Basics of Test Ideals

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1 Thursday, September 9

The purpose of the next three seminars is to develop an analogue in positive characteristic for the log canonical threshold and multiplier ideals. These were introduced in my GSO talk on Friday, September 3, but for those who did not attend, we'll quickly go through the main points here. I have added much more detail here than what we went through in the seminar.

1.1 Motivation

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, called the *irrelevant ideal*. Consider any polynomial $f \in \mathfrak{m}$. By definition, we have that the origin $\mathbf{0} = (0, 0, \ldots, 0)$ is on the zero set of f, i.e. the variety defined by f. If we suppose that f is in a higher power of $\mathfrak{m}$, we say that the origin is a singularity of f. The goal is to explain and measure the behavior of f near $\mathbf{0}$.

This question alone has Analytic and Geometric approaches. The Geometric approach involves learning about canonical divisors, birational maps, effective divisors, resolution of singularities, etc. This would take too much time to go into, so we stick to the analytic approach to this problem, which yields a more than sufficient understanding for the problem.

For this, we consider how fast the function

$$\begin{aligned} \mathbb{C}^n = \mathbb{R}^{2n} &\longrightarrow \mathbb{R} \\ z &\longmapsto \tfrac{1}{|f(z)|} \end{aligned}$$

"blows up" near the origin. Thinking like an Analyst, we could ask if this function's square is integrable in a sufficiently small neighborhood of the origin, i.e. if it is L_2. In fact, with our setup, the integral

$$\int \frac{1}{|f|^2}$$

never converges in any small ball around $\mathbf{0}$. However, what if we try to control how fast $\frac{1}{|f|}$ is blowing up by raising it to a small positive power $\lambda \in \mathbb{R}_{\geq 0}$? This will effectively dampen the behavior of f around the origin and we can ask the same question, for which the answer changes a bit. Indeed, for sufficiently small values of λ, the integral

$$\int \frac{1}{|f|^{2\lambda}}$$

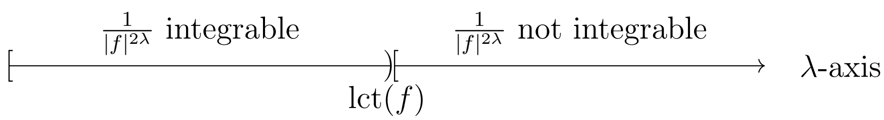
converges in a sufficiently small ball around the origin.[1] If we find a $\lambda \in \mathbb{R}_{\geq 0}$ for which this integral converges, it should not be surprising that replacing λ with $\lambda + \varepsilon$ should not affect the convergence, barring $\varepsilon \geq 0$ is small enough.

You might have noticed the subtle use of "$\geq$" involving ε. This is a very important piece of the puzzle. It turns out that as you vary λ, you find that there is a critical value at which the integral suddenly stops converging. This critical value is the log canonical threshold of f, sometimes called the complex singularity exponent.[2]

Definition 1.1. The log canonical threshold of f (at $\mathbf{0}$) is defined to be

$$\operatorname{lct}(f) = \sup\left\{ \lambda \in \mathbb{R}_{\geq 0} \;\middle|\; \text{there exists a neighborhood } \mathbf{0} \in B \text{ s.t. } \int_B \frac{1}{|f|^{2\lambda}} < \infty \right\}.$$

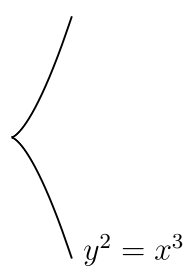
The picture below depicts the situation along the λ-axis: for small λ, the function $\frac{1}{|f|^{\lambda}}$ is L_2, but for large enough values of λ it fails to be L_2.



Now, what are the nice properties of $\operatorname{lct}(f)$? First, by our discussion above, i.e. that $\frac{1}{|f|^{\lambda}}$ fails to be L_2 when $\lambda = 1$ and is when $\lambda = 0$, we have that $\operatorname{lct}(f) \in (0, 1]$. In fact, whenever f is smooth at the origin, we have that $\operatorname{lct}(f) = 1$.[3] We now look at some values of $\operatorname{lct}(f)$.

Examples

1. Let $f = x^3 - y^2$, commonly referred to as the *cuspidal cubic*. Here is a picture of its zero set:



[1] If you don't believe this at first, let $\lambda = 0$.

[2] There is a way to define this invariant at any point x on the zero set of f, but for simplicity's sake, we are only considering the origin.

[3] The converse does not hold, there are f which are not smooth at $\mathbf{0}$, yet have $\operatorname{lct}(f) = 1$. For example, $f = xy$.

One can compute that $\operatorname{lct}(f) = \frac{5}{6}$.

2. Let $f = x_1^{a_1} \cdots x_n^{a_n}$, a monomial. Using a change of variables to polar coordinates and using Fubini's Theorem, you can show that $\int \frac{1}{|f|^{2\lambda}}$ converges whenever

$$\lambda < \frac{1}{a_i} \qquad \text{for all } i.$$

Thus, it follows that $\operatorname{lct}(f) = \min \left\{ \frac{1}{a_i} \right\}$. For a full computation, check out [BFS13, Example 2.2].

In the first example, notice that $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$. This is not a coincidence.

Proposition 1.2. Let $f = x^a + y^b$ where $(a, b) = 1$. Then $\operatorname{lct}(f) = \frac{1}{a} + \frac{1}{b}$.

There is also a nice result about $\operatorname{lct}(f)$ when f is a homogeneous polynomial.

Proposition 1.3. Let f be a homogeneous polynomial of degree d in n variables. Then

$$\operatorname{lct}(f) = \begin{cases} \frac{n}{d} & \text{if } d \geq n \\ 1 & \text{o.w.} \end{cases}$$

Now, something that you might have picked up on in all of these examples and propositions is that $\operatorname{lct}(f)$ has been rational for every single one of these. You might think that this is a coincidence because we are choosing "nice" examples. In fact, it isn't a coincidence. It is a corollary of Hironaka's Theorem on the existence of resolution of singularities over $\mathbb{C}$ that $\operatorname{lct}(f)$ is a rational number for all $f \in R$. For more explanation, check out [BFS13, Section 2.2].

