A Width–2 Boundary Program for Excluding Off–Axis Quartets with a Certified Tail Criterion and a Finite–Height Front–End (v34)

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Abstract

This document is an audit-honest evolution (v34) of the width–2 boundary program. Relative to v33, the decisive change is the *UE–Gate audit*: the pointwise upper-envelope derivation as written supports a δ^1 prefactor (not $\delta^{3/2}$) against $\sup_{\partial B} |E'/E|$. Under the current collar/local-window mechanism, this makes the local penalty term δ–inert, so the η–absorption tail closure does *not* go through.

Accordingly, v34 reframes the core output as a *certified tail criterion* (a forcing-vs-envelope inequality at each height) and treats η–absorption as *conditional* on a genuinely stronger upper-envelope mechanism achieving an effective exponent $p > 1$ (or on a redesign that avoids the sup-based collar blow-up). All v33 improvements are retained: restored forcing constant provenance ($K_{\text{alloc}} = 3 + 8\sqrt{3}$), hardened Bridge 1 via Dirichlet outer factorization and uniqueness, and an explicit unconditional local-window count bound $N_{\text{loc}}(m) \leq 1.01 \log m + 17$ for $m \geq 10$.

The reproducibility pack is updated to record the UE exponent p explicitly so that exponent drift cannot silently occur in the certificate/harness layer.

Contents

Executive Proof Status

Status (v34): This version incorporates the Batch 3 UE–Gate audit across the v33 post-pivot spine. The manuscript is now *internally consistent* with the proved pointwise upper-envelope scaling: Lemma 10.3 yields a δ^1 prefactor against $\sup_{\partial B} |E'/E|$. Under the current collar/local-window program, this leaves a δ–inert local contribution of size $\frac{2C_{\text{up}}}{\kappa} N_{\text{up}}(m)$, where $N_{\text{up}}(m) = 1.01 \log m + 17$ is the available unconditional majorant, and therefore the *η–absorption tail closure is blocked* (see Remark 10.11 and Lemma 10.12).

Consequently, v34 reframes the core output as a *tail criterion*: at each height m, absence of off-axis quartets follows from an explicit forcing-vs-envelope inequality (Theorem 11.1). Uniform tail closure is treated as an *open UE–Gate problem* (obtain an effective exponent $p > 1$ against the local term, or redesign the envelope architecture to avoid the sup-based collar blow-up).

Retained from v33:

1. *Forcing constant provenance:* $K_{\text{alloc}} = 3 + 8\sqrt{3}$ is carried consistently.
2. *Bridge 1 hypotheses hardened:* outer factor via the Dirichlet problem on the box, and zero-free inner collapse via Dirichlet uniqueness.
3. *Local term explicit:* $N_{\text{loc}}(m) \le 1.01 \log m + 17$ for $m \ge 10$.
4. *κ–admissibility hygiene:* existence of admissible $\delta \le \delta_0$ and monotone safety under δ–shrinking.

Open proof-grade blockers (v34):

1. *UE–Gate redesign (primary):* produce an analytic mechanism that yields an effective δ–power $p > 1$ against the *local* term, or avoid the collar blow-up in the envelope endpoint (see Remark 10.11). (G–4, G–5)

2. *Residual envelope ledger:* Lemma 7.2 is upgraded to a δ–uniform proof sketch with explicit dependency control, but still imports a standard RH-free bound for ζ'/ζ with local-zero subtraction; this must be either fully proved in-text or cited in a referee-acceptable way. (G–1, G–12)

3. *Front-end dependence:* the finite-height hypothesis remains an external input (Appendix B). (G–11)

Reproducibility posture (v34): numerical artifacts are retained only as an *audit harness*. The v34 repro pack records the UE exponent p explicitly to prevent silent exponent drift and can be used to evaluate the tail inequality given any proposed constant ledger.

Part I
Reader's Guide / Definitions and Reduction

1 Width–2 normalization

Define the width–2 objects

$$u := 2s, \qquad \zeta_2(u) := \zeta\left(\tfrac{u}{2}\right), \qquad \Lambda_2(u) := \pi^{-u/4}\,\Gamma\left(\tfrac{u}{4}\right)\,\zeta\left(\tfrac{u}{2}\right). \tag{1}$$

Then Λ_2 is entire and satisfies the functional equation

$$\Lambda_2(u) = \Lambda_2(2-u). \tag{2}$$

We recenter at $u = 1$:

$$v := u - 1, \qquad E(v) := \Lambda_2(1+v). \tag{3}$$

The functional equation becomes the evenness relation

$$E(v) = E(-v), \tag{4}$$

and complex conjugation gives $E(\overline{v}) = \overline{E(v)}$.

2 Heights and horizontal displacement (RH–free)

Let $\rho = \beta + i\gamma$ be any nontrivial zero of $\zeta(s)$ (no assumption on β). In width–2 we write

$$u_\rho := 2\rho = (1+a) + im, \qquad a := 2\beta - 1 \in (-1,1), \qquad m := 2\gamma > 0. \tag{5}$$

Thus RH is equivalent to $a = 0$ for every nontrivial zero.

3 Quartet symmetry in width–2

The functional equation and conjugation imply that any off–axis zero with parameters (a,m) produces a quartet

$$\{\,1 \pm a \pm im\,\} \subset \{u \in \mathbb{C} : \Lambda_2(u) = 0\}. \tag{6}$$

In the centered v–coordinate this becomes $\{\pm a \pm im\} \subset \{v \in \mathbb{C} : E(v) = 0\}$.

4 Finite-height front-end after lowering the tail anchor

Once the tail anchor is lowered to $m_\star$, the analytic tail argument covers all $m \ge m_\star$. The remaining region corresponds to classical heights

$$0 < \mathrm{Im}(s) < H_0 := m_\star/2. \tag{7}$$

In v31 we take $m_\star = 10$, hence $H_0 = 5$.

Definition 4.1 (Front-end statement)**.** We say that *RH holds up to height* H_0 if every nontrivial zero $\rho = \beta + i\gamma$ with $0 < \gamma \le H_0$ satisfies $\beta = 1/2$.

Remark 4.2 (How v31 discharges the front-end). The required statement for v31 is RH up to height $H_0 = 5$. This is a tiny special case of published rigorous verifications of RH to enormous heights. For example, Platt–Trudgian prove RH for all zeros with $0 < \gamma \le 3 \cdot 10^{12}$ using interval arithmetic, which immediately implies RH up to $H_0 = 5$. Appendix B records this discharge in a pinned JSON file.

Part II
Self-Contained Boundary Program and Tail Closure

5 Aligned boxes and the $\delta(m)$ scale

Let $m > 0$ and $\alpha \in (0, 1]$. Fix a parameter $\eta \in (0, 1)$ and define the *nominal* box scale

$$\delta_0 = \delta_0(m, \alpha) := \frac{\eta\alpha}{(\log m)^2}. \tag{8}$$

We will work with aligned boxes $B(\alpha, m, \delta)$ at scales $0 < \delta \le \delta_0$. By default one may take $\delta = \delta_0$, but later (Definition 10.5) we allow shrinking δ to enforce κ–admissibility; this is non-circular and monotone-safe (Lemmas 10.6 and 11.2).

Define the (width–2) box centered at $\alpha + im$ by

$$B(\alpha, m, \delta) := \{\, v \in \mathbb{C} : |\mathrm{Re}\, v - \alpha| \le \delta,\ |\mathrm{Im}\, v - m| \le \delta \,\}. \tag{9}$$

We will also use the symmetric dial centers $v_\pm := \pm\alpha + im$.

6 Local factor and finiteness

For a fixed $m > 0$, let

$$Z(m) := \{\, \rho : E(\rho) = 0,\ |\mathrm{Im}\, \rho - m| \le 1 \,\} \tag{10}$$

(zeros counted with multiplicity). Define the local zero factor and residual:

$$Z_{\mathrm{loc}}(v) := \prod_{\rho \in Z(m)} (v - \rho)^{m_\rho}. \tag{11}$$

$$F(v) := \frac{E(v)}{Z_{\mathrm{loc}}(v)}. \tag{12}$$

Lemma 6.1 (Finiteness of $Z_{\rm loc}$)**.** *For each fixed $m > 0$ the set $Z(m)$ is finite; hence $Z_{\rm loc}$ is a finite product and F is meromorphic globally and analytic in any neighborhood of $\partial B(\alpha, m, \delta)$ that contains no zeros of E.*

Proof. Nontrivial zeros of ζ satisfy $0 < \beta < 1$, hence in the v–coordinate one has $\operatorname{Re} v \in (-1, 1)$ for all nontrivial zeros. Therefore the set $\{|\operatorname{Im} v - m| \le 1\} \cap \{|\operatorname{Re} v| \le 1\}$ is compact. Since E is entire and its zeros are discrete, only finitely many zeros can lie in this compact set. □

7 Residual envelope bound and the constants ledger

Remark 7.1 (Constant gate for the residual term (what is and is not assumed)). The tail criterion uses a bound of the form

$$\sup_{v \in \partial B(\alpha,m,\delta)} \left| \frac{F'(v)}{F(v)} \right| \;\le\; C_1 \log m + C_2,$$

with constants that must be (i) unconditional (not RH-equivalent) and (ii) uniform in $(\alpha, \delta, \eta, \kappa)$ once $m \ge 10$ and $0 < \alpha \le 1$. The proof below reduces this to standard RH-free bounds for ζ'/ζ in the critical strip with local-zero subtraction, plus a Stirling-type bound for Γ'/Γ.

