The Geometric Intuition of Regularity

Trevor Arrigoni

Abstract: Frequently in commutative algebra, we are interested in studying some type of singularity. In colloquial terms, we view the word "singularity" as a moniker for something not being smooth. For instance, a singularity of a surface defined by $z = f(x, y)$ could be a point p such that f is not differentiable at p. But, how do we discuss this in the commutative algebra setting? Typically we will discuss a ring being "singular" if it deviates from being regular. But why are regular rings so special? In short, a regular ring is the algebraic analogue to that of a differentiable manifold. Thus, locally, regular rings are very well-behaved. In this talk, we introduce the idea of a regular local ring and develop the ideas for why we should consider this type of object as something which is smooth in some sense. To do so, we develop some intuition from classical algebraic geometry by looking at the local ring at a point on an irreducible variety over an algebraically closed field.

This talk is highly motivated by the Regular Rings section of Karen Smith's characteristic p techniques lecture notes.

Setup

In this talk, every ring is commutative with unit. When we say a ring $(R, \mathfrak{m}, k)$ is local, we imply R is Noetherian. If $(R, \mathfrak{m}, k)$ has a unique maximal ideal, but may or may not be Noetherian, we say R is *quasi-local*.

If $I = \langle f_1, \dots, f_t \rangle$ is an ideal of the polynomial ring $k[x_1, \dots, x_n]$, we denote by $\mathbb{V}(I)$, the vanishing set or *algebraic variety* defined by I. Explicitly, $\mathbb{V}(I)$ is the set

$$\mathbb{V}(f_1, \dots, f_n) = \mathbb{V}(I) = \{p \in k^n \mid f_i(p) = 0 \text{ for all } i\}.$$

When $I = \langle f \rangle$ is a principal ideal of $k[x_1, \dots, x_n]$, we denote the variety defined by f as $\mathbb{V}(f)$ and call it a hypersurface. In generality, if $I \subseteq R$ is an ideal of a ring R, one defines $\mathbb{V}(I) \subseteq \operatorname{Spec}(R)$ as the set of prime ideals which contain I and then such collections of prime ideals to be the closed sets of a topology, called the Zariski topology. We will not need this generalization though.

Generally speaking, we say a set of points $V \subseteq k^n$ is an algebraic variety if it is the common vanishing set of a collection of polynomials $\{f_1, \dots, f_t\}$.[1] Then, the *coordinate ring* of V, denoted by $k[V]$, is the quotient ring

$$k[V] := \frac{k[x_1, \dots, x_n]}{\langle f_1, \dots, f_t \rangle}.$$

[1]Hilbert's Basis Theorem ensures that one can always find a finite set. Further, it is not hard to prove that the ideal $I_V = \langle f_1, \dots, f_t \rangle$ is a radical ideal.

1 Localization

To begin this talk, we need the construction of localization. The process of localization is in big comparison to considering open neighborhoods at points in a topology. In fact, by looking at the Zariski Topology, one can actually quantify that statement. Let's define what we mean by localization.

Definition 1 (Localization)**.** Let R be a ring and $W \subseteq R \setminus \{0\}$ be a multiplicatively closed subset of R.[a] The **localization** of R at W is defined to be the set

$$W^{-1}R := \left\{ \frac{r}{w} \middle| r \in R, w \in W \right\}.$$

The localization $W^{-1}R$ is a ring with operations

$$\frac{r}{w} + \frac{s}{u} = \frac{ru + sw}{uw} \qquad \text{and} \qquad \frac{r}{w} \cdot \frac{s}{u} = \frac{rs}{uw}.$$

[a]We define W to be a multiplicatively closed set if $1 \in W$ and if $w_1, w_2 \in W$, then $w_1 w_2 \in W$.

Remark 1: Similar to the construction of $\mathbb{Q}$ using the nonzero elements of $\mathbb{Z}$, the localization $W^{-1}R$ is a set of equivalence classes. We say that $\frac{r}{w}, \frac{s}{u} \in W^{-1}R$ are equal if there exists $t \in W$ such that

$$t(ru - sw) = 0.$$ [2]

Remark 2: Note that there is a natural embedding of R into $W^{-1}R$ via the ring homomorphism ϕ_W that sends $\phi_W(r) = \frac{r}{1}$. This maps ϕ_W is called the **natural localization map** and gives a way of viewing R as sitting inside $W^{-1}R$.

