MA172: Calculus II

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Chapter 1

Differentiation and Integration

Broadly speaking, differential calculus is the study of instantaneous change. Early on in a first calculus course, students learn that the derivative of a function at a point measures the slope of the line tangent at that point; the slope of the tangent line at a point is simply limit of the slopes of the secant lines passing through the specified point, and these slopes measure the average rate of change of the function. Consequently, the derivative measures the instantaneous change of a function. Bearing this in mind, calculus is immediately applicable in a wide range of fields — from physics and engineering to biology, chemistry, and medicine. Conversely, it is the aim of integral calculus to quantify change over time given the instantaneous rate of change. Combined, differential and integral calculus constitute an indispensable tool in many applied sciences today.

1.1 Limits and Continuity

Calculus is the study of change in functions. Essentially, a **function** is simply a rule that assigns to each input x one and only one output $y = f(x)$. Often, in this course, we will simply consider **real functions**, i.e., functions that are defined such that their inputs and outputs are **real numbers**. We are unwittingly very familiar with real numbers: the real numbers $\mathbb{R}$ include zero, all positive and negative whole numbers, all positive and negative rational numbers (or fractions), all positive and negative square roots of positive rational numbers, and transcendental numbers like π and e.

We will use the notation $f : \mathbb{R} \to \mathbb{R}$ to express that f is a function whose **domain** is the real numbers $\mathbb{R}$ and whose **codomain** is the real numbers $\mathbb{R}$. Explicitly, the domain of a function is the set of all possible inputs of a function, and the codomain of a function is the set of all possible outputs of the function. Even more, the collection of all possible outputs of a function is the **range** of the function. We will adopt the **set-builder** notation for the domain and range of a function f.

$D_f = \{x \in \mathbb{R} \mid f(x) \text{ is a real number}\}$ consists of real numbers x such that $f(x)$ is a real number.

$R_f = \{f(x) \in \mathbb{R} \mid x \in D_f\}$ consists of real numbers $f(x)$ such that x lies in the domain of f.

Example 1.1.1. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. By definition, this function outputs the real number x that is input. We refer to this as the **identity function** on the real numbers. Consequently, the domain of f is $D_f = \mathbb{R}$ because the output of any real number is a real number, and the range of f is $R_f = \mathbb{R}$ because every real number is the output of itself.

Caution: the domain of a real function might not be all real numbers; the range of a real function might not be all real numbers, either, as our next pair of examples illustrate.

Example 1.1.2. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. By definition, this function outputs the square x^2 of the real number x that is input. Certainly, the square of any real number is a real number, hence the domain of f is $D_f = \mathbb{R}$; on the other hand, the only real numbers that are the square of another real number are the non-negative real numbers. Explicitly, for any real number x, the real number $f(x) = x^2$ is a non-negative real number, i.e., we have that $x^2 \geq 0$. Consequently, the codomain of f is $\mathbb{R}$, but the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Example 1.1.3. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. By definition, this function outputs the square root $\sqrt{x}$ of the real number x that is input. We cannot take the square root of a negative real number, hence the domain of f consists of all non-negative real numbers, i.e., we have that $D_f = \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$; on the other hand, every non-negative real number can be realized as the square root of a non-negative real number. Explicitly, for any non-negative real number y, the real number y^2 satisfies that $y = \sqrt{y^2} = f(y^2)$. Consequently, the codomain of f is $\mathbb{R}$, but once again, the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Generally, the restrictions on the domain of a real function consist of the following situations.

(a.) We cannot divide by zero.

(b.) We cannot take the even root of a negative real number.

(c.) We cannot take the logarithm of a non-positive real number.

Occasionally, it is necessary to split the domain or the range of a function into distinct chunks of the real number line. By the above rule, the domain of the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ consists of all nonzero real numbers. Consequently, we can certainly realize the domain of f as $D_f = \{x \in \mathbb{R} \mid x \neq 0\}$, but it is sometimes more convenient to describe this set using the **union** symbol $\cup$. Put simply, the union symbol $\cup$ functions as the logical connective "or." Clearly, a nonzero real number is either positive or negative, hence we can partition the domain of f into those real numbers that are positive and those real numbers that are negative. We achieve this with the union symbol as $D_f = \{x \in \mathbb{R} \mid x > 0\} \cup \{x \in \mathbb{R} \mid x < 0\}$. Even more, we learn in college algebra (or earlier) that the set of real numbers x satisfying the **inequalities** $x > 0$ and $x < 0$ can be described respectively using the **open intervals** $(0, \infty)$ and $(-\infty, 0)$. Consequently, in **interval notation**, the domain of the real function $f(x) = x^{-1}$ is given by $D_f = (-\infty, 0) \cup (0, \infty)$.

Exercise 1.1.4. Compute the domain and range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Exercise 1.1.5. Compute the domain and range of the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^{-3}$.

Exercise 1.1.6. Compute the domain and range of the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \frac{1}{\ln(x)}$.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ whose domain is D_f. Given any real number a in D_f, we say that the **limit** of $f(x)$ as x approaches a is the quantity L (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$. Put another way, the quantity L can be made arbitrarily close to the value of $f(x)$ by taking x to be sufficiently close in value to a. Conveniently, if the quantity L exists, then we write $L = \lim_{x \to a} f(x)$.

Example 1.1.7. Let us compute the limit of $f(x) = x^2$ as x approaches $a = 1$ using the definition. Computing the limit is essentially like playing a game of limbo: we are handed a real number $\varepsilon > 0$ (the limbo bar), and our challenge is to find a real number $\delta > 0$ such that $|x^2 - 1| < \varepsilon$ whenever we assume that $|x - 1| < \delta$. Of course, we are at liberty to take δ as small as necessary to ensure that $|x^2 - 1| < \varepsilon$. We may therefore assume that $0 < \delta \leq 1$. Considering that $x^2 - 1 = (x-1)(x+1)$, if we assume that $|x - 1| < \delta \leq 1$, then we must have that $0 < x < 2$, from which it follows that $|x+1| \leq |x| + 1 = x + 1 < 3$ by the Triangle Inequality. Consequently, we have that

$$|x^2 - 1| = |(x-1)(x+1)| = |x-1||x+1| < 3\delta.$$

Last, if we wish to have that $|x^2 - 1| < \varepsilon$, then we should choose δ to be the minimum of 1 and $\frac{\varepsilon}{3}$.

One-sided limits can be defined analogously to the limit above: the **left-hand limit** of $f(x)$ as x approaches a is the quantity L^- (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $-\delta < x - a < 0$ implies that $|f(x) - L^-| < \varepsilon$. Likewise, the **right-hand limit** of $f(x)$ as x approaches a is the quantity L^+ (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $0 < x - a < \delta$ implies that $|f(x) - L^+| < \varepsilon$.

$L^- = \lim_{x \to a^-} f(x)$ is the symbolic way to express the left-hand limit of $f(x)$ as x approaches a.

$L^+ = \lim_{x \to a^+} f(x)$ is the symbolic way to express the right-hand limit of $f(x)$ as x approaches a.

Ultimately, the two-sided limit exists if and only if the left- and right-hand limits exist and are equal; thus, the two-sided limit is equal to the common value of the left- and right-hand limits.

$$L^- = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = L^+$$

Graphically, it is possible to compute the two-sided limit L of some functions $f(x)$ as x approaches a by tracing one's finger along the graph of $f(x)$ from the left- and right-hand sides.

Example 1.1.8. Let us graphically compute the limit of $f(x) = x^2$ as x approaches $a = 1$. Using the graph of $f(x) = x^2$, we find that the limit is 1. Particularly, if we trace the graph with our left pointer finger, moving from left to right toward the point $x = 1$, our finger stops at $y = f(1) = 1$. Likewise, if we trace the graph with our right pointer finger moving from right to left toward $x = 1$, our finger stops at $y = f(1) = 1$. Put in the language of calculus, we have that $L^- = 1 = L^+$.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a real number a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Explicitly, we require three things to be true of the function $f(x)$ in this case.

1.) We must have that f is defined at the real number a, i.e., $f(a)$ must be in the range of f.

2.) We must have that $\lim_{x \to a^-} f(x) = f(a)$, i.e., the left-hand limit of f at a must be $f(a)$.

3.) We must have that $\lim_{x \to a^+} f(x) = f(a)$, i.e., the right-hand limit of f at a must be $f(a)$.

Consequently, if any of these criteria is violated, then the function f cannot be continuous at a.

Example 1.1.9. One of the easiest ways to detect that a function is not continuous at a real number a is to observe that the function is not defined at a. Explicitly, the function $f(x) = \frac{1}{x}$ is not continuous at $a = 0$ because the domain of f excludes $a = 0$ (since we cannot divide by zero).