Now, this definition of log canonical threshold leads to a finer invariant called the *multiplier ideals* of f. In essence, we will be attaching ideals to f^λ for all $\lambda \geq 0$.

Definition 1.4. Fix $f \in R = \mathbb{C}[x_1, \ldots, x_n]$. For every $\lambda \in \mathbb{R}_{\geq 0}$, we define the *multiplier ideal* of f^λ as

$$\mathcal{J}(f^\lambda) = \left\{ h \in R \;\middle|\; \text{there exists a neighborhood } \mathbf{0} \in B \text{ s.t. } \int_B \frac{|h|^2}{|f|^{2\lambda}} < \infty \right\}.$$

The multiplier ideals of f consist of functions that can be used as "multipliers" to dampen the singularity of $\frac{1}{|f|^\lambda}$ to make the function $\frac{|h|}{|f|^\lambda} \in L_2$. It is easy to check that the set $\mathcal{J}(f^\lambda)$ is in fact an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$.

We now list some useful properties of the multiplier ideals $\mathcal{J}(f^\lambda)$.

Proposition 1.5. Fix a polynomial f and view the multiplier ideals as varying over λ. Then the following hold:

(1) If $\lambda \in \mathbb{R}_{\geq 0}$ is sufficiently small, then $\mathcal{J}(f^\lambda) = \langle 1 \rangle = R$.

(2) If $\lambda < \lambda'$, then $\mathcal{J}(f^\lambda) \supseteq \mathcal{J}(f^{\lambda'})$.

(3) The log canonical threshold of f is

$$\operatorname{lct}(f) = \sup\{\lambda \mid \mathcal{J}(f^\lambda) = \langle 1 \rangle\}.$$

(4) Fix $\lambda \geq 0$. For $\varepsilon > 0$ small enough,[4] we have $\mathcal{J}(f^\lambda) = \mathcal{J}(f^{\lambda+\varepsilon})$.

(5) There exist distinguished $\lambda \in \mathbb{R}_+$ such that $\mathcal{J}(f^\lambda) \neq \mathcal{J}(f^{\lambda-\varepsilon})$ for all $\varepsilon > 0$.

All of these properties are easy to show once you have the right geometric definitions. For now though, we can take them as given.

The distinguished, or critical, values described in (5) give rise to a sequence of invariants which we view as a refinement of the log canonical threshold. In fact, the log canonical threshold is the smallest term of this sequence. Formally:

Definition 1.6. The *jumping numbers* of $f \in \mathbb{C}[x_1, \ldots, x_n]$ are the positive real numbers λ such that $\mathcal{J}(f^\lambda) \subsetneq \mathcal{J}(f^{\lambda-\varepsilon})$ for all $\varepsilon > 0$.

Proposition 1.7. The jumping numbers of $f \in \mathbb{C}[x_1, \ldots, x_n]$ are discrete[5] and rational.

Even though the sequence of jumping numbers is an infinite sequence, the following theorem, called Skoda's Theorem,[6] tells us the sequence is determined by finitely many of them.

Theorem 1.8. (Skoda's Theorem, [Laz11, Theorem 9.6.21]) Fix a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then

$$\mathcal{J}(f^{\lambda+1}) = \langle f \rangle \mathcal{J}(f^\lambda),$$

for $\lambda \geq 0$. In particular, a positive real number λ is a jumping number if and only if $\lambda + 1$ is a jumping number.

Remark: Fix a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$. Using the previously mentioned Hironaka's theorem, one can show that the set of jumping numbers for f are a subset of

$$\left\{ \frac{k_i + m}{a_i} \right\}_{m \in \mathbb{N}} \qquad \text{for some } k_i, a_i \in \mathbb{N}.$$

It is usually quite subtle and difficult to figure out which of these "candidates" are actually jumping numbers for f. In 2 dimensions, it turns out that the answer is intimately related to some nice geometry.

Note: For those who took Daniel's 831 class last spring, the jumping numbers of f in the interval $(0, 1]$ are always (negative of) roots of the Bernstein-Sato polynomial, which is the monic polynomial generator in $R[s]$ for which there is a Bernstein-Sato functional equation. To see what this is and more about $\mathcal{D}$-modules, check out Coutinho's book on $\mathcal{D}$-modules. Now, there can be more roots to this polynomial, but the negatives of these jumping numbers always show up as roots. It is this behavior that motivates the Algebraic story.

[4] How small is small enough depends on λ

[5] By this, we mean in every finite interval there are only finitely many jumping numbers.

[6] It was originally called the Briançon-Skoda Theorem, however, their seminal result regarding integral closures got more attention.

1.2 Move to Algebraic Approach: Characteristic p

One might ask now, how does an Algebraist start viewing all this. The idea is to look at reducing the polynomials modulo p. There is a technical way to do this via tensor products with $\mathbb{Z}/p\mathbb{Z}$, but we will leave that for another day. Instead, we will focus on the case where $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Then, we denote by $f \bmod p$, or f_p, the polynomial which takes all the coefficients of f and replaces them with their remainder modulo p.

The idea is that we should be able to assess the singularity of $f \bmod p$ at $\mathbf{0}$ for every p prime. Then, the properties of this singularity for p sufficiently large should tell us about the properties of the singularity for f. This idea is much more technical than it leads on, but it at least shows that we need to be well-versed with the world of positive characteristic.

Thus, for the rest of this talk, we will just introduce basic definitions and tools that we will need to construct our analogues of log canonical threshold and multiplier ideals. First, what do we even mean by positive characteristic?

1.3 Basic notions of characteristic p

This section is highly motivated by the first sections of lecture notes by Schwede and Smith, see [Sch17] and [Smi19] for more details.

Definition 1.9. Let R be a ring[7]. We say R is a ring of positive characteristic $p > 0$ if there is an embedding

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow R.$$

Equivalently, it means that $p \cdot 1_R = 1_R + 1_R + \cdots + 1_R = 0$.

Almost of all of the time, we care about the case where p is a prime. From now on, whenever we write p, it is a prime.
Examples:

- $R = \mathbb{F}_p$ - The finite field of p elements
- $R = \mathbb{F}_{p^e}$ - The finite field of p^e-many elements
- $R = \mathbb{F}_p[x]$
- $R = \mathrm{Frac}(\mathbb{F}_p[x]) = \left\{ \dfrac{f}{g} \;\middle|\; f, g \in \mathbb{F}_p[x], g \neq 0 \right\}$
- $R = \mathbb{F}_p[x_1, \ldots, x_n]/I$, where $I \subseteq \mathbb{F}_p[\mathbf{x}]$ is any ideal.