Remark 3:(Universal Property of Localization) Let R be a ring and $W \subseteq R \setminus \{0\}$ any multiplicatively closed set. Take any $w \in W$ and consider the element $\frac{1}{w} \in W^{-1}R$. Then we have that

$$\frac{w}{1} \cdot \frac{1}{w} = \frac{w}{w} = \frac{1}{1} = 1.$$

Thus, every element of W becomes a unit in $W^{-1}R$. In fact, one can think of the localization $W^{-1}R$ as the smallest ring extension of R containing the inverses of the elements of R. It adjoins inverses for W in the freest way possible. One way to see this is through the isomorphism

$$W^{-1}R \cong R[x_w \mid w \in W] / \langle w \cdot x_w - 1 \mid w \in W \rangle \cong R\left[\frac{1}{w} \middle| w \in W \right].$$

Another way to see this is via the **Universal property of localization**. Suppose $f : R \to S$ is any ring homomorphism such that $f(w)$ is a unit in S for all $w \in W$. The universal property of localization says that there exists a unique ring homomorphism $\alpha : W^{-1}R \to S$ such that the follows diagram commutes

[2]The addition of an extra multiple from W is there to make sure this is an equivalence relation. It may not be obvious, but this extra element ensures transitivity. If R is a domain, it is not hard to prove that you can always set $t = 1$.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\phi_W"'] & S \\ W^{-1}R \arrow[ur, dashed, "\exists!\alpha"'] & \end{tikzcd}$$

Basically, the universal property of localization says that if you create ANY way to invert the elements of W (i.e. by a ring homomorphism $f : R \to S$), it must factor through the most natural way to invert the elements of W, the localization $W^{-1}R$.

Examples: Let's see some examples of localizations:

- Set $R = \mathbb{Z}$ and $W = \mathbb{Z} \setminus \{0\}$. The localization $W^{-1}R$ is exactly $\mathbb{Q}$.
- Generally, suppose R is any domain and set W to be the set of nonzero elements of R. Then $W^{-1}R = \mathrm{Frac}(R)$, the fraction field of R.
- If R is any ring, set W to be the set of nonzero divisors of R. The localization $W^{-1}R$, often denoted $Q(R)$, is called the **total ring of fractions** of R.
- Let $R = \mathbb{Z}$ and $W = \{1, 2, 4, 8, 16, \dots\}$, so that W is the set of powers of 2. In this formulation, we have
$$W^{-1}R = \mathbb{Z}_2 = \left\{ \frac{m}{2^n} \middle| m \in \mathbb{Z}, n \geq 0 \right\}.$$
- More generally, suppose R is any ring and $f \in R$ is any element which is not nilpotent. Define $W \coloneqq \{1, f, f^2, f^3, \dots\}$. Then the localization of R at W, denoted R_f, becomes
$$W^{-1}R = R_f = \left\{ \frac{r}{f^n} \middle| r \in R,\ n \geq 0 \right\}.$$
- Let $R = \mathbb{C}[x]$ and set W to be the powers of x. It follows that
$$W^{-1}R = R_x = \mathbb{C}[x]_x = \left\{ \frac{f}{x^n} \middle| f \in \mathbb{C}[x],\ n \geq 0 \right\} = \mathbb{C}[x, x^{-1}].$$

Main Example: One more example that is one of the main objects of study in Commuative Algebra is the localization at a prime ideal. Let R be any ring and $\mathfrak{p} \in \mathrm{Spec}(R)$. By definition of prime ideal, if we take two elements $r, s \notin \mathfrak{p}$, then the product $rs \notin \mathfrak{p}$. Since 0 is in every ideal and 1 is not in any prime ideal, it follows that $W \coloneqq R \setminus \mathfrak{p}$ is a multiplicatively closed set of R. The **localization of R at the prime ideal $\mathfrak{p}$** is the ring

$$W^{-1}R = R_\mathfrak{p} = \left\{ \frac{r}{w} \middle| r \in R, w \in R \setminus \mathfrak{p} \right\}.$$

