

# A Height–Local Width–2 Program for Excluding Off–Axis Quartets with an Analytic Tail and a Rigorous Certified Criterion

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## Abstract

The paper is organized in three parts: **Part I** (Reader’s Guide) reduces RH to a height–local target  $a(m) = 0$  in the width–2 frame and records non–load–bearing scaffolding; **Part II** gives a self–contained, boundary–only analytic proof that at each nontrivial height the tilt vanishes  $a(m) = 0$  via a disc–based  $L^2$  upper envelope and an  $L^2$  lower envelope (allocation + restricted contour + Jensen), plus an optional certified Outer/Rouché path; **Part III** promotes the toolbox identities to structural corollaries and presents a deterministic, prime–locked generator of the ordinates.

## Contents

### Part I — Reader’s Guide / Motivation, Reduction & Implications

**What this section is (and is not).** *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered “ $a$ –lens” that measures horizontal tilt at a fixed height, and reduces RH to the height–local target  $a(m) = 0$  for each nontrivial height  $m$ . It also records a structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become *corollaries* after Part II.

*What it does not do.* It contains no analytic estimates and no proofs. The hinge–unitarity fact and all bounds are proved later. This Guide is not used by the analytic part.

#### 1) Modulated frames and the width–2 pivot

For  $f > 0$  define the modulated family  $\zeta_f(s) := \zeta(s/f)$  with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)} \Gamma\left(\frac{s}{2f}\right) \zeta_f(s),$$

so  $\Lambda_f$  is entire and satisfies  $\Lambda_f(s) = \Lambda_f(f - s)$ . Equivalently,  $\zeta_f(s) = A_f(s) \zeta_f(f - s)$  with  $A_f(s)A_f(f - s) \equiv 1$ .

**Width–2 normalization.** Put  $u := (2/f)s$ . Then

$$\zeta_2(u) := \zeta(u/2), \quad \Lambda_2(u) := \pi^{-u/4} \Gamma(u/4) \zeta(u/2), \quad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads  $\zeta_2(u) = A_2(u) \zeta_2(2 - u)$ . In the open strip  $0 < \operatorname{Re} u < 2$  and  $\operatorname{Im} u \neq 0$ ,  $A_2$  is analytic and nonvanishing.

**Partner map.** On  $\operatorname{Im} u > 0$ , FE + conjugation gives the involution  $J(u) = 2 - \bar{u}$ , swapping the two column points at the same height.

**Hinge unitarity (deferred).** The statement “ $|\chi_2(u)| = |A_2(u)|^{-1} = 1$ ” iff  $\operatorname{Re} u = 1$  is proved in Part II (Hinge–Unitarity, Theorem ??; see also Appendix ??).

## 2) Centered $a$ -lens and the quartet

Let  $v := u - 1$  and  $E(v) := \Lambda_2(1 + v)$ . Then  $E(v) = E(-v) = \overline{E(\bar{v})}$ . A “nontrivial height”  $m > 0$  means  $m$  occurs as the imaginary part of a nontrivial zero  $s = \frac{1}{2} + im/2$ . At fixed  $m > 0$ , set

$$U_R(m; a) = 1 + a + im, \quad U_L(m; a) = 1 - a + im, \quad a \in [0, 1].$$

In the centered frame, the dial points are  $\pm(a + im)$ ; the partner map  $J$  swaps  $U_R \leftrightarrow U_L$ . Conjugation plus FE reflection generate the quartet  $\{1 \pm a \pm im\}$ .

## 3) Why width-2: slope invariance

If the columns collapse at height  $m$  ( $a = 0$ ), the point is  $u = 1 + im$  and its slope is  $\text{Im } u / \text{Re } u = m$ . Rescaling to any frame  $s = (f/2)u$  preserves slope:

$$\frac{\text{Im } s}{\text{Re } s} = \frac{(f/2)m}{f/2} = m.$$

## 4) Height-local reduction of RH

Fix  $m > 0$  and write  $U_R = 1 + a + im$ ,  $U_L = 1 - a + im$ . The following equivalent algebraic forms are used:

- (PHU-1)  $\text{Re } U_R = \text{Re } U_L \iff a = 0$ .
- (PHU-2)  $\text{Im } U_R / \text{Re } U_R = \text{Im } U_L / \text{Re } U_L \iff a = 0$ .
- (PHU-3)  $U_R = U_L = 1 + im$ .

Thus RH  $\iff$  for every nontrivial height  $m > 0$ ,  $a(m) = 0$ .

## 5) Box alignment and hand-off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta := \eta \alpha / (\log m)^2, \quad \eta \in (0, 1).$$

When  $\alpha = \pm a$ , the dials  $\pm(a + im)$  lie on the horizontal centerline. *What Part II does.* Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three-lines tools, Stirling-class envelopes, explicit control of  $\zeta'/\zeta$  away from zeros), Part II shows any off-axis quartet forces a boundary lower bound larger than an explicit upper bound, hence  $a(m) = 0$ .

## 6) Parity gating and selection devices (interpretive only)

In width-2,

$$\zeta_2(u) = A_2(u) \zeta_2(2 - u), \quad A_2(u) = 2^{u/2} \pi^{u/2-1} \sin\left(\frac{\pi u}{4}\right) \Gamma\left(1 - \frac{u}{2}\right).$$

On  $0 < \text{Re } u < 2$ ,  $\text{Im } u \neq 0$ , the prefactor  $A_2(u)$  is nonzero; its sine zeros lie on the real axis only. Thus *inside* the open strip only  $\zeta_2$  can vanish (nontrivial), while the trivial ladder is confined to  $\text{Re } u$ . This motivates an odd/even split on the integer lattice via

$$P_{\text{odd}}(n) = \frac{1-\cos(\pi n)}{2}, \quad P_{\text{even}}(n) = \frac{1+\cos(\pi n)}{2}.$$

We assign the nontrivial stream to odd slots and the trivial ladder to even slots. (Interpretive; not used in Part II.)

## 7) Toolbox → structural consequences (after the theorem)

The items (columns/templates, canonical stream, single-frequency collapse, self-indexed recurrence, curvature extractor, seed→rectifier) become *Structural Corollaries in Part III* once Part II proves  $a(m) = 0$ . No toolbox component is used as an input in Part II.

## Part II — Self-Contained Boundary-Only Contradiction on Aligned Boxes

In the width-2 centered frame  $u = 2s$ ,  $v = u - 1$ , let  $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$  and  $E(v) = \Lambda_2(1+v)$ . We present a boundary-only, height-local program to exclude off-axis quartets  $\{\pm a \pm im\}$  via two complementary routes:

- (1) an *analytic tail*, uniform in  $\alpha \in (0, 1]$ , using only: (i) explicit short-side forcing  $\geq \pi/2$ ; (ii) a residual bound for  $F = E/Z_{\text{loc}}$  with perimeter factor  $8\delta$ ; (iii) a disc-based,  $L^2$  boundary-to-midpoint estimate with *shape-only* constants;
- (2) a rigorous *Outer/Rouché Certification Path* (optional): interval arithmetic on  $\partial B$  + validated Poisson + Lipschitz grid→continuum enclosure  $\Rightarrow \sup_{\partial B} |E - G_{\text{out}}| / |G_{\text{out}}| < 1 \Rightarrow$  zero-free box, followed by Bridge 1 (inner collapse  $W \equiv e^{i\theta}$ ) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. All constants in the upper/lower envelope are *shape-only* (independent of  $m, \alpha, a$ ); residual constants are symbolic and optionally instantiated in Appendix ??.

