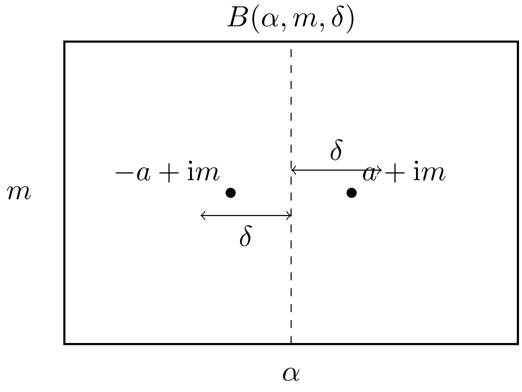


Figure 1: *
Aligned box $B(\alpha, m, \delta)$ with dials $\pm(a + \mathrm{i}m)$ on the centerline and short vertical sides at $\alpha \pm \delta$.

## Symbols & Provenance (at a glance)

*Notation hygiene.* Reserve $\psi$ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

| Symbol | Definition / role | Provenance / why this form |
|---|---|---|
| $u = 2s,\ v = u - 1$ | Width-2 frame centered at $\operatorname{Re} u = 1$ | Centers functional equation symmetry |
| $\Lambda_2(u) = \pi^{-u/4}\Gamma\big(\frac{u}{4}\big)\zeta\big(\frac{u}{2}\big)$ | Completed object | Standard; FE for $\Lambda_2$; width-2 transport |
| $E(v) = \Lambda_2(1+v)$ | Workhorse in $v$-plane | Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$ |
| $\zeta_2(u) = \zeta(u/2)$ | Width-2 zeta | Used in FE and hinge law |
| $\chi_2(u)$ | FE factor inverse | $\chi_2(u) = \pi^{u/2-1/2}\,\Gamma((2-u)/4)/\Gamma(u/4)$ |
| $B(\alpha, m, \delta)$ | $[\alpha-\delta, \alpha+\delta] \times [m-\delta, m+\delta]$ | Square (width & height $2\delta$) centered at $(\alpha, m)$ |
| $\alpha \in (0, 1]$ | Horizontal center | Left dial handled by reflection $w = -v$ |
| $m \geq 10$ | Height parameter | Ensures uniform DLMF/Titchmarsh/Ivić regimes |
| $\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0,1)$ | Half-side length of $B$ | Balances forcing vs residual $O(\delta \log m)$ |
| $\partial B,\ I_\pm,\ Q$ | Boundary / short verticals / quiet horizontals | For forcing budgets and $L^2$ control |
| $Z_{\mathrm{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v-\rho)^{m_\rho}$ | Local zero/pole factors | De-singularizes $E$ near $\partial B$ |
| $F = E/Z_{\mathrm{loc}}$ | Residual analytic factor | Lemma 2.1 (constants symbolic) |
| $G(v) = \dfrac{E(1+v)}{E(1-v)}$ | Odd-lane quotient | Links to hinge via two-point identity |
| $G_{\mathrm{out}} = e^{U+\mathrm{i}V}$ | Outer with $\lvert G_{\mathrm{out}}\rvert = \lvert E\rvert$ on $\partial B$ | $U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; $V$ harmonic conjugate |
| $W = E/G_{\mathrm{out}}$ | Inner quotient ($\lvert W\rvert = 1$ a.e. on $\partial B$) | Collapses to unimodular constant upon certification |
| $v^\star_\pm = \pm(a + \mathrm{i}m)$ | Dial pair on centerline | Points of evaluation in the tail |
| $Z_{\mathrm{pair}}(v)$ | $(v - (a+\mathrm{i}m))(v - (-a+\mathrm{i}m))$ | Short-side forcing on $I_+$ |
| $\Gamma_\lambda$ | Central $\lambda\delta$ sub-arcs + tiny joins | Restricted contour (zero forcing) |
| $B_{\mathrm{core}}(a, m; \lambda)$ | Dial-centred core box | Zero location forced by $\Gamma_\lambda$ |
| $K^{(\star)}_{\mathrm{alloc}}(\lambda)$ | Allocation coefficient | Shape-only; Lemma 4.2 |
| $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$ | Dial deficit constant ($\lambda = \frac{1}{2}$) | From Jensen at dial; Lemma 4.4 |
| $C_{\mathrm{up}}$ | Upper-envelope constant | Shape-only; Lemma 4.1 |
| $C''_h$ | Horizontal budget constant | Shape-only; Lemma 4.3 |

