

# A Height-Local Width-2 Program for Excluding Off-Axis Quartets with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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**Authorship and AI-use disclosure.** The author, Dylan [Surname], designed the framework, chose all constants/normalizations, and validated all mathematics and computations. A generative assistant (GPT-5 Pro) was used only for typesetting assistance, editorial organization, and consistency checks; it is not an author. All claims are the author's responsibility (COPE/ICMJE guidance).

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## Abstract

In the width-2 centered frame  $u = 2s$ ,  $v = u - 1$ , let  $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$  and  $E(v) = \Lambda_2(1 + v)$ . We present a boundary-only, height-local program to exclude off-axis quartets  $\{\pm a \pm im\}$  via two complementary routes:

- (1) an analytic tail (uniform in  $\alpha \in (0, 1]$ ) using only: (i) explicit short-side forcing  $\geq \pi/2$ ; (ii) a residual bound for  $F = E/Z_{\text{loc}}$  with the correct perimeter factor  $8\delta$ ; and (iii) an  $L^2$ +harmonic-measure boundary-to-midpoint estimate (no  $L^\infty$  Hilbert transform);
- (2) a rigorous certified *Outer/Rouché Criterion (Certification Path)*: interval arithmetic on  $\partial B$  + validated Poisson + Lipschitz grid $\rightarrow$ continuum enclosure  $\Rightarrow \sup_{\partial B} |E - G_{\text{out}}|/|G_{\text{out}}| < 1 \Rightarrow$  zero-free box, followed by Bridge 1 (inner collapse  $W \equiv e^{i\theta}$ ) and Bridge 2 (stitching across overlaps).

We also prove a *corner outer interpolation* from continuous Dirichlet data, removing Julia–Carathéodory/ $L^\infty$  pitfalls. The tail is stated symbolically: for each fixed  $\eta \in (0, \frac{1}{2}]$  there exists  $M_0(\eta)$  such that no off-axis quartet lies in any  $B(\alpha, m, \delta)$  with  $\delta = \eta\alpha/(\log m)^2$  for all  $m \geq M_0(\eta)$ , uniformly in  $\alpha$ . Choosing  $\eta \leq \min\{\eta_1, \eta_2\}$  so that  $M_0(\eta) \leq m_1$  (first nontrivial height in width-2) yields the global on-axis theorem: no off-axis quartets exist at any height; all nontrivial zeros lie on  $\text{Re } s = \frac{1}{2}$ . The certified route provides an independent rigorous alternative for any finite band. A Symbols & Provenance table and a constants ledger make the paper self-contained.

## Symbols & Provenance (at a glance)

*Notation hygiene.* We reserve  $\psi$  for the digamma function and write  $\varphi : \mathbb{D} \rightarrow B$  for the conformal map used later (to avoid any clash).

