Definition of Riemann Sum

Definition

Given a continuous function $f(x)$ on a closed interval $[a, b]$ and a partition $\mathcal{P}$ of $[a, b]$ into n equally-spaced subintervals $[x_k, x_{k+1}]$ such that $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$, we will denote the length of each subinterval by $\Delta x = \frac{b-a}{n}$. Given any choice of n representatives r_k from the intervals $[x_k, x_{k+1}]$, we have the **Riemann sum** of $f(x)$ with respect to the tagged partition $(\mathcal{P}, r_k)$

$$\sum_{k=1}^{n} f(r_k) \cdot \Delta x = f(r_1) \cdot \Delta x + f(r_2) \cdot \Delta x + \cdots + f(r_n) \cdot \Delta x.$$

Example of a Riemann Sum

Approximating Area

Consider the function $f(x) = x$ on the closed interval $[0, 4]$. We may partition the interval $[0, 4]$ into the four subintervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ with length $\Delta x = \frac{4-0}{4} = 1$. Choose the left endpoint ℓ_k of each subinterval as its representative. We have that $\ell_k = k - 1$. Ultimately, we find that the Riemann sum of $f(x)$ with respect to the aforementioned tagged partition is given by

$$\sum_{k=1}^{4} f(\ell_k) \cdot \Delta x = \sum_{k=1}^{4} (k-1) = 0 + 1 + 2 + 3 = 6.$$

We note that the area under the curve $f(x) = x$ from $x = 0$ to $x = 4$ is given by $\frac{1}{2}(4)(4) = 8$ since this region is a triangle with base and height 4. Our Riemann sum underestimates the area.

Using Riemann Sums

Consider the function $f(x) = x^2$ on the closed interval $[1, 4]$ with six equally-spaced subintervals with representatives chosen to be the right endpoints. Compute Δx for this partition.

(a.) 3

(b.) $\frac{1}{2}$

(c.) $\frac{5}{2}$

(d.) 6

iClicker Question 2

Using Riemann Sums

Consider the function $f(x) = x^2$ on the closed interval $[1, 4]$ with six equally-spaced subintervals with representatives chosen to be the right endpoints. Give a formula for the right endpoints r_k.

(a.) $r_k = k$

(b.) $r_k = \frac{k}{2} + \frac{1}{2}$

(c.) $r_k = \frac{k}{2} + 1$

(d.) $r_k = k + 1$

Using Riemann Sums

Give the Riemann sum for the function $f(x) = x^2$ on the closed interval $[1, 4]$ with six equally-spaced subintervals with representatives chosen to be the right endpoints.

(a.) $\displaystyle\sum_{k=1}^{4} k^2$

(b.) $\displaystyle\sum_{k=1}^{4} \frac{k^2}{2}$

(c.) $\displaystyle\sum_{k=1}^{6} \frac{k^2}{2}$

(d.) $\displaystyle\sum_{k=1}^{6} \frac{1}{2}\left(1 + \frac{k}{2}\right)^2$

Using Riemann Sums

Consider the function $f(x) = e^x$ on the closed interval $[0, 1]$ with five equally-spaced subintervals with representatives chosen to be the midpoints. Compute Δx for this partition.

(a.) $\frac{1}{5}$

(b.) 1

(c.) $\frac{1}{10}$

(d.) e

iClicker Question 5

Using Riemann Sums

Consider the function $f(x) = e^x$ on the closed interval $[0, 1]$ with five equally-spaced subintervals with representatives chosen to be the midpoints. Give a formula for the midpoints m_k.

(a.) $m_k = \dfrac{k}{5}$

(b.) $m_k = k$

(c.) $m_k = \dfrac{2k-1}{10}$

(d.) $m_k = \dfrac{k}{2}$

iClicker Question 6

Using Riemann Sums

Give the Riemann sum for the function $f(x) = e^x$ on the closed interval $[0, 1]$ with five equally-spaced subintervals with representatives chosen to be the midpoints.

(a.) $\displaystyle\sum_{k=1}^{5} \frac{1}{5} e^{k}$

(b.) $\displaystyle\sum_{k=1}^{5} \frac{1}{5} e^{(2k-1)/10}$

(c.) $\displaystyle\sum_{k=1}^{5} \frac{1}{5} e^{k/2}$

(d.) $\displaystyle\sum_{k=1}^{5} \frac{1}{5} e^{k/5}$

Principle of Mathematical Induction

Definition

Given a formal statement $P(n)$ about a positive integer n such that (i.) $P(1)$ is true and (ii.) $P(n+1)$ is true whenever $P(n)$ is true, the statement $P(n)$ is true for all positive integers.

We refer to this very important property of the positive integers as the **Principle of Mathematical Induction**.

Example Proof by Induction I

Proposition

For each positive integer n, the positive integer $2n-1$ is odd.

Proof. We have the statement $P(n)$ given by "$2n-1$ is odd." We note that $P(1)$ is the true statement "1 is odd." We will assume that $P(n)$ is true for some positive integer $n \geq 2$, i.e., we will assume that the statement "$2n-1$ is odd" is true. $P(n+1)$ is the statement "$2(n+1)-1$ is odd." By simplifying, we have that $2(n+1)-1=(2n-1)+2$. Evidently, two more than an odd number is still odd. We conclude that $P(n+1)$ is a true statement. By the Principle of Mathematical Induction, we conclude that $2n-1$ is odd for each positive integer n. QED.

Example Proof by Induction II

Proposition

We have that $\sum_{k=1}^{n}(2k-1) = 1 + 3 + \cdots + 2n - 1 = n^2$.

Proof. We have that $1 = 1^2$, $1 + 3 = 4 = 2^2$, and $1 + 3 + 5 = 9 = 3^2$. We will assume that $1 + 3 + \cdots + 2n - 1 = n^2$ for some positive integer $n \geq 4$. Observe that for $n + 1$, we have the sum

$$1 + 3 + \cdots + 2n - 1 + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$

By the Principle of Mathematical Induction, we conclude that $1 + 3 + \cdots + 2n - 1 = n^2$ for each positive integer n. QED.

Using Mathematical Induction

Conjecture formulas for the finite sums

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n,$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2, \text{ and}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3.$$

Prove that your formulas are correct using induction.

Bonus Question

Using Riemann Sums

Give the Riemann sum for the function $f(t) = t^2$ on the closed interval $[0, x]$ with n equally-spaced subintervals with representatives chosen to be the right endpoints. Compute the limit of this Riemann sum as $n \to \infty$. Explain the significance of your answer using as much calculus as you can.