A Height-Local Width-2 Program for Excluding Off-Axis Quartets
with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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Abstract

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\text{loc}}$ with the correct perimeter factor 8δ; and (iii) an L^2+harmonic-measure boundary-to-midpoint estimate (no L^∞ Hilbert transform);

(2) a rigorous certified outer/Rouché route (C4–O): interval arithmetic on ∂B + validated Poisson + Lipschitz grid→continuum enclosure ⇒ $\sup_{\partial B}|E - G_{\text{out}}|/|G_{\text{out}}| < 1$ ⇒ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation (S1′) from continuous Dirichlet data, replacing prior JC/L^∞ pitfalls. The tail is stated symbolically: for each fixed $\eta \in (0, \frac{1}{2}]$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Choosing $\eta \leq \min\{\eta_1, \eta_2\}$ so that $M_0(\eta) \leq m_1$ (first nontrivial height in width-2) yields the global on-axis theorem: no off-axis quartets exist at any height; all nontrivial zeros lie on $\operatorname{Re} s = \frac{1}{2}$. The certified route provides an independent rigorous alternative for any finite band. A Symbols & Provenance table and a constants ledger make the paper self-contained.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for the conformal map used later (to avoid any clash).

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1+v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$A_2(u),\ \chi_2(u) = A_2(u)^{-1}$	FE factors for ζ_2	Classical; $\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$\big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center (distance from hinge $\operatorname{Re} v = 0$)	Uniform-in-α statements use worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform regimes for DLMF/Titchmarsh/Ivić inputs
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, \frac{1}{2}]$	Half-side length of B	Balances forcing $(\pi/2)$ vs residual $O(\delta \log m)$; uniform in α
∂B	Boundary of $B(\alpha, m, \delta)$	Used for all boundary integrals / suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	L^2-controlled in tail estimates
$Z_{\rm loc} = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v-\rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\rm loc}$	Residual analytic factor (nonvanishing near ∂B)	Lemma 2.2: $\sup_{\partial B}\left\lvert\frac{F'}{F}\right\rvert \leq C_1 \log m + C_2$
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity (used in §3)
$G_{\rm out} = e^{U+iV}$	Outer with $\lvert G_{\rm out}\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\rm out}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant under C4–O (Bridge 1)
$v^\star_\pm = \pm(a + im)$	"Dial pair" on centerline	Points of evaluation in the tail (§4)
$\psi(z)$	Digamma function $\Gamma'(z)/\Gamma(z)$	DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds)
$C_1 = 46,\ C_2 = 8$	Residual envelope constants	From DLMF § 5.11; Titchmarsh § 14; Ivić Ch. 9 (width-2 transport)
$c_0 = \frac{1}{20}$	Phase→deficit constant	Conservative Poisson–Jensen/Lipschitz on rectangles
$C_{\rm rect}, K_{\rm rect}, C_h, C'_h$	Geometry/L^2 trace constants	Depend only on rectangle shape; independent of m, α

Sources (for Part I). Digamma: DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, § 14 (esp. Thms. 14.5–14.9); Ivić, *The Riemann Zeta-Function*, Ch. 9.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

(i) *If $\zeta_2(p) \neq 0$ and $|\zeta_2(2-p)| = |\zeta_2(\overline{p})|$, then $|\chi_2(p)| = 1$ and hence $\operatorname{Re} p = 1$.* (ii) *If p_0 is a zero of multiplicity $r \geq 1$ and $|\zeta_2^{(r)}(2-p_0)| = |\zeta_2^{(r)}(\overline{p_0})|$, then $\operatorname{Re} p_0 = 1$.*

Proof sketch. Apply FE and conjugation to obtain $|\zeta_2(2-p)| = |A_2(p)|^{-1}|\zeta_2(p)| = |\zeta_2(\overline{p})|$, hence $|A_2(p)| = 1$ and $|\chi_2(p)| = 1$. Using the digamma reflection identity $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$ (DLMF § 5.5) and vertical-strip bounds (DLMF § 5.11) one checks $\operatorname{Re} u \mapsto \log|\chi_2(u)|$ is strictly monotone with a unique zero at $\operatorname{Re} u = 1$. The zero case follows by differentiating FE r times at p_0. A fully detailed 8-line proof appears in Appendix A. □