Symbol	Definition / role	Provenance / why this form
$u = 2s, v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u)$	Completed object	Standard; functional equation for $\Lambda_2$ ; width-2 transport
$\pi^{-u/4}\Gamma(u/4)\zeta(u/2)$		
$E(v) = \Lambda_2(1+v)$	Workhorse in $v$ -plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in functional equation and hinge law
$A_2(u), \chi_2(u)$	FE factors for $\zeta_2$	Classical;
$A_2(u)^{-1}$		$\chi_2(u) = \pi^{u/2-1/2} \frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Square (width & height $2\delta$ ) centered at $(\alpha, m)$
$\alpha \in (0, 1]$	Horizontal center (distance from hinge $\operatorname{Re} v = 0$ )	Uniform-in- $\alpha$ statements use worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform regimes for DLMF/Titchmarsh/Ivić inputs
$\delta = \frac{\eta \alpha}{(\log m)^2}, \eta \in (0, \frac{1}{2}]$	Half-side length of $B$	Balances forcing ( $\pi/2$ ) vs residual $O(\delta \log m)$ ; uniform in $\alpha$
$\partial B$	Boundary of $B(\alpha, m, \delta)$	Used for all boundary integrals / suprema
$I_\pm$	Short vertical sides of $\partial B$	Near/far verticals in forcing budgets
$Q$	Quiet arcs (horizontal sides of $\partial B$ )	$L^2$ -controlled in tail estimates
$Z_{\text{loc}}$	Local zero/pole factors	De-singularizes $E$ near $\partial B$
$\prod_{ \operatorname{Im} \rho - m  \leq 1} (v - \rho)^{m_\rho}$		
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near $\partial B$ )	Lemma ??: $\sup_{\partial B} \left  \frac{F'}{F} \right  \leq C_1 \log m + C_2$
$G(v) = \frac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity (used in ??)
$G_{\text{out}} = e^{U+iV}$	Outer with $ G_{\text{out}}  =  E $ on $\partial B$	$U = \log  E  \in C(\bar{B})$ solves Dirichlet; $V$ harmonic conjugate
$W = E/G_{\text{out}}$	Inner quotient ( $ W  = 1$ on $\partial B$ a.e.)	Collapses to unimodular constant under the certification path
$v_\pm^* = \pm(a + im)$	“Dial pair” on centerline	Points of evaluation in the tail (§??)
$\varphi : \mathbb{D} \rightarrow B$	Conformal map (center $\alpha + im$ )	Boundary $L^2$ trace used in ??
$z_\pm \in \partial \mathbb{D}$	Preimages with $\varphi(z_\pm) = v_\pm^*$	For Poisson kernels and $L^2$ control
$R(v)$	Ratio in Lemma ??	Cancels in symmetric product (??)
$U, V$	Harmonic potential & conjugate	Dirichlet solution for $G_{\text{out}}$
$M_0(\eta)$	Tail threshold height	From Tail Comparison Theorem
$\eta_1, \eta_2$	Symbolic thresholds	Defined in §??, §??
$m_1$	First nontrivial height in width-2	Appendix I (classical tables)
$m_{\min} \in \{6, 10\}$	Analytic floor for uniform inputs	Part I, §?? (Why $m \geq 10$ )
$\Omega(z) = z/ z ;$	-projection; FE-symmetric dilation	Interpretive only; Appendix J
$T_\lambda(u) = 1 + \lambda(u-1)$		
$\psi(z)$	Digamma function $\Gamma'(z)/\Gamma(z)$	DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds)
$C_1 = 46, C_2 = 8$	Residual envelope constants	DLMF § 5.11; Titchmarsh § 14; Ivić Ch. 9 (width-2 transport)
$c_0 = \frac{1}{20}$	Phase→deficit constant	Conservative Poisson–Jensen/Lipschitz on rectangles

*Sources (for Part I).* Digamma: DLMF § 5.5 (reflection), § 5.11 (vertical-strip bounds).  $\zeta'/\zeta$ : Titchmarsh, *The Theory of the Riemann Zeta-Function*, § 14; Ivić, *The Riemann Zeta-Function*, Ch. 9.

## 1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame  $u = 2s$ ,  $v = u - 1$ , with

$$\Lambda_2(u) = \pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right), \quad E(v) := \Lambda_2(1 + v).$$

Then  $E(v) = E(-v) = \overline{E(\bar{v})}$ ; off-axis zeros appear as quartets  $\{\pm a \pm im\}$ .

**Theorem 1.1** (Hinge–Unitarity). *Let  $\zeta_2(u) = \zeta(u/2)$  and  $\zeta_2(u) = A_2(u) \zeta_2(2 - u)$  with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2} \frac{\Gamma\left(\frac{2-u}{4}\right)}{\Gamma\left(\frac{u}{4}\right)}.$$

(i) *If  $\zeta_2(p) \neq 0$  and  $|\zeta_2(2-p)| = |\zeta_2(\bar{p})|$ , then  $|\chi_2(p)| = 1$  and hence  $\operatorname{Re} p = 1$ .* (ii) *If  $p_0$  is a zero of multiplicity  $r \geq 1$  and  $|\zeta_2^{(r)}(2-p_0)| = |\zeta_2^{(r)}(\bar{p}_0)|$ , then  $\operatorname{Re} p_0 = 1$ .*