One can check that $R_\mathfrak{p}$ is a quasi-local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$, the expansion of $\mathfrak{p}$ into $R_\mathfrak{p}$ via the natural localization map.[3] Explicitly, we will have

$$\mathfrak{p}R_\mathfrak{p} = \left\{ \frac{r}{w} \middle| r \in \mathfrak{p}, w \in W \right\}.$$

[3]Noetherianity is preserved under localization, meaning that $W^{-1}R$ is Noetherian if R is. However, Noetherianity is not a local property – the converse very importantly does NOT hold: There exists a ring where every localization is Noetherian, yet it is not. An example is the ring which is the direct product of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. Regardless, we will use the forward direction of this statement to say that if R is a Noetherian ring, then $R_\mathfrak{p}$ will always be a local ring.

Remark 4:(Prime ideals of a localization) Using the natural localization map $\phi_W : R \to W^{-1}R$ described in Remark 2, one can describe a 1-1 correspondence between $\mathrm{Spec}(W^{-1}R)$ and a special subset of $\mathrm{Spec}(R)$. In fact, the 1-1 correspondence is

$$\begin{aligned} \left\{\text{prime ideals of } W^{-1}R\right\} &\longleftrightarrow \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \cap W = \emptyset\} \\ \mathfrak{q} &\longmapsto \phi_W^{-1}(\mathfrak{q}) = \mathfrak{q} \cap R \\ \langle \phi_W(\mathfrak{q}) \rangle = \mathfrak{q}W^{-1}R &\longleftarrow\!\shortmid \mathfrak{q} \end{aligned}$$

Explicitly, this correspondence gives us the following results:

Proposition 2. Let R be a ring and W be a multiplicative closed set of R. Then

1. $\dim(W^{-1}R) \le \dim(R)$
2. $\dim(R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) = \sup\{n \mid \exists P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}, P_i \in \mathrm{Spec}(R)\}$

2 Local Rings

From now on, $(R, \mathfrak{m}, k)$ is a local ring (recall this implies Noetherian) with unique maximal ideal $\mathfrak{m}$ and residue field $k \cong R/\mathfrak{m}$. Recall that in a local ring, the (Krull) dimension of R is given by $\dim(R) = \mathrm{ht}(\mathfrak{m})$. Since R is Noetherian, it follows that the dimension is finite. Unless otherwise stated, set $d = \dim(R)$. In a local ring, there is another formulation of the dimension

Proposition 3. If $(R, \mathfrak{m}, k)$ is a local ring, then the dimension of R is finite and

$$\dim(R) = \text{ least number of generators for an ideal } I \text{ with } \sqrt{I} = \mathfrak{m}.$$

There is a systematic way to construct an ideal I which satisfies this proposition. In fact, if $\dim(R) = 0$, then $\sqrt{0} = \mathfrak{m}$ since $\mathfrak{m}$ will be the unique minimal prime in that setting. If $\dim(R) > 0$, it follows from the *Prime Avoidance Lemma* that there exists an element $x_1 \in \mathfrak{m}$ which is not contained in any minimal prime of R. Since x_1 is not contained in any minimal prime of R, it follows that

$$\dim(R/\langle x_1 \rangle) = \dim(R) - 1$$

and this quotient is also a local ring with maximal ideal $\mathfrak{m} + \langle x_1 \rangle$. If $\dim(R/\langle x_1 \rangle) = 0$, we are done. If the dimension is not zero, it follows once again by Prime Avoidance that there exists $x_2 + \langle x_1 \rangle \in \mathfrak{m} + \langle x_1 \rangle$ not contained in any minimal prime of $R/\langle x_1 \rangle$. Then, the ring $R/\langle x_1, x_2 \rangle$ is local with dimension

$$\dim(R/\langle x_1, x_2 \rangle) = \dim(R/\langle x_1 \rangle) - 1 = \dim(R) - 2.$$

Continue this process. Since R is Noetherian, we must stop at some finite step. At this point, we will have an ideal $I = \langle x_1, \dots, x_t \rangle$ such that

- $\dim(R/\langle x_1 \rangle) = \dim(R) - 1$

- $\dim(R/\langle x_1, \ldots, x_i \rangle) = \dim(R) - i$
- $\dim(R/I) = \dim(R) - t = d - t$

