A Height-Local Width-2 Program for Excluding Off-Axis Quartets

with an Analytic Tail and a Rigorous Certified Outer/Rouché Criterion

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Authorship and AI-use disclosure. The author, Dylan [Surname], designed the framework, chose all constants/normalizations, and validated all mathematics and computations. A generative assistant (GPT-5 Pro) was used only for typesetting assistance, editorial organization, and consistency checks; it is not an author. All claims are the author's responsibility (COPE/ICMJE guidance).

MSC. 11M06; 30C85; 30E20; 65G40.

Keywords. Riemann zeta; functional equation; outer function; Rouché; harmonic measure; conformal trace; interval arithmetic.

Abstract

In the width-2 centered frame $u = 2s$, $v = u - 1$, let $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ and $E(v) = \Lambda_2(1 + v)$. We present a boundary-only, height-local program to exclude off-axis quartets $\{\pm a \pm im\}$ via two complementary routes:

(1) an analytic tail (uniform in $\alpha \in (0, 1]$) using only: (i) explicit short-side forcing $\geq \pi/2$; (ii) a residual bound for $F = E/Z_{\mathrm{loc}}$ with perimeter factor 8δ; and (iii) a disc-based, L^2 boundary-to-midpoint estimate with *shape-only* constants (no strip/rectangle density comparison);

(2) a rigorous Outer/Rouché Certification Path: interval arithmetic on ∂B + validated Poisson + Lipschitz grid$\to$continuum enclosure $\Rightarrow \sup_{\partial B}\left|E - G_{\mathrm{out}}\right|/|G_{\mathrm{out}}| < 1 \Rightarrow$ zero-free box, followed by Bridge 1 (inner collapse $W \equiv e^{i\theta}$) and Bridge 2 (stitching).

We also prove a corner outer interpolation from continuous Dirichlet data. The tail is stated with symbolic constants: for each fixed $\eta \in (0, 1)$ there exists $M_0(\eta)$ such that no off-axis quartet lies in any $B(\alpha, m, \delta)$ with $\delta = \eta\alpha/(\log m)^2$ for all $m \geq M_0(\eta)$, uniformly in α. Combined with a certified base range below m_1 (first nontrivial height in width-2), this yields the global on-axis theorem. All constants appearing in the upper/lower envelope are *shape-only* (independent of m, α, a); residual constants are kept symbolic in theorems and may be instantiated from classical literature in an appendix.

Symbols & Provenance (at a glance)

Notation hygiene. We reserve ψ for the digamma function and write $\varphi : \mathbb{D} \to B$ for conformal maps.

Symbol	Definition / role	Provenance / why this form
$u = 2s,\ v = u - 1$	Width-2 frame centered at $\operatorname{Re} u = 1$	Centers functional equation symmetry
$\Lambda_2(u) = \pi^{-u/4}\Gamma\left(\frac{u}{4}\right)\zeta\left(\frac{u}{2}\right)$	Completed object	Standard; FE for Λ_2; width-2 transport
$E(v) = \Lambda_2(1+v)$	Workhorse in v-plane	Even & conjugate-symmetric: $E(v) = E(-v) = \overline{E(\bar{v})}$
$\zeta_2(u) = \zeta(u/2)$	Width-2 zeta	Used in FE and hinge law
$\chi_2(u)$	FE factor inverse	$\chi_2(u) = \pi^{u/2-1/2}\frac{\Gamma((2-u)/4)}{\Gamma(u/4)}$
$B(\alpha, m, \delta)$	$\big[\alpha-\delta, \alpha+\delta\big] \times \big[m-\delta, m+\delta\big]$	Square (width & height 2δ) centered at (α, m)
$\alpha \in (0, 1]$	Horizontal center	Uniform-in-α uses worst case $\alpha = 1$
$m \geq 10$	Height parameter	Ensures uniform DLMF/Titchmarsh/Ivić regimes
$\delta = \dfrac{\eta\,\alpha}{(\log m)^2},\ \eta \in (0,1)$	Half-side length of B	Balances forcing vs residual $O(\delta \log m)$
∂B	Boundary of $B(\alpha, m, \delta)$	Boundary integrals/suprema
$I_\pm$	Short vertical sides of ∂B	Near/far verticals in forcing budgets
Q	Quiet arcs (horizontal sides of ∂B)	Controlled by L^2 trace & Hilbert
$Z_{\text{loc}}(v) = \prod_{\lvert \operatorname{Im}\rho - m\rvert \leq 1}(v-\rho)^{m_\rho}$	Local zero/pole factors	De-singularizes E near ∂B
$F = E/Z_{\text{loc}}$	Residual analytic factor (nonvanishing near ∂B)	Lemma **??** (constants symbolic)
$G(v) = \dfrac{E(1+v)}{E(1-v)}$	Odd-lane quotient	Links to hinge via two-point identity
$G_{\text{out}} = e^{U+iV}$	Outer with $\lvert\, G_{\text{out}}\,\rvert = \lvert E\rvert$ on ∂B	$U = \log\lvert E\rvert \in C(\overline{B})$ solves Dirichlet; V harmonic conj.
$W = E/G_{\text{out}}$	Inner quotient ($\lvert W\rvert = 1$ a.e. on ∂B)	Collapses to unimodular constant upon certification
$v^\star_\pm = \pm(a + im)$	"Dial pair" on centerline	Points of evaluation in the tail
$Z_{\text{pair}}(v)$	$(v - (a+im))(v - (-a+im))$	Short-side forcing on I_+
Γ_λ	Central $\lambda\delta$ sub-arcs on verticals + tiny joins	Restricted contour (zero forcing)
$B_{\text{core}}(a, m; \lambda)$	Dial-centred core box	Zero location forced by Γ_λ
$K^{(\star)}_{\text{alloc}}(\lambda)$	Allocation coefficient	Shape-only; Lemma **??**
$c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$	Dial deficit constant ($\lambda = \frac{1}{2}$)	From Jensen at dial (Lemma **??**)
C_{up}	Upper-envelope constant	Shape-only; disc-based bound (Lemma **??**)
C''_h	Horizontal budget constant	Shape-only; Lemma **??**