## Symbols & Provenance (at a glance)

Symbol	Definition / role	Provenance / rationale
$u = 2s, v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\frac{\Lambda_2(u)}{\pi^{-u/4}\Gamma(u/4)\zeta(u/2)}$	= Completed object	Standard; FE for $\Lambda_2$ ; width-2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in $v$ -plane	Even & conjugate symmetrical: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2} \frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Square centered at $(\alpha, m)$ , side $2\delta$
$\alpha \in (0, 1]$	Horizontal center	Worst case $\alpha = 1$ ; left dial via reflection $w = -v$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \frac{\eta \alpha}{(\log m)^2}, \eta \in (0, 1)$	Half-side length of $B$	Balances forcing vs. residual $O(\delta \log m)$
$\partial B, I_{\pm}, Q$	Boundary, short verticals, horizontals	Boundary integrals/suprema; quiet arcs
$Z_{\text{loc}}(v)$ $\prod_{ \operatorname{Im} \rho - m  \leq 1} (v - \rho)^{m_\rho}$	= Local zero/pole factors	De-singularizes $E$ near $\partial B$
$F = E/Z_{\text{loc}}$	Residual analytic factor	Lemma ?? (constants symbolic)
$G(v) = \frac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge; two-point identity
$G_{\text{out}} = e^{U+iV}$	Modulus-outer with $ G_{\text{out}}  =  E $ on $\partial B$	$U = \log  E $ solves Dirichlet; $V$ harmonic conjugate
$W = E/G_{\text{out}}$	Inner quotient ( $ W  = 1$ a.e. on $\partial B$ )	Collapses to unimodular constant upon certification
$v_{\pm}^* = \pm(a + im)$	Dial pair	Points of evaluation in the tail on centerline
$Z_{\text{pair}}(v)$	$(v - (a + im))(v - (-a + im))$	Short-side forcing on $I_+$
$\Gamma_\lambda$	Central $\lambda\delta$ sub-arcs + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by $\Gamma_\lambda$
$K_{\text{alloc}}^{(*)}(\lambda)$	Allocation coefficient	Shape-only; Lemma ??
$c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$	Dial deficit constant ( $\lambda = \frac{1}{2}$ )	From Jensen at dial; Lemma ??
$C_{\text{up}}$	Upper-envelope constant	Disc-based bound; Lemma ??
$C_h''$	Horizontal budget constant	Shape-only; Lemma ??

*Sources.* Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds).  $\zeta'/\zeta$ : Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

## 1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame  $u = 2s$ ,  $v = u - 1$ , with

$$\Lambda_2(u) = \pi^{-u/4} \Gamma\left(\frac{u}{4}\right) \zeta\left(\frac{u}{2}\right), \quad E(v) := \Lambda_2(1+v).$$

Then  $E(v) = E(-v) = \overline{E(\bar{v})}$  and off-axis zeros appear as quartets  $\{\pm a \pm im\}$  by the FE symmetry plus conjugation.

**Theorem 1.1** (Hinge–Unitarity). *Let  $\zeta_2(u) = \zeta(u/2)$  and  $\zeta_2(u) = A_2(u) \zeta_2(2-u)$  with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2} \frac{\Gamma\left(\frac{2-u}{4}\right)}{\Gamma\left(\frac{u}{4}\right)}.$$

For each fixed  $t \neq 0$ , define  $f(\sigma) = \log |\chi_2(\sigma + it)|$ . Then

$$f'(\sigma) = \frac{1}{2} \log \pi - \frac{1}{2} \operatorname{Re} \psi\left(\frac{\sigma+it}{4}\right) - \frac{1}{4} \operatorname{Re} \left[ \pi \cot\left(\frac{\pi}{4}(\sigma+it)\right) \right].$$

Moreover,

$$|\operatorname{Re}[\pi \cot(x+iy)]| \leq \frac{\pi}{\cosh(2y)-1},$$

indeed using  $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$  and  $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$  one obtains  $\cot(x+iy) = \frac{\sin(2x) - i \sinh(2y)}{\cosh(2y) - \cos(2x)}$  with real denominator; hence  $|\operatorname{Re}[\pi \cot(x+iy)]| = \pi \frac{|\sin(2x)|}{\cosh(2y) - \cos(2x)} \leq \frac{\pi}{\cosh(2y)-1}$ . With  $x = \frac{\pi}{4}\sigma$ ,  $y = \frac{\pi}{4}|t|$ , for  $|t| \geq m_1/2$  (Appendix ??) the cotangent term is negligible, and vertical-strip bounds give  $\operatorname{Re} \psi\left(\frac{\sigma+it}{4}\right) \geq \log\left(\frac{|t|}{4}\right) - \frac{2}{|t|}$ . Hence  $f'(\sigma) < 0$  on  $\mathbb{R}$  for such  $t$ . Since  $f(1) = 0$ , we have  $|\chi_2(u)| = 1$  iff  $\operatorname{Re} u = 1$ . A short proof is also recorded in Appendix ??.

## 2 Boxes, de-singularization, residual control, and forcing

Fix  $m \geq 10$ ,  $\alpha \in (0, 1]$ , and

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \quad \delta = \frac{\eta \alpha}{(\log m)^2}, \quad \eta \in (0, 1). \quad (2.1)$$

**Lemma 2.1** (Short boxes stay in  $\operatorname{Re} v > 0$ ). *For  $m \geq 10$  and any  $\eta \in (0, 1)$ , one has  $\delta < \alpha$  and  $B(\alpha, m, \delta) \subset \{\operatorname{Re} v > 0\}$ , uniformly in  $\alpha \in (0, 1]$ .*

*Proof.* Since  $\eta/(\log m)^2 < 1$ , we have  $\delta = \alpha \eta/(\log m)^2 < \alpha$ , so the left edge is  $\alpha - \delta > 0$ .  $\square$

**De-singularization on  $\partial B$ .** Let

$$Z_{\text{loc}}(v) = \prod_{\rho: |\operatorname{Im} \rho - m| \leq 1} (v - \rho)^{m_\rho}, \quad F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (2.2)$$

Then  $F$  is analytic and zero-free on a neighborhood of  $\partial B$  (all local zeros/poles with  $|\operatorname{Im} \rho - m| \leq 1$  have been removed). If a zero/pole meets  $\partial B$ , shrink  $\delta$  by  $1 - \varepsilon$  or shift  $\alpha$  by  $O(\delta)$ ; all bounds below are stable under  $O(\delta)$  changes.