By the Prime Avoidance Lemma, the only way this process can stop is if $\dim(R/I) = d - t = 0$, which implies $t = d$. Thus, we have

$$\dim(R/\langle x_1, \ldots, x_t \rangle) = 0,$$

and so $\mathfrak{m} + I$ is a minimal prime of R/I. Thus, as the only minimal prime, $\mathfrak{m} + I$ is equal to the nilradical of R/I. Thus

$$\mathfrak{m} + I = \sqrt{\langle 0 \rangle + I} \qquad \Longrightarrow \qquad \mathfrak{m} = \sqrt{I},$$

when lifting back to R. When this happens, we call the generators of I, the $x_1, \ldots, x_d$ with $d = \dim(R)$ which generate $\mathfrak{m}$ up to radical, a **system of parameters** (s.o.p.) for R. This construction is also a sufficient proof of the following statement:

Proposition 4. Let $(R, \mathfrak{m}, k)$ be a local ring. Then a system of parameters exists.

Remark 5: By the construction, one sees quite readily that any regular sequence $x_1, \ldots, x_t$ contained in the maximal ideal must be part of a system of parameters for $(R, \mathfrak{m}, k)$. One of the conditions for being a regular sequence is that each x_i is a nonzero divisor on $R/\langle x_1, \ldots, x_{i-1} \rangle$. This is equivalent to x_i not being in any associated prime of $\langle x_1, \ldots, x_{i-1} \rangle$. Via primary decomposition, one proves that

$$\{\text{minimal primes of } \langle x_1, \ldots, x_{i-1} \rangle\} \subset \{\text{associated primes of } \langle x_1, \ldots, x_{i-1} \rangle\}.$$

Thus, it follows that $x_1, \ldots, x_t$ will abide by the conditions of being part of a s.o.p. Thus, we recover the result

$$\operatorname{depth}_{\mathfrak{m}}(R) = \operatorname{depth}(R) \leq \dim(R).$$

There are two deficiencies in this formulation for any local ring. The first being whether or not a regular sequence can be a s.o.p. In fact, when this happens, we say that R is Cohen-Macaulay (CM). Formally, we define it as follows:

Definition 5 (Cohen-Macaulay Ring)**.** Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:

1. $\operatorname{depth}(R) = \dim(R)$.
2. Some system of parameters is a regular sequence.
3. Every system of parameters is a regular sequence.
4. The Koszul Complex is exact for every system of parameters.

When any of these equivalent conditions is satisfied, we say that R is *Cohen-Macaulay*.

2.1 Regular Local Rings

We just saw that the Cohen-Macaulay condition takes care of the deficiency of a regular sequence only being *part* of a system of parameters. The next deficiency is that, even in a CM ring, a regular sequence only generates the maximal ideal up to radical. What if it's not only up to radical?

Definition 6 (Regular Local Ring)**.** Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. We say that R is *Regular* if there exists a regular sequence $x_1, \ldots, x_d \in \mathfrak{m}$ such that

$$\langle x_1, \ldots, x_d \rangle = \mathfrak{m}.$$

There is another way to define the concept of a regular ring, one which will be more useful for our discussion. We first recall the ever important Nakayama's Lemma

Theorem 7 (One of the versions of Nakayama's Lemma)**.** Let $(R, \mathfrak{m}, k)$ be a quasi-local ring and M be any finitely generated R-module. Then the module $M/\mathfrak{m}M$ is a finite dimensional k-vector space and $\overline{x_1}, \ldots, \overline{x_n} \in M/\mathfrak{m}M$ is a k-basis for $M/\mathfrak{m}M$ if and only if $x_1, \ldots, x_n \in M$ are a minimal set of generators for M over R.

Moreover, the minimal number of generators for M as an R-module, denoted $\mu_R(M)$, is unique and is equal to $\dim_k(M/\mathfrak{m}M)$.

Remark 6: Given a local ring $(R, \mathfrak{m}, k)$, we can view $\mathfrak{m}$ as a finitely generated R-module. By Nakayama's Lemma, we can understand some of the structure of $\mathfrak{m}$ via the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, which we call the **Zariski cotangent space**. As a finitely generated k-vector space, it has another finitely generated k-vector space naturally associated to it, its dual space $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, called the **Zariski tangent space**. By Nakayama's Lemma, it follows that

$$\mu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

We call this number the **embedding dimension** of R. This embedding dimension gives us another way of defining a regular local ring: a local ring $(R, \mathfrak{m}, k)$ is regular local if $\dim(R) = d = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. In other words, a regular local ring is one whose Krull dimension equals its embedding dimension. Let's view this in an example.