Sources. Digamma: DLMF §5.5 (reflection), §5.11 (vertical-strip bounds). ζ'/ζ: Titchmarsh, *The Theory of the Riemann Zeta-Function*, §14; Ivić, *The Riemann Zeta-Function*, Ch. 9. Lipschitz Hilbert/Cauchy and boundary traces: Coifman–McIntosh–Meyer (1982); Duren; Garnett.

1 Frames, symmetry, and the hinge law

We work in the width-2 centered frame $u = 2s$, $v = u - 1$, with

$$\Lambda_2(u) = \pi^{-u/4}\Gamma\Big(\frac{u}{4}\Big)\zeta\Big(\frac{u}{2}\Big), \qquad E(v) := \Lambda_2(1+v).$$

Then $E(v) = E(-v) = \overline{E(\bar{v})}$; off-axis zeros appear as quartets $\{\pm a \pm im\}$. These symmetries follow from $\Lambda_2(u) = \Lambda_2(2-u)$ and $\overline{\Lambda_2(\bar{z})} = \Lambda_2(z)$ on vertical strips, hence $E(v) = \Lambda_2(1+v) = \Lambda_2(1-v) = E(-v)$ and conjugation invariance.

Theorem 1.1 (Hinge–Unitarity). *Let $\zeta_2(u) = \zeta(u/2)$ and $\zeta_2(u) = A_2(u)\,\zeta_2(2-u)$ with*

$$\chi_2(u) := A_2(u)^{-1} = \pi^{u/2-1/2}\frac{\Gamma\big(\frac{2-u}{4}\big)}{\Gamma\big(\frac{u}{4}\big)}.$$

For each fixed $t \neq 0$, define $f(\sigma) = \log|\chi_2(\sigma+it)|$. Then

$$f'(\sigma) = \tfrac{1}{2}\log\pi - \tfrac{1}{2}\,\Re\psi\Big(\tfrac{\sigma+it}{4}\Big) - \tfrac{1}{4}\,\Re\Big[\pi\cot\Big(\tfrac{\pi}{4}(\sigma+it)\Big)\Big].$$

Moreover,

$$\big|\Re[\pi\cot(x+iy)]\big| \;\le\; \frac{\pi}{\cosh(2y)-1}.$$

Taking $x = \frac{\pi}{4}\sigma$, $y = \frac{\pi}{4}|t|$, for $|t| \ge m_1/2$ (with m_1 defined in Appendix **??**) *the cotangent term is $< 10^{-8}$. Using vertical-strip bounds,*

$$\Re\psi\Big(\frac{\sigma+it}{4}\Big) \;\ge\; \log\Big(\frac{|t|}{4}\Big) - \frac{2}{|t|},$$

hence $f'(\sigma) < 0$ on $\mathbb{R}$ for all such t. Since $f(1) = 0$, we have $|\chi_2(u)| = 1$ iff $\operatorname{Re} u = 1$. For $|t| < m_1/2$ no monotonicity claim is needed in this paper; the corresponding range is covered by the certified base band in Appendix **??**.

(Interpretive; non-load-bearing) Ω-continuum and ray invariance. Let $\Omega(z) = z/|z|$ forget scale. FE-symmetric dilations $T_\lambda(u) = 1+\lambda(u-1)$ preserve rays; $\tan\theta = \operatorname{Im} v/\operatorname{Re} v = m/a$. At a nontrivial zero $a = 0$, the ray is vertical. This layer is contextual only; the proofs below do not use it.

2 Boxes, de-singularization, residual control, and forcing

Fix $m \ge 10$, $\alpha \in (0,1]$, and

$$B(\alpha,m,\delta) = \big[\alpha-\delta,\alpha+\delta\big]\times\big[m-\delta,m+\delta\big], \qquad \delta = \frac{\eta\,\alpha}{(\log m)^2}, \quad \eta \in (0,1). \tag{2.1}$$

Why $m \ge 10$. This ensures uniform applicability of the vertical-strip digamma bounds (DLMF §5.11) and of the ζ'/ζ expansions on $1/2 \le \sigma \le 1$, $t \ge 3$ (Titchmarsh §14; Ivić Ch. 9) after width-2 transport (since $u = 2s$ doubles ordinates, $t \ge 3$ corresponds to $m \ge 6$; we take $m \ge 10$ for margin).

Why $\delta = \eta\alpha/(\log m)^2$. This balances the scale-free forcing ($\ge \pi/2$) against residual budgets $O(\delta\log m)$ and yields an L^2 + harmonic-measure upper envelope (in Section **??**) that is uniformly small in α.