Lemma 7.2 (Residual envelope inequality (δ-uniform))**.** *Fix $m \ge 10$ and $\alpha \in (0, 1]$. Let $\eta \in (0, 1]$ and set the nominal width $\delta_0 := \eta\alpha/(\log m)^2$. Let $\delta \in (0, \delta_0]$ and set $B := B(\alpha, m, \delta)$.*

Define E, $Z_{\rm loc}$ and $F := E/Z_{\rm loc}$ as in §6 (equations (11)*–*(12)*). Assume* boundary-contact *on ∂B (i.e. $E \ne 0$ on ∂B; hence F is holomorphic on a neighborhood of ∂B). Then there exist absolute constants $C_1, C_2 > 0$ (independent of $m, \alpha, \delta, \eta, \kappa$ and of the zero configuration) such that*

$$\sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right| \;\le\; C_1 \log m + C_2.$$

Proof sketch with explicit dependency control. Write $u := 1 + v$ and $s := u/2 = (1 + v)/2 = \sigma + it$. For $v \in \partial B(\alpha, m, \delta)$ we have $\operatorname{Re}(s) \in [0, 1]$ and

$$\operatorname{Im}(s) = \frac{\operatorname{Im}(v)}{2} \in \Big[\frac{m}{2} - \frac{\delta}{2}, \frac{m}{2} + \frac{\delta}{2}\Big].$$

Since $m \ge 10$ and $\delta \le \delta_0 \le 1/(\log 10)^2 < 1/5$, we have $\operatorname{Im}(s) \asymp m$ uniformly in δ.

1) Log-derivative identity in the s-frame. From $\Lambda_2(u) = \pi^{-u/4}\Gamma(u/4)\zeta(u/2)$ we obtain for $u = 1 + v$:

$$\frac{E'(v)}{E(v)} = -\frac{1}{4}\log \pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{1+v}{4}\Big) + \frac{1}{2}\frac{\zeta'}{\zeta}(s).$$

Moreover, since $v = 2s - 1$, the local factor derivative satisfies

$$\frac{Z_{\rm loc}'(v)}{Z_{\rm loc}(v)} = \sum_{\rho \in Z(m)} \frac{m_\rho}{v - \rho} = \frac{1}{2} \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s},$$

where $Z_s(m)$ denotes the corresponding multiset of nontrivial zeros $\rho_s = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma - \frac{m}{2}| \le \frac{1}{2}$.

Therefore

$$\frac{F'(v)}{F(v)} = \frac{E'(v)}{E(v)} - \frac{Z_{\rm loc}'(v)}{Z_{\rm loc}(v)} = -\frac{1}{4}\log \pi + \frac{1}{4}\frac{\Gamma'}{\Gamma}\Big(\frac{1+v}{4}\Big) + \frac{1}{2}\bigg(\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{m_{\rho_s}}{s - \rho_s}\bigg).$$

2) RH-free residual bound for ζ'/ζ with local-zero subtraction. A standard "local-zero decomposition" (unconditional) asserts that there exist absolute constants A_ζ, B_ζ such that for $0 \le \sigma \le 1$ and $t \ge 5$,

$$\Big|\frac{\zeta'}{\zeta}(\sigma+it) - \sum_{|\gamma-t|\le 1} \frac{1}{(\sigma+it)-\rho}\Big| \;\le\; A_\zeta \log(t+2) + B_\zeta. \tag{$\star$}$$

(For a self-contained route, ($\star$) can be derived from the Hadamard product for $\xi(s)$ plus a Riemann–von Mangoldt bound for $N(T)$; otherwise cite a standard reference.)

For $v \in \partial B$ we have $|t - \frac{m}{2}| \le \delta/2 < 1/10$, hence every zero in $Z_s(m)$ satisfies $|\gamma - t| \le 1$ and is included in the sum in ($\star$). Thus

$$\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s-\rho_s} = \Big(\frac{\zeta'}{\zeta}(s) - \sum_{|\gamma-t|\le 1} \frac{1}{s-\rho}\Big) + \sum_{\substack{|\gamma-t|\le 1 \\ |\gamma-\frac{m}{2}|>1/2}} \frac{1}{s-\rho}.$$

In the remaining sum we have $|\gamma - t| \ge 1/2 - |t - \frac{m}{2}| \ge 2/5$, hence $|s-\rho| \ge 2/5$ and each term has modulus $\le 5/2$. The number of zeros with $|\gamma - t| \le 1$ is bounded by the manuscript's explicit local window majorant (Lemma 10.9) at height $\asymp m$, so this difference-of-windows sum is $\ll \log m$.

Combining these bounds yields absolute constants $A_{\mathrm{res}}, B_{\mathrm{res}}$ such that

$$\Big|\frac{\zeta'}{\zeta}(s) - \sum_{\rho_s \in Z_s(m)} \frac{1}{s-\rho_s}\Big| \;\le\; A_{\mathrm{res}} \log m + B_{\mathrm{res}},$$

uniformly for all $v \in \partial B$ and all $\delta \in (0, \delta_0]$.

3) Gamma factor bound (Stirling, uniform in δ). For $z = (1+v)/4$ we have $\mathrm{Re}(z) \in [1/4, 3/4]$ and $|\,\mathrm{Im}(z)| \asymp m$. A uniform Stirling-type bound gives

$$\Big|\frac{\Gamma'}{\Gamma}(z)\Big| \le \log(|\,\mathrm{Im}(z)| + 2) + C_\Gamma \;\le\; \log(m+2) + C_\Gamma,$$

with an absolute constant C_Γ.

4) Conclusion. Insert the bounds from (2)–(3) into the identity in (1), and absorb harmless constants into (C_1, C_2). All constants are independent of $(\alpha, \delta, \eta, \kappa)$ because: (i) σ ranges over a fixed compact interval $[0,1]$, (ii) $t \asymp m$ with $m \ge 10$ uniformly for $\delta \le \delta_0$, and (iii) the difference-of-windows sum is controlled by Lemma 10.9, which is unconditional. □

Remark 7.3 (Hard gate / certificates (v34)). The tail harness in Appendix A uses explicit numerical interval enclosures for the constant ledger (e.g. $C_1, C_2, C_{\mathrm{up}}, C_h'', \kappa$) stored in `v34_repro_pack/v34_constants_m10.j`
It evaluates the tail inequality for a pinned parameter choice and records the UE exponent p explicitly. This is an *audit harness* only: it does not certify that the constants file is correct, and it does not, by itself, yield a uniform tail closure. An unconditional proof therefore still requires a referee-acceptable certification of the analytic constant ledger, and a resolution of the UE–Gate (Remark 10.11).

8 Short-side forcing

Assume an off-axis pair at height m with displacement $a > 0$ exists. On an aligned box with $\alpha = a$, the two upper zeros in the centered v–plane are at $v = \pm a + im$. The pair factor

$$Z_{\mathrm{pair}}(v) := (v - (a+im))(v - (-a+im)) \tag{13}$$

produces a large phase rotation on the near vertical side.

Lemma 8.1 (Short-side forcing lower bound)**.** *Let $I_+ := \{\alpha + iy : |y - m| \le \delta\}$ with $|\alpha - a| \le \delta$. Then*

$$\Delta_{I_+} \arg Z_{\text{pair}} = 2\arctan\left(\frac{\delta}{|\alpha - a|}\right) + 2\arctan\left(\frac{\delta}{\alpha + a}\right) \ge \frac{\pi}{2}. \tag{14}$$

9 Outer factorization and the inner quotient (Bridge 1)

We work on a fixed box $B = B(\alpha, m, \delta)$ and write B° for its interior. Assume the boundary-contact convention: E has no zeros on ∂B.

Lemma 9.1 (Dirichlet outer factor on a box)**.** *Let $B = B(\alpha, m, \delta)$ be the closed rectangle and B° its interior. Assume E is holomorphic on a neighborhood of $\overline{B}$ and $E \ne 0$ on ∂B. Then $\log|E| \in C(\partial B)$. Let $U \in C(\overline{B}) \cap \mathrm{Harm}(B^\circ)$ be the unique solution of the Dirichlet problem with boundary data $U|_{\partial B} = \log|E|$. Since B° is simply connected, there exists a harmonic conjugate V on B° (unique up to an additive constant) such that $U + iV$ is holomorphic on B°. Define*

$$G_{\text{out}}(v) := \exp(U(v) + iV(v)), \qquad v \in B^\circ.$$

Then G_{out} is holomorphic and zero-free on B°, satisfies $|G_{\text{out}}(v)| = e^{U(v)}$ for $v \in B^\circ$, and

$$\lim_{z \to \xi,\ z \in B^\circ} |G_{\text{out}}(z)| = |E(\xi)| \qquad (\xi \in \partial B).$$

Proof. Continuity of $\log|E|$ on ∂B follows from $E \ne 0$ on ∂B. Existence and uniqueness of U on a rectangle are standard. Since B° is simply connected, U admits a harmonic conjugate V on B°, unique up to an additive constant. The function $U + iV$ is holomorphic, hence so is $G_{\text{out}} = \exp(U + iV)$, and it is zero-free. Finally $|G_{\text{out}}| = e^U$ on B°, and by continuity of U on $\overline{B}$ we have $e^{U(\xi)} = |E(\xi)|$ on ∂B, yielding the boundary modulus identity in interior-limit form. □

Define on B° the inner quotient

$$W(v) := \frac{E(v)}{G_{\text{out}}(v)}.$$

Then W is holomorphic on B° and $|W| = 1$ on ∂B in the sense of interior limits in modulus.