Working in characteristic $p > 0$ is not the most intuitive; however, the following lemma is the heart of it all.

Lemma 1.10. Let R be a commutative ring of prime characteristic p. The map $F : R \to R$ defined by $r \mapsto r^p$ is a ring homomorphism.

[7] Unless otherwise stated, the word 'ring' means commutative with unity.

Proof. Since R is commutative, $F(rs) = F(r)F(s)$ is straight-forward. The meat of this statement comes from looking at how this map behaves on sums. By the Binomial Theorem, we have

$$\begin{aligned} F(r+s) &= (r+s)^p \\ &= r^p + \binom{p}{1} r^{p-1}s + \cdots + \binom{p}{p-1} rs^{p-1} + s^p. \end{aligned}$$

It is not hard to see from the definition that $\binom{p}{i}$ is divisible by p whenever $1 \leq i \leq p-1$. So ALL of the "mixed terms" vanish since R is characteristic p. Therefore, $F(r+s) = r^p + s^p = F(r) + F(s)$. $\square$

The p-th power map defined in Lemma 1.10 is called the *Frobenius Endomorphism*, or *Frobenius map*, of R. It is the crucial tool to working in characteristic p. The main heuristic for life in characteristic p is that the Frobenius map detects all types of singularities. We can even see one immediately by the following lemma:

Lemma 1.11. Let R be a ring of prime characteristic p. Then F is injective if and only if R is a reduced ring (i.e. has no nonzero nilpotents[8]).

Proof. Suppose F is injective and let $x \in R$ be a nilpotent element. Then there is an $n \in \mathbb{N}$ such that $x^n = 0$. Choose $e > 0$ such that $p^e \geq n$. Since $x^n = 0$, it follows that $x^{p^e} = 0$. But notice that $F^e = F \circ F \circ \cdots \circ F$ sends $x \mapsto x^{p^e} = 0$. Since F (and hence all of its iterates) is injective, we have that $x = 0$. Thus, R is reduced.

Suppose R is reduced and let $x \in R$ be in the kernel of F. Then $F(x) = x^p = 0$, so x is nilpotent. Since R is reduced, it follows that $x = 0$. $\square$

Because of this Lemma, we tend to work only with reduced rings, e.g. domains. Frobenius being injective makes life a bit easier. When working in characteristic p, we don't always deal with Frobenius explicitly. So, how do we actually view it? For this, we have three main ways. For these explanations, take R to be a reduced ring.[9]

- $R^p \subseteq R$: Let R^p denote the subring of pth powers of R. Explicitly, it is

  $$R^p = \{x^p \mid x \in R\}.$$

  It isn't hard to see that R^p is the image of Frobenius. This means that the map $R \to R^p$ defined by $x \mapsto x^p$ is a ring isomorphism. It follows that $F : R \to R$ must factor through the inclusion $R^p \hookrightarrow R$, and can be identified with this inclusion.

- $R \subseteq R^{1/p}$: Here we have two ways to define the ring $R^{1/p}$. The standard way is to formally create symbols $x^{1/p}$ for every $x \in R$ that act like pth roots.[10] Consider the

[8] An element $x \in R$ is nilpotent if there exists $n \in \mathbb{N}$ such that $x^n = 0$.

[9] If you want something explicit, use $R = \mathbb{F}_p[x]$.

[10] Another way is the following: If R is a domain, consider the algebraic closure of the fraction field of R. In that algebraic closure, define $R^{1/p}$ to be the smallest subring containing all the pth roots of the elements of R.

map $R^{1/p} \to R$ which sends $x^{1/p} \to x$. Once again, this is a ring isomorphism. In particular, the Frobenius map on $R^{1/p}$ has image R (considered as a subring of $R^{1/p}$). Thus, F can be viewed as the inclusion $R \subseteq R^{1/p}$.

- F_*R: Anytime we have a ring homomorphism $\varphi : R \to S$, we can view S as an R-module. The ring action of R on S is defined to be

$$r \cdot s = \varphi(r)s,$$

where you send r into S via φ and then do the multiplication inside S.

This means that we can view R as an R-module via Frobenius. It can be pretty confusing to write R for this module, so there are a few options:

  - Use $R^{1/p}$ instead (at least when R is reduced)
  - Think of R as a module over R^p where the action is now just multiplication in R
  - Rewrite R as F_*R (this notation is borrowed from the language of sheaf theory)

Generally, the idea of viewing the target of a ring homomorphism as a module over the source is called restriction of scalars. It can also be applied to modules. If M is an R-module, we denote by F_*M to be the same Abelian group as M but with the ring action $x \cdot m = x^p m$.

For our purposes, we will usually be thinking of R as a module over R^p. Though, there are instances where we need to talk about the pth roots.

Definition 1.12. Let R be a ring of characteristic $p > 0$. We say R is F-finite if R is a finitely generated module over R^p.

Examples:

1. (Polynomial ring in one variable). Consider $R = \mathbb{F}_p[x]$. Notice that $R^p = \mathbb{F}_p[x^p]$. It follows that a finite generating set for R as an R^p-module is

$$1, x, x^2, \ldots, x^{p-1}.$$

Thus, R is F-finite. Further, these generators are linearly independent (with scalars coming from R^p) and so we say the set $\{1, x, \ldots, x^{p-1}\}$ is a *free basis* for R over R^p. Equivalently, we say that R is a *free* R^p-module of rank p.[11]

2. (Polynomial ring in n variables). Let $R = \mathbb{F}_p[x_1, \ldots, x_n]$. The image of Frobenius is $R^p = \mathbb{F}_p[x_1^p, \ldots, x_n^p]$. Similar to the case of one variable, R is a free R^p-module of rank p^n with free basis being the monomials of R which are not in R^p. Explicitly, the free basis can be written as the set

$$\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mid 0 \leq a_i \leq p - 1\}.$$

[11]The concept of a finitely generated free module is completely analogous to finite dimensional vector spaces. The idea of a free basis means that every element of the free module can be written uniquely as a linear combination of those basis elements.

