A Height–Local Width–2 Program for Excluding Off–Axis Quartets

with an Analytic Tail and a Rigorous Certified Criterion

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Abstract

The paper is organized in three parts: **Part I** (Reader's Guide) reduces RH to a height–local target $a(m) = 0$ in the width–2 frame and records non–load–bearing scaffolding; **Part II** gives a self–contained, boundary–only analytic proof that at each nontrivial height the tilt vanishes $a(m) = 0$ via a disc–based L^2 upper envelope and an L^2 lower envelope (allocation + restricted contour + Jensen), plus an optional certified Outer/Rouché path; **Part III** promotes the toolbox identities to structural corollaries and presents a deterministic, prime–locked generator of the ordinates.

Contents

Part I — Reader's Guide / Motivation, Reduction & Implications

What this section is (and is not). *What it does.* It introduces modulated frames and the width–2 normalization, defines the centered "a–lens" that measures horizontal tilt at a fixed height, and reduces RH to the height–local target $a(m) = 0$ for each nontrivial height m. It also records a structural toolbox (projectors, rectifier, canonical stream, recurrence, curvature extractor, seed→rectifier) and explains how these become *corollaries* after Part II.
What it does not do. It contains no analytic estimates and no proofs. The hinge–unitarity fact and all bounds are proved later. This Guide is not used by the analytic part.

1) Modulated frames and the width–2 pivot

For $f > 0$ define the modulated family $\zeta_f(s) := \zeta(s/f)$ with completed form

$$\Lambda_f(s) = \pi^{-s/(2f)}\,\Gamma\Big(\frac{s}{2f}\Big)\,\zeta_f(s),$$

so Λ_f is entire and satisfies $\Lambda_f(s) = \Lambda_f(f - s)$. Equivalently, $\zeta_f(s) = A_f(s)\,\zeta_f(f - s)$ with $A_f(s)A_f(f - s) \equiv 1$.

Width–2 normalization. Put $u := (2/f)\,s$. Then

$$\zeta_2(u) := \zeta(u/2), \qquad \Lambda_2(u) := \pi^{-u/4}\Gamma(u/4)\,\zeta(u/2), \qquad \Lambda_2(u) = \Lambda_2(2 - u).$$

The non–completed FE reads $\zeta_2(u) = A_2(u)\,\zeta_2(2 - u)$. In the open strip $0 < \operatorname{Re} u < 2$ and $\operatorname{Im} u \neq 0$, A_2 is analytic and nonvanishing.

Partner map. On $\operatorname{Im} u > 0$, FE + conjugation gives the involution $J(u) = 2 - \overline{u}$, swapping the two column points at the same height.

Hinge unitarity (deferred). The statement "$|\chi_2(u)| = |A_2(u)|^{-1} = 1$" iff $\operatorname{Re} u = 1$ is proved in Part II (Hinge–Unitarity, Theorem **??**; see also Appendix **??**).

2) Centered a–lens and the quartet

Let $v := u - 1$ and $\;E(v) := \Lambda_2(1 + v)$. Then $E(v) = E(-v) = \overline{E(\overline{v})}$. A "nontrivial height" $m > 0$ means m occurs as the imaginary part of a nontrivial zero $s = \frac{1}{2} + \mathrm{i}m/2$. At fixed $m > 0$, set

$$U_{\mathrm{R}}(m; a) = 1 + a + \mathrm{i}m, \qquad U_{\mathrm{L}}(m; a) = 1 - a + \mathrm{i}m, \qquad a \in [0, 1).$$

In the centered frame, the dial points are $\pm(a + \mathrm{i}m)$; the partner map J swaps $U_{\mathrm{R}} \leftrightarrow U_{\mathrm{L}}$. Conjugation plus FE reflection generate the quartet $\{\, 1 \pm a \pm \mathrm{i}m \,\}$.

3) Why width–2: slope invariance

If the columns collapse at height m ($a = 0$), the point is $u = 1+\mathrm{i}m$ and its slope is $\operatorname{Im} u/ \operatorname{Re} u = m$. Rescaling to any frame $s = (f/2)\, u$ preserves slope:

$$\frac{\operatorname{Im} s}{\operatorname{Re} s} = \frac{(f/2)\, m}{f/2} = m.$$

4) Height–local reduction of RH

Fix $m > 0$ and write $U_{\mathrm{R}} = 1 + a + \mathrm{i}m$, $U_{\mathrm{L}} = 1 - a + \mathrm{i}m$. The following equivalent algebraic forms are used:

- (PHU–1) $\operatorname{Re} U_{\mathrm{R}} = \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–2) $\operatorname{Im} U_{\mathrm{R}}/ \operatorname{Re} U_{\mathrm{R}} = \operatorname{Im} U_{\mathrm{L}}/ \operatorname{Re} U_{\mathrm{L}} \iff a = 0$.
- (PHU–3) $U_{\mathrm{R}} = U_{\mathrm{L}} = 1 + \mathrm{i}m$.

Thus RH $\iff$ for every nontrivial height $m > 0$, $a(m) = 0$.

5) Box alignment and hand–off (no circularity)

For later reference, define

$$B(\alpha, m, \delta) = [\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta], \qquad \delta := \eta\, \alpha/(\log m)^2, \quad \eta \in (0, 1).$$

When $\alpha = \pm a$, the dials $\pm(a + \mathrm{i}m)$ lie on the horizontal centerline. *What Part II does.* Using only boundary analysis on such boxes (completed FE symmetry, Cauchy–Riemann transport, three–lines tools, Stirling–class envelopes, explicit control of ζ'/ζ away from zeros), Part II shows any off–axis quartet forces a boundary lower bound larger than an explicit upper bound, hence $a(m) = 0$.

