1 Graded Rings and Modules

Definition 1.1. A ring R is *graded* if there exists a collection of subgroups $\{R_i\}_{i\in\mathbb{Z}}$ such that

a) $R = \bigoplus_{i\in\mathbb{Z}} R_i$

b) $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$

Notably, R_0 is a ring, and R is an R_0 algebra.

Definition 1.2. If $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is a graded algebra and $r \in R_i$, we say r is *homogeneous* of degree i. If r has unique expression $r = \sum_i r_i$ where each $r_i \in R_i$, we call the r_i the *homogeneous components* of r, and the above expression the *homogeneous decomposition* of r.

The classic example of a graded algebra is below:

Example 1.3. Let $R = k[x_1, \cdots, x_m]$ be the polynomial ring in m variables, and let R_i be the set of all homogeneous polynomials of degree i for $i \geq 0$, and $R_i = \{0\}$ for $i < 0$. Then $\{R_i\}$ is a grading for R. In this case, the R_i are the polynomials of degree i, and are a k-vector space spanned by monomials of degree i. We note that our idea of homogeneous polynomials coincides with the definition above.

Definition 1.4. Let $R = \bigoplus R_i$ be a graded ring. Then a *grading* on an R-module M is a collection of R-submodules $\{M_k\}$ of M such that

a) $M = \bigoplus_{k\in\mathbb{Z}} M_k$

b) $R_kM_l \subseteq M_{k+l}$ for all $k, l \in \mathbb{Z}$.

Note that if M is finitely generated and graded, then M has a finite set of generators consisting of homogeneous elements: we take the homogeneous components of each generator.

We now introduce the concept of a graded submodule.

Definition 1.5. A submodule N of M is *graded* or *homogeneous* if whenever $n \in N$ has decomposition $n = \sum_i n_i$ into its homogeneous components (that is, $n_i \in M_i$), then $n_i \in N$.

Example 1.6. To see that this definition of a homogeneous submodule coincides with our intuition in polynomial rings, consider the ideal $I = \langle x^2 - y^3\rangle$, which we might not consider as "homogeneous", as it is not generated by a homogeneous polynomial. Furthermore, consider $f = (x - y)(x^2 - y^3) \in I$. Then f has homogeneous decomposition

$$f = (x^3 - x^2y) - (xy^3 + y^4).$$

But if $x^3 - x^2y \in I$, then we must have $a(y)$ and $b(y)$ that satisfy

$$(x^3 - x^2y) = (a(y)x + b(y))(x^2 - y^3),$$

and solving for $a(y)$ and $b(y)$ leads to a contradiction.

On the other hand consider $J = \langle x^2y - y^3\rangle$ and let $g = (x - y^2)(x^2y - y^3)$. Then g has the following decomposition into homogeneous components:

$$g = (x^3y - y^3x) + (-x^2y^3 + y^5).$$

We observe that $x^3y - y^3x = x(x^2y - y^3) \in J$ and $x^2y^3 + y^5 = -y^2(x^2 - y^3) \in J$.

We would like for our intuition for what a homogeneous ideal is to coincide with the above definition for a homogeneous submodule. The following theorem shows that they do:

Proposition 1.7. *Let $N \leq M$ be a submodule of a graded R-module M. Then the following are equivalent:*

a) N is a homogeneous submodule of M (every homogeneous component of some $n \in N$ also lies in N)

b) N is generated by homogeneous elements of M

c) The collection $\{(M_k + N)/N\}_{k\in\mathbb{Z}}$ is a grading of M/N.

Also note that if I is homogeneous in R, then IM is a homogeneous submodule of M, and M/IM is also a graded module. With this equivalence, we also note that if M is a finitely generated G-module, we can pick a generating set of homogeneous elements.

We also state an important variation of Nakayama's Lemma, though we do not explicitly use it in this talk.

Lemma 1.8. *Let G be a graded k-algebra, $I \subset G$ an ideal generated by homogeneous elements of positive degree, M a graded G-module, and $N \leq M$ a graded submodule. If the grading of M/N is bounded below, and if*

$$M = N + IM,$$

then $M = N$.