Notation: Let $\varphi : R \to S$ be any ring homomorphism and $I \subseteq R$ an ideal of R. It is rare that the image $\varphi(I) \subseteq S$ is an ideal of S. However, we can expand it to an ideal of S, denoted IS. Called the *expansion* of I to S, this ideal is

$$IS = \left\{ \sum_{\text{finite}} s_i f(r_i) \;\middle|\; s_i \in S, r_i \in R \right\}.$$

Specializing this to a ring R of prime characteristic p, we denote by $I^{[p]}$ the expansion of I via Frobenius. Explicitly, we have that

$$I^{[p]} = \langle x^p \mid x \in I \rangle.$$

We call $I^{[p]}$ the pth *Frobenius power* of I. In particular, if $I = \langle f_1, \ldots, f_m \rangle$ is a finitely generated ideal of R, then

$$I^{[p]} = \langle f_1^p, \ldots, f_m^p \rangle.$$

Similarly, we denote by $I^{[p^e]}$ the ideal generated by the p^e-th powers of I. We think of this ideal as the expansion of I via the (iterated) map $F^e : R \to R$, which we call the p^eth Frobenius power of I. Similarly, if $I = \langle f_1, \ldots, f_m \rangle$ is finitely generated, we have

$$I^{[p^e]} = \langle f_1^{p^e}, \ldots, f_m^{p^e} \rangle.$$

Remark: We would like to be able to say that

$$x \in I \qquad \Longleftrightarrow \qquad x^p \in I^{[p]}.$$

However, this is not always true. For it to be true, we usually need F to be a (faithfully) flat map,[12] in which case R is free over its pth powers. In the cases we care about, that being polynomial rings, power series rings, and any localizations of these rings, F will always be flat. What we will be defining and discussing for the next two talks will commute with localization and completion, so we will tend to only consider the case where R is regular. In particular, we will explicitly discuss the case where $R = \mathbb{F}_p[x_1, \ldots, x_n]$.

To end off this first talk, we want to state why Frobenius being flat is such an important property. In general, regularity of a ring is a very hard property to show.[13] However, in characteristic p, it relies only on checking the behvavior of Frobenius.

Theorem 1.13 (Kunz' Theorem, [Kun69])**.** If R is a Noetherian ring of characteristic $p > 0$, then R is regular if and only if F is flat.

In general, the direction that regular implies flat is relatively easy to show, see [Sch17, Section 1.3] for 2 proofs of this direction. However, the other direction requires some MASSIVE machinery. The proof given in [Smi19] requires the Cohen Structure Theorem for

[12] A ring map $\varphi : R \to S$ is said to be *flat* if S is flat as an R-module. This means that applying the functor $S \otimes_R -$ preserves injections. To learn more about this, check out Chapter 2 of Atiyah-MacDonald. Or wait until Chris' talk on the intro to module theory. For our purposes, it is enough to trust that flatness lets us take the p-th roots of both sides of the statement $x^p \in I^{[p]}$.

[13] In some sense, regularity of a ring means the ring is smooth in some sense, i.e. it is free of singularities. It is generally ok to hear a ring is regular and think only of polynomial rings over a field.

Complete Local rings, the idea of Lech independence, Krull's Intersection Theorem, and much more. In 2017, Bhargav Bhatt and Peter Scholze supplied a shorter, alternative proof of this direction using the tools of perfectoid rings. In particular, they used the actions of perfection and tilting, the idea of global dimension, and triangulated categories for their proof. See [Sch17, Section 1.7] or [BS17].

2 Thursday, September 16

The goal of today's talk is to develop the tools to define test ideals, which will be our analogue of multiplier ideals. For this, we will look at defining what the $[1/q]$-Frobenius Power of an ideal should be. This will closely follow the treatment of [BMS08].

2.1 Setup

For this talk, our setup is the following:

- $p > 0$ will denote a prime number
- q will denote any power of p, i.e. $q = p^e$ for some e
- $R = \mathbb{F}_p[x_1, \dots, x_n]$
- $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$
- $\mathfrak{a}$ will denote any ideal of R

For what follows, the results will hold true for any F-finite regular ring. One of the important pieces to this will be that if R is regular (i.e. Frobenius is flat), every ideal $\mathfrak{a}$ is equal to its tight closure, see [HH90]. In other words, if $u, f \in R$ have that $uf^q \in \mathfrak{a}^{[q]}$ and $u \neq 0$, then $f \in \mathfrak{a}$.

2.2 $[1/q]$-Frobenius Powers

Definition 2.1. For an ideal $\mathfrak{a}$ and q a power of p, we let $\mathfrak{a}^{[1/q]}$ denote the unique ideal J of R minimal such that

$$\mathfrak{a} \subseteq J^{[q]}.$$

The uniqueness in the definition above comes from the regularity of R. By Kunz' Theorem, we have that R is a a finitely generated, flat module over R^p. In this setting, there is a nice property for the q-th Frobenius powers. Namely, if $\{J_i\}_i$ is a family of ideals of R, then

$$\left(\bigcap_i J_i\right)^{[q]} = \bigcap_i J_i^{[q]}.$$

Let's do an example of $[1/q]$ powers.

Example: Let $R = \mathbb{F}_5[x, y]$. Let's compute $\mathfrak{a}^{[1/q]}$ where $\mathfrak{a} = \langle x^{74}, y^{72} \rangle$ and $q = p^2$. The idea behind this is that the ideal $\mathfrak{a}^{[1/p]}$ should be as close as possible to the ideal where we take the 25-th root of each of the generators. Explicitly, it should be that

$$\mathfrak{a}^{[1/25]} \text{ `` = '' } \langle x^{74/25}, y^{72/25} \rangle$$

But this ideal doesn't exist in R, though we can round down the exponents to obtain an ideal of R which "contains" these elements. Thus, we have that

$$\mathfrak{a}^{[1/q]} = \langle x^2, y^2 \rangle.$$

Another way to view the situation above is to take the exponents and use the division algorithm with q. In that case, we'd have

$$74 = 2q + 24, \qquad 72 = 2q + 22.$$

Then, we just pull off the quotient to get the generators. This actually motivates the formula for how to compute $[1/q]$ powers. This proposition is actually easier to state and prove in generality, and we do as such.