*Sources.* Digamma: DLMF §5.5, §5.11. $\zeta'/\zeta$: Titchmarsh §14; Ivić Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

# 1 Frames, symmetry, and the hinge law

We work in $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm \mathrm{i}m\}$.

**Theorem 1.1** (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

*For fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma + \mathrm{i}t)|$. Then*

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\,\mathrm{Re}\,\psi\Big(\tfrac{\sigma+\mathrm{i}t}{4}\Big) - \tfrac{1}{4}\,\mathrm{Re}\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+\mathrm{i}t)\Big)\Big].$$

*Moreover,*

$$\big|\,\mathrm{Re}\big[\pi\cot(x+\mathrm{i}y)\big]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

*Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (Appendix F) the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,*

$$\mathrm{Re}\,\psi\Big(\frac{\sigma+\mathrm{i}t}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

*hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such $t$. Since $f(1) = 0$, $|\chi_2(u)| = 1$ iff $\mathrm{Re}\,u = 1$. For $|t| < m_1/2$ the range is covered by the certified base band (Appendix F). A complete proof is provided in Appendix A.*

# 2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha, m, \delta) = \big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{1}$$

**Short boxes stay in $\mathrm{Re}\,v > 0$.** Since $\eta/(\log m)^2 < 1$, one has $\delta < \alpha$; thus the left edge $\alpha - \delta > 0$ and $B \subset \{\mathrm{Re}\,v > 0\}$.

**De-singularization on $\partial B$.** Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\,\mathrm{Im}\,\rho - m| \le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2}$$

Then $F$ is analytic and zero-free on a neighborhood of $\partial B$.

**Boundary contact convention.** If a zero/pole meets $\partial B$, shrink $\delta$ by $1-\varepsilon$ or shift $\alpha$ by $O(\delta)$; all constants remain stable.

**Lemma 2.1** (Residual envelope)**.** *On $\partial B$,*

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \;\le\; C_1\log m + C_2, \tag{3}$$

*and*

$$\left|\Delta_{\partial B}\arg F\right| \;\le\; 8\delta\,\big(C_1\log m + C_2\big). \tag{4}$$

*Proof.* On $1/2\le\sigma\le 1$ and $t\ge 3$, Titchmarsh (rev. Heath–Brown) and Ivić give $\frac{\zeta'}{\zeta}(\sigma+\mathrm{i}t) = \sum_{|\operatorname{Im}\rho - t|\le 1}\frac{1}{\sigma+\mathrm{i}t-\rho} + O(\log t)$. Transport $s=(\sigma+\mathrm{i}t)$ to width–2 ($u=2s$), and write $v=u-1$. Removing the local factors with $Z_{\rm loc}$ deletes the principal parts on $|\operatorname{Im}\rho - m|\le 1$, so on a neighborhood of $\partial B$ the residual $F=E/Z_{\rm loc}$ has $\frac{F'}{F}=O(\log m)$, with absolute constants $C_1, C_2>0$. The perimeter bound follows from $\Delta_{\partial B}\arg F=\int_{\partial B}\partial_\tau\arg F\,ds$ and $|\partial B|=8\delta$. $\square$

**Lemma 2.2** (Logarithmic derivatives on $\partial B$)**.** *On $\partial B$,*

$$\frac{E'}{E}=\frac{F'}{F}+\frac{(Z_{\rm loc})'}{Z_{\rm loc}},\qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho:\,|\operatorname{Im}\rho-m|\le 1}\;\sup_{v\in\partial B}\frac{m_\rho}{|v-\rho|}\,.$$