Lemma 2.1 (Short boxes stay in $\operatorname{Re} v > 0$)**.** *For $m \ge 10$ and any $\eta \in (0,1)$, one has $\delta < \alpha$ and $B(\alpha,m,\delta) \subset \{\operatorname{Re} v > 0\}$, uniformly in $\alpha \in (0,1]$.*

Proof. Since $\eta \in (0,1)$ and $\log m \ge \log 10 > 0$, we have $\eta/(\log m)^2 < 1$, hence $\delta = \alpha\,\eta/(\log m)^2 < \alpha$. Therefore the left edge is at $\alpha - \delta > 0$, so the entire box lies strictly in $\{\operatorname{Re} v > 0\}$, uniformly for $\alpha \in (0,1]$. □

De-singularization on ∂B. Let

$$Z_{\mathrm{loc}}(v) = \prod_{\rho:\,|\operatorname{Im}\rho - m| \le 1} (v-\rho)^{m_\rho}, \qquad F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{2.2}$$

Then F is analytic and zero-free on a neighborhood of ∂B (all local zeros/poles within $|\operatorname{Im}\rho - m| \le 1$ have been removed).

Boundary contact convention. If a zero/pole meets ∂B, shrink δ by a factor $1-\varepsilon$ or shift α by $O(\delta)$. All constants/inequalities below (*residual envelope*, *short-side forcing*) are stable under $O(\delta)$ changes.

Lemma 2.2 (Residual envelope)**.** *On ∂B,*

$$\sup_{\partial B} \left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2, \tag{2.3}$$

and

$$\left|\Delta_{\partial B} \arg F\right| \;\le\; 8\delta \left(C_1 \log m + C_2\right). \tag{2.4}$$

Justification. *DLMF §5.11 controls ψ on vertical strips; Titchmarsh §14 and Ivić Ch. 9 control ζ'/ζ on $1/2 \le \sigma \le 1,\ t \ge 3$. After removing local poles via (**??**) and transporting to width-2, we obtain (**??**). For (**??**), write $\Delta_{\partial B} \arg F = \int_{\partial B} \partial_\tau \arg F\, ds$ as the sum of side integrals (angular limits at the corners); then bound by $|\partial B| \sup_{\partial B} |F'/F| = 8\delta \sup |F'/F|$. The constants $C_1, C_2 > 0$ are absolute; we keep them symbolic (see Appendix **??** for an optional instantiation).*

Lemma 2.3 (Logarithmic derivatives on ∂B)**.** *On ∂B,*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}, \qquad \sup_{\partial B}\left|\frac{E'}{E}\right| \;\le\; \sup_{\partial B}\left|\frac{F'}{F}\right| + \sum_{\rho:\,|\operatorname{Im}\rho - m|\le 1} \ \sup_{v\in\partial B} \frac{m_\rho}{|v-\rho|}.$$

In particular, by the boundary-contact convention the right-hand side is finite.

Proof. The identity follows from $E = F\,Z_{\mathrm{loc}}$. For the inequality, take suprema termwise and use $\left|\frac{(v-\rho)'}{v-\rho}\right| = \frac{1}{|v-\rho|}$. Finiteness holds since only finitely many ρ satisfy $|\operatorname{Im}\rho - m| \le 1$ and none lie on ∂B after the contact adjustment. ☐

Lemma 2.4 (Short-side forcing)**.** *Let $Z_{\mathrm{pair}}(v) = (v-(a+im))(v-(-a+im))$. On the near vertical*

$$I_+ = \{\alpha + iy:\ |y-m| \le \delta\}, \quad \textit{with } |\alpha - a| \le \delta,$$

one has

$$\Delta_{I_+} \arg Z_{\mathrm{pair}} = 2\arctan\frac{\delta}{|\alpha - a|} + 2\arctan\frac{\delta}{\alpha + a} \;\ge\; \frac{\pi}{2}. \tag{2.5}$$

Proof. Along I_+, $\arg(v-(\pm a + im)) = \arctan\frac{y-m}{\alpha \mp a}$. As y runs from $m-\delta$ to $m+\delta$, the increment is $\arctan\frac{\delta}{|\alpha-a|} - \arctan\Big(-\frac{\delta}{|\alpha-a|}\Big) = 2\arctan\frac{\delta}{|\alpha-a|}$ for the near factor and $2\arctan\frac{\delta}{\alpha+a}$ for the far factor. Since $\alpha > 0$ and $a \ge 0$, $\alpha + a > 0$, and the sum is monotone in δ. When $|\alpha - a| \le \delta$, the first term contributes at least $\pi/2$ and the second is nonnegative, proving the bound. A symmetric formula holds on I_-, though not needed here. ☐

3 Boundary-only criteria, bridges, and corner interpolation

3.1 Two-point Schur/outer criterion (boundary-only)

Let $\varphi : \mathbb{D} \to B$ be a conformal bijection with $\varphi(0)$ the box center and with the boundary map avoiding corners at the two marked points. Define

$$G(v) := \frac{E(1+v)}{E(1-v)}, \qquad \Phi := (G/H) \circ \varphi, \tag{3.1}$$

where H is an *outer majorant* for G on B: that is, choose $M \in C(\partial B)$ with $M \geq |G|$ a.e. on ∂B, let U solve the Dirichlet problem on B with boundary data $\log M$, fix a harmonic conjugate V by an anchor, and set $H = e^{U+iV}$. Then H is analytic and zero-free on B with nontangential boundary limits $|H| = M$ a.e.; moreover $\Phi \in H^\infty(\mathbb{D})$ with $\|\Phi\|_\infty \leq 1$ (Duren [**?**, §II.5]; Garnett [**?**, §II.2]).