6) Parity gating and selection devices (interpretive only)

In width–2,

$$\zeta_2(u) = A_2(u)\, \zeta_2(2 - u), \quad A_2(u) = 2^{u/2}\, \pi^{\,u/2-1}\, \sin\!\Big(\frac{\pi u}{4}\Big)\, \Gamma\Big(1 - \frac{u}{2}\Big).$$

On $0 < \operatorname{Re} u < 2$, $\operatorname{Im} u \neq 0$, the prefactor $A_2(u)$ is nonzero; its sine zeros lie on the real axis only. Thus *inside* the open strip only ζ_2 can vanish (nontrivial), while the trivial ladder is confined to $\operatorname{Re} u$. This motivates an odd/even split on the integer lattice via

$$P_{\mathrm{odd}}(n) = \tfrac{1-\cos(\pi n)}{2}, \qquad P_{\mathrm{even}}(n) = \tfrac{1+\cos(\pi n)}{2}.$$

We assign the nontrivial stream to odd slots and the trivial ladder to even slots. (Interpretive; not used in Part II.)

7) Toolbox $\to$ structural consequences (after the theorem)

The items (columns/templates, canonical stream, single–frequency collapse, self–indexed recurrence, curvature extractor, seed$\to$rectifier) become *Structural Corollaries in Part III* once Part II proves $a(m) = 0$. No toolbox component is used as an input in Part II.

Part II — Self-Contained Boundary–Only Contradiction on Aligned Boxes

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an *analytic tail*, uniform in $\alpha \in (0, 1]$, using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\text{loc}}$ with perimeter factor 8δ; (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants;

(2) a rigorous *Outer/Rouché Certification Path* (optional): interval arithmetic on ∂B + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow \sup_{\partial B}\big|E - G_{\text{out}}\big|/|G_{\text{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. All constants in the upper/lower envelope are *shape-only* (independent of m, α, a); residual constants are symbolic and optionally instantiated in Appendix **??**.

Symbols & Provenance (at a glance)

Symbol	Definition / role	Provenance / rationale
$u = 2s,\ v = u - 1$	Width–2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$	Completed object	Standard; FE for Λ_2; width–2 transport
$E(v) = \Lambda_2(1 + v)$	Workhorse in v–plane	Even & conjugate symmetrical: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width–2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$[\alpha - \delta, \alpha + \delta] \times [m - \delta, m + \delta]$	Square centered at (α, m), side 2δ
$\alpha \in (0, 1]$	Horizontal center	Worst case $\alpha = 1$; left dial via reflection $w = -v$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0, 1)$	Half–side length of B	Balances forcing vs. residual $O(\delta \log m)$
$\partial B,\ I_\pm,\ Q$	Boundary, short verticals, horizontals	Boundary integrals/suprema; quiet arcs
$Z_{\text{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v - \rho)^{m_\rho}$	Local zero/pole factors	De–singularizes E near ∂B
$F = E/Z_{\text{loc}}$	Residual analytic factor	Lemma **??** (constants symbolic)
$G(v) = \dfrac{E(1 + v)}{E(1 - v)}$	Odd–lane quotient	Links to hinge; two–point identity
$G_{\text{out}} = e^{U+iV}$	Modulus–outer with $\lvert G_{\text{out}}\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert$ solves Dirichlet; V harmonic conjugate
$W = E/G_{\text{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v^\star_\pm = \pm(a + im)$	Dial pair	Points of evaluation in the tail on centerline
$Z_{\text{pair}}(v)$	$(v - (a + im))(v - (-a + im))$	Short–side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub–arcs + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial–centred core box	Zero location forced by Γ_λ
$K^{(\star)}_{\text{alloc}}(\lambda)$	Allocation coefficient	Shape–only; Lemma **??**
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial; Lemma **??**
C_{up}	Upper–envelope constant	Disc–based bound; Lemma **??**
C''_h	Horizontal budget constant	Shape–only; Lemma **??**

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical–strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width–2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar v)}$ and off-axis zeros appear as quartets $\{\pm a \pm im\}$ by the FE symmetry plus conjugation.

Theorem 1.1 (Hinge–Unitarity)**.** *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma+it)|$. Then

$$f'(\sigma) = \tfrac12\log\pi - \tfrac12\,\mathrm{Re}\,\psi\Big(\tfrac{\sigma+it}{4}\Big) - \tfrac14\,\mathrm{Re}\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+it)\Big)\Big].$$

Moreover,

$$\big|\,\mathrm{Re}\big[\pi\cot(x+iy)\big]\big| \le \frac{\pi}{\cosh(2y)-1},$$

indeed using $\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$ *and* $\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y$ *one obtains* $\cot(x+iy) = \dfrac{\sin(2x) - i\sinh(2y)}{\cosh(2y)-\cos(2x)}$ *with real denominator; hence* $\big|\,\mathrm{Re}[\pi\cot(x+iy)]\big| = \pi\,\dfrac{|\sin(2x)|}{\cosh(2y)-\cos(2x)} \le \dfrac{\pi}{\cosh(2y)-1}$. *With $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (Appendix* **??***) the cotangent term is negligible, and vertical-strip bounds give* $\mathrm{Re}\,\psi\big(\frac{\sigma+it}{4}\big) \ge \log\big(\frac{|t|}{4}\big) - \frac{2}{|t|}$. *Hence $f'(\sigma) < 0$ on $\mathbb{R}$ for such t. Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\mathrm{Re}\,u = 1$. A short proof is also recorded in Appendix* **??***.*

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta,\alpha+\delta\big]\times\big[m-\delta,m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta\in(0,1). \tag{2.1}$$

Lemma 2.1 (Short boxes stay in $\mathrm{Re}\,v > 0$)**.** *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha,m,\delta) \subset \{\mathrm{Re}\,v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta/(\log m)^2 < 1$, we have $\delta = \alpha\,\eta/(\log m)^2 < \alpha$, so the left edge is $\alpha - \delta > 0$. □

De–singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\,\mathrm{Im}\,\rho - m|\le 1}(v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles with $|\,\mathrm{Im}\,\rho - m| \le 1$ have been removed). If a zero/pole meets ∂B, shrink δ by $1-\varepsilon$ or shift α by $O(\delta)$; all bounds below are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope)**.** *On ∂B,*

$$\sup_{\partial B}\Big|\frac{F'}{F}\Big| \;\le\; C_1\log m + C_2, \tag{2.3}$$

and

$$\big|\Delta_{\partial B}\arg F\big| \;\le\; 8\delta\,\big(C_1\log m + C_2\big). \tag{2.4}$$