Proposition 2.2 ([BMS08], Prop. 2.5)**.** Let R be a Noetherian[14] ring which is free over R^q with $e_1, \ldots, e_N$ a free basis for R over R^q. Let $\mathfrak{a} = \langle f_1, \ldots, f_s \rangle$. Writing each generator in terms of its free basis, we have

$$f_i = \sum_{j=1}^{N} \alpha_{i,j}^q e_j \tag{1}$$

where each $\alpha_{i,j} \in R$. Then, we have

$$\mathfrak{a}^{[1/q]} = \langle \alpha_{i,j} \mid i \leq s, j \leq N \rangle.$$

Proof. It follows from Equation (1) that $\mathfrak{a} \subseteq \langle \alpha_{i,j} \mid i, j \rangle^{[q]}$. Since $\mathfrak{a}^{[1/q]}$ is the smallest ideal with this property, we get the forward containment $\subseteq$ for our proposition.

For the reverse inclusion we use the free-ness to employ some linear algebra techniques and that R is Noetherian to write a finite generating set for our ideals. Namely, suppose $\mathfrak{a} \subseteq J^{[q]}$ for some ideal $J \subseteq R$. As R is Noetherian, J is finitely generated, say

$$J = \langle g_1, \ldots, g_m \rangle.$$

As $\mathfrak{a} \subseteq J^{[q]}$, it follows that each generator f_i is in $J^{[q]}$. Fix i, then we can write

$$f_i = \sum_{k=1}^{m} \beta_k g_k^q,$$

for some $\beta_k \in R$. Recall that $e_1, \ldots, e_N$ is a free basis for R over R^q. This gives us access to the dual basis $e_1^*, \ldots, e_N^*$ for $\mathrm{Hom}_{R^q}(R, R^q)$, where $e_j^*(r)$ pulls off the coefficient of e_j when r is written in terms of the free basis. Thus, $e_j^*(f_i) = \alpha_{i,j}^q$. On the other hand, we have

$$e_j^*(f_i) = e_j^* \left(\sum_{k=1}^{m} \beta_k g_k^q \right) = \sum_{k=1}^{m} e_j^*(\beta_k g_k^q).$$

[14]Two equivalent characterizations of a Noetherian ring that we will be using is (1) Every ideal is finitely generated and (2) Every ascending chain of ideals stabilizes.

As each e_j^* is linear over R^q, the g_k^q can be factored out to get

$$e_j^*(f_i) = \sum_{k=1}^{m} g_k^q e_j^*(\beta_k) \in J^{[q]}.$$

Thus, $\alpha_{i,j} \in J^{[q]}$ for all i and j, which gives us $\alpha_{i,j} \in J$ for all i, j by the Remark on page 8. It follows that

$$\langle \alpha_{i,j} \mid i, j \rangle \subseteq J.$$

Therefore, by definition of $\mathfrak{a}^{[1/q]}$, we have the reserve inclusion in the proposition and

$$\langle \alpha_{i,j} \mid i, j \rangle = \mathfrak{a}^{[1/q]}$$

□

Note: The proposition above says that the description of $\mathfrak{a}^{[1/q]}$ does not depend on the choice of free basis nor on the chosen generators for $\mathfrak{a}$. This proposition also yields an algorithm for finding $[1/q]$ powers. Namely,

- Find a generating set for the ideal
- Write each generator in terms of the free basis
- Pull off the coefficients (without the q-th power)

Let's list and prove some properties of these $[1/q]$ powers.

Lemma 2.3. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R and q, q' be powers of p. Then

(i) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathfrak{a}^{[1/q]} \subseteq \mathfrak{b}^{[1/q]}$.

(ii) $(\mathfrak{a} \cap \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cap \mathfrak{b}^{[1/q]}$, and $(\mathfrak{a} + \mathfrak{b})^{[1/q]} = \mathfrak{a}^{[1/q]} + \mathfrak{b}^{[1/q]}$.

(iii) $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]} \subseteq \mathfrak{a}^{[1/q]} \cdot \mathfrak{b}^{[1/q]}$

(iv) $(\mathfrak{a}^{[q']})^{[1/q]} := \mathfrak{a}^{[q'/q]} \subseteq (\mathfrak{a}^{[1/q]})^{[q']}$.

(v) $\mathfrak{a}^{[1/qq']} \subseteq (\mathfrak{a}^{[1/q]})^{[1/q']}$.

(vi) $\mathfrak{a}^{[1/q]} \subseteq (\mathfrak{a}^{q'})^{[1/qq']}$.

Proof. Statement (i) follows from definition, and both forward containments in (ii) follow from (i). For the reverse containment in (ii), we notice that by definition $\mathfrak{a} \subseteq (\mathfrak{a}^{[1/q]})^{[q]}$. Thus, by (i) and R being characteristic p, we have

$$\mathfrak{a} + \mathfrak{b} \subseteq (\mathfrak{a}^{[1/q]})^{[q]} + (\mathfrak{b}^{[1/q]})^{[q]} = (\mathfrak{a} + \mathfrak{b})^{[q]}.$$

Statement (iii) follows from the minimality in the definition of $(\mathfrak{a} \cdot \mathfrak{b})^{[1/q]}$. Statement (iv) requires the flatness of F, so see [BMS08] for that proof. For statement (v), we use the definition to get

$$\mathfrak{a} \subseteq (\mathfrak{a}^{[1/q]})^{[q]} \subseteq \left(\left((\mathfrak{a}^{[1/q]})^{[q]} \right)^{[1/q']} \right)^{[q']}.$$

Then, by (iv), we can swap the middle two bracket powers to obtain a larger ideal, and so

$$\mathfrak{a} \subseteq (\mathfrak{a}^{[1/q]})^{[q]} \subseteq \left(\left((\mathfrak{a}^{[1/q]})^{[1/q']} \right)^{[q]} \right)^{[q']} = \left((\mathfrak{a}^{[1/q]})^{[1/q']} \right)^{[qq']}.$$

Lastly, for (vi), we notice that (iv) gives us

$$\mathfrak{a}^{[1/q]} = (\mathfrak{a}^{[q']})^{[qq']},$$

and this is contained in $(\mathfrak{a}^{q'})^{[1/qq']}$ by (i) $\square$

2.3 Definition of (Generalized) Test Ideals

Now that we know how to take $[1/q]$ powers, we can start looking at how these "play" with ordinary powers of ideals. Effectively, we will be looking at p-rational[15] powers of ideals. Explicitly, we will be caring about the ideals

$$(\mathfrak{a}^r)^{[1/p^e]},$$

where r is a positive integer.