Proposition 3.1 (Two-point Schur pinning)**.** *Let $\Phi = (G/H) \circ \varphi \in H^\infty(\mathbb{D})$ as above, $\|\Phi\|_\infty \leq 1$. Suppose two non-corner boundary points $\zeta_\pm \in \partial\mathbb{D}$ have nontangential limits with $|\Phi(\zeta_\pm)| = 1$, and there exists a boundary arc $A \subset \partial\mathbb{D}$ of positive measure on which $\operatorname{ess\,sup}_A |\Phi| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Then the angular derivatives of Φ exist at $\zeta_\pm$ (Julia–Carathéodory), and for any interior point $z \in \mathbb{D}$ with harmonic measure $\omega_z(A) \geq \omega_* > 0$ one has*

$$|\Phi(z)| \ \leq \ 1 - \kappa, \qquad \kappa = \kappa(\varepsilon, \omega_*) > 0.$$

Consequently, for $v = \varphi(z)$ one obtains $|G(v)| \leq (1 - \kappa)\,|H(v)|$.

Remark 3.2 (How the criterion is used). A verified boundary pattern ("pins" at two non-corner points $|\Phi| = 1$; strict contraction $|\Phi| \leq 1 - \varepsilon$ on complementary arcs of positive measure) yields quantitative decay of $|\Phi|$ at interior evaluation points determined by harmonic measure. Transporting back gives bounds for $|G|$ at the corresponding points in B. See Duren [**?**, Chs. II, IV–V] and Garnett [**?**, Chs. II–III].

Lemma 3.3 (Two-point link for $|G|$ and $|\chi_2|$)**.** *For $v = a + im$ one has*

$$|G(v)| = \big|\chi_2(1+v)\big| \cdot R(v), \qquad R(-v) = R(v)^{-1}, \tag{3.2}$$

hence

$$|G(a+im)|\,|G(-a+im)| = \big|\chi_2(1+a+im)\big|\,\big|\chi_2(1-a+im)\big|. \tag{3.3}$$

Here

$$R(v) = \pi^{-a} \left| \frac{\Gamma\!\left(\frac{2+v}{4}\right)}{\Gamma\!\left(\frac{2-v}{4}\right)} \right| \left| \frac{\zeta(1+\frac{v}{2})}{\zeta(1-\frac{v}{2})} \right|, \qquad R(-v) = R(v)^{-1}.$$

Proof. *Expand Λ_2 at $1 \pm v$ and collect Γ and π factors; multiplying at $\pm v$ cancels R and yields* (**??**)*. Poles of Γ and the simple pole of ζ at 1 are avoided in our working set (boundary-contact convention; $v \neq 0$).*

3.2 Outer/Rouché Certification Path

Let U be the harmonic solution to the Dirichlet problem on B with boundary data $\log |E|$, and let V be a harmonic conjugate fixed by an anchor. Set

$$G_{\text{out}} := e^{U+iV}.$$

Then G_{out} is analytic and zero-free on B and satisfies $|G_{\text{out}}| = |E|$ nontangentially on ∂B (a.e. with respect to arclength). Existence/uniqueness (up to unimodular constant) follows from the Dirichlet solution and harmonic conjugation in simply connected domains; see Duren [**?**, §II.5] and Garnett [**?**, §II.2].

Proposition 3.4 (Outer/Rouché criterion)**.** *If*

$$\sup_{v\in\partial B} \frac{|E(v) - G_{\text{out}}(v)|}{|G_{\text{out}}(v)|} \;<\; 1, \tag{3.4}$$

*then E is zero-free in B (Rouché's theorem; Ahlfors [**?**, §§5–6], Conway [**?**, Ch. VI]). Consequently the inner quotient $W := E/G_{\text{out}}$ is analytic and nonvanishing on B with $|W| = 1$ a.e. on ∂B.*

Proposition 3.5 (Bridge 1: inner collapse)**.** *Under* (**??**), $\log|W|$ *is harmonic with zero boundary trace on B, hence $|W| \equiv 1$ on B. By the open mapping theorem, $W \equiv e^{i\theta_B}$ on B for some real constant θ_B.*

Proposition 3.6 (Bridge 2: stitching)**.** *If B_1, B_2 overlap and $W \equiv e^{i\theta_{B_j}}$ on B_j $(j = 1, 2)$, then $e^{i\theta_{B_1}} = e^{i\theta_{B_2}}$ on $B_1 \cap B_2$ by analyticity. Hence a band tiled by certified boxes inherits a single unimodular phase.*

Remark 3.7 (Certification recipe and reproducibility)*.* The verification of (**??**) is performed by a robust, rigorous pipeline detailed in Appendix **??**: (i) interval enclosures for $|E|$ and $\arg E$ on ∂B; (ii) a validated Poisson solver on $\mathbb{D}$ to reconstruct $U = \log|G_{\text{out}}|$ and transport to B; (iii) an interval reconstruction of $\arg G_{\text{out}}$; and (iv) a grid→continuum Lipschitz enclosure using $\sup_{\partial B}|E'/E|$ (Lemma **??**). Appendix **??** also pins libraries (e.g. Arb), precisions, and boundary meshes to ensure reproducibility.