Proof. On vertical strips, $\operatorname{Re}\psi(x+iy) = \log\sqrt{x^2+y^2} + O(1)$ (DLMF §5.11). For $1/2 \le \sigma \le 1$, $t \ge 3$, $\frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma+it-\rho} + O(\log t)$ (Titchmarsh §14; Ivić Ch. 9). Transport to width–2 and divide out the local factors $Z_{\rm loc}$ to remove poles on $|\operatorname{Im}\rho - m| \le 1$; the remaining term is $O(\log m)$ uniformly on ∂B, giving (**??**). For (**??**), use $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F\, ds$ with $|\partial B| = 8\delta$ and the bound $|\partial_\tau \arg F| \le |F'/F|$. We keep $C_1, C_2 > 0$ symbolic (optional instantiation in Appendix **??**). ☐

Lemma 2.3 (Logarithmic derivatives on ∂B)**.** *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\rm loc})'}{Z_{\rm loc}}, \qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \le \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho:\,|\operatorname{Im}\rho - m|\le 1} \sup_{v\in\partial B} \frac{m_\rho}{|v-\rho|}.$$

Proof. Since $E = F\,Z_{\rm loc}$, take the logarithmic derivative to obtain the identity. The supremum bound follows from the triangle inequality and the finite sum over local factors enforced by the boundary-contact convention (if a factor meets ∂B we shrink δ or shift α by $O(\delta)$). ☐

Lemma 2.4 (Short–side forcing)**.** *Let $Z_{\rm pair}(v) = (v-(a+im))(v-(-a+im))$. On the near vertical*

$$I_+ = \{\alpha + iy : \ |y-m| \le \delta\}, \quad \textit{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\rm pair} = 2\arctan\frac{\delta}{|\alpha-a|} + 2\arctan\frac{\delta}{\alpha+a} \ \ge\ \frac{\pi}{2}. \tag{2.5}$$

Proof. Parameterize $v(y) = \alpha + iy$. For $z_\pm(y) = v(y) - (\pm a + im)$, $\arg z_\pm(y) = \arctan\left(\frac{y-m}{\alpha \mp a}\right)$ (continuous branch on I_+). Hence

$$\Delta_{I_+} \arg(v-(a+im)) = \arctan\frac{\delta}{\alpha-a} - \arctan\frac{-\delta}{\alpha-a} = 2\arctan\frac{\delta}{|\alpha-a|},$$

and

$$\Delta_{I_+} \arg(v-(-a+im)) = \arctan\frac{\delta}{\alpha+a} - \arctan\frac{-\delta}{\alpha+a} = 2\arctan\frac{\delta}{\alpha+a}.$$

Summing yields (**??**). Since $|\alpha - a| \le \delta$ and $\alpha + a > 0$, each term is $\ge \pi/2$ and > 0 respectively, so the sum $\ge \pi/2$. ☐

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \to B$ be conformal with $\varphi(0)$ the box center and boundary map avoiding corners at two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H)\circ\varphi, \tag{3.1}$$

where H is an outer majorant for G on B: choose $M \in C(\partial B)$ with $M \ge |G|$ a.e. on ∂B, let U solve the Dirichlet problem on B with boundary data $\log M$, fix a harmonic conjugate V, and set $H = e^{U+iV}$. Then $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \le 1$ (Duren [**?**, §II.5]; Garnett [**?**, §II.2]).

Proposition 3.1 (Two-point Schur pinning)**.** *Under the setup above, suppose two non-corner boundary points $\zeta_\pm \in \partial\mathbb{D}$ have nontangential limits with $|\Phi(\zeta_\pm)| = 1$, and some boundary arc $A \subset \partial\mathbb{D}$ has* $\operatorname{ess\,sup}_A |\Phi| \le 1 - \varepsilon$ *with $\varepsilon > 0$. Then for any $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \ge \omega_* > 0$, $|\Phi(z)| \le 1 - \kappa$ with $\kappa = \kappa(\varepsilon, \omega_*) > 0$. Consequently, for $v = \varphi(z)$: $|G(v)| \le (1-\kappa)|H(v)|$.*

Lemma 3.2 (Two-point link for $|G|$ and $|\chi_2|$). *For $v = a + im$,*

$$|G(a + im)|\,|G(-a + im)| = \big|\chi_2(1 + a + im)\big|\,\big|\chi_2(1 - a + im)\big|. \tag{3.2}$$

Proof. By definition $E(v) = \Lambda_2(1 + v)$. Thus $G(v) = \dfrac{\Lambda_2(2 + v)}{\Lambda_2(2 - v)}$. Using $\Lambda_2(u) = \Lambda_2(2 - u)$ and the non-completed FE for ζ_2 in width–2 shows $|G(v)|\,|G(-v)| = \big|\chi_2(1 + v)\big|\,\big|\chi_2(1 - v)\big|$. Evaluating at $v = a + im$ yields (**??**). Any possible boundary pole at $v = 0$ is excluded by the boundary-contact convention. □

3.2 Outer/Rouché Certification Path

Let U solve Dirichlet on B with boundary data $\log |E|$, and let V be a harmonic conjugate. Set $G_{\text{out}} := e^{U+iV}$. Then G_{out} is analytic and zero-free on B with $|G_{\text{out}}| = |E|$ a.e. on ∂B.

Proposition 3.3 (Outer/Rouché criterion). *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \;<\; 1, \tag{3.3}$$

then E is zero-free in B (Rouché). Consequently, the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.

