

# A Width-2 Boundary Program for Excluding Off-Axis Quartets with a Certified Tail Criterion and a Finite-Height Front-End (v38)

Dylan Anthony Dupont

v38 compiled: 2026-01-25

## Abstract

This document is a truth-latching *architecture* build (v38) of the width-2 boundary program. It retains the v36 NO-GO guardrails (exponent budget, UE scaling NO-GO for pointwise/sup endpoints, and constant-limited forcing) and it installs an  $S5'$  toolkit for *phase/winding* endpoints (Section 12): a collar-safe phase increment calculus, a residual phase budget, and a buffered boundary phase endpoint  $\tilde{D}_B(W)$  with a direct  $\pi/2$  forcing lemma via the argument principle (Definition 12.19, Lemma 12.23).

**Header note (no drift):** v38 makes *no* claim of uniform tail closure and *no* claim of the Riemann Hypothesis. The remaining analytic frontier is explicit and single: invent a budget-eligible endpoint mechanism that is simultaneously (i) forceable and (ii)  $\delta$ -shrinkable under the nominal scale (OPEN box in Section 12).

For global hygiene, v35 corrected completion/holomorphy; v38 retains this convention: the working function is the entire width-2 completion  $\Xi_2(u) := \xi(u/2)$ , recentered as  $E(v) := \Xi_2(1+v)$ , so all uses of “ $E$  is entire” are literally true.

## Contents

I	Reader’s Guide / Definitions and Reduction	2
1	Width-2 normalization	2
2	Heights and horizontal displacement (RH-free)	3
3	Quartet symmetry in width-2	3
4	Finite-height front-end after lowering the tail anchor	3
II	Self-Contained Boundary Program and Tail Closure	4
5	Aligned boxes and the $\delta(m)$ scale	4
6	Local factor and finiteness	4
7	Residual envelope bound and the constants ledger	5
8	Short-side forcing	6

<b>9</b>	<b>Outer factorization and the inner quotient (Bridge 1)</b>	<b>7</b>
<b>10</b>	<b>Shape-only invariance and the envelope constants</b>	<b>8</b>
10.1	Local factor split and collar control . . . . .	10
10.2	Horizontal non-forcing budget in residual form . . . . .	14
<b>11</b>	<b>The explicit tail inequality (post-pivot)</b>	<b>15</b>
<b>12</b>	<b>S5' frontier: phase/winding endpoint redesign (open)</b>	<b>17</b>
12.1	S5' phase endpoints: winding / argument-increment toolkit . . . . .	21
12.2	Baseline NO-GO results for naive non-pointwise endpoints . . . . .	24
<b>13</b>	<b>Global RH from a finite front-end + the tail criterion family</b>	<b>26</b>
<b>A</b>	<b>Discarded closure routes (as of v38)</b>	<b>26</b>
A.1	D1: Pointwise UE endpoint $\sup_{\partial B}  E'/E  + \text{collar} + \eta\text{-absorption}$ (S1/S1') . . . . .	26
A.2	D2: Shrinking the local window / short-interval zero counts . . . . .	27
A.3	D3: “Make forcing grow with $m$ ” within single-box forcing . . . . .	27
A.4	D4: “Boundary modulus implies interior zero-freeness” converse . . . . .	27
A.5	D5: Absolute $L^r$ log-derivative endpoints (NO-GO) . . . . .	27
A.6	D6: Projecting out the local kernel span (NO-GO) . . . . .	27
<b>B</b>	<b>S6 harness: explicit-formula interpretation (non-closure)</b>	<b>28</b>
B.1	Frame mapping: $v$ -displacement and the real part $\beta$ . . . . .	28
B.2	Explicit formula: off-axis zeros as amplitude leaks . . . . .	28
B.3	What S5' would mean in this language . . . . .	28
<b>C</b>	<b>Tail harness bundle and reproducibility (v38)</b>	<b>29</b>
C.1	What the tail checks prove (and what they do not) . . . . .	29
C.2	SHA-256 table (exact artifacts) . . . . .	29
C.3	Commands . . . . .	29
C.4	Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false) . .	29
C.5	Bundle files (verbatim) . . . . .	30
<b>D</b>	<b>Finite-height front-end certificate (literature-based)</b>	<b>40</b>
	<b>References</b>	<b>43</b>

## Executive Proof Status

**Status (v38):** v38 is an *architecture* build: it preserves the decisive v36 NO-GO constraints and installs an S5' phase/winding endpoint toolkit (Section 12) so that future work is forced onto a single explicit frontier and cannot drift back to the (now invalid) v33 absorption narrative or into naive endpoint redesigns. The main unconditional output remains the *tail criterion family* (Theorem 11.1) together with a finite-height front-end (Definition 4.1). No claim of uniform tail closure is made in this version.

**Proof-grade NO-GO constraints (now explicit):**

1. *Exponent budget obstruction.* Under the nominal scaling  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ , a local/collar blow-up exponent  $\theta$  and UE exponent  $p$  must satisfy  $p - \theta \geq \frac{1}{2}$  for uniform  $\eta$ -shrinking closure (Theorem 10.13 and Theorem 12.5). In the present pointwise collar bound one has  $\theta = 1$ .
2. *UE scaling obstruction (pointwise/sup).* With the pointwise/sup endpoint  $\sup_{\partial B} |E'/E|$  and shape-only constants, one cannot achieve any UE exponent  $p > 1$  (Lemma 10.15). The proved pointwise bound has  $p = 1$ .
3. *Forceability gate + constant-limited forcing.* The current single-box forcing architecture forces only the dial deviation  $D_B(W)$  and the available forcing margin is  $O(1)$  (Lemma 8.2). Any S5 endpoint redesign is invalid unless it dominates  $D_B(W)$  or supplies a new forcing lemma (Remark 12.10).
4. *Boundary modulus has no converse.* The condition  $|W| = 1$  on  $\partial B$  does not exclude interior zeros; Bridge 1 is strictly one-directional (Remark 9.4).
5. *Absolute  $L^r(\partial B)$  log-derivative endpoints are dead.* In any absolute  $L^r$  endpoint class, the best possible UE exponent collapses to  $p(r) = 1 - 1/r$  (Lemma 12.26), while the collar/local split has the same exponent  $\theta(r) = 1 - 1/r$  (Proposition 12.27), so  $p(r) - \theta(r) = 0$  and the local term is  $\delta$ -inert.
6. *Projection endpoints are dead under current forcing.* Endpoints that annihilate the local kernel span cannot control the forced dial deviation  $D_B(W)$  without a new forcing link (Lemma 12.29).

**Immediate consequence:** the v33/v32 style  $\eta$ -absorption closure route based on the pointwise/sup upper envelope and the collar is *formally discarded* in v35 and remains discarded in v36 (Appendix A). Any future closure must change the envelope endpoint and/or the local interface.

**Completion / holomorphy hygiene (fixed):** the working function is the entire width-2 completion  $\Xi_2(u) := \xi(u/2)$  and  $E(v) := \Xi_2(1+v)$  (Section 1). All uses of “ $E$  is entire” are now literally correct.

#### Open proof-grade blockers (v38):

1. **S5 UE redesign (primary frontier).** Replace the pointwise/sup UE endpoint by a non-pointwise functional that controls the same dial deviation  $D_B(W)$  while avoiding the  $\delta^{-1}$  collar blow-up; see the S5 acceptance criterion (Remark 12.4) and the baseline NO-GO filters (Subsection 12.2) (Section 12). (G-4/G-5 in the prior register.)
2. *Residual envelope ledger.* Lemma 7.2 still imports a standard RH-free bound for  $\zeta'/\zeta$  with local-zero subtraction; this must be proved in-text or cited in a referee-acceptable way with explicit constants. (G-1/G-12.)
3. *Front-end dependence.* The finite-height hypothesis remains an external input (Appendix D). (G-11.)

**Reproducibility posture (v38):** numerical artifacts remain an *audit harness* only (Appendix C). The v36 repro pack is hardened to record endpoint metadata (functional class + exponent budget parameters + forceability mode). Missing fields render a certificate invalid (Appendix C), so future redesigns cannot silently mismatch the forcing chain.

## Part I

# Reader's Guide / Definitions and Reduction

## 1 Width-2 normalization

Let  $s$  denote the usual complex variable for  $\zeta(s)$ . We pass to the width-2 coordinate

$$u := 2s, \quad \zeta_2(u) := \zeta(u/2).$$

Define the width-2 completed zeta

$$\Lambda_2(u) := \pi^{-u/4} \Gamma(u/4) \zeta(u/2).$$

Then  $\Lambda_2$  is *meromorphic* (simple poles at  $u = 0$  and  $u = 2$ ) and satisfies the functional equation

$$\Lambda_2(u) = \Lambda_2(2 - u).$$

Define the entire completion

$$\Xi_2(u) := \xi(u/2) = \frac{u(u-2)}{8} \Lambda_2(u),$$

so that  $\Xi_2$  is entire and obeys  $\Xi_2(u) = \Xi_2(2 - u)$ .

We recenter at  $u = 1$ :

$$v := u - 1, \quad E(v) := \Xi_2(1 + v).$$

Then  $E$  is entire and satisfies the evenness relation

$$E(v) = E(-v),$$

and complex conjugation gives  $E(\bar{v}) = \overline{E(v)}$ .

*Remark 1.1* (Zeros). The zeros of  $\Xi_2(u) = \xi(u/2)$  are exactly the *nontrivial* zeros of  $\zeta(s)$  under the map  $u = 2s$ , with multiplicity. All boxes used in the tail program lie at heights  $m \geq 10$ , so the only zeros that can occur in the relevant local windows are nontrivial zeros.

## 2 Heights and horizontal displacement (RH-free)

Let  $\rho = \beta + i\gamma$  be any nontrivial zero of  $\zeta(s)$  (no assumption on  $\beta$ ). In width-2 we write

$$u_\rho := 2\rho = (1 + a) + im, \quad a := 2\beta - 1 \in (-1, 1), \quad m := 2\gamma > 0. \quad (1)$$

Thus RH is equivalent to  $a = 0$  for every nontrivial zero.

## 3 Quartet symmetry in width-2

The functional equation and conjugation imply that any off-axis zero with parameters  $(a, m)$  produces a quartet

$$\{1 \pm a \pm im\} \subset \{u \in \mathbb{C} : \Xi_2(u) = 0\}. \quad (2)$$

In the centered  $v$ -coordinate this becomes  $\{\pm a \pm im\} \subset \{v \in \mathbb{C} : E(v) = 0\}$ .

## 4 Finite-height front-end after lowering the tail anchor

Once the tail anchor is lowered to  $m_\star$ , the analytic tail argument covers all  $m \geq m_\star$ . The remaining region corresponds to classical heights

$$0 < \text{Im}(s) < H_0 := m_\star/2. \quad (3)$$

In v31 we take  $m_\star = 10$ , hence  $H_0 = 5$ .

**Definition 4.1** (Front-end statement). We say that *RH holds up to height  $H_0$*  if every nontrivial zero  $\rho = \beta + i\gamma$  with  $0 < \gamma \leq H_0$  satisfies  $\beta = 1/2$ .

*Remark 4.2* (How v31 discharges the front-end). The required statement for v31 is RH up to height  $H_0 = 5$ . This is a tiny special case of published rigorous verifications of RH to enormous heights. For example, Platt–Trudgian prove RH for all zeros with  $0 < \gamma \leq 3 \cdot 10^{12}$  using interval arithmetic, which immediately implies RH up to  $H_0 = 5$ . Appendix D records this discharge in a pinned JSON file.

## Part II

# Self-Contained Boundary Program and Tail Closure

## 5 Aligned boxes and the $\delta(m)$ scale

Let  $m > 0$  and  $\alpha \in (0, 1]$ . Fix a parameter  $\eta \in (0, 1)$  and define the *nominal* box scale

$$\delta_0 = \delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}. \quad (4)$$

We will work with aligned boxes  $B(\alpha, m, \delta)$  at scales  $0 < \delta \leq \delta_0$ . By default one may take  $\delta = \delta_0$ , but later (Definition 10.5) we allow shrinking  $\delta$  to enforce  $\kappa$ -admissibility; this is non-circular and monotone-safe (Lemmas 10.6 and 11.2).

Define the (width-2) box centered at  $\alpha + im$  by

$$B(\alpha, m, \delta) := \{v \in \mathbb{C} : |\text{Re } v - \alpha| \leq \delta, |\text{Im } v - m| \leq \delta\}. \quad (5)$$

We will also use the symmetric dial centers  $v_\pm := \pm\alpha + im$ .

## 6 Local factor and finiteness

For a fixed  $m > 0$ , let

$$Z(m) := \{\rho : E(\rho) = 0, |\text{Im } \rho - m| \leq 1\} \quad (6)$$

(zeros counted with multiplicity). Define the local zero factor and residual:

$$Z_{\text{loc}}(v) := \prod_{\rho \in Z(m)} (v - \rho)^{m_\rho}. \quad (7)$$

$$F(v) := \frac{E(v)}{Z_{\text{loc}}(v)}. \quad (8)$$

**Lemma 6.1** (Finiteness of  $Z_{\text{loc}}$ ). *For each fixed  $m > 0$  the set  $Z(m)$  is finite; hence  $Z_{\text{loc}}$  is a finite product and  $F$  is meromorphic globally and analytic in any neighborhood of  $\partial B(\alpha, m, \delta)$  that contains no zeros of  $E$ .*

*Proof.* Nontrivial zeros of  $\zeta$  satisfy  $0 < \beta < 1$ , hence in the  $v$ -coordinate one has  $\text{Re } v \in (-1, 1)$  for all nontrivial zeros. Therefore the set  $\{|\text{Im } v - m| \leq 1\} \cap \{|\text{Re } v| \leq 1\}$  is compact. Since  $E$  is entire and its zeros are discrete, only finitely many zeros can lie in this compact set.  $\square$

## 7 Residual envelope bound and the constants ledger

*Remark 7.1* (Constant gate for the residual term (what is and is not assumed)). The tail criterion uses a bound of the form

$$\sup_{v \in \partial B(\alpha, m, \delta)} \left| \frac{F'(v)}{F(v)} \right| \leq C_1 \log m + C_2,$$

with constants that must be (i) unconditional (not RH-equivalent) and (ii) uniform in  $(\alpha, \delta, \eta, \kappa)$  once  $m \geq 10$  and  $0 < \alpha \leq 1$ . The proof below reduces this to standard RH-free bounds for  $\zeta'/\zeta$  in the critical strip with local-zero subtraction, plus a Stirling-type bound for  $\Gamma'/\Gamma$ .

**Lemma 7.2** (Residual envelope inequality ( $\delta$ -uniform)). *Fix  $m \geq 10$  and  $\alpha \in (0, 1]$ . Let  $\eta \in (0, 1]$  and set the nominal width  $\delta_0 := \eta\alpha/(\log m)^2$ . Let  $\delta \in (0, \delta_0]$  and set  $B := B(\alpha, m, \delta)$ .*

*Define  $E$ ,  $Z_{\text{loc}}$  and  $F := E/Z_{\text{loc}}$  as in §6 (equations (7)–(8)). Assume boundary-contact on  $\partial B$  (i.e.  $E \neq 0$  on  $\partial B$ ; hence  $F$  is holomorphic on a neighborhood of  $\partial B$ ). Then there exist absolute constants  $C_1, C_2 > 0$  (independent of  $m, \alpha, \delta, \eta, \kappa$  and of the zero configuration) such that*

$$\sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right| \leq C_1 \log m + C_2.$$

*Proof sketch with explicit dependency control.* Write  $u := 1 + v$  and  $s := u/2 = (1 + v)/2 = \sigma + it$ . For  $v \in \partial B(\alpha, m, \delta)$  we have  $\text{Re}(s) \in [0, 1]$  and

$$\text{Im}(s) = \frac{\text{Im}(v)}{2} \in \left[ \frac{m}{2} - \frac{\delta}{2}, \frac{m}{2} + \frac{\delta}{2} \right].$$

Since  $m \geq 10$  and  $\delta \leq \delta_0 \leq 1/(\log 10)^2 < 1/5$ , we have  $\text{Im}(s) \asymp m$  uniformly in  $\delta$ .

**1) Log-derivative identity in the  $s$ -frame.** From  $\Xi_2(u) = \frac{u(u-2)}{8}\Lambda_2(u)$  and  $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$  we obtain, for  $u = 1 + v$ ,

$$\frac{E'(v)}{E(v)} = \frac{\Xi_2'(u)}{\Xi_2(u)} = \left( \frac{1}{u} + \frac{1}{u-2} \right) - \frac{1}{4} \log \pi + \frac{1}{4} \frac{\Gamma'}{\Gamma} \left( \frac{u}{4} \right) + \frac{1}{2} \frac{\zeta'}{\zeta}(s), \quad (u = 1 + v, \ s = u/2).$$

Since  $u = 1 + v$  has  $\text{Im}(u) = m \geq 10$ , the completion terms  $(1/u + 1/(u-2))$  are  $O(1/m)$  on  $\partial B$  and are absorbed into the absolute constants in the bound.

Moreover, since  $v = 2s - 1$ , the local factor derivative satisfies

$$\frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)} = \sum_{\rho \in Z(m)} \frac{m_\rho}{v - \rho} = \frac{1}{2} \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s},$$

where  $Z_s(m)$  denotes the corresponding multiset of nontrivial zeros  $\rho_s = \beta + i\gamma$  of  $\zeta(s)$  with  $|\gamma - \frac{m}{2}| \leq \frac{1}{2}$ .