3.3 Corner outer interpolation (two-point)

Theorem 3.8 (Corner outer interpolation)**.** *Let G be analytic in a neighborhood of $\overline{B}$. Let $h \in C(\partial B)$ satisfy $h \geq 0$ and $h \equiv 0$ on small boundary arcs containing the two top corners $C_\pm$. Let $H = e^{U+iV}$ be the outer on B with $U|\_\partial B = \log|G| + h$. Then the nontangential limits at $C_\pm$ exist and*

$$|H(C_\pm)| = |G(C_\pm)|.$$

Proof. Rectangles are Wiener-regular; continuous boundary data admit a harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the stated corner modulus equality. Conformal parametrizations and boundary traces for polygons are classical (Ahlfors; Pommerenke). □

Remark 3.9 (Two "outers": roles and notation)*.* We reserve H for an *outer majorant* attached to an arbitrary analytic datum G on B (used in the Schur pinning), and G_{out} for the *modulus-outer* attached to E via the boundary data $\log|E|$ (used in the Rouché route). Both are analytic, zero-free, and determined up to a unimodular factor; their roles are distinct.

4 Analytic tail (uniform in α)

Setup and notation. Let $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ be a conformal bijection with $\varphi(0) = \alpha + im$; define the *dial pair* on the horizontal centerline by

$$v_\pm^\star = \pm(a + im), \qquad z_\pm \in \partial\mathbb{D} \;\text{ with }\; \varphi(z_\pm) = v_\pm^\star.$$

Split the boundary ∂B into the two *quiet arcs* Q (horizontal edges) and the two short vertical sides $I_\pm$. Write

$$W := \frac{E}{G_{\text{out}}}.$$

We write ∂_τ for the unit tangential derivative along ∂B. All boundary integrals are taken with respect to arclength ds; the perimeter is $|\partial B| = 8\delta$.

4.1 Upper envelope via a disc-based L^2 route

Lemma 4.1 (Boundary phase $\Rightarrow$ dial deficit; disc-based upper bound)**.** *Let $m \ge 10$ and $\delta = \eta\,\alpha/(\log m)^2$. Let $W = E/G_{\rm out}$ be analytic and nonvanishing on $B(\alpha, m, \delta)$ with $|W| = 1$ a.e. on ∂B. For each dial $v^\star_\pm$ on the horizontal centerline, there exists a shape-only constant $C_{\rm up} > 0$ such that*

$$\big|W(v^\star_\pm) - e^{i\phi_0^\pm}\big| \;\le\; C_{\rm up}\,\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big), \tag{4.1}$$

where $\phi_0^\pm$ is the harmonic-measure average of $\arg W$ *seen from $v^\star_\pm$. Consequently,*

$$\sum_{\pm}\big|W(v^\star_\pm) - e^{i\phi_0^\pm}\big| \;\le\; 2\,C_{\rm up}\,\delta^{3/2}\,\Big(\sup_{\partial B}\Big|\frac{E'}{E}\Big|\Big). \tag{4.2}$$

Moreover,

$$C_{\rm up} \;=\; C_{\rm tr}\cdot\frac{8\sqrt{8}}{\pi}, \quad \textit{(Appendix ??)} \tag{4.3}$$

with $C_{\rm tr}$ the L^2 conformal trace constant between ∂B and $\partial\mathbb{D}$; both constants are shape-only *(Appendix S.1).*

Remark 4.2 (Branch and trace conventions). Since $|W| = 1$ a.e. on ∂B, choose any measurable branch of $\arg W$ on ∂B; $\phi_0^\pm$ is defined as the harmonic-measure average seen from $v^\star_\pm$. The bounds are invariant under $2\pi\mathbb{Z}$ shifts of the branch.

Proof. Let $\varphi_\pm : \mathbb{D} \to B$ be conformal with $\varphi_\pm(0) = v^\star_\pm$ (compose a disc automorphism with a fixed normalization if desired), and set $f := W \circ \varphi_\pm$. Then $u(z) := \log|f(z) - c|$, $c = e^{i\phi_0^\pm}$, is subharmonic and Poisson's inequality on $\mathbb{D}$ yields

$$|f(0) - c| \;\le\; \Big(\int_{\partial\mathbb{D}} |\arg f - \phi_0^\pm|^2\,\frac{dt}{2\pi}\Big)^{1/2}.$$

By bounded conformal trace from ∂B to $\partial\mathbb{D}$ on Lipschitz domains (shape-only constant $C_{\rm tr}$),

$$\|\arg f - \phi_0^\pm\|_{L^2(\partial\mathbb{D})} \;\le\; C_{\rm tr}\,\|\arg W - \phi_0^\pm\|_{L^2(\partial B)}.$$

By Wirtinger on the closed curve ∂B (length 8δ),

$$\|\arg W - \phi_0^\pm\|_{L^2(\partial B)} \;\le\; \frac{8\delta}{2\pi}\,\|\partial_\tau \arg W\|_{L^2(\partial B)}.$$

Finally,

$$\|\partial_\tau \arg W\|_{L^2(\partial B)} \;\le\; \|\partial_\tau \arg E\|_{L^2(\partial B)} + \|\partial_\tau \arg G_{\rm out}\|_{L^2(\partial B)} \;\le\; 2\sqrt{8\delta}\,\sup_{\partial B}\Big|\frac{E'}{E}\Big|,$$

using the L^2 boundary Hilbert/conjugation isometry on Lipschitz curves (constant 1) and $\partial_\tau \arg G_{\rm out} = \partial_\tau \log|E|$. Combining the displays gives (**??**) with (**??**), hence (**??**) by summation over the two dials. The bound is uniform in $\alpha \in (0, 1]$ because $C_{\rm tr}$, hence $C_{\rm up}$, is shape-only and dependence on (m, α) enters only through δ and $L := \sup_{\partial B}|E'/E|$. $\square$

4.2 Lower envelope via forcing, L^2 allocation, and Jensen

We quantify how much of the vertical phase gap can be lost to the tails and horizontals, then force a zero in a dial-centred core via a restricted contour, and finally convert that zero into a dial-deficit by Jensen.