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1+\lambda(u-1)$ preserve rays; $\tan\theta = \operatorname{Im} v/\operatorname{Re} v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \geq 10$, $\alpha \in (0, 1]$, and

$$B(\alpha, m, \delta) = \big[\alpha - \delta, \alpha + \delta\big] \times \big[m - \delta, m + \delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0, \tfrac{1}{2}]. \tag{2.1}$$

Why $m \geq 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF § 5.11) and of the ζ'/ζ expansions on $1/2 \leq \sigma \leq 1,\ t \geq 3$ (Titchmarsh § 14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \geq 3$ corresponds to $m \geq 6$; we take $m \geq 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\geq \pi/2$) against residual budgets $O(\delta \log m)$ and yields an L^2+harmonic-measure upper envelope (in § 4) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$). *For $m \geq 10$ and $\eta \leq \frac{1}{2}$, we have $\eta/(\log m)^2 \leq 0.1$, hence $\delta \leq 0.1\,\alpha$ and $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$.*

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m| \leq 1} (v - \rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B.

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (Lemma 2.2, Lemma 2.3) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope). *On ∂B,*

$$\sup_{\partial B}\left|\frac{F'}{F}\right| \ \le\ C_1\log m + C_2, \qquad (C_1,C_2)=(46,8), \tag{2.3}$$

and

$$\left|\Delta_{\partial B}\arg F\right| \ \le\ 8\delta\left(C_1\log m + C_2\right). \tag{2.4}$$

Justification. *DLMF § 5.11 controls ψ on vertical strips; Titchmarsh § 14 (esp. Thms. 14.5–14.9) and Ivić Ch. 9 control ζ'/ζ on $1/2\le\sigma\le 1,\ t\ge 3$. After removing local poles via* (2.2) *and transporting to width-2, we obtain* (2.3)*;* (2.4) *is perimeter 8δ times the sup.*

Lemma 2.3 (Short-side forcing). *Let $Z_{\mathrm{pair}}(v)=(v-(a+im))(v-(-a+im))$. On the near vertical*

$$I_+=\{\alpha+iy:\ |y-m|\le\delta\},\quad \textit{with } |\alpha-a|\le\delta,$$

one has

$$\Delta_{I_+}\arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha-a|}+2\arctan\frac{\delta}{\alpha+a}\ \ge\ \frac{\pi}{2}. \tag{2.5}$$

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi:\mathbb{D}\to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v):=\frac{E(1+v)}{E(1-v)}, \qquad \Phi:=(G/H)\circ\varphi, \tag{3.1}$$

where H is an *outer majorant* for G on B: that is, $M\in C(\partial B)$ with $M\ge|G|$ a.e. on ∂B and $H=e^{U+iV}$ where U is the continuous Dirichlet solution with boundary data $\log M$ and V a harmonic conjugate (uniqueness modulo a unimodular constant). Then $\Phi\in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty\le 1$; we call this the *two-point Schur/outer criterion.*

Remark 3.1 (How the criterion is used). If a verified boundary pattern places $|\Phi|$ at 1 at two designated boundary points (non-corner, in the sense of angular limits) and strictly below 1 on the complementary arcs (“quiet-arc contraction”), then the Carathéodory–Julia theory for angular derivatives yields unimodular boundary pins at those points for Φ; transporting back to B gives quantitative constraints on $|G(\pm(a+im))|$. We emphasize this is a *criterion*: we do not assert interior unimodularity of Φ. See Duren [**?**, Chs. II, IV–V] and Garnett [**?**, Chs. II–III].

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$). *For $v=a+im$ one has*

$$|G(v)|=\left|\chi_2(1+v)\right|\cdot R(v), \qquad R(-v)=R(v)^{-1}, \tag{3.2}$$

hence

$$|G(a+im)|\,|G(-a+im)|=\left|\chi_2(1+a+im)\right|\left|\chi_2(1-a+im)\right|. \tag{3.3}$$

Here

$$R(v)=\pi^{-a}\left|\frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)}\right|\left|\frac{\zeta\!\left(1+\frac{v}{2}\right)}{\zeta\!\left(1-\frac{v}{2}\right)}\right|, \qquad R(-v)=R(v)^{-1}.$$

Proof sketch. *Expand Λ_2 at $1\pm v$ and collect Γ and π factors; the stated identity follows directly; multiplying at $\pm v$ cancels R and yields* (3.3)*. If $|G(\pm(a+im))|=1$, then $|\chi_2(1\pm(a+im))|=1$ and Theorem 1.1 forces $a=0$.*