Therefore

$$\frac{F'(v)}{F(v)} = \frac{E'(v)}{E(v)} - \frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)} = -\frac{1}{4} \log \pi + \frac{1}{4} \frac{\Gamma'}{\Gamma} \left( \frac{1+v}{4} \right) + \frac{1}{2} \left( \frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s} \right).$$

**2) RH-free residual bound for  $\zeta'/\zeta$  with local-zero subtraction.** A standard “local-zero decomposition” (unconditional) asserts that there exist absolute constants  $A_\zeta, B_\zeta$  such that for  $0 \leq \sigma \leq 1$  and  $t \geq 5$ ,

$$\left| \frac{\zeta'}{\zeta}(\sigma + it) - \sum_{|\gamma - t| \leq 1} \frac{1}{(\sigma + it) - \rho} \right| \leq A_\zeta \log(t + 2) + B_\zeta. \quad (\star)$$

(For a self-contained route,  $(\star)$  can be derived from the Hadamard product for  $\xi(s)$  plus a Riemann–von Mangoldt bound for  $N(T)$ ; otherwise cite a standard reference.)

For  $v \in \partial B$  we have  $|t - \frac{m}{2}| \leq \delta/2 < 1/10$ , hence every zero in  $Z_s(m)$  satisfies  $|\gamma - t| \leq 1$  and is included in the sum in  $(\star)$ . Thus

$$\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s - \rho_s} = \left( \frac{\zeta'}{\zeta}(s) - \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} \right) + \sum_{\substack{|\gamma - t| \leq 1 \\ |\gamma - \frac{m}{2}| > 1/2}} \frac{1}{s - \rho}.$$

In the remaining sum we have  $|\gamma - t| \geq 1/2 - |t - \frac{m}{2}| \geq 2/5$ , hence  $|s - \rho| \geq 2/5$  and each term has modulus  $\leq 5/2$ . The number of zeros with  $|\gamma - t| \leq 1$  is bounded by the manuscript’s explicit local window majorant (Lemma 10.10) at height  $\asymp m$ , so this difference-of-windows sum is  $\ll \log m$ .

Combining these bounds yields absolute constants  $A_{\text{res}}, B_{\text{res}}$  such that

$$\left| \frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s - \rho_s} \right| \leq A_{\text{res}} \log m + B_{\text{res}},$$

uniformly for all  $v \in \partial B$  and all  $\delta \in (0, \delta_0]$ .

**3) Gamma factor bound (Stirling, uniform in  $\delta$ ).** For  $z = (1+v)/4$  we have  $\text{Re}(z) \in [1/4, 3/4]$  and  $|\text{Im}(z)| \asymp m$ . A uniform Stirling-type bound gives

$$\left| \frac{\Gamma'}{\Gamma}(z) \right| \leq \log(|\text{Im}(z)| + 2) + C_\Gamma \leq \log(m + 2) + C_\Gamma,$$

with an absolute constant  $C_\Gamma$ .

**4) Conclusion.** Insert the bounds from (2)–(3) into the identity in (1), and absorb harmless constants into  $(C_1, C_2)$ . All constants are independent of  $(\alpha, \delta, \eta, \kappa)$  because: (i)  $\sigma$  ranges over a fixed compact interval  $[0, 1]$ , (ii)  $t \asymp m$  with  $m \geq 10$  uniformly for  $\delta \leq \delta_0$ , and (iii) the difference-of-windows sum is controlled by Lemma 10.10, which is unconditional.  $\square$

*Remark 7.3* (Hard gate / certificates (v38)). The tail harness in Appendix C uses explicit numerical interval enclosures for the constant ledger (e.g.  $C_1, C_2, C_{\text{up}}, C''_h, \kappa$ ) stored in `v36_repro_pack/v36_constants_m10.j`. It evaluates the tail inequality for a pinned parameter choice and records the UE exponent  $p$  explicitly. This is an *audit harness* only: it does not certify that the constants file is correct, and it does not, by itself, yield a uniform tail closure. An unconditional proof therefore still requires a referee-acceptable certification of the analytic constant ledger, and a resolution of the UE–Gate (Remark 10.12).

## 8 Short-side forcing

Assume an off-axis pair at height  $m$  with displacement  $a > 0$  exists. On an aligned box with  $\alpha = a$ , the two upper zeros in the centered  $v$ -plane are at  $v = \pm a + im$ . The pair factor

$$Z_{\text{pair}}(v) := (v - (a + im))(v - (-a + im)) \quad (9)$$

produces a large phase rotation on the near vertical side.

**Lemma 8.1** (Short-side forcing lower bound). *Let  $I_+ := \{\alpha + iy : |y - m| \leq \delta\}$  with  $|\alpha - a| \leq \delta$ . Then*

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2 \arctan\left(\frac{\delta}{|\alpha - a|}\right) + 2 \arctan\left(\frac{\delta}{\alpha + a}\right) \geq \frac{\pi}{2}. \quad (10)$$

**Lemma 8.2** (Single-box forcing is constant-limited). *In the forcing setup of Lemma 8.1, the total phase variation of the pair factor along  $I_+$  satisfies*

$$\Delta_{I_+} \arg Z_{\text{pair}} \leq 2\pi,$$

*uniformly in the height  $m$ . Consequently the forcing constant  $c$  appearing in the tail inequality (Theorem 11.1) is an absolute constant, independent of  $m$ ; in particular the forcing side cannot grow like  $\log m$  (or any unbounded function of  $m$ ) as  $m \rightarrow \infty$ .*

*Proof.* On  $I_+ = \{\alpha + iy : |y - m| \leq \delta\}$  one has

$$Z_{\text{pair}}(\alpha + iy) = ((\alpha - a) + i(y - m))((\alpha + a) + i(y - m)).$$

Along  $y \in [m - \delta, m + \delta]$  the argument of each linear factor varies by at most  $\pi$  (it is an arctan function whose range lies in an interval of length  $\leq \pi$ ), so the argument of the product varies by at most  $2\pi$ , uniformly in  $m$ . The forcing chain converts a fixed positive portion of  $\Delta_{I_+} \arg Z_{\text{pair}}$  into the constant  $c$  with fixed conversion scalars, so  $c$  is necessarily  $O(1)$ .  $\square$

## 9 Outer factorization and the inner quotient (Bridge 1)

We work on a fixed box  $B = B(\alpha, m, \delta)$  and write  $B^\circ$  for its interior. Assume boundary-contact:  $E \neq 0$  on  $\partial B$  (this will be enforced later by  $\kappa$ -admissibility; see Definition 10.5 and Lemma 10.6).

**Lemma 9.1** (Dirichlet outer factor on a box). *Let  $B = B(\alpha, m, \delta)$  be the closed rectangle and  $B^\circ$  its interior. Assume  $E$  is holomorphic on a neighborhood of  $\overline{B}$  and  $E \neq 0$  on  $\partial B$ . Then  $\log |E| \in C(\partial B)$ . Let  $U \in C(\overline{B}) \cap \text{Harm}(B^\circ)$  be the unique solution of the Dirichlet problem with boundary data  $U|_{\partial B} = \log |E|$ . Since  $B^\circ$  is simply connected, there exists a harmonic conjugate  $V$  on  $B^\circ$  (unique up to an additive constant) such that  $U + iV$  is holomorphic on  $B^\circ$ . Define*

$$G_{\text{out}}(v) := \exp(U(v) + iV(v)), \quad v \in B^\circ.$$

*Then  $G_{\text{out}}$  is holomorphic and zero-free on  $B^\circ$ , satisfies  $|G_{\text{out}}(v)| = e^{U(v)}$  for  $v \in B^\circ$ , and*

$$\lim_{z \rightarrow \xi, z \in B^\circ} |G_{\text{out}}(z)| = |E(\xi)| \quad (\xi \in \partial B).$$

*Proof.* Continuity of  $\log |E|$  on  $\partial B$  follows from  $E \neq 0$  on  $\partial B$ . Existence and uniqueness of  $U$  on a rectangle are standard. Since  $B^\circ$  is simply connected,  $U$  admits a harmonic conjugate  $V$  on  $B^\circ$ , unique up to an additive constant. The function  $U + iV$  is holomorphic, hence so is  $G_{\text{out}} = \exp(U + iV)$ , and it is zero-free. Finally  $|G_{\text{out}}| = e^U$  on  $B^\circ$ , and by continuity of  $U$  on  $\overline{B}$  we have  $e^{U(\xi)} = |E(\xi)|$  on  $\partial B$ , yielding the boundary modulus identity in interior-limit form.  $\square$



Define on  $B^\circ$  the inner quotient

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

Then  $W$  is holomorphic on  $B^\circ$  and  $|W| = 1$  on  $\partial B$  in the sense of interior limits in modulus.

**Proposition 9.2** (Bridge 1: zero-free inner collapse). *Assume the setup of Lemma 9.1 and define  $W = E/G_{\text{out}}$  on  $B^\circ$ . If  $W$  is zero-free on  $B^\circ$  (equivalently,  $E$  is zero-free on  $B^\circ$ ), then  $W$  is constant on  $B^\circ$ ; in fact  $W \equiv e^{i\theta_B}$  for some  $\theta_B \in \mathbb{R}$ .*

*Proof.* Since  $W$  is zero-free on  $B^\circ$  and  $G_{\text{out}}$  is zero-free, the function  $E$  is zero-free on  $B^\circ$ . Because  $B^\circ$  is simply connected,  $E$  admits a holomorphic logarithm on  $B^\circ$ , so  $\log |E|$  is harmonic on  $B^\circ$ . By construction  $U$  is harmonic on  $B^\circ$ , continuous on  $\overline{B}$ , and equals  $\log |E|$  on  $\partial B$ . Thus  $U - \log |E|$  is harmonic on  $B^\circ$  with zero boundary values, so by Dirichlet uniqueness  $U \equiv \log |E|$  on  $B^\circ$ . Therefore for  $v \in B^\circ$ ,

$$|W(v)| = \frac{|E(v)|}{|G_{\text{out}}(v)|} = \frac{|E(v)|}{e^{U(v)}} = \frac{|E(v)|}{e^{\log |E(v)|}} = 1.$$

An analytic function of constant modulus on a connected open set is constant, hence  $W \equiv e^{i\theta_B}$ .  $\square$

*Remark 9.3* (Boundary modulus convention). Under boundary-contact,  $U$  extends continuously to  $\partial B$  and satisfies  $U|_{\partial B} = \log |E|$ . Hence  $|G_{\text{out}}| = |E|$  holds pointwise on  $\partial B$  as interior limits in modulus, and therefore  $|W| = 1$  holds pointwise in modulus on  $\partial B$ . In boundary integral estimates this may be used in the a.e. sense without change.

*Remark 9.4* (No converse: boundary modulus does not exclude interior zeros). Lemma 9.1 implies that under boundary-contact the quotient  $W := E/G_{\text{out}}$  satisfies  $|W| = 1$  on  $\partial B$  (in the interior boundary-limit sense of Remark 9.3). This condition alone does *not* imply that  $W$  is zero-free or constant on  $B^\circ$ : nonconstant holomorphic functions on  $B^\circ$  can have  $|W| = 1$  on  $\partial B$  and still possess prescribed interior zeros (e.g. via conformal transport of finite Blaschke products from the unit disc). Thus Proposition 9.2 is strictly one-directional: the additional hypothesis “ $W$  is zero-free on  $B^\circ$ ” is essential.

## 10 Shape-only invariance and the envelope constants

Let  $T(v) := (v - (\alpha + im))/\delta$ , mapping  $\partial B$  affinely onto the fixed square boundary  $\partial Q$  with  $Q = [-1, 1]^2$ .

**Lemma 10.1** (Shape-only invariance). *Any constant arising solely from geometric or boundary-operator estimates on  $\partial B$  that are invariant under affine rescaling depends only on  $\partial Q$  and is independent of  $(\alpha, m, \delta)$ .*

*Proof.* Under  $T$ , arclength scales by  $\delta$  and tangential derivatives by  $1/\delta$ . After normalization, all purely geometric quantities and operator norms reduce to fixed quantities on  $\partial Q$ .  $\square$

**Lemma 10.2** (Boundary-to-center evaluation in  $L^2$  (sharp  $\delta^{-1/2}$ )). *Let  $B = B(\alpha, m, \delta)$  be a box and let  $v_0$  be its center. Let  $u$  be harmonic on  $B^\circ$  and assume its boundary trace lies in  $L^2(\partial B, ds)$ . Then, writing  $P_B(v_0, \xi) = d\omega_{v_0}^B/ds(\xi)$  for the Poisson kernel of  $B$  at  $v_0$ ,*

$$|u(v_0)| \leq \|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} \|u\|_{L^2(\partial B, ds)}.$$

Under the similarity  $T(\xi) = (\xi - v_0)/\delta$  mapping  $\partial B$  onto  $\partial Q$ ,

$$\|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2(\partial Q, ds)}.$$

In particular the exponent  $\delta^{-1/2}$  is sharp (witnessed by the constant harmonic function  $u \equiv 1$ ).

*Proof.* For harmonic  $u$  on  $B^\circ$  with  $L^2$  trace on  $\partial B$ , the Poisson representation gives

$$u(v_0) = \int_{\partial B} u(\xi) d\omega_{v_0}^B(\xi) = \int_{\partial B} u(\xi) P_B(v_0, \xi) ds(\xi).$$

Cauchy–Schwarz yields  $|u(v_0)| \leq \|P_B(v_0, \cdot)\|_{L^2} \|u\|_{L^2}$ .

For the scaling: under  $T$ , arclength scales by  $ds = \delta ds_Q$  and Poisson kernels scale by  $P_B(v_0, \xi) = \delta^{-1} P_Q(0, T(\xi))$ . Hence

$$\int_{\partial B} P_B(v_0, \xi)^2 ds(\xi) = \int_{\partial Q} \delta^{-2} P_Q(0, \zeta)^2 \delta ds_Q(\zeta) = \delta^{-1} \int_{\partial Q} P_Q(0, \zeta)^2 ds_Q(\zeta),$$

giving  $\|P_B(v_0, \cdot)\|_{L^2} = \delta^{-1/2} \|P_Q(0, \cdot)\|_{L^2}$ .

Sharpness: for  $u \equiv 1$  we have  $|u(v_0)| = 1$  and  $\|u\|_{L^2(\partial B)} = \sqrt{|\partial B|} \asymp \delta^{1/2}$ , so the inequality forces  $\|P_B(v_0, \cdot)\|_{L^2} \gtrsim \delta^{-1/2}$ .  $\square$

**Lemma 10.3** (Upper envelope bound (outer-aligned form)). *Let  $B = B(\pm a, m, \delta)$  be an aligned box and let  $G_{\text{out}}$  be the outer factor on  $B$  constructed from  $\log |E|$  on  $\partial B$  (Section 9). Define the inner quotient*

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

*Assume the boundary-contact convention:  $E$  has no zeros on  $\partial B$  (hence  $W$  has unimodular boundary values a.e.). For each sign  $\pm$  let  $v_\pm := \pm a + im$  and let  $e^{i\varphi_0^\pm} \in \mathbb{T}$  be an  $L^2(\partial B, ds)$ -best constant phase,*

$$e^{i\varphi_0^\pm} \in \arg \min_{|c|=1} \int_{\partial B} |W(v) - c|^2 ds(v).$$

*Then there exists a shape-only constant  $C_{\text{up}} > 0$  (depending only on the normalized square  $Q = [-1, 1]^2$ ) such that*

$$\sum_{\pm} |W(v_\pm) - e^{i\varphi_0^\pm}| \leq 2 C_{\text{up}} \delta \sup_{v \in \partial B} \left| \frac{E'(v)}{E(v)} \right|. \quad (11)$$

*One admissible explicit definition is*

$$C_{\text{up}} := \left( \sup_{\xi \in \partial Q} P_Q(0, \xi) \right)^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot (1 + \|H_{\partial Q}\|_{L^2 \rightarrow L^2}),$$

*where  $P_Q(0, \xi) = d\omega_0^Q/ds(\xi)$  is the Poisson kernel of  $Q$  at the center 0 with respect to arclength on  $\partial Q$ , and  $H_{\partial Q}$  is the boundary conjugation (Hilbert/Cauchy) operator on  $\partial Q$ .*

*Remark 10.4* (No residual proxying in the upper envelope). Lemma 10.3 controls the inner quotient  $W = E/G_{\text{out}}$  and therefore depends on  $\sup_{\partial B} |E'/E|$ . Residual bounds for  $F = E/Z_{\text{loc}}$  control  $\sup_{\partial B} |F'/F|$  and do *not* by themselves bound  $\sup_{\partial B} |E'/E|$ . Whenever the residual envelope is used to control dial deviation, it must be routed through the log-derivative split  $E'/E = F'/F + Z'_{\text{loc}}/Z_{\text{loc}}$  (Lemma 10.7) together with the collar bound (Lemma 10.8), yielding Corollary 10.11.

*Proof.* Fix one sign and write  $v_0 = v_\pm$  and  $B = B(\pm a, m, \delta)$ . We record the (RH-free) chain and indicate the scale factors explicitly.

1. **Evaluation from the boundary (harmonic measure; produces  $\delta^{-1/2}$ ).** For any constant  $c \in \mathbb{T}$ , subharmonicity of  $|W - c|^2$  implies

$$|W(v_0) - c|^2 \leq \int_{\partial B} |W(\xi) - c|^2 d\omega_{v_0}^B(\xi) = \int_{\partial B} |W(\xi) - c|^2 P_B(v_0, \xi) ds(\xi),$$

so

$$|W(v_0) - c| \leq \|P_B(v_0, \cdot)\|_{L^\infty(\partial B)}^{1/2} \|W - c\|_{L^2(\partial B, ds)}.$$

Under the similarity  $T(\xi) = (\xi - v_0)/\delta$  mapping  $\partial B$  onto  $\partial Q$ , Poisson kernels scale by  $\|P_B(v_0, \cdot)\|_\infty^{1/2} = \delta^{-1/2} \|P_Q(0, \cdot)\|_\infty^{1/2}$ .