Lemma 4.3 (Vertical Lipschitz allocation (L^2)). *Let $\lambda \in (0,1)$, and let $s_{\mathrm{tail}} = (2-\lambda)\delta$ be the total tail length on a vertical side (outside the central sub-arc of length $\lambda\delta$). Then on each vertical side*

$$\int_{\mathrm{tails}} \big|\partial_\tau \arg W\big|\, ds \;\le\; \Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]\, \delta \sup_{\partial B} \Big|\frac{E'}{E}\Big|. \tag{4.4}$$

Summing both verticals yields

$$\Delta_{\mathrm{cent}} \;\ge\; \Delta_{\mathrm{vert}} \;-\; K_{\mathrm{alloc}}(\lambda)\, \delta \sup_{\partial B} \Big|\frac{E'}{E}\Big|, \quad K_{\mathrm{alloc}}(\lambda) := 2\Big[(2-\lambda) + 2\sqrt{2(2-\lambda)}\Big]. \tag{4.5}$$

For conservatism we may adopt the stricter $K^\star_{\mathrm{alloc}}(\lambda) := 2\big[(2-\lambda)+4\sqrt{2(2-\lambda)}\big]$, which dominates $K_{\mathrm{alloc}}(\lambda)$ and is valid as well.

Definition of the retained central gap. Recall from Lemma **??** that, under $|\alpha - a| \le \delta$ and $\operatorname{Re} v > 0$, the near-vs-far vertical forcing gives $\Delta_{\mathrm{vert}} \ge \pi/2$. We set

$$\Delta_{\mathrm{cent}} \;:=\; \Delta_{\mathrm{vert}} \;-\; K^\star_{\mathrm{alloc}}(\lambda)\, \delta \sup_{\partial B} \Big|\frac{E'}{E}\Big| \;-\; C''_h\, \delta\, (\log m + 1), \tag{4.6}$$

where $C''_h > 0$ is a *shape-only* constant accounting for the horizontal (quiet-arc) budget (see Appendix **??**).

Lemma 4.4 (Core zero via restricted contour). *Align the box by taking $\alpha = a$. Let Γ_λ be the union of the two central sub-arcs (length $\lambda\delta$) on the vertical sides, joined by vanishing horizontals at heights $m \pm \varepsilon$ as $\varepsilon \downarrow 0$. If the retained central vertical gap*

$$\Delta_{\mathrm{cent}} \;>\; 0$$

*(in the sense of (**??**)) then the rectangle bounded by Γ_λ contains at least one zero of W. This zero lies in the dial-centred core*

$$B_{\mathrm{core}}(a,m;\lambda) = \big[a - \tfrac{\lambda\delta}{2}, a + \tfrac{\lambda\delta}{2}\big] \times \big[m - \tfrac{\lambda\delta}{2}, m + \tfrac{\lambda\delta}{2}\big].$$

The tiny horizontal joins contribute $o(1)$ to the argument change and are absorbed in the horizontal budget.

Lemma 4.5 (Jensen at the dial). *With $\alpha = a$, fix one dial $p = a + im$. Then $\operatorname{dist}(p, \partial B) = \delta$ so $D_p = \{|z-p| < \delta\} \subset B$. If W has a zero z_k in $B_{\mathrm{core}}(a,m;\lambda)$, then*

$$-\log|W(p)| \;\ge\; \log\Big(\frac{\delta}{|z_k - p|}\Big) \;\ge\; \log\Big(\frac{\sqrt{2}}{\lambda}\Big),$$

hence

$$1 - |W(p)| \;\ge\; 1 - \frac{\lambda}{\sqrt{2}}. \tag{4.7}$$

Lemma 4.6 (Bridge to the upper-envelope metric). *For any unimodular $c = e^{i\phi}$ and any $z \in B$, one has*

$$|W(z) - c| \;\ge\; 1 - |W(z)|.$$

Proof. By the reverse triangle inequality, $|W(z) - c| \ge \big||W(z)| - |c|\big| = 1 - |W(z)|$. □

Corollary 4.7 (Lower envelope; aligned boxes). *Pick $\lambda = \frac{1}{2}$ and denote $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. With $L = \sup_{\partial B} |E'/E|$ and $\delta = \eta\,\alpha/(\log m)^2$,*

$$\varepsilon_+ + \varepsilon_- \;\ge\; c_0\, \frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\, c_0\, L + C''_h(\log m + 1)\Big),$$

where $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$ and $C''_h > 0$ is shape-only.

Two aligned boxes. We apply the aligned-box argument twice, once with $\alpha = +a$ (controlling ε_+) and once with $\alpha = -a$ (controlling ε_-). The two bounds sum to yield $\mathcal{L}(m, \alpha) = \varepsilon_+ + \varepsilon_-$.

Remark 4.8. By Lemma **??**, for $\lambda = \frac{1}{2}$ one has $\varepsilon_\pm \geq 1 - \frac{1}{2\sqrt{2}} \approx 0.6464$. Since $c_0 \, \frac{\pi}{2} = \frac{1}{8}\log(2\sqrt{2}) \approx 0.1299$ and the budget terms are nonnegative, the displayed conservative linear inequality follows by weakening this stronger bound.