Proposition 9.2 (Bridge 1: zero-free inner collapse)**.** *Assume the setup of Lemma 9.1 and define $W = E/G_{\text{out}}$ on B°. If W is zero-free on B° (equivalently, E is zero-free on B°), then W is constant on B°; in fact $W \equiv e^{i\theta_B}$ for some $\theta_B \in \mathbb{R}$.*

Proof. Since W is zero-free on B° and G_{out} is zero-free, the function E is zero-free on B°. Because B° is simply connected, E admits a holomorphic logarithm on B°, so $\log|E|$ is harmonic on B°. By construction U is harmonic on B°, continuous on $\overline{B}$, and equals $\log|E|$ on ∂B. Thus $U - \log|E|$ is harmonic on B° with zero boundary values, so by Dirichlet uniqueness $U \equiv \log|E|$ on B°. Therefore for $v \in B^\circ$,

$$|W(v)| = \frac{|E(v)|}{|G_{\text{out}}(v)|} = \frac{|E(v)|}{e^{U(v)}} = \frac{|E(v)|}{e^{\log|E(v)|}} = 1.$$

An analytic function of constant modulus on a connected open set is constant, hence $W \equiv e^{i\theta_B}$. □

Remark 9.3 (Boundary modulus convention). Under boundary-contact, U extends continuously to ∂B and satisfies $U|_{\partial B} = \log|E|$. Hence $|G_{\text{out}}| = |E|$ holds pointwise on ∂B as interior limits in modulus, and therefore $|W| = 1$ holds pointwise in modulus on ∂B. In boundary integral estimates this may be used in the a.e. sense without change.

10 Shape-only invariance and the envelope constants

Let $T(v) := (v - (\alpha + im))/\delta$, mapping ∂B affinely onto the fixed square boundary ∂Q with $Q = [-1, 1]^2$.

Lemma 10.1 (Shape-only invariance)**.** *Any constant arising solely from geometric or boundary-operator estimates on ∂B that are invariant under affine rescaling depends only on ∂Q and is independent of (α, m, δ).*

Proof. Under T, arclength scales by δ and tangential derivatives by $1/\delta$. After normalization, all purely geometric quantities and operator norms reduce to fixed quantities on ∂Q. □

Lemma 10.2 (Boundary-to-center evaluation in L^2 (sharp $\delta^{-1/2}$))**.** *Let $B = B(\alpha, m, \delta)$ be a box and let v_0 be its center. Let u be harmonic on B° and assume its boundary trace lies in $L^2(\partial B, ds)$. Then, writing $P_B(v_0, \xi) = d\omega_{v_0}^B/ds(\xi)$ for the Poisson kernel of B at v_0,*

$$|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)}\ \|u\|_{L^2(\partial B, ds)}.$$

Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q,

$$\|P_B(v_0, \cdot)\|_{L^2(\partial B, ds)} = \delta^{-1/2}\ \|P_Q(0, \cdot)\|_{L^2(\partial Q, ds)}.$$

In particular the exponent $\delta^{-1/2}$ is sharp (witnessed by the constant harmonic function $u \equiv 1$).

Proof. For harmonic u on B° with L^2 trace on ∂B, the Poisson representation gives

$$u(v_0) = \int_{\partial B} u(\xi)\, d\omega_{v_0}^B(\xi) = \int_{\partial B} u(\xi)\, P_B(v_0, \xi)\, ds(\xi).$$

Cauchy–Schwarz yields $|u(v_0)| \le \|P_B(v_0, \cdot)\|_{L^2}\|u\|_{L^2}$.

For the scaling: under T, arclength scales by $ds = \delta\, ds_Q$ and Poisson kernels scale by $P_B(v_0, \xi) = \delta^{-1} P_Q(0, T(\xi))$. Hence

$$\int_{\partial B} P_B(v_0, \xi)^2\, ds(\xi) = \int_{\partial Q} \delta^{-2} P_Q(0, \zeta)^2\, \delta\, ds_Q(\zeta) = \delta^{-1} \int_{\partial Q} P_Q(0, \zeta)^2\, ds_Q(\zeta),$$

giving $\|P_B(v_0, \cdot)\|_{L^2} = \delta^{-1/2}\|P_Q(0, \cdot)\|_{L^2}$.

Sharpness: for $u \equiv 1$ we have $|u(v_0)| = 1$ and $\|u\|_{L^2(\partial B)} = \sqrt{|\partial B|} \asymp \delta^{1/2}$, so the inequality forces $\|P_B(v_0, \cdot)\|_{L^2} \gtrsim \delta^{-1/2}$. □

Lemma 10.3 (Upper envelope bound (outer-aligned form))**.** *Let $B = B(\pm a, m, \delta)$ be an aligned box and let G_{out} be the outer factor on B constructed from $\log |E|$ on ∂B (Section 9). Define the inner quotient*

$$W(v) := \frac{E(v)}{G_{\mathrm{out}}(v)}.$$

Assume the boundary-contact convention: E has no zeros on ∂B (hence W has unimodular boundary values a.e.). For each sign $\pm$ let $v_\pm := \pm a + im$ and let $e^{i\varphi_0^\pm} \in \mathbb{T}$ be an $L^2(\partial B, ds)$-best constant phase,

$$e^{i\varphi_0^\pm} \in \arg\min_{|c|=1} \int_{\partial B} |W(v) - c|^2\, ds(v).$$

Then there exists a shape-only *constant* $C_{\mathrm{up}} > 0$ *(depending only on the normalized square* $Q = [-1,1]^2$*) such that*

$$\sum_{\pm} \big| W(v_\pm) - e^{i\varphi_0^\pm} \big| \;\le\; 2\, C_{\mathrm{up}}\, \delta \sup_{v\in\partial B} \Big| \frac{E'(v)}{E(v)} \Big|. \tag{15}$$

One admissible explicit definition is

$$C_{\mathrm{up}} := \Big(\sup_{\xi\in\partial Q} P_Q(0,\xi) \Big)^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2\to L^2} \big),$$

where $P_Q(0,\xi) = d\omega_0^Q/ds(\xi)$ *is the Poisson kernel of* Q *at the center* 0 *with respect to arclength on* ∂Q*, and* $H_{\partial Q}$ *is the boundary conjugation (Hilbert/Cauchy) operator on* ∂Q*.*

Remark 10.4 (No residual proxying in the upper envelope). Lemma 10.3 controls the inner quotient $W = E/G_{\mathrm{out}}$ and therefore depends on $\sup_{\partial B} |E'/E|$. Residual bounds for $F = E/Z_{\mathrm{loc}}$ control $\sup_{\partial B} |F'/F|$ and do *not* by themselves bound $\sup_{\partial B} |E'/E|$. Whenever the residual envelope is used to control dial deviation, it must be routed through the log-derivative split $E'/E = F'/F + Z'_{\mathrm{loc}}/Z_{\mathrm{loc}}$ (Lemma 10.7) together with the collar bound (Lemma 10.8), yielding Corollary 10.10.

Proof. Fix one sign and write $v_0 = v_\pm$ and $B = B(\pm a, m, \delta)$. We record the (RH-free) chain and indicate the scale factors explicitly.

1. **Evaluation from the boundary (harmonic measure; produces $\delta^{-1/2}$).** For any constant $c \in \mathbb{T}$, subharmonicity of $|W - c|^2$ implies

   $$|W(v_0) - c|^2 \le \int_{\partial B} |W(\xi) - c|^2 \, d\omega_{v_0}^B(\xi) = \int_{\partial B} |W(\xi) - c|^2 \, P_B(v_0,\xi)\, ds(\xi),$$

   so

   $$|W(v_0) - c| \le \|P_B(v_0,\cdot)\|_{L^\infty(\partial B)}^{1/2} \, \|W - c\|_{L^2(\partial B, ds)}.$$

   Under the similarity $T(\xi) = (\xi - v_0)/\delta$ mapping ∂B onto ∂Q, Poisson kernels scale by $\|P_B(v_0,\cdot)\|_\infty^{1/2} = \delta^{-1/2}\, \|P_Q(0,\cdot)\|_\infty^{1/2}$.