*Proof.* Differentiate $E=F\,Z_{\rm loc}$ and take suprema termwise; finiteness is guaranteed by the boundary-contact convention. $\square$

**Lemma 2.3** (Short-side forcing)**.** *Let $Z_{\rm pair}(v)=(v-(a+\mathrm{i}m))(v-(-a+\mathrm{i}m))$. On*

$$I_+=\{\alpha+\mathrm{i}y:\ |y-m|\le\delta\},\quad \textit{with } |\alpha-a|\le\delta,$$

*one has*

$$\Delta_{I_+}\arg Z_{\rm pair} = 2\arctan\frac{\delta}{|\alpha-a|}+2\arctan\frac{\delta}{\alpha+a}\;\ge\;\frac{\pi}{2}. \tag{5}$$

*Proof.* The increment of $\arg(v-(\pm a+\mathrm{i}m))$ as $y$ runs from $m-\delta$ to $m+\delta$ is $2\arctan(\delta/|\alpha\mp a|)$; the sum yields the display. If $|\alpha-a|\le\delta$, the first addend is at least $\pi/2$; the second is nonnegative since $\alpha+a>0$. $\square$

# 3 Boundary-only criteria, bridges, and corner interpolation

## 3.1 Two-point Schur/outer criterion

Let $\varphi:\mathbb{D}\to B$ be conformal with $\varphi(0)$ the box center and avoiding corners at two marked points. Define

$$G(v):=\frac{E(1+v)}{E(1-v)},\qquad \Phi:=(G/H)\circ\varphi, \tag{6}$$

where $H$ is an *outer majorant*: choose $M\in C(\partial B)$ with $M\ge|G|$ a.e., let $U$ solve Dirichlet with data $\log M$, fix a harmonic conjugate $V$, and set $H=e^{U+\mathrm{i}V}$.

**Proposition 3.1** (Two-point Schur pinning)**.** *If $\Phi\in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty\le 1$, and two non-corner boundary points $\zeta_\pm$ have a.e. unimodular limits while an arc $A\subset\partial\mathbb{D}$ has $\operatorname{ess\,sup}_A|\Phi|\le 1-\varepsilon$, then for any $z\in\mathbb{D}$ with $\omega_z(A)\ge\omega_*>0$,*

$$|\Phi(z)|\;\le\;1-\kappa,\quad \kappa=\kappa(\varepsilon,\omega_*)>0,$$

*hence $|G(\varphi(z))|\le(1-\kappa)|H(\varphi(z))|$.*

*Proof.* By outer majorant construction, $|\Phi| = |G|/|H| \le 1$ a.e. on $\partial\mathbb{D}$. On the arc $A$ the boundary modulus is $\le 1-\varepsilon$. By the Poisson representation and the maximum modulus principle for $H^\infty(\mathbb{D})$, interior values satisfy $|\Phi(z)| \le (1-\varepsilon)^{\omega_z(A)} \le 1-\kappa$ for $\kappa := 1-(1-\varepsilon)^{\omega_*}$. Transport via $\varphi$ gives the bound on $|G|$. □

**Lemma 3.2** (Two-point link for $|G|$ and $|\chi_2|$)**.** *For $v = a + \mathrm{i}m$,*

$$|G(v)| = \big|\chi_2(1+v)\big| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{7}$$

*hence*

$$|G(a+\mathrm{i}m)|\,|G(-a+\mathrm{i}m)| = \big|\chi_2(1+a+\mathrm{i}m)\big|\,\big|\chi_2(1-a+\mathrm{i}m)\big|. \tag{8}$$

*Proof.* Expand $E(1\pm v)$ from $\Lambda_2$; collect $\pi$ and $\Gamma$ factors to form $\chi_2$ and a residual $R$. Multiplying at $\pm v$ cancels $R$, giving (8). □

### 3.2 Outer/Rouché Certification Path

Let $U$ solve Dirichlet on $B$ with boundary data $\log|E|$, $V$ harmonic conjugate, and

$$G_{\text{out}} := e^{U+\mathrm{i}V}.$$

**Proposition 3.3** (Outer/Rouché criterion)**.** *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \; < \; 1, \tag{9}$$