**Lemma 2.2** (Residual envelope). *On  $\partial B$ ,*

$$\sup_{\partial B} \left| \frac{F'}{F} \right| \leq C_1 \log m + C_2, \quad (2.3)$$

and

$$|\Delta_{\partial B} \arg F| \leq 8\delta (C_1 \log m + C_2). \quad (2.4)$$

*Proof.* On vertical strips,  $\operatorname{Re} \psi(x + iy) = \log \sqrt{x^2 + y^2} + O(1)$  (DLMF §5.11). For  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$ ,  $\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im} \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$  (Titchmarsh §14; Ivić Ch. 9). Transport to width-2 and divide out the local factors  $Z_{\text{loc}}$  to remove poles on  $|\operatorname{Im} \rho - m| \leq 1$ ; the remaining term is  $O(\log m)$  uniformly on  $\partial B$ , giving (??). For (??), use  $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F ds$  with  $|\partial B| = 8\delta$  and the bound  $|\partial_\tau \arg F| \leq |F'/F|$ . We keep  $C_1, C_2 > 0$  symbolic (optional instantiation in Appendix ??).  $\square$

**Lemma 2.3** (Logarithmic derivatives on  $\partial B$ ). *On  $\partial B$ ,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}, \quad \sup_{\partial B} \left| \frac{E'}{E} \right| \leq \sup_{\partial B} \left| \frac{F'}{F} \right| + \sum_{\rho: |\operatorname{Im} \rho - m| \leq 1} \sup_{v \in \partial B} \frac{m_\rho}{|v - \rho|}.$$

*Proof.* Since  $E = F Z_{\text{loc}}$ , take the logarithmic derivative to obtain the identity. The supremum bound follows from the triangle inequality and the finite sum over local factors enforced by the boundary-contact convention (if a factor meets  $\partial B$  we shrink  $\delta$  or shift  $\alpha$  by  $O(\delta)$ ).  $\square$

**Lemma 2.4** (Short-side forcing). *Let  $Z_{\text{pair}}(v) = (v - (a + im))(v - (-a + im))$ . On the near vertical*

$$I_+ = \{\alpha + iy : |y - m| \leq \delta\}, \quad \text{with } |\alpha - a| \leq \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan \frac{\delta}{|\alpha - a|} + 2 \arctan \frac{\delta}{\alpha + a} \geq \frac{\pi}{2}. \quad (2.5)$$

*Proof.* Parameterize  $v(y) = \alpha + iy$ . For  $z_\pm(y) = v(y) - (\pm a + im)$ ,  $\arg z_\pm(y) = \arctan(\frac{y-m}{\alpha \mp a})$  (continuous branch on  $I_+$ ). Hence

$$\Delta_{I_+} \arg(v - (a + im)) = \arctan \frac{\delta}{\alpha - a} - \arctan \frac{-\delta}{\alpha - a} = 2 \arctan \frac{\delta}{|\alpha - a|},$$

and

$$\Delta_{I_+} \arg(v - (-a + im)) = \arctan \frac{\delta}{\alpha + a} - \arctan \frac{-\delta}{\alpha + a} = 2 \arctan \frac{\delta}{\alpha + a}.$$

Summing yields (??). Since  $|\alpha - a| \leq \delta$  and  $\alpha + a > 0$ , each term is  $\geq \pi/2$  and  $> 0$  respectively, so the sum  $\geq \pi/2$ .  $\square$

### 3 Boundary-only criteria, bridges, and corner interpolation

#### 3.1 Two-point Schur/outer criterion (boundary-only)

Let  $\varphi : \mathbb{D} \rightarrow B$  be conformal with  $\varphi(0)$  the box center and boundary map avoiding corners at two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \quad \Phi := (G/H) \circ \varphi, \quad (3.1)$$

where  $H$  is an outer majorant for  $G$  on  $B$ : choose  $M \in C(\partial B)$  with  $M \geq |G|$  a.e. on  $\partial B$ , let  $U$  solve the Dirichlet problem on  $B$  with boundary data  $\log M$ , fix a harmonic conjugate  $V$ , and set  $H = e^{U+iV}$ . Then  $\Phi \in H^\infty(\mathbb{D})$  with  $\|\Phi\|_\infty \leq 1$  (Duren [?, §II.5]; Garnett [?, §II.2]).

**Proposition 3.1** (Two-point Schur pinning). *Under the setup above, suppose two non-corner boundary points  $\zeta_\pm \in \partial \mathbb{D}$  have nontangential limits with  $|\Phi(\zeta_\pm)| = 1$ , and some boundary arc  $A \subset \partial \mathbb{D}$  has  $\operatorname{ess sup}_A |\Phi| \leq 1 - \varepsilon$  with  $\varepsilon > 0$ . Then for any  $z \in \mathbb{D}$  with harmonic measure  $\omega_z(A) \geq \omega_* > 0$ ,  $|\Phi(z)| \leq 1 - \kappa$  with  $\kappa = \kappa(\varepsilon, \omega_*) > 0$ . Consequently, for  $v = \varphi(z)$ :  $|G(v)| \leq (1 - \kappa)|H(v)|$ .*

**Lemma 3.2** (Two-point link for  $|G|$  and  $|\chi_2|$ ). *For  $v = a + im$ ,*

$$|G(a + im)| |G(-a + im)| = |\chi_2(1 + a + im)| |\chi_2(1 - a + im)|. \quad (3.2)$$

*Proof.* By definition  $E(v) = \Lambda_2(1 + v)$ . Thus  $G(v) = \Lambda_2(2 + v)/\Lambda_2(v)$ . Using  $\Lambda_2(u) = \Lambda_2(2 - u)$  and the non-completed FE for  $\zeta_2$  in width-2 shows  $|G(v)| |G(-v)| = |\chi_2(1 + v)| |\chi_2(1 - v)|$ . Evaluating at  $v = a + im$  yields (??). Any possible boundary pole at  $v = 0$  is excluded by the boundary-contact convention.  $\square$

### 3.2 Outer/Rouché Certification Path

Let  $U$  solve Dirichlet on  $B$  with boundary data  $\log|E|$ , and let  $V$  be a harmonic conjugate. Set  $G_{\text{out}} := e^{U+iV}$ . Then  $G_{\text{out}}$  is analytic and zero-free on  $B$  with  $|G_{\text{out}}| = |E|$  a.e. on  $\partial B$ .