2. **Poincaré/Wirtinger on ∂B (produces δ).** For the L^2-best constant $c = e^{i\varphi_0^\pm}$ and $|\partial B| = 8\delta$, periodic Poincaré on a loop of length 8δ gives

   $$\|W - c\|_{L^2(\partial B)} \le \frac{|\partial B|}{2\pi} \, \|\partial_s W\|_{L^2(\partial B)} = \frac{4\delta}{\pi} \, \|\partial_s W\|_{L^2(\partial B)}.$$

3. **Outer factor control (no δ; uses bounded boundary conjugation).** Write $\log G_{\mathrm{out}} = U + i\widetilde{U}$ with $U|_{\partial B} = \log|E|$ and $\widetilde{U} = H_{\partial B} U$. Differentiating tangentially, $\partial_s \log G_{\mathrm{out}} = \partial_s U + i\, H_{\partial B}(\partial_s U)$. Since $\log W = \log E - \log G_{\mathrm{out}}$,

   $$\|\partial_s \log W\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2\to L^2}\big) \, \|\partial_s \log E\|_{L^2(\partial B)} \le \big(1 + \|H_{\partial B}\|_{L^2\to L^2}\big) \, \Big\| \frac{E'}{E} \Big\|_{L^2(\partial B)}.$$

   On ∂B we have $|W| = 1$ a.e., hence $|\partial_s W| = |\partial_s \log W|$.

4. **L^2 to sup (produces $\delta^{1/2}$).** Using $|\partial B| = 8\delta$,

   $$\Big\| \frac{E'}{E} \Big\|_{L^2(\partial B)} \le \sqrt{|\partial B|} \, \sup_{\partial B} \Big| \frac{E'}{E} \Big| = \sqrt{8\delta} \, \sup_{\partial B} \Big| \frac{E'}{E} \Big|.$$

Combining the four steps yields

$$|W(v_0) - e^{i\varphi_0^\pm}| \le \|P_Q(0,\cdot)\|_\infty^{1/2} \cdot \frac{4}{\pi} \cdot \sqrt{8} \cdot \big(1 + \|H_{\partial Q}\|_{L^2\to L^2}\big) \cdot \delta \sup_{\partial B} \Big|\frac{E'}{E}\Big|,$$

where we used the similarity invariance $\|H_{\partial B}\|_{L^2\to L^2} = \|H_{\partial Q}\|_{L^2\to L^2}$. Summing over $\pm$ gives (15). $\square$

10.1 Local factor split and collar control

Definition 10.5 (Collar-admissible aligned boxes)**.** Fix once and for all a collar parameter $\kappa \in (0, 1/10)$. An aligned box $B = B(\alpha, m, \delta)$ is called *κ–admissible* if

$$\mathrm{dist}\big(\partial B,\ \mathcal{Z}(E)\big) \ge \kappa\delta.$$

Given any nominal scale $\delta_0 > 0$ and any center, there exists some $0 < \delta \le \delta_0$ for which κ–admissibility holds (Lemma 10.6). Whenever a chosen box is not κ–admissible, we shrink δ until κ–admissibility holds. Moreover the assembled tail inequality is monotone-safe under such δ–shrinking (Lemma 11.2).

Lemma 10.6 (Existence of a κ–admissible shrink)**.** *Fix $\kappa \in (0, 1/10)$ and a center $v_0 \in \mathbb{C}$. For every $\delta_0 > 0$ there exists $\delta' \in (0, \delta_0]$ such that the closed box*

$$B(v_0, \delta') := \{\, v \in \mathbb{C} :\ \|v - v_0\|_\infty \le \delta' \,\}$$

satisfies

$$\mathrm{dist}\big(\partial B(v_0, \delta'),\ \mathcal{Z}(E)\big) \ge \kappa\delta'.$$

In particular, given (α, m) and nominal $\delta_0 = \eta\alpha/(\log m)^2$, one may always choose a scale $0 < \delta \le \delta_0$ for which $B(\alpha, m, \delta)$ is κ–admissible.

Proof. Zeros of the entire function E are isolated. Choose $\varepsilon > 0$ such that $\mathcal{Z}(E) \cap \{\, 0 < \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) = 0$) or such that $\mathcal{Z}(E) \cap \{\, \|v - v_0\|_\infty \le \varepsilon \,\}$ is empty (if $E(v_0) \ne 0$). Set $\delta' := \min\{\delta_0, \varepsilon/(1+\kappa)\}$. Then every boundary point satisfies $\|v - v_0\|_\infty = \delta'$. Any zero $\rho \in \mathcal{Z}(E)$ is either $\rho = v_0$ (in which case $\mathrm{dist}(v, \rho) = \delta' \ge \kappa\delta'$) or satisfies $\|\rho - v_0\|_\infty \ge \varepsilon$ (in which case $\mathrm{dist}(v, \rho) \ge \varepsilon - \delta' \ge \kappa\delta'$). Therefore $\mathrm{dist}(\partial B(v_0, \delta'), \mathcal{Z}(E)) \ge \kappa\delta'$. $\square$

Lemma 10.7 (Log-derivative decomposition)**.** *With Z_{loc} and F as in* (11) *and* (12)*, one has on any region where E and Z_{loc} are holomorphic and nonvanishing (in particular on ∂B under the boundary-contact convention)*

$$\frac{E'}{E} = \frac{F'}{F} + \frac{Z'_{\mathrm{loc}}}{Z_{\mathrm{loc}}}.$$

Lemma 10.8 (Buffered local factor bound on ∂B)**.** *Let $B = B(\alpha, m, \delta)$ be κ–admissible in the sense of Definition 10.5. Then*

$$\sup_{v\in\partial B} \left|\frac{Z'_{\mathrm{loc}}(v)}{Z_{\mathrm{loc}}(v)}\right| \le \frac{N_{\mathrm{loc}}(m)}{\kappa\,\delta},$$

where $N_{\mathrm{loc}}(m)$ counts zeros of E in the local window used to define Z_{loc}, with multiplicity.

Lemma 10.9 (Explicit local window zero count)**.** *Let $N(T)$ denote the number of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \le T$, counted with multiplicity. Then for every $T \ge 5$,*

$$N(T+1) - N(T-1) \le 1.01 \log T + 17. \tag{16}$$

Consequently, for every $m \ge 10$,

$$N_{\mathrm{loc}}(m) \le 1.01 \log m + 17. \tag{17}$$

Proof. By [7, Theorem 1.1], for every $x \ge e$,

$$\Big| N(x) - \frac{x}{2\pi} \log\Big(\frac{x}{2\pi e}\Big) \Big| \le 0.10076 \log x + 0.24460 \log\log x + 8.08344.$$

Let $M(x) := \frac{x}{2\pi} \log(\frac{x}{2\pi e})$, so $M'(x) = \frac{1}{2\pi} \log(\frac{x}{2\pi})$. For $T \ge 5$ we have $\log(T \pm 1) \le \log(2T)$ and $\log\log x \le \log x$ for $x \ge e$, hence

$$N(T+1) - N(T-1) \le (M(T+1) - M(T-1)) + 2(0.10076 + 0.24460) \log(2T) + 2 \cdot 8.08344.$$

Moreover

$$M(T+1) - M(T-1) = \int_{T-1}^{T+1} M'(x)\, dx \le \int_{T-1}^{T+1} \frac{1}{2\pi} \log x \, dx \le \frac{1}{\pi} \log(2T).$$

Combining these bounds gives $N(T+1) - N(T-1) \le 1.00903 \log T + 16.8663 \le 1.01 \log T + 17$, establishing (16). Finally, in width–2 one has $m = 2T$. The local window $|\operatorname{Im} \rho - m| \le 1$ corresponds to $|\gamma - T| \le 1/2$ in the s–plane, so $N_{\mathrm{loc}}(m) = N(T + \frac{1}{2}) - N(T - \frac{1}{2}) \le N(T+1) - N(T-1)$, yielding (17). □

Corollary 10.10 (Outer-aligned upper envelope in residual+local form)**.** *Let B be κ–admissible. Assume the residual envelope bound of Lemma 7.2, i.e.* $\sup_{\partial B} |F'/F| \le L(m) := C_1 \log m + C_2$*. Then*

$$\sum_{\pm} |W(v_\pm) - e^{i\varphi_0^\pm}| \le 2C_{\mathrm{up}} \left(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa} \right) \le 2C_{\mathrm{up}} \left(\delta L(m) + \frac{1.01 \log m + 17}{\kappa} \right).$$

Remark 10.11 (UE exponent gate and the local term)*.* Lemma 10.3 is the *only* step in the envelope chain that generates a positive power of δ in front of $\sup_{\partial B} |E'/E|$. More generally, if one could prove an upper-envelope bound of the form

$$\sum_{\pm} \big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \le 2C_{\mathrm{up}}\, \delta^p \sup_{\partial B} \Big|\frac{E'}{E}\Big| \qquad (p > 0),$$

then, after the log-derivative split $E'/E = F'/F + Z_{\mathrm{loc}}'/Z_{\mathrm{loc}}$ and the collar bound $\sup_{\partial B} |Z_{\mathrm{loc}}'/Z_{\mathrm{loc}}| \le N_{\mathrm{loc}}(m)/(\kappa\delta)$, the local contribution would scale as $\delta^{p-1} N_{\mathrm{loc}}(m)/\kappa$. Hence *$\eta$–absorption can suppress the local term only when $p > 1$.* In the proved pointwise form of Lemma 10.3, one has $p = 1$, so the collar/local term survives as $(2C_{\mathrm{up}}/\kappa)\, N_{\mathrm{loc}}(m)$, independent of δ.