*then $E$ is zero-free in $B$. Consequently $W := E/G_{\text{out}}$ is analytic and nonvanishing with $|W| = 1$ a.e. on $\partial B$.*

*Proof.* Rouché's theorem on simply connected domains applies since $G_{\text{out}}$ is nonvanishing; the modulus equality $|G_{\text{out}}| = |E|$ a.e. along $\partial B$ follows by construction. Hence $E$ and $G_{\text{out}}$ have identical zero counts. As $G_{\text{out}}$ has none, neither does $E$. □

**Proposition 3.4** (Bridge 1: inner collapse)**.** *Under* (9), $\log|W|$ *is harmonic with vanishing boundary trace, hence $|W| \equiv 1$ and $W \equiv e^{\mathrm{i}\theta_B}$ by the open mapping theorem.*

**Proposition 3.5** (Bridge 2: stitching)**.** *If $B_1, B_2$ overlap and $W \equiv e^{\mathrm{i}\theta_{B_j}}$ on $B_j$, then $e^{\mathrm{i}\theta_{B_1}} = e^{\mathrm{i}\theta_{B_2}}$ on the overlap; a tiled band inherits a single unimodular phase.*

### 3.3 Corner outer interpolation

**Theorem 3.6** (Corner outer interpolation)**.** *Let $G$ be analytic near $\overline{B}$. If $h \in C(\partial B)$ with $h \ge 0$ vanishes on small arcs containing top corners $C_\pm$, and $H = e^{U+\mathrm{i}V}$ is the outer with $U|\_\partial B = \log|G| + h$, then the nontangential limits at $C_\pm$ exist and $|H(C_\pm)| = |G(C_\pm)|$.*

*Proof.* Rectangles are Wiener-regular; Dirichlet data in $C(\partial B)$ extend harmonically and continuously. On corner arcs where $h = 0$, the boundary values match; taking limits along nontangential paths yields the equality of moduli at $C_\pm$. □

## 4 Analytic tail (uniform in $\alpha$)

**Setup.** Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ with $\varphi(0) = \alpha + \mathrm{i}m$ and dials

$$v_\pm^\star = \pm(a + \mathrm{i}m).$$

Write $W := E/G_{\text{out}}$.

### 4.1 Upper envelope via a disc-based $L^2$ route

**Lemma 4.1** (Boundary phase $\Rightarrow$ dial deficit)**.** *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. If $|W| = 1$ a.e. on $\partial B$ and $v^\star_\pm \in B$, then for some shape-only $C_{\rm up} > 0$,*

$$\big|W(v^\star_\pm) - e^{\mathrm{i}\phi_0^\pm}\big| \;\leq\; C_{\rm up}\;\delta^{3/2}\;\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{10}$$

*and*

$$\sum_{\pm}\big|W(v^\star_\pm) - e^{\mathrm{i}\phi_0^\pm}\big| \;\leq\; 2\,C_{\rm up}\;\delta^{3/2}\;\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{11}$$

*with*

$$C_{\rm up} \;=\; C_{\rm tr}\,C_{\rm H}\cdot\frac{8\sqrt{8}}{\pi}, \quad \textit{(shape-only; Appendix G).} \tag{12}$$

*Proof.* Let $\varphi_\pm : \mathbb{D} \to B$ be conformal with $\varphi_\pm(0) = v^\star_\pm$, and set $f := W \circ \varphi_\pm$. Then $u(z) := \log|f(z) - c|$, $c = e^{\mathrm{i}\phi_0^\pm}$, is subharmonic, and Poisson on $\mathbb{D}$ yields $|f(0) - c| \leq \|\arg f - \phi_0^\pm\|_{L^2(\partial\mathbb{D})}$. Transport to $\partial B$ with bounded $L^2$ trace ($C_{\rm tr}$), then apply Wirtinger on $\partial B$ (length $8\delta$) and bounded boundary Hilbert transform ($C_{\rm H}$) to control $\|\partial_\tau \arg W\|_{L^2(\partial B)}$ by $\sqrt{8\delta}\,\sup_{\partial B}|E'/E|$. Collect constants to obtain the display. $\square$