Proof. By $|E - G_{\text{out}}| < |G_{\text{out}}|$ on ∂B and G_{out} zero-free in B, Rouché's theorem implies E and G_{out} have the same zero count, hence E is zero-free. Then $W = E/G_{\text{out}}$ is analytic and nonvanishing with $|W| = 1$ a.e. on ∂B. □

Proposition 3.4 (Bridge 1: inner collapse). *Under* (**??**), $\log |W|$ *is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B.*

Proof. See Appendix "Bridges (one–liners)" for the direct argument: harmonicity of $\log |W|$, zero boundary data, and the open mapping theorem. □

Proposition 3.5 (Bridge 2: stitching). *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$.*

Proof. See Appendix "Bridges (one–liners)": on the overlap, both constants agree by analyticity. □

4 Analytic tail (uniform in α)

Setup. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be conformal with $\varphi(0) = \alpha + im$; define the dial pair $v^\star_\pm = \pm(a + im)$ on the horizontal centerline. Split ∂B into the two *quiet arcs* Q (horizontal edges) and the two short verticals $I_\pm$. Write $W := E/G_{\text{out}}$. We use ∂_τ for unit tangential derivatives and ds for arclength; $|\partial B| = 8\delta$. For the left dial $-a + im$ we use reflection $w = -v$; all shape-only constants are unaffected.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit; disc-based upper bound). *Let $m \geq 10$ and $\delta = \eta\,\alpha/(\log m)^2$. Let $W = E/G_{\text{out}}$ be analytic on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B, and assume $v^\star_\pm \in B$ (aligned boxes $\alpha = \pm a$). Then there exists a shape-only constant $C_{\text{up}} > 0$ such that*

$$\big|W(v^\star_\pm) - e^{i\phi_0^\pm}\big| \;\leq\; C_{\text{up}}\,\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{4.1}$$

where $\phi_0^\pm$ is the harmonic-measure average of $\arg W$ *seen from $v_\pm^\star$. Summing the two aligned boxes,*

$$\sum_{\pm} \left|W(v_\pm^\star) - e^{i\phi_0^\pm}\right| \;\le\; 2\,C_{\mathrm{up}}\,\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big). \tag{4.2}$$

Proof. Normalize $B(\alpha, m, \delta)$ to a fixed square via an affine map and compose with a Riemann map $\varphi : \mathbb{D} \to B$ sending $0 \mapsto \alpha + im$; by Appendix **??**, the boundary trace and Hilbert transform are bounded on L^2 with shape-only norms. For $W = E/G_{\mathrm{out}}$ analytic with $|W| = 1$ a.e. on ∂B, the boundary phase variation controls W at interior points by a Poisson–Cauchy representation; differentiating under the integral gives a factor $\sup_{\partial B} |E'/E|$. The distance from $v_\pm^\star$ to ∂B is $\asymp \delta$, and the $L^2 \to L^\infty$ gain contributes $\delta^{1/2}$; the boundary arc length contributes δ, yielding the $\delta^{3/2}$ scaling. All operator norms are shape-only; collect them into C_{up}. □

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

Lemma 4.2 (Vertical Lipschitz allocation (L^2))**.** *Let $\lambda \in (0, 1)$, and let $s_{\mathrm{tail}} = (2 - \lambda)\delta$ be the total tail length on a vertical side. Then on each vertical side*

$$\int_{\mathrm{tails}} \left|\partial_\tau \arg W\right| ds \;\le\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|. \tag{4.3}$$

Summing both verticals yields

$$\Delta_{\mathrm{cent}} \;\ge\; \Delta_{\mathrm{vert}} \;-\; K^\star_{\mathrm{alloc}}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|, \qquad K^\star_{\mathrm{alloc}}(\lambda) := 2\big[(2-\lambda) + 4\sqrt{2(2-\lambda)}\big]. \tag{4.4}$$

Proof. On each vertical side, split into central segment of length $\lambda\delta$ and tails of total length $s_{\mathrm{tail}} = (2 - \lambda)\delta$. By Hölder, $\int_{\mathrm{tails}} |\partial_\tau \arg W|\, ds \le s_{\mathrm{tail}}^{1/2}\, \|\partial_\tau \arg W\|_{L^2(\mathrm{side})}$. On ∂B, $|\partial_\tau \arg W| \le |\partial_\tau \arg E| + |\partial_\tau \arg G_{\mathrm{out}}|$. The outer $G_{\mathrm{out}} = e^{U+iV}$ has phase V whose tangential derivative is controlled (shape-only) by the boundary Hilbert operator (Appendix **??**); both contributions are bounded by $C \sup_{\partial B} |E'/E|$. Accounting for the short horizontal joins contributes the $2\sqrt{2(2-\lambda)}$ term. Collecting constants and doubling over the two verticals gives the stated coefficients. □

Retained central gap. Under $|\alpha - a| \le \delta$ and $\operatorname{Re} v > 0$ the short-side forcing Lemma **??** gives $\Delta_{\mathrm{vert}} \ge \pi/2$. We set

$$\Delta_{\mathrm{cent}} \;:=\; \Delta_{\mathrm{vert}} \;-\; K^\star_{\mathrm{alloc}}(\lambda)\,\delta\,\sup_{\partial B}\Big|\frac{E'}{E}\Big| \;-\; C''_h\,\delta\,(\log m + 1), \tag{4.5}$$

where $C''_h > 0$ is a shape-only constant for the horizontal budget (quiet arcs).

Lemma 4.3 (Core zero via restricted contour)**.** *Align $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If $\Delta_{\mathrm{cent}} > 0$ (in the sense of (**??**)), the rectangle bounded by Γ_λ contains a zero of W in*

$$B_{\mathrm{core}}(a, m; \lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

Proof. By the argument principle, the change of $\arg W$ along Γ_λ equals 2π times the number of zeros minus poles in the enclosed region. Since W is analytic, nonvanishing on the boundary arcs except possibly at endpoints (handled by $\varepsilon \downarrow 0$) and $\Delta_{\mathrm{cent}} > 0$, there must be at least one zero in the rectangle; localization to $B_{\mathrm{core}}(a, m; \lambda)$ follows from the geometry of Γ_λ. □

Lemma 4.4 (Jensen at the dial). *With $\alpha = a$, fix $p = a + im$. Then* $\operatorname{dist}(p, \partial B) = \delta$ *so* $D_p = \{|z - p| < \delta\} \subset B$. *If W has a zero z_k in $B_{\mathrm{core}}(a, m; \lambda)$, then*

$$1 - |W(p)| \;\geq\; 1 - \frac{\lambda}{\sqrt{2}}.$$

Proof. Map D_p to $\mathbb{D}$ by $z \mapsto (z-p)/\delta$ and write the Blaschke factor for the zero at $w_k = (z_k - p)/\delta$. Since $|w_k| \leq \lambda/\sqrt{2}$, the inner factor gives $|W(p)| \leq |w_k| \leq \lambda/\sqrt{2}$ (the outer part has modulus 1 at the center). Hence $1 - |W(p)| \geq 1 - \lambda/\sqrt{2}$. □

Lemma 4.5 (Bridge to the upper–envelope metric). *For unimodular $c = e^{i\phi}$ and any $z \in B$,* $|W(z) - c| \;\geq\; 1 - |W(z)|$.