Lemma 10.12 (η–absorption obstruction under the pointwise UE exponent $p = 1$)**.** *Assume the hypotheses of Corollary 10.10. Then for every $\delta \le \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$,*

$$\sum_{\pm} \big|W(v_\pm) - e^{i\varphi_0^\pm}\big| \le 2C_{\mathrm{up}} \Big(\delta L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa} \Big).$$

In particular, letting $\eta \downarrow 0$ (hence $\delta \downarrow 0$) only suppresses the residual term $\delta L(m)$; the local term $N_{\rm loc}(m)/\kappa$ does not *decay with η. Therefore any absorption-style closure that attempts to force the envelope side small by choosing η must additionally verify a* separate *inequality of the form*

$$\frac{2C_{\rm up}}{\kappa} N_{\rm loc}(m) \ < \ c$$

at the relevant anchor height(s), where c is the forcing constant in (18)*.*

Proof. The displayed bound is exactly Corollary 10.10 with the corrected UE exponent $p = 1$. As $\eta \to 0$ one has $\delta_0 \to 0$ and hence $\delta L(m) \to 0$, while $N_{\rm loc}(m)/\kappa$ is unchanged. Since the forcing lower bound in the tail inequality tends to c as $\delta \downarrow 0$, the strict inequality requires the stated necessary condition at the anchor. $\square$

10.2 Horizontal non-forcing budget in residual form

Definition 10.13 (Horizontal non-forcing phase budget)**.** Let $B = B(\pm a, m, \delta)$ be an aligned box and let $F = E/Z_{\rm loc}$ be the residual factor. Assume F is holomorphic and zero-free on a neighborhood of ∂B. Let $H_\pm$ denote the top and bottom edges of ∂B:

$$H_+ := \{x + i(m+\delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}, \qquad H_- := \{x + i(m-\delta) : \ x \in [\pm a - \delta, \pm a + \delta]\}.$$

Define

$$\Delta_{\rm nonforce}(B) := \int_{H_+} |\partial_s \arg F| \, ds + \int_{H_-} |\partial_s \arg F| \, ds.$$

Lemma 10.14 (Horizontal budget (residual form; audit-grade))**.** *In the setting of Definition 10.13,*

$$\Delta_{\rm nonforce}(B) \le 4\delta \sup_{v \in \partial B} \left| \frac{F'(v)}{F(v)} \right|.$$

Consequently, if $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2$*, then*

$$\Delta_{\rm nonforce}(B) \le C''_h \, \delta \, (\log m + 1), \qquad C''_h := 4 \max\{C_1, \, C_2\}.$$

Proof. On either horizontal edge, $|\partial_s \arg F| \le |F'/F|$ pointwise. Each edge has length 2δ, hence each integral is bounded by $2\delta \sup_{\partial B} |F'/F|$. Summing top and bottom gives the first inequality, and the second follows from $\sup_{\partial B} |F'/F| \le C_1 \log m + C_2 \le \max\{C_1, C_2\}(\log m + 1)$. $\square$

11 The explicit tail inequality (post-pivot)

For $m \ge 10$ we use the growth surrogate

$$L(m) := C_1 \log m + C_2,$$

with constants as in Lemma 7.2. For the local window term we use the explicit majorant from Lemma 10.9:

$$N_{\rm up}(m) := 1.01 \log m + 17 \ \text{ so that } \ N_{\rm loc}(m) \le N_{\rm up}(m) \quad (m \ge 10).$$

For a parameter $\eta \in (0, 1)$ and a dial displacement $\alpha \in (0, 1]$ define the *nominal* scale

$$\delta_0 := \delta_0(m, \alpha) := \frac{\eta \alpha}{(\log m)^2}.$$

Fix a collar parameter $\kappa \in (0, 1/10)$ as in Definition 10.5. For each (m, α) we choose any scale $0 < \delta \le \delta_0$ such that the aligned boxes $B = B(\pm\alpha, m, \delta)$ are κ–admissible; existence is guaranteed by Lemma 10.6. By Lemma 11.2, shrinking δ only helps in the tail inequality, so it is safe to treat δ_0 as the worst-case scale in one-height reductions.

Theorem 11.1 (Tail inequality (criterion form; pointwise UE exponent $p = 1$))**.** *Fix $m \ge 10$ and $\eta \in (0, 1)$. Assume:*

1. *the forcing lemma producing the positive constant*

$$c_0 := \frac{3\log 2}{8\pi}, \qquad c := \frac{3\log 2}{16}, \qquad K_{\mathrm{alloc}} := 3 + 8\sqrt{3};$$

2. *the residual envelope bound (Lemma 7.2) providing C_1, C_2;*
3. *the audit-grade horizontal budget bound (Lemma 10.14), giving a constant C_h'' independent of (α, m, δ);*
4. *the explicit local window bound (Lemma 10.9) providing the majorant $N_{\mathrm{up}}(m) = 1.01 \log m + 17$.*

Then for every $\alpha \in (0, 1]$ and every κ–admissible aligned box $B = B(\pm\alpha, m, \delta)$, absence of off–axis quartets at height m follows from the strict inequality

$$2C_{\mathrm{up}}\Big(\delta L(m) \;+\; \frac{N_{\mathrm{up}}(m)}{\kappa}\Big) \;<\; c \;-\; \delta\Big(K_{\mathrm{alloc}}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big). \tag{18}$$

Proof sketch / bookkeeping. The forcing side is unchanged from v31. The only post-pivot modification is on the upper-envelope side: Lemma 10.3 bounds dial deviation in terms of $\sup_{\partial B} |E'/E|$. Applying the log-derivative split (Lemma 10.7), the residual envelope for $\sup_{\partial B} |F'/F| \le L(m)$ (Lemma 7.2), and the collar bound $\sup_{\partial B} |Z_{\mathrm{loc}}'/Z_{\mathrm{loc}}| \le N_{\mathrm{loc}}(m)/(\kappa\delta)$ (Lemma 10.8) yields

$$\sup_{\partial B} \Big|\frac{E'}{E}\Big| \le L(m) + \frac{N_{\mathrm{loc}}(m)}{\kappa\delta} \le L(m) + \frac{N_{\mathrm{up}}(m)}{\kappa\delta}.$$

Plugging this into Lemma 10.3 gives the left-hand side of (18). The right-hand side is the forcing lower bound, with the horizontal non-forcing term bounded by Lemma 10.14. □

Lemma 11.2 (Monotonicity under δ–shrinking)**.** *Fix $m \ge 10$, $\alpha \in (0, 1]$, and constants $C_{\mathrm{up}}, \kappa, c, c_0, K_{\mathrm{alloc}}, C_h'', C_1, C$ Let $L(m) = C_1 \log m + C_2$ and $N_{\mathrm{up}}(m) = 1.01 \log m + 17$. For $\delta \in (0, 1]$ define*

$$\mathrm{LHS}(\delta) := 2C_{\mathrm{up}}\Big(\delta L(m) \;+\; \frac{N_{\mathrm{up}}(m)}{\kappa}\Big), \qquad \mathrm{RHS}(\delta) := c \;-\; \delta\Big(K_{\mathrm{alloc}}\, c_0\, L(m) + C_h''\,(\log m + 1)\Big).$$

Then $\mathrm{LHS}(\delta)$ *is (weakly) increasing in δ and* $\mathrm{RHS}(\delta)$ *is (weakly) decreasing. Consequently, if* $\mathrm{LHS}(\delta_0) < \mathrm{RHS}(\delta_0)$ *for some $\delta_0 \in (0, 1]$, then* $\mathrm{LHS}(\delta) < \mathrm{RHS}(\delta)$ *holds for every $\delta \in (0, \delta_0]$.*

Proof. For $\delta > 0$, the map $\delta \mapsto \delta L(m)$ is increasing and the term $N_{\mathrm{up}}(m)/\kappa$ is independent of δ, hence $\mathrm{LHS}(\delta)$ is (weakly) increasing. The bracketed factor in $\mathrm{RHS}(\delta)$ is nonnegative and independent of δ, so $\mathrm{RHS}(\delta)$ decreases linearly in δ. □

Lemma 11.3 (Worst case in α is $\alpha = 1$ at the nominal scale)**.** *Fix $m \ge 10$ and $\eta \in (0, 1)$. Define the nominal scale $\delta_0(m, \alpha) = \eta\alpha/(\log m)^2$. Consider the tail inequality* (18) *evaluated at $\delta = \delta_0(m, \alpha)$. Then the left-hand side is (weakly)* increasing *in $\alpha \in (0, 1]$, while the right-hand side is (weakly)* decreasing. *Therefore it suffices to verify* (18) *at $\alpha = 1$ and $\delta = \delta_0(m, 1)$. If one later shrinks $\delta \le \delta_0(m, \alpha)$ to enforce κ–admissibility, the inequality only becomes easier (Lemma 11.2).*

Proof. With $\delta = \delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, the only α-dependence in the left-hand side is through the factor $\delta L(m)$, which is increasing in α, so the left-hand side increases. The right-hand side equals $c - \delta \cdot \Xi(m)$ for a nonnegative factor $\Xi(m)$ independent of α, hence it decreases. $\square$

Remark 11.4 (No one-height reduction in m under the pointwise UE exponent $p = 1$). In v33, the (claimed) $\delta^{3/2}$ prefactor in Lemma 10.3 made the local contribution scale like $\delta^{1/2}N_{\mathrm{up}}(m)$ at the nominal choice $\delta_0(m,\alpha) = \eta\alpha/(\log m)^2$, leading to an expression essentially independent of m and enabling a one-height reduction. After the UE–Gate audit, Lemma 10.3 provides only the pointwise exponent $p = 1$, so the tail left-hand side contains the δ–inert term $(2C_{\mathrm{up}}/\kappa)\,N_{\mathrm{up}}(m)$. With the explicit majorant $N_{\mathrm{up}}(m) = 1.01\log m + 17$, this term is *increasing* in m. Therefore a one-height reduction in m is not available under the current pointwise envelope mechanism: the tail criterion must be controlled as a family in m, or the UE–Gate must be cleared by a strengthened envelope mechanism (Remark 10.11).