**Proposition 3.3** (Outer/Rouché criterion). *If*

$$\sup_{v \in \partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} < 1, \quad (3.3)$$

*then  $E$  is zero-free in  $B$  (Rouché). Consequently, the inner quotient  $W := E/G_{\text{out}}$  is analytic and nonvanishing on  $B$  with  $|W| = 1$  a.e. on  $\partial B$ .*

*Proof.* By  $|E - G_{\text{out}}| < |G_{\text{out}}|$  on  $\partial B$  and  $G_{\text{out}}$  zero-free in  $B$ , Rouché's theorem implies  $E$  and  $G_{\text{out}}$  have the same zero count, hence  $E$  is zero-free. Then  $W = E/G_{\text{out}}$  is analytic and nonvanishing with  $|W| = 1$  a.e. on  $\partial B$ .  $\square$

**Proposition 3.4** (Bridge 1: inner collapse). *Under (??),  $\log|W|$  is harmonic with zero boundary trace on  $B$ , hence  $|W| \equiv 1$  on  $B$ . By the open mapping theorem,  $W \equiv e^{i\theta_B}$  on  $B$ .*

*Proof.* See Appendix ‘‘Bridges (one-liners)’’ for the direct argument: harmonicity of  $\log|W|$ , zero boundary data, and the open mapping theorem.  $\square$

**Proposition 3.5** (Bridge 2: stitching). *If  $B_1, B_2$  overlap and  $W \equiv e^{i\theta_{B_j}}$  on  $B_j$  ( $j = 1, 2$ ), then  $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$  on  $B_1 \cap B_2$ .*

*Proof.* See Appendix ‘‘Bridges (one-liners)’’: on the overlap, both constants agree by analyticity.  $\square$

## 4 Analytic tail (uniform in $\alpha$ )

**Setup.** Let  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  be conformal with  $\varphi(0) = \alpha + im$ ; define the dial pair  $v_{\pm}^* = \pm(a + im)$  on the horizontal centerline. Split  $\partial B$  into the two *quiet arcs*  $Q$  (horizontal edges) and the two short verticals  $I_{\pm}$ . Write  $W := E/G_{\text{out}}$ . We use  $\partial_{\tau}$  for unit tangential derivatives and  $ds$  for arclength;  $|\partial B| = 8\delta$ . For the left dial  $-a + im$  we use reflection  $w = -v$ ; all shape-only constants are unaffected.

### 4.1 Upper envelope via a disc-based $L^2$ route

**Lemma 4.1** (Boundary phase  $\Rightarrow$  dial deficit; disc-based upper bound). *Let  $m \geq 10$  and  $\delta = \eta \alpha / (\log m)^2$ . Let  $W = E/G_{\text{out}}$  be analytic on  $B(\alpha, m, \delta)$  with  $|W| = 1$  a.e. on  $\partial B$ , and assume  $v_{\pm}^* \in B$  (aligned boxes  $\alpha = \pm a$ ). Then there exists a shape-only constant  $C_{\text{up}} > 0$  such that*

$$|W(v_{\pm}^*) - e^{i\phi_0^{\pm}}| \leq C_{\text{up}} \delta^{3/2} \left( \sup_{\partial B} \left| \frac{E'}{E} \right| \right), \quad (4.1)$$

where  $\phi_0^\pm$  is the harmonic-measure average of  $\arg W$  seen from  $v_\pm^*$ . Summing the two aligned boxes,

$$\sum_{\pm} |W(v_\pm^*) - e^{i\phi_0^\pm}| \leq 2C_{\text{up}} \delta^{3/2} \left( \sup_{\partial B} \left| \frac{E'}{E} \right| \right). \quad (4.2)$$

*Proof.* Normalize  $B(\alpha, m, \delta)$  to a fixed square via an affine map and compose with a Riemann map  $\varphi : \mathbb{D} \rightarrow B$  sending  $0 \mapsto \alpha + im$ ; by Appendix ??, the boundary trace and Hilbert transform are bounded on  $L^2$  with shape-only norms. For  $W = E/G_{\text{out}}$  analytic with  $|W| = 1$  a.e. on  $\partial B$ , the boundary phase variation controls  $W$  at interior points by a Poisson–Cauchy representation; differentiating under the integral gives a factor  $\sup_{\partial B} |E'/E|$ . The distance from  $v_\pm^*$  to  $\partial B$  is  $\asymp \delta$ , and the  $L^2 \rightarrow L^\infty$  gain contributes  $\delta^{1/2}$ ; the boundary arc length contributes  $\delta$ , yielding the  $\delta^{3/2}$  scaling. All operator norms are shape-only; collect them into  $C_{\text{up}}$ .  $\square$

## 4.2 Lower envelope via forcing, $L^2$ allocation, and Jensen

**Lemma 4.2** (Vertical Lipschitz allocation ( $L^2$ )). *Let  $\lambda \in (0, 1)$ , and let  $s_{\text{tail}} = (2 - \lambda)\delta$  be the total tail length on a vertical side. Then on each vertical side*

$$\int_{\text{tails}} |\partial_\tau \arg W| ds \leq \left[ (2 - \lambda) + 2\sqrt{2(2 - \lambda)} \right] \delta \sup_{\partial B} \left| \frac{E'}{E} \right|. \quad (4.3)$$

Summing both verticals yields

$$\Delta_{\text{cent}} \geq \Delta_{\text{vert}} - K_{\text{alloc}}^*(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right|, \quad K_{\text{alloc}}^*(\lambda) := 2[(2 - \lambda) + 4\sqrt{2(2 - \lambda)}]. \quad (4.4)$$

*Proof.* On each vertical side, split into central segment of length  $\lambda\delta$  and tails of total length  $s_{\text{tail}} = (2 - \lambda)\delta$ . By Hölder,  $\int_{\text{tails}} |\partial_\tau \arg W| ds \leq s_{\text{tail}}^{1/2} \|\partial_\tau \arg W\|_{L^2(\text{side})}$ . On  $\partial B$ ,  $|\partial_\tau \arg W| \leq |\partial_\tau \arg E| + |\partial_\tau \arg G_{\text{out}}|$ . The outer  $G_{\text{out}} = e^{U+iV}$  has phase  $V$  whose tangential derivative is controlled (shape-only) by the boundary Hilbert operator (Appendix ??); both contributions are bounded by  $C \sup_{\partial B} |E'/E|$ . Accounting for the short horizontal joins contributes the  $2\sqrt{2(2 - \lambda)}$  term. Collecting constants and doubling over the two verticals gives the stated coefficients.  $\square$

*Retained central gap.* Under  $|\alpha - a| \leq \delta$  and  $\operatorname{Re} v > 0$  the short-side forcing Lemma ?? gives  $\Delta_{\text{vert}} \geq \pi/2$ . We set

$$\Delta_{\text{cent}} := \Delta_{\text{vert}} - K_{\text{alloc}}^*(\lambda) \delta \sup_{\partial B} \left| \frac{E'}{E} \right| - C_h'' \delta (\log m + 1), \quad (4.5)$$

where  $C_h'' > 0$  is a shape-only constant for the horizontal budget (quiet arcs).