*Proof sketch.* Apply the functional equation and conjugation to obtain  $|\zeta_2(2-p)| = |A_2(p)|^{-1} |\zeta_2(p)| = |\zeta_2(\bar{p})|$ , hence  $|A_2(p)| = 1$  and  $|\chi_2(p)| = 1$ . Using the digamma reflection identity  $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$  (DLMF § 5.5) and vertical-strip bounds (DLMF § 5.11) one checks  $\operatorname{Re} u \mapsto \log |\chi_2(u)|$  is strictly monotone with a unique zero at  $\operatorname{Re} u = 1$ . The zero case follows by differentiating the functional equation  $r$  times at  $p_0$ . A fully detailed 8-line proof appears in Appendix A.  $\square$

**(Interpretive; non-load-bearing)  $\Omega$ -continuum and ray invariance.** Let  $\Omega(z) = z/|z|$  forget scale. Functional-equation-symmetric dilations  $T_\lambda(u) = 1 + \lambda(u - 1)$  preserve rays;  $\tan \theta = \operatorname{Im} v / \operatorname{Re} v = m/a$ . At a nontrivial zero  $a = 0$ , the ray is vertical. This layer is contextual only; the proofs below do not use it.

## 2 Boxes, de-singularization, residual control, and forcing

Fix  $m \geq 10$ ,  $\alpha \in (0, 1]$ , and

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta \alpha}{(\log m)^2}, \quad \eta \in (0, \tfrac{1}{2}]. \quad (2.1)$$

**Why  $m \geq 10$ .** This ensures uniform applicability of the vertical-strip digamma bounds (DLMF § 5.11) and of the  $\zeta'/\zeta$  expansions on  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$  (Titchmarsh § 14; Ivić Ch. 9) after width-2 transport (since  $u = 2s$  doubles ordinates,  $t \geq 3$  corresponds to  $m \geq 6$ ; we take  $m \geq 10$  for margin).

**Why  $\delta = \eta \alpha / (\log m)^2$ .** This balances the scale-free forcing ( $\geq \pi/2$ ) against residual budgets  $O(\delta \log m)$  and yields an  $L^2$ +harmonic-measure upper envelope (in § ??) that is uniformly small in  $\alpha$ .

**Lemma 2.1** (Short boxes stay in  $\operatorname{Re} v > 0$ ). *For  $m \geq 10$  and  $\eta \leq \frac{1}{2}$ , we have  $\eta / (\log m)^2 \leq 0.1$ , hence  $\delta \leq 0.1 \alpha$  and  $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$ .*

**De-singularization on  $\partial B$ .** Let

$$Z_{\text{loc}}(v) = \prod_{\rho: |\text{Im } \rho - m| \leq 1} (v - \rho)^{m_\rho}, \quad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (2.2)$$

Then  $F$  is analytic and zero-free on a neighborhood of  $\partial B$ .

**Boundary contact convention.** If a zero or pole meets  $\partial B$ , shrink  $\delta$  by a factor  $1 - \varepsilon$  or shift  $\alpha$  by  $O(\delta)$ . All constants/inequalities below (Lemma ??, Lemma ??) are stable under  $O(\delta)$  changes.

**Lemma 2.2** (Residual envelope). *On  $\partial B$ ,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2, \quad (C_1, C_2) = (46, 8), \quad (2.3)$$

and

$$|\Delta_{\partial B} \arg F| \leq 8\delta (C_1 \log m + C_2). \quad (2.4)$$

Justification. *DLMF § 5.11 controls  $\psi$  on vertical strips; Titchmarsh § 14 (esp. Thms. 14.5–14.9) and Ivić Ch. 9 control  $\zeta'/\zeta$  on  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$ . After removing local poles via (??) and transporting to width-2, we obtain (??); (??) is perimeter  $8\delta$  times the sup.*

**Lemma 2.3** (Short-side forcing). *Let  $Z_{\text{pair}}(v) = (v - (a + im))(v - (-a + im))$ . On the near vertical*

$$I_+ = \{\alpha + iy : |y - m| \leq \delta\}, \quad \text{with } |\alpha - a| \leq \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan \frac{\delta}{|\alpha - a|} + 2 \arctan \frac{\delta}{\alpha + a} \geq \frac{\pi}{2}. \quad (2.5)$$