Example 1.1.10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ that is defined **piecewise** as follows.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and} \\ -1 & \text{if } x < 0 \end{cases}$$

Graphically, if we trace our fingers along f from the left-hand side, when we arrive at $a = 0$ from the left-hand side, we find that the limiting value here is -1; however, if we trace our fingers along f from the right-hand side, when we arrive at $a = 0$ from the right-hand side, we find that the limiting value here is 1. Consequently, the function $f(x)$ is not continuous at $a = 0$.

Example 1.1.11. Let us prove by definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous for all real numbers a. Observe that f is defined piecewise as follows.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0 \end{cases}$$

Consequently, it suffices to show that $g(x) = x$ and $h(x) = -x$ are everywhere continuous. Given real numbers $\varepsilon_1, \varepsilon_2 > 0$, we must find real numbers $\delta_1, \delta_2 > 0$ such that $|x - a| < \varepsilon_1$ whenever $|x - a| < \delta_1$ and $|-x - (-a)| < \varepsilon_2$ whenever $|x - a| < \delta_2$. Considering that the absolute value is multiplicative, we have that $|-x - (-a)| = |-x + a| = |-(x - a)| = |x - a|$, we may simply take the real numbers $\delta_1 = \varepsilon_1$ and $\delta_2 = \varepsilon_2$. We conclude that $g(x) = x$ and $h(x) = -x$ are continuous for all real numbers a so that $f(x) = |x|$ is continuous for all nonzero real numbers by the piecewise definition of $f(x)$ prescribed above. We are done as soon as we show that

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x|.$$

By continuity of the functions $g(x)$ and $h(x)$ and by definition of $|x|$, the left-hand limit is given by $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} h(x) = h(0) = 0$, and the right-hand limit is $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} g(x) = g(0) = 0$.

Generally, continuity can be defined as a property of a function on any **subset** of its domain, i.e., on any collection of real numbers that lie in the domain. Often, we will consider functions that are continuous on their entire domain, but it is possible that a function is not continuous at some point in its domain. We say that a function f is **discontinuous** at a real number a if f is not continuous at the real number a. By the above three criteria, we can classify these **discontinuities**.

- We say that f has a **removable discontinuity** at a real number a if a is not in the domain of f but the left- and right-hand limits of f at a exist and are equal, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
- We say that f has a **jump discontinuity** at a real number a if both of the left- and right-hand limits of f at a exist but are not equal, i.e., $\lim_{x \to a^-} f(x) = L^- \neq L^+ = \lim_{x \to a^+} f(x)$.
- We say that f has an **essential discontinuity** at a real number a if either the left- or the right-hand limit of f at a does not exist, i.e., either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist.

Often, if a function f is continuous for every real number in its domain D_f, we will say that the function is **continuous** on its domain. Explicitly, if the domain of a function f is all real numbers and f is continuous on its domain, then we will say that f is **everywhere continuous**. Graphically, we may detect that a function is continuous if we can draw it without lifting our pencil.

Example 1.1.12. We can graph $|x|$ without lifting our pencil, hence it is everywhere continuous.

Example 1.1.13. We cannot graph x^{-2} without lifting our pencil at $x = 0$, hence x^{-2} is not continuous at $a = 0$. On the other hand, for all real numbers a other than $a = 0$, we can graph this function without lifting our pencil, hence x^{-2} is continuous on its domain $(-\infty, 0) \cup (0, \infty)$.

Continuous functions abound: **polynomial** functions such as $x^3 - 2x^2 + x - 7$ and **exponential** functions such as e^x are defined for all real numbers and are everywhere continuous. Likewise, the **trigonometric** functions $\sin(x)$ and $\cos(x)$ are defined for all real numbers and are everywhere continuous. **Logarithmic** functions such as $\ln(x)$ and $\log(x)$ and **algebraic** functions such as $\sqrt{x}$ and $x^{3/2}$ are defined for all positive real numbers and are continuous on their domains. Further, addition, subtraction, multiplication, division, composition, and any finite combination of these operations on continuous functions result in functions that are typically continuous on their domains.

1.2 Differentiation and L'Hôpital's Rule

Given any real numbers a and $h > 0$ and any real function $f(x)$ such that $f(a)$ and $f(a+h)$ are defined, consider the closed interval $[a, a+h]$ consisting of all real numbers x with $a \leq x \leq a+h$. We define the **secant line** of $f(x)$ over this interval as the line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$. Observe that the slope of the secant line is given by the **difference quotient**

$$Q_a(h) = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

By taking the limit of $Q_a(h)$ as h approaches 0, we obtain the **derivative** of $f(x)$ at a

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Of course, this limit might not exist; however, if it does, we interpret it geometrically as the slope of the line tangent to $f(x)$ at the point $(a, f(a))$. Given that the quantity $f'(a)$ exists, we say that $f(x)$ is **differentiable** at a. One fundamental interpretation of the derivative in the context of a function that measures something physical (e.g., velocity) is as the instantaneous rate of change.

Exercise 1.2.1. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^3$.

Exercise 1.2.2. Use the limit definition of the derivative to compute $g'(x)$ for $g(x) = \frac{1}{x}$.

Exercise 1.2.3. Use the limit definition of the derivative to compute $h'(x)$ for $h(x) = \sqrt{x}$.

One of the most important properties of differentiable real functions is the following.

Proposition 1.2.4. *If a real function f is differentiable at a real number a, then f is continuous at a. Explicitly, a function that is differentiable at a point in its domain is necessarily continuous there. Conversely, there exists a function that is continuous but not differentiable on its domain.*

Proof. We will assume that f is differentiable at a real number a. Consequently, the limit

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Using the substitution $x = a + h$, we have that $h = x - a$. Crucially, under this substitution, the limit of any function $g(h)$ as h approaches 0 is equal to the limit of the function $g(x - a)$ as x approaches a. (Verify this by definition of the limit.) Consequently, the following identity holds.

$$f'(a) = \lim_{x \to a} Q_a(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Considering that $x - a$ is a polynomial function, it is continuous at a, and we conclude that

$$\lim_{x \to a} (x - a) = a - a = 0.$$

Using the fact that the limit of a product is the product of limits (when both limits exist),

$$0 = f'(a) \cdot \lim_{x \to a} (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot x - a = \lim_{x \to a} [f(x) - f(a)]$$

yields the result that $\lim_{x \to a} f(x) = \lim_{x \to a} [f(a) + f(x) - f(a)] = \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] = f(a)$.

Conversely, the function $|x|$ is continuous on its domain, but it is not differentiable at $a = 0$: indeed, by Example 1.1.10, the piecewise function $f(x)$ satisfying that $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$ is not continuous because the left- and right-hand limits do not agree at 0. One can readily verify that this function is exactly the derivative of $|x|$, hence the claim holds. □

Computing limits by definition is even more tedious than it looks, but luckily, there are plenty of tools that allow us to compute derivatives of functions without ever touching a limit. Particularly,

- the **Power Rule** says that if $f(x) = x^r$ for some real number r, then $f'(x) = rx^{r-1}$;
- the **Product Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x);$$

- the **Quotient Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g'(x)]^2}; \text{ and}$$

- the **Chain Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f \circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = [f' \circ g(x)] \cdot g'(x).$$

Computing the limit of a function that is continuous is quite easy: we may simply "plug and chug;" however, there exist functions that are not continuous. Even worse, when evaluating limits, we can encounter situations that result in an **indeterminate form** when the limit is the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Theorem 1.2.5 (L'Hôpital's Rule)**.** *Given any real functions $f(x)$ and $g(x)$ that are differentiable for all real numbers x such that $a < x < b$ (with the possible exception of one point $x = c$ for some real number $a \leq c \leq b$), consider the following conditions.*

(1.) *We have that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$.*

(2.) *We have that $g'(x) \neq 0$ for any real number x such that $a < x < b$ and $x \neq c$.*

(3.) *We have that $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists.*

Granted that each of the above conditions holds, it follows that $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \lim_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Exercise 1.2.6. Compute the limit of $f(x) = \dfrac{\ln(x)}{x^3 - 1}$ as x approaches $a = 1$.

Exercise 1.2.7. Compute the limit of $g(x) = (2x - \pi)\sec(x)$ as x approaches $a = \frac{\pi}{2}$ from the left.

Exercise 1.2.8. Compute the limit of $h(x) = \dfrac{\sin(x)}{\sin(x) + \tan(x)}$ as x approaches $a = 0$.

Exercise 1.2.9. If $\frac{d}{dx}\sin(x) = \cos(x)$, compute the limit of $f(x) = \dfrac{\sin(x)}{x}$ as x approaches $a = 0$.