Lemma 2.4. If r, r', e and e' are such that $r/p^e \geq r'/p^{e'}$ with $e' \geq e$, then

$$(\mathfrak{a}^r)^{[1/p^e]} \subseteq (\mathfrak{a}^{r'})^{[1/p^{e'}]}.$$

Proof. Notice that $r' \leq r/p^{e'-e}$. Thus, $\mathfrak{a}^{r'} \supseteq \mathfrak{a}^{rp^{e'-e}}$. It follows by Lemma 2.3(i, vi) that

$$(\mathfrak{a}^r)^{[1/p^e]} \subseteq (\mathfrak{a}^{rp^{e'-e}})^{[1/p^e p^{e'-e}]} = (\mathfrak{a}^{rp^{e'-e}})^{[1/p^{e'}]} \subseteq (\mathfrak{a}^{r'})^{[1/p^{e'}]}$$

$\square$

Let's take a quick diversion to see show what we are aiming at here. For any positive number λ, there exists a *monotonically decreasing* sequence of p-rational numbers converging to λ. Explicitly, consider the sequence $\frac{\lceil \lambda p^e \rceil}{p^e}$, where $\lceil x \rceil$ is the smallest integer which is greater than or equal to x. To give a quick explanation for what has been asserted about this sequence, it is helpful to think about how to evaluate it. Take λ and write it in its base p expansion,

$$\lambda = \sum_{n=-\infty}^{\infty} a_i p^i = \ \ldots a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots,$$

where $0 \leq a_i \leq p-1$. Then λp^e will "move the decimal point right e places." Explicitly,

$$\lambda p^e = \sum_{i=-\infty}^{N} a_i p^{i+e} = \ \ldots a_{3-e} a_{2-e} a_{1-e} a_{-e} . a_{-1-e} a_{-2-e} \ldots$$

Then, we want to round up this value. If λp^e is an integer, then $\lambda = \frac{c}{p^e}$ for some integer c and for $e' \gg 0$ the sequence in question will be constant at λ. If not, the round up will cut

[15] We say a rational number x is p-rational if it looks like $x = \frac{c}{p^e}$ where $e \geq 0$ and $c \in \mathbb{Z}$.

off this expansion at the decimal point at replace a_{-e} with $a_{-e}+1$. Dividing now by p^e, we will shift the decimal point left e-many places leaving us with

$$\frac{\lceil \lambda p^e \rceil}{p^e} = \frac{1}{p^e} + \sum_{i=-e}^{N} a_i p^i.$$

Whenever $e \to \infty$, this representation makes it quicker to see that it is monotonically decreasing and is converging to λ.

Now, how is this going to help us? Consider the sequence of ideals

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

Notice that replacing e with $e+1$ gives us a relation $\lceil \lambda p^e \rceil / p^e \geq \lceil \lambda p^{e+1} \rceil / p^{e+1}$ with $e+1 \geq e$. Thus, by Lemma 2.4, we have an ascending chain of ideals

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\left\lceil \lambda p^{e+1} \right\rceil}\right)^{[1/p^{e+1}]}.$$

This allows us to define a (generalized) test ideal.

Definition 2.5. Fix $\lambda \geq 0$. Then the (generalized) test ideal of $\mathfrak{a}$ with exponent λ is defined to be

$$\tau(\mathfrak{a}^\lambda) \coloneqq \bigcup_{e>0} \left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

Remark: We have that R is a Noetherian ring, so ascending chains of ideals stabilize. In particular, the union above stablizes after finitely many steps. Thus, we can say that

$$\tau(\mathfrak{a}^\lambda) = (\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}, \qquad \text{where } e \gg 0.$$

Connection to Differential Operators: There are other equivalent formulations of test ideals in the case of principal ideals, or hypersurfaces. Namely if $\lambda = \frac{c}{p^e}$ is a p-rational number and $\mathfrak{a} = \langle f \rangle$, then

$$\tau(f^\lambda) = \operatorname{Im}[\operatorname{Hom}_R(R^{1/p^e}, R) \to R, \text{ defined by evaluation at } f^\lambda].$$

To be specific, one can determine the test ideal of f^λ by taking any R-linear map $\varphi : R^{1/p^e} \to R$, noticing that $f^\lambda = (f^c)^{1/p^e} \in R^{1/p^e}$, looking at the value of $\varphi(f^\lambda)$, and throwing that into our set. The collection of all of those will be the test ideal of f^λ.

This is actually a really cool and interesting formulation. Notice that any R-linear map $R^{1/p^e} \to R$ can be realized as an R^{p^e}-linear map $R \to R^{p^e}$. As $R^{p^e} \subseteq R$, we can think of this map as an R^{p^e}-linear map $R \to R$. This means that this map is in $\operatorname{End}_{R^{p^e}}(R)$. This gives us a connection to differential operators. It is due to A. Yekuteli in [Yek92] that for any F-finite ring of characteristic p, the ring of $\mathbb{F}_p$-linear differential operators, denoted $\mathcal{D}_{R/\mathbb{F}_p}$, is

$$\mathcal{D}_{R/\mathbb{F}_p} = \bigcup_{e \in \mathbb{N}} \operatorname{End}_{R^{p^e}}(R).$$

Thus, one can view $\tau(f^\lambda)$ as the p-th roots of the image of f^c over all differential operators of R which have image in R^{p^e}.

3 Thursday, September 23

In this talk, the goal is to show why test ideals are characteristic p analogues for multiplier ideals. First, we have the following properties, which follow almost immediately from Lemma 2.3, the properties of the $[1/q]$-Frobenius Powers.