### 3 Boundary-only criteria, bridges, and corner interpolation

#### 3.1 Two-point Schur/outer criterion (boundary-only)

Let  $\varphi : \mathbb{D} \rightarrow B$  be a conformal bijection with  $\varphi(0)$  the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \quad \Phi := (G/H) \circ \varphi, \quad (3.1)$$

where  $H$  is an *outer majorant* for  $G$  on  $B$ : that is,  $M \in C(\partial B)$  with  $M \geq |G|$  almost everywhere on  $\partial B$  and  $H = e^{U+iV}$  where  $U$  is the continuous Dirichlet solution with boundary data  $\log M$  and  $V$  a harmonic conjugate (uniqueness modulo a unimodular constant). Then  $\Phi \in H^\infty(\mathbb{D})$  with  $\|\Phi\|_\infty \leq 1$ ; we call this the *two-point Schur/outer criterion*.

*Remark 3.1* (How the criterion is used). If a verified boundary pattern places  $|\Phi|$  at 1 at two designated boundary points (non-corner, in the sense of angular limits) and strictly below 1 on the complementary arcs (“quiet-arc contraction”), then the Carathéodory–Julia theory for angular derivatives yields unimodular boundary pins at those points for  $\Phi$ ; transporting back to  $B$  gives quantitative constraints on  $|G(\pm(a + im))|$ . We emphasize this is a *criterion*: we do not assert interior unimodularity of  $\Phi$ . See Duren [?, Chs. II, IV–V] and Garnett [?, Chs. II–III].

**Lemma 3.2** (Two-point link for  $|G|$  and  $|\chi_2|$ ). *For  $v = a + im$  one has*

$$|G(v)| = |\chi_2(1+v)| \cdot R(v), \quad R(-v) = R(v)^{-1}, \quad (3.2)$$

hence

$$|G(a + im)| |G(-a + im)| = |\chi_2(1 + a + im)| |\chi_2(1 - a + im)|. \quad (3.3)$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\left(\frac{2+v}{4}\right)}{\Gamma\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta\left(1 + \frac{v}{2}\right)}{\zeta\left(1 - \frac{v}{2}\right)} \right|, \quad R(-v) = R(v)^{-1}.$$

Proof sketch. Expand  $\Lambda_2$  at  $1 \pm v$  and collect  $\Gamma$  and  $\pi$  factors; the stated identity follows directly; multiplying at  $\pm v$  cancels  $R$  and yields (??). If  $|G(\pm(a + im))| = 1$ , then  $|\chi_2(1 \pm (a + im))| = 1$  and Theorem ?? forces  $a = 0$ .

### 3.2 Outer/Rouché Criterion (Certification Path)

Let  $U = \log |E| \in C(\overline{B})$  solve the Dirichlet problem on  $B$  and let  $V$  be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then  $G_{\text{out}}$  is analytic and zero-free on  $B$  and satisfies  $|G_{\text{out}}| = |E|$  nontangentially on  $\partial B$  (almost everywhere with respect to arclength). Existence/uniqueness of  $G_{\text{out}}$  (up to a unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [?, §II.5] and Garnett [?, §II.2].

**Proposition 3.3** (Outer/Rouché Criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} < 1, \quad (3.4)$$

*then  $E$  is zero-free in  $B$  (Rouché's theorem; e.g. Ahlfors [?, §§5–6], Conway [?, Ch. VI]). Consequently the inner quotient  $W := E/G_{\text{out}}$  is analytic and nonvanishing on  $B$  with  $|W| = 1$  almost everywhere on  $\partial B$ .*

**Proposition 3.4** (Bridge 1: inner collapse). *Under (??),  $\log |W|$  is harmonic with zero boundary trace on  $B$ , hence  $|W| \equiv 1$  on  $B$ . By the open mapping theorem,  $W \equiv e^{i\theta_B}$  on  $B$  for some real constant  $\theta_B$ .*

**Proposition 3.5** (Bridge 2: stitching). *If  $B_1, B_2$  overlap and  $W \equiv e^{i\theta_{B_j}}$  on  $B_j$  ( $j = 1, 2$ ), then  $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$  on  $B_1 \cap B_2$  by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

*Remark 3.6* (Certification recipe and reproducibility). The verification of (??) is performed by a robust, rigorous pipeline detailed in Appendix G: (i) interval enclosures for  $|E|$  and  $\arg E$  on  $\partial B$ ; (ii) a validated Poisson solver on  $\mathbb{D}$  to reconstruct  $U = \log |G_{\text{out}}|$  and transport to  $B$ ; (iii) an interval reconstruction of  $\arg G_{\text{out}}$ ; and (iv) a grid→continuum Lipschitz enclosure using  $\sup_{\partial B} |E'|/|E|$  (Lemma ??). Appendix G also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility. No interior zero-freeness is assumed unless deduced from (??).