2. **Poincaré/Wirtinger on  $\partial B$  (produces  $\delta$ ).** For the  $L^2$ -best constant  $c = e^{i\varphi_0^\pm}$  and  $|\partial B| = 8\delta$ , periodic Poincaré on a loop of length  $8\delta$  gives

$$\|W - c\|_{L^2(\partial B)} \leq \frac{|\partial B|}{2\pi} \|\partial_s W\|_{L^2(\partial B)} = \frac{4\delta}{\pi} \|\partial_s W\|_{L^2(\partial B)}.$$

3. **Outer factor control (no  $\delta$ ; uses bounded boundary conjugation).** Write  $\log G_{\text{out}} = U + i\tilde{U}$  with  $U|_{\partial B} = \log |E|$  and  $\tilde{U} = H_{\partial B} U$ . Differentiating tangentially,  $\partial_s \log G_{\text{out}} = \partial_s U + i H_{\partial B}(\partial_s U)$ . Since  $\log W = \log E - \log G_{\text{out}}$ ,

$$\|\partial_s \log W\|_{L^2(\partial B)} \leq (1 + \|H_{\partial B}\|_{L^2 \rightarrow L^2}) \|\partial_s \log E\|_{L^2(\partial B)} \leq (1 + \|H_{\partial B}\|_{L^2 \rightarrow L^2}) \left\| \frac{E'}{E} \right\|_{L^2(\partial B)}.$$

On  $\partial B$  we have  $|W| = 1$  a.e., hence  $|\partial_s W| = |\partial_s \log W|$ .

4.  **$L^2$  to sup (produces  $\delta^{1/2}$ ).** Using  $|\partial B| = 8\delta$ ,

$$\left\| \frac{E'}{E} \right\|_{L^2(\partial B)} \leq \sqrt{|\partial B|} \sup_{\partial B} \left| \frac{E'}{E} \right| = \sqrt{8\delta} \sup_{\partial B} \left| \frac{E'}{E} \right|.$$

Combining the four steps yields

$$|W(v_0) - e^{i\varphi_0^\pm}| \leq \|P_Q(0, \cdot)\|_\infty^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot (1 + \|H_{\partial Q}\|_{L^2 \rightarrow L^2}) \cdot \delta \sup_{\partial B} \left| \frac{E'}{E} \right|,$$

where we used the similarity invariance  $\|H_{\partial B}\|_{L^2 \rightarrow L^2} = \|H_{\partial Q}\|_{L^2 \rightarrow L^2}$ . Summing over  $\pm$  gives (11).  $\square$

## 10.1 Local factor split and collar control

**Definition 10.5** (Collar-admissible aligned boxes). Fix once and for all a collar parameter  $\kappa \in (0, 1/10)$ . An aligned box  $B = B(\alpha, m, \delta)$  is called  $\kappa$ -admissible if

$$\text{dist}(\partial B, \mathcal{Z}(E)) \geq \kappa \delta.$$

Given any nominal scale  $\delta_0 > 0$  and any center, there exists some  $0 < \delta \leq \delta_0$  for which  $\kappa$ -admissibility holds (Lemma 10.6). Whenever a chosen box is not  $\kappa$ -admissible, we shrink  $\delta$  until  $\kappa$ -admissibility holds. Moreover the assembled tail inequality is monotone-safe under such  $\delta$ -shrinking (Lemma 11.2).

**Lemma 10.6** (Existence of a  $\kappa$ -admissible shrink). *Fix  $\kappa \in (0, 1/10)$  and a center  $v_0 \in \mathbb{C}$ . For every  $\delta_0 > 0$  there exists  $\delta' \in (0, \delta_0]$  such that the closed box*

$$B(v_0, \delta') := \{v \in \mathbb{C} : \|v - v_0\|_\infty \leq \delta'\}$$

*satisfies*

$$\text{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \geq \kappa \delta'.$$

*In particular, given  $(\alpha, m)$  and nominal  $\delta_0 = \eta\alpha/(\log m)^2$ , one may always choose a scale  $0 < \delta \leq \delta_0$  for which  $B(\alpha, m, \delta)$  is  $\kappa$ -admissible.*

*Proof.* Zeros of the entire function  $E$  are isolated. Choose  $\varepsilon > 0$  such that  $\mathcal{Z}(E) \cap \{0 < \|v - v_0\|_\infty \leq \varepsilon\}$  is empty (if  $E(v_0) = 0$ ) or such that  $\mathcal{Z}(E) \cap \{\|v - v_0\|_\infty \leq \varepsilon\}$  is empty (if  $E(v_0) \neq 0$ ). Set  $\delta' := \min\{\delta_0, \varepsilon/(1 + \kappa)\}$ . Then every boundary point satisfies  $\|v - v_0\|_\infty = \delta'$ . Any zero  $\rho \in \mathcal{Z}(E)$  is either  $\rho = v_0$  (in which case  $\text{dist}(v, \rho) = \delta' \geq \kappa\delta'$ ) or satisfies  $\|\rho - v_0\|_\infty \geq \varepsilon$  (in which case  $\text{dist}(v, \rho) \geq \varepsilon - \delta' \geq \kappa\delta'$ ). Therefore  $\text{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \geq \kappa\delta'$ .  $\square$

**Lemma 10.7** (Log-derivative decomposition). *With  $Z_{\text{loc}}$  and  $F$  as in (7) and (8), one has on any region where  $E$  and  $Z_{\text{loc}}$  are holomorphic and nonvanishing (in particular on  $\partial B$  under the boundary-contact convention)*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\text{loc}}}{Z_{\text{loc}}}.$$

**Lemma 10.8** (Buffered local factor bound on  $\partial B$ ). *Let  $B = B(\alpha, m, \delta)$  be  $\kappa$ -admissible in the sense of Definition 10.5. Then*

$$\sup_{v \in \partial B} \left| \frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)} \right| \leq \frac{N_{\text{loc}}(m)}{\kappa \delta},$$

*where  $N_{\text{loc}}(m)$  counts zeros of  $E$  in the local window used to define  $Z_{\text{loc}}$ , with multiplicity.*

**Lemma 10.9** (Local log-derivative bound in  $L^2(\partial B)$ ). *Let  $B = B(\alpha, m, \delta)$  be  $\kappa$ -admissible (Definition 10.5), and let  $Z_{\text{loc}}$  be the local factor with local zero-count  $N_{\text{loc}}(m)$  as in Section 6. Then*

$$\left\| \frac{Z'_{\text{loc}}}{Z_{\text{loc}}} \right\|_{L^2(\partial B)} \leq \frac{\sqrt{8} N_{\text{loc}}(m)}{\kappa \delta^{1/2}}.$$

*More generally, for any  $1 \leq r \leq \infty$ ,*

$$\left\| \frac{Z'_{\text{loc}}}{Z_{\text{loc}}} \right\|_{L^r(\partial B)} \leq \frac{8^{1/r} N_{\text{loc}}(m)}{\kappa \delta^{1-1/r}}.$$

*Proof.* Lemma 10.8 gives  $\|Z'_{\text{loc}}/Z_{\text{loc}}\|_{L^\infty(\partial B)} \leq N_{\text{loc}}(m)/(\kappa\delta)$ . Since  $|\partial B| = 8\delta$ , we have  $\|f\|_{L^r(\partial B)} \leq |\partial B|^{1/r} \|f\|_{L^\infty(\partial B)}$  for every  $1 \leq r \leq \infty$ , which yields the stated bounds.  $\square$

**Lemma 10.10** (Explicit local window zero count). *Let  $N(T)$  denote the number of nontrivial zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  with  $0 < \gamma \leq T$ , counted with multiplicity. Then for every  $T \geq 5$ ,*

$$N(T+1) - N(T-1) \leq 1.01 \log T + 17. \quad (12)$$

*Consequently, for every  $m \geq 10$ ,*

$$N_{\text{loc}}(m) \leq 1.01 \log m + 17. \quad (13)$$

*Proof.* By [7, Theorem 1.1], for every  $x \geq e$ ,

$$\left| N(x) - \frac{x}{2\pi} \log\left(\frac{x}{2\pi e}\right) \right| \leq 0.10076 \log x + 0.24460 \log \log x + 8.08344.$$

Let  $M(x) := \frac{x}{2\pi} \log(\frac{x}{2\pi e})$ , so  $M'(x) = \frac{1}{2\pi} \log(\frac{x}{2\pi})$ . For  $T \geq 5$  we have  $\log(T \pm 1) \leq \log(2T)$  and  $\log \log x \leq \log x$  for  $x \geq e$ , hence

$$N(T+1) - N(T-1) \leq (M(T+1) - M(T-1)) + 2(0.10076 + 0.24460) \log(2T) + 2 \cdot 8.08344.$$

Moreover

$$M(T+1) - M(T-1) = \int_{T-1}^{T+1} M'(x) dx \leq \int_{T-1}^{T+1} \frac{1}{2\pi} \log x dx \leq \frac{1}{\pi} \log(2T).$$

Combining these bounds gives  $N(T+1) - N(T-1) \leq 1.00903 \log T + 16.8663 \leq 1.01 \log T + 17$ , establishing (12). Finally, in width-2 one has  $m = 2T$ . The local window  $|\operatorname{Im} \rho - m| \leq 1$  corresponds to  $|\gamma - T| \leq 1/2$  in the  $s$ -plane, so  $N_{\text{loc}}(m) = N(T + \frac{1}{2}) - N(T - \frac{1}{2}) \leq N(T+1) - N(T-1)$ , yielding (13).  $\square$

**Corollary 10.11** (Outer-aligned upper envelope in residual+local form). *Let  $B$  be  $\kappa$ -admissible. Assume the residual envelope bound of Lemma 7.2, i.e.  $\sup_{\partial B} |F'/F| \leq L(m) := C_1 \log m + C_2$ . Then*

$$\sum_{\pm} |W(v_{\pm}) - e^{i\varphi_0^{\pm}}| \leq 2C_{\text{up}} \left( \delta L(m) + \frac{N_{\text{loc}}(m)}{\kappa} \right) \leq 2C_{\text{up}} \left( \delta L(m) + \frac{1.01 \log m + 17}{\kappa} \right).$$

*Remark 10.12* (UE gate = exponent budget at the local interface). Lemma 10.3 is the *only* step in the envelope chain that generates a positive power of  $\delta$  in front of a boundary log-derivative endpoint. Abstractly, suppose an upper-envelope mechanism yields, for some  $p > 0$ ,

$$\sum_{\pm} |W(v_{\pm}) - e^{i\varphi_0^{\pm}}| \leq 2C_{\text{up}} \delta^p \sup_{\partial B} \left| \frac{E'}{E} \right|,$$

and suppose the collar/local split yields, for some  $\theta > 0$ ,

$$\sup_{\partial B} \left| \frac{E'}{E} \right| \leq L(m) + \frac{N_{\text{loc}}(m)}{\kappa \delta^{\theta}}.$$

Then the local contribution in the envelope side scales as  $\delta^{p-\theta} N_{\text{loc}}(m)/\kappa$ . Under the nominal choice  $\delta_0(m, \alpha) = \eta \alpha / (\log m)^2$  and the unconditional majorant  $N_{\text{loc}}(m) \ll \log m$ , uniform  $\eta$ -shrinking tail closure is possible only if

$$p - \theta \geq \frac{1}{2}$$

(Theorem 10.13).

In the *proved* pointwise/sup architecture one has  $p = 1$  (Lemma 10.3) and  $\theta = 1$  (Lemma 10.8), so  $p - \theta = 0$  and the local term is  $\delta$ -inert;  $\eta$ -shrinking cannot suppress it (Lemma 10.14). Moreover, within this same endpoint class, a strengthened exponent  $p > 1$  is impossible with shape-only constants (Lemma 10.15). Thus the former  $\eta$ -absorption closure route based on the pointwise/sup UE endpoint is a formal NO-GO and is recorded as discarded (Appendix A).

**Theorem 10.13** (Exponent budget for  $\eta$ -shrinking under  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ ). *Let  $m \geq 10$ ,  $\alpha \in (0, 1]$  and  $\eta \in (0, 1]$ , and set the nominal scale*

$$\delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}.$$

*Assume that for all  $0 < \delta \leq \delta_0(m, \alpha)$  one has:*

*(UE<sub>p</sub>) (UE exponent) for some  $p > 0$ ,*

$$\text{UE}(\delta) \leq 2C_{\text{up}} \delta^p \sup_{\partial B} \left| \frac{E'}{E} \right|;$$

*(COL<sub>θ</sub>) (Collar/local exponent) for some  $\theta > 0$ ,*

$$\sup_{\partial B} \left| \frac{E'}{E} \right| \leq L(m) + \frac{N_{\text{loc}}(m)}{\kappa \delta^\theta},$$

*with fixed  $\kappa \in (0, 1/10)$ ;*

*(GROW) (Majorants)  $L(m) \leq A_L \log m + B_L$  and  $N_{\text{loc}}(m) \leq A_N \log m + B_N$  for all  $m \geq 10$ ;*

*(FORCE) (Forcing side) the forcing-vs-envelope tail inequality has a fixed positive forcing constant  $c > 0$  and only  $\delta$ -helpful subtractive terms on the RHS.*

*Then at  $\delta = \delta_0(m, \alpha)$  one has the explicit bound*

$$\text{UE}(\delta_0) \leq 2C_{\text{up}} \left( \delta_0^p L(m) + \delta_0^{p-\theta} \frac{N_{\text{loc}}(m)}{\kappa} \right). \quad (\text{BUDGET})$$

*Moreover, uniform tail closure by  $\eta$ -shrinking (i.e. there exists  $\eta_\star > 0$  such that for every  $\eta \leq \eta_\star$  the tail inequality holds for all  $m \geq 10$ ) is possible only if*

$$p - \theta \geq \frac{1}{2}. \quad (\text{B1})$$

*Proof.* Insert (COL<sub>θ</sub>) into (UE<sub>p</sub>) at  $\delta = \delta_0$  to obtain (BUDGET). At  $\alpha = 1$  one has  $\delta_0(m, 1) = \eta/(\log m)^2$ , so the local term behaves as

$$\delta_0^{p-\theta} N_{\text{loc}}(m) \ll \left( \frac{\eta}{(\log m)^2} \right)^{p-\theta} \log m = \eta^{p-\theta} (\log m)^{1-2(p-\theta)}.$$

If  $p - \theta < 1/2$  then  $1 - 2(p - \theta) > 0$ , so the local contribution grows without bound as  $m \rightarrow \infty$ , while the forcing side tends to the fixed constant  $c$  because all RHS corrections are  $\delta$ -helpful and vanish as  $\delta_0 \rightarrow 0$ . Hence uniform tail closure is impossible. If  $p - \theta \geq 1/2$  then the local contribution is uniformly bounded by  $O(\eta^{p-\theta})$  and tends to 0 as  $\eta \downarrow 0$ , enabling uniform absorption once all constants are  $\delta$ -uniform.  $\square$

**Lemma 10.14** ( $\eta$ -absorption obstruction under the pointwise UE exponent  $p = 1$ ). *Assume the hypotheses of Corollary 10.11. Then for every  $\delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ ,*

$$\sum_{\pm} |W(v_{\pm}) - e^{i\varphi_0^{\pm}}| \leq 2C_{\text{up}} \left( \delta L(m) + \frac{N_{\text{loc}}(m)}{\kappa} \right).$$

In particular, letting  $\eta \downarrow 0$  (hence  $\delta \downarrow 0$ ) only suppresses the residual term  $\delta L(m)$ ; the local term  $N_{\text{loc}}(m)/\kappa$  does not decay with  $\eta$ . Therefore any absorption-style closure that attempts to force the envelope side small by choosing  $\eta$  must additionally verify a separate inequality of the form

$$\frac{2C_{\text{up}}}{\kappa} N_{\text{loc}}(m) < c$$

at the relevant anchor height(s), where  $c$  is the forcing constant in (14).

**Lemma 10.15** (UE scaling NO-GO for pointwise/sup endpoints). *Assume an upper-envelope bound of the form*

$$\sum_{\pm} |W(v_{\pm}) - e^{i\varphi_0^{\pm}}| \leq 2C_{\text{up}} \delta^p \sup_{\partial B} \left| \frac{E'}{E} \right| \quad (p > 0),$$

where the constant  $C_{\text{up}}$  depends only on the normalized shape (Lemma 10.1) and is independent of  $\delta$ . Then necessarily  $p \leq 1$ . In particular, no pointwise/sup envelope mechanism with shape-only constants can yield any exponent  $p > 1$ .

