MATH 115: Brief Guide to Riemann Sums

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Let $f(x)$ be a function that is non-negative on the closed interval $[a, b]$. Let R denote area under the curve $f(x)$ on the specified interval. Consider partitioning $[a, b]$ into n equally spaced subintervals. Each interval has length $\Delta x = \frac{b-a}{n}$. Choose representative points $x_1, x_2, \ldots, x_n$ from each interval so that no two distinct representatives are in the same subinterval of the partition. Like before, we may approximate the area under the curve of $f(x)$ as

$$\sum_{k=1}^{n}[f(x_k) \cdot \Delta x] = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x.$$

We refer to this sum as the **Riemann sum** of $f(x)$ on the interval $[a, b]$. We define the area of the region R as the limit of the Riemann sum as the number of subintervals approaches infinity.

Definition. Given a continuous function $f(x)$ that is non-negative on the interval $[a, b]$, the area of the region under $f(x)$ is given by

$$\lim_{n \to \infty} \sum_{k=1}^{n}[f(x_k) \cdot \Delta x] = \lim_{n \to \infty}[f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x],$$

where $x_1, x_2, \ldots, x_n$ are arbitrary points in the n subintervals of $[a, b]$ of length $\Delta x = \frac{b-a}{n}$.

We have seen in class that the Riemann sum is the prototype of the definite integral. Lucky for us, all of the functions that we encounter in MATH 115 can be integrated by the standard integration methods that we have learned in class, so we have had no need to use Riemann sums to compute definite integrals; however, this is not the case with the majority of examples that we face in real-world applications. Often, numerical integration (i.e., integration by Riemann sums and other fancy things like cubic splines) is the best way to tackle real-world problems. Like we have seen, more rectangles give us a better approximation by Riemann sums, but obviously, more rectangles also make the calculations untenable. (Consider taking a sum of four things versus a sum of a hundred things by hand.) Computers allow us to automate these computations.

Quadratic Riemann Sum. Find an approximation of the area of the region R under the graph of the function $f(x) = 4 - x^2$ on the interval $[-1, 2]$. Use $n = 6$ subintervals. Choose the representative points to be the left endpoints of the subintervals.

Solution. We note that $\Delta x = \frac{2-(-1)}{6} = \frac{3}{6} = \frac{1}{2}$. Our representative points are the left endpoints, hence we must find a formula ℓ_k for the left endpoint of the kth subinterval. Certainly, our first left endpoint will be -1, hence we have that $\ell_1 = -1$. Each time we move from one subinterval to the next, we must travel a distance of $\frac{1}{2}$ units to arrive at the next left endpoint, so we have that $\ell_k = \frac{1}{2}k + C$ for some real number C. We have two equations, so we can solve for C. We have that

$$-1 = \ell_1 = \frac{1}{2} \cdot 1 + C,$$

from which it follows that $C = -\frac{3}{2}$. We conclude that $\ell_k = \frac{1}{2}k - \frac{3}{2}$. Using a Riemann sum,

$$\text{area}(R) \approx \sum_{k=1}^{6} f(\ell_k) \cdot \Delta x = \sum_{k=1}^{6} \left[4 - \left(\frac{1}{2}k - \frac{3}{2}\right)^2\right] \cdot \frac{1}{2} = 9.625. \qquad \diamond$$

We can actually automate these Riemann sums with the summation function of the TI-83 (or any newer model) calculator! Use the following algorithm to compute the above Riemann sum.

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1.) Click "MATH."
2.) Click "0" to pull up the summation function.
3.) Click the left parenthesis to input K as the variable of summation.
4.) Click "1" to begin the summation at K = 1.
5.) On the directional pad, tab up to the upper limit of summation.
6.) Click "6" to end the summation at K = 6.
7.) On the directional pad, tab right to input the summand.
8.) Input the summand as follows: "(4-((1/2)K-(3/2))^2)*(1/2)."
Use the combination "ALPHA" + "left parenthesis" to obtain the variable K.
9.) Click "Enter" to evaluate the sum. We obtain a value of 9.625, as desired.
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Rational Riemann Sum. Find an approximation of the area of the region R under the graph of the function $f(x) = \frac{1}{x}$ on the interval $[1, 3]$. Use $n = 4$ subintervals. Choose the representative points to be the right endpoints of the subintervals.