Lemma 3.1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Then

(i) If $\lambda < \gamma$, then $\tau(\mathfrak{a}^\gamma) \subseteq \tau(\mathfrak{a}^\lambda)$

(ii) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$

(iii) $\tau\left((\mathfrak{a} \cap \mathfrak{b})^\lambda\right) \subseteq \tau(\mathfrak{a}^\lambda) \cap \tau(\mathfrak{b}^\lambda)$ and $\tau(\mathfrak{a}^\lambda) + \tau(\mathfrak{b}^\lambda) \subseteq \tau\left((\mathfrak{a} + \mathfrak{b})^\lambda\right)$

(iv) $\tau\left((\mathfrak{a} \cdot \mathfrak{b})^\lambda\right) \subseteq \tau(\mathfrak{a}^\lambda) \cdot \tau(\mathfrak{b}^\lambda)$.

3.1 Hypersurface Case

From here on out, we will only consider the hypersurface case, i.e. $\mathfrak{a} = \langle f \rangle$. We will lay out and prove all the analogous statements to multiplier ideals in this special case. Most of the results we say from here will be true for any ideal $\mathfrak{a}$. Though before doing this, let's fix some notation.

Notation: Let $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$ be a vector. We use multi-nomial notation to say

$$\mathbf{x}^v = x_1^{v_1} \cdots x_n^{v_n}.$$

We also write $|v|$ to be the sum of the coordinates. Thus, $|v| = \sum v_i$. Using this notation, we can write any mulitvariate polynomial as

$$f = \sum_{\substack{v \in \mathbb{N}^n \\ |v| \leq m}} c_v \mathbf{x}^v$$

Proposition 3.2 (Right Constancy)**.** Let $\mathfrak{a} = \langle f \rangle \subseteq R$. Then, there exists $\epsilon > 0$ such that whenever $\lambda \leq \lambda' \leq \lambda + \varepsilon$

$$\tau(f^\lambda) = \tau(f^{\lambda'}).$$

Proof. Let $e \gg 0$ such that

$$\tau(f^\lambda) = \left(f^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}.$$

In this case, notice that we can compute $\tau(f^\lambda)$ as

$$f^{\lceil \lambda p^e \rceil} = \sum_{\substack{v \in \mathbb{N}^n \\ |v| \leq m}} c_v \mathbf{x}^v.$$

For each v, write each as $(v_1, \ldots, v_n)$. Then, for each i, use the division algorithm to write $v_i = z_i p^e + r_i$. Let z_v be the vector of all z_i and r_v be the vector of all the remainders r_i. Then

$$f^{\lceil \lambda p^e \rceil} = \sum_{\substack{v \in \mathbb{N}^n \\ |v| \leq m}} c_v (\mathbf{x}^{z_v})^{p^e} \mathbf{x}^{r_v}.$$

Thus, by Proposition 2.2, we have

$$\left(f^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} = \langle \mathbf{x}^{z_v} \mid c_v \neq 0 \rangle.$$

It follows that if we choose $\varepsilon > 0$ small enough so that $\lceil \lambda p^e \rceil = \lceil (\lambda + \varepsilon) p^e \rceil$. Now, if $e \gg 0$ then

$$\tau(f^\lambda) = \tau(f^{\lambda+\varepsilon}).$$

□

Since we do not prove constancy on the left, we introduce a definition for when this occurs. Similar to the characteristic 0 case, we call these F-jumping numbers.

Definition 3.3 (F-Jumping Numbers)**.** A positive real number λ is an *F-jumping number* for an ideal $\mathfrak{a}$ if $\tau(\mathfrak{a}^\lambda) \neq \tau(\mathfrak{a}^{\lambda-\varepsilon})$ for all $\varepsilon > 0$.

Unless explicitly mentioned, we make the convention that 0 is also an F-jumping number. For the rest of this talk, we will discuss the behavior of these invariants. To start with, we have an analogue of Skoda's Theorem (cf. Theorem 1.8) for test ideals.

Theorem 3.4. ([BMS08, Proposition 2.25]) Let $\lambda \geq 1$. Then, we have

$$\tau(f^\lambda) = \langle f \rangle \cdot \tau(f^{\lambda-1}).$$

In particular, λ is an F-jumping number for f if and only if $\lambda - 1$ is an F-jumping number for f.

Proof. For the forward containment, let $e \gg 0$ such that

$$\tau(f^\lambda) = \left(f^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \qquad \text{and } \tau(f^{\lambda-1}) = \left(f^{\lceil (\lambda-1) p^e \rceil}\right)^{[1/p^e]}.$$

Rewriting λ as $(\lambda - 1) + 1$, we have

$$\begin{aligned} \tau(f^\lambda) &= \left(f^{\lceil (\lambda-1)p^e \rceil + p^e}\right)^{[1/p^e]} \\ &= \left(f^{\lceil (\lambda-1)p^e \rceil} f^{p^e}\right)^{[1/p^e]} \\ &\subseteq \left(f^{\lceil (\lambda-1)p^e \rceil}\right)^{[1/p^e]} \cdot \langle f \rangle \\ &= \langle f \rangle \cdot \tau(f^{\lambda-1}), \end{aligned}$$

where the "$\subseteq$" in the third line comes from Lemma (2.3)(iii). Now, for the backward containment, we claim that for all $r \geq p^e$, we have

$$\langle f \rangle \cdot \left(f^{r-p^e}\right)^{[1/p^e]} \subseteq (f^r)^{[1/p^e]}.$$

Notice that this is equivalent to

$$\left(f^{r-p^e}\right)^{[1/p^e]} \subseteq \left((f^r)^{[1/p^e]} : f\right)$$

where $(I : J) = \{r \in R \mid rJ \subseteq I\}$ for I, J ideals of R. Raising both sides to the $[p^e]$-th Frobenius power, this is equivalent to having

$$f^{r-p^e} \in \left((f^r)^{[1/p^e]} : f\right)^{[p^e]} = \left(f^r : f^{p^e}\right).$$

Since this last statement is obviously true, our claim must be true for all $r \geq p^e$. Since $\lambda \geq 1$, it follows that $\lceil \lambda p^e \rceil \geq p^e$. Hence, replacing r with $\lceil \lambda p^e \rceil$ in the claim and taking $e \gg 0$ to be large enough, we have

$$\langle f \rangle \cdot \tau(f^{\lambda-1}) \subseteq \tau(f^\lambda).$$

□

Remark: For the general statement where $\mathfrak{a}$ is any ideal, it holds true whenever $\lambda \geq m$ and $\mathfrak{a}$ can be generated by m elements. Further, in [BMS09, Lemma 2.27], it is proven that the test ideals for $\mathfrak{a}$ are invariant under integral closure. In other words, they prove that $\tau(\mathfrak{a}^\lambda) = \tau(\overline{\mathfrak{a}}^\lambda)$. From that statement, it follows that the general statement holds whenever m is taken to be the *analytic spread* of $\mathfrak{a}$, i.e., the minimal number of generators for any ideal with the same integral closure as $\mathfrak{a}$. The analytic spread of an ideal has a lot of geometric implications and interpretations dealing with affine charts and blowup algebras associated to the variety defined by $\mathfrak{a}$.