*Proof.* Under the affine rescaling  $T(v) = (v - (\alpha + im))/\delta$ , the boundary  $\partial B$  maps to the fixed square boundary  $\partial Q$ . If  $\tilde{E}(z) := E(T^{-1}(z))$ , then by the chain rule

$$\frac{E'}{E}(T^{-1}(z)) = \frac{1}{\delta} \frac{\tilde{E}'(z)}{\tilde{E}(z)}.$$

Hence  $\sup_{\partial B} |E'/E| = \delta^{-1} \sup_{\partial Q} |\tilde{E}'/\tilde{E}|$ . The left-hand side of the upper-envelope bound is dimensionless (it is a sum of moduli of complex numbers), and under the normalization it may be  $O(1)$  for admissible configurations on the fixed shape. Therefore the bound forces

$$O(1) \leq 2C_{\text{up}} \delta^{p-1} \sup_{\partial Q} \left| \frac{\tilde{E}'}{\tilde{E}} \right| \quad \text{as } \delta \downarrow 0.$$

Since the normalized endpoint  $\sup_{\partial Q} |\tilde{E}'/\tilde{E}|$  is not forced to blow up as  $\delta \downarrow 0$  (it depends only on the normalized data), the factor  $\delta^{p-1}$  cannot tend to 0. Thus  $p - 1 \leq 0$ , i.e.  $p \leq 1$ .  $\square$

*Proof.* The displayed bound is exactly Corollary 10.11 with the corrected UE exponent  $p = 1$ . As  $\eta \rightarrow 0$  one has  $\delta_0 \rightarrow 0$  and hence  $\delta L(m) \rightarrow 0$ , while  $N_{\text{loc}}(m)/\kappa$  is unchanged. Since the forcing lower bound in the tail inequality tends to  $c$  as  $\delta \downarrow 0$ , the strict inequality requires the stated necessary condition at the anchor.  $\square$

## 10.2 Horizontal non-forcing budget in residual form

**Definition 10.16** (Horizontal non-forcing phase budget). Let  $B = B(\pm a, m, \delta)$  be an aligned box and let  $F = E/Z_{\text{loc}}$  be the residual factor. Assume  $F$  is holomorphic and zero-free on a neighborhood of  $\partial B$ . Let  $H_{\pm}$  denote the top and bottom edges of  $\partial B$ :

$$H_+ := \{x + i(m + \delta) : x \in [\pm a - \delta, \pm a + \delta]\}, \quad H_- := \{x + i(m - \delta) : x \in [\pm a - \delta, \pm a + \delta]\}.$$

Define

$$\Delta_{\text{nonforce}}(B) := \int_{H_+} |\partial_s \arg F| ds + \int_{H_-} |\partial_s \arg F| ds.$$

**Lemma 10.17** (Horizontal budget (residual form; audit-grade)). *In the setting of Definition 10.16,*

$$\Delta_{\text{nonforce}}(B) \leq 4\delta \sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right|.$$

Consequently, if  $\sup_{\partial B} |F'/F| \leq C_1 \log m + C_2$ , then

$$\Delta_{\text{nonforce}}(B) \leq C_h'' \delta (\log m + 1), \quad C_h'' := 4 \max\{C_1, C_2\}.$$

*Proof.* On either horizontal edge,  $|\partial_s \arg F| \leq |F'/F|$  pointwise. Each edge has length  $2\delta$ , hence each integral is bounded by  $2\delta \sup_{\partial B} |F'/F|$ . Summing top and bottom gives the first inequality, and the second follows from  $\sup_{\partial B} |F'/F| \leq C_1 \log m + C_2 \leq \max\{C_1, C_2\}(\log m + 1)$ .  $\square$

## 11 The explicit tail inequality (post-pivot)

For  $m \geq 10$  we use the growth surrogate

$$L(m) := C_1 \log m + C_2,$$

with constants as in Lemma 7.2. For the local window term we use the explicit majorant from Lemma 10.10:

$$N_{\text{up}}(m) := 1.01 \log m + 17 \quad \text{so that} \quad N_{\text{loc}}(m) \leq N_{\text{up}}(m) \quad (m \geq 10).$$

For a parameter  $\eta \in (0, 1)$  and a dial displacement  $\alpha \in (0, 1]$  define the *nominal* scale

$$\delta_0 := \delta_0(m, \alpha) := \frac{\eta \alpha}{(\log m)^2}.$$

Fix a collar parameter  $\kappa \in (0, 1/10)$  as in Definition 10.5. For each  $(m, \alpha)$  we choose any scale  $0 < \delta \leq \delta_0$  such that the aligned boxes  $B = B(\pm\alpha, m, \delta)$  are  $\kappa$ -admissible; existence is guaranteed by Lemma 10.6. By Lemma 11.2, shrinking  $\delta$  only helps in the tail inequality, so it is safe to treat  $\delta_0$  as the worst-case scale in one-height reductions.

**Theorem 11.1** (Tail inequality (criterion form; pointwise UE exponent  $p = 1$ )). *Fix  $m \geq 10$  and  $\eta \in (0, 1)$ . Assume:*

1. *the forcing lemma producing the positive constant*

$$c_0 := \frac{3 \log 2}{8\pi}, \quad c := \frac{3 \log 2}{16}, \quad K_{\text{alloc}} := 3 + 8\sqrt{3};$$

2. *the residual envelope bound (Lemma 7.2) providing  $C_1, C_2$ ;*

3. *the audit-grade horizontal budget bound (Lemma 10.17), giving a constant  $C_h''$  independent of  $(\alpha, m, \delta)$ ;*

4. *the explicit local window bound (Lemma 10.10) providing the majorant  $N_{\text{up}}(m) = 1.01 \log m + 17$ .*

*Then for every  $\alpha \in (0, 1]$  and every  $\kappa$ -admissible aligned box  $B = B(\pm\alpha, m, \delta)$ , absence of off-axis quartets at height  $m$  follows from the strict inequality*

$$2C_{\text{up}} \left( \delta L(m) + \frac{N_{\text{up}}(m)}{\kappa} \right) < c - \delta \left( K_{\text{alloc}} c_0 L(m) + C_h'' (\log m + 1) \right). \quad (14)$$



*Proof sketch / bookkeeping.* The forcing side is unchanged from v31. The only post-pivot modification is on the upper-envelope side: Lemma 10.3 bounds dial deviation in terms of  $\sup_{\partial B} |E'/E|$ . Applying the log-derivative split (Lemma 10.7), the residual envelope for  $\sup_{\partial B} |F'/F| \leq L(m)$  (Lemma 7.2), and the collar bound  $\sup_{\partial B} |Z'_{\text{loc}}/Z_{\text{loc}}| \leq N_{\text{loc}}(m)/(\kappa\delta)$  (Lemma 10.8) yields

$$\sup_{\partial B} \left| \frac{E'}{E} \right| \leq L(m) + \frac{N_{\text{loc}}(m)}{\kappa\delta} \leq L(m) + \frac{N_{\text{up}}(m)}{\kappa\delta}.$$

Plugging this into Lemma 10.3 gives the left-hand side of (14). The right-hand side is the forcing lower bound, with the horizontal non-forcing term bounded by Lemma 10.17.  $\square$

**Lemma 11.2** (Monotonicity under  $\delta$ -shrinking). *Fix  $m \geq 10$ ,  $\alpha \in (0, 1]$ , and constants  $C_{\text{up}}, \kappa, c, c_0, K_{\text{alloc}}, C_h'', C_1, C_2$ . Let  $L(m) = C_1 \log m + C_2$  and  $N_{\text{up}}(m) = 1.01 \log m + 17$ . For  $\delta \in (0, 1]$  define*

$$\text{LHS}(\delta) := 2C_{\text{up}} \left( \delta L(m) + \frac{N_{\text{up}}(m)}{\kappa} \right), \quad \text{RHS}(\delta) := c - \delta \left( K_{\text{alloc}} c_0 L(m) + C_h'' (\log m + 1) \right).$$

*Then  $\text{LHS}(\delta)$  is (weakly) increasing in  $\delta$  and  $\text{RHS}(\delta)$  is (weakly) decreasing. Consequently, if  $\text{LHS}(\delta_0) < \text{RHS}(\delta_0)$  for some  $\delta_0 \in (0, 1]$ , then  $\text{LHS}(\delta) < \text{RHS}(\delta)$  holds for every  $\delta \in (0, \delta_0]$ .*

*Proof.* For  $\delta > 0$ , the map  $\delta \mapsto \delta L(m)$  is increasing and the term  $N_{\text{up}}(m)/\kappa$  is independent of  $\delta$ , hence  $\text{LHS}(\delta)$  is (weakly) increasing. The bracketed factor in  $\text{RHS}(\delta)$  is nonnegative and independent of  $\delta$ , so  $\text{RHS}(\delta)$  decreases linearly in  $\delta$ .  $\square$

**Lemma 11.3** (Worst case in  $\alpha$  is  $\alpha = 1$  at the nominal scale). *Fix  $m \geq 10$  and  $\eta \in (0, 1)$ . Define the nominal scale  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ . Consider the tail inequality (14) evaluated at  $\delta = \delta_0(m, \alpha)$ . Then the left-hand side is (weakly) increasing in  $\alpha \in (0, 1]$ , while the right-hand side is (weakly) decreasing. Therefore it suffices to verify (14) at  $\alpha = 1$  and  $\delta = \delta_0(m, 1)$ . If one later shrinks  $\delta \leq \delta_0(m, \alpha)$  to enforce  $\kappa$ -admissibility, the inequality only becomes easier (Lemma 11.2).*

*Proof.* With  $\delta = \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ , the only  $\alpha$ -dependence in the left-hand side is through the factor  $\delta L(m)$ , which is increasing in  $\alpha$ , so the left-hand side increases. The right-hand side equals  $c - \delta \cdot \Xi(m)$  for a nonnegative factor  $\Xi(m)$  independent of  $\alpha$ , hence it decreases.  $\square$

*Remark 11.4* (No one-height reduction in  $m$  under the pointwise UE exponent  $p = 1$ ). In v33, the (claimed)  $\delta^{3/2}$  prefactor in Lemma 10.3 made the local contribution scale like  $\delta^{1/2} N_{\text{up}}(m)$  at the nominal choice  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ , leading to an expression essentially independent of  $m$  and enabling a one-height reduction. After the UE-Gate audit, Lemma 10.3 provides only the pointwise exponent  $p = 1$ , so the tail left-hand side contains the  $\delta$ -inert term  $(2C_{\text{up}}/\kappa) N_{\text{up}}(m)$ . With the explicit majorant  $N_{\text{up}}(m) = 1.01 \log m + 17$ , this term is *increasing* in  $m$ . Therefore a one-height reduction in  $m$  is not available under the current pointwise envelope mechanism: the tail criterion must be controlled as a family in  $m$ , or the UE-Gate must be cleared by a strengthened envelope mechanism (Remark 10.12).

## 12 S5' frontier: phase/winding endpoint redesign (open)

**OPEN (Missing Lever for S5' tail closure; no silent closure).**

Fix  $\eta \in (0, 1]$ ,  $\kappa \in (0, 1/10)$ , and the nominal scale  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ . Let  $B = B(\pm\alpha, m, \delta)$  be any  $\kappa$ -admissible aligned box with  $0 < \delta \leq \delta_0(m, \alpha)$ . Let  $G_{\text{out}}$  be the Dirichlet outer factor on  $B^\circ$  and  $W := E/G_{\text{out}}$  the corresponding quotient. Let  $\tilde{D}_B(W)$  be the buffered boundary phase endpoint (Def. 12.19).

**FORCE (already proved).** If  $W$  has a zero in  $B_{\kappa/2}^\circ$  then  $\tilde{D}_B(W) \geq \pi/2$  (Lemma 12.23).

**ENVELOPE (missing).** To activate Theorem 12.6, it remains to prove a phase-class UE+split bound of the form

$$\tilde{D}_B(W) \leq C_{\text{up}} \delta^p \left( \text{Res}(m) + C_{\text{loc}} \delta^{-\theta} G(N_{\text{eff}}(m, \delta), \kappa) \right),$$

uniformly for all  $m \geq 10$ , all  $\alpha \in (0, 1]$ , and all  $\kappa$ -admissible  $0 < \delta \leq \delta_0(m, \alpha)$ , with  $C_{\text{up}}, C_{\text{loc}}$  independent of  $\delta$  and with an explicit growth model  $G(n, \kappa) \leq C_G \kappa^{-u} n^q$ .

**BUDGET GATE (non-negotiable).** The declared exponent tuple  $(p, \theta, q)$  must satisfy the acceptance gate:

$$2p \geq 1, \quad 2(p - \theta) \geq q, \quad p - \theta > 0,$$

otherwise uniform  $\eta$ -shrinking at  $\delta = \delta_0(m, \alpha)$  is impossible under constant-limited forcing.

**LOCAL REDESIGN (one of these is decisive).** Because phase is  $\delta$ -inert per zero (Lemma 12.17), the default choice  $N_{\text{eff}}(m, \delta) = N_{\text{loc}}(m) \sim \log m$  forces  $q = 1$ . Any one of the following would be decisive: (i) a phase-class UE mechanism achieving  $p \geq 1/2$  while keeping  $\theta = 0$  in the same endpoint class; (ii) a micro-window clustering bound  $N_{\text{eff}}(m, \delta_0) = O(1)$  (so  $q = 0$  effectively); (iii) a pair-isolation mechanism: only the forced pair contributes  $O(1)$  to  $\tilde{D}_B$  while the remainder contributes  $O(\delta \log m)$ .

**NO CLAIM POLICY.** Until the ENVELOPE bound and one decisive LOCAL redesign item are proved in-text (or cited referee-grade), the manuscript makes *no* claim of tail closure and *no* claim of RH from S5'.

**Lemma 12.1** (Endpoint-only NO-GO:  $\theta = 0$  per pole forbids any  $p > 0$  UE gain). *Fix  $\kappa \in (0, 1/10)$ . Let  $B = B(\alpha, m, \delta)$  be an aligned box and assume  $\kappa$ -admissibility so that  $\tilde{D}_B(W)$  is defined (Definition 12.19). Let  $E$  be holomorphic on a neighborhood of  $\bar{B}$  with outer factorization  $E = G_{\text{out}} W$  on  $B$ .*

*Assume  $\Phi_B$  is a boundary functional acting on the trace of  $E'/E$  and that there exist constants  $C_{\text{loc}} = C_{\text{loc}}(\kappa)$  and  $u \geq 0$  such that, for every  $\rho \in B_{\kappa/2}$  and the test function  $E_\rho(v) := v - \rho$ , one has the per-pole bound*

$$\Phi_B \left( \frac{E'_\rho}{E_\rho} \right) = \Phi_B \left( \frac{1}{v - \rho} \right) \leq C_{\text{loc}} \kappa^{-u},$$

*uniformly in  $\delta$  (this is the strong form of “ $\theta = 0$  per pole”).*

*Then there do not exist constants  $C_{\text{up}} > 0$  and  $p > 0$  (independent of  $\delta$ ) such that the phase-class UE inequality*

$$\tilde{D}_B(W) \leq C_{\text{up}} \delta^p \Phi_B \left( \frac{E'}{E} \right)$$

*holds for all such boxes  $B$  and all such holomorphic  $E$ . In fact, for  $E = E_\rho(v) = v - \rho$  with*

$\rho \in B_{\kappa/2}$ , the inequality forces

$$\frac{\pi}{2} \leq C_{\text{up}} C_{\text{loc}} \kappa^{-u} \delta^p,$$

which fails for sufficiently small  $\delta$ .

*Proof.* Fix  $\rho \in B_{\kappa/2}$  and take  $E(v) = E_\rho(v) = v - \rho$ . Then  $E$  has a zero in  $B_{\kappa/2}^\circ$ , hence its quotient  $W = E/G_{\text{out}}$  also has a zero there (since  $G_{\text{out}}$  is holomorphic and zero-free on  $B^\circ$ ). By Lemma 12.23 this implies  $\tilde{D}_B(W) \geq \pi/2$ . On the other hand  $E'/E = 1/(v - \rho)$ , so by hypothesis  $\Phi_B(E'/E) \leq C_{\text{loc}} \kappa^{-u}$  uniformly in  $\delta$ . Substituting into the claimed UE bound gives  $\pi/2 \leq C_{\text{up}} C_{\text{loc}} \kappa^{-u} \delta^p$ , contradicting  $p > 0$  as  $\delta \rightarrow 0$ .  $\square$

*Remark 12.2* (Consequence: endpoint-only  $\theta = 0$  cannot yield any  $p > 0$  gain). Any attempt to close the S5' envelope obligation by choosing an endpoint class  $\Phi_B$  that is  $\delta$ -inert on each local Cauchy kernel  $(v - \rho)^{-1}$  (“ $\theta = 0$  per pole”) cannot produce a  $\delta^p$  UE gain with  $p > 0$  via a purely local analytic argument: the test input  $E(v) = v - \rho$  contradicts such a bound (Lemma 12.1). Therefore any successful phase-class UE inequality with  $p \geq 1/2$  must incorporate additional structure that defeats the one-pole model (e.g. forcing redesign or pair-isolation/cancellation in the local factor).

**Lemma 12.3** (LOCAL isolation needed to beat the one-pole obstruction). *Fix  $\kappa \in (0, 1/10)$  and let  $\Phi_B$  be the endpoint class targeted in S5'. To obtain any UE gain  $p > 0$  with a local exponent  $\theta < p$  in the S5' budget theorem, it is necessary to prove a structural statement of the following form:*

*Whenever  $B = B(\alpha, m, \delta)$  is  $\kappa$ -admissible at the nominal scale  $\delta \leq \delta_0(m, \alpha)$  and  $Z_{\text{loc}}$  is the local factor of  $E$  on  $B$ , there exists a factorization*

$$Z_{\text{loc}} = Z_{\text{forced}} \cdot Z_{\text{rest}}$$

*such that (i)  $Z_{\text{forced}}$  contains only  $O(1)$  zeros (the “forced pair”), and (ii) in the chosen endpoint class one has a  $\delta$ -small bound*

$$\Phi_B\left(\frac{Z'_{\text{rest}}}{Z_{\text{rest}}}\right) \leq C \kappa^{-u} (\log m)^{q_{\text{eff}}} \delta^{-(\theta_{\text{eff}})} \quad \text{with} \quad \theta_{\text{eff}} < p$$

*(and ideally  $\theta_{\text{eff}} = 0$  with  $q_{\text{eff}} = 0$  or with an extra  $\delta$  factor).*

*Remark 12.4* (S5 acceptance criterion (budget calculus; no drift)). Any proposed S5 redesign must specify a boundary functional  $\Phi_B$  (shape-invariant;  $\delta$ -normalized) and prove two explicit inequalities uniformly for all  $m \geq 10$ , all  $\alpha \in (0, 1]$ , and all  $\kappa$ -admissible  $0 < \delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ :

1. **(S5-UE)** a forceable upper-envelope bound

$$D_B(W) \leq C_{\text{up}} \delta^p \Phi_B(E'/E)$$

with an explicit exponent  $p > 0$  and  $\delta$ -uniform constant  $C_{\text{up}}$ ;

2. **(S5-LOC)** a local/collar bound in the same endpoint class

$$\Phi_B(Z'_{\text{loc}}/Z_{\text{loc}}) \leq C_{\text{loc}} \delta^{-\theta} G(N_{\text{loc}}(m), \kappa)$$

with explicit  $\theta \geq 0$  and an explicit growth model for  $G$  (e.g.  $G(n, \kappa) \ll \kappa^{-u} n^q$ ).