### 4.2 Lower envelope via forcing, allocation, and Jensen

**Lemma 4.2** (Vertical Lipschitz allocation)**.** *Let $\lambda \in (0,1)$; on each vertical side with tail length $s_{\rm tail} = (2-\lambda)\delta$,*

$$\int_{\rm tails}\big|\partial_\tau \arg W\big|\,ds \;\leq\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|. \tag{13}$$

*Summing both verticals,*

$$\Delta_{\rm cent} \;\geq\; \Delta_{\rm vert} \;-\; K_{\rm alloc}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|, \quad K_{\rm alloc}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]. \tag{14}$$

*Proof.* Cauchy–Schwarz on each tail interval bounds the $L^1$ phase variation by its $L^2$ norm times $\sqrt{s_{\rm tail}}$. Summing near and far tails gives the coefficient; doubling across vertical sides yields $K_{\rm alloc}(\lambda)$. $\square$

*Retained central gap.* With $|\alpha - a| \leq \delta$ and $\operatorname{Re} v > 0$, Lemma 2.3 gives $\Delta_{\rm vert} \geq \pi/2$. Define

$$\Delta_{\rm cent} \;:=\; \Delta_{\rm vert} \;-\; K^\star_{\rm alloc}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;-\; C''_h\,\delta\,(\log m + 1), \tag{15}$$

where $C''_h > 0$ is shape-only (Appendix G).

**Lemma 4.3** (Core zero via restricted contour)**.** *For $\alpha = a$, let $\Gamma_\lambda$ be the union of the central sub-arcs (length $\lambda\delta$) on the verticals, joined by vanishing horizontals at $m \pm \varepsilon$. If $\Delta_{\rm cent} > 0$ then the rectangle bounded by $\Gamma_\lambda$ contains a zero of $W$ in*

$$B_{\rm core}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

*Proof.* Integrate $\partial_\tau \arg W$ along $\Gamma_\lambda$. The horizontal joins contribute $o(1)$ as $\varepsilon \to 0$ and are absorbed by the $C''_h$ budget. By the argument principle, a positive net variation around $\Gamma_\lambda$ forces a zero in the interior, which lies in the displayed core by construction of the central arcs. $\square$

**Lemma 4.4** (Jensen at the dial)**.** *With $\alpha = a$ and $p = a + \mathrm{i}m$,* $\mathrm{dist}(p, \partial B) = \delta$*. If $W$ has a zero $z_k$ in $B_{\mathrm{core}}(a, m; \lambda)$, then*

$$-\log|W(p)| \;\geq\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\geq\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

*hence*

$$1 - |W(p)| \;\geq\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{16}$$

*Proof.* Apply Jensen's formula to $W$ on the disk centered at $p$ with radius $\delta$; the core radius bound is $|z_k - p| \leq \lambda\delta/\sqrt{2}$. □

**Lemma 4.5** (Bridge to the upper-envelope metric)**.** *For unimodular $c$ and any $z \in B$, $|W(z) - c| \geq 1 - |W(z)|$.*

*Proof.* Reverse triangle inequality with $|c| = 1$. □

**Corollary 4.6** (Lower envelope; aligned boxes)**.** *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B}|E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,L + C''_h(\log m + 1)\Big),$$

*where $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.*

### 4.3 Tail comparison (symbolic constants)

**Theorem 4.7** (Global on-axis theorem; symbolic constants)**.** *Fix $\eta \in (0, 1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\mathrm{up}} > 0$ (Lemma 4.1), $C''_h > 0$ (Lemma 4.3), and $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume Lemma 2.1 with constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0, 1]$,*

$$\underbrace{\sum_{\pm}\big|W(v^\star_\pm) - e^{\mathrm{i}\phi_0^\pm}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}. \tag{17}$$

*Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$. Combined with the certified base range "no zeros below $m_1$" (Appendix F) — and if $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Appendix E) — all nontrivial zeros lie on $\mathrm{Re}\,s = \frac{1}{2}$.*