We now prove the main result of [BMS08] in the hypersurface case and $R = \mathbb{F}_p[x_1, \ldots, x_n]$.

Theorem 3.5. ([BMS08, Theorem 3.1] Let $f \in R$.

(i) The set of F-jumping numbers of f is discrete (i.e., in every finite interval there are only finitely many F-jumping numbers).

(ii) Every F-jumping number of f is rational.

Proof. We do not prove (i) as it appeals to using the fact that localization commutes with $[1/q]$-Frobenius Powers. Instead we only prove (ii).

Assume (i) and suppose $\lambda > 0$ is an F-jumping number. We may assume that no $p^e\lambda$ is an integer since that would clearly mean λ is rational. Let e_0 be large enough so that $p^e\lambda > 1$. It follows from Theorem 3.4 that $\{p^e\lambda\}$ is an F-jumping number for all $e \geq e_0$, where $\{x\} \coloneqq x - \lfloor x \rfloor$ is the *fractional part* of x. By definition, notice that each $\{p^e\lambda\} \in [0, 1)$. Thus, by (i) of this theorem, there are only finitely many numbers. As a result, we can find $e_1 \neq e_2$ such that $\{p^{e_1}\lambda\} = \{p^{e_2}\lambda\}$. It follows that $p^{e_1}\lambda - p^{e_2}\lambda = (p^{e_1} - p^{e_2})\lambda$ is an integer, and hence λ is rational. □

3.2 F-pure Threshold

To finish off this series of talks, let's briefly discuss the characteristic p analogue of the log canonical threshold, called the F-pure Threshold. First, notice that whenever $\lambda = 0$, we have that $\tau(f^\lambda) = \langle 1 \rangle$. Thus, by Right Constancy of test ideals, we have that $\tau(f^\lambda) = \langle 1 \rangle$ whenever λ is sufficiently close to zero. However, by Theorem 3.4, it follows that $\tau(f^1) = \langle f \rangle$. Thus, there must exist a minimal number λ in $(0, 1]$ where $\tau(f^\lambda) \neq \langle 1 \rangle$. As with the log canonical threshold, we make this a definition.

Definition 3.6 (F-pure Threshold)**.** Let $f \in R$. The F-pure threshold of f, denoted $\mathbf{fpt}(f)$ is

$$\begin{aligned}\mathbf{fpt}(f) &\coloneqq \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \tau(f^\lambda) = \langle 1 \rangle\} \\ &= \min\{\lambda \in \mathbb{R}_{\geq 0} \mid \tau(f^\lambda) \neq \langle 1 \rangle\}\end{aligned}$$

In other words, $\mathbf{fpt}(f)$ is the smallest positive F-jumping number of f.

In the hypersurface case, Theorem 3.4 tells us that $\mathbf{fpt}(f) \in (0, 1]$. This is also true in the general case. This follows from the similar deduction that if $\lambda > 0$ is a F-jumping number, then $\{p^e\lambda\}$ is a F-jumping number for all $e \gg 0$. As with the log canonical threshold of a polynomial explained in the very first section, $\mathbf{fpt}(f)$ is meant to be a measure the singularities of f. Let's look at a quick example.

Example. Let $f = x^2 - y^3 \in \mathbb{F}_p[x, y]$. In order to see what $\mathbf{fpt}(f)$ is, we look at how to compute the test ideals. Notice that if $\frac{r}{p^e}$ is a positive p-rational number, then

$$f^r = (x^2 - y^3)^r = \sum_{i=0}^{r} \binom{r}{i} x^{2i} y^{3(r-i)}.$$

Since there are no relations between the terms of this polynomial, we have that $\mathbf{fpt}(f) \geq \frac{r}{p^e}$ if and only if there is an index i such that

$$2^i < p^e, \qquad 3(r-i) < p^e, \qquad \text{and } \binom{r}{i} \not\equiv 0 \bmod p. \tag{2}$$

First, we see that if $2r < p^e$, then $i = r$ fulfills the three conditions above. It follows that $\mathbf{fpt}(f) \geq \frac{1}{2}$, regardless of p. In fact, if $p = 2$, then $\mathbf{fpt}(f) = \frac{1}{2}$.

On the other hand, if $\frac{r}{p^e} \geq \frac{5}{6}$, then condition (2) is never satisfied. It follows that $\mathbf{fpt}(f) \leq \frac{5}{6}$. We withhold the rest of the computation of $\mathbf{fpt}(f)$. To see the details, refer to Daniel's thesis [Her11].

We end by comparing the F-pure threshold to the log canonical threshold in the most natural setting. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Since $\mathbb{Z} \subseteq \mathbb{C}$, we can view f as a polynomial over $\mathbb{C}$ and compute its log canonical threshold. Then, reducing f modulo p (i.e., reduce each of its coefficients modulo p), we can calculate the F-pure threshold of f mod p in $\mathbb{F}_p[x_1, \ldots, x_n]$.

Because both of these invariants measure the singularity of f and we can think of $\mathbb{Z}$ as the limit of $\mathbb{Z}/p\mathbb{Z}$ as $p \to \infty$, we'd like to have a relationship between the values of $\mathbf{fpt}(f \bmod p)$ and $\operatorname{lct}(f)$. The following theorem gives a partial answer.

Theorem 3.7. Fix $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Then,

- $\mathbf{fpt}(f \bmod p) \leq \operatorname{lct}(f)$ for all $p \gg 0$
- $\lim\limits_{p \to \infty} \mathbf{fpt}(f \bmod p) = \operatorname{lct}(f)$.

The proof of the theorem above is the culmination of the work of the Japanese school of tight closure. For the proofs, see [HW02] and [HY03].

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