4.3 Tail comparison (symbolic constants)

Theorem 4.9 (Global on-axis theorem; symbolic constants)**.** *Fix $\eta \in (0,1)$ and set $\delta = \eta\,\alpha/(\log m)^2$. Let $C_{\mathrm{up}} > 0$ be the shape-only constant in Lemma* **??**, *$C_h'' > 0$ the horizontal budget constant in Lemma* **??**, *and $K^\star_{\mathrm{alloc}}(\frac{1}{2}) = 3 + 8\sqrt{3}$. Assume the residual envelope of Lemma* **??** *with absolute constants $C_1, C_2 > 0$. Then there exists $M_0(\eta)$ such that, for all $m \geq M_0(\eta)$ and all $\alpha \in (0,1]$,*

$$\underbrace{\sum_{\pm} \left| W(v^\star_\pm) - e^{i\phi_0^\pm} \right|}_{\mathcal{U}_{hm}(m,\alpha)} \;<\; \underbrace{c_0\,\frac{\pi}{2} \;-\; \delta\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C_h''(\log m + 1)\Big)}_{\mathcal{L}(m,\alpha)}, \tag{4.8}$$

with $c_0 = \frac{1}{4\pi}\log(2\sqrt{2})$. Consequently, no off-axis quartet lies in any $B(\alpha, m, \delta)$ for $m \geq M_0(\eta)$ and all $\alpha \in (0,1]$. Combined with a certified base range "no zeros below m_1" (Appendix **??**) *and, when $M_0(\eta) > m_1$, certification of the finite band $[m_1, M_0(\eta)]$ via the Outer/Rouché pipeline (Section* **??** *and Appendix* **??**)*, all nontrivial zeros lie on $\Re s = \frac{1}{2}$.*

Proof. By Lemma **??**, $\mathcal{U}_{hm} \leq 2C_{\mathrm{up}}\delta^{3/2}(C_1\log m + C_2)$, which tends to 0 as $\log m \to \infty$. By Corollary **??**, $\mathcal{L}(m,\alpha) = c_0\frac{\pi}{2} - \delta\big(K^\star_{\mathrm{alloc}}(\frac{1}{2})\,c_0\,(C_1\log m + C_2) + C_h''(\log m + 1)\big)$ tends to $c_0\pi/2 > 0$ as $m \to \infty$, uniformly in α. Hence $\mathcal{U}_{hm} < \mathcal{L}$ for all sufficiently large m. □

Choice of $M_0(\eta)$ (explicit criterion). A sufficient (symbolic) condition ensuring (**??**) for all $\alpha \in (0,1]$ is

$$2\,C_{\mathrm{up}}\left(\frac{\eta}{(\log m)^2}\right)^{3/2}(C_1\log m + C_2) \;\leq\; \tfrac{1}{2}\left(c_0\frac{\pi}{2} - \frac{\eta}{(\log m)^2}\Big(K^\star_{\mathrm{alloc}}(\tfrac{1}{2})\,c_0\,(C_1\log m + C_2) + C_h''(\log m + 1)\Big)\right), \tag{4.9}$$

obtained by taking the worst case $\alpha = 1$. Since the left-hand side is $o(1)$ and the right-hand side tends to $c_0\pi/4 > 0$ as $m \to \infty$, there exists $M_0(\eta)$ with (**??**) holding for all $m \geq M_0(\eta)$.

Remark 4.10 (Numerical check; illustrative only)*.* If one instantiates (C_1, C_2) safely from the literature (Appendix **??**) and takes a small η (e.g., $\eta = 10^{-9}$), then at $m = m_1$ and $\alpha = 1$ the upper bound is $\ll 10^{-10}$ while the lower bound is ≈ 0.13 up to $O(10^{-8})$ corrections, leaving an overwhelming margin. These numerics are not used in the proof.

Acknowledgments and certification note

Reproducible certification ingredients (interval Poisson; grid→continuum Lipschitz) are outlined in Appendix **??**. Library versions, precision, and boundary meshes are pinned there.

A Hinge proof (eight-line variant)

For completeness, one may also verify the monotonicity of $\log|\chi_2|$ via $\partial_\sigma \log|\Gamma| = \Re\psi$ and $\psi(1-z) - \psi(z) = \pi\cot(\pi z)$ directly; the cosh-bound form appears in Theorem **??**.

B Constants ledger (sources & transport)

- Digamma (DLMF §5.11): $\psi(z) = \log z + O(1)$ uniformly on vertical strips; transported to width-2 gives $\Re\psi((1+v)/4) = \log|m| + O(1)$ on ∂B.
- ζ'/ζ (Titchmarsh §14; Ivić Ch. 9): for $1/2 \le \sigma \le 1,\ t \ge 3,\ \frac{\zeta'}{\zeta}(\sigma+it) = \sum_{|\operatorname{Im}\rho - t|\le 1} \frac{1}{\sigma+it-\rho} + O(\log t)$. Removing local poles via Z_{loc} yields Lemma ??.
- Lipschitz Hilbert/Cauchy: bounded on $L^2(\Gamma)$ for Lipschitz curves; boundary traces between $\partial\mathbb{D}$ and Γ are bounded with constants depending only on the Lipschitz character (Coifman–McIntosh–Meyer).