12 UE–Gate diagnostics and (conditional) absorption

At fixed (m,α), the tail inequality (18) is a strict forcing-vs-envelope condition. With the *proved* pointwise UE exponent $p = 1$, the envelope side has the form

$$2C_{\mathrm{up}}\Big(\delta L(m) + \frac{N_{\mathrm{up}}(m)}{\kappa}\Big),$$

so η–shrinking (which only shrinks $\delta_0 = \eta\alpha/(\log m)^2$) can suppress the residual term $\delta L(m)$ but *cannot* suppress the local term $N_{\mathrm{up}}(m)/\kappa$ (Lemma 10.12). Thus v33-style η–absorption is blocked unless one can separately verify the η-independent gate $(2C_{\mathrm{up}}/\kappa)\,N_{\mathrm{loc}}(m) < c$ at the relevant anchor height(s).

More generally, if a strengthened upper-envelope mechanism achieved an effective exponent $p > 1$ in the sense of Remark 10.11, then the local term would scale as $\delta^{p-1}N_{\mathrm{up}}(m)/\kappa$ and would decay as $\delta \downarrow 0$, making an absorption-style closure plausible.

Proposition 12.1 (Conditional η–absorption at a fixed anchor under a strengthened UE exponent $p > 1$)**.** *Assume that, for some $p > 1$, the upper-envelope step admits the strengthened form in Remark 10.11 with the same constant C_{up}, and that all other constants in* (18) *are finite. Fix an anchor height $m_\star \geq 10$ and evaluate* (18) *at $(m,\alpha) = (m_\star, 1)$ with the nominal scale $\delta_0(m_\star, 1) = \eta/(\log m_\star)^2$. Then there exists $\eta_\star(m_\star, p) > 0$ such that* (18) *holds at $(m_\star, 1)$ for every $\eta \in (0, \eta_\star]$.*

Proof. Under the strengthened exponent $p > 1$, the envelope side becomes

$$2C_{\mathrm{up}}\Big(\delta_0^p L(m_\star) + \delta_0^{p-1}\frac{N_{\mathrm{up}}(m_\star)}{\kappa}\Big) = A\,\eta^p + B\,\eta^{p-1},$$

for finite constants A, B depending on $(m_\star, p)$ and the constant ledger. The forcing side equals $c - D\,\eta$ for a finite D. Since $p > 1$, we have $\eta^p \to 0$, $\eta^{p-1} \to 0$, and $\eta \to 0$ as $\eta \downarrow 0$, so the strict inequality holds for all sufficiently small η. $\square$

Remark 12.2 (Uniformity caveat)*.* Proposition 12.1 is an *anchor* statement at a fixed height. Uniform tail closure for all $m \geq 10$ additionally requires an m–uniform mechanism (e.g. a monotonicity reduction, or a verified-band argument), and is not supplied by the pointwise UE exponent $p = 1$.

13 Global RH from a finite front-end + the tail criterion family

Theorem 13.1 (Global closure (criterion-first logical form)). *Assume:*

1. *(Front-end) All nontrivial zeros with* $0 < \mathrm{Im}(s) \leq 5$ *lie on the critical line.*
2. *(Tail criterion) Fix some* $\eta \in (0, 1)$ *and* $\kappa \in (0, 1/10)$*, and assume the analytic inputs Lemmas 10.3–10.9 and Lemma 10.14 with finite constants. Assume moreover that for every* $m \geq 10$ *and every* $\alpha \in (0, 1]$ *there exists a* κ*–admissible scale* $0 < \delta \leq \delta_0(m, \alpha) = \eta\alpha/(\log m)^2$ *such that the strict tail inequality* (18) *holds.*

Then all nontrivial zeros of $\zeta(s)$ *lie on the critical line.*

Proof. For each $m \geq 10$, Theorem 11.1 turns the strict inequality (18) into exclusion of off-axis quartets at height m. By the tail criterion hypothesis, no off-axis quartets exist at any height $m \geq 10$. By the front-end hypothesis, there are no off-axis zeros below height 5. Hence there are no off-axis zeros at any height, so every nontrivial zero lies on the critical line. □

Remark 13.2 (Role of computations and the repro pack (v34)). Appendix A provides a small interval-arithmetic harness that evaluates the tail inequality for pinned parameters and a pinned constant ledger. In v34 this is used only for audit purposes (e.g. exponent tracking), not as a proof substitute.

A Tail harness bundle and reproducibility (v34)

A.1 What the tail checks prove (and what they do not)

Each tail check records the statement:

> Given a constants file that provides interval enclosures for $(C_1, C_2, C_{\mathrm{up}}, C''_h, \kappa)$, the chosen parameters (m, η, α), and the recorded UE exponent p, the harness computes interval bounds for the left-hand side LHS and right-hand side RHS in (18) and reports whether the strict separation $\mathrm{LHS}_{\mathrm{hi}} < \mathrm{RHS}_{\mathrm{lo}}$ holds.

It does *not* certify that the constants file is correct.

A.2 SHA–256 table (exact artifacts)

The file `v34_repro_pack/SHA256SUMS.txt` is the canonical hash list.

```
cb9f61fd9a605ba1c2df478eb3ee304e3186367bd1078b246ee3137fa8d21e1d  v34_generate_frontend_certificate
    .py
74d35532c4055d8d35a8d4d17b168d4ecdf5ff918a4388c92dd08200e7fd84c7  v34_generate_tail_check.py
76f736fa4e6722a8c929f625968c046ad68befda20186e1ba17f6082c8f0c30a  v34_verify_frontend_certificate.
    py
df56ed5456631b9226ddbdee89d630d53b6c1811ad599f30df8bfdbe88126ae9  v34_verify_tail_check.py
7af1e99771af76174c65be9990ede2c7a42a893ca820b27f0dc19903b52a5262  v34_constants_m10.json
f018e4cafde46e68bb4bfb7504a0aa6c798fa090353d4e8b4168550e5cffd9db  v34_frontend_certificate.json
be518b6df8ea07913577dd89c078ec473a855c62c82f18a2ef17bfc18b8eb195  v34_tail_check_m10.json
8a0f1b9411e64f469cae6adebf50447bb4cd5c93b1199b1c7e308d262aee0ab7  README.md
c1debbda3583dbaf0dc7120684ba89c457fef1227f4aa13504b21cf11e029acb  v34_frontend_verifier_output.txt
3b2c15c47e9eaadc4edbac24296091f41e9c04062de51536469584fc1783a307
    v34_tail_check_verifier_output_m10.txt
```

A.3 Commands

From the directory v34_repro_pack/:

1. sha256sum -c SHA256SUMS.txt
2. python3 v34_verify_tail_check.py --constants v34_constants_m10.json --certificate v34_tail_check_m10.json
3. python3 v34_verify_frontend_certificate.py --certificate v34_frontend_certificate.json

A.4 Expected verifier output: $m = 10$ (verbatim; may report strict inequality as false)

```
LHS_hi =
    850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076
RHS_lo =
    0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351
STRICT (LHS_hi < RHS_lo) = False
REGEN_MATCH = True
INEQUALITY_STRICT = False
CERT_REPORTED_PASS = False
OK
```