Example: Let $k = \overline{k}$ be an algebraically closed field and consider the point $p = (\lambda_1, \ldots, \lambda_n) \in k^n$. It is a classical result in the formulation of classical algebraic geometry that the maximal ideals of the polynomial ring $k[x_1, \ldots, x_n]$ are in 1-1 correspondence with the points of k^n. It follows from this correspondence that there is a unique maximal ideal associated to the point p, which we denote $\mathfrak{m}_p$. Explicitly, we have that

$$\mathfrak{m}_p = \langle x_1 - \lambda_1, \ldots, x_n - \lambda_n \rangle .$$

Thus, to p, we can associate the local ring at p to be the ring $R = k[x_1, \ldots, x_n]_{\mathfrak{m}_p}$. By constructing this local ring R, we are zeroing in on the polynomials which vanish at p. In fact, this localization inverted every polynomial which does NOT vanish at p. Said a different way, R is the ring of all rational functions defined at p, often called the set of *regular functions* at p. Let's try to describe the Zariski cotangent space of R, which we recall is $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let f be any polynomial of $k[x_1, \dots, x_n]$. We recall from calculus, that we can build the Taylor series of f expanded around p, so that

$$\begin{aligned} f =& f(p) + \left(\frac{\partial f}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots \frac{\partial f}{\partial x_n}(p)(x_n - \lambda_n) \right) \\ &+ \left(\frac{1}{2!} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(p)(x_i - \lambda_i)^2 + \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)(x_i - \lambda_i)(x_j - \lambda_j) \right) \\ &+ \text{ higher order terms,} \end{aligned}$$

where the operator $\frac{1}{m!}\frac{\partial^m}{\partial x_i^m}$ makes sense over any field k. We highlighted the first-order term because this is the portion we want to focus on. Denote the colored portion above, the best linear approximation of f at p, by $d_p f$, which is the differential of f at p. Suppose further that f vanishes at p, which implies $f \in \mathfrak{m}_p$. Then, we have that the differential of f at p is

$$d_p f := \frac{\partial f}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots \frac{\partial f}{\partial x_n}(p)(x_n - \lambda_n) \in \mathfrak{m}_p$$

Thus, there is a k-linear map that picks off the coefficients of this differential. Namely, the map is

$$\mathfrak{m}_p \longrightarrow k^n \qquad\qquad f \longmapsto d_p f \longmapsto \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right).$$

Further, it is quick to see via product rule that the kernel of this map is all of $\mathfrak{m}_p^2$. Thus, it induces a map

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow k^n \qquad\qquad f + \mathfrak{m}_p^2 \longmapsto d_p f \longmapsto \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right),$$

which is readily seen to be an isomorphism. Thus, every minimal generator of $\mathfrak{m}_p/\mathfrak{m}_p^2$, which look like $(x_i - \lambda_i)$, are identified in $\mathfrak{m}_p/\mathfrak{m}_p^2$ with the element $d_p x_i$, which implies that $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = n = \dim(R)$. Thus, R is a regular local ring where the maximal ideal is

$$\mathfrak{m}_p R = \left\langle \frac{x_1 - \lambda_1}{1}, \dots, \frac{x_n - \lambda_n}{1} \right\rangle.$$

Let's now state some basic examples of regular local rings:

- A regular local ring of dimension 0 is a field since the maximal ideal with be generated by 0.
- A regular local ring of dimension 1 is the same as a DVR.[4]

As stated in the abstract, one wants to think of a regular local ring as one which is free of singularity. They are the most well-behaved rings we have which are not fields. The most basic type of singularity is that of a zero divisor. Thus, we can show that regular local rings do not have these:

[4]We say $(R, \mathfrak{m}, k)$ is a DVR if it is a local PID. In this case, we will have $\mathfrak{m} = \langle t \rangle$ be a principal ideal where t is called the *uniformizer* of R.

Proposition 8. Every regular local ring is a domain.

Proof. We induce on $d = \dim(R)$. The base case is easy. If $\dim(R) = 0$, then R is a field, which is obviously a domain.