Caution: Unfortunately, the above example is not a valid proof of this limit identity: in fact, this limit identity is needed to prove that $\frac{d}{dx}\sin(x) = \cos(x)$, so in order to prove this identity in a rigorous and non-circular manner, we must use tools from trigonometry and the Squeeze Theorem.

1.3 Implicit Differentiation

Curves in the Cartesian plane can be represented by an equation involving a function of two variables. Explicitly, we are familiar with such curves as $xy = 1$ and $y - x^2 = 0$; they are respectively the functions $y = f(x) = x^{-1}$ and $y = g(x) = x^2$. We refer to the functions $f(x)$ and $g(x)$ as the **explicit** forms of the curves. Unfortunately, it is not possible to write every curve in the Cartesian plane as a function of one variable: curves such as the unit circle $x^2 + y^2 = 1$ or the hyperbola $y^2 - x = 0$ cannot be represented as functions because they fail the **Vertical Line Test**; however, we will see throughout this semester that these curves provide important models in calculus. Curves that do not admit closed-form expressions of the form $y = f(x)$ can be written **implicitly**.

Under certain conditions, it is possible to find a "small enough" region in the Cartesian plane in which an implicit curve can be represented by a function; thus, in this "window," the slope and tangent line of such curves are well-defined. Consequently, we may define the **implicit derivative** by assuming that y is a function of x (on some "small window" in the plane) with derivative $y' = \frac{dy}{dx}$.

Example 1.3.1. Compute $\frac{dy}{dx}$ for the unit circle $x^2 + y^2 = 1$.

Solution. Considering the variable y as some function $y = f(x)$ of x and using the convention that $y' = \frac{dy}{dx}$, we may invoke the Chain Rule in order to determine that

$$0 = \frac{d}{dx}1 = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy'.$$

Crucially, each time the derivative operator $\frac{d}{dx}$ encounters the variable y, we differentiate y as we would the function $y = f(x)$ that represents y locally. Consequently, if y is nonzero, then

$$\frac{dy}{dx} = y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

Otherwise, the tangent line does not exist if $y = 0$ because $2x + 2yy' = 0$ has no solution if $y = 0$. $\diamond$

Example 1.3.2. Compute $\frac{dy}{dx}$ for the parabola $y^2 - x = 0$.

Solution. By the Chain Rule applied to $y = f(x)$, we have that

$$0 = \frac{d}{dx}0 = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

so that $\frac{dy}{dx} = y' = (2y)^{-1}$ for all points (x, y) on the hyperbola such that y is nonzero. $\diamond$

1.4 Exponential and Logarithmic Functions

Given any positive real number a, the **exponential** function with **base** a is given by $\exp_a(x) = a^x$. Crucially, the most important exponential function is simply $\exp(x) = e^x$: here, the base is **Euler's number** $e \approx 2.72$. Later, we will concern ourselves with the definition of Euler's number; for now, we need only recall the following properties of exponential functions for any real numbers x and y.

1.) $a^{x+y} = a^x a^y$

2.) $a^{x-y} = a^x a^{-y}$

3.) $a^{xy} = (a^x)^y$

4.) $(ab)^x = a^x b^x$ for any real number $b > 0$

We do not yet have the machinery available to use to prove the following, but it is true that

$$\frac{d}{dx}e^x = e^x.$$

Considering that $e^x > 0$ for all real numbers x, it follows that e^x is a strictly increasing function, hence it passes the **Horizontal Line Test** and must therefore admit an **inverse** function; we refer to this function as the **natural logarithmic** function $\ln(x)$. Put another way, we have that

$$e^{\ln(x)} = x \text{ for all real numbers } x > 0 \text{ and } \ln(e^x) = x \text{ for all real numbers } x.$$

Observe that the range of e^x is $(0, \infty)$, hence the domain of $\ln(x)$ is $(0, \infty)$. Conversely, the domain of e^x is $(-\infty, \infty)$, hence the range of $\ln(x)$ is $(-\infty, \infty)$. We will also simply assert that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

We may also deduce the following properties of logarithmic functions for any real numbers $x, y > 0$.

1.) $\log_a(xy) = \log_a(x) + \log_a(y)$

2.) $\log_a(x^r) = r\log_a(x)$ for all real numbers r

3.) $\log_a(xy^{-1}) = \log_a(x) - \log_a(y)$

4.) $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

Even more, for any real number $a > 0$, the exponential function $\exp_a(x) = a^x$ is differentiable for all real numbers x. Further, observe that $y = a^x$ is strictly positive for all real numbers x, hence the function $\ln(y) = x\ln(a)$ is well-defined. Using the Chain Rule, we find that

$$\frac{1}{y}\cdot y' = \frac{d}{dx}\ln(y) = \frac{d}{dx}[x\ln(a)] = \ln(a)\cdot\frac{d}{dx}x = \ln(a) \text{ and } \frac{d}{dx}a^x = \frac{d}{dx}y = \frac{dy}{dx} = y' = y\ln(a) = a^x\ln(a).$$

By a similar rationale as before, one can define the **logarithmic** function $\log_a(x)$ **base** a for any positive real number a as the function inverse of a^x; its domain is $(0, \infty)$, and its range is $(-\infty, \infty)$.

Exercise 1.4.1. Compute the derivative of $y = \log_a(x)$ by using the fact that $a^y = x$.

1.5 Inverse Trigonometric Functions

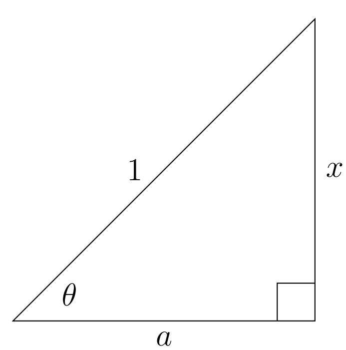
Even though the trigonometric functions like $\sin(x)$, $\cos(x)$, and $\tan(x)$ are **periodic**, we can find a region on the x-axis in which these functions pass the Horizontal Line Test and admit function inverses. Explicitly, the inverse trigonometric functions are denoted as follows.

$$\begin{array}{lll} \arcsin(x) = \sin^{-1}(x) & \text{domain: } [-1, 1] & \text{range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \arccos(x) = \cos^{-1}(x) & \text{domain: } [-1, 1] & \text{range: } [0, \pi] \\ \arctan(x) = \tan^{-1}(x) & \text{domain: } (-\infty, \infty) & \text{range: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}$$

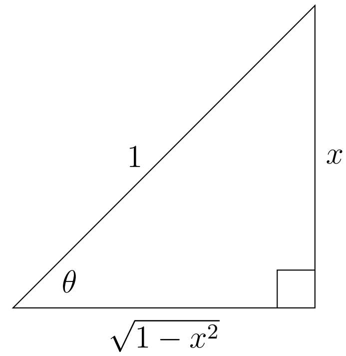
Considering that the input of the sine function is an angle, the output of the arcsine function is an angle. Consequently, if $x = \sin(\theta)$, then it follows by definition that $\theta = \arcsin(x)$ so that

$$\frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx}.$$

Observe that $\sin(\theta)$ is the ratio of the opposite side and the hypotenuse of a right triangle, so we may construct a right triangle whose opposite side has length x and whose hypotenuse has length 1 in order to obtain $\sin(\theta) = x$. Our right triangle therefore has the following form.



By the **Pythagorean Theorem**, we must have that $x^2 + a^2 = 1$ so that $a = \sqrt{1 - x^2}$.

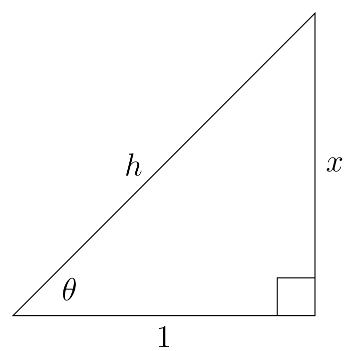


Using the Chain Rule, we can compute $\frac{d\theta}{dx}$. Explicitly, we have that

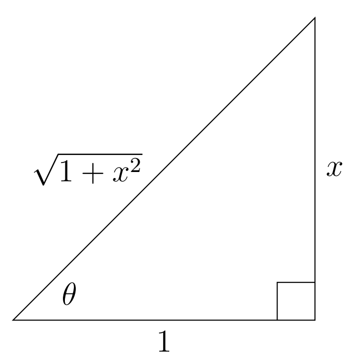
$$\cos(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\sin(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}.$$

Exercise 1.5.1. Use a right triangle involving 1, x, and $\sqrt{1-x^2}$ to compute $\frac{d}{dx}\arccos(x)$.

Using a similar idea as the one we employed to compute the derivative of $\arcsin(x)$ and $\arccos(x)$, we will set up a triangle with $\tan(\theta) = x$. Observe that $\tan(\theta)$ is the ratio of the opposite side and the adjacent side of a right triangle, so we may construct a right triangle whose opposite side has length x and whose adjacent side has length 1 in order to obtain $\tan(\theta) = x$.