A.5 Bundle files (verbatim)


```
{
  "certificate_version": "v34",
  "created_utc": "2026-01-22T00:00:00Z",
  "m_band": "10",
  "eta": "1e-14",
  "alpha_worst": "1",
  "kappa": "0.05",
  "intervals": {
    "C1": {
      "lo": "15.1",
      "hi": "15.2"
    },
    "C2": {
      "lo": "37.3",
      "hi": "37.4"
    },
    "C_up": {
      "lo": "1100",
      "hi": "1100.5"
    },
    "C_hpp": {
      "lo": "1100",
      "hi": "1100.5"
    }
  },
  "notes": [
    "Demo-only intervals carried forward from v31-style scaffolding; replace with audit-proven
    enclosures when G-1/G-12 are closed.",
    "The verifier/generator implement directed-rounding interval arithmetic with Python's decimal
    module.",
    "The local-window majorant N_up(m)=1.01*log(m)+17 is hard-coded from Lemma Nloc-logm in
    manuscript_v34.",
```

```
    "UE_exponent_p is recorded explicitly to prevent exponent drift across versions."
  ],
  "UE_exponent_p": "1"
}
```

```
{
  "certificate_version": "v34",
  "m_band": "10",
  "eta": "1e-14",
  "alpha": "1",
  "kappa": "0.05",
  "UE_exponent_p": "1",
  "prec": 90,
  "pi_interval": {
    "lo": "3.14159265358979323846264338327950288419716939937510",
    "hi": "3.14159265358979323846264338327950288419716939937511"
  },
  "logm_interval": {
    "lo":
    "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721",
    "hi":
    "2.30258509299404568401799145468436420760110148862877297603332790096757260967735248023599721"
  },
  "delta_interval": {
    "lo":
    "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197468E
    -15",
    "hi":
    "1.88611697011613929219960829965060873665900545176220488941908879591085361622963010761197469E
    -15"
  },
  "L_interval": {
    "lo":
    "72.0690349042100898286716709657338995347766324782944719381032513046103464061280224515635578",
    "hi":
    "72.3992934135094943970734701112023359555367426271573492357065840947071036670957576995871576"
  },
  "Nup_interval": {
    "lo":
    "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383571",
    "hi":
    "19.3256109439239861408581713692312078496771125035150607057936611799772483357741260050383572"
  },
  "kappa_interval": {
    "lo": "0.05",
    "hi": "0.05"
  },
  "lhs_interval": {
    "lo":
    "850326.881532655689245144996236820608990676883579823939491593791503565415726034904366571925",
    "hi":
    "850713.393751534170474909289208547785595304819451816083078159425660980262730419140960240076"
  },
  "rhs_interval": {
    "lo":
    "0.129965096347944215724970679716013192260392769855133829588479000426675277738893819116146351",
```

```
    "hi":
    "0.129965096347948199005209691457697222838229716361348668790498959759558243588339370271250251"
  },
  "derived_constants": {
    "ln2_interval": {
      "lo":
    "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327",
      "hi":
    "0.693147180559945309417232121458176568075500134360255254120680009493393621969694715605863327"
    },
    "c_interval": {
      "lo":
    "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099373",
      "hi":
    "0.129965096354989745515731022773408106514156275192547860147627501780011304119317759176099375"
    },
    "c0_interval": {
      "lo":
    "0.0827383500572443475236711620442491341185086557736206913728528561387020242248387512851407512",

      "hi":
    "0.0827383500572443475236711620442491341185086557736209547372007536994885577445868650239268751"
    },
    "Kalloc_interval": {
      "lo":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491695",
      "hi":
    "16.8564064605510183482195707320469789355424420304830450244464558356154641352704002966491696"
    }
  },
  "pass": false
}
```


```
#!/usr/bin/env python3
"""
v34_generate_tail_check.py

Deterministically generates v34_tail_certificate_m10.json from v34_constants_m10.json using
directed-rounding interval arithmetic implemented with Python's decimal module.

This generator is intended to be auditable: no network access, no randomness, and no external
    libraries.

Tail inequality evaluation (for given inputs):
  LHS(delta) < RHS(delta), where
  LHS(delta) = 2*C_up*( delta^(3/2)*L(m) + delta^(1/2)*N_up(m)/kappa )
  RHS(delta) = c - delta*( Kalloc*c0*L(m) + C_hpp*(log(m)+1) )

with
  L(m)    = C1*log(m) + C2,
  N_up(m) = 1.01*log(m) + 17,
  c  = (3 ln 2)/16,
  c0 = (3 ln 2)/(8 pi),
  Kalloc = 3 + 8*sqrt(3).

Usage:
```

```
  python3 v34_generate_tail_check.py v34_constants_m10.json v34_tail_certificate_m10.json
"""

import json
import sys
from dataclasses import dataclass
from decimal import Decimal, getcontext, localcontext, ROUND_FLOOR, ROUND_CEILING


# ---- Fixed enclosure for pi (50 decimal places) ----
# pi = 3.14159265358979323846264338327950288419716939937510...
PI_LO = Decimal("3.14159265358979323846264338327950288419716939937510")
PI_HI = Decimal("3.14159265358979323846264338327950288419716939937511")


@dataclass
class Interval:
    lo: Decimal
    hi: Decimal

    def __post_init__(self) -> None:
        if self.lo > self.hi:
            raise ValueError(f"Bad interval: {self.lo} > {self.hi}")


def ctx(prec: int, rounding):
    c = getcontext().copy()
    c.prec = prec
    c.rounding = rounding
    return c


def iv(lo: str, hi: str | None = None) -> Interval:
    if hi is None:
        hi = lo
    return Interval(Decimal(lo), Decimal(hi))


def add(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo + b.lo
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi + b.hi
    return Interval(lo, hi)


def sub(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo - b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi - b.lo
    return Interval(lo, hi)


def mul(a: Interval, b: Interval, prec: int) -> Interval:
    with localcontext(ctx(prec, ROUND_FLOOR)):
        cands_lo = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        lo = min(cands_lo)
    with localcontext(ctx(prec, ROUND_CEILING)):
```

```
        cands_hi = [a.lo*b.lo, a.lo*b.hi, a.hi*b.lo, a.hi*b.hi]
        hi = max(cands_hi)
    return Interval(lo, hi)


def div(a: Interval, b: Interval, prec: int) -> Interval:
    if b.lo <= 0 <= b.hi:
        raise ZeroDivisionError("Interval division by an interval containing 0.")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        rlo = Decimal(1) / b.hi
    with localcontext(ctx(prec, ROUND_CEILING)):
        rhi = Decimal(1) / b.lo
    return mul(a, Interval(rlo, rhi), prec)


def sqrt(a: Interval, prec: int) -> Interval:
    if a.lo < 0:
        raise ValueError("sqrt of negative interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.sqrt()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.sqrt()
    return Interval(lo, hi)


def ln(a: Interval, prec: int) -> Interval:
    if a.lo <= 0:
        raise ValueError("ln of nonpositive interval")
    with localcontext(ctx(prec, ROUND_FLOOR)):
        lo = a.lo.ln()
    with localcontext(ctx(prec, ROUND_CEILING)):
        hi = a.hi.ln()
    return Interval(lo, hi)


def pow_3_2(a: Interval, prec: int) -> Interval:
    return mul(a, sqrt(a, prec), prec)


def compute(constants: dict, prec: int = 90) -> dict:
    m = iv(constants["m_band"])
    eta = iv(constants["eta"])
    alpha = iv(constants["alpha_worst"])
    kappa = iv(constants["kappa"])

    p = str(constants.get("UE_exponent_p", "1"))

    C1 = iv(constants["intervals"]["C1"]["lo"], constants["intervals"]["C1"]["hi"])
    C2 = iv(constants["intervals"]["C2"]["lo"], constants["intervals"]["C2"]["hi"])
    Cup = iv(constants["intervals"]["C_up"]["lo"], constants["intervals"]["C_up"]["hi"])
    Chpp = iv(constants["intervals"]["C_hpp"]["lo"], constants["intervals"]["C_hpp"]["hi"])

    logm = ln(m, prec)
    delta = div(mul(eta, alpha, prec), mul(logm, logm, prec), prec)

    # L(m) = C1*logm + C2
    L = add(mul(C1, logm, prec), C2, prec)

    # N_up(m) = 1.01*logm + 17
```

```
Nup = add(mul(iv("1.01"), logm, prec), iv("17"), prec)

# ln 2
ln2 = ln(iv("2"), prec)

# c = (3 ln 2)/16
c = div(mul(iv("3"), ln2, prec), iv("16"), prec)

# c0 = (3 ln 2)/(8 pi), pi enclosed
pi = Interval(PI_LO, PI_HI)
c0 = div(mul(iv("3"), ln2, prec), mul(iv("8"), pi, prec), prec)

# Kalloc = 3 + 8 sqrt(3)
sqrt3 = sqrt(iv("3"), prec)
Kalloc = add(iv("3"), mul(iv("8"), sqrt3, prec), prec)

logm_plus1 = add(logm, iv("1"), prec)

# UE exponent p: LHS = 2*Cup*(delta^p * L + delta^(p-1) * Nup/kappa).
# We support p="1" (pointwise UE proved in v34) and p="3/2" (hypothetical strengthened gate).
if p in ("1", "1.0", "1.00"):
    local_term = div(Nup, kappa, prec)              # delta^(p-1)=1
    residual_term = mul(delta, L, prec)             # delta^p = delta
elif p in ("3/2", "1.5", "1.50"):
    sqrt_delta = sqrt(delta, prec)
    local_term = mul(sqrt_delta, div(Nup, kappa, prec), prec)     # delta^(1/2)
    residual_term = mul(mul(delta, sqrt_delta, prec), L, prec)    # delta^(3/2)
else:
    raise ValueError(f"Unsupported UE_exponent_p={p!r}; use '1' or '3/2'.")

lhs = mul(mul(iv("2"), Cup, prec), add(residual_term, local_term, prec), prec)

# RHS = c - delta*(Kalloc*c0*L + Chpp*(logm+1))
term1 = mul(mul(Kalloc, c0, prec), L, prec)
term2 = mul(Chpp, logm_plus1, prec)
rhs = sub(c, mul(delta, add(term1, term2, prec), prec), prec)

passed = (lhs.hi < rhs.lo)

return {
    "prec": prec,
    "UE_exponent_p": p,
    "pi_interval": {"lo": str(PI_LO), "hi": str(PI_HI)},
    "logm_interval": {"lo": str(logm.lo), "hi": str(logm.hi)},
    "delta_interval": {"lo": str(delta.lo), "hi": str(delta.hi)},
    "L_interval": {"lo": str(L.lo), "hi": str(L.hi)},
    "Nup_interval": {"lo": str(Nup.lo), "hi": str(Nup.hi)},
    "kappa_interval": {"lo": str(kappa.lo), "hi": str(kappa.hi)},
    "lhs_interval": {"lo": str(lhs.lo), "hi": str(lhs.hi)},
    "rhs_interval": {"lo": str(rhs.lo), "hi": str(rhs.hi)},
    "derived_constants": {
        "ln2_interval": {"lo": str(ln2.lo), "hi": str(ln2.hi)},
        "c_interval": {"lo": str(c.lo), "hi": str(c.hi)},
        "c0_interval": {"lo": str(c0.lo), "hi": str(c0.hi)},
        "Kalloc_interval": {"lo": str(Kalloc.lo), "hi": str(Kalloc.hi)},
    },
    "pass": bool(passed),
}
```

```
def main() -> int:
    if len(sys.argv) != 3:
        print("Usage: v34_generate_tail_check.py constants.json tail_check.json", file=sys.stderr)
        return 2

    with open(sys.argv[1], "r", encoding="utf-8") as f:
        constants = json.load(f)

    out = {
        "certificate_version": "v34",
        "m_band": constants["m_band"],
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        "UE_exponent_p": constants.get("UE_exponent_p", "1"),
    }
    out.update(compute(constants, prec=90))

    with open(sys.argv[2], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[2])
    print("[generate] PASS =", out["pass"])
    print("[generate] lhs_interval.hi =", out["lhs_interval"]["hi"])
    print("[generate] rhs_interval.lo =", out["rhs_interval"]["lo"])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