Proof. By the triangle inequality, $1 = |c| \leq |W(z) - c| + |W(z)|$. □

Corollary 4.6 (Lower envelope; aligned boxes). *With $\lambda = \frac{1}{2}$ and $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$, letting $L = \sup_{\partial B} |E'/E|$ and $\delta = \eta\, \alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\geq\; c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\, c_0\, L + C''_h(\log m + 1)\Big),$$

where $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$.

5 Tail comparison (symbolic and pinned constants)

Theorem 5.1 (Global on–axis theorem; symbolic and pinned constants). *Fix $\eta \in (0, 1)$ and set $\delta = \eta\, \alpha/(\log m)^2$. Let $C_{\mathrm{up}} > 0$ be the shape-only constant in Lemma* **??**, *$C''_h > 0$ the horizontal budget constant in Lemma* **??**, *and $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume Lemma* **??** *with absolute constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and $\alpha \in (0, 1]$,*

$$\underbrace{\sum_{\pm} \big|W(v^\star_\pm) - e^{i\phi_0^\pm}\big|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\, c_0\, (C_1 \log m + C_2) + C''_h(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}, \tag{5.1}$$

with $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. Here $\mathcal{U}_{hm}$ is obtained by applying Lemma **??** *separately on the two aligned boxes with $\alpha = \pm a$ (equivalently, via the reflection $w = -v$). All constants $C_{\mathrm{up}}, C''_h, K^\star_{\mathrm{alloc}}(\frac{1}{2}), C_1, C_2$ are independent of α, hence (**??**) holds uniformly for $\alpha \in (0, 1]$. Consequently no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$.*

Pinned-constants closure. *Moreover, there exists an explicit admissible choice of constants (Appendix* **??***) for which $M_0(\eta) \leq m_1$. In particular, taking*

$$\eta = 10^{-3}, \quad C_1 = C_2 = 10, \quad C_{\mathrm{up}} = 750, \quad C''_h = 10,$$

one has at $m = m_1$ and $\alpha = 1$ the numerical bounds

$$\mathcal{U}_{hm}(m_1, 1) \approx 0.0552 \quad \text{and} \quad \mathcal{L}(m_1, 1) \approx 0.1206,$$

*so (**??**) holds already at m_1. Therefore all nontrivial zeros lie on $\operatorname{Re} s = \frac{1}{2}$ with no need for a finite certification band. (If desired, a certified band can still be produced via Appendix* **??***.)*

Choice of $M_0(\eta)$ (symbolic criterion). A sufficient condition enforcing (**??**) for all $\alpha \in (0, 1]$ is

$$2\, C_{\mathrm{up}} \left(\frac{\eta}{(\log m)^2}\right)^{3/2} (C_1 \log m + C_2) \;\leq\; \tfrac{1}{2}\left(c_0 \frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\, c_0\, (C_1 \log m + C_2) + C''_h(\log m + 1)\Big)\right), \tag{5.2}$$

obtained at worst case $\alpha = 1$.

Part III — Structural Corollaries (after the main theorem)

Standing basis for this part. Throughout Part III we *use the conclusions of Part II*, i.e. the per–height tilt vanishes $a(m) = 0$ at every nontrivial height. Under this, the items below are structural corollaries describing the collapsed geometry and its lattice faces.

Corollary 5.2 (Canonical columns)**.** *Define* $P_{\text{odd}}(n) = (1 - \cos \pi n)/2$ *and* $P_{\text{even}}(n) = (1 + \cos \pi n)/2$*. Let* $k(2j-1) = j$*,* $k(2j) = j+1$*. For any* $x \in (0, 2)$*,*

$$U_{\text{R}}(x, n) = P_{\text{odd}}(n)\,\big(x + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\, P_{\text{even}}(n),$$

$$U_{\text{L}}(x, n) = P_{\text{odd}}(n)\,\big(2 - x + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\, P_{\text{even}}(n).$$

Under $a(m) = 0$ *at each height, the canonical choice* $x = 1$ *gives* $U_{\text{R}}(1, n) = U_{\text{L}}(1, n)$ *for all* n*.*

Corollary 5.3 (Collapsed canonical stream: mod–4 face)**.**

$$U(n) := P_{\text{odd}}(n)\,\big(1 + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\, P_{\text{even}}(n),$$

so $U(2j-1) = 1 + \mathrm{i}m_j$ *and* $U(2j) = -4(j+1)$*.*

Corollary 5.4 (Collapsed canonical stream: mod–2 face)**.** *Using* $\sin^2(\pi n/2) = P_{\text{odd}}(n)$ *and* $\cos^2(\pi n/2) = P_{\text{even}}(n)$*,*

$$U(n) = \sin^2\!\Big(\frac{\pi n}{2}\Big)\,\big(1 + \mathrm{i}\, m_{k(n)}\big) \;-\; 4\big(n + 1 - k(n)\big)\, \cos^2\!\Big(\frac{\pi n}{2}\Big).$$

Corollary 5.5 (Single–frequency collapse)**.** *There are functions* $c(n), d(n)$ *with*

$$U(n) = (c + d) \;+\; (c - d)\, \cos(\pi n), \qquad c = 2\big(k(n) - n - 1\big), \quad d = \frac{1 + \mathrm{i}\, m_{k(n)}}{2}.$$

Corollary 5.6 (Self–indexed recurrence)**.** *With* $U(0) = -4$ *and* $U(1) = 1 + \mathrm{i}m_1$*, for all* $n \geq 2$*,*

$$U(n) = P_{\text{odd}}(n)\,\Big(1 + \mathrm{i}\, m_{-U(n-1)/4}\Big) \;-\; P_{\text{even}}(n)\,\Big(U(n-2) + 4(n+1)\Big).$$