Given a graded module, it would make sense to study each graded piece to gain a better understanding of the module as a whole. A natural thing to study is its "size", which we can quantify with the *length* of the graded piece. Before we proceed, we review a few facts about length.

1.1 Length

We define length and recall a few basic facts.

Definition 1.9. If M is an R-module, we define a *composition series* of M to be a chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that M_i/M_{i-1} has no nontrivial submodules for all i.

One can show that any two composition series of M have the same length. Thus we define the *length* of an R-module $\ell_R(M)$ to be the length of a composition series of M. When there is no ambiguity regarding R, we simply write $\ell(M)$ to denote the length of M.

We now state without proof a few properties about the length of a module, as well as properties regarding Noetherian and Artinian modules.

Proposition 1.10. *a) A module M has finite length if and only if M is Noetherian and Artinian. In particular, a ring R has finite length if and only if R is Artinian.*

b) If R is Noetherian (or Artinian) and M is a finitely generated R-module, then M is Noetherian (or Artinian).

c) Length is an additive function over an exact sequence. That is, given an exact sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0,$$

we have $\ell(M')+\ell(M'') = \ell(M)$. In particular, if N is a submodule of M, then $\ell(N)+\ell(M/N) = \ell(M)$.

d) If R is a field and M an R-module, then the length of M is its dimension over R as a vector space.

With these facts in mind, we now proceed to constructing the Hilbert Series, a useful tool for studying graded rings and modules.

2 Hilbert Series

Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be graded. In general, R is an R_0-algebra. Now consider the case where R is *finitely generated* over R_0. Then we can regard $R = R_0[x_1, \cdots, x_m]$, with the x_i homogeneous of degree d_i. Moreover, if M is a finitely generated R-module, one can show that each M_i is a finitely generated R_0 module. For if M is generated by homogeneous elements $\{m_1, \cdots, m_s\}$ where each m_j is homogeneous of degree α_j, then M_i is generated as an R_0 module by (the finitely many) elements of the form

$$x_1^{\beta_1} \cdots x_m^{\beta_m} m_j$$

where $d_1\beta_1 + \cdots d_m\beta_m + \alpha_j = i$.

In the case that R_0 is an Artinian (and hence Noetherian) ring, the M_i being finitely generated R_0 modules imply the M_i are both Artinian and Noetherian. Equivalently, R_0 being Artinian guarantees finite length of each M_i.

Definition 2.1. When the above conditions are satisfied, we define the *Hilbert Function* $\chi_M : \mathbb{N} \to \mathbb{N}$ where $\chi_M(i) = \ell_{R_0}(M_i)$.

For the rest of this section, I will further restrict R_0 to be a *field* k. Then we can regard each submodule M_i as a finite dimensional vector space over k. In this special case, we rewrite the Hilbert function $\chi_M(i) = \dim_k(M_i)$.

Example 2.2. Let $R = k[x_1, \cdots, x_n]$ be the usual polynomial ring over k, where each indeterminate has degree 1. Regard R as a module over itself. The dimension of M_i is the number of monomials in R of degree k. Using a combinatorial proof, we find that

$$\dim_k(M_i) = \binom{i+n-1}{n-1}$$

for $k \geq 0$.

In order to find the Hilbert function of more general graded modules, we need to use the following lemma.