Once again, by the Pythagorean Theorem, we find that $h^2 = x^2 + 1^2$ so that $h = \sqrt{1+x^2}$.

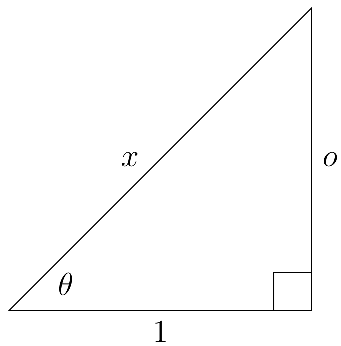


Using the Chain Rule, we can compute $\frac{d\theta}{dx}$. Explicitly, we have that

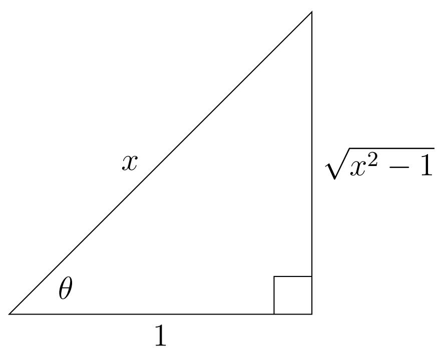
$$\sec^2(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\tan(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arctan(x) = \frac{d\theta}{dx} = \cos^2(\theta) = \frac{1}{1+x^2}.$$

Exercise 1.5.2. Use a right triangle involving 1, x, and $\sqrt{1+x^2}$ to compute $\frac{d}{dx}\operatorname{arccot}(x)$.

Last but not least, we will set up a triangle with $\sec(\theta) = x$. Observe that $\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of a right triangle, so we obtain the following diagram.



Once again, by the Pythagorean Theorem, we find that $x^2 = o^2 + 1^2$ so that $o = \sqrt{x^2 - 1}$.



Using the Chain Rule, we can compute $\dfrac{d\theta}{dx}$. Explicitly, we have that

$$\sec(\theta)\tan(\theta)\cdot\frac{d\theta}{dx} = \frac{d}{dx}\sec(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\operatorname{arcsec}(x) = \frac{d\theta}{dx} = \cos(\theta)\cot(\theta) = \frac{1}{x\sqrt{x^2-1}}.$$

Exercise 1.5.3. Use a right triangle involving 1, x, and $\sqrt{x^2-1}$ to compute $\frac{d}{dx}\operatorname{arccsc}(x)$.

1.6 Antidifferentiation

Considering that a derivative is a rate of change, it is natural in the applied sciences to begin with a rate of change and use it to estimate the net change of a process over time. Explicitly, if we observe that the velocity of a body is given by a function $f(x)$ over some interval of time, then we may seek a function $F(x)$ such that $F'(x) = f(x)$ over this interval of time. Given that such a function $F(x)$ exists and satisfies that $F'(x) = f(x)$, we refer to $F(x)$ as an **antiderivative** of $f(x)$.

Exercise 1.6.1. Prove that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Exercise 1.6.2. Prove that the function $G(x) = x\ln(x) - x$ is an antiderivative of $g(x) = \ln(x)$.

Exercise 1.6.3. Prove that the function $H(x) = xe^x - e^x$ is an antiderivative of $h(x) = xe^x$.

Observe that for any antiderivative $F(x)$ of a function $f(x)$, there exists a family of antiderivatives indexed by the real numbers. Particularly, the function $G(x) = F(x) + C$ is an antiderivative of $f(x)$ for every real number C. Even more, by the **Mean Value Theorem**, every antiderivative of $f(x)$ is of the form $F(x) + C$ for some antiderivative $F(x)$ of $f(x)$ and some real number C. Consequently, we may define the **general antiderivative** or **indefinite integral** of $f(x)$ to be

$$\int f(x)\, dx = F(x) + C$$

for any real number C. By the familiar derivative rules, we obtain

- the **Power Rule**, i.e., $\int x^r\, dx = \frac{1}{r+1} x^{r+1} + C$ for all real numbers $r \neq -1$ and
- the **Chain Rule**, i.e., $\int f'(g(x))g'(x)\, dx = f(g(x)) + C$.

Further, indefinite integration is **linear**: for all real functions $f(x)$ and $g(x)$, we have

- the **Multiples Rule** $\int kf(x)\, dx = k\big(\int f(x)\, dx\big)$ for all real numbers k and
- the **Sum Rule** $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

Exercise 1.6.4. Compute the indefinite integral of $f(x) = x^{-1}$.

Exercise 1.6.5. Compute the indefinite integral of $g(x) = 2xe^{x^2}$.

Exercise 1.6.6. Compute the indefinite integral of $h(x) = 2\sin(x)\cos(x)$.

Circling back to the opening remarks of this section, we will assume that the velocity of a body over an interval of time is a continuous function $v(t)$. Even more, suppose that we note the position $s(t)$ of the particle at time $t = 0$, i.e., the quantity $s(0)$ is known. Considering that $s'(t) = v(t)$, it follows that $s(t)$ must differ from $\int v(t)\, dt$ by a constant C that depends on the quantity $s(0)$. We refer to this scenario as an **initial value problem** of the **differential equation** $s'(t) = v(t)$.

Example 1.6.7. Consider the velocity function $v(t) = 3t^2 - 4t + 2$ of a body whose position $s(t)$ at time $t = 0$ is given by $s(0) = 7$. Give an explicit formula for $s(t)$.

Solution. Observe that $s(t) = \int v(t)\, dt = \int 3t^2\, dt - \int 4t\, dt = \int 2\, dt = t^3 - 2t^2 + 2t + C$. By plugging in our initial value of $s(0) = 7$, we find that $7 = s(0) = C$ so that $s(t) = t^3 - 2t^2 + 2t + 7$. $\diamond$

Exercise 1.6.8. Consider tossing a ball upward with an initial velocity of 48 feet per second and constant acceleration of -32 feet per second from the edge of a cliff of height 432 feet. Compute the maximum height of the ball; then, find the time it takes for the ball to reach the ground.

1.7 Computing Area Bounded by a Curve of One Variable

Continuing in the theme of extrapolating data from intermittent observations, suppose that we observe the velocity $v(t)$ of a particle over a period of time $0 \leq t \leq 25$, taking care to mark down the velocity of the particle every five seconds. Consider along these lines the following table.

t	0	5	10	15	20	25
$v(t)$	25	31	35	43	47	46

We can roughly approximate the total distance traveled by the body for $0 \leq t \leq 25$ by assuming (incorrectly) that the body maintains a constant velocity each time we see it. Computing the total distance travelled by the particle during our observation amounts to finding the **displacement** of the body over each time interval and adding these quantities together. Explicitly, we have that

$$\text{total distance traveled} = 25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 + 47 \cdot 5 + 46 \cdot 5 = 1135.$$

Certainly, we can improve this estimation by taking more measurements: even recording one more observation will give us a better understanding of the behavior of the particle over the specified interval of time. Better yet, the more observations we record, the more accurate our understanding of the total distance traveled; however, this also requires adding more numbers together. Consequently, it will be convenient to develop notation to take sums of arbitrarily large quantities of data.

Let us assume for the moment that we have a collection of n real numbers $a_1, a_2, \dots, a_n$ for some positive integer n. Certainly, the sum of these real numbers can be realized as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

We refer to this as **sigma notation**: indeed, the Greek letter sigma Σ is used as a mnemonic device for "sum"; the subscript $i = 1$ denotes the **index of summation** and informs us of the first term a_1 in our collection of data; and the superscript n tells us that the sum terminates with the last term a_n in our collection of data. We refer to the real number a_i as the ith **summand** for each integer $1 \leq i \leq n$; the entire sum $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called a **finite sum**.

Often, we will consider finite sums whose ith summand can be conveniently expressed in **closed-form**. Explicitly, this means that there exists a function $f(x)$ such that $a_i = f(i)$.

Example 1.7.1. Consider the finite sum $1 + 2 + 3 + \cdots + 10$ of the first ten positive integers. Observe that the ith summand is simply the positive integer i, hence we have that $a_i = i$ and

$$1 + 2 + 3 + \cdots + 10 = \sum_{i=1}^{10} i.$$

Crucially, we point out another way to **index** the given sum — namely, we have that

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = 0 + 1 + 2 + 3 + \cdots + 10 = \sum_{i=0}^{10} i.$$

Often, if a sum involves a summand of zero, we will simply omit it (unless it is more convenient to include it). We could have also written this sum in a third way as follows.