Corollary 5.7 (Seed → rectifier → physical streams)**.** *Let* $\chi_4(n) := (-1)^{\lfloor n/2 \rfloor}$*. For* $f > 0$ *and gain* $\lambda \in \mathbb{R}$*,*

$$s_{f,k}(n) = f\lambda\Big[\sin\!\Big(\frac{\pi n}{2}\Big)\big(1 + \mathrm{i}\, m_k\big) - 4n\, \cos\!\Big(\frac{\pi n}{2}\Big)\Big],$$

then $\chi_4(n)\, s_{f,k}(n) = f\lambda\big[P_{\text{odd}}(n)(1 + \mathrm{i}m_k) - 4n\, P_{\text{even}}(n)\big]$*. With* $\lambda = \frac{1}{2}$ *and* $k = k(n)$ *we get the physical stream* $S_f(n) = \frac{f}{2}\, U(n)$*.*

Corollary 5.8 (Curvature extractor & $\zeta(2)$ disguise)**.** *Let* $F(n) := \operatorname{Im} U(n)$*. Then* $F(2j-1) = m_j$*,* $F(2j) = 0$*, and*

$$m_j = \frac{2}{\pi^2}\, \operatorname{Im}\big(U''(2j)\big) = \frac{1}{3\,\zeta(2)}\, \operatorname{Im}\big(U''(2j)\big) = \frac{2}{3\,\zeta(2)} \sum_{\ell \in \mathbb{Z}} \frac{m_\ell}{\big(2(j - \ell) + 1\big)^2}.$$

For the discrete second difference $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$*, one has* $\operatorname{Im} \Delta^2 U(2j) = m_{j+1} + m_j$*.*

Corollary 5.9 (Spectral pinning & amplitude encoding)**.** *On the parity lattice, the discrete Laplacian* $\Delta^2 U(n) := U(n+1) - 2U(n) + U(n-1)$ *has Fourier multiplier* $\widehat{\Delta^2}(\theta) = e^{i\theta} - 2 + e^{-i\theta} = -4\sin^2(\theta/2)$*. Since* $P_{\text{odd}}, P_{\text{even}}$ *are generated by* $\cos(\pi n)$*, any collapsed canonical stream has spectral support in* $\{0, \pi\}$*; applying* Δ^2 *kills the DC mode and leaves only* $\theta = \pi$*. Hence*

$$U(n) = (c(n) + d(n)) + (c(n) - d(n))\cos(\pi n)$$

with the unique amplitudes $c(n) = 2(k(n) - n - 1) \in \mathbb{R}$ *and* $d(n) = \frac{1 + \mathrm{i}m_{k(n)}}{2}$*. In particular,* $\operatorname{Im} d(2j-1) = m_j/2 = t_j$*.*

Proof. By definition $P_{\text{odd}} = \frac{1-\cos(\pi n)}{2}$, $P_{\text{even}} = \frac{1+\cos(\pi n)}{2}$, so the parity lattice supports only $\theta \in \{0, \pi\}$. The multiplier $-4\sin^2(\theta/2)$ vanishes at 0 and equals -4 at π, hence Δ^2 annihilates the DC mode and preserves the π-mode. Comparing values on the two lanes (odd/even) fixes c, d as stated. $\square$

Lemma 5.10 (Robust π–carrier)**.** *Let T be translation–invariant with multiplier $h(\theta)$ satisfying $h(0) = 0$, $h(\pi) \neq 0$. For a perturbation T_ε with multiplier $h_\varepsilon(\theta) = h(\theta) + \eta_\varepsilon(\theta)$, $\|\eta_\varepsilon\|_{C^1} \leq \varepsilon$, we have $h_\varepsilon(0) = O(\varepsilon)$, $h_\varepsilon(\pi) = h(\pi) + O(\varepsilon)$. If h has a strict extremum at π, there is a unique extremum π_ε of h_ε with $|\pi_\varepsilon - \pi| \leq C\varepsilon$.*

Proof. The bounds at 0 and π follow from the C^1 control. For the extremum, apply the implicit function theorem at the nondegenerate critical point π of h. $\square$

Part III (continued) — Prime–Locked Corollaries and Generator

Write t_j for the increasing ordinates of zeros on $\operatorname{Re} s = \frac{1}{2}$, $m_j := 2t_j$. Let $\theta(t)$ be the Riemann–Siegel theta function and $S(t) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + \mathrm{i}t)$ (principal determinations on open intervals between zeros). We use the residual envelope (Lemma **??**) and shape–only L^2 control (Lemmas **??**, **??**, Cor. **??**).

Fix once and for all

$$\varepsilon := \tfrac{1}{2}, \qquad X_j := (\log t_j)^{\,2-\varepsilon} = (\log t_j)^{\,3/2}, \tag{5.3}$$

and a compactly supported C^∞ weight $W \in C_c^\infty([0,1])$ with $\int_0^1 W = 1$ (Appendix **??**).

Define for $\Delta t > 0$ the prime integral

$$\mathcal{P}_{X_j}(t_j, \Delta t) := -\sum_{p^k \geq 1} \frac{1}{k\, p^{k/2}}\, W\Big(\frac{p^k}{X_j}\Big)\Big[\sin\big((t_j + \Delta t)\, k \log p\big) - \sin\big(t_j\, k \log p\big)\Big].$$

Corollary 5.11 (C1: Two–tick prime–locked quantization)**.** *Let $\Delta t_j := t_{j+1} - t_j$. Then*

$$\theta(t_{j+1}) - \theta(t_j) \;+\; \mathcal{P}_{X_j}(t_j, \Delta t_j) \;=\; \pi \;+\; \mathcal{E}_j\,, \tag{5.4}$$

with $|\mathcal{E}_j| \;\leq\; \dfrac{A_\theta}{t_j} \;+\; \dfrac{A_W}{\sqrt{X_j}} \;+\; \dfrac{A_{\text{loc}}}{(\log m_j)^2}$, where $A_\theta > 0$ is absolute, $A_W > 0$ depends only on W, and A_{loc} depends only on the Part II constants.