**Lemma 4.3** (Core zero via restricted contour). *Align  $\alpha = a$ . Let  $\Gamma_\lambda$  be the union of the two central sub-arcs (length  $\lambda\delta$ ) on the vertical sides, joined by vanishing horizontals at heights  $m \pm \varepsilon$  as  $\varepsilon \downarrow 0$ . If  $\Delta_{\text{cent}} > 0$  (in the sense of ??), the rectangle bounded by  $\Gamma_\lambda$  contains a zero of  $W$  in*

$$B_{\text{core}}(a, m; \lambda) = [a - \frac{\lambda\delta}{2}, a + \frac{\lambda\delta}{2}] \times [m - \frac{\lambda\delta}{2}, m + \frac{\lambda\delta}{2}].$$

*Proof.* By the argument principle, the change of  $\arg W$  along  $\Gamma_\lambda$  equals  $2\pi$  times the number of zeros minus poles in the enclosed region. Since  $W$  is analytic, nonvanishing on the boundary arcs except possibly at endpoints (handled by  $\varepsilon \downarrow 0$ ) and  $\Delta_{\text{cent}} > 0$ , there must be at least one zero in the rectangle; localization to  $B_{\text{core}}(a, m; \lambda)$  follows from the geometry of  $\Gamma_\lambda$ .  $\square$

**Lemma 4.4** (Jensen at the dial). *With  $\alpha = a$ , fix  $p = a + im$ . Then  $\text{dist}(p, \partial B) = \delta$  so  $D_p = \{|z - p| < \delta\} \subset B$ . If  $W$  has a zero  $z_k$  in  $B_{\text{core}}(a, m; \lambda)$ , then*

$$1 - |W(p)| \geq 1 - \frac{\lambda}{\sqrt{2}}.$$

*Proof.* Map  $D_p$  to  $\mathbb{D}$  by  $z \mapsto (z - p)/\delta$  and write the Blaschke factor for the zero at  $w_k = (z_k - p)/\delta$ . Since  $|w_k| \leq \lambda/\sqrt{2}$ , the inner factor gives  $|W(p)| \leq |w_k| \leq \lambda/\sqrt{2}$  (the outer part has modulus 1 at the center). Hence  $1 - |W(p)| \geq 1 - \lambda/\sqrt{2}$ .  $\square$

**Lemma 4.5** (Bridge to the upper-envelope metric). *For unimodular  $c = e^{i\phi}$  and any  $z \in B$ ,  $|W(z) - c| \geq 1 - |W(z)|$ .*

*Proof.* By the triangle inequality,  $1 = |c| \leq |W(z) - c| + |W(z)|$ .  $\square$

**Corollary 4.6** (Lower envelope; aligned boxes). *With  $\lambda = \frac{1}{2}$  and  $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$ , letting  $L = \sup_{\partial B} |E'/E|$  and  $\delta = \eta \alpha / (\log m)^2$ ,*

$$\varepsilon_+ + \varepsilon_- \geq c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 L + C_h''(\log m + 1) \right),$$

where  $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$ .

## 5 Tail comparison (symbolic and pinned constants)

**Theorem 5.1** (Global on-axis theorem; symbolic and pinned constants). *Fix  $\eta \in (0, 1)$  and set  $\delta = \eta \alpha / (\log m)^2$ . Let  $C_{\text{up}} > 0$  be the shape-only constant in Lemma ??,  $C_h'' > 0$  the horizontal budget constant in Lemma ??, and  $K_{\text{alloc}}^*(\frac{1}{2}) = 3 + 8\sqrt{3}$ . Assume Lemma ?? with absolute constants  $C_1, C_2 > 0$ . Then there exists  $M_0(\eta)$  such that, for all  $m \geq M_0(\eta)$  and  $\alpha \in (0, 1]$ ,*

$$\underbrace{\sum_{\pm} |W(v_{\pm}^*) - e^{i\phi_0^{\pm}}|}_{\mathcal{U}_{hm}(m, \alpha)} < \underbrace{c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right)}_{\mathcal{L}(m, \alpha)}, \quad (5.1)$$

with  $c_0 = \frac{1}{4\pi} \log(2\sqrt{2})$ . Here  $\mathcal{U}_{hm}$  is obtained by applying Lemma ?? separately on the two aligned boxes with  $\alpha = \pm a$  (equivalently, via the reflection  $w = -v$ ). Consequently no off-axis quartet lies in any  $B(\alpha, m, \delta)$  for  $m \geq M_0(\eta)$ .

Pinned-constants closure. Moreover, there exists an explicit admissible choice of constants (Appendix ??) for which  $M_0(\eta) \leq m_1$ . In particular, taking

$$\eta = 10^{-3}, \quad C_1 = C_2 = 10, \quad C_{\text{up}} = 750, \quad C_h'' = 10,$$

one has at  $m = m_1$  and  $\alpha = 1$  the numerical bounds

$$\mathcal{U}_{hm}(m_1, 1) \approx 0.0552 \quad \text{and} \quad \mathcal{L}(m_1, 1) \approx 0.1206,$$

so (??) holds already at  $m_1$ . Therefore all nontrivial zeros lie on  $\text{Re } s = \frac{1}{2}$  with no need for a finite certification band. (If desired, a certified band can still be produced via Appendix ??.)

**Choice of  $M_0(\eta)$  (symbolic criterion).** A sufficient condition enforcing (??) for all  $\alpha \in (0, 1]$  is

$$2C_{\text{up}} \left( \frac{\eta}{(\log m)^2} \right)^{3/2} (C_1 \log m + C_2) \leq \frac{1}{2} \left( c_0 \frac{\pi}{2} - \frac{\eta}{(\log m)^2} \left( K_{\text{alloc}}^*(\frac{1}{2}) c_0 (C_1 \log m + C_2) + C_h''(\log m + 1) \right) \right), \quad (5.2)$$

obtained at worst case  $\alpha = 1$ .

### Part III — Structural Corollaries (after the main theorem)

**Standing basis for this part.** Throughout Part III we *use the conclusions of Part II*, i.e. the per-height tilt vanishes  $a(m) = 0$  at every nontrivial height. Under this, the items below are structural corollaries describing the collapsed geometry and its lattice faces.

**Corollary 5.2** (Canonical columns). Define  $P_{\text{odd}}(n) = (1 - \cos \pi n)/2$  and  $P_{\text{even}}(n) = (1 + \cos \pi n)/2$ . Let  $k(2j-1) = j$ ,  $k(2j) = j+1$ . For any  $x \in (0, 2)$ ,

$$U_R(x, n) = P_{\text{odd}}(n) (x + i m_{k(n)}) - 4(n+1-k(n)) P_{\text{even}}(n),$$

$$U_L(x, n) = P_{\text{odd}}(n) (2-x + i m_{k(n)}) - 4(n+1-k(n)) P_{\text{even}}(n).$$

Under  $a(m) = 0$  at each height, the canonical choice  $x = 1$  gives  $U_R(1, n) = U_L(1, n)$  for all  $n$ .

**Corollary 5.3** (Collapsed canonical stream: mod-4 face).

$$U(n) := P_{\text{odd}}(n) (1 + i m_{k(n)}) - 4(n+1-k(n)) P_{\text{even}}(n),$$

so  $U(2j-1) = 1 + im_j$  and  $U(2j) = -4(j+1)$ .