Lemma 2.3. *Suppose the assumptions above still hold. Let $x \in R$ be homogeneous of degree d. Suppose further that x is a nonzero divisor of M (in this case we note that x is M-regular). Then*

$$\chi_{M/xM}(i) = \chi_M(i) - \chi_M(i-d).$$

Proof. We consider the exact sequence

$$0 \longrightarrow M_{i-d} \xrightarrow{x\cdot} M_i \longrightarrow (M/xM)_i \longrightarrow 0,$$

then take the dimension over the sequence. □

Example 2.4. We now use the above formula to determine the Hilbert function for graded ring $R = k[x_1, \cdots, x_n]/\langle F \rangle$, where F is a homogeneous polynomial of degree d. If we let $M = k[x_1, \cdots, x_n]$, then

$$\chi_R(i) = \chi_M(i) - \chi_M(i-d).$$

If $i \geq 0$ but $i - d < 0$ (i.e. $0 \leq i < d$), then $\chi_M(i-d) = 0$ and $\chi_R(i) = \chi_M(i) = \binom{n+i-1}{n-1}$. If $i - d > 0$ (i.e. $i > d$), then

$$\chi_R(i) = \binom{i+n-1}{n-1} - \binom{i-d+n-1}{n-1}.$$

Example 2.5. Using similar logic, if F and G are homogeneous polynomials in $k[x,y]$ of degree p and q respectively and $\gcd(F,G)=1$, we claim the Hilbert function for $R' = k[x,y]/\langle F, G\rangle$ is

$$\chi_{R'}(i) = \begin{cases} i+1 & 0 \le i < p \\ p & p \le i < q \\ p+q-i-1 & q \le i < p+q \\ 0 & i \ge p+q \end{cases}$$

We would like to use the preceding lemma again, where $R' = R/GR$. This is valid because G being relatively prime from F implies that if $Gf \in \langle F\rangle$for any $f \in k[x,y]$, F divides Gf and hence divides f. Thus $f \in \langle F\rangle$, and G is a nonzerodivisor in $k[x,y]/\langle F\rangle$. In fact, we note that (F,G) is an M-regular sequence. Then by the preceding lemma,

$$\chi_{R'}(i) = \chi_R(i) - \chi_R(i-q).$$

The rest of the proof is casework.

We remark that the technique of considering both $\chi_M(i)$ and $\chi_m(i-d)$ is a common one, and formalized by the idea of "twisting".

Definition 2.6. Let $M = \bigoplus M_i$ be a graded module. Then $M(-d) := \bigoplus M_{i-d}$ is a *twisting* of M of degree d.

One can think of twisting a module as shifting its graded pieces by d. We also note that $\chi_M(i-d) = \chi_{M(-d)}(i)$.

Definition 2.7. Let $R = \bigoplus R_i$ be a graded ring, R a finitely generated algebra over R_0, and M a finitely generated graded G-module. If R_0 is a field, we call the *Hilbert Series* of M

$$H(M,t) := \sum_{i\in\mathbb{Z}} \dim_k(M_i)t^i.$$

Remark 2.8. We could also relax the condition on R_0 to be an Artinian ring. In this case, we would replace dimension of M_i in the above formula with the length $\ell_{R_0}(M_i)$.

Example 2.9. If $M = k[x,y,z]$ with the usual grading, then

$$H(M,t) = \sum_{i\in\mathbb{Z}} \binom{2+k}{2} t^i = 1 + 3t + 6t^2 + 10t^3 + \cdots$$

Using Newton's negative binomial formula, we observe that $H(M,t) = (1-t)^{-3}$. The fact that the Hilbert Series can be written as a rational function is no coincidence, as we will see shortly.

Example 2.10. If $M = k[x,y]/\langle x^2, y^4\rangle$, one can either use example 2.5 or compute from first principles that

$$H(M,t) = 1 + 2t + 2t^2 + 2t^3 + t^4.$$

Notably, $H(M,t)$ is a polynomial.