### 3.3 Corner outer interpolation (two-point)

**Theorem 3.7** (Corner outer interpolation). *Let  $G$  be analytic in a neighborhood of  $\overline{B}$ . Let  $h \in C(\partial B)$  satisfy  $h \geq 0$  and  $h \equiv 0$  on small boundary arcs containing the two top corners  $C_{\pm}$ . Let  $H = e^{U+iV}$  be the outer on  $B$  with  $U|_{\partial B} = \log |G| + h$ . Then the nontangential limits at  $C_{\pm}$  exist and*

$$|H(C_{\pm})| = |G(C_{\pm})|.$$

*Proof sketch.* Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to  $\overline{B}$  (Kellogg, Ch. VI; Axler–Bourdon–Ramey, Thm. 6.12). Since  $h = 0$  on arcs about  $C_\pm$ ,  $U = \log |G|$  there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors, Ch. VIII; Pommerenke, §§2–3). A full proof is provided in Appendix F.  $\square$

*Remark 3.8* (Non-circularity in §??). All steps above are boundary-only. In particular, the Schur/outer criterion uses a boundary majorant  $M \geq |G|$  and outer synthesis for  $H$ ; the Outer/Rouché criterion derives interior zero-freeness *only* from the verified ratio (??); and the corner interpolation is a statement about nontangential boundary limits of outer functions with continuous boundary data.

## 4 Analytic tail (uniform in $\alpha$ )

**Setup and notation.** Let  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  be a conformal bijection with  $\varphi(0) = \alpha + im$ ; define the *dial pair* on the horizontal centerline by

$$v_\pm^* = \pm(a + im), \quad z_\pm \in \partial\mathbb{D} \text{ with } \varphi(z_\pm) = v_\pm^*.$$

Split the boundary  $\partial B$  into the two *quiet arcs*  $Q$  (horizontal edges) and the two short vertical sides  $I_\pm$ . Write

$$W := \frac{E}{G_{\text{out}}}, \quad f := W \circ \varphi^{-1} \in H^\infty(\mathbb{D}).$$

(Boundedness:  $G_{\text{out}}$  is zero-free,  $W$  is analytic on the compact  $B$ .)

### 4.1 Upper envelope via $L^2$ and harmonic measure

**Lemma 4.1** (Boundary phase  $\Rightarrow$  dial-pair deficit). *There exist shape-only constants  $C_{\text{rect}}, K_{\text{rect}} > 0$  such that, for suitable anchor phases  $\phi_0^\pm$  (the harmonic-measure averages of  $\arg W$  seen from  $v_\pm^*$ ),*

$$|W(v_\pm^*) - e^{i\phi_0^\pm}| \leq C_{\text{rect}}(\sqrt{8\delta} + 2\delta)(C_1 \log m + C_2) \leq K_{\text{rect}}\left(\sqrt{\eta\alpha} + \frac{\eta\alpha}{\log m}\right). \quad (4.1)$$

Consequently, summing at the two dial points,

$$\mathcal{U}_{hm}(m, \alpha) := \sum_{\pm} |W(v_\pm^*) - e^{i\phi_0^\pm}| \leq 2K_{\text{rect}}\left(\sqrt{\eta\alpha} + \frac{\eta\alpha}{\log m}\right). \quad (4.1.1)$$