Suppose $\dim(R) = d > 0$ and assume every regular local ring of dimension $d - 1$ is a domain. Since $d > 0$, it follows that $\mathfrak{m}$ is not a minimal prime of R. Thus, by Prime avoidance, there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is not contained in any minimal prime of R. It follows that $\dim(R/\langle x\rangle)$ drops by one, as does its embedding dimension. Thus, $R/\langle x\rangle$ is a regular local ring, and hence a domain via the induction hypothesis. Thus, $\langle x\rangle$ is a prime ideal of R with height equal to 1. Thus, there exists a minimal prime $P \subsetneq \langle x\rangle$.

We claim that $P = 0$. In fact, take any $y \in P \subsetneq \langle x\rangle$. Then, we can write $y = rx$ for some $r \in R$. As $x \notin P$, it must be that $r \in P$. Thus, we have that $P \subset xP$. As the other containment trivially holds, we have that $xP = P$. Therefore, by Nakayama's Lemma, we have $P = 0$, which implies R is a domain. $\square$

We record the main "trick" of this proof as a useful lemma:

Lemma 9. Let $(R, \mathfrak{m}, k)$ be a local ring and let $x \in \mathfrak{m}$ be an element not contained in any minimal prime of R. If $x \notin \mathfrak{m}^2$, then R is regular if and only if $R/\langle x\rangle$ is regular. Conversely, if both R and $R/\langle x\rangle$ are regular, then $x \notin \mathfrak{m}^2$.

2.2 Regularity in the Classical Setting

Fix $k = \overline{k}$ to be an algebraically closed field.[5] Consider the irreducible hypersurface $V = \mathbb{V}(f) \subseteq k^n$, so that f is an irreducible polynomial in $k[x_1, \ldots, x_n]$ and V is its zero set. As in the previous example, we can linearize f at a point $p \in V$ by looking at polynomial

$$d_p f = \frac{\partial f}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots \frac{\partial f}{\partial x_n}(p)(x_n - \lambda_n).$$

If we look at the vanishing set of $d_p f$, denoted $T_pV = \mathbb{V}(d_p f)$, we obtain the tangent plane approximation to V at p. As this is defined by a single linear equation, it is a linear space of dimension $n - 1$, provided that $d_p f$ is not the zero polynomial. In this case, we see that the tangent space of dimension $n - 1$ nicely approximates the $n - 1$ dimensional variety V near p. When this happens, we say that p is a *smooth point* of V. However, if $d_p f = 0$, then $T_pV = \mathbb{V}(d_p f) = \mathbb{V}(0)$ will be the whole space k^n, which is pretty clearly not a good approximation of V at p. This is the non-smooth or *singular* setting.

Example: Let's do an explicit example of finding the set of smooth points on a hypersurface $V = \mathbb{V}(f)$, called the smooth locus of V. Consider $f = x^2 + y^2 - z^2$, so that $V = \mathbb{V}(f) \subseteq k^3$ is a cone. Take any point $p = (a, b, c)$ on V, which implies that $f(p) = 0$. Then, the tangent space approximation of V at p is given by

$$\mathbb{V}(d_p f) = \mathbb{V}(2a(x - a) + 2b(y - b) - 2c(z - c)) \subseteq k^3.$$

[5] Recall that this also includes the case where k is of prime characteristic $p > 0$ and perfect, i.e. $k = k^p$.

Notice that this linear variety is 2 dimensional except when $p = (0,0,0)$ is the origin.[6]

Consider the coordinate ring $k[V] = k[x_1, \ldots, x_n]/\langle f\rangle$. Localize this ring at $\mathfrak{m}_p$ to create the local ring of V at p, which is often denoted $\mathcal{O}_{V,p}$. Let's look at the Zariski cotangent space of $\mathcal{O}_{V,p}$. Notice that