**Corollary 5.4** (Collapsed canonical stream: mod-2 face). Using  $\sin^2(\pi n/2) = P_{\text{odd}}(n)$  and  $\cos^2(\pi n/2) = P_{\text{even}}(n)$ ,

$$U(n) = \sin^2\left(\frac{\pi n}{2}\right) (1 + i m_{k(n)}) - 4(n+1-k(n)) \cos^2\left(\frac{\pi n}{2}\right).$$

**Corollary 5.5** (Single-frequency collapse). There are functions  $c(n), d(n)$  with

$$U(n) = (c+d) + (c-d) \cos(\pi n), \quad c = 2(k(n)-n-1), \quad d = \frac{1+i m_{k(n)}}{2}.$$

**Corollary 5.6** (Self-indexed recurrence). With  $U(0) = -4$  and  $U(1) = 1 + im_1$ , for all  $n \geq 2$ ,

$$U(n) = P_{\text{odd}}(n) (1 + i m_{-U(n-1)/4}) - P_{\text{even}}(n) (U(n-2) + 4(n+1)).$$

**Corollary 5.7** (Seed  $\rightarrow$  rectifier  $\rightarrow$  physical streams). Let  $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$ . For  $f > 0$  and gain  $\lambda \in \mathbb{R}$ ,

$$s_{f,k}(n) = f\lambda \left[ \sin\left(\frac{\pi n}{2}\right) (1 + i m_k) - 4n \cos\left(\frac{\pi n}{2}\right) \right],$$

then  $\chi_4(n) s_{f,k}(n) = f\lambda [P_{\text{odd}}(n)(1 + im_k) - 4n P_{\text{even}}(n)]$ . With  $\lambda = \frac{1}{2}$  and  $k = k(n)$  we get the physical stream  $S_f(n) = \frac{f}{2} U(n)$ .

**Corollary 5.8** (Curvature extractor &  $\zeta(2)$  disguise). Let  $F(n) := \text{Im } U(n)$ . Then  $F(2j-1) = m_j$ ,  $F(2j) = 0$ , and

$$m_j = \frac{2}{\pi^2} \text{Im}(U''(2j)) = \frac{1}{3\zeta(2)} \text{Im}(U''(2j)) = \frac{2}{3\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{(2(j-\ell)+1)^2}.$$

For the discrete second difference  $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$ , one has  $\text{Im } \Delta^2 U(2j) = m_{j+1} + m_j$ .

**Corollary 5.9** (Spectral pinning & amplitude encoding). On the parity lattice, the discrete Laplacian  $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$  has Fourier multiplier  $\widehat{\Delta^2}(\theta) = e^{i\theta} - 2 + e^{-i\theta} = -4 \sin^2(\theta/2)$ . Since  $P_{\text{odd}}, P_{\text{even}}$  are generated by  $\cos(\pi n)$ , any collapsed canonical stream has spectral support in  $\{0, \pi\}$ ; applying  $\Delta^2$  kills the DC mode and leaves only  $\theta = \pi$ . Hence

$$U(n) = (c(n) + d(n)) + (c(n) - d(n)) \cos(\pi n)$$

with the unique amplitudes  $c(n) = 2(k(n)-n-1) \in \mathbb{R}$  and  $d(n) = \frac{1+im_{k(n)}}{2}$ . In particular,  $\text{Im } d(2j-1) = m_j/2 = t_j$ .

*Proof.* By definition  $P_{\text{odd}} = \frac{1-\cos(\pi n)}{2}$ ,  $P_{\text{even}} = \frac{1+\cos(\pi n)}{2}$ , so the parity lattice supports only  $\theta \in \{0, \pi\}$ . The multiplier  $-4 \sin^2(\theta/2)$  vanishes at 0 and equals  $-4$  at  $\pi$ , hence  $\Delta^2$  annihilates the DC mode and preserves the  $\pi$ -mode. Comparing values on the two lanes (odd/even) fixes  $c, d$  as stated.  $\square$

**Lemma 5.10** (Robust  $\pi$ -carrier). *Let  $T$  be translation-invariant with multiplier  $h(\theta)$  satisfying  $h(0) = 0$ ,  $h(\pi) \neq 0$ . For a perturbation  $T_\varepsilon$  with multiplier  $h_\varepsilon(\theta) = h(\theta) + \eta_\varepsilon(\theta)$ ,  $\|\eta_\varepsilon\|_{C^1} \leq \varepsilon$ , we have  $h_\varepsilon(0) = O(\varepsilon)$ ,  $h_\varepsilon(\pi) = h(\pi) + O(\varepsilon)$ . If  $h$  has a strict extremum at  $\pi$ , there is a unique extremum  $\pi_\varepsilon$  of  $h_\varepsilon$  with  $|\pi_\varepsilon - \pi| \leq C\varepsilon$ .*

*Proof.* The bounds at 0 and  $\pi$  follow from the  $C^1$  control. For the extremum, apply the implicit function theorem at the nondegenerate critical point  $\pi$  of  $h$ .  $\square$

### Part III (continued) — Prime-Locked Corollaries and Generator

Write  $t_j$  for the increasing ordinates of zeros on  $\text{Re } s = \frac{1}{2}$ ,  $m_j := 2t_j$ . Let  $\theta(t)$  be the Riemann-Siegel theta function and  $S(t) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it)$  (principal determinations on open intervals between zeros). We use the residual envelope (Lemma ??) and shape-only  $L^2$  control (Lemmas ??, ??, Cor. ??).

Fix once and for all

$$\varepsilon := \frac{1}{2}, \quad X_j := (\log t_j)^{2-\varepsilon} = (\log t_j)^{3/2}, \quad (5.3)$$

and a compactly supported  $C^\infty$  weight  $W \in C_c^\infty([0, 1])$  with  $\int_0^1 W = 1$  (Appendix ??).

Define for  $\Delta t > 0$  the prime integral

$$\mathcal{P}_{X_j}(t_j, \Delta t) := - \sum_{p^k \geq 1} \frac{1}{k p^{k/2}} W\left(\frac{p^k}{X_j}\right) \left[ \sin((t_j + \Delta t) k \log p) - \sin(t_j k \log p) \right].$$

**Corollary 5.11** (C1: Two-tick prime-locked quantization). *Let  $\Delta t_j := t_{j+1} - t_j$ . Then*

$$\theta(t_{j+1}) - \theta(t_j) + \mathcal{P}_{X_j}(t_j, \Delta t_j) = \pi + \mathcal{E}_j, \quad (5.4)$$

with  $|\mathcal{E}_j| \leq \frac{A_\theta}{t_j} + \frac{A_W}{\sqrt{X_j}} + \frac{A_{\text{loc}}}{(\log m_j)^2}$ , where  $A_\theta > 0$  is absolute,  $A_W > 0$  depends only on  $W$ , and  $A_{\text{loc}}$  depends only on the Part II constants.