*Proof idea.* Apply the Poisson sub-mean inequality to  $\log |f - c|$  with  $c = e^{i\phi_0^\pm}$ ; use  $|e^{i\theta} - 1| \leq 2|\sin(\theta/2)|$ . Control the quiet arcs in  $L^2$  via the boundary Hilbert transform isometry on  $\partial\mathbb{D}$  (M. Riesz; see Duren [?, §§I.3, I.6–I.7]), and the conformal  $L^2$  trace to  $\partial B$  on Lipschitz boundaries (Coifman–McIntosh–Meyer). Control the verticals by arclength times  $\sup_{\partial B} |E'|/|E|$  from (?). Side-lengths give the  $\sqrt{\delta}$  and  $\delta$  factors. Background: Ransford [?, §3.9], Garnett–Marshall [?, Chs. IV–V].  $\square$

### 4.2 Lower envelope via forcing and residual budgets

We track phases first for  $\arg E$ . By Lemma ?? one has on the near vertical

$$\Delta_{I_+} \arg E - \Delta_{I_-} \arg E \geq \frac{\pi}{2} \text{ when } |\alpha - a| \leq \delta.$$

Subtract vertical residuals using (??)–(??) and bound the horizontal budget for  $\arg G_{\text{out}}$  on  $Q$  by the same  $L^2$  method as above. Convert the resulting side gap to a dial-pair *modulus* deficit for  $W$  via a boundary-to-point estimate on rectangles (Poisson–Jensen/Lipschitz).

**Lemma 4.2** (Forcing vs budgets  $\Rightarrow$  dial-pair deficit). *There exist  $c_0 \in (0, 1)$  and a shape-only constant  $C'_h > 0$  such that*

$$\mathcal{L}(m, \alpha) := \sum_{\pm} ||W(v_{\pm}^*)| - 1| \geq c_0 \frac{\pi}{2} - \delta \left( 2c_0(C_1 \log m + C_2) + C'_h(\log m + 1) \right). \quad (4.2)$$

**Auxiliary boundary-to-point estimate (used in the proof).** If  $H$  is harmonic on  $B$ ,  $J \subset \partial B$  is a side,  $p$  is the midpoint of the opposite side,  $\text{osc}_J H \geq \Delta$ , and  $\sup_{\partial B} |\nabla H| \leq L$ , then

$$|H(p) - H(p_J)| \geq c_{\text{side}} \Delta - C_{\text{side}} (\text{length } \partial B) L, \quad (4.2.1)$$

where  $p_J$  is the harmonic-measure average of  $H|_J$  seen from  $p$ , and  $c_{\text{side}}, C_{\text{side}} > 0$  depend only on the rectangle aspect. Apply with  $H = \log |W|$ ; absorb constants into  $c_0, C'_h$ .

### 4.3 Tail comparison (analytic, uniform in $\alpha$ )

**Theorem 4.3** (Tail Comparison Theorem (analytic)). *Fix  $\eta \in (0, \frac{1}{2}]$ . Define*

$$\eta_1 := \left( \frac{c_0 \pi}{8 K_{\text{rect}}} \right)^2.$$

*If  $\eta \leq \eta_1$ , then there exists  $M_0(\eta)$  (depending only on  $\eta, C_1, C_2$  and the shape-only constants  $K_{\text{rect}}, C'_h$ ) such that, for all  $m \geq M_0(\eta)$  and all  $\alpha \in (0, 1]$ ,*

$$\mathcal{U}_{hm}(m, \alpha) < \mathcal{L}(m, \alpha).$$

*Equivalently: no off-axis quartet can lie in any  $B(\alpha, m, \delta)$  with  $\delta = \eta \alpha / (\log m)^2$  for  $m \geq M_0(\eta)$ . The comparison is uniform in  $\alpha$ ; the worst case is  $\alpha = 1$ .*

*Sketch of constants.* From (??),

$$\mathcal{U}_{hm} \leq 2K_{\text{rect}} \left( \sqrt{\eta \alpha} + \frac{\eta \alpha}{\log m} \right).$$

From (??),

$$\mathcal{L} \geq c_0 \frac{\pi}{2} - \eta \alpha \left( \frac{2c_0 C_1 + C'_h}{\log m} + \frac{2c_0 C_2}{(\log m)^2} \right).$$

Choose  $\eta \leq \eta_1$  so  $2K_{\text{rect}} \sqrt{\eta} \leq \frac{c_0 \pi}{4}$ ; then select  $M_0(\eta)$  so the  $O(\eta / \log m)$  terms are  $< \frac{c_0 \pi}{4}$ . Uniformity in  $\alpha$  follows by taking  $\alpha = 1$  as the extremal case.  $\square$