Proposition 2.11.
- *If $x \in R$ is homogeneous of degree d and a nonzerodivisor on R, then*

$$H(M/xM, t) = (1-t^d)H(M,t).$$

- *Hilbert series are additive over exact sequences.*

PROOF. These results follows from what we have proven about the Hilbert function. For

$$\begin{aligned} H(M/xM,t) &= \sum_i \chi_{M/xM}(i)t^i \\ &= \sum_i (\chi_M(i) - \chi_M(i-d))t^i \\ &= \sum_i \chi_M(i)t^i - \sum_i \chi_M(i-d)t^{i-d}t^d \\ &= H(M,t) - H(M,t)t^d \\ &= (1-t^d)H(M,t) \end{aligned}$$

Furthermore, if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, is an exact sequence of graded G-modules, then $\chi_R(M_i) = \chi_R(M_i') + \chi_R(M_i'')$. Thus

$$H(M,t) = \sum_i \chi_R(M_i)t^i = \sum_i (\chi_R(M_i') + \chi_R(M_i''))t^i = H(M',t) + H(M'',t).$$

□

We would now like to determine some structure to the Hilbert series. Is it meromorphic? Does it have any essential singularities? The following theorem gives a closed form of the Hilbert series.

Theorem 2.12 (Hilbert-Serre). *Let $R = \bigoplus_i R_i$ be a graded ring, and $M = \bigoplus_i M_i$ a graded R-module. Assume $R_0 = k$ a field and R a finitely generated algebra over k. Also assume the grading on M is bounded below by some integer n_0. Then*

$$H(M,t) = \frac{f(t)t^{n_0}}{(1-t^{k_1})\cdots(1-t^{k_r})}$$

where $f(t) \in \mathbb{Z}[t]$, $f(0) \neq 0$.

PROOF. Once again we first remark that this proof remains unchanged if R_0 is artinian. Let $R = k[x_1, \cdots, x_r]$ with $x_i \in R_{k_i}$. We proceed with induction on n. First assume $n = 0$. Then $R = k$, and M is a finite dimensional vector space over k. Suppose the set $\{m_1, \cdots, m_s\}$ form a basis of homogeneous elements of M, where $m_i \in M_{l_i}$. WLOG suppose $l_1 \leq l_2 \leq \cdots \leq l_s$. Then $M_i = 0$ for $i > l_s$ and $M_i = 0$ for $i < l_1$. Then

$$H(M,t) = \sum_{i=l_1}^{l_s} \dim_k(M_i)t^i = t^{l_1}\sum_{i=0}^{l_s-l_1} \dim_k(M_{i+l_1})t^i,$$

which is of the desired form.

Now suppose the theorem holds for $r-1$. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow M(-k_r) \xrightarrow{x_r\cdot} M \longrightarrow C \to 0,$$

where K is the kernel of $\cdot x_r$ and C is the cokernel of $\cdot x_r$. Then x_r is in the annihilator of K and C, so both K and C can be considered as $R/x_rR \simeq k[x_1, \cdots, x_{r-1}]$ modules. Then by the inductive step, $H(K,t)$ and $H(L,t)$ have the desired form. Using additivity of the exact sequence,

$$H(C,t) - H(K,t) = \frac{t^{n_0}(f_C(t) - f_K(t))}{\prod_{i=1}^{k_{r-1}}} = H(M,t) - H(M(-k_r),t).$$

But we have already seen that $H(M(-k_r), t) = t^{k_r} H_M(t)$. Substituting this back into the above equation and solving for $H(M,t)$ gives us the desired formula. $\square$

In particular, if R is positively graded, the Hilbert series $H(M,t)$ is the quotient of a polynomial in $\mathbb{Z}[t]$ that does not vanish at 0 and $\prod(1-t^{k_i})$.

We now give a corollary under stricter conditions. We also show that for sufficiently large n (for which we have a strict lower bound), $\dim_k(M_n)$ is a polynomial. We phrase the corollary more precisely below.