The redesign is budget-viable for uniform  $\eta$ -shrinking closure under  $\delta_0$  only if the S5 Budget Theorem yields  $2(p-\theta) \geq q$  (and  $p-\theta > 0$  for shrinkability). If  $p-\theta < 0$ , the standard  $\kappa$ -admissible shrink policy is unsafe (shrinking  $\delta$  can increase the envelope term) and must be redesigned.

Finally, the forcing chain remains phrased in terms of  $D_B(W)$ ; therefore S5 must include either  $\Phi_B \geq D_B(W)$  on all admissible boxes or a new forcing lemma that lower-bounds  $\Phi_B$  directly (Remark 12.10).

**Theorem 12.5** (S5 Budget Theorem (general endpoint functional)). *Fix  $\eta \in (0, 1]$  and  $\kappa \in (0, 1/10)$  and define the nominal scale  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ . Let  $\Phi_B$  be a boundary functional and assume that for every  $m \geq 10$ ,  $\alpha \in (0, 1]$ , and every  $\kappa$ -admissible  $0 < \delta \leq \delta_0(m, \alpha)$  one has:*

- (i) **(S5-UE)**  $D_B(W) \leq C_{\text{up}} \delta^p \Phi_B(E'/E)$  for some  $p > 0$  and  $\delta$ -uniform constant  $C_{\text{up}}$ ;
- (ii) **(S5-SPLIT)**  $\Phi_B(E'/E) \leq \text{Res}(m) + \Phi_B(Z'_{\text{loc}}/Z_{\text{loc}})$ ;
- (iii) **(S5-LOC)**  $\Phi_B(Z'_{\text{loc}}/Z_{\text{loc}}) \leq C_{\text{loc}} \delta^{-\theta} G(N_{\text{loc}}(m), \kappa)$  for some  $\theta \geq 0$  and  $\delta$ -uniform  $C_{\text{loc}}$ .

Assume moreover that  $N_{\text{loc}}(m) \leq A_N \log m + B_N$  and  $\text{Res}(m) \leq A_L(\log m)^{r_L} + B_L$  for absolute constants, and that for some  $q, u \geq 0$  one has the growth model

$$G(n, \kappa) \leq C_G \kappa^{-u} n^q \quad (n \geq 1),$$

with  $C_G$  independent of  $(m, \alpha, \delta)$ .

Then at the nominal choice  $\delta = \delta_0(m, \alpha)$ ,

$$D_B(W) \leq C_{\text{up}} \left( \delta_0^p \text{Res}(m) + C_{\text{loc}} \delta_0^{p-\theta} G(N_{\text{loc}}(m), \kappa) \right). \quad (15)$$

Furthermore:

1. **(Uniformity in  $m$ )** The local contribution in (15) is uniformly bounded in  $m \geq 10$  only if

$$2(p-\theta) \geq q. \quad (16)$$

2. **( $\eta$ -shrinking)** If (16) holds and  $p-\theta > 0$ , then

$$\sup_{m \geq 10} \delta_0(m, 1)^{p-\theta} G(N_{\text{loc}}(m), \kappa) = O(\eta^{p-\theta}),$$

so the local penalty can be made arbitrarily small by choosing  $\eta$  sufficiently small.

3. **( $\delta$ -shrink monotonicity)** If  $p \geq 0$  and  $p-\theta \geq 0$ , then the right-hand side of (15) is non-decreasing in  $\delta$  (for fixed  $m, \alpha$ ); hence replacing  $\delta_0$  by a smaller  $\kappa$ -admissible  $\delta \leq \delta_0$  can only improve the inequality. If  $p-\theta < 0$ ,  $\kappa$ -shrinking can worsen the envelope term.

*Proof.* Combine (S5-UE) with (S5-SPLIT) and (S5-LOC) to obtain

$$D_B(W) \leq C_{\text{up}} \delta^p \text{Res}(m) + C_{\text{up}} C_{\text{loc}} \delta^{p-\theta} G(N_{\text{loc}}(m), \kappa).$$

Set  $\delta = \delta_0(m, \alpha)$  to obtain (15).

For the local contribution at  $\alpha = 1$  use  $\delta_0 = \eta/(\log m)^2$ , the growth model  $G(n, \kappa) \leq C_G \kappa^{-u} n^q$ , and  $N_{\text{loc}}(m) \ll \log m$  to get

$$\delta_0^{p-\theta} G(N_{\text{loc}}(m), \kappa) \ll \kappa^{-u} \eta^{p-\theta} (\log m)^{-2(p-\theta)} (\log m)^q = \kappa^{-u} \eta^{p-\theta} (\log m)^{q-2(p-\theta)}.$$

This is uniformly bounded in  $m$  only if  $q - 2(p-\theta) \leq 0$ , i.e. (16). If additionally  $p-\theta > 0$ , the factor  $\eta^{p-\theta}$  yields  $\eta$ -shrinking.

Finally, the monotonicity claim follows because  $\delta \mapsto \delta^p$  and  $\delta \mapsto \delta^{p-\theta}$  are nondecreasing on  $(0, \infty)$  exactly when  $p \geq 0$  and  $p-\theta \geq 0$ .  $\square$

**Theorem 12.6** (S5' closure from a forceable phase endpoint). *Fix  $\kappa \in (0, 1/10)$  and  $\eta \in (0, 1)$  and define  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ . Let  $\tilde{D}_B$  be a boundary phase endpoint functional assigned to each  $\kappa$ -admissible aligned box  $B = B(\pm a, m, \delta)$  and its boundary quotient  $W$ . Assume:*

*(S5'-FORCE) Under an off-axis quartet at height  $m$  with displacement  $a > 0$ , there exists an aligned  $\kappa$ -admissible box  $B$  (with  $\alpha \approx a$ ) such that  $\tilde{D}_B(W) \geq c_{\text{force}} - \delta \Xi(m)$  with  $c_{\text{force}} > 0$  absolute and  $\Xi(m) \geq 0$  explicit.*

*(S5'-UE+SPLIT) For every  $\kappa$ -admissible aligned box,*

$$\tilde{D}_B(W) \leq C_{\text{up}} \delta^p \left( \text{Res}(m) + C_{\text{loc}} \delta^{-\theta} G(N_{\text{loc}}(m), \kappa) \right),$$

*where  $p > 0$ ,  $\theta \geq 0$ , and  $C_{\text{up}}, C_{\text{loc}}$  are  $\delta$ -uniform, and  $G(n, \kappa) \leq C_G \kappa^{-u} n^q$  for fixed  $u, q \geq 0$ .*

*Suppose additionally that  $2p \geq 1$ ,  $2(p - \theta) \geq q$ , and  $p - \theta > 0$ . Then there exists  $\eta_* \in (0, 1)$  (depending on the displayed constants and  $\kappa$ ) such that for every  $\eta \in (0, \eta_*]$  the S5' tail inequality holds at  $\delta = \delta_0(m, \alpha)$  for all  $m \geq 10$  and all  $\alpha \in (0, 1]$ , and hence no off-axis quartet exists at any height  $m \geq 10$ . Combined with any finite-height front-end, this implies RH.*

**Remark 12.7** (S5' acceptance gate for phase endpoints (no drift)). Any proposed S5' endpoint built from boundary phase data (e.g.  $\Delta \arg$  or an integral of  $\Im(\log\text{-derivative})$ ) must declare its exponent budget data  $(p, \theta, q)$  in the schematic bound

$$\tilde{D}_B(W) \leq C_{\text{up}} \delta^p \left( \text{Res}(m) + C_{\text{loc}} \delta^{-\theta} G(N_{\text{loc}}(m), \kappa) \right),$$

and must satisfy the uniformity/shrink conditions of Theorem 12.6:  $2p \geq 1$ ,  $2(p - \theta) \geq q$ , and  $p - \theta > 0$ . Pure  $\Delta \arg$  endpoints have  $p = 0$  and are rejected. Any phase endpoint whose proof reduces to an absolute  $L^r(\partial B)$  estimate for  $E'/E$  is also rejected by Lemma 12.26 and Proposition 12.27.

At fixed  $(m, \alpha)$  the tail inequality (14) is a strict forcing-vs-envelope condition. In v38 (inherited from v36) the combination of Theorem 10.13, Lemma 10.15, and Lemma 8.2 formally rules out the former “ $\eta$ -absorption” closure route based on the pointwise/sup endpoint  $\sup_{\partial B} |E'/E|$  together with the pointwise collar bound.

**What must change.** The forcing chain produces a lower bound for the *dial deviation*

$$D_B(W) := \sum_{\pm} |W(v_{\pm}) - e^{i\varphi_0^{\pm}}|$$

appearing in Lemma 10.3. In the current architecture this deviation is upper-bounded by a pointwise endpoint  $\delta \sup_{\partial B} |E'/E|$ , which (via the collar) introduces the sharp  $\delta^{-1}$  blow-up. To obtain a tail closure mechanism one must redesign the envelope endpoint and/or the local interface so that the exponent budget  $p - \theta \geq \frac{1}{2}$  is met *uniformly in  $m$* .

**Remark 12.8** (Forcing compatibility for redesigned endpoints). The existing forcing chain lower-bounds  $D_B(W)$  (via the pair-factor phase rotation) by a fixed constant  $c$  up to  $\delta$ -small corrections. If one proposes a redesigned envelope endpoint  $\Phi_B$  (non-pointwise, e.g. an  $L^2$  or energy functional), then the current forcing lower bound is useful only if it implies a corresponding lower bound for  $\Phi_B$ . A sufficient (and simplest) compatibility condition is:

$$\Phi_B \geq D_B(W) \quad \text{for all admissible boxes and quotients } W,$$

so that the forcing lower bound propagates unchanged. If this domination fails, then a *new forcing lemma* must be proved that lower-bounds  $\Phi_B$  directly.

**Lemma 12.9** (Forceability transfer by domination). *Let  $B$  be a  $\kappa$ -admissible aligned box and  $W$  the associated boundary quotient. Suppose a boundary endpoint functional  $\Phi_B$  satisfies*

$$\Phi_B \geq D_B(W) \quad \text{for all admissible } (B, W).$$

*Then the existing single-box forcing lower bound for  $D_B(W)$  implies the same forcing lower bound for  $\Phi_B$  with no change in the forcing constants.*

**Remark 12.10** (Forceability gate for S5 endpoints (NO-GO unless met)). The current forcing architecture (Section 8) forces only the dial deviation  $D_B(W)$  by an  $O(1)$  constant up to  $\delta$ -small deductions (Lemma 8.2). Consequently, any S5 redesign that replaces  $D_B(W)$  by a different endpoint  $\tilde{D}_B$  (or  $\Phi_B$ ) is *invalid* unless it proves either:

- (i)  $\tilde{D}_B \geq D_B(W)$  for all admissible boxes/quotients (domination transfer), or
- (ii) a new forcing lemma that lower-bounds  $\tilde{D}_B$  directly under an off-axis quartet.

Without (i) or (ii), the forcing half of the tail inequality becomes logically disconnected from the envelope half.

## 12.1 S5' phase endpoints: winding / argument-increment toolkit

**Definition 12.11** (Phase increment on a boundary arc). Let  $\Gamma \subset \mathbb{C}$  be a piecewise  $C^1$  oriented curve and let  $f$  be holomorphic on an open neighborhood of  $\Gamma$  with  $f(v) \neq 0$  for all  $v \in \Gamma$ . Define the phase increment of  $f$  along  $\Gamma$  by

$$\Delta_\Gamma \arg f := \Im \int_\Gamma \frac{f'(v)}{f(v)} dv.$$

(Equivalently,  $\Delta_\Gamma \arg f$  is the total change of a continuous branch of  $\arg f$  along  $\Gamma$ .)

**Lemma 12.12** (Phase increment identity and vertical specialization). *Under the hypotheses of Definition 12.11, the phase increment is additive under concatenation of curves and satisfies:*

1. *If  $\Gamma = \Gamma_1 \cup \Gamma_2$  (oriented concatenation), then  $\Delta_\Gamma \arg f = \Delta_{\Gamma_1} \arg f + \Delta_{\Gamma_2} \arg f$ .*
2. *If  $\Gamma$  is the vertical segment  $I_+ := \{\alpha + iy : |y - m| \leq \delta\}$  oriented upward, then*

$$\Delta_{I_+} \arg f = \Im \int_{m-\delta}^{m+\delta} \frac{f'(\alpha + iy)}{f(\alpha + iy)} i dy = \int_{m-\delta}^{m+\delta} \Re \left( \frac{f'(\alpha + iy)}{f(\alpha + iy)} \right) dy.$$

**Remark 12.13** (Parentheses hygiene for phase endpoints). For non-horizontal arcs, one must distinguish

$$\Im \int_\Gamma \frac{f'}{f} dv \quad \text{from} \quad \int_\Gamma \Im \left( \frac{f'}{f} \right) dv.$$

Only the former is a phase increment. This distinction is essential on vertical segments where  $dv = i dy$ .

**Definition 12.14** (Shifted near-vertical segment). Let  $B = B(\alpha, m, \delta)$  be an aligned box and let  $\lambda \in (0, 1)$ . Define the shifted segment

$$I_{+, \lambda} := \{\alpha + \lambda\delta + iy : |y - m| \leq \delta\},$$

oriented upward. (*This lies strictly inside  $B$ . It is separated from the unshifted vertical line  $I_+ = \{\alpha + iy : |y - m| \leq \delta\}$  by horizontal distance  $\lambda\delta$ , and its distance to  $\partial B$  is at least  $(1 - \lambda)\delta$ .)*

**Lemma 12.15** (Phase split on  $I_{+, \lambda}$ ). *Let  $B = B(\alpha, m, \delta)$  be  $\kappa$ -admissible and aligned, and let  $I_{+, \lambda}$  be as in Definition 12.14. Assume  $E$ ,  $Z_{\text{loc}}$  and  $F := E/Z_{\text{loc}}$  are holomorphic and nonvanishing on an open neighborhood of  $I_{+, \lambda}$ . Then*

$$\Delta_{I_{+, \lambda}} \arg E = \Delta_{I_{+, \lambda}} \arg F + \Delta_{I_{+, \lambda}} \arg Z_{\text{loc}}.$$

Moreover,

$$|\Delta_{I_{+, \lambda}} \arg F| \leq 2\delta \sup_{v \in I_{+, \lambda}} \left| \frac{F'(v)}{F(v)} \right| \leq 2\delta \sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right|.$$

**Corollary 12.16** (Residual phase budget ( $\delta$ -uniform)). *Assume the residual envelope bound of Lemma 7.2, i.e.  $\sup_{\partial B} |F'/F| \leq C_1 \log m + C_2$  on every  $\kappa$ -admissible aligned box. Then, for every  $\lambda \in (0, 1)$ ,*

$$|\Delta_{I_{+, \lambda}} \arg F| \leq 2\delta (C_1 \log m + C_2).$$

**Lemma 12.17** (Local phase is  $\delta$ -inert on line segments (per-zero contribution is  $O(1)$ )). *Let  $S \subset \mathbb{C}$  be any oriented line segment and let  $\rho \notin S$ . Choose a continuous branch of  $\arg(v - \rho)$  along  $S$ . Then*

$$\left| \Im \int_S \frac{dv}{v - \rho} \right| = |\arg(v_1 - \rho) - \arg(v_0 - \rho)| \leq \pi,$$

where  $v_0, v_1$  are the endpoints of  $S$ . Consequently, writing  $Z_{\text{loc}}(v) = \prod_{\rho \in Z_{\text{loc}}(m)} (v - \rho)^{m_\rho}$ , for any segment  $S$  avoiding  $Z_{\text{loc}}(m)$  one has

$$\left| \Im \int_S \frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)} dv \right| = |\Delta_S \arg Z_{\text{loc}}| \leq \pi N_{\text{loc}}(m).$$

**Corollary 12.18** (Prototype phase upper bound (residual + local)). *Under the hypotheses of Lemma 12.15 and Corollary 12.16,*

$$|\Delta_{I_{+, \lambda}} \arg E| \leq 2\delta (C_1 \log m + C_2) + |\Delta_{I_{+, \lambda}} \arg Z_{\text{loc}}|.$$

*In particular, the residual contribution is  $O(\delta \log m)$  while the local contribution is  $\delta$ -inert in the phase class.*

**Definition 12.19** (Buffered boundary phase endpoint). *Let  $B = B(\alpha, m, \delta)$  be an aligned box and assume  $\kappa$ -admissibility:  $\text{dist}(\partial B, Z(E)) \geq \kappa\delta$ . Let  $G_{\text{out}}$  be the Dirichlet outer factor on  $B^\circ$  and  $W := E/G_{\text{out}}$  the inner quotient. Define the buffered inner rectangle*

$$B_{\kappa/2} := \{v \in B : \text{dist}(v, \partial B) \geq \frac{\kappa\delta}{2}\},$$

and write its oriented boundary as  $\partial B_{\kappa/2} = \bigcup_{j=1}^4 S_j$  (counterclockwise). Define the sidewise phase increment

$$\Delta_{S_j} \arg W := \Im \int_{S_j} \frac{W'(v)}{W(v)} dv,$$

and the phase endpoint

$$\tilde{D}_B(W) := \max_{1 \leq j \leq 4} |\Delta_{S_j} \arg W|.$$

**Lemma 12.20** (Collar nonvanishing for buffered phase endpoints). *Let  $B = B(\alpha, m, \delta)$  be  $\kappa$ -admissible:  $\text{dist}(\partial B, Z(E)) \geq \kappa\delta$ . Let  $B_{\kappa/2}$  be the buffered inner rectangle from Definition 12.19. Then*

$$\text{dist}(\partial B_{\kappa/2}, Z(E)) \geq \frac{\kappa\delta}{2}.$$

In particular, if  $G_{\text{out}}$  is the Dirichlet outer factor on  $B^\circ$  and  $W = E/G_{\text{out}}$ , then both  $G_{\text{out}}$  and  $W$  are holomorphic and nonvanishing on an open neighborhood of  $\partial B_{\kappa/2}$ , so the phase increments  $\Delta_{S_j} \arg W$  are well-defined (no branch crossing).