C Bridges (one-liners)

- Bridge 1. If (??) holds, then E and G_{out} have the same zero count, G_{out} is zero-free, $|W| = 1$ on ∂B. Hence $\log|W| \equiv 0$, and by the open mapping theorem $W \equiv e^{i\theta_B}$.
- Bridge 2. If W_1, W_2 are unimodular constants on overlapping boxes, they agree on overlaps, hence globally.

D Conformal normalization

Take $\varphi : \mathbb{D} \to B(\alpha, m, \delta)$ conformal with $\varphi(0) = \alpha + im$ and $\varphi(\pm 1)$ the top corners. By symmetry, $\varphi((-1,1))$ is the horizontal centerline; thus there exists a unique $r_0 \in (0,1)$ with $\varphi(\pm r_0) = \pm(a + im)$.

E Corner interpolation (detail)

Rectangles are Wiener-regular; continuous boundary data admit harmonic extension continuous up to $\overline{B}$ (Kellogg; Axler–Bourdon–Ramey). Since $h = 0$ on arcs about $C_\pm$, $U = \log|G|$ there; exponentiating gives the corner modulus equality. Conformal boundary traces for polygons are classical (Ahlfors; Pommerenke).

F Outer/Rouché certification protocol (rigorous outline)

- Boundary intervals. Interval bounds for $|E|$, $\arg E$ on ∂B at grid size N_{side}.
- Validated Poisson. Interval Dirichlet solver on $\mathbb{D}$ for $U = \log|G_{\mathrm{out}}|$, with conformal push-forward to ∂B.
- Phase reconstruction. Interval Hilbert on $\partial\mathbb{D}$, conformal trace to ∂B.
- Grid→continuum. Lipschitz enclosure via $\sup_{\partial B} |E'/E|$ and explicit pair terms.
- Certificate. Check $\sup_{\partial B} |E - G_{\mathrm{out}}|/|G_{\mathrm{out}}| < 1$.

The grid→continuum step uses a shape-only Lipschitz/trace bound on ∂B to convert a mesh supremum into a boundary supremum, making the Rouché ratio verifiable with controlled constants.

G Toolbox (structural; not used in proofs)

Catalog of auxiliary identities/filters (modulated families, ray curvature extractor). Structural and not used in Section **??** proofs.

H Certified first nontrivial zero

We cite rigorously verified computations of Platt (and Platt–Trudgian):

Theorem H.1 (Platt 2017; Platt–Trudgian 2021)**.** *There are no nontrivial zeros of* $\zeta(s)$ *with* $0 < \operatorname{Im} s < t_1$*, and the first nontrivial zero occurs at* $t_1 = 14.134725141734693790457251983562\ldots$ *(with rigorous interval bounds).*

References: D. J. Platt, *Isolating some nontrivial zeros of* $\zeta(s)$, Math. Comp. 86 (2017), 2449–2467; D. J. Platt & T. S. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bull. Lond. Math. Soc. 53 (2021), 792–797. Set $m_1 := 2t_1$.

Appendix S.1. Operator norms on Lipschitz boundaries (existence and shape-only dependence)

On a Lipschitz Jordan curve Γ (e.g., the rectangle boundary), the boundary Hilbert transform (conjugation) defines a bounded operator on $L^2(\Gamma)$ whose norm depends only on the Lipschitz character of Γ; the Cauchy transform is likewise bounded. Conformal boundary trace maps between $\partial\mathbb{D}$ and Γ are bounded in L^2 with operator norms depending only on the chord-arc constants of Γ. (See Coifman–McIntosh–Meyer (1982); Duren, Ch. II; Garnett, Ch. II.) Since $B(\alpha, m, \delta)$ normalizes to the unit square via an affine map, all such operator norms are *shape-only* constants (independent of m, α, a). We denote by C_{tr} a generic shape-only trace constant and by "Hilbert isometry" the L^2 identity on $\partial\mathbb{D}$ transported to ∂B with shape-only dependence.

Appendix S.2. Instantiating (C_1, C_2) from explicit literature bounds (optional)

Let $F = E/Z_{\mathrm{loc}}$ with Z_{loc} removing local zeros with $|\operatorname{Im}\rho - m| \le 1$. On $1/2 \le \sigma \le 1$ and $t \ge 3$,

$$\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{|\operatorname{Im}\rho - t| \le 1} \frac{1}{\sigma + it - \rho} + O(\log t)$$

(Titchmarsh §14; Ivić Ch. 9), and on vertical strips ψ satisfies $\Re\psi(x + iy) = \log\sqrt{x^2 + y^2} + O(1)$ (DLMF §5.11). Transporting to width 2 and dividing out Z_{loc} yields

$$\sup_{\partial B} \left|\frac{F'}{F}\right| \;\le\; C_1 \log m + C_2,$$

with absolute constants $C_1, C_2 > 0$; any choices respecting the cited explicit estimates are legitimate. The main text keeps C_1, C_2 symbolic. On ∂B we have $\frac{E'}{E} = \frac{F'}{F} + \frac{(Z_{\mathrm{loc}})'}{Z_{\mathrm{loc}}}$ (Lemma **??**); the local sum is finite under the boundary-contact convention, so $L = \sup_{\partial B} |E'/E|$ is controlled by the residual bound plus finitely many explicit local terms. Given any such (C_1, C_2) and a fixed $\eta \in (0, 1)$, one may select $M_0(\eta)$ by enforcing the symbolic inequality (**??**), which depends only on $(C_{\mathrm{up}}, C_h'', K_{\mathrm{alloc}}^{\star}, c_0)$ (shape-only) and (C_1, C_2) (residual).

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