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = (1 + 2 + 3 + \cdots + 20) - (11 + 12 + 13 + \cdots + 20) = \sum_{i=1}^{20} i - \sum_{i=11}^{20} i.$$

Example 1.7.2. Consider the finite sum $1 + 4 + 9 + \cdots + 100$ of squares of the first ten positive integers in which the ith summand is simply the positive integer i^2. We have that $a_i = i^2$ and

$$1 + 4 + 9 + \cdots + 100 = \sum_{i=1}^{10} i^2.$$

Example 1.7.3. Express the finite sum $1^3 + 2^3 + 3^3 + \cdots + 1000^3$ of cubes of the first 1000 positive integers in summation notation, identifying the closed-form expression for the ith summand a_i.

Quite importantly, finite sums admit a convenient arithmetic of their own.

Proposition 1.7.4 (Properties of Finite Sums)**.** *Given any positive integer n and any real numbers $a_1, \dots, a_n, b_1, \dots, b_n$, and C, the following identities hold.*

(i.) (Empty Sum Law) *We have that $\sum_{i=n}^m a_i = 0$ for all integers $m < n$.*

(ii.) (Constant Sum Formula) *We have that $\sum_{i=m}^n C = C(n - m + 1)$ for all integers $m \leq n$.*

(iii.) (Linearity of a Finite Sum I) *We have that $\sum_{i=1}^n Ca_i = C(\sum_{i=1}^n a_i)$.*

(iv.) (Linearity of a Finite Sum II) *We have that $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$.*

One can easily prove the above formulas by expanding and comparing the expressions on both sides of the equation. We will not endeavor to prove the following identities because these details are beyond the scope of this course; however, they will be indispensable in what follows.

Proposition 1.7.5. *Consider any positive integer n.*

(i.) *We have that* $\displaystyle\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

(ii.) *We have that* $\displaystyle\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

(iii.) *We have that* $\displaystyle\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Going back to our example of tracking a particle over a period of time, if we know the velocity $v(t)$ of the particle at any time $0 \leq t \leq 25$, then we can approximate the total distance traveled by the particle by recording the velocity a positive integer n times and computing the total displacement of the particle over each interval of time. Explicitly, if we observe the particle for some real numbers $0 = t_0 < t_1 < \cdots < t_n = 25$ and we assume that the particle has constant velocity $v(t_i)$ for each integer $0 \leq i \leq n$, then the total distance traveled by the particle between time t_{i-1} and time t_i is given by the real number $\Delta t_i = t_i - t_{i-1}$ and the total displacement of the particle on this closed interval $[t_{i-1}, t_i]$ is $v(t_i)\,\Delta t_i$ (rate $\times$ time). Consequently, in sigma notation, we have that

$$\text{total distance traveled} = v(t_1)\,\Delta t_1 + v(t_2)\,\Delta t_2 + \cdots + v(t_n)\,\Delta t_n = \sum_{i=1}^n v(t_i)\,\Delta t_i.$$

By viewing the points $(t_i, v(t_i))$ as lying on the graph of the velocity curve $v(t)$, we may recognize $\sum_{i=1}^n v(t_i)\,\Delta t_i$ as an approximation of the area between the curve $v(t)$ and the t-axis, i.e., the net area bounded by the curve $v(t)$ of one variable. We will now generalize this idea.

Consider any real function $f(x)$ that is continuous on a closed and bounded interval $[a, b]$. Choose any positive integer n; then, choose n real numbers $a = x_0 < x_1 < \cdots < x_n = b$. Consider the closed and bounded intervals $[x_{i-1}, x_i]$ for each integer $1 \leq i \leq n$. We refer to the collection $\mathcal{P}$ of such closed and bounded intervals as a **partition** of $[a, b]$, and we denote by $\Delta x_i = x_i - x_{i-1}$ the length of the interval $[x_{i-1}, x_i]$. Choosing **sample points** x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$ yields a so-called **tagged partition** $(\mathcal{P}, x_i^*)$ consisting of closed and bounded intervals and sample points within them. We associate to each tagged partition a **Riemann sum** (or **Riemann approximation**)

$$\sum_{i=1}^n f(x_i^*)\, \Delta x_i = f(x_1^*)\, \Delta x_1 + f(x_2^*)\, \Delta x_2 + \cdots + f(x_n^*)\, \Delta x_n.$$

Geometrically, we may realize $f(x_i^*)$ as the height of a rectangle with base Δx_i, hence the above Riemann sum provides an approximation of the **net area** bounded by the curve $f(x)$ over the closed interval $[a, b]$. Common tagged partitions are formed by taking x_i^* to be the left- or right-**endpoint** or the **midpoint** of $[x_{i-1}, x_i]$. Each of these tagged partitions uses $n + 1$ **equally-spaced** points $a = x_0 < x_1 < \cdots < x_n = b$; the common length of each interval $[x_{i-1}, x_i]$ is Δx. Considering that

$$b - a = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = \sum_{i=1}^n \Delta x_i = \sum_{i=1}^n \Delta x = n\, \Delta x$$

by the second part of Proposition 1.7.4, we conclude that $\Delta x = \frac{b-a}{n}$.

- We denote by $\mathcal{L}_n$ the **left-endpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = \ell_i = a + (i-1)\, \Delta x$.
- We denote by $\mathcal{R}_n$ the **right-endpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = r_i = a + i\, \Delta x$.
- We denote by $\mathcal{M}_n$ the **midpoint Riemann approximation** of the function $f(x)$ on the closed interval $[a, b]$ with $\Delta x_i = \Delta x = \frac{b-a}{n}$ and sample points $x_i^* = m_i = a + \frac{2i-1}{2}\, \Delta x$.

Example 1.7.6. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $f(x) = x$ on the closed and bounded interval $[0, 4]$ using four equally-spaced points.

Solution. By recognizing that $a = 0$ and $b = 4$, the length of each interval of the partition is

$$\Delta x_i = \Delta x = \frac{4-0}{4} = \frac{4}{4} = 1.$$

Consequently, the left-endpoint approximation satisfies that $\ell_i = 0 + (i-1)1 = i - 1$; the right-endpoint approximation satisfies that $r_i = 0 + i = i$; and the midpoint approximation satisfies that $m_i = 0 + \frac{2i-1}{2}(1) = \frac{2i-1}{2}$ for each integer $1 \leq i \leq 4$. We conclude therefore that the following hold.

$$\mathcal{L}_4 = \sum_{i=1}^4 f(\ell_i)\, \Delta x = \sum_{i=1}^4 \ell_i = \sum_{i=1}^4 (i-1) = \sum_{i=1}^4 i - \sum_{i=1}^4 1 = \frac{4(4+1)}{2} - 4 = 6$$

$$\mathcal{R}_4 = \sum_{i=1}^4 f(r_i)\, \Delta x = \sum_{i=1}^4 r_i = \sum_{i=1}^4 i = \frac{4(4+1)}{2} = 10$$

$$\mathcal{M}_4 = \sum_{i=1}^4 f(m_i)\, \Delta x = \sum_{i=1}^4 \frac{2i-1}{2} = \sum_{i=1}^4 i - \sum_{i=1}^4 \frac{1}{2} = 10 - \frac{1}{2}(4) = 8$$

◇

Example 1.7.7. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $g(x) = x^2$ on the closed and bounded interval $[0, 1]$ using five equally-spaced points.

Solution. Like before, we find that $a = 0$ and $b = 1$ so that the length of each interval is

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}.$$

By the above, the left-endpoint approximation uses the sample points $\ell_i = 0 + (i-1)\,\Delta x = \frac{i-1}{5}$; the right-endpoint approximation uses the sample points $r_i = 0 + i\Delta x = \frac{i}{5}$; and the midpoint approximation uses the sample points $m_i = 0 + \frac{2i-1}{2}\Delta x = \frac{2i-1}{10}$. We conclude that

$$\mathcal{L}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{\ell_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^2 = \frac{1}{5}\left(0 + \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right) = \frac{30}{75} = \frac{6}{25}$$

$$\mathcal{R}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{r_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i}{5}\right)^2 = \frac{1}{5}\left(\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right) = \frac{55}{75} = \frac{11}{25}$$

$$\mathcal{M}_5 = \sum_{i=1}^{5} g(m_i)\,\Delta x = \sum_{i=1}^{5} \frac{m_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{2i-1}{10}\right)^2 = \frac{1}{5}\left(\frac{1}{100} + \frac{9}{100} + \frac{25}{100} + \frac{49}{100} + \frac{81}{100}\right) = \frac{33}{100} \quad \diamond$$

Exercise 1.7.8. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $h(x) = x^3$ on the closed and bounded interval $[0, 2]$ using eight equally-spaced points.