**Corollary 12.21** (Local term on the buffered boundary phase endpoint class). *Assume the hypotheses of Definition 12.19 and write  $\partial B_{\kappa/2} = \bigcup_{j=1}^4 S_j$ . Then the local factor satisfies*

$$\max_{1 \leq j \leq 4} \left| \Im \int_{S_j} \frac{Z'_{\text{loc}}(v)}{Z_{\text{loc}}(v)} dv \right| \leq \pi N_{\text{loc}}(m).$$

Equivalently,

$$\max_{1 \leq j \leq 4} |\Delta_{S_j} \arg Z_{\text{loc}}| \leq \pi N_{\text{loc}}(m).$$

**Lemma 12.22** (Refined per-zero phase bound by horizontal separation). *Let  $S \subset \mathbb{C}$  be a line segment of length  $|S|$  and let  $\rho \notin S$ . Write  $d := \text{dist}(\rho, S)$ . Then*

$$\left| \Im \int_S \frac{dv}{v - \rho} \right| \leq \min \left\{ \pi, \frac{|S|}{d} \right\}.$$

**Lemma 12.23** (Phase forcing from an interior zero). *Assume the setup of Definition 12.19. If  $W$  has at least one zero in  $B_{\kappa/2}^\circ$  (equivalently  $E$  has at least one zero there), then*

$$\tilde{D}_B(W) \geq \frac{\pi}{2}.$$

*Proof.* Since  $W$  is holomorphic and nonvanishing on a neighborhood of  $\partial B_{\kappa/2}$ , the argument principle gives

$$\oint_{\partial B_{\kappa/2}} \frac{W'(v)}{W(v)} dv = 2\pi i N,$$

where  $N \geq 1$  is the number of zeros of  $W$  in  $B_{\kappa/2}^\circ$ , counted with multiplicity. Taking imaginary parts and decomposing  $\partial B_{\kappa/2}$  into four sides yields

$$\sum_{j=1}^4 \Delta_{S_j} \arg W = 2\pi N.$$

Hence

$$\tilde{D}_B(W) \geq \frac{1}{4} \left| \sum_{j=1}^4 \Delta_{S_j} \arg W \right| = \frac{\pi N}{2} \geq \frac{\pi}{2}.$$

□

**Corollary 12.24** (Forcing hypothesis is automatic for the buffered phase endpoint). *Let  $\tilde{D}_B$  be the buffered boundary phase endpoint of Definition 12.19. Then whenever  $W$  has a zero in  $B_{\kappa/2}^\circ$  one has  $\tilde{D}_B(W) \geq \pi/2$ . In particular, in Theorem 12.6 the forcing condition (S5'–FORCE) may be taken with  $c_{\text{force}} = \pi/2$  and  $\Xi(m) \equiv 0$  whenever the contradiction endpoint is  $\tilde{D}_B$ .*

*Proof.* This is exactly Lemma 12.23. □



*Remark 12.25* (Forceability gate for phase endpoints). The single-box forcing chain in this manuscript supplies a lower bound only for a *forced endpoint*. For the buffered phase endpoint  $\tilde{D}_B$  this lower bound is provided by Lemma 12.23 (and recorded as a hypothesis-discharge in Corollary 12.24). Consequently, any proposed S5' contradiction endpoint  $\Phi_B$  is admissible only if either:

1.  $\Phi_B(W) \geq \tilde{D}_B(W)$  on every  $\kappa$ -admissible aligned box (forcing transfers), or
2. a new forcing lemma is proved that lower-bounds  $\Phi_B(W)$  directly under an interior zero/off-axis quartet.

Without such a link, forcing and envelope are logically disconnected.

## 12.2 Baseline NO-GO results for naive non-pointwise endpoints

The S5 goal is to replace the pointwise/sup endpoint in Lemma 10.3 by a non-pointwise functional that still controls the same dial deviation  $D_B(W)$  appearing in the forcing chain. The next two results prevent drift into two large endpoint classes that cannot work under the present  $D_B(W)$  target and the v36 local split/collar interface (unchanged from v35).

**Lemma 12.26** (Absolute  $L^r$  endpoint scaling collapse). *Let  $B = B(\pm a, m, \delta)$  be an aligned box and let  $G_{\text{out}}$  and  $W = E/G_{\text{out}}$  be as in Lemma 10.3. Assume boundary contact so that  $W$  has unimodular boundary values a.e. Fix  $r \in [1, \infty]$  and write  $L^r(\partial B)$  for  $L^r(\partial B, ds)$ . Then there exists a shape-only constant  $C_r > 0$  (depending only on the normalized square  $Q = [-1, 1]^2$ ) such that for each sign  $\pm$ ,*

$$|W(v_{\pm}) - e^{i\varphi_0^{\pm}}| \leq C_r \delta^{1-1/r} \left\| \frac{E'}{E} \right\|_{L^r(\partial B)}. \quad (17)$$

*In particular, any upper-envelope mechanism whose endpoint is an absolute  $L^r(\partial B)$  norm of  $E'/E$  cannot have a  $\delta$ -prefactor exponent exceeding  $p(r) = 1 - 1/r$  within this endpoint class.*

*Proof.* Repeat the proof of Lemma 10.3 with  $L^2$  replaced by  $L^r$  throughout. Evaluation from the boundary gives  $|W(v_{\pm}) - c| \leq \|P_B(v_{\pm}, \cdot)\|_{L^q} \|W - c\|_{L^r}$  for  $1/r + 1/q = 1$ , and under affine rescaling  $\|P_B\|_{L^q} \asymp \delta^{-1/r}$ . Boundary Poincaré in  $L^r$  yields  $\|W - c\|_{L^r} \leq C'_r \delta \|\partial_s W\|_{L^r}$  with a shape-only constant  $C'_r$ , and outer factor control bounds  $\|\partial_s W\|_{L^r}$  by a shape-only constant times  $\|E'/E\|_{L^r}$ . Choosing  $c = e^{i\varphi_0^{\pm}}$  gives (17), with overall factor  $\delta^{-1/r} \cdot \delta = \delta^{1-1/r}$ .  $\square$

**Proposition 12.27** (NO-GO: absolute  $L^r$  log-derivative endpoints cannot clear the UE-Gate). *Assume in addition that  $B$  is  $\kappa$ -admissible and hence the pointwise collar bound holds:  $\sup_{\partial B} |Z'_{\text{loc}}/Z_{\text{loc}}| \leq N_{\text{loc}}(m)/(\kappa\delta)$  (Lemma 10.8). Then for every  $r \in [1, \infty]$ ,*

$$\delta^{1-1/r} \left\| \frac{Z'_{\text{loc}}}{Z_{\text{loc}}} \right\|_{L^r(\partial B)} \leq 8^{1/r} \frac{N_{\text{loc}}(m)}{\kappa},$$

*independent of  $\delta$ . In particular, under the nominal scale  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$  and the unconditional majorant  $N_{\text{loc}}(m) \ll \log m$ , uniform  $\eta$ -shrinking cannot suppress the local term within any envelope mechanism whose endpoint is an absolute  $L^r(\partial B)$  norm of  $E'/E$ .*

*Proof.* Use  $|\partial B| = 8\delta$  and  $\|f\|_{L^r} \leq |\partial B|^{1/r} \|f\|_{L^\infty}$  to get

$$\left\| \frac{Z'_{\text{loc}}}{Z_{\text{loc}}} \right\|_{L^r(\partial B)} \leq (8\delta)^{1/r} \cdot \frac{N_{\text{loc}}(m)}{\kappa\delta} = 8^{1/r} \frac{N_{\text{loc}}(m)}{\kappa \delta^{1-1/r}}.$$

Multiply by  $\delta^{1-1/r}$ .  $\square$

*Remark 12.28* (Implication for S5 endpoint design). Lemmas 12.26–12.27 rule out the entire family of S5 proposals that attempt to replace  $\sup_{\partial B} |E'/E|$  by an *absolute*  $L^r(\partial B)$  norm of  $E'/E$  while keeping the same  $\kappa$ -collar local control. Any viable S5 redesign must instead (i) exploit cancellation (argument-principle style *signed* endpoints) and/or (ii) move to a less singular boundary object (e.g. endpoints built from  $\log |E|$  / BMO-type control).

**Lemma 12.29** (NO–GO: local-kernel projection endpoints cannot control  $D_B(W)$  without a new forcing link). *Fix an aligned box  $B$  and consider an endpoint functional of the form*

$$\Phi_B(E) := \|(I - \Pi_B)(E'/E)\|_{X(\partial B)}$$

*for some normed boundary space  $X(\partial B)$  and a bounded projection  $\Pi_B$  satisfying  $\Pi_B(Z'_{\text{loc}}/Z_{\text{loc}}) = Z'_{\text{loc}}/Z_{\text{loc}}$  whenever  $Z_{\text{loc}}$  is the local factor associated to  $B$  (so that the local term is annihilated under the split  $E'/E = F'/F + Z'_{\text{loc}}/Z_{\text{loc}}$ ). Then there is no universal inequality of the form*

$$D_B(W) \leq C \delta^p \Phi_B(E)$$

*(valid for all forcing-aligned boxes under the boundary-contact convention), for any fixed  $p > 0$  and constant  $C$ , unless one supplies a new forcing link that lower-bounds  $\Phi_B$  directly under an off-axis quartet.*

*Proof.* This is a structural counterexample: in the class of holomorphic functions  $E$  obeying the boundary-contact convention, take  $E = Z_{\text{loc}}$  on a box for which  $Z_{\text{loc}}$  has a zero at one of the dial points  $v_{\pm}$ . Then  $F \equiv 1$  and  $E'/E = Z'_{\text{loc}}/Z_{\text{loc}}$ , so by assumption  $(I - \Pi_B)(E'/E) = 0$  and hence  $\Phi_B(E) = 0$ . However  $G_{\text{out}}$  is zero-free, so  $W = E/G_{\text{out}}$  shares the same interior zeros as  $E$  and  $W(v_{\pm}) = 0$  for at least one sign, giving  $D_B(W) \geq 1$ . Thus no inequality  $D_B(W) \leq C \delta^p \Phi_B(E)$  can hold from these hypotheses alone; any attempt to use such an endpoint must replace  $D_B(W)$  as the forced object and provide a forcing-transfer lemma (Remark 12.10).  $\square$

*Remark 12.30* (Consequence for S5 searches). Lemmas 12.27 and 12.29 close two broad endpoint classes: (i) absolute  $L^r(\partial B)$  norms of  $E'/E$  (including  $L^2$ ) under the current collar interface, and (ii) endpoints that annihilate the local kernel span while still targeting the forced dial deviation  $D_B(W)$ . Any viable S5 redesign must introduce a genuinely new local-interface input and/or a new forcing-compatible endpoint.

**S5 design targets (open).** A future closure route (S5) should provide a non-pointwise endpoint  $\Phi_B$  and a UE-type inequality of the schematic form

$$D_B(W) \leq C_{\text{up}} \delta^p \Phi_B(E) \quad (p > 0),$$

together with a local/residual split of  $\Phi_B(E)$  whose local contribution scales as  $\delta^{-\theta}$  with  $\theta < p - \frac{1}{2}$ , or more generally satisfies the exponent budget of Theorem 10.13. The point is *not* to recover the specific exponent  $\frac{3}{2}$  from older drafts, but to obtain any effective gain  $p - \theta > \frac{1}{2}$  with proof-grade uniformity.

*Remark 12.31* (Recorded open lemmas (S5 checklist)). A proof-grade S5 implementation would minimally require:

1. **(S5–UE)** a redesigned upper-envelope inequality with a forceable endpoint  $\Phi_B$ ;
2. **(S5–RES)** a  $\delta$ -uniform residual envelope bound in the same endpoint class;
3. **(S5–LOC)** a collar/local bound in the same endpoint class that avoids the pointwise  $\delta^{-1}$  blow-up;
4. **(S5–FORCE)** either  $\Phi_B \geq D_B(W)$  or a new forcing lemma as in Remark 12.8.

## 13 Global RH from a finite front-end + the tail criterion family

**Theorem 13.1** (Global closure (criterion-first logical form)). *Assume:*

1. (Front-end) All nontrivial zeros with  $0 < \text{Im}(s) \leq 5$  lie on the critical line.
2. (Tail criterion) Fix some  $\eta \in (0, 1)$  and  $\kappa \in (0, 1/10)$ , and assume the analytic inputs Lemmas 10.3–10.10 and Lemma 10.17 with finite constants. Assume moreover that for every  $m \geq 10$  and every  $\alpha \in (0, 1]$  there exists a  $\kappa$ -admissible scale  $0 < \delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$  such that the strict tail inequality (14) holds.

Then all nontrivial zeros of  $\zeta(s)$  lie on the critical line.

*Proof.* For each  $m \geq 10$ , Theorem 11.1 turns the strict inequality (14) into exclusion of off-axis quartets at height  $m$ . By the tail criterion hypothesis, no off-axis quartets exist at any height  $m \geq 10$ . By the front-end hypothesis, there are no off-axis zeros below height 5. Hence there are no off-axis zeros at any height, so every nontrivial zero lies on the critical line.  $\square$

*Remark 13.2* (Role of computations and the repro pack (v38)). Appendix C provides a small interval-arithmetic harness that evaluates the tail inequality for pinned parameters and a pinned constant ledger. In v36 this is used only for audit purposes (e.g. exponent tracking), not as a proof substitute.

## A Discarded closure routes (as of v38)

This appendix records closure routes that were explored in earlier iterations (v32–v34) but are now ruled out *under the currently proved inputs*. The purpose is to prevent future drift: these routes should not be re-opened unless a genuinely new analytic input (e.g. an S5 endpoint redesign) is supplied.

### A.1 D1: Pointwise UE endpoint $\sup_{\partial B} |E'/E|$ + collar + $\eta$ -absorption (S1/S1')

The former absorption narrative attempted to close the tail family by shrinking  $\eta$  in the nominal scale  $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ . In the pointwise/sup architecture the UE step has exponent  $p = 1$  (Lemma 10.3) and the collar/local split has exponent  $\theta = 1$  (Lemma 10.8), so the local contribution is  $\delta$ -inert and cannot be suppressed by  $\eta$  (Lemma 10.14). More strongly, the exponent budget (Theorem 10.13) shows that uniform  $\eta$ -shrinking requires  $p - \theta \geq \frac{1}{2}$ , while the scaling NO-GO (Lemma 10.15) forbids any  $p > 1$  within this endpoint class. Finally, the forcing margin is constant-limited in the single-box architecture (Lemma 8.2), so one cannot compensate by “making forcing grow with  $m$ ”.

**Proposition A.1** (Historical record: formal anchor absorption under a hypothetical strengthened UE exponent). *This proposition is not used in v36. It is recorded only to document the logical shape of the discarded absorption idea.*

*Assume that, for some  $p > 1$ , an upper-envelope step admits the strengthened form*

$$D_B(W) \leq 2C_{\text{up}} \delta^p \sup_{\partial B} \left| \frac{E'}{E} \right|$$

*with the same constant ledger, and that all other constants in (14) are finite. Fix an anchor height  $m_\star \geq 10$  and evaluate (14) at  $(m, \alpha) = (m_\star, 1)$  with the nominal scale  $\delta_0(m_\star, 1) = \eta/(\log m_\star)^2$ . Then there exists  $\eta_\star(m_\star, p) > 0$  such that (14) holds at  $(m_\star, 1)$  for every  $\eta \in (0, \eta_\star]$ .*

Warning: *within the pointwise/sup endpoint class, Lemma 10.15 forbids any  $p > 1$ , so this proposition cannot be invoked without an S5 redesign.*

*Proof.* Under a strengthened exponent  $p > 1$ , the envelope side becomes  $A\eta^p + B\eta^{p-1}$  for finite constants  $A, B$  depending on  $(m_*, p)$  and the constant ledger, while the forcing side equals  $c - D\eta$  for a finite  $D$ . Since  $p > 1$ , one has  $\eta^p \rightarrow 0$ ,  $\eta^{p-1} \rightarrow 0$ , and  $\eta \rightarrow 0$  as  $\eta \downarrow 0$ , so the strict inequality holds for all sufficiently small  $\eta$ .  $\square$

## A.2 D2: Shrinking the local window / short-interval zero counts

A tempting workaround is to replace the fixed local window  $|\gamma - t| \leq 1$  in the residual/collar interface by a shrinking window  $|\gamma - t| \leq \delta^\beta$  to reduce the local term. However, without additional analytic input, available RH-free methods control  $N(t+1) - N(t-1)$  at unit scale and do *not* provide a proof-grade bound for  $N(t+\delta^\beta) - N(t-\delta^\beta)$  as  $\delta \downarrow 0$ . Thus v36 does not pursue window-shrinking as a substitute for the missing UE gain.

## A.3 D3: “Make forcing grow with $m$ ” within single-box forcing

Because  $\Delta_{I_+} \arg Z_{\text{pair}} \leq 2\pi$  uniformly (Lemma 8.2), the forcing constant  $c$  in the tail inequality is  $O(1)$ . Any attempt to obtain a forcing side that grows like  $\log m$  (or any unbounded function of  $m$ ) would require a different forcing architecture (not the v36 single-box forcing chain).

## A.4 D4: “Boundary modulus implies interior zero-freeness” converse

Under boundary-contact, the quotient  $W = E/G_{\text{out}}$  satisfies  $|W| = 1$  on  $\partial B$  (Remark 9.3), but this has no converse implication toward zero-freeness or constancy (Remark 9.4). Therefore, any closure route that implicitly treats  $|W| = 1$  as “almost zero-free” is invalid.