Corollary 2.13. *Under the above condtions, let $R = k[x_1, \cdots, x_r]$ with $x_i \in R_1$. Then $H(M,t)$ can be written uniquely in the form*

$$H(M,t) = \frac{e(t)t^{n_0}}{(1-t)^s}$$

with $e(t) \in \mathbb{Z}[t]$, with $e(0), e(1) \neq 0$ and $0 \leq s \leq r$. Moreover, for $n \geq deg(e(t)) + n_0$, there exists a polynomial $h_M(n)$ of degree $s-1$ and leading coefficient $\frac{e(1)}{(s-1)!}$ such that

$$dim(M_n) = h_M(n).$$

Proof. By setting $k_i = 1$ for all i in the expression for $H(M,t)$ in the last theorem, and by factoring out every power of $(1-t)$ from the polynomial $f(t)$ in the numerator, we obtain

$$H(M,t) = \frac{e(t)t^{n_0}}{(1-t)^s}.$$

If $s < 0$, then 1 is a root of the Hilbert series. That would imply that $0 = H(M,1) = \sum_i \dim(M_i)$, a contradiction. Thus $s \geq 0$.

To show that $\dim(M_n)$ can be expressed as a polynomial in n for sufficiently large n, first write $e(t) = \sum_{i=1}^{h} e_i t^i$, and write

$$1/(1-t)^s = \sum_n \binom{s+n-1}{s-1} t^n.$$

We then substitute both of these expressions into the above formula for $H(M,t)$ and study the t^n term:

$$\sum_{i=0}^{h} e_i t^i \binom{s+(n-n_0-i)-1}{s-1} t^{(n-n_0)-i} t^{n_0}.$$

We now claim that the above coefficient $\displaystyle\sum_{i=0}^{h} e_i \binom{n-n_0-i+s-1}{s-1}$ satisfies the properties stated in the corollary. Let $h_M(n)$ denote the t^n coefficient we just wrote. Each binomial coefficient in $h_M(n)$ the product of $s-1$ terms, each a monic linear polynomial in n. Thus we can write

$$\binom{n-n_0-i+s-1}{s-1} = \frac{e_i n^{s-1} + \text{“lower order terms”}}{(s-1)!}.$$

Thus the leading term of $H_M(n)$ is

$$\sum_{i=0}^{h} \frac{e_i}{(s-1)!} n^{s-1} = \frac{e(1)}{(s-1)!} n^{s-1}.$$

$\square$

We call the polynomial $h_M(x)$ the *Hilbert-Samuel Polynomial.*

Example 2.14. We return to the case where $M = R = k[x_1, \cdots, x_r]$, where x_i have degree d_i. One can show with a simplified version of the above proof that if we consider R as an R-module, the Hilbert Series takes the form

$$H(M,t) = \frac{t^{n_0}}{\prod_{i=1}^{r}(1-t^{d_i})},$$

where in this case $f(t) = 1$. In particular, when each x_i has degree 1 and G is positively graded,

$$H(M,t) = \frac{1}{(1-t)^r}.$$

While it is not immediately obvious that this was the Hilbert series we first found, we can show the two forms are the same:

$$\sum_i \binom{r+i-1}{r-1} t^i = \sum_i \binom{-i}{r} (-1)^i t^i = (1-t)^{-r}$$

by Newton's negative binomial formula.

We now check the second part of this corollary. In this case, $deg(e(t)) = n_0 = 0$, so we expect $\dim(M_n)$ to be a polynomial expression in n of degree $r-1$ for $n \geq 0$. But this can be verified by inspecting the binomial coefficient

$$\binom{r+n-1}{r-1} = \frac{\overbrace{(n+r-1)\cdots(n+1)}^{r-1}}{(r-1)!}.$$

3 Multiplicity

Before we define multiplicities, we will need the following proposition:

Proposition 3.1. *Let $R = \bigoplus R_i$ be a graded ring. The following are equivalent:*

a) R is Noetherian.

b) R_0 is Noetherian and $R_+ = \bigoplus_{i\geq 1} R_i$ is finitely generated ideal.

c) R_0 is Noetherian and $R \simeq R_0[x_1, \cdots, x_n]/I$ with x_i having degree k_i, where I is a homogeneous ideal.

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal ($\sqrt{I} = \mathfrak{m}$), and let M be a finitely generated R-module. We define the *associated graded ring* of R with respect to I as

$$gr_I R := \bigoplus_{n\geq 0} I^n/I^{n+1},$$

where $I^0 = R$. Given $x \in I^n/I^{n+1}$ and $y \in I^m/I^{m+1}$, we note that $xy \in I^{m+n}$. We define their product naturally to be $(x + I^{n+1})(y + I^{m+1}) = xy + I^{n+m+1}$.