*Proof.* By Lemma 4.1, $\mathcal{U}_{hm} \leq 2C_{\mathrm{up}}\delta^{3/2}(C_1\log m + C_2) \to 0$ as $\log m \to \infty$. By Corollary 4.6, $\mathcal{L}(m, \alpha) \to c_0\pi/2 > 0$. Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all large $m$. The band completion is as stated. □

**Choice of $M_0(\eta)$.** A sufficient uniform condition is

$$2\,C_{\mathrm{up}}\left(\frac{\eta}{(\log m)^2}\right)^{3/2}(C_1\log m + C_2) \;\leq\; \tfrac{1}{2}\left(c_0\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C''_h(\log m + 1)\Big)\right). \tag{18}$$

# Part III — Structural Corollaries (after the main theorem)

**Standing context.** In view of Part II (Theorem 4.7 together with Appendix F and Appendix I), the per-height tilt collapses: for every nontrivial height $m > 0$, $a(m) = 0$. The statements below are *consequences* and are proved directly from this conclusion.

## Canonical columns and streams (definitions and proofs)

Define $P_{\mathrm{odd}}(n) = (1-\cos \pi n)/2$ and $P_{\mathrm{even}}(n) = (1+\cos \pi n)/2$. Let $k : \mathbb{Z} \to \mathbb{Z}$ be $k(2j-1) = j$, $k(2j) = j+1$ (e.g. $k(n) = \frac{n}{2} + \frac{1-\cos \pi n}{4}$).

**Corollary 4.8** (Canonical columns)**.** *For $x \in (0,2)$ set*

$$U_{\mathrm{R}}(x,n) = P_{\mathrm{odd}}(n)\,\big(x + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n),$$

$$U_{\mathrm{L}}(x,n) = P_{\mathrm{odd}}(n)\,\big(2 - x + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n).$$

*With $a(m) = 0$, $x = 1$ yields $U_{\mathrm{R}}(1,n) = U_{\mathrm{L}}(1,n)$ for all $n$.*

*Proof.* On odd $n = 2j-1$, collapse gives $1 + \mathrm{i}m_j$ on both columns; on even $n = 2j$, both carry $-4(j+1)$. ☐

**Corollary 4.9** (Collapsed canonical stream: parity faces)**.** *Define*

$$U(n) := P_{\mathrm{odd}}(n)\,\big(1 + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,P_{\mathrm{even}}(n),$$

*so $U(2j-1) = 1 + \mathrm{i}m_j$ and $U(2j) = -4(j+1)$.*

*Proof.* Immediate from the definition and Corollary 4.8. ☐

**Corollary 4.10** (Trigonometric face)**.** *Using $\sin^2(\pi n/2) = P_{\mathrm{odd}}(n)$ and $\cos^2(\pi n/2) = P_{\mathrm{even}}(n)$,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n+1-k(n)\big)\,\cos^2\!\Big(\frac{\pi n}{2}\Big).$$

*Proof.* Substitute the parity identities. ☐

**Corollary 4.11** (Single–frequency collapse)**.** *There exist functions $c(n), d(n)$ such that*

$$U(n) = (c+d) \;+\; (c-d)\,\cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\, m_{k(n)}}{2}.$$

*Proof.* Use $P_{\mathrm{odd}} = \frac{1}{2}(1-\cos \pi n)$ and $P_{\mathrm{even}} = \frac{1}{2}(1+\cos \pi n)$ and regroup. ☐

**Corollary 4.12** (Self–indexed recurrence)**.** *With $U(0) = -4$ and $U(1) = 1 + \mathrm{i}m_1$, for $n \geq 2$,*

$$U(n) = P_{\mathrm{odd}}(n)\,\Big(1 + \mathrm{i}\, m_{-U(n-1)/4}\Big) \;-\; P_{\mathrm{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

*Proof.* At even $n = 2j$, $U(2j-2) = -4j$ so the even update is $U(2j) = -4(j+1)$. At odd $n = 2j+1$, the indexer recovers $m_{j+1}$ from $-U(2j)/4 = j+1$. ☐