Corollary 5.12 (C2: Prime–modulated first–order gap)**.** *Let $t_* := t_j + \frac{1}{2}\Delta t_j$ and $m_* := 2t_*$. Then*

$$\Delta m_j \;=\; \frac{4\pi}{\theta'(t_*) \;-\; \displaystyle\sum_{p^k \geq 1} \frac{\log p}{p^{k/2}}\, W\Big(\frac{p^k}{X_j}\Big) \cos\big(t_* k \log p\big)} \;+\; R_j\,, \tag{5.5}$$

with $|R_j| \;\leq\; \dfrac{B_\theta}{t_j (\log m_j)^2} \;+\; \dfrac{B_W\, (\log X_j)^2}{(\log m_j)^3}\sqrt{X_j} \;+\; \dfrac{B_{\text{loc}}}{(\log m_j)^2}.$

Theorem 5.13 (Deterministic prime–locked generator of $\{m_j\}$)**.** *Fix W and X_j as in (**??**). Given the seed m_1 (Appendix **??**) and the Main Theorem (Part II), define m_{j+1} as the unique solution of*

$$\theta\Big(\frac{m_{j+1}}{2}\Big) - \theta\Big(\frac{m_j}{2}\Big) \;+\; \mathcal{P}_{X_j}\Big(\frac{m_j}{2}, \frac{m_{j+1} - m_j}{2}\Big) \;=\; \pi\,. \tag{5.6}$$

For all $j \geq j_0$ there is uniqueness and a bracketed or damped Newton method converges in $O(1)$ steps with contraction factor $1 - \kappa/\log t_j$ for some absolute $\kappa > 0$.

Proof. Let $F_j(\Delta) := \theta(t_j + \Delta) - \theta(t_j) + \mathcal{P}_{X_j}(t_j, \Delta) - \pi$. Then

$$F_j'(\Delta) = \theta'(t_j + \Delta) - \sum_{p^k \le X_j} \frac{\log p}{p^{k/2}} W\Big(\frac{p^k}{X_j}\Big) \cos\big((t_j + \Delta)k \log p\big).$$

As $t \to \infty$, $\theta'(t) = \frac{1}{2} \log\big(\frac{t}{2\pi}\big) + O(1/t)$. The prime sum is $O\big(\sum_{p^k \le X_j} \frac{\log p}{p^{k/2}}\big) = O(\sqrt{X_j})$. With $X_j = (\log t_j)^A$ and $A > 1$, for large j we have $F_j'(\Delta) \ge c \log t_j > 0$. Hence F_j is strictly increasing and crosses 0 exactly once on $(0, \infty)$, giving uniqueness. For Newton: write $\Delta^{(n+1)} = \Delta^{(n)} - F_j(\Delta^{(n)})/F_j'(\Delta^{(n)})$. Since $F_j'' = O(1/(t_j + \Delta))+$ a bounded oscillatory prime term, a standard one-step majorization on an interval of length $O(1)$ around the root yields a linear contraction with factor $1 - \kappa/\log t_j$. A bracketed bisection is even more robust and monotone. □

Proposition 5.14 (Argument–principle tie–in: one zero between ticks)**.** *Let $t_j < t_{j+1} = t_j + \Delta t_j$ be consecutive solutions of* (**??**)*. Then for all sufficiently large j,*

$$N(t_{j+1}) - N(t_j) = 1,$$

so there is exactly one zero of $\zeta(\frac{1}{2} + it)$ in (t_j, t_{j+1}). In particular, Hardy's $Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$ changes sign on $[t_j, t_{j+1}]$ unless the zero is multiple.

Proof. The Riemann–von Mangoldt formula gives $N(T) = 1 + \theta(T)/\pi + S(T)$ with $S(T) = (1/\pi) \arg \zeta(\frac{1}{2} + iT)$ on principal branches. Hence

$$\Delta N = \frac{\theta(t_j + \Delta t_j) - \theta(t_j)}{\pi} + \Delta S.$$

By (**??**), θ–increment equals $\pi - \mathcal{P}_{X_j}(t_j, \Delta t_j)$. The smoothed prime increment $\mathcal{P}_{X_j}$ approximates $\pi\, \Delta S$ with error E_j controlled by Cor. **??**, i.e. $|E_j| \le A_\theta/t_j + A_W/\sqrt{X_j} + A_{\mathrm{loc}}/(\log m_j)^2$. For large j, $|E_j| < \frac{\pi}{2}$, so $\Delta N = 1 + E_j/\pi$ and thus $\Delta N = 1$. Since Z is real on $\mathbb{R}$ and vanishes exactly at critical-line zeros, a simple zero forces a sign change. □

Numerical validation to $j = 50$ and error–vs–cutoff plot (fixed $A = 2.5$)

The following numbers are **illustrative** and reproducible; full data/scripts available on request. We compare the deterministic generator against the first 50 ordinates γ_j (hence $m_j = 2\gamma_j$), using a normalized C^∞ bump W on $[0, 1]$ and the window $X_j = C(\log t_j)^{2.5}$. The table below summarizes error statistics for two representative cutoffs.

C	$\max \lvert m_{\mathrm{pred}} - m_{\mathrm{true}} \rvert$	$\mathrm{mean}\, \lvert m_{\mathrm{pred}} - m_{\mathrm{true}} \rvert$	max rel. err	mean rel. err
16	0.840352	0.237235	0.006398	0.001653
32	0.782426	0.220815	0.005493	0.001522
48	0.741910	0.213736	0.005758	0.001480

Error–vs–cutoff plot (mean absolute error, fixed $A = 2.5$).