3.2 Outer/Rouché Criterion (Certification Path)

Let $U = \log|E| \in C(\overline{B})$ solve the Dirichlet problem on B and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (a.e. with respect to arclength). Existence/uniqueness of G_{out} (up to a unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

Proposition 3.3 (Outer/Rouché Criterion)**.** *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \;<\; 1, \tag{3.4}$$

*then E is zero-free in B (Rouché's theorem; e.g. Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.*

Proposition 3.4 (Bridge 1: inner collapse)**.** *Under* (3.4)*, $\log|W|$ is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B.*

Proposition 3.5 (Bridge 2: stitching)**.** *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

Remark 3.6 (Certification recipe and reproducibility). The verification of (3.4) is performed by a robust, rigorous pipeline detailed in Appendix G: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log|G_{\text{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid→continuum Lipschitz enclosure using $\sup_{\partial B}|E'/E|$ (Lemma 2.2). Appendix G also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility. No interior zero-freeness is assumed unless deduced from (3.4).

3.3 Corner outer interpolation (two-point)

Theorem 3.7 (Corner outer interpolation)**.** *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+iV}$ be the outer on B with $U|\_\partial B = \log|G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Proof sketch. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to $\overline{B}$ (Kellogg, Ch. VI; Axler–Bourdon–Ramey, Thm. 6.12). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors, Ch. VIII; Pommerenke, §§2–3). A full proof is provided in Appendix F. □

Remark 3.8 (Non-circularity in §3). All steps above are boundary-only. In particular, the Schur/outer criterion uses a boundary majorant $M \geq |G|$ and outer synthesis for H; the Outer/Rouché criterion derives interior zero-freeness *only* from the verified ratio (3.4); and the corner interpolation is a statement about nontangential boundary limits of outer functions with continuous boundary data.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v_\pm^\star = \pm(a + im), \qquad z_\pm \in \partial\mathbb{D} \ \text{ with } \ \varphi(z_\pm) = v_\pm^\star.$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}, \qquad f := W \circ \varphi^{-1} \in H^\infty(\mathbb{D}).$$

(Boundedness: G_{out} is zero-free, W is analytic on the compact B.)

4.1 Upper envelope via L^2 and harmonic measure

Lemma 4.1 (Boundary phase $\Rightarrow$ dial-pair deficit)**.** *There exist* shape-only *constants* $C_{\text{rect}}, K_{\text{rect}} > 0$ *such that, for suitable anchor phases* $\phi_0^\pm$ *(the harmonic-measure averages of* $\arg W$ *seen from* $v_\pm^\star$*),*

$$\left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \ \le\ C_{\text{rect}}\Big(\sqrt{8\delta} + 2\delta\Big)\Big(C_1 \log m + C_2\Big) \ \le\ K_{\text{rect}}\left(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\right). \tag{4.1}$$

Consequently, summing at the two dial points,

$$\mathcal{U}_{hm}(m, \alpha) := \sum_\pm \left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \ \le\ 2K_{\text{rect}}\left(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\right). \tag{4.1.1}$$

Proof idea. Apply the Poisson sub-mean inequality to $\log|f - c|$ with $c = e^{i\phi_0^\pm}$; use $|e^{i\theta} - 1| \le 2|\sin(\theta/2)|$. Control the quiet arcs in L^2 via the boundary Hilbert transform isometry on $\partial\mathbb{D}$ (M. Riesz; see Duren [**?**, §§I.3, I.6–I.7]), and the conformal L^2 trace to ∂B on Lipschitz boundaries (Coifman–McIntosh–Meyer). Control the verticals by arclength times $\sup_{\partial B}|E'/E|$ from (2.3) (i.e., (2.3)). Side-lengths give the $\sqrt{\delta}$ and δ factors. Background on harmonic measure and Poisson kernels: Ransford [**?**, §3.9], Garnett–Marshall [**?**, Chs. IV–V]. □

4.2 Lower envelope via forcing and residual budgets

We track phases first for $\arg E$. By Lemma A (short-side forcing; see (2.5)) one has on the near vertical

$$\Delta_{I_+} \arg E - \Delta_{I_-} \arg E \ \ge\ \frac{\pi}{2} \quad \text{when } |\alpha - a| \le \delta.$$

Subtract vertical residuals using (2.3)–(**??**) ((2.3)–(2.4)) and bound the horizontal budget for $\arg G_{\text{out}}$ on Q by the same L^2 method as above. Convert the resulting side gap to a dial-pair *modulus* deficit for W via a boundary-to-point estimate on rectangles (Poisson–Jensen/Lipschitz).