We also note that given $x + I^{n+1} \in I^n/I^{n+1}$, we can write x as the linear combination of monomials of the form $r \cdot a_1 \cdots a_k$ where $a_i \in I$ and $r \in R$. By definition of multiplication on $gr_I R$, we have

$$x + I^{n+1} = (r + I)(a_1 + I)\cdots(a_n + I).$$

Thus $gr_I R$ is a R/I algebra generated by elements in I/I^2. We now let $G := gr_I R$, and $G_n := I^n/I^{n+1}$. Using this notation we have shown that G is a G_0 algebra generated by elements in G_1.

Lemma 3.2. *If the above assumptions, G is Noetherian and a finitely generated G_0 algebra.*

PROOF. Since R is local, R/I is Artinian and hence Noetherian. Also, R is noetherian, and hence I is finitely generated. Thus the ideal $G_+ = \bigoplus_{n\geq 1} I^n/I^{n+1}$ is finitely generated too. The conclusion follows from Proposition 3.1. $\square$

Definition 3.3. Let $I \subseteq R$ and M be as above. Define

$$\mathfrak{M}(I) := \bigoplus_{n\geq 0} I^n M/I^{n+1}M.$$

We claim $\mathfrak{M}(I)$ is a G-module if we define multiplicaton in the natural way, where $(x + I^{n+1}) \cdot (ym+I^{m+1}M) := xy+I^{n+m+1}M$. Under this assumption we observe that $\mathfrak{M}(I)$ is a graded G-module generated by elements of degree zero. For any $xm + I^{n+1}M \in I^nM/I^{n+1}M$ where $x \in I^n$ can be expressed as

$$xm + I^{n+1}M = (x + I^{n+1})(m + IM),$$

where $x+I^n \in I^n/I^{n+1} \subseteq G$ and $m+IM \in M/IM$, which is a term of degree zero. We also note that $M/IM = R/I \otimes_R M$ is the tensor product of two Noetherian modules, which is Noetherian. Thus M/IM is a finitely generated R/I module. Since $\mathfrak{M}(I) = G \cdot (M/IM)$ we conclude that $\mathfrak{M}(I)$ is a finitely generated G module.

To summarize, we have determined that $\mathfrak{M}(I)$ is finitely generated over $G := gr_I R$, where G is a finitely generated algebra over Artinian ring $G_0 = R/I$. These are exactly the conditions we used to prove the existence of the Hilbert Samuel polynomial. We do so now in this special case.

Corollary 3.4. *$\ell(I^nM/I^{n+1}M) = Q(n)$ is a polynomial in n for $n >> 0$ of degree at most $\mu(I) - 1$, where $\mu(I) = \ell(I/\mathfrak{m}I)$ is the minimal number of generators of the ideal I.*

PROOF. By Corollary 2.13, such a polynomial $Q(n)$ exists for sufficiently large n, and is of degree at most $\mu(G_1) - 1 = \mu(I/I^2) - 1$. By Nakayama's lemma on local rings,

$$\mu(I/I^2) = \ell((I/I^2)/\mathfrak{m}(I/I^2)) = \ell(I/\mathfrak{m}I),$$

where the second equality follows from the third isomorphism theorem. Finally, $\ell(I/\mathfrak{m}I) = \mu(I)$, giving us the equality $\mu(G_1) - 1 = \mu(I) - 1$. $\square$

Corollary 3.5. *With the same assumptions as above, we have that $\ell(M/I^nM)$ is a polynomial in n for sufficiently large n, and has degree at most $\mu(I)$.*