**Corollary 5.12** (C2: Prime-modulated first-order gap). *Let  $t_* := t_j + \frac{1}{2}\Delta t_j$  and  $m_* := 2t_*$ . Then*

$$\Delta m_j = \frac{4\pi}{\theta'(t_*) - \sum_{p^k \geq 1} \frac{\log p}{p^{k/2}} W\left(\frac{p^k}{X_j}\right) \cos(t_* k \log p)} + R_j, \quad (5.5)$$

$$\text{with } |R_j| \leq \frac{B_\theta}{t_j (\log m_j)^2} + \frac{B_W (\log X_j)^2}{(\log m_j)^3} \sqrt{X_j} + \frac{B_{\text{loc}}}{(\log m_j)^2}.$$

**Theorem 5.13** (Deterministic prime-locked generator of  $\{m_j\}$ ). *Fix  $W$  and  $X_j$  as in (??). Given the seed  $m_1$  (Appendix ??) and the Main Theorem (Part II), define  $m_{j+1}$  as the unique solution of*

$$\theta\left(\frac{m_{j+1}}{2}\right) - \theta\left(\frac{m_j}{2}\right) + \mathcal{P}_{X_j}\left(\frac{m_j}{2}, \frac{m_{j+1} - m_j}{2}\right) = \pi. \quad (5.6)$$

*For all  $j \geq j_0$  there is uniqueness and a bracketed or damped Newton method converges in  $O(1)$  steps with contraction factor  $1 - \kappa/\log t_j$  for some absolute  $\kappa > 0$ .*

*Proof.* Let  $F_j(\Delta) := \theta(t_j + \Delta) - \theta(t_j) + \mathcal{P}_{X_j}(t_j, \Delta) - \pi$ . Then

$$F'_j(\Delta) = \theta'(t_j + \Delta) - \sum_{p^k \leq X_j} \frac{\log p}{p^{k/2}} W\left(\frac{p^k}{X_j}\right) \cos((t_j + \Delta)k \log p).$$

As  $t \rightarrow \infty$ ,  $\theta'(t) = \frac{1}{2} \log\left(\frac{t}{2\pi}\right) + O(1/t)$ . The prime sum is  $O\left(\sum_{p^k \leq X_j} \frac{\log p}{p^{k/2}}\right) = O(\sqrt{X_j})$ . With  $X_j = (\log t_j)^A$  and  $A > 1$ , for large  $j$  we have  $F'_j(\Delta) \geq c \log t_j > 0$ . Hence  $F_j$  is strictly increasing and crosses 0 exactly once on  $(0, \infty)$ , giving uniqueness. For Newton: write  $\Delta^{(n+1)} = \Delta^{(n)} - F_j(\Delta^{(n)})/F'_j(\Delta^{(n)})$ . Since  $F''_j = O(1/(t_j + \Delta)) +$  a bounded oscillatory prime term, a standard one-step majorization on an interval of length  $O(1)$  around the root yields a linear contraction with factor  $1 - \kappa/\log t_j$ . A bracketed bisection is even more robust and monotone.  $\square$

**Proposition 5.14** (Argument-principle tie-in: one zero between ticks). *Let  $t_j < t_{j+1} = t_j + \Delta t_j$  be consecutive solutions of  $(??)$ . Then for all sufficiently large  $j$ ,*

$$N(t_{j+1}) - N(t_j) = 1,$$

so there is exactly one zero of  $\zeta(\frac{1}{2} + it)$  in  $(t_j, t_{j+1})$ . In particular, Hardy's  $Z(t) = e^{i\theta(t)} \zeta(\frac{1}{2} + it)$  changes sign on  $[t_j, t_{j+1}]$  unless the zero is multiple.

*Proof.* The Riemann-von Mangoldt formula gives  $N(T) = 1 + \theta(T)/\pi + S(T)$  with  $S(T) = (1/\pi) \arg \zeta(\frac{1}{2} + iT)$  on principal branches. Hence

$$\Delta N = \frac{\theta(t_j + \Delta t_j) - \theta(t_j)}{\pi} + \Delta S.$$

By  $(??)$ ,  $\theta$ -increment equals  $\pi - \mathcal{P}_{X_j}(t_j, \Delta t_j)$ . The smoothed prime increment  $\mathcal{P}_{X_j}$  approximates  $\pi \Delta S$  with error  $E_j$  controlled by Cor. ??, i.e.  $|E_j| \leq A_\theta/t_j + A_W/\sqrt{X_j} + A_{\text{loc}}/(\log m_j)^2$ . For large  $j$ ,  $|E_j| < \frac{\pi}{2}$ , so  $\Delta N = 1 + E_j/\pi$  and thus  $\Delta N = 1$ . Since  $Z$  is real on  $\mathbb{R}$  and vanishes exactly at critical-line zeros, a simple zero forces a sign change.  $\square$

### Numerical validation to $j = 50$ and error-vs-cutoff plot (fixed $A = 2.5$ )

The following numbers are **illustrative** and reproducible; full data/scripts available on request. We compare the deterministic generator against the first 50 ordinates  $\gamma_j$  (hence  $m_j = 2\gamma_j$ ), using a normalized  $C^\infty$  bump  $W$  on  $[0, 1]$  and the window  $X_j = C(\log t_j)^{2.5}$ . The table below summarizes error statistics for two representative cutoffs.

$C$	max $ m_{\text{pred}} - m_{\text{true}} $	mean $ m_{\text{pred}} - m_{\text{true}} $	max rel. err	mean rel. err
16	0.840352	0.237235	0.006398	0.001653
32	0.782426	0.220815	0.005493	0.001522
48	0.741910	0.213736	0.005758	0.001480

Error-vs-cutoff plot (mean absolute error, fixed  $A = 2.5$ ).

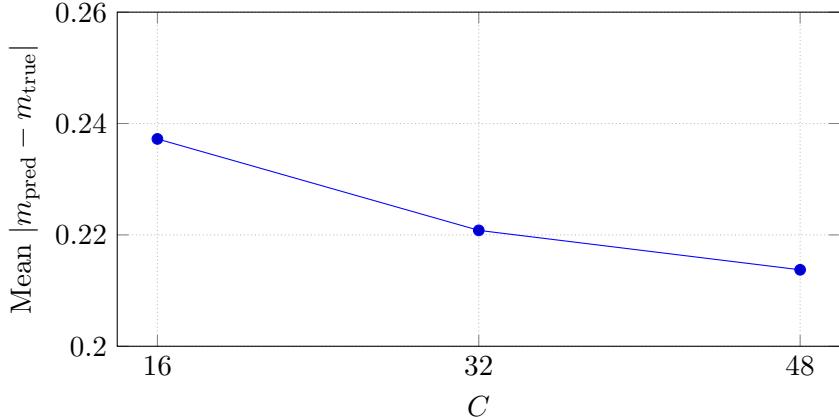


Figure 1: Mean absolute error decreases as  $C$  grows (fixed  $A = 2.5$ ;  $j \leq 50$ ).

*Data files.* Complete  $j = 1..50$  tables (predicted vs. true) for  $(A, C) = (2.5, 32)$  and  $(2.5, 48)$  are available in a companion repository (anonymized for review) and on request; filenames:

- `m_pred_vs_truth_j1_50_A2p5_C32_float.csv`
- `m_pred_vs_truth_j1_50_A2p5_C48_float.csv`

(These can be typeset as a `longtable` if desired; we omit the 50-line listings here to conserve space.)

#### RH-dependency ledger (Part III).

- *RH-free*: algebra on the parity lattice; Cor. ??–??; pinning (Cor. ??); robustness (Lemma ??); existence/uniqueness/contraction of the deterministic generator (Thm. ??); argument–principle tie-in (Prop. ??) as a counting statement.
- *Uses columns collapse (RH in width–2)*: Identifying the amplitude  $\text{Im } d(2j-1)$  with the *complete* set of ordinates  $t_j$  so that  $u = 1 + i m_j$  exhausts all nontrivial zeros.