By allowing the number of sample points to grow arbitrarily large, the error of approximating the area bounded by a curve of one variable by a Riemann sum shrinks to zero, hence we define

$$\text{area bounded by the curve } f(x) \text{ on the closed and bounded interval } [a, b] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i,$$

where x_i^* are sample points of a partition $\mathcal{P}$ of $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$.

Example 1.7.9. Compute the area bounded by $f(x) = x^2$ on the closed interval $[0, 1]$.

Solution. Crucially, the above definition of the area does not depend on the choice sample points x_i^* or the partition $\mathcal{P}$ of $[0, 1]$, so we may carefully construct these to make things as convenient as possible. Given any choice of equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$, we have that $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. We may choose the right-endpoint approximation so that $x_i^* = \frac{i}{n}$ and

$$\mathcal{R}_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

by the second part of Proposition 1.7.5. By taking the limit as $n \to \infty$, we conclude that

$$\text{area bounded by } x^2 \text{ on } [0, 1] = \lim_{n\to\infty} \mathcal{R}_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}. \quad \diamond$$

1.8 Definite Integration

Given any real function $f(x)$ and any real numbers a and b, consider any collection of points $(x_n, f(x_n))$ on the graph of $f(x)$ with $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$. Each of the closed and bounded intervals $[x_{i-1}, x_i]$ gives rise to a partition $\mathcal{P}$ of the closed interval $[a, b]$, and we may choose sample points x_i^* for each integer $1 \leq i \leq n$ such that $x_{i-1} \leq x_i^* \leq x_i$ and $x_1^* < x_2^* < \cdots < x_n^*$. Crucially, we are not assuming here that the points $x_0, x_1, \dots, x_n$ are equally-spaced, hence we may denote $\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\}$. We define

$$\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\, \Delta x_i$$

as the **definite integral** of $f(x)$ on the closed and bounded interval $[a, b]$. Provided that the above limit exists, we say that $f(x)$ is **integrable** on $[a, b]$. We refer to the function $f(x)$ in this case as the **integrand**; the real numbers a and b are the **limits of integration**. By our work in the previous section, we may interpret the definite integral $\int_a^b f(x)\, dx$ as the net area bounded by $f(x)$: indeed, $\sum_{i=1}^n f(x_i^*)\, \Delta x_i$ is a Riemann sum representing rectangles of height $f(x_i^*)$ and width Δx_i.

Example 1.8.1. Express the following as the definite integral of a function on the interval $[1, 8]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2}\, \Delta x_i$$

Solution. Considering that we do not know the partition $\mathcal{P}$ or the sample points x_i^*, there is not much we can do other than recognize the function $f(x)$. Comparing the limit with the definition above, we recognize that $f(x) = \sqrt{2x + x^2}$ so that the limit in question is $\int_1^8 \sqrt{2x + x^2}\, dx$. $\diamond$

Exercise 1.8.2. Express the following as the definite integral of a function on the interval $[0, \pi]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n x_i^* \sin(x_i^*)\, \Delta x_i$$

Often, it is most simple to work with a **regular partition** $\mathcal{P}$, i.e., a partition of $[a, b]$ with $n+1$ equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$ such that $\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x = \frac{b-a}{n}$. Under this identification, we have that $\Delta x_1 = x_1 - x_0$ so that $x_1 = x_0 + \Delta x_1 = a + \Delta x$, from which it follows that $x_2 = x_1 + \Delta x_2 = (a + \Delta x) + \Delta x = a + 2\Delta x$ and $x_i = a + i\Delta x$ for each integer $1 \leq i \leq n$. Choosing our sample points such that $x_i^* = x_i = a + i\Delta x$ and using the fact that

$$\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\} = \Delta x = \frac{b-a}{n}$$

approaches zero if and only if n approaches ∞, we conclude that

$$\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\, \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^n f(a + i\Delta x) \left(\frac{b-a}{n} \right).$$

Example 1.8.3. Express the following as the definite integral of a function on a closed interval.

$$\lim_{n \to \infty} \sum_{i=1}^n \cos\left(-\pi + i\frac{2\pi}{n} \right) \left(\frac{2\pi}{n} \right)$$

Solution. Considering that $\Delta x = \frac{2\pi}{n} = \frac{b-a}{n}$ and $a = -\pi$, we must have $b = a + n\Delta x = -\pi + 2\pi = \pi$. Even more, the integrand is $\cos(x)$, hence the limit describes the quantity $\int_{-\pi}^{\pi} \cos(x)\, dx$. ◇

Before we endeavor to compute any definite integrals by the limit definition provided above, it is conceptually important to note that the definite integral can be computed by hand in some cases without appealing to any limits. Explicitly, for any real numbers c and d, we have that $\int_a^b (cx+d)\, dx$ represents the net area bounded by the line $cx + d$ and the coordinate axes. Consequently, this area can be computed geometrically as a linear combination of areas of triangles and rectangles.

Exercise 1.8.4. Compute the definite integral $\int_{-2}^{3} (3x - 2)\, dx$ using geometry.

Exercise 1.8.5. Compute the definite integral $\int_{-3}^{2} (5 - 2x)\, dx$ using geometry.

Likewise, for any function of the form $y = f(x) = \sqrt{r^2 - x^2}$, it follows that $x^2 + y^2 = r^2$ yields a circle of radius r, hence we can determine an integral of the form $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$.

Exercise 1.8.6. Compute the definite integral $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$ using geometry.

Often, we will deal with definite integrals that cannot be computed by geometry; for now, if we encounter this situation, we can sometimes use the limit definition of the definite integral.

Example 1.8.7. Compute the definite integral $\int_0^1 x^2\, dx$ as the limit of a Riemann approximation as the number n of subintervals tends to infinity.

Solution. Considering that $a = 0$ and $b = 1$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so that $a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$. Consequently, it follows that

$$\int_0^1 x^2\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}. \qquad \diamond$$

Example 1.8.8. Compute the definite integral $\int_0^3 (x^3 - 6x)\, dx$ as the limit of a Riemann approximation as the number n of subintervals tends to infinity.

Solution. Considering that $a = 0$ and $b = 3$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so that $a + i\Delta x = 0 + \frac{3i}{n} = \frac{3i}{n}$. Consequently, it follows that

$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) = \lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} \left(\frac{27i^3}{n^2} - 18i\right).$$

Granted that the limit of each of these Riemann sums exists, the limit of their difference is given by the difference of their limits, hence it suffices to compute these limits separately.

$$\lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} \frac{81i^3}{n^2} = \lim_{n \to \infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n \to \infty} \frac{81}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{81}{4}$$

$$\lim_{n \to \infty} \frac{3}{n^2} \sum_{i=1}^{n} 18i = \lim_{n \to \infty} \frac{54}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{54}{n^2} \cdot \frac{n(n+1)}{2} = \frac{54}{2} = \frac{108}{4}$$

Consequently, we have that $\int_0^3 (x^3 - 6x)\,dx = \frac{81}{4} - \frac{108}{4} = -\frac{27}{4}$. ◇

Based on the definition of the definite integrals and the summation properties outlined in the previous section, we can extrapolate the following properties of definite integrals.

Proposition 1.8.9 (Properties of Definite Integrals)**.** *Given any real function $f(x)$ that is integrable on a closed and bounded interval $[a, b]$, the following properties hold for $\int_a^b f(x)\,dx$.*

(i.) (Empty Integral Law) $\int_a^a f(x)\,dx = 0$

(ii.) (Reversing the Limits of Integration) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii.) (Additivity of Adjacent Intervals) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all real numbers c*

(iv.) (Constant Integral Formula) $\int_a^b k\,dx = k(b-a)$ *for all real numbers k*

(v.) (Linearity of a Definite Integral I) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ *for all real numbers k*

(vi.) (Linearity of a Definite Integral II) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 1.8.10. Compute the definite integral $\int_0^1 (3x^2 + 4)\,dx$.

Solution. By appealing to Example 1.8.7 and Proposition 1.8.9, we have that

$$\int_0^1 (3x^2 + 4)\,dx = \int_0^1 3x^2\,dx + \int_0^1 4\,dx = 3\int_0^1 x^2\,dx + 4(1-0) = 3\left(\frac{1}{3}\right) + 4 = 5.$$ ◇

Example 1.8.11. Given any pair of real functions $f(x)$ and $g(x)$ such that $\int_{-1}^1 f(x)\,dx = 2$ and $\int_{-1}^1 g(x)\,dx = -1$, compute the definite integral $\int_{-1}^1 [3f(x) - g(x)]\,dx$.