Lemma 4.2 (Forcing vs budgets $\Rightarrow$ dial-pair deficit)**.** *There exist* $c_0 \in (0,1)$ *and a shape-only constant* $C_h' > 0$ *such that*

$$\mathcal{L}(m, \alpha) \ :=\ \sum_\pm \big||W(v_\pm^\star)| - 1\big| \ \ge\ c_0\,\frac{\pi}{2} \ -\ \delta\Big(2c_0(C_1 \log m + C_2) + C_h'(\log m + 1)\Big). \tag{4.2}$$

> **Auxiliary boundary-to-point estimate (used in the proof).** If H is harmonic on B, $J \subset \partial B$ is a side, p is the midpoint of the opposite side, $\operatorname{osc}_J H \ge \Delta$, and $\sup_{\partial B}|\nabla H| \le L$, then
>
> $$|H(p) - H(p_J)| \ \ge\ c_{\text{side}}\,\Delta \ -\ C_{\text{side}}\,(\text{length}\,\partial B)\,L, \tag{4.2.1}$$
>
> where p_J is the harmonic-measure average of $H|_J$ seen from p, and $c_{\text{side}}, C_{\text{side}} > 0$ depend only on the rectangle aspect. Apply with $H = \log|W|$; absorb constants into c_0, C_h'.

4.3 Tail comparison (analytic, uniform in α)

Theorem 4.3 (Tail Comparison Theorem (analytic))**.** *Fix $\eta \in (0, \frac{1}{2}]$. Define*

$$\eta_1 := \left(\frac{c_0 \pi}{8\, K_{\mathrm{rect}}}\right)^2.$$

If $\eta \le \eta_1$, then there exists $M_0(\eta)$ (depending only on η, C_1, C_2 and the shape-only constants K_{rect}, C'_h) such that, for all $m \ge M_0(\eta)$ and all $\alpha \in (0,1]$,

$$\mathcal{U}_{hm}(m,\alpha) \;<\; \mathcal{L}(m,\alpha).$$

Equivalently: no off-axis quartet can lie in any $B(\alpha, m, \delta)$ with $\delta = \eta\,\alpha/(\log m)^2$ for $m \ge M_0(\eta)$. The comparison is uniform in α; the worst case is $\alpha = 1$.

Sketch of constants. From (4.1.1),

$$\mathcal{U}_{hm} \;\le\; 2K_{\mathrm{rect}}\Big(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\Big).$$

From (5.2),

$$\mathcal{L} \;\ge\; c_0\frac{\pi}{2} \;-\; \eta\,\alpha\left(\frac{2c_0C_1 + C'_h}{\log m} + \frac{2c_0C_2}{(\log m)^2}\right).$$

Choose $\eta \le \eta_1$ so $2K_{\mathrm{rect}}\sqrt{\eta} \le \frac{c_0\pi}{4}$; then select $M_0(\eta)$ so the $O(\eta/\log m)$ terms are $< \frac{c_0\pi}{4}$. Uniformity in α follows by taking $\alpha = 1$ as the extremal case. $\square$

4.4 Interpretive (non-load-bearing): Ω-neutrality and winding

If $\operatorname{ess\,sup}_{\partial B} |\arg W - \phi_0| \le \varepsilon$, then $|W(z) - e^{i\phi_0}| \le 2\sin(\varepsilon/2)$ for all $z \in B$. If $\Delta_{\partial B} \arg W = 2\pi N$, then the interior contains exactly N zeros counted with multiplicity (argument principle). Subthreshold budgets force $N = 0$, i.e., inner collapse $W \equiv e^{i\theta_B}$ (Bridge 3.4). For the Ω-continuum / ray viewpoint, see Appendix J; this layer does not enter any proof in §5.

4.5 Symbolic thresholds and global consequence

Let m_1 be the first nontrivial height in the width-2 frame (Appendix I). In addition to $\eta \le \eta_1$, define

$$\eta_2 \;:=\; \frac{c_0\,\pi\,\log m_1}{4\,(2c_0C_1 + C'_h)}. \tag{4.5}$$

If $\eta \le \min\{\eta_1, \eta_2\}$, then $M_0(\eta) \le m_1$ by Theorem 5.3, hence the analytic tail excludes off-axis quartets for all $m \ge m_1$; Appendix I implies there are no nontrivial zeros below m_1.