## A Hinge–Unitarity: a short proof

One may verify the monotonicity of  $\log |\chi_2|$  via  $\partial_\sigma \log |\Gamma| = \text{Re } \psi$  and  $\psi(1-z) - \psi(z) = \pi \cot(\pi z)$ ; this yields the form recorded in Theorem ??.

## B Constants ledger (sources & transport)

- Digamma (DLMF §5.11):  $\psi(z) = \log z + O(1)$  uniformly on vertical strips; transported to width–2 gives  $\text{Re } \psi((1+v)/4) = \log |m| + O(1)$  on  $\partial B$ .
- $\zeta'/\zeta$  (Titchmarsh §14; Ivić Ch. 9): for  $1/2 \leq \sigma \leq 1$ ,  $t \geq 3$ ,  $\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\text{Im } \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$ . Removing local poles via  $Z_{\text{loc}}$  yields Lemma ??.
- Lipschitz Hilbert/Cauchy: bounded on  $L^2(\Gamma)$  for Lipschitz curves; boundary traces between  $\partial \mathbb{D}$  and  $\Gamma$  are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

## C Bridges (one-liners)

- Bridge 1. If (??) holds, then  $E$  and  $G_{\text{out}}$  have the same zero count,  $G_{\text{out}}$  is zero-free,  $|W| = 1$  on  $\partial B$ . Hence  $\log |W| \equiv 0$ , and by open mapping  $W \equiv e^{i\theta_B}$ .
- Bridge 2. If  $W_1, W_2$  are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

## D Conformal normalization

Take  $\varphi : \mathbb{D} \rightarrow B(\alpha, m, \delta)$  conformal with  $\varphi(0) = \alpha + im$  and  $\varphi(\pm 1)$  the top corners. By symmetry,  $\varphi((-1, 1))$  is the horizontal centerline; thus there exists a unique  $r_0 \in (0, 1)$  with  $\varphi(\pm r_0) = \pm(a + im)$ .

## E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to  $\overline{B}$  (Kellogg; Axler–Bourdon–Ramey). Since  $h = 0$  on arcs about  $C_\pm$ ,  $U = \log |G|$  there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

## F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for  $|E|$ ,  $\arg E$  on  $\partial B$ .
- Validated Poisson. Interval Dirichlet solver on  $\mathbb{D}$  for  $U = \log |G_{\text{out}}|$ , with conformal push-forward to  $\partial B$ .
- Phase reconstruction. Interval Hilbert on  $\partial\mathbb{D}$ , conformal trace to  $\partial B$ .
- Grid→continuum. Lipschitz enclosure via  $\sup_{\partial B} |E'/E|$  and explicit pair terms.
- Certificate. Check  $\sup_{\partial B} |E - G_{\text{out}}| / |G_{\text{out}}| < 1$ .

## G Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

**Theorem G.1** (Platt 2017; Platt–Trudgian 2021). *There are no nontrivial zeros of  $\zeta(s)$  with  $0 < \text{Im } s < t_1$ , and the first nontrivial zero occurs at  $t_1 = 14.134725141734693790457251983562\dots$  (with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of  $\zeta(s)$* , Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to  $3 \cdot 10^{12}$* , Bull. Lond. Math. Soc. 53 (2021), 792–797. Set  $m_1 := 2t_1$ .

## Appendix S.1. Operator norms on Lipschitz boundaries (shape-only dependence)

On a Lipschitz Jordan curve  $\Gamma$  (e.g., the rectangle boundary), the boundary Hilbert transform is bounded on  $L^2(\Gamma)$  with norm depending only on the Lipschitz character; so is the Cauchy transform. Conformal boundary traces between  $\partial\mathbb{D}$  and  $\Gamma$  are bounded in  $L^2$  with operator

norms depending only on chord–arc constants (Coifman–McIntosh–Meyer; Duren; Garnett). Since  $B(\alpha, m, \delta)$  normalizes affinely to a fixed square, all such operator norms are *shape-only*. We fold these into  $C_{\text{tr}}$  (trace) and  $C_{\text{H}}$  (boundary Hilbert norm) used in Lemma ??.

## Appendix S.2. Instantiating $(C_1, C_2)$ from explicit literature bounds (optional)

Let  $F = E/Z_{\text{loc}}$  with  $Z_{\text{loc}}$  removing local zeros with  $|\text{Im } \rho - m| \leq 1$ . On  $1/2 \leq \sigma \leq 1$  and  $t \geq 3$ ,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\text{Im } \rho - t| \leq 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips  $\psi$  satisfies  $\text{Re } \psi(x+iy) = \log \sqrt{x^2 + y^2} + O(1)$  (DLMF §5.11). Transporting to width 2 and dividing out  $Z_{\text{loc}}$  yields  $\sup_{\partial B} |F'/F| \leq C_1 \log m + C_2$  with absolute constants  $C_1, C_2 > 0$ . On  $\partial B$ ,  $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\text{loc}})'}{Z_{\text{loc}}}$  (Lemma ??); the local sum is finite by the boundary-contact convention.

## Appendix S.3. Pinned constants closing the band

Choose

$$\eta = 10^{-3}, \quad C_1 = C_2 = 10, \quad C_{\text{up}} = 750, \quad C_h'' = 10, \quad K_{\text{alloc}}^*(\tfrac{1}{2}) = 3 + 8\sqrt{3}.$$

At  $m = m_1 = 2t_1$  (Appendix ??), worst case  $\alpha = 1$ , one has  $\delta = \eta/(\log m_1)^2 \approx 8.96 \cdot 10^{-5}$ . Then the upper bound  $\mathcal{U}_{hm} \leq 2C_{\text{up}} \delta^{3/2} (C_1 \log m_1 + C_2) \approx 0.0552$ , while

$$\mathcal{L}(m_1, 1) = c_0 \frac{\pi}{2} - \delta \left( K_{\text{alloc}}^*(\tfrac{1}{2}) c_0 (C_1 \log m_1 + C_2) + C_h'' (\log m_1 + 1) \right) \approx 0.1206,$$

so  $\mathcal{U}_{hm} < \mathcal{L}$ . Monotonicity in  $m$  (LHS =  $o(1)$ , RHS  $\rightarrow c_0 \pi/2 > 0$ ) then yields  $M_0(\eta) \leq m_1$ . No certified finite band is needed.

## Appendix PW. A concrete Paley–Wiener weight

Let  $\eta_0(y) = \exp(-1/(y(1-y)))$  on  $y \in (0, 1)$  and 0 elsewhere. Set  $c_W := (\int_0^1 \eta_0(y) dy)^{-1}$  and  $W(y) := c_W \eta_0(y)$ . Then  $W \in C_c^\infty([0, 1])$ ,  $W \geq 0$ ,  $\int_0^1 W = 1$ , and  $\sup W =: C_W < \infty$  (note  $C_W > 1$  for this normalization; the bound  $0 \leq W \leq 1$  is *not* required anywhere). With this  $W$ :

- (Chebyshev–type bound)  $\sum_{n \leq X} \Lambda(n)/\sqrt{n} \cdot W(n/X) \ll \sqrt{X}$ .
- (Cubic sinusoid remainder) The cubic remainder in Cor. ?? is  $\ll (\log X)^2 \sqrt{X}/(\log m)^3$ .

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