Solution. By appealing to Proposition 1.8.9, we have that

$$\begin{aligned}\int_{-1}^1 [3f(x) - g(x)]\,dx &= \int_{-1}^1 (3f(x) + [-g(x)])\,dx \\ &= \int_{-1}^1 3f(x)\,dx + \int_{-1}^1 [-g(x)]\,dx \\ &= 3\int_{-1}^1 f(x)\,dx - \int_{-1}^1 g(x)\,dx = 3(2) - (-1) = 7.\end{aligned}$$ ◇

Example 1.8.12. Given any real function $f(x)$ such that $\int_0^4 f(x)\,dx = 1$, $\int_{-2}^3 f(x)\,dx = 3$, and $\int_{-2}^0 f(x)\,dx = 5$, compute the definite integral $\int_3^4 f(x)\,dx$.

Solution. By appealing to Proposition 1.8.9, we have that

$$\begin{aligned}\int_3^4 f(x)\,dx &= \int_3^{-2} f(x)\,dx + \int_{-2}^4 f(x)\,dx \\ &= -\int_{-2}^3 f(x)\,dx + \int_{-2}^4 f(x)\,dx \\ &= -\int_{-2}^3 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_0^4 f(x)\,dx = -3+5+1 = 3.\end{aligned}$$ ◇

1.9 The Fundamental Theorem of Calculus

Calculus can be divided into two topics — differentiation and integration — that are connected by the Fundamental Theorem of Calculus. Essentially, the Fundamental Theorem of Calculus says that differentiation and integration are inverse operations: if $f(x)$ is continuous on an open interval, then $f(x)$ admits an antiderivative by the definite integral, and conversely, the definite integral of $f(x)$ over a closed interval measures the **net change** of any antiderivative over that interval.

Theorem 1.9.1 (Fundamental Theorem of Calculus, Part I)**.** *Given any real function $f(x)$ that is integrable with a continuous antiderivative $F(x)$ on a closed interval $[a, b]$, we have that*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Even more, this quantity measures the net area bounded by the curve $f(x)$ from $x = a$ to $x = b$.

Proof. Observe that the quantity $F(b)-F(a)$ measures the net change of $F(x)$ on the closed interval $[a, b]$. Given any collection of n real numbers $a = x_0 < x_1 < \cdots < x_n = b$, we have that

$$F(b) - F(a) = F(b) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(a)$$

by adding and subtracting $F(x_i)$ for each integer $1 \leq i \leq n-1$. Grouping each consecutive pair of differences and using the fact that $a = x_0$ and $b = x_n$, it follows that

$$F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})].$$

By the Mean Value Theorem applied to $F(x)$, for each integer $1 \leq i \leq n$, there exists a real number x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$ and $F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1})$. By assumption that $F(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$, we have that $F'(x) = f(x)$, hence we can rewrite each of these equations as $F(x_i) - F(x_{i-1}) = f(x_i^*)\,\Delta x_i$ for the quantity $\Delta x_i = x_i - x_{i-1}$. Going back to our above displayed equation with this new identity, we have that

$$F(b) - F(a) = \sum_{i=1}^n f(x_i^*)\,\Delta x_i.$$

By taking the limit as n approaches ∞ on both sides, we conclude the desired result that

$$F(b) - F(a) = \lim_{n \to \infty} [F(b) - F(a)] = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i = \int_a^b f(x) \, dx. \qquad \square$$

Consequently, if $v(t)$ measures the velocity of a particle over time, then the (definite) integral of $v(t)$ over $[a, b]$ measures the total distance travelled by the particle from time $t = a$ to time $t = b$.

Exercise 1.9.2. Compute the net area bounded by the curve $f(x) = x^3$ from $x = -1$ to $x = 1$.

Exercise 1.9.3. Compute the net area bounded by the curve $g(x) = \sin(x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

Exercise 1.9.4. Compute the net area bounded by the curve $h(x) = \frac{1}{x}$ from $x = 1$ to $x = e$.

Remark 1.9.5. Like we previously mentioned, if $F(x)$ is an antiderivative of a real function $f(x)$ on a closed interval $[a, b]$, the Mean Value Theorem implies that every antiderivative of $f(x)$ over $[a, b]$ is of the form $F(x) + C$ for some real number C. Consequently, the choice of antiderivative of $f(x)$ does not matter when it comes to computing the definite integral of $f(x)$ on $[a, b]$:

$$\int_a^b f(x) \, dx = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

holds for all real numbers C by the Fundamental Theorem of Calculus, Part I.

One other way to interpret the first part of the Fundamental Theorem of Calculus is as follows.

Corollary 1.9.6 (Net Change Theorem)**.** *Given any differentiable function $f(x)$ on an open interval (a, b) such that $f(a)$ and $f(b)$ are defined, we have that*

$$f(b) - f(a) = \int_a^b f'(x) \, dx.$$

Put another way, the net change of $f(x)$ over the closed interval $[a, b]$ is $\int_a^b f'(x) \, dx$.

Exercise 1.9.7. Consider a leaky water heater that loses $2 + 5t$ gallons of water per hour for each hour after 7 AM. Compute the total amount of water leaked between the time of 9 AM and 12 PM.

Exercise 1.9.8. Consider any medication that disperses into a patient's bloodstream at a rate of $50 - 2\sqrt{t}$ milligrams per hour from the time it is administered. Compute the amount of medication dispersed into a patient's bloodstream one hour after it is administered. Given that one full dose is 50 milligrams, what percentage of the dose reaches the patient's bloodstream in an hour?

Exercise 1.9.9. Consider any particle that moves with velocity $t^3 - 10t^2 + 24t$ meters per second after initial observation at time $t = 0$. Compute the total displacement of and the total distance travelled by the particle from time $t = 0$ to time $t = 6$; then, compare the values.

Conversely, the second part of the Fundamental Theorem of Calculus states that every continuous function on a closed interval $[a, b]$ admits an antiderivative in the form of a definite integral.

Theorem 1.9.10 (Fundamental Theorem of Calculus, Part II)**.** *Given any real function $f(x)$ that is continuous on a closed interval $[a, b]$, for all real numbers $a < x < b$, we have that*

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

Proof. Considering that $f(x)$ is continuous on $[a, b]$, it is integrable on $[a, b]$, hence we may define

$$F(x) = \int_a^x f(t)\, dt$$

for all real numbers $a \leq x \leq b$. We must demonstrate that for all real numbers $a < x < b$, the limit

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

exists. By the second and third parts of Proposition 1.8.9, it follows that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt = \int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt = \int_x^{x+h} f(t)\, dt.$$

By the **Mean Value Theorem for Definite Integrals**, there exists a real number c (depending upon h) such that $x < c < x + h$ and $\int_x^{x+h} f(t)\, dt = f(c)[(x+h) - x] = f(c)h$ so that

$$f(c) = \frac{F(x+h) - F(x)}{h}.$$

Considering that $f(x)$ is continuous on the closed interval $[a, b]$, it follows that

$$f\left(\lim_{h \to 0} c\right) = \lim_{h \to 0} f(c) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x),$$

hence it suffices to compute the limit of c as h approaches 0. By the Squeeze Theorem, we have

$$x = \lim_{h \to 0} x \leq \lim_{h \to 0} c \leq \lim_{h \to 0} (x + h) = x$$

so that $\lim\limits_{h \to 0} c = x$ and $F'(x) = f(x)$ for all real numbers $a < x < b$, as desired. □

Exercise 1.9.11. Compute the derivative of $\int_0^x \sin(t)\, dt$ for any real number $x > 0$.

Exercise 1.9.12. Compute the derivative of $\int_{-1}^x e^t\, dx$ for any real number $x > -1$.

Exercise 1.9.13. Compute the derivative of $\int_1^x \ln(t)\, dt$ for any real number $x > 1$.

Exercise 1.9.14. Given any differentiable real functions $f(x)$, $g(x)$, and $h(x)$, use the Fundamental Theorem of Calculus, Part II and the Chain Rule for derivatives to prove that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Exercise 1.9.15. Compute the derivative of $\int_0^{x^2} \sin(\cos(t))\, dt$ for any real number $x > 0$.

Exercise 1.9.16. Compute the derivative of $\int_{\ln(x)}^{10} \sqrt{t^2 + 1}\, dt$ for any real number $0 < x < e^{10}$.

Exercise 1.9.17. Compute the derivative of $\int_{x^3}^{x^2} \sqrt{t}\, dt$ for any real number $0 < x < 1$.

1.10 u-Substitution

Until now, we have managed to find the antiderivatives of many functions by viewing antidifferentiation as the inverse to differentiation (in the sense of the Fundamental Theorem of Calculus, Part I) and subsequently using the appropriate analog of the familiar rules for differentiation such as the Power Rule and the Chain Rule. Explicitly, given any real number $r \neq -1$, we have that

$$\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C$$

by the Power Rule. Further, for any differentiable functions $f(x)$ and $g(x)$, we have that

$$\int f'(g(x)) g'(x) \, dx = f(g(x)) + C$$

by the Chain Rule. Essentially, if we make the assignment $u = g(x)$, then it follows that $\frac{du}{dx} = g'(x)$ and $f'(g(x))g'(x) = f'(u)\frac{du}{dx}$. Conventionally, this relationship is written as $du = g'(x)\,dx$ so that $f'(g(x))g'(x)\,dx = f'(u)\,du$. Considering that $f(u)$ is an antiderivative of $f'(u)$, it follows that

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C.$$

Colloquially, we refer to this technique (and its broader applications) as u-**substitution**.