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = \langle x-a, y-b, z-c\rangle / \langle x-a, y-b, z-c\rangle^2.$$

inside the ring $k[x,y,z]_{\mathfrak{m}_p}/(\langle f\rangle + \mathfrak{m}_p^2)$. Notice that, at the origin $p = (0,0,0)$, we have that this ring is

$$\mathcal{O}_{V,p}/\mathfrak{m}_p^2 \cong k[x,y,z]/\langle x,y,z\rangle^2,$$

and $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the 3-dimensional k-vector space spanned by the classes of x, y, z. However, if p is not the origin, notice that

$$\frac{k[x,y,z]_{\mathfrak{m}_p}}{\langle x^2+y^2-z^2\rangle + \mathfrak{m}_p^2} \cong \frac{k[x-a, y-b, z-c]}{\langle d_p f\rangle + \mathfrak{m}_p^2}.$$

What does this mean? Without loss of generality, suppose $c \neq 0$. It follows that we can solve for $z-c$ in terms of $x-a$ and $y-b$. Thus, $\mathfrak{m}_p/\mathfrak{m}_p^2$ will be the 2-dimensional k-vector space spanned by $x-a$ and $y-b$. More generally, the Zariski cotangent space will be generated by 2 of $x-a = d_p x$, $y-b = d_p y$, or $z-c = d_p z$ corresponding to which coordinates of p are nonzero.

The previous example can be formalized into what's called the **Jacobian Criterion for Smoothness**: Let $p \in k^n$ be a point on an irreducible variety $V \subseteq k^n$ of dimension d. Then, $V = \mathbb{V}(I_V)$ where I_V is a radical ideal of $k[x_1, \ldots, x_n]$. Then, we can choose generators for I_V, say

$$I_V = \langle f_1, \ldots, f_t\rangle.$$

We can then create the best linear approximation to V at p by looking at

$$T_pV := \mathbb{V}(d_pf_1, \ldots, d_pf_t).$$

With this formulation we see that T_pV, where p is the point $p = (\lambda_1, \ldots, \lambda_n)$, is the solution set of

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f}{\partial x_2}(p) & \cdots & \frac{\partial f_2}{\partial x_n}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \cdots & \frac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_t}{\partial x_1}(p) & \frac{\partial f_t}{\partial x_2}(p) & \cdots & \frac{\partial f_t}{\partial x_n}(p) \end{pmatrix} \begin{pmatrix} x_1-\lambda_1 \\ x_2-\lambda_2 \\ \vdots \\ x_n-\lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, we say a point $p \in V$ is smooth if and only if T_pV is a d-dimensional variety, matching the dimension of V. But then, recall by the Rank-Nullity Theorem that the dimension of T_pV

[6] We also assume that the characteristic of k is not 2. In the case where the characteristic is 2, then $f = (x+y+z)^2$ and $\mathbb{V}(f) = \mathbb{V}((x+y+z)^2)$. It follows from here that $d_pf = 0$ for all $p \in V$, so that V has no smooth points.

is $n - \text{rank } J_p$, where J_p is the Jacobian matrix (the matrix of partial derivatives) evaluated at p. Thus, p is a smooth point on V if and only if $n - \text{rank } J_p$ equals the dimension of V. It is not hard to see that the dimension of T_pV is at least the dimension of V. Therefore, this gives us a criterion for smooth points based on the rank of J_p. Explicity, we can say that p is a smooth point of a d-dimensional variety V if and only if

$$\text{rank } J_p < n - d,$$

which is described by the vanishing of all the $(n-d) \times (n-d)$ minors of the Jacobian matrix J evaluated at p. In essence, the Jacobian criterion for smoothness can be summarized as follows:

Theorem 10. The singular locus of an equidimensional variety $V \subseteq k^n$, where $V = \mathbb{V}(I_V)$ and I_V is the radical ideal generated by $f_1, \ldots, f_t$, is the Zariski-closed subset[7]

$$V \cap \mathbb{V}(\text{all } c \times c \text{ minors of } J) \subseteq V,$$

with $c = n - \text{dim}V$.

Example: Consider the ring $k[x, y, z]/\langle x^2y - z^2\rangle$, which is a ring of dimension 2. The Jacobian matrix is the row vector $J = \begin{bmatrix} 2xy & x^2 & 2z \end{bmatrix}$. The singular locus of $V = \mathbb{V}(x^2y - z^2)$ is then determined by the intersection of V and the vanishing of the 1×1 minors of J, which by definition is $\mathbb{V}(2xy, x^2, 2z) \cap \mathbb{V}(x^2y - z^2) = \mathbb{V}(2xy, x^2, 2z, x^2y - z^2)$. It is not terribly hard to see that, regardless of the characteristic of k, the singular locus is nothing more than $\mathbb{V}(x, z)$, which is simply the y-axis.