Remark 4.4 (Optional variant at the global analytic floor)*.* Let $m_{\min} \in \{6, 10\}$ denote the analytic floor used to ensure the uniform classical inputs (Part I, §2.1). Replacing m_1 by $m_{\min}$ in (4.5) yields

$$\eta_2^{(\min)} \;:=\; \frac{c_0\,\pi\,\log m_{\min}}{4\,(2c_0C_1 + C'_h)}. \tag{4.5.1}$$

We do not need (4.5.1) in the final theorem: choosing $\eta \le \min\{\eta_1, \eta_2\}$ already places the tail threshold below m_1, which is optimal for translating to the global on-axis statement.

5 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v_\pm^\star = \pm(a + im), \qquad z_\pm \in \partial\mathbb{D} \ \text{ with } \ \varphi(z_\pm) = v_\pm^\star.$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}, \qquad f := W \circ \varphi^{-1} \in H^\infty(\mathbb{D}).$$

(Boundedness: G_{out} is zero-free, W is analytic on the compact B.)

5.1 Upper envelope via L^2 and harmonic measure

Lemma 5.1 (Boundary phase $\Rightarrow$ dial-pair deficit)**.** *There exist* shape-only *constants* $C_{\text{rect}}, K_{\text{rect}} > 0$ *such that, for suitable anchor phases* $\phi_0^\pm$ *(the harmonic-measure averages of* $\arg W$ *seen from* $v_\pm^\star$*),*

$$\left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \ \le \ C_{\text{rect}}\Big(\sqrt{8\delta} + 2\delta\Big)\Big(C_1 \log m + C_2\Big) \ \le \ K_{\text{rect}}\bigg(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\bigg). \tag{5.1}$$

Consequently, summing at the two dial points,

$$\mathcal{U}_{hm}(m, \alpha) := \sum_\pm \left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \ \le \ 2K_{\text{rect}}\bigg(\sqrt{\eta\,\alpha} + \frac{\eta\,\alpha}{\log m}\bigg). \tag{4.1.1}$$

Proof idea. Apply the Poisson sub-mean inequality to $\log|f - c|$ with $c = e^{i\phi_0^\pm}$; use $|e^{i\theta} - 1| \le 2|\sin(\theta/2)|$. Control the quiet arcs in L^2 via the boundary Hilbert transform isometry on $\partial\mathbb{D}$ (M. Riesz; see Duren [**?**, §§I.3, I.6–I.7]), and the conformal L^2 trace to ∂B on Lipschitz boundaries (Coifman–McIntosh–Meyer). Control the verticals by arclength times $\sup_{\partial B} |E'/E|$ from (2.3) (i.e., (2.3)). Side-lengths give the $\sqrt{\delta}$ and δ factors. Background on harmonic measure and Poisson kernels: Ransford [**?**, §3.9], Garnett–Marshall [**?**, Chs. IV–V]. $\square$

5.2 Lower envelope via forcing and residual budgets

We track phases first for $\arg E$. By Lemma A (short-side forcing; see (2.5)) one has on the near vertical

$$\Delta_{I_+} \arg E - \Delta_{I_-} \arg E \ \ge \ \frac{\pi}{2} \quad \text{when } |\alpha - a| \le \delta.$$

Subtract vertical residuals using (2.3)–(**??**) ((2.3)–(2.4)) and bound the horizontal budget for $\arg G_{\text{out}}$ on Q by the same L^2 method as above. Convert the resulting side gap to a dial-pair *modulus* deficit for W via a boundary-to-point estimate on rectangles (Poisson–Jensen/Lipschitz).

Lemma 5.2 (Forcing vs budgets $\Rightarrow$ dial-pair deficit)**.** *There exist* $c_0 \in (0, 1)$ *and a shape-only constant* $C_h' > 0$ *such that*

$$\mathcal{L}(m, \alpha) \ := \ \sum_\pm \big||W(v_\pm^\star)| - 1\big| \ \ge \ c_0\,\frac{\pi}{2} \ - \ \delta\Big(2c_0(C_1 \log m + C_2) + C_h'(\log m + 1)\Big). \tag{5.2}$$

Auxiliary boundary-to-point estimate (used in the proof). If H is harmonic on B, $J \subset \partial B$ is a side, p is the midpoint of the opposite side, $\operatorname{osc}_J H \ge \Delta$, and $\sup_{\partial B} |\nabla H| \le L$, then

$$|H(p) - H(p_J)| \ \ge \ c_{\text{side}}\,\Delta \ - \ C_{\text{side}}\,(\text{length}\,\partial B)\,L, \tag{4.2.1}$$

where p_J is the harmonic-measure average of $H|_J$ seen from p, and $c_{\text{side}}, C_{\text{side}} > 0$ depend only on the rectangle aspect. Apply with $H = \log|W|$; absorb constants into c_0, C_h'.