## A.5 D5: Absolute $L^r$ log-derivative endpoints (NO-GO)

Replacing the pointwise endpoint  $\sup_{\partial B} |E'/E|$  by an *absolute* boundary  $L^r(\partial B)$  norm of  $E'/E$  does not improve the exponent budget: Lemma 12.26 forces the UE prefactor exponent to be  $p(r) = 1 - 1/r$ , while Proposition 12.27 shows the local/collar contribution has the same exponent  $\theta(r) = 1 - 1/r$ , hence  $p(r) - \theta(r) = 0$  and the local leakage is  $\delta$ -inert.

## A.6 D6: Projecting out the local kernel span (NO-GO)

A tempting idea is to define an endpoint by projecting  $E'/E$  off the span of local Cauchy kernels so that the local term vanishes. Lemma 12.29 shows this cannot control the forced dial deviation  $D_B(W)$  without changing the contradiction endpoint or supplying a new forcing link.

*Supporting documentation for D6 (not a viable endpoint under current forcing).* The next definition and lemmas formalize the projection setup and the exact cancellation of the local term. They are recorded only to document the mechanism behind the NO-GO.

**Definition A.2** (Local Cauchy subspace and  $L^2$  projection (supporting documentation)). Let  $B = B(\alpha, m, \delta)$  be  $\kappa$ -admissible and let  $Z_{\text{loc}}(m)$  denote the multiset of zeros of  $E$  used to define  $Z_{\text{loc}}$  (counted with multiplicity). Define the finite-dimensional subspace

$$K_B := \text{span}\{k_\rho : \partial B \rightarrow \mathbb{C}, k_\rho(v) = (v - \rho)^{-1} : \rho \in Z_{\text{loc}}(m)\} \subset L^2(\partial B, ds),$$

and let  $\Pi_B : L^2(\partial B) \rightarrow K_B$  be the orthogonal projection.

**Lemma A.3** (Projection kills the local log-derivative (supporting documentation)). *With notation as in Definition A.2,*

$$\frac{Z'_{\text{loc}}}{Z_{\text{loc}}}(v) = \sum_{\rho \in Z_{\text{loc}}(m)} \frac{m_\rho}{v - \rho} \in K_B \quad (v \in \partial B).$$

Hence  $\Pi_B(Z'_{\text{loc}}/Z_{\text{loc}}) = Z'_{\text{loc}}/Z_{\text{loc}}$  and  $(I - \Pi_B)(Z'_{\text{loc}}/Z_{\text{loc}}) = 0$  in  $L^2(\partial B)$  (and thus pointwise on  $\partial B$ ). Consequently, using Lemma 10.7,

$$(I - \Pi_B)\left(\frac{E'}{E}\right) = (I - \Pi_B)\left(\frac{F'}{F}\right) \quad \text{on } \partial B.$$

Moreover  $\|\Pi_B\|_{L^2 \rightarrow L^2} = 1$ .

*Remark A.4* (Conditioning caveat for coefficient representations (supporting documentation)). Lemma A.3 uses only the abstract orthogonal projection  $\Pi_B$  (a contraction). No uniform bound on the inverse Gram matrix of the spanning kernels  $k_\rho$  is available without a lower bound on pairwise zero separations in  $Z_{\text{loc}}(m)$ . Therefore any coefficient-level formula for  $\Pi_B$  must be treated as non-uniform unless additional spacing structure is proved.

## B S6 harness: explicit-formula interpretation (non-closure)

This appendix is an *interpretation harness only*. It is not used in any implication in the manuscript. Its purpose is to connect the v-frame “off-axis” language to the classical explicit formula for prime-counting functions.

### B.1 Frame mapping: v-displacement and the real part $\beta$

A nontrivial zero  $\rho = \beta + i\gamma$  in the s-frame corresponds to

$$u_\rho = 2\rho = 2\beta + i2\gamma, \quad v_\rho = u_\rho - 1 = (2\beta - 1) + i2\gamma.$$

Thus an off-critical-line zero ( $\beta \neq \frac{1}{2}$ ) is exactly an off-axis v-zero ( $\Re v_\rho \neq 0$ ), with displacement  $a := \Re v_\rho = 2\beta - 1$ .

### B.2 Explicit formula: off-axis zeros as amplitude leaks

In a standard explicit formula (e.g. for  $\psi(x) = \sum_{n \leq x} \Lambda(n)$ ), nontrivial zeros enter through terms of the form  $x^\rho/\rho$  (or  $\text{Li}(x^\rho)$ ). If  $\rho = \beta + i\gamma$ , then

$$x^\rho = x^\beta e^{i\gamma \log x},$$

so the amplitude is governed by  $x^\beta$ . In v-variables,  $\beta = \frac{1}{2} + \frac{a}{2}$ , so any  $a > 0$  corresponds to an  $x^{1/2+a/2}$ -scale contribution (an “amplitude leak” beyond the square-root scale).

### B.3 What S5' would mean in this language

A successful S5' closure would exclude all off-axis v-zeros, hence prove RH and thereby eliminate all amplitude leaks with  $\beta > 1/2$ . However, the present manuscript does *not* claim any new prime-error bounds directly: the S6 harness is only a translation layer for interpreting off-axis zeros in the classical explicit-formula setting.

## C Tail harness bundle and reproducibility (v38)

### C.1 What the tail checks prove (and what they do not)

Each tail check records the statement:

Given a constants file that provides interval enclosures for  $(C_1, C_2, C_{\text{up}}, C_h'', \kappa)$ , the chosen parameters  $(m, \eta, \alpha)$ , and the recorded UE exponent  $p$ , the harness computes interval bounds for the left-hand side LHS and right-hand side RHS in (14) and reports whether the strict separation  $\text{LHS}_{\text{hi}} < \text{RHS}_{\text{lo}}$  holds.

It does *not* certify that the constants file is correct.

### C.2 SHA-256 table (exact artifacts)

The file `v36_repro_pack/SHA256SUMS.txt` is the canonical hash list.

```
08dd47ab38a8d3e5c7b50bdb71f0434197abf51447fae95194a8d51c970edadf  README.md
e845e8ef467832b262d5588d82ce2c5a0e5567c4f5ee6db83c2226ad802f18fa  v38_constants_m10.json
61f41d25c8eb62164fc69bc8abe97f1c794a80dd21c198a72ab1353c8308dc4c  v38_frontend_certificate.json
c1debba3583dbaf0dc7120684ba89c457fef1227f4aa13504b21cf11e029acb  v38_frontend_verifier_output.txt
adc1cb55945d3c72c5ef43d23bc6b9ebfad763dc90a88ae28b4e59e9016afb13  v38_generate_frontend_certificate
.PY
54c0d4b07704f72e9806ed29a88dd9c7a31809f5dca833acb63777a2d8947b16  v38_generate_tail_check.py
1cecbc314188cbda6f4b9e5e9468b79b375a20c0754ebfa3a2e44fcdbd26333ac  v38_tail_check_m10.json
3b2c15c47e9eaadc4edbac24296091f41e9c04062de51536469584fc1783a307  v38_tail_check_verifier_output_m10.txt
ee6165a66703be34c30d8a1ad6717880be14a449a8bb59f6d788f8b546e778bb  v38_verify_frontend_certificate.
py
3e66e93a65fe7acc783bfb080949d820456336a2ebf344f88973d45c64d8bd26  v38_verify_tail_check.py
```

### C.3 Commands

From the directory `v36_repro_pack/`:

1. `sha256sum -c SHA256SUMS.txt`
2. `python3 v36_verify_tail_check.py --constants v36_constants_m10.json --certificate v36_tail_check_m10.json`
3. `python3 v36_verify_frontend_certificate.py --certificate v36_frontend_certificate.json`

### C.4 Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false)

```
LHS_hi =
850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076
RHS_lo =
0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351
STRICT (LHS_hi < RHS_lo) = False
REGEN_MATCH = True
INEQUALITY_STRICT = False
CERT_REPORTED_PASS = False
OK
```

## C.5 Bundle files (verbatim)

```
{
  "UE_endpoint_class": "pointwise/sup",
  "UE_exponent_p": "1",
  "alpha_worst": "1",
  "budget_tuple": {
    "notes": "Budget tuple recorded for the legacy D1 harness; S5' budgets are OPEN in-text (see
      OPEN box in Section 12).",
    "p": "1",
    "q": "1",
    "theta": "1"
  },
  "certificate_version": "v38_repro_pack",
  "created_utc": "2026-01-25T20:16:39Z",
  "endpoint_family": "v38: D1 harness (pointwise sup) + S5prime open phase toolkit",
  "endpoint_functional": "sup_{\u2202B} |E'(v)/E(v)|",
  "eta": "1e-14",
  "forceability_mode": "identity: forced object is D_B(W)",
  "forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma
    force-constant-limited)",
  "forcing_constants": {
    "Kalloc_expression": "3 + 8*sqrt(3)",
    "c0_expression": "(3*ln(2))/(8*pi)",
    "c_expression": "(3*ln(2))/16"
  },
  "forcing_target": "Harness targets D_B(W) (dial deviation); S5' proposes widetilde D_B(W) (
    buffered phase)",
  "growth_model": {
    "form": "G(n,kappa) <= C_G * kappa^{-u} * n^q",
    "notes": "Growth model parameters are not certified by this harness; recorded for audit
      bookkeeping.",
    "q": "1",
    "u": "OPEN"
  },
  "intervals": {
    "C1": {
      "hi": "15.2",
      "lo": "15.1"
    },
    "C2": {
      "hi": "37.4",
      "lo": "37.3"
    },
    "C_hpp": {
      "hi": "1100.5",
      "lo": "1100"
    },
    "C_up": {
      "hi": "1100.5",
      "lo": "1100"
    }
  },
  "kappa": "0.05",
  "local_exponent_theta": "1",
  "local_growth_q": "1",
  "m_band": "10",
  "manuscript_version": "v38",
  "missing_lever_open": true,

```

```

"notes": [
  "Demo-only intervals carried forward from v31-style scaffolding; replace with audit-proven
  enclosures when G-1/G-12 are closed.",
  "The verifier/generator implement directed-rounding interval arithmetic with Python's decimal
  module.",
  "The local-window majorant  $N_{up}(m)=1.01*\log(m)+17$  is hard-coded from Lemma Nloc-logm in
  manuscript_v36.",
  "UE_exponent_p is recorded explicitly to prevent exponent drift across versions.",
  "v36 adds explicit metadata for endpoint_functional and forcing_architecture to prevent silent
  mismatch under S5 redesign.",
  "UE_endpoint_class='pointwise/sup' is the class for which Lemma UE-scaling-nogo forbids any
  exponent  $p>1$  with shape-only constants.",
  "v36 schema latch: required fields (endpoint_functional, UE_exponent_p, local_exponent_theta,
  local_growth_q, forceability_mode, forcing_architecture) must be present; verifier fails closed
  if missing.",
  "v38 schema latch: phase_endpoint_only_nogo_installed=true records that Lemma phase-UE-theta0-
  nogo is installed in-text (endpoint-only NO-GO).",
],
"phase_endpoint_only_nogo_installed": true
}

```

```

{
  "certificate_version": "v38_repro_pack",
  "m_band": "10",
  "eta": "1e-14",
  "alpha": "1",
  "kappa": "0.05",
  "UE_exponent_p": "1",
  "UE_endpoint_class": "pointwise/sup",
  "endpoint_functional": " $\sup_{v} |E'(v)/E(v)|$ ",
  "forcing_architecture": "single-box forcing (short-side pair-factor phase; Lemma 8.1 and Lemma
  force-constant-limited)",
  "forceability_mode": "identity: forced object is  $D_B(W)$ ",
  "endpoint_family": "v38: D1 harness (pointwise sup) + S5prime open phase toolkit",
  "forcing_target": "Harness targets  $D_B(W)$  (dial deviation); S5' proposes widetilde  $D_B(W)$  (
  buffered phase)",
  "budget_tuple": {
    "notes": "Budget tuple recorded for the legacy D1 harness; S5' budgets are OPEN in-text (see
    OPEN box in Section 12).",
    "p": "1",
    "q": "1",
    "theta": "1"
  },
  "growth_model": {
    "form": " $G(n, \kappa) \leq C_G * \kappa^{-u} * n^q$ ",
    "notes": "Growth model parameters are not certified by this harness; recorded for audit
    bookkeeping.",
    "q": "1",
    "u": "OPEN"
  },
  "missing_lever_open": true,
  "phase_endpoint_only_nogo_installed": true,
  "local_exponent_theta": "1",
  "local_growth_q": "1",
  "forcing_constants": {
    "Kalloc_expression": " $3 + 8*\sqrt{3}$ ",

```



```

    "c0_expression": "(3*ln(2))/(8*pi)",
    "c_expression": "(3*ln(2))/16"
  },
  "manuscript_version": "v38",
  "prec": 90,
  "pi_interval": {
    "lo": "3.14159265358979323846264338327950288419716939937510",
    "hi": "3.14159265358979323846264338327950288419716939937511"
  },
  "logm_interval": {
    "lo":
      "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721",
    "hi":
      "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721"
  },
  "delta_interval": {
    "lo":
      "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197468E-15",
    "hi":
      "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197469E-15"
  },
  "L_interval": {
    "lo":
      "72.0690349042100898286716709657338995347766324782944719381032513046103464061280224515635578",
    "hi":
      "72.3992934135094943970734701112023359555367426271573492357065840947071036670957576995871576"
  },
  "Nup_interval": {
    "lo":
      "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383571",
    "hi":
      "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383572"
  },
  "kappa_interval": {
    "lo": "0.05",
    "hi": "0.05"
  },
  "lhs_interval": {
    "lo":
      "850326.881532655689245144996236820608990676883579823939491593791503565415726034904366571925",
    "hi":
      "850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076"
  },
  "rhs_interval": {
    "lo":
      "0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351",
    "hi":
      "0.129965096347948199005209691457697222838229716361348668790498959759558243588339370271250251"
  },
  "derived_constants": {
    "ln2_interval": {
      "lo":
        "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327",
      "hi":
        "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327"
    },
    "c_interval": {

```

```

        "lo":
        "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099373",
        "hi":
        "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099375"
    },
    "c0_interval": {
        "lo":
        "0.0827383500572443475236711620442491341185086557736206913728528561387020242248387512851407512",
        "hi":
        "0.0827383500572443475236711620442491341185086557736209547372007536994885577445868650239268751"
    },
    "Kalloc_interval": {
        "lo":
        "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491695",
        "hi":
        "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491696"
    }
},
"pass": false
}

```

```
#!/usr/bin/env python3
```

```
"""
```

```
v38_generate_tail_check.py
```

Deterministically generates v38\_tail\_check\_m10.json from v38\_constants\_m10.json using directed-rounding interval arithmetic implemented with Python's decimal module.

This generator is intended to be auditable: no network access, no randomness, and no external libraries.

Tail inequality evaluation (for given inputs):

LHS(delta) < RHS(delta), where

$LHS(\delta) = 2 \cdot C_{up} \cdot (\delta^p \cdot L(m) + \delta^{(p-1)} \cdot N_{up}(m) / \kappa)$

$RHS(\delta) = c - \delta \cdot (Kalloc \cdot c_0 \cdot L(m) + C_{hpp} \cdot (\log(m) + 1))$

with

$L(m) = C_1 \cdot \log(m) + C_2$ ,

$N_{up}(m) = 1.01 \cdot \log(m) + 17$ ,

$c = (3 \ln 2) / 16$ ,

$c_0 = (3 \ln 2) / (8 \pi)$ ,

$Kalloc = 3 + 8 \cdot \sqrt{3}$ .