With these calculations in hand, we turn our attention to why we consider regularity to be the same as smoothness in commutative algebra.

Theorem 11. Let $\mathcal{O}_{V,p}$ be the local ring of a point p on a variety V over an algebraically closed field k. Then $\mathcal{O}_{V,p}$ is a regular local ring if and only if p is a smooth point on V.

Proof. Via a change of coordinates, we can assume $p \in k^n$ is the origin. Then, $\mathcal{O}_{V,p}$ can be described as

$$\mathcal{O}_{V,p} = \frac{k[x_1, \ldots, x_n]_{\mathfrak{m}}}{\langle f_1, \ldots, f_t\rangle},$$

where $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ is the unique homogeneous maximal ideal and $I_V = \langle f_1, \ldots, f_t\rangle$ is a radical ideal such that $V = \mathbb{V}(f_1, \ldots, f_t) \subseteq k^n$. Denote the maximal ideal of $\mathcal{O}_{V,p}$ as $\overline{\mathfrak{m}}$. We have already seen that the map

$$k[x_1, \ldots, x_n] \longrightarrow \qquad f \longmapsto d_pf \longmapsto \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\right)$$

identifies $\mathfrak{m}/\mathfrak{m}^2$ with k^n as k-vectors spaces. Notice that this map sends

$$I_V \longmapsto \text{span}_k\{d_pf \mid f \in I_V\} = \text{span}_k\{d_pf_1, \ldots, d_pf_t\}.$$

[7] A set $S \subseteq k^n$ is Zariski-closed if it is a closed set in the Zariski Topology, i.e. $S = \mathbb{V}(f_1, \ldots, f_t)$ is the common zero set of some collection of polynomials. It is important to note that any Zariski-closed set is also closed in the usual Euclidean Topology on k^n.

It follows that there is an induced isomorphism

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \frac{\mathfrak{m}}{I_V + \mathfrak{m}^2} \longrightarrow k^n/\mathrm{span}_k\{d_p f_1, \ldots, d_p f_t\} \cong (\ker J_p)^* ,$$

where $(\ker J_p)^*$ denotes the dual space of this kernel. Since the left-hand side is the Zariski cotangent space of $\mathcal{O}_{V,p}$, it follows that the condition for $\mathcal{O}_{V,p}$ being regular is the same as the smoothness condition for V at p, namely that this dimension is $n - \dim \mathcal{O}_{V,p} = n - \dim V$. $\square$

Remark 7: The condition of algebraically closed is VERY important. Consider $k = \mathbb{F}_p(t)$, the field of rational functions in t over the finite field of p elements, and $R = k[x]/\langle x^p - t\rangle$. It is clear that R is regular since it is a field $R \cong k(t^{1/p})$. However, look at the Jacobian matrix for the polynomial $f = x^p - t$. Since k is characteristic p, we have that $\frac{df}{dx} = px^{p-1} = 0$. Thus, the rank of the Jacobian matrix is 0, not the 1 we would expect for a smooth variety. In fact, even though R is regular, the variety $\mathbb{V}(x^p - t)$ is not smooth over k.

Remark 8: Even though this intuition of a regular local ring being smooth is fairly simple to grasp, it is not easy to show that a local ring is regular in general. In fact, it is a theorem of Auslander-Buchsbaum and Serre that describes regularity as a homological condition:

Theorem 12. Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following are equivalent:

1. R is regular.
2. The residue field $k = R/\mathfrak{m}$ has a finite free resolution.
3. Every R-module has a finite free resolution.

Thus, to prove regularity in general, you have to compute the projective dimension of the residue field. Not such a simple task to do by hand. It is remarkable, however, that there is a very simple way of describing regularity in the prime characteristic setting. It is a seminal result by Kunz which says:

Theorem 13. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Then R is regular if and only if R is a finite free R^p-module.

Moreover, if $\dim(R) = d$ and $[k : k^p] = n$, then we have that R is a free R^p-module of rank np^d.

In essence, in characteristic $p > 0$, regular local rings are so nice – they are so free of singularity – that they are legitimately free when thought of as an R-module via Frobenius. They are as nice as finitely generated vector spaces.