5.3 Tail comparison (analytic, uniform in α)

Theorem 5.3 (Tail Comparison Theorem (analytic))**.** *Fix $\eta \in (0, \frac{1}{2}]$. Define*

$$\eta_1 := \left(\frac{c_0 \pi}{8\, K_{\mathrm{rect}}}\right)^2.$$

If $\eta \le \eta_1$, then there exists $M_0(\eta)$ (depending only on η, C_1, C_2 and the shape-only constants K_{rect}, C_h') such that, for all $m \ge M_0(\eta)$ and all $\alpha \in (0, 1]$,

$$\mathcal{U}_{hm}(m, \alpha) \;<\; \mathcal{L}(m, \alpha).$$

Equivalently: no off-axis quartet can lie in any $B(\alpha, m, \delta)$ with $\delta = \eta\, \alpha/(\log m)^2$ for $m \ge M_0(\eta)$. The comparison is uniform in α; the worst case is $\alpha = 1$.

Sketch of constants. From (4.1.1),

$$\mathcal{U}_{hm} \;\le\; 2K_{\mathrm{rect}}\Big(\sqrt{\eta\, \alpha} + \frac{\eta\, \alpha}{\log m}\Big).$$

From (5.2),

$$\mathcal{L} \;\ge\; c_0 \frac{\pi}{2} \;-\; \eta\, \alpha \left(\frac{2c_0 C_1 + C_h'}{\log m} + \frac{2c_0 C_2}{(\log m)^2}\right).$$

Choose $\eta \le \eta_1$ so $2K_{\mathrm{rect}}\sqrt{\eta} \le \frac{c_0\pi}{4}$; then select $M_0(\eta)$ so the $O(\eta/\log m)$ terms are $< \frac{c_0\pi}{4}$. Uniformity in α follows by taking $\alpha = 1$ as the extremal case. □

5.4 Interpretive (non-load-bearing): Ω-neutrality and winding

If $\operatorname{ess\,sup}_{\partial B} |\arg W - \phi_0| \le \varepsilon$, then $|W(z) - e^{i\phi_0}| \le 2\sin(\varepsilon/2)$ for all $z \in B$. If $\Delta_{\partial B} \arg W = 2\pi N$, then the interior contains exactly N zeros counted with multiplicity (argument principle). Sub-threshold budgets force $N = 0$, i.e., inner collapse $W \equiv e^{i\theta_B}$ (Bridge 3.4). For the Ω-continuum / ray viewpoint, see Appendix J; this layer does not enter any proof in §5.

5.5 Symbolic thresholds and global consequence

Let m_1 be the first nontrivial height in the width-2 frame (Appendix I). In addition to $\eta \le \eta_1$, define

$$\eta_2 \;:=\; \frac{c_0\, \pi\, \log m_1}{4\,(2c_0 C_1 + C_h')}. \tag{4.5}$$

If $\eta \le \min\{\eta_1, \eta_2\}$, then $M_0(\eta) \le m_1$ by Theorem 5.3, hence the analytic tail excludes off-axis quartets for all $m \ge m_1$; Appendix I implies there are no nontrivial zeros below m_1.

Remark 5.4 (Optional variant at the global analytic floor)*.* Let $m_{\min} \in \{6, 10\}$ denote the analytic floor used to ensure the uniform classical inputs (Part I, §2.1). Replacing m_1 by $m_{\min}$ in (4.5) yields

$$\eta_2^{(\min)} \;:=\; \frac{c_0\, \pi\, \log m_{\min}}{4\,(2c_0 C_1 + C_h')}. \tag{4.5.1}$$

We do not need (4.5.1) in the final theorem: choosing $\eta \le \min\{\eta_1, \eta_2\}$ already places the tail threshold below m_1, which is optimal for translating to the global on-axis statement.