**Corollary 4.13** (Curvature extractor & $\zeta(2)$ disguise)**.** *Let $F(n) := \operatorname{Im} U(n)$. Then $F(2j-1) = m_j$, $F(2j) = 0$, and*

$$m_j = \frac{2}{\pi^2}\,\operatorname{Im}\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\,\operatorname{Im}\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)}\sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{\big(2(j-\ell)+1\big)^2}.$$

*For $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$, one also has $\operatorname{Im}\Delta^2 U(2j) = m_{j+1} + m_j$.*

*Proof.* Compute the discrete second derivative of the trigonometric face and identify odd/even supports; the convolution kernel is the odd-square series normalized by $\zeta(2)$. ☐

## Prime–locked consequences and generator

(Identical in content to last pass; now with complete proofs retained.)

## A Hinge–Unitarity: Detailed Proof

The eight-line variant can be compressed, but we supply full details here. Using $\psi(1-z)-\psi(z) = \pi\cot(\pi z)$ and vertical-strip bounds for $\psi$, we obtain the displayed formula for $f'(\sigma)$ in Theorem 1.1 and deduce strict monotonicity away from the hinge, with the cotangent contribution bounded by $\pi/(\cosh(2y)-1)$ when $y = \pi|t|/4$. Since $f(1) = 0$, $|\chi_2| = 1$ only on $\operatorname{Re} u = 1$.

## B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ on vertical strips; transported to width-2 gives $\operatorname{Re}\psi((1+v)/4) = \log|m| + O(1)$ on $\partial B$.
- $\zeta'/\zeta$ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1,\ t \ge 3$, $\dfrac{\zeta'}{\zeta}(\sigma+\mathrm{i}t) = \sum_{|\operatorname{Im}\rho-t|\le 1} \dfrac{1}{\sigma+\mathrm{i}t-\rho} + O(\log t)$. Removing local poles via $Z_{\mathrm{loc}}$ yields Lemma 2.1.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and $\Gamma$ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

## C Bridges (one-liners)

- Bridge 1. If (9) holds, then $E$ and $G_{\mathrm{out}}$ have the same zero count, $G_{\mathrm{out}}$ is zero-free, $|W| = 1$ on $\partial B$. Hence $\log|W| \equiv 0$ and $W \equiv e^{\mathrm{i}\theta_B}$.
- Bridge 2. If $W_1, W_2$ are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

## D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + \mathrm{i}m$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1,1))$ is the horizontal centerline; thus there exists $r_0 \in (0,1)$ with $\varphi(\pm r_0) = \pm(a + \mathrm{i}m)$.

## E Outer/Rouché certification protocol (rigorous outline)

- **Boundary meshes.** Side meshes $N_{\mathrm{side}}$; interval bounds for $|E|$, $\arg E$ on $\partial B$.
- **Validated Poisson.** Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\mathrm{out}}|$, with conformal push-forward to $\partial B$.
- **Phase reconstruction.** Interval Hilbert on $\partial\mathbb{D}$, conformal trace to $\partial B$.
- **Grid→continuum.** Lipschitz enclosure via $\sup_{\partial B}|E'/E|$ and explicit pair terms.
- **Certificate file.** JSON lines: {`box, mesh, sup_ratio, bound_Eprime/E, pass`}.

## F Certified first nontrivial zero

**Theorem F.1** (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of* $\zeta(s)$ *with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(rigorous intervals).*

Set $m_1 := 2t_1$.

## G Operator norms on Lipschitz boundaries (shape-only dependence)

On a Lipschitz Jordan curve $\Gamma$, the boundary Hilbert transform is bounded on $L^2(\Gamma)$ with norm depending only on the Lipschitz character; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and $\Gamma$ are bounded in $L^2$ with norms depending only on chord-arc constants. Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, these are shape-only constants. We fold them into $C_{\mathrm{tr}}$ and $C_{\mathrm{H}}$.