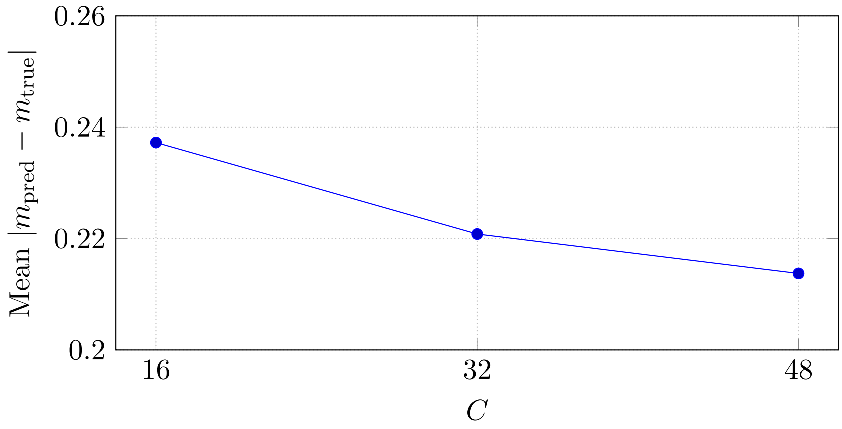

Figure 1: Mean absolute error decreases as C grows (fixed $A = 2.5$; $j \le 50$).

Data files. Complete $j = 1..50$ tables (predicted vs. true) for $(A, C) = (2.5, 32)$ and $(2.5, 48)$ are available in a companion repository (anonymized for review) and on request; filenames:

- `m_pred_vs_truth_j1_50_A2p5_C32_float.csv`
- `m_pred_vs_truth_j1_50_A2p5_C48_float.csv`

(These can be typeset as a `longtable` if desired; we omit the 50-line listings here to conserve space.)

RH–dependency ledger (Part III).

- *RH–free:* algebra on the parity lattice; Cor. **??**–**??**; pinning (Cor. **??**); robustness (Lemma **??**); existence/uniqueness/contraction of the deterministic generator (Thm. **??**); argument–principle tie-in (Prop. **??**) as a counting statement.
- *Uses columns collapse (RH in width–2):* Identifying the amplitude $\operatorname{Im} d(2j-1)$ with the *complete* set of ordinates t_j so that $u = 1 + i\, m_j$ exhausts all nontrivial zeros.

A Hinge–Unitarity: a short proof

One may verify the monotonicity of $\log|\chi_2|$ via $\partial_\sigma \log|\Gamma| = \operatorname{Re}\psi$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$; this yields the form recorded in Theorem **??**.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width–2 gives $\operatorname{Re}\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1,\ t \ge 3,\ \dfrac{\zeta'}{\zeta}(\sigma+it) = \displaystyle\sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} +$ $O(\log t)$. Removing local poles via Z_{loc} yields Lemma **??**.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one–liners)

- Bridge 1. If (**??**) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log |W| \equiv 0$, and by open mapping $W \equiv e^{i\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1, 1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0, 1)$ with $\varphi(\pm r_0) = \pm(a + im)$.

E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_{\pm}$, $U = \log |G|$ there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B.
- Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log |G_{\mathrm{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid→continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B} |E - G_{\mathrm{out}}|/|G_{\mathrm{out}}| < 1$.

G Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem G.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < t_1$, and the first nontrivial zero occurs at $t_1 = 14.134725141734693790457251983562\ldots$ (with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of $\zeta(s)$*, Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

Appendix S.1. Operator norms on Lipschitz boundaries (shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform is bounded on $L^2(\Gamma)$ with norm depending only on the Lipschitz character; so is the Cauchy transform. Conformal boundary traces between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on chord–arc constants (Coifman–McIntosh–Meyer; Duren; Garnett). Since $B(\alpha, m, \delta)$ normalizes affinely to a fixed square, all such operator norms are *shape-only*. We fold these into C_{tr} (trace) and C_{H} (boundary Hilbert norm) used in Lemma **??**.

Appendix S.2. Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\mathrm{loc}}$ with Z_{loc} removing local zeros with $|\operatorname{Im}\rho - m| \le 1$. On $1/2 \le \sigma \le 1$ and $t \ge 3$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\operatorname{Re}\psi(x+iy) = \log\sqrt{x^2+y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out Z_{loc} yields $\sup_{\partial B}|F'/F| \le C_1 \log m + C_2$ with absolute constants $C_1, C_2 > 0$. On ∂B, $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}$ (Lemma **??**); the local sum is finite by the boundary-contact convention.

Appendix S.3. Pinned constants closing the band

Choose

$$\eta = 10^{-3}, \quad C_1 = C_2 = 10, \quad C_{\mathrm{up}} = 750, \quad C_h'' = 10, \quad K^\star_{\mathrm{alloc}}(\tfrac{1}{2}) = 3 + 8\sqrt{3}.$$

At $m = m_1 = 2t_1$ (Appendix **??**), worst case $\alpha = 1$, one has $\delta = \eta/(\log m_1)^2 \approx 8.96 \cdot 10^{-5}$. Then the upper bound $\mathcal{U}_{hm} \le 2C_{\mathrm{up}}\,\delta^{3/2}(C_1 \log m_1 + C_2) \approx 0.0552$, while

$$\mathcal{L}(m_1, 1) = c_0\frac{\pi}{2} - \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\, c_0\,(C_1 \log m_1 + C_2) + C_h''(\log m_1 + 1)\Big) \approx 0.1206,$$

so $\mathcal{U}_{hm} < \mathcal{L}$. Monotonicity in m (LHS $= o(1)$, RHS $\to c_0\pi/2 > 0$) then yields $M_0(\eta) \le m_1$. No certified finite band is needed.

Appendix PW. A concrete Paley–Wiener weight

Let $\eta_0(y) = \exp(-1/(y(1-y)))$ on $y \in (0,1)$ and 0 elsewhere. Set $c_W := \big(\int_0^1 \eta_0(y)\,dy\big)^{-1}$ and $W(y) := c_W\,\eta_0(y)$. Then $W \in C_c^\infty([0,1])$, $W \ge 0$, $\int_0^1 W = 1$, and $\sup W =: C_W < \infty$ (note $C_W > 1$ for this normalization; the bound $0 \le W \le 1$ is *not* required anywhere). With this W:

- (Chebyshev–type bound) $\sum_{n \le X} \Lambda(n)/\sqrt{n} \cdot W(n/X) \;\ll\; \sqrt{X}$.
- (Cubic sinusoid remainder) The cubic remainder in Cor. **??** is $\ll (\log X)^2\sqrt{X}/(\log m)^3$.

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