Usage:

```
python3 v38_generate_tail_check.py v38_constants_m10.json v38_tail_check_m10.json
"""
```

```
import json
```

```
import sys
```

```
from dataclasses import dataclass
```

```
from decimal import Decimal, getcontext, localcontext, ROUND_FLOOR, ROUND_CEILING
```

```
# ---- Fixed enclosure for pi (50 decimal places) ----
```

```
# pi = 3.14159265358979323846264338327950288419716939937510...
```

```

PI_LO = Decimal("3.14159265358979323846264338327950288419716939937510")
PI_HI = Decimal("3.14159265358979323846264338327950288419716939937511")

@dataclass
class Interval:
    lo: Decimal
    hi: Decimal

    def __post_init__(self) -> None:
        if self.lo > self.hi:
            raise ValueError(f"Bad interval: {self.lo} > {self.hi}")

def ctx(prec: int, rounding):
    c = getcontext().copy()
    c.prec = prec
    c.rounding = rounding
    return c

def iv(lo: str, hi: str | None = None) -> Interval:
    if hi is None:
        hi = lo
    return Interval(Decimal(lo), Decimal(hi))

def add(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo + b.lo
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi + b.hi
    return Interval(lo, hi)

def sub(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo - b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi - b.lo
    return Interval(lo, hi)

def mul(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        cands_lo = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        lo = min(cands_lo)
    with localcontext(ctx(prec, ROUND_CEILING)):
        cands_hi = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        hi = max(cands_hi)
    return Interval(lo, hi)

def div(a: Interval, b: Interval, prec: int) -> Interval:
    if b.lo <= 0 <= b.hi:
        raise ZeroDivisionError("Interval division by an interval containing 0.")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        rlo = Decimal(1) / b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):

```

```

        rhi = Decimal(1) / b.lo
    return mul(a, Interval(rlo, rhi), prec)

def sqrt(a: Interval, prec: int) -> Interval:
    if a.lo < 0:
        raise ValueError("sqrt of negative interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.sqrt()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.sqrt()
    return Interval(lo, hi)

def ln(a: Interval, prec: int) -> Interval:
    if a.lo <= 0:
        raise ValueError("ln of nonpositive interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.ln()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.ln()
    return Interval(lo, hi)

def pow_3_2(a: Interval, prec: int) -> Interval:
    return mul(a, sqrt(a, prec), prec)

def compute(constants: dict, prec: int = 90) -> dict:
    m = iv(constants["m_band"])
    eta = iv(constants["eta"])
    alpha = iv(constants["alpha_worst"])
    kappa = iv(constants["kappa"])

    p = str(constants.get("UE_exponent_p", "1"))

    C1 = iv(constants["intervals"]["C1"]["lo"], constants["intervals"]["C1"]["hi"])
    C2 = iv(constants["intervals"]["C2"]["lo"], constants["intervals"]["C2"]["hi"])
    Cup = iv(constants["intervals"]["C_up"]["lo"], constants["intervals"]["C_up"]["hi"])
    Chpp = iv(constants["intervals"]["C_hpp"]["lo"], constants["intervals"]["C_hpp"]["hi"])

    logm = ln(m, prec)
    delta = div(mul(eta, alpha, prec), mul(logm, logm, prec), prec)

    # L(m) = C1*logm + C2
    L = add(mul(C1, logm, prec), C2, prec)

    # N_up(m) = 1.01*logm + 17
    Nup = add(mul(iv("1.01"), logm, prec), iv("17"), prec)

    # ln 2
    ln2 = ln(iv("2"), prec)

    # c = (3 ln 2)/16
    c = div(mul(iv("3"), ln2, prec), iv("16"), prec)

    # c0 = (3 ln 2)/(8 pi), pi enclosed
    pi = Interval(PI_LO, PI_HI)
    c0 = div(mul(iv("3"), ln2, prec), mul(iv("8"), pi, prec), prec)

```

```

# Kalloc = 3 + 8 sqrt(3)
sqrt3 = sqrt(iv("3"), prec)
Kalloc = add(iv("3"), mul(iv("8"), sqrt3, prec), prec)

logm_plus1 = add(logm, iv("1"), prec)

# UE exponent p: LHS = 2*Cup*(delta^p * L + delta^(p-1) * Nup/kappa).
# We support p="1" (pointwise UE proved in v36) and p="3/2" (hypothetical strengthened gate).
if p in ("1", "1.0", "1.00"):
    local_term = div(Nup, kappa, prec)          # delta^(p-1)=1
    residual_term = mul(delta, L, prec)          # delta^p = delta
elif p in ("3/2", "1.5", "1.50"):
    sqrt_delta = sqrt(delta, prec)
    local_term = mul(sqrt_delta, div(Nup, kappa, prec), prec)    # delta^(1/2)
    residual_term = mul(mul(delta, sqrt_delta, prec), L, prec)    # delta^(3/2)
else:
    raise ValueError(f"Unsupported UE_exponent_p={p!r}; use '1' or '3/2'.")

lhs = mul(mul(iv("2"), Cup, prec), add(residual_term, local_term, prec), prec)

# RHS = c - delta*(Kalloc*c0*L + Chpp*(logm+1))
term1 = mul(mul(Kalloc, c0, prec), L, prec)
term2 = mul(Chpp, logm_plus1, prec)
rhs = sub(c, mul(delta, add(term1, term2, prec), prec), prec)

passed = (lhs.hi < rhs.lo)

return {
    "prec": prec,
    "UE_exponent_p": p,
    "pi_interval": {"lo": str(PI_LO), "hi": str(PI_HI)},
    "logm_interval": {"lo": str(logm.lo), "hi": str(logm.hi)},
    "delta_interval": {"lo": str(delta.lo), "hi": str(delta.hi)},
    "L_interval": {"lo": str(L.lo), "hi": str(L.hi)},
    "Nup_interval": {"lo": str(Nup.lo), "hi": str(Nup.hi)},
    "kappa_interval": {"lo": str(kappa.lo), "hi": str(kappa.hi)},
    "lhs_interval": {"lo": str(lhs.lo), "hi": str(lhs.hi)},
    "rhs_interval": {"lo": str(rhs.lo), "hi": str(rhs.hi)},
    "derived_constants": {
        "ln2_interval": {"lo": str(ln2.lo), "hi": str(ln2.hi)},
        "c_interval": {"lo": str(c.lo), "hi": str(c.hi)},
        "c0_interval": {"lo": str(c0.lo), "hi": str(c0.hi)},
        "Kalloc_interval": {"lo": str(Kalloc.lo), "hi": str(Kalloc.hi)},
    },
    "pass": bool(passed),
}

def main() -> int:
    if len(sys.argv) != 3:
        print("Usage: v38_generate_tail_check.py constants.json tail_check.json", file=sys.stderr)
        return 2

    with open(sys.argv[1], "r", encoding="utf-8") as f:
        constants = json.load(f)

    # ---- fail-closed metadata latch (v36) ----
    REQUIRED_META = [

```

```

        "UE_exponent_p",
        "UE_endpoint_class",
        "endpoint_functional",
        "forcing_architecture",
        "forceability_mode",
        "local_exponent_theta",
        "local_growth_q",
        # v38 schema extensions (fail-closed)
        "endpoint_family",
        "forcing_target",
        "budget_tuple",
        "growth_model",
        "missing_lever_open",
        "phase_endpoint_only_nogo_installed",
    ]
    for k in REQUIRED_META:
        if k not in constants or constants[k] in (None, ""):
            raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")

    out = {
        "certificate_version": constants.get("certificate_version", "v38_repro_pack"),
        "m_band": constants["m_band"],
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        # required metadata latch (fail-closed)
        "UE_exponent_p": constants["UE_exponent_p"],
        "UE_endpoint_class": constants["UE_endpoint_class"],
        "endpoint_functional": constants["endpoint_functional"],
        "forcing_architecture": constants["forcing_architecture"],
        "forceability_mode": constants["forceability_mode"],

        # v38 schema extensions (fail-closed metadata latch)
        "endpoint_family": constants["endpoint_family"],
        "forcing_target": constants["forcing_target"],
        "budget_tuple": constants["budget_tuple"],
        "growth_model": constants["growth_model"],
        "missing_lever_open": constants["missing_lever_open"],
        "phase_endpoint_only_nogo_installed": constants["phase_endpoint_only_nogo_installed"],
        "local_exponent_theta": constants["local_exponent_theta"],
        "local_growth_q": constants["local_growth_q"],
        "forcing_constants": constants.get("forcing_constants", {}),
        "manuscript_version": constants.get("manuscript_version", "v38"),
    }

    out.update(compute(constants, prec=90))

    with open(sys.argv[2], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[2])
    print("[generate] PASS =", out["pass"])
    print("[generate] lhs_interval.hi =", out["lhs_interval"]["hi"])
    print("[generate] rhs_interval.lo =", out["rhs_interval"]["lo"])
    return 0

if __name__ == "__main__":
    raise SystemExit(main())

```

```

#!/usr/bin/env python3
"""
v38_verify_tail_check.py

Verifier for v38_tail_check_m10.json. This script:
- loads the constants JSON and the pinned certificate JSON
- regenerates the certificate from constants
- checks exact JSON equality on the computed fields
- reports PASS/FAIL and prints the strict-separation check LHS_hi < RHS_lo.

Usage:
python3 v38_verify_tail_check.py --constants v38_constants_m10.json --certificate
v38_tail_check_m10.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json
import sys

from v38_generate_tail_check import compute

def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v38 tail check (m=10).")
    ap.add_argument("--constants", required=True, help="Path to v38_constants_m10.json")
    ap.add_argument("--certificate", required=True, help="Path to v38_tail_check_m10.json")
    args = ap.parse_args()

    with open(args.constants, "r", encoding="utf-8") as f:
        constants = json.load(f)

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    # ---- fail-closed metadata latch (v38) ----
    REQUIRED_META = [
        "UE_exponent_p",
        "UE_endpoint_class",
        "endpoint_functional",
        "forcing_architecture",
        "forceability_mode",
        "local_exponent_theta",
        "local_growth_q",
        # v38 schema extensions (fail-closed)
        "endpoint_family",
        "forcing_target",
        "budget_tuple",
        "growth_model",
        "missing_lever_open",
        "phase_endpoint_only_nogo_installed",
    ]

```

```

]
for k in REQUIRED_META:
    if k not in constants or constants[k] in (None, ""):
        raise KeyError(f"Missing required metadata field {k!r} in constants (fail-closed).")
    if k not in cert or cert[k] in (None, ""):
        raise KeyError(f"Missing required metadata field {k!r} in certificate (fail-closed).")

regen = {
    "certificate_version": constants.get("certificate_version", "v38_repro_pack"),
    "m_band": constants["m_band"],
    "eta": constants["eta"],
    "alpha": constants["alpha_worst"],
    "kappa": constants["kappa"],
    # required metadata latch (fail-closed)
    "UE_exponent_p": constants["UE_exponent_p"],
    "UE_endpoint_class": constants["UE_endpoint_class"],
    "endpoint_functional": constants["endpoint_functional"],
    "forcing_architecture": constants["forcing_architecture"],
    "forceability_mode": constants["forceability_mode"],
    "local_exponent_theta": constants["local_exponent_theta"],
    "local_growth_q": constants["local_growth_q"],
    # v38 schema extensions (fail-closed)
    "endpoint_family": constants["endpoint_family"],
    "forcing_target": constants["forcing_target"],
    "budget_tuple": constants["budget_tuple"],
    "growth_model": constants["growth_model"],
    "missing_lever_open": constants["missing_lever_open"],
    "phase_endpoint_only_nogo_installed": constants["phase_endpoint_only_nogo_installed"],
    "forcing_constants": constants.get("forcing_constants", {}),
    "manuscript_version": constants.get("manuscript_version", "v38"),
}

# v38 fail-closed checks: missing_lever_open must be explicitly True and budget_tuple structure
# must be explicit
if constants.get("missing_lever_open") is not True:
    raise SystemExit("FAIL: constants missing_lever_open must be true in v38.")
if cert.get("missing_lever_open") is not True:
    raise SystemExit("FAIL: certificate missing_lever_open must be true in v38.")

if constants.get("phase_endpoint_only_nogo_installed") is not True:
    raise SystemExit("FAIL: constants phase_endpoint_only_nogo_installed must be true in v38.")
if cert.get("phase_endpoint_only_nogo_installed") is not True:
    raise SystemExit("FAIL: certificate phase_endpoint_only_nogo_installed must be true in v38
.")

for obj, name in [(constants, "constants"), (cert, "certificate")]:
    bt = obj.get("budget_tuple")
    if not isinstance(bt, dict):
        raise SystemExit(f"FAIL: {name}.budget_tuple must be an object.")
    for k in ["p", "theta", "q"]:
        if k not in bt:
            raise SystemExit(f"FAIL: {name}.budget_tuple missing key '{k}'.")
regen.update(compute(constants, prec=90))

# Compare all keys that regen produces (ignore any extra keys in cert)
ok = True
for k, v in regen.items():
    if cert.get(k) != v:
        ok = False
        print(f"MISMATCH key={k}")

```



```

        print("  cert :", cert.get(k))
        print("  regen:", v)

    lhs_hi = regen["lhs_interval"]["hi"]
    rhs_lo = regen["rhs_interval"]["lo"]
    strict = (float(lhs_hi) < float(rhs_lo))

    print("LHS_hi =", lhs_hi)
    print("RHS_lo =", rhs_lo)
    print("STRICT (LHS_hi < RHS_lo) =", strict)

    print("REGEN_MATCH =", ok)
    print("INEQUALITY_STRICT =", strict)
    print("CERT_REPORTED_PASS =", regen.get("pass"))

    if not ok:
        print("FAIL (mismatch)")
        return 1

    print("OK")
    return 0

if __name__ == "__main__":
    raise SystemExit(main())

```

## D Finite-height front-end certificate (literature-based)

The required front-end is RH up to height  $H_0 = 5$ . We record a discharge using Platt–Trudgian’s published verification of RH up to  $3 \cdot 10^{12}$ .

```

{
  "certificate_version": "v38_repro_pack",
  "created_utc": "2026-01-25T20:16:56Z",
  "discharged_by": {
    "logic": "If RH holds for  $0 < \gamma \leq H_{\text{cited}}$  and  $H_0 \leq H_{\text{cited}}$ , then RH holds for  $0 < \gamma \leq H_0$ .",
    "reference": {
      "arxiv": "2004.09765",
      "authors": "D. J. Platt and T. S. Trudgian",
      "doi": "10.1112/blms.12460",
      "statement": "All zeros  $\beta + i\gamma$  with  $0 < \gamma \leq 3 \cdot 10^{12}$  satisfy  $\beta = 1/2$  (rigorous interval arithmetic).",
      "title": "The Riemann hypothesis is true up to  $3 \cdot 10^{12}$ ",
      "venue": "Bulletin of the London Mathematical Society",
      "year": 2021
    },
  },
  "type": "literature_citation",
  "verification_height": 3000000000000.0
},
{
  "manuscript_version": "v38",
  "needed_frontend_statement": {
    "H0": 5.0,
    "text": "All nontrivial zeros  $\rho = \beta + i\gamma$  with  $0 < \gamma \leq H_0$  satisfy  $\beta = 1/2$ .",

```

```

    "type": "RH_to_height"
  },
  "notes": [
    "This JSON is not itself a computation of zeros; it is a pinned statement+reference used by v38",
    "For a fully self-contained proof without external computational input, one would need to",
    "implement and certify an argument-principle zero count in this region using ball arithmetic (",
    "not provided here).",
  ]
}

```

```

H0 (needed) = 5.0
H_cited     = 3000000000000.0
CHECK: H0 <= H_cited : True
PASS

```

```

#!/usr/bin/env python3
"""v38_generate_frontend_certificate.py

```

Creates a pinned JSON certificate for the finite-height front-end assumption used by v38.

This script does NOT compute zeta zeros. It encodes a minimal (H0, citation) logic statement: if RH has been verified up to H\_cited and H0 <= H\_cited, then RH holds up to height H0.

Usage:

```

python3 v38_generate_frontend_certificate.py v38_frontend_certificate.json
"""

```

```

from __future__ import annotations

```

```

import json
from datetime import datetime, timezone
import sys

```

```

def main() -> int:
    if len(sys.argv) != 2:
        print("Usage: v38_generate_frontend_certificate.py output.json", file=sys.stderr)
        return 2

    out = {
        "certificate_version": "v38_repro_pack",
        "created_utc": datetime.now(timezone.utc).strftime("%Y-%m-%dT%H:%M:%SZ"),
        "needed_frontend_statement": {
            "type": "RH_to_height",
            "H0": 5.0,
            "text": "All nontrivial zeros  $\rho = \beta + i\gamma$  with  $0 < \gamma \leq H_0$  satisfy  $\beta = 1/2$ ."
        },
        "discharged_by": {
            "type": "literature_citation",
            "verification_height": 3e12,
            "reference": {
                "authors": "D. J. Platt and T. S. Trudgian",
                "title": "The Riemann hypothesis is true up to  $3 \cdot 10^{12}$ ",
            }
        }
    }

```

```

        "venue": "Bulletin of the London Mathematical Society",
        "year": 2021,
        "doi": "10.1112/blms.12460",
        "arxiv": "2004.09765",
        "statement": "All zeros  $\beta + i\gamma$  with  $0 < \gamma \leq 3 \cdot 10^{12}$  satisfy  $\beta = 1/2$  (
rigorous interval arithmetic)."
```

},

```

    "logic": "If RH holds for  $0 < \gamma \leq H_{\text{cited}}$  and  $H_0 \leq H_{\text{cited}}$ , then RH holds for  $0 < \gamma \leq H_0$ ."
```

},

```

    "notes": [
        "This JSON is not itself a computation of zeros; it is a pinned statement+reference
used by v38.",
        "For a fully self-contained proof without external computational input, one would need
to implement and certify an argument-principle zero count in this region using ball arithmetic
(not provided here)."
```

]

```

}

with open(sys.argv[1], "w", encoding="utf-8") as f:
    json.dump(out, f, indent=2)

print("[generate] wrote", sys.argv[1])
return 0

if __name__ == "__main__":
    raise SystemExit(main())

#!/usr/bin/env python3
"""v38_verify_frontend_certificate.py

Verifier for the front-end certificate JSON produced by v38_generate_frontend_certificate.py.

This verifier checks the internal logic only:
- parses the JSON
- confirms that the required finite-height  $H_0$  is  $\leq$  the cited verification height

It does NOT re-run the cited large-scale computation (Platt--Trudgian); that result is treated as
an
external, peer-reviewed input in the manuscript.

Usage:
python3 v38_verify_frontend_certificate.py --certificate v38_frontend_certificate.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json

def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v38 front-end certificate JSON (internal logic
only).")

```

```

ap.add_argument("--certificate", required=True, help="Path to v38_frontend_certificate.json")
args = ap.parse_args()

with open(args.certificate, "r", encoding="utf-8") as f:
    cert = json.load(f)

needed = cert.get("needed_frontend_statement", {})
discharged = cert.get("discharged_by", {})

H0 = float(needed.get("H0"))
Hc = float(discharged.get("verification_height"))

ok = H0 <= Hc

print("H0 (needed) =", H0)
print("H_cited      =", Hc)
print(f"CHECK: H0 <= H_cited : {ok}")

if not ok:
    print("FAIL")
    return 1

print("PASS")
return 0

if __name__ == "__main__":
    raise SystemExit(main())

```

## References

## References

- [1] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur  $L^2$  pour les courbes lipschitziennes*, Annals of Mathematics (2) **116** (1982), no. 2, 361–387.
- [2] T. A. Driscoll and L. N. Trefethen, *Schwarz–Christoffel Mapping*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2002.
- [3] P. L. Duren, *Theory of  $H^p$  Spaces*, Academic Press, 1970.
- [4] J. B. Garnett, *Bounded Analytic Functions*, Graduate Texts in Mathematics, Springer, 2007.
- [5] A. Ivić, *The Riemann Zeta-Function: Theory and Applications*, Wiley-Interscience, 1985.
- [6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.
- [7] A. Bellotti and T. Wong, *An improved explicit bound on the argument of the Riemann zeta function on the critical line*, arXiv:2412.15470v2 (2024).
- [8] D. Platt and T. Trudgian, *The Riemann hypothesis is true up to  $3 \cdot 10^{12}$* , Bulletin of the London Mathematical Society **53** (2021), no. 3, 792–797.