PROOF. By studying the short exact sequence

$$0 \longrightarrow I^kM/(I^{k+1}M) \longrightarrow M/(I^{k+1}M) \longrightarrow M/(I^kM) \longrightarrow 0,$$

we note that $\ell(M/(I^{k+1}M)) = \ell(I^kM/(I^{k+1}M)) + \ell(M/I^kM)$. By induction, we obtain the formula

$$\ell(M/I^nM) = \sum_{j=0}^{n-1} \ell(I^jM/I^{j+1}M).$$

Then $\ell(M/I^nM)$ is the sum of at least n polynomials of degree at most $\mu(I)-1$, and we are done. $\square$

Definition 3.6. We denote $P_{I,M}(n)$ to be the polynomial such that

$$P_{I,M}(n) = \ell(M/I^nM)$$

for $n >> 0$. We call $P_{I,M}(n)$ the *Hilbert Polynomial*

A well-known result about the Hilbert polynomial is that its degree is equal to the Krull dimension of M. The proof of this statement is omitted.

Definition 3.7. Let $(R, \mathfrak{m}, k)$ be a d-dimensional (Krull) noetherian local ring, and let I be an $\mathfrak{m}$-primary ideal. Suppose M is a finitely generated R-module. Then the *Multiplicity* of M with respect to I is

$$e(I; M) := \lim_{n\to\infty} \frac{d!\ell(M/I^nM)}{n^d}.$$

We note that if $\dim(R) = \dim(M)$ we are studying the leading term of the Hilbert polynomial, multiplying by an extra factor of $d!$. The presence of this $d!$ is to counterbalance a factor of of a factorial in the denominator, the existence of which we saw when constructing the Hilbert-Samuel polynomial.

We end these notes by proving a few elementary results about the multiplicity of M.

Lemma 3.8. *The multiplicity $e(I; M) = 0$ iff dim$M <$ dim$R$.*

Proof. Let $s = \dim M$ and $d = \dim R$. In general $s \le d$. Note that

$$\ell(M/I^nM) = \frac{b_s}{s!}n^s + O(n^{s-1}),$$

where we use the fact that the degree of the Hilbert polynomial is $\dim M$. Then

$$e(I; M) = \frac{d!\ell(M/I^nM)}{n^d} = \frac{d!b_s}{s!}n^{d-s} + O(n^{d-s-1}).$$

Taking the limit as $n \to \infty$ shows that if $d > s$, $e(I; M) = 0$, and if $d = s$, then $e(I; M) = b_s \neq 0$ is the leading coefficient of the Hilbert polynomial. □

Thus if the annihilator of M over R is a minimal prime (in particular if M is torsion free), then the multiplicity of M with respect to I is nonzero.

Lemma 3.9. *If t is a positive integer, then $e(I^t; M) = e(I; M)t^d$.*

Proof. If $\dim M < \dim R$, then $e(I^t; M) = e(I; M) = 0$ and equality holds. So we can assume $\dim M = \dim R = d$. Then

$$\ell(M/I^nM) = \frac{e(I; M)}{d!}n^d + O(n^{d-1})$$

Then

$$\ell(M/(I^t)^nM) = \frac{e(I; M)}{d!}(tn)^d + O(n^{d-1}) = \frac{e(I; M)t^d}{d!}n^d + O(n^{d-1}).$$

Comparing leading coefficients gives us the desired inequality. □

If I is an ideal of R, we define the multiplicity of I, $e(I) := e(I; R)$, considering R as an R-module. If $(R, \mathfrak{m}, k)$ is local, then we define the multiplicity of the ring $e(R) := e(\mathfrak{m})$, the multiplicity of its maximal ideal.

Lemma 3.10. *If $(R, \mathfrak{m}, k)$ is local and Artinian, then $e(R) = \ell(R)$.*

Proof. Since R is Artinian, $\mathfrak{m}^n = 0$ for sufficiently large n (otherwise we would have nonterminating descending chain of ideals. We also recall that $\dim(R) = 0$. Then

$$e(R) = \lim_{n\to\infty} \frac{\ell(R/\mathfrak{m}^nR)}{0!}n^0 = \ell(R).$$

□