```
#!/usr/bin/env python3
"""
v34_verify_tail_check.py

Verifier for v34_tail_check_m10.json. This script:
- loads the constants JSON and the pinned certificate JSON
- regenerates the certificate from constants
- checks exact JSON equality on the computed fields
- reports PASS/FAIL and prints the strict-separation check LHS_hi < RHS_lo.

Usage:
  python3 v34_verify_tail_check.py --constants v34_constants_m10.json --certificate
    v34_tail_check_m10.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json
import sys
```

```python
from v34_generate_tail_check import compute


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v34 tail check (m=10).")
    ap.add_argument("--constants", required=True, help="Path to v34_constants_m10.json")
    ap.add_argument("--certificate", required=True, help="Path to v34_tail_check_m10.json")
    args = ap.parse_args()

    with open(args.constants, "r", encoding="utf-8") as f:
        constants = json.load(f)

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    regen = {
        "certificate_version": "v34",
        "m_band": constants["m_band"],
        "eta": constants["eta"],
        "alpha": constants["alpha_worst"],
        "kappa": constants["kappa"],
        "UE_exponent_p": constants.get("UE_exponent_p", "1"),
    }
    regen.update(compute(constants, prec=90))

    # Compare all keys that regen produces (ignore any extra keys in cert)
    ok = True
    for k, v in regen.items():
        if cert.get(k) != v:
            ok = False
            print(f"MISMATCH key={k}")
            print("  cert :", cert.get(k))
            print("  regen:", v)

    lhs_hi = regen["lhs_interval"]["hi"]
    rhs_lo = regen["rhs_interval"]["lo"]
    strict = (float(lhs_hi) < float(rhs_lo))

    print("LHS_hi =", lhs_hi)
    print("RHS_lo =", rhs_lo)
    print("STRICT (LHS_hi < RHS_lo) =", strict)

    print("REGEN_MATCH =", ok)
    print("INEQUALITY_STRICT =", strict)
    print("CERT_REPORTED_PASS =", regen.get("pass"))

    if not ok:
        print("FAIL (mismatch)")
        return 1

    print("OK")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

B Finite-height front-end certificate (literature-based)

The required front-end is RH up to height $H_0 = 5$. We record a discharge using Platt–Trudgian's published verification of RH up to $3 \cdot 10^{12}$.


```
{
  "certificate_version": "v34",
  "created_utc": "2026-01-22T00:00:00Z",
  "needed_frontend_statement": {
    "type": "RH_to_height",
    "H0": 5.0,
    "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
  },
  "discharged_by": {
    "type": "literature_citation",
    "verification_height": 3000000000000.0,
    "reference": {
      "authors": "D. J. Platt and T. S. Trudgian",
      "title": "The Riemann hypothesis is true up to 3*10^12",
      "venue": "Bulletin of the London Mathematical Society",
      "year": 2021,
      "doi": "10.1112/blms.12460",
      "arxiv": "2004.09765",
      "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (rigorous
    interval arithmetic)."
    },
    "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=H0."
  },
  "notes": [
    "This JSON is not itself a computation of zeros; it is a pinned statement+reference used by v31
    .",
    "For a fully self-contained proof without external computational input, one would need to
    implement and certify an argument-principle zero count in this region using ball arithmetic (
    not provided here)."
  ]
}
```


```
H0 (needed) = 5.0
H_cited     = 3000000000000.0
CHECK: H0 <= H_cited : True
PASS
```

```
#!/usr/bin/env python3
"""v34_generate_frontend_certificate.py

Creates a pinned JSON certificate for the finite-height front-end assumption used by v34.

This script does NOT compute zeta zeros. It encodes a minimal (H0, citation) logic statement:
if RH has been verified up to H_cited and H0 <= H_cited, then RH holds up to height H0.

Usage:
```

```
  python3 v34_generate_frontend_certificate.py v34_frontend_certificate.json
"""

from __future__ import annotations

import json
from datetime import datetime, timezone
import sys


def main() -> int:
    if len(sys.argv) != 2:
        print("Usage: v34_generate_frontend_certificate.py output.json", file=sys.stderr)
        return 2

    out = {
        "certificate_version": "v34",
        "created_utc": datetime.now(timezone.utc).strftime("%Y-%m-%dT%H:%M:%SZ"),
        "needed_frontend_statement": {
            "type": "RH_to_height",
            "H0": 5.0,
            "text": "All nontrivial zeros rho=beta+i gamma with 0<gamma<=H0 satisfy beta=1/2."
        },
        "discharged_by": {
            "type": "literature_citation",
            "verification_height": 3e12,
            "reference": {
                "authors": "D. J. Platt and T. S. Trudgian",
                "title": "The Riemann hypothesis is true up to 3*10^12",
                "venue": "Bulletin of the London Mathematical Society",
                "year": 2021,
                "doi": "10.1112/blms.12460",
                "arxiv": "2004.09765",
                "statement": "All zeros beta+i gamma with 0<gamma<=3*10^12 satisfy beta=1/2 (
    rigorous interval arithmetic)."
            },
            "logic": "If RH holds for 0<gamma<=H_cited and H0<=H_cited, then RH holds for 0<gamma<=
    H0."
        },
        "notes": [
            "This JSON is not itself a computation of zeros; it is a pinned statement+reference
    used by v34.",
            "For a fully self-contained proof without external computational input, one would need
    to implement and certify an argument-principle zero count in this region using ball arithmetic
    (not provided here)."
        ]
    }

    with open(sys.argv[1], "w", encoding="utf-8") as f:
        json.dump(out, f, indent=2)

    print("[generate] wrote", sys.argv[1])
    return 0


if __name__ == "__main__":
    raise SystemExit(main())

#!/usr/bin/env python3
```

```
"""v34_verify_frontend_certificate.py

Verifier for the front-end certificate JSON produced by v34_generate_frontend_certificate.py.

This verifier checks the internal logic only:
- parses the JSON
- confirms that the required finite-height H0 is <= the cited verification height

It does NOT re-run the cited large-scale computation (Platt--Trudgian); that result is treated as
    an
external, peer-reviewed input in the manuscript.

Usage:
  python3 v34_verify_frontend_certificate.py --certificate v34_frontend_certificate.json

Exit codes:
- 0 on PASS
- nonzero on FAIL
"""

from __future__ import annotations

import argparse
import json


def main() -> int:
    ap = argparse.ArgumentParser(description="Verify v34 front-end certificate JSON (internal logic
     only).")
    ap.add_argument("--certificate", required=True, help="Path to v34_frontend_certificate.json")
    args = ap.parse_args()

    with open(args.certificate, "r", encoding="utf-8") as f:
        cert = json.load(f)

    needed = cert.get("needed_frontend_statement", {})
    discharged = cert.get("discharged_by", {})

    H0 = float(needed.get("H0"))
    Hc = float(discharged.get("verification_height"))

    ok = H0 <= Hc

    print("H0 (needed) =", H0)
    print("H_cited     =", Hc)
    print(f"CHECK: H0 <= H_cited : {ok}")

    if not ok:
        print("FAIL")
        return 1

    print("PASS")
    return 0


if __name__ == "__main__":
    raise SystemExit(main())
```

References

References

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[8] D. Platt and T. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bulletin of the London Mathematical Society **53** (2021), no. 3, 792–797.