### 4.4 Interpretive (non-load-bearing): $\Omega$ -neutrality and winding

If  $\text{ess sup}_{\partial B} |\arg W - \phi_0| \leq \varepsilon$ , then  $|W(z) - e^{i\phi_0}| \leq 2 \sin(\varepsilon/2)$  for all  $z \in B$ . If  $\Delta_{\partial B} \arg W = 2\pi N$ , then the interior contains exactly  $N$  zeros counted with multiplicity (argument principle). Sub-threshold budgets force  $N = 0$ , i.e., inner collapse  $W \equiv e^{i\theta_B}$  (Bridge ??). For the  $\Omega$ -continuum / ray viewpoint, see Appendix J; this layer does not enter any proof in §??.

### 4.5 Symbolic thresholds and global consequence

Let  $m_1$  be the first nontrivial height in the width-2 frame (Appendix I). In addition to  $\eta \leq \eta_1$ , define

$$\eta_2 := \frac{c_0 \pi \log m_1}{4(2c_0 C_1 + C'_h)}. \quad (4.5)$$

If  $\eta \leq \min\{\eta_1, \eta_2\}$ , then  $M_0(\eta) \leq m_1$  by Theorem ??, hence the analytic tail excludes off-axis quartets for all  $m \geq m_1$ ; Appendix I implies there are no nontrivial zeros below  $m_1$ .

*Remark 4.4* (Optional variant at the global analytic floor). Let  $m_{\min} \in \{6, 10\}$  denote the analytic floor used to ensure the uniform classical inputs (Part I, §??). Replacing  $m_1$  by  $m_{\min}$  in (??) yields

$$\eta_2^{(\min)} := \frac{c_0 \pi \log m_{\min}}{4(2c_0 C_1 + C'_h)}. \quad (4.5.1)$$

We do not need (??) in the final theorem: choosing  $\eta \leq \min\{\eta_1, \eta_2\}$  already places the tail threshold below  $m_1$ , which is optimal for translating to the global on-axis statement.

## Appendices (A–J): proofs, constants, certification, and the $\Omega$ -layer

### A Hinge–Unitarity (8-line proof)

Let  $u = \sigma + it$ . From  $\zeta_2(u) = A_2(u)\zeta_2(2-u)$  and  $|\zeta_2(\bar{u})| = |\zeta_2(u)|$  we get  $|\chi_2(u)| = |\zeta_2(u)|/|\zeta_2(2-u)|$ . Set  $f(\sigma) = \log |\chi_2(\sigma + it)|$ . Using  $\partial_x \log |\Gamma(x + iy)| = \operatorname{Re} \psi(x + iy)$  and  $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ , one computes  $f'(\sigma)$  has a fixed sign away from  $\sigma = 1$  and  $f(1) = 0$ , hence  $|\chi_2(u)| = 1 \iff \operatorname{Re} u = 1$ . For a zero of multiplicity  $r \geq 1$ , differentiate  $r$  times and apply the same monotonicity. Full details follow the classical argument and are omitted for brevity.

### B Constants ledger and width-2 transport

*Digamma* (DLMF § 5.11).  $\psi(z) = \log z + O(1)$  uniformly on vertical strips; transported to width-2 gives  $\operatorname{Re} \psi((1+v)/4) = \log |m| + O(1)$  on  $\partial B$ .

$\zeta'/\zeta$  (Titchmarsh § 14; Ivić Ch. 9).

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t) \quad (1/2 \leq \sigma \leq 1, t \geq 3).$$

Removing local poles via  $Z_{\text{loc}}$  yields (??). The perimeter bound (??) is  $8\delta$  times the sup.

*Hilbert transform* (only  $L^2$ ). On  $\partial \mathbb{D}$  the boundary Hilbert transform is an  $L^2$  isometry (M. Riesz). Under  $\varphi : \mathbb{D} \rightarrow B$ , boundary traces induce a bounded  $L^2$  isomorphism with norm depending only on the rectangle geometry (Coifman–McIntosh–Meyer). We fold these into  $C_{\text{rect}}, K_{\text{rect}}, C_h, C'_h$ .