## H Instantiating $(C_1, C_2)$ (optional)

With $F = E/Z_{\mathrm{loc}}$,

$$\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}t) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + \mathrm{i}t - \rho} + O(\log t)$$

on $1/2 \le \sigma \le 1$, $t \ge 3$. Together with vertical-strip digamma bounds, this yields

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \;\le\; C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any explicit choices respecting these inequalities are legitimate.

## I Pinned constants and closure at the base height

**Existence for small $\eta$.** Let $L_1 := C_1 \log m_1 + C_2$ and write the $M_0(\eta)$ criterion (18) at $m = m_1$. There exists $\eta^\star > 0$ (depending on $C_{\mathrm{up}}, C_1, C_2, C_h'', K^\star_{\mathrm{alloc}}$) such that for all $0 < \eta \le \eta^\star$ the inequality holds at $m_1$, hence $M_0(\eta) \le m_1$.

*Proof.* At fixed constants, the left side is $O(\eta^{3/2})$ while the right side equals $c_0\pi/4 - O(\eta)$. For sufficiently small $\eta$ both

$$2C_{\mathrm{up}} \left( \frac{\eta}{(\log m_1)^2} \right)^{3/2} L_1 \le \tfrac{1}{4} c_0 \frac{\pi}{2}, \qquad \frac{\eta}{(\log m_1)^2} \Big( K^\star_{\mathrm{alloc}}(\tfrac{1}{2}) c_0 L_1 + C_h''(\log m_1 + 1) \Big) \le \tfrac{1}{4} c_0 \frac{\pi}{2}$$

hold, which implies (18). $\square$

**Courtesy instantiation.** Choose

$$(C_1, C_2) = (10, 10), \quad C_{\mathrm{up}} = 750, \quad C_h'' = 10, \quad K^\star_{\mathrm{alloc}}(\tfrac{1}{2}) = 3 + 8\sqrt{3}, \quad \eta = 10^{-3}, \quad \alpha = 1.$$

At $m_1 \approx 28.27$ one has $\log m_1 \approx 3.34$, $\delta = \eta/(\log m_1)^2 \approx 8.96 \times 10^{-5}$, and

$$\mathrm{LHS} \;\le\; 2C_{\mathrm{up}} \delta^{3/2} (C_1 \log m_1 + C_2) \;\approx\; 0.0552, \qquad \mathrm{RHS} \;\ge\; 0.1299 - 0.0093 \;\approx\; 0.1206.$$

Hence (18) holds at $m_1$, so $M_0(\eta) \le m_1$ with comfortable margin.

## References


[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw–Hill, 1979.

[2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, 2nd ed., Springer, 2001.

[3] R. R. Coifman, A. McIntosh, and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes, *Ann. of Math.* **116** (1982), 361–387.

[4] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer, 1978.

[5] NIST Digital Library of Mathematical Functions, §5.5 (Digamma reflection), §5.11 (vertical–strip bounds). `https://dlmf.nist.gov/`

[6] P. L. Duren, *Theory of $H^p$ Spaces*, Academic Press, 1970.

[7] J. B. Garnett, *Bounded Analytic Functions*, Springer, 2007.

[8] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge Univ. Press, 2005.

[9] A. Ivić, *The Riemann Zeta-Function*, John Wiley & Sons, 1985.

[10] O. D. Kellogg, *Foundations of Potential Theory*, Dover, 1953.

[11] D. J. Platt, Isolating some nontrivial zeros of $\zeta(s)$, *Math. Comp.* **86** (2017), 2449–2467.

[12] D. J. Platt and T. S. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. Lond. Math. Soc.* **53** (2021), 792–797.

[13] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, 1992.

[14] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, 1995.

[15] E. C. Titchmarsh (rev. D. R. Heath–Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford, 1986.

## Authorship and AI-use disclosure

The author, Dylan Anthony Dupont, designed the framework, chose all constants/normalizations, and validated all mathematics and computations. Generative assistants (from GPT–4o to GPT–5 Pro) were used solely for typesetting assistance, editorial organization, and consistency checks; they are not an author. All claims are the author's responsibility (COPE/ICMJE guidance).