Solution. We note that $\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$. Our representative points are the right endpoints, hence we must find a formula r_k for the right endpoint of the kth subinterval. We note that our first right endpoint will be $r_1 = \frac{3}{2}$. Each time we move from one subinterval to the next, we must travel a distance of $\frac{1}{2}$ units to arrive at the next right endpoint, so we have that $r_k = \frac{1}{2}k + C$ for some real number C. We have two equations, so we can solve for C. We have that

$$\frac{3}{2} = r_1 = \frac{1}{2} \cdot 1 + C,$$

from which it follows that $C = 1$. We conclude that $r_k = \frac{1}{2}k + 1$. Using a Riemann sum,

$$\text{area}(R) \approx \sum_{k=1}^{4} f(r_k) \cdot \Delta x = \sum_{k=1}^{4} \frac{1}{\frac{1}{2}k + 1} \cdot \frac{1}{2} = 0.95. \qquad \diamond$$

Use the following algorithm to compute the above Riemann sum.

```
1.) Click "MATH."
2.) Click "0" to pull up the summation function.
3.) Click the left parenthesis to input K as the variable of summation.
4.) Click "1" to begin the summation at K = 1.
5.) On the directional pad, tab up to the upper limit of summation.
6.) Click "4" to end the summation at K = 4.
7.) On the directional pad, tab right to input the summand.
8.) Input the summand as follows: "(1/((1/2)K+1))*(1/2)."
Use the combination "ALPHA" + "left parenthesis" to obtain the variable K.
9.) Click "Enter" to evaluate the sum. We obtain a value of 0.95, as desired.
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Exponential Riemann Sum. Find an approximation of the area of the region R under the graph of the function $f(x) = e^x$ on the interval $[0, 3]$. Use $n = 5$ subintervals. Choose the representative points to be the midpoints of the subintervals.

Solution. We note that $\Delta x = \frac{3-0}{5} = \frac{3}{5}$. Our representative points are the midpoints, hence we must find a formula $\overline{x}_k$ for the midpoint of the kth subinterval. We note that our first midpoint will be $\overline{x}_1 = \frac{1}{2}(\frac{3}{5} + 0) = \frac{3}{10}$. Each time we move from our present subinterval to the next, we must travel a distance of $\frac{3}{5}$ units to arrive at the next midpoint, so we have that $\overline{x}_k = \frac{3}{5}k + C$ for some real number C. We have two equations, so we can solve for C. We have that

$$\frac{3}{10} = \overline{x}_1 = \frac{3}{5} \cdot 1 + C,$$

from which it follows that $C = -\frac{3}{10}$. We conclude that $\overline{x}_k = \frac{3}{5}k - \frac{3}{10}$. Using a Riemann sum,

$$\text{area}(R) \approx \sum_{k=1}^{5} f(\overline{x}_k) \cdot \Delta x = \sum_{k=1}^{5} e^{\frac{3}{5}k - \frac{3}{10}} \cdot \frac{3}{5} \approx 18.8. \qquad \diamond$$

Use the following algorithm to compute the above Riemann sum.

```
1.) Click "MATH."
2.) Click "0" to pull up the summation function.
3.) Click the left parenthesis to input K as the variable of summation.
4.) Click "1" to begin the summation at K = 1.
5.) On the directional pad, tab up to the upper limit of summation.
6.) Click "5" to end the summation at K = 5.
7.) On the directional pad, tab right to input the summand.
8.) Input the summand as follows: "e^((3/5)K-(3/10))*(3/5)."
Use the combination "ALPHA" + "left parenthesis" to obtain the variable K.
9.) Click "Enter" to evaluate the sum. We obtain a value of approximately 18.8.
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