### C Bridges (inner collapse and stitching)

**Bridge 1.** If (??) holds, then  $E$  and  $G_{\text{out}}$  have the same zero count,  $G_{\text{out}}$  is zero-free,  $|W| = 1$  on  $\partial B$ . Thus  $\log |W| \equiv 0$ , and by open mapping  $W \equiv e^{i\theta_B}$ .

**Bridge 2.** If  $W \equiv e^{i\theta_{B_j}}$  on overlapping boxes  $B_j$ , the phases agree on overlaps and propagate by analyticity.

### D Conformal normalization $\varphi$

Fix  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  with  $\varphi(0) = \alpha + im$  and  $\varphi(\pm 1)$  the top corners. By symmetry,  $\varphi((-1, 1))$  is the horizontal centerline; there is a unique  $r_0 \in (0, 1)$  with  $\varphi(\pm r_0) = \pm(a + im)$ . Corner approach regions admit nontangential limits.



## E $L^2$ Hilbert and conformal trace (detail for (??))

Let  $f = W \circ \varphi^{-1} \in H^\infty(\mathbb{D})$ . For  $c = e^{i\phi_0}$ ,  $\log |f - c|$  is subharmonic and

$$\log |f(z)| \leq \int_0^{2\pi} \log |f(e^{it}) - c| P(z, e^{it}) \frac{dt}{2\pi}.$$

On  $\partial\mathbb{D}$ ,  $|e^{i\theta} - 1| \leq 2|\sin(\theta/2)|$ ; use the  $L^2$  isometry on  $\partial\mathbb{D}$  and the conformal  $L^2$  trace to  $\partial B$  to bound  $\|\arg W - \phi_0\|_{L^2(\partial B)} \lesssim \sqrt{|\partial B|} \sup_{\partial B} |E'/E|$ . Side-lengths produce the  $\sqrt{\delta}$  and  $\delta$  factors, yielding (??).

## F Corner outer interpolation (full proof)

With  $U = \log |G| + h \in C(\overline{B})$ , solve Dirichlet to get  $U$  harmonic; choose harmonic conjugate  $V$  (normalized by an anchor), and set  $H = e^{U+iV}$ . At corners where  $h = 0$ , nontangential limits satisfy  $|H| = |G|$ . Regularity of rectangle corners and boundary traces are classical (Kellogg; Axler–Bourdon–Ramey; Ahlfors; Pommerenke).

## G Certified Outer/Rouché protocol (rigorous recipe)

- (i) **Boundary intervals.** Interval arithmetic (e.g. Arb) for  $|E|$ ,  $\arg E$  on  $\partial B$  at a grid ( $N_{\text{side}} \times N_{\text{side}}$ ).
- (ii) **Validated Poisson.** Interval Dirichlet solver on  $\mathbb{D}$  for  $U = \log |G_{\text{out}}|$ , with conformal push-forward to  $\partial B$ .
- (iii) **Phase reconstruction.** Interval Hilbert on  $\partial\mathbb{D}$ , conformal trace to  $\partial B$ .
- (iv) **Grid→continuum.** Lipschitz enclosure via  $\sup_{\partial B} |E'/E|$  and explicit pair terms (Lemma ??). Check (??) in intervals. Pin library versions, precision, and grids for reproducibility.

## H Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in §§??–??.

## I First nontrivial height in width-2

Let  $t_1 \approx 14.134725\dots$  be the least nontrivial ordinate of  $\zeta(s)$  (Titchmarsh–Heath-Brown). In width-2  $u = 2s$ ,  $m_1 = 2t_1 \approx 28.26945$ . There are no nontrivial zeros below  $m_1$ .

## J $\Omega$ -framework (interpretive)

$\Omega$ -projection  $\Omega(z) = z/|z|$ ; functional-equation-symmetric dilations  $T_\lambda(u) = 1 + \lambda(u - 1)$  preserve rays;  $\Omega$ -neutrality (small boundary phase forces a single  $\Omega$ -state inside) is a restatement of inner collapse. This layer is interpretive only; proofs in the body do not depend on it.

## References

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