Exercise 1.10.1. Compute the indefinite integral of $f(x) = (x+1)^{100}$.

Exercise 1.10.2. Compute the indefinite integral of $g(x) = x\cos(2x^2)$.

Exercise 1.10.3. Compute the indefinite integral of $h(x) = x^2 e^{x^3}$.

Exercise 1.10.4. Compute the indefinite integral of $k(x) = x\sqrt{2x-1}$.

Even more, the technique of u-substitution can be used to evaluate definite integrals. Explicitly, suppose that $f'(x)$ is integrable on the closed interval $[g(a), g(b)]$ and $f'(g(x))g'(x)$ is integrable on the closed interval $[a, b]$. By performing the substitution $u = g(x)$, we have that $du = g'(x)\,dx$ and $f'(g(x))g'(x)\,dx = f'(u)\,du$. Even more, if $x = a$, then $u = g(a)$, and if $x = b$, then $u = g(b)$ so that

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du.$$

Exercise 1.10.5. Compute the definite integral $\int_0^1 x^4(x^5-1)^{10}\,dx$.

Exercise 1.10.6. Compute the definite integral $\int_{-\pi/4}^{\pi/4} 2x\sec^2(x^2)\,dx$

Exercise 1.10.7. Compute the definite integral $\int_1^e \frac{\ln(x)}{x}\,dx$.

We say that a real function $f(x)$ is **even** if it holds that $f(-x) = f(x)$ for all real numbers x in the domain of f such that $-x$ is in the domain of f. Consequently, the polynomial $3x^4 - x^2 + 2$ and the trigonometric function $\cos(x)$ are even functions. Conversely, we say that $f(x)$ is **odd** if it holds that $f(-x) = -f(x)$ for all real numbers x in the domain of f such that $-x$ is in the domain of f. We note that the polynomial $4x^5 + x + 1$ and the trigonometric function $\sin(x)$ are odd functions. We refer to the property that a function is even or odd as the **parity** of the function. We note that a function need not have parity, as illustrated by the fact that $f(x) = x^2 + x$ does not satisfy either $f(-x) = f(x)$ or $f(-x) = -f(x)$; however, the parity of a function is always well-defined.

Exercise 1.10.8. Explain whether $f(x) = \tan(x)$ is even, odd, or neither.

Exercise 1.10.9. Explain whether $g(x) = x^2 e^x$ is even, odd, or neither.

Exercise 1.10.10. Explain whether $h(x) = \sin^2(x)$ is even, odd, or neither.

Proposition 1.10.11 (Properties of Function Parity)**.** *Consider any real functions $f(x)$ and $g(x)$.*

(i.) (Preservation of Parity Under Nonzero Scalar Multiple) *If $f(x)$ has parity, then for all nonzero real numbers α, the scalar multiple $\alpha f(x)$ of $f(x)$ by α has the same parity as $f(x)$.*

(ii.) (Preservation of Parity Under Sum) *If $f(x)$ and $g(x)$ have the same parity, then their sum $f(x) + g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(iii.) (Preservation of Parity Under Product) *If $f(x)$ and $g(x)$ have the same parity, then their product $f(x)g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(iv.) (Products of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their product $f(x)g(x)$ is an odd function.*

(v.) (Preservation of Parity Under Quotient) *If $f(x)$ and $g(x)$ have the same parity, then their quotient $f(x)/g(x)$ has the same parity as both $f(x)$ and $g(x)$.*

(vi.) (Quotients of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their quotient $f(x)/g(x)$ is an odd function.*

(vii.) (Preservation of Parity Under Composition) *If $f(x)$ and $g(x)$ have the same parity, then their composite $f(g(x))$ has the same parity as both $f(x)$ and $g(x)$.*

(viii.) (Composition of Functions of Opposite Parity) *If $f(x)$ and $g(x)$ have opposite parity, then their composite $f(g(x))$ is an even function.*

(ix.) (Parity of the Derivative of a Function) *If $f(x)$ is differentiable and $f(x)$ has parity, then the derivative $f'(x)$ has the opposite parity of $f(x)$.*

Proposition 1.10.12 (Definite Integral of an Even Function on a Symmetric Interval)**.** *Consider any even real function $f(x)$ that is integrable on a closed interval $[-a, a]$. We have that*

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Proof. By the third property of Proposition 1.8.9, it follows that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Consider the substitution $u = -x$ with $du = -dx$. By assumption that $f(-x) = f(x)$, we have that

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)\,dx = \int_{a}^{0} f(u)(-du) = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

Consequently, by the above two displayed equations, the desired identity holds. $\square$

Proposition 1.10.13 (Definite Integral of an Odd Function on a Symmetric Interval)**.** *Consider any odd real function $f(x)$ that is integrable on a closed interval $[-a, a]$. We have that*

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Proof. By the third property of Proposition 1.8.9, it follows that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

By assumption that $f(-x) = -f(x)$, the substitution $u = -x$ with $du = -dx$ yields that

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} -f(-x)\,dx = \int_{a}^{0} -f(u)(-du) = \int_{a}^{0} f(u)\,du = -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx.$$

Consequently, by the above two displayed equations, the desired identity holds. □

1.11 Integration by Parts

We turn our attention next to an analog of the Product Rule for antidifferentiation. We adopt the shorthand notation $u = f(x)$ and $v = g(x)$ for some differentiable functions $f(x)$ and $g(x)$ so that $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$ or $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. By the Product Rule, we have that

$$\frac{d}{dx}[uv] = \frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Considering that uv is clearly an antiderivative of $\frac{d}{dx}[uv] = f(x)g'(x) + g(x)f'(x)$, it follows that

$$uv = \int [f(x)g'(x) + g(x)f'(x)]\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = \int u\,dv + \int v\,du.$$

By rearranging, we obtain an analog to the Product Rule for antidifferentiation.

Theorem 1.11.1 (Integration by Parts)**.** *Given any differentiable functions $u = f(x)$ and $v = g(x)$, we will denote $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. We have that*

$$\int u\,dv = uv - \int v\,du.$$

Colloquially, we refer to this technique as the method of **integration by parts** because the rule allows us to identify two parts of the integrand — namely, u and dv — in such a way that

(i.) the antiderivative of u is difficult to determine and its derivative du is simpler;

(ii.) the antiderivative v of dv is readily obtained; and

(iii.) the antiderivative of $v\,du$ is known or can be found by the method of integration by parts.

Exercise 1.11.2. Use integration by parts to compute the antiderivative of $x\cos(x)$.

Exercise 1.11.3. Use integration by parts to compute the antiderivative of $\ln(x)$.

Exercise 1.11.4. Use integration by parts to compute the antiderivative of xe^x.

Once again, the advantage of the method of integration by parts is that it allows us to trade an expression $u\,dv$ that is difficult to antidifferentiate for an expression $v\,du$ whose antiderivative is known or can be found by integration by parts. Consequently, we may identify families of functions whose antiderivatives are unknown to us at this time — e.g., logarithmic and inverse trigonometric functions — and use these as candidates for u. On the other hand, we may identify functions whose antiderivatives are easily found — e.g., algebraic, trigonometric, and exponential functions — and use these as candidates for dv. Ultimately, this gives rise to the following acronym.

Logarithmic Inverse Trigonometric Algebraic Trigonometric Exponential

Essentially, this acronym is intended to help us remember how to prioritize the assignments of u and dv to our integrand: if the function is further left on the list, then it should be made u; if the function is further right on the list, it should be made dv. Consequently, we have the following.

Algorithm 1.11.5 (Using LIATE)**.** Given any pair of functions $f(x)$ and $g(x)$ such that

(a.) $f(x)$ is a logarithmic, inverse trigonometric, or algebraic function and

(b.) $g(x)$ is an algebraic, trigonometric, or exponential function,

in order to compute $\int f(x)g(x)\,dx$, we may assign $u = f(x)$ and $dv = g(x)\,dx$.

Exercise 1.11.6. Use integration by parts once to compute the antiderivative of $x^3 \ln(x)$.

Exercise 1.11.7. Use integration by parts twice to compute the antiderivative of $x^2 \sin(x)$.

Exercise 1.11.8. Use integration by parts three times to compute the antiderivative of $x^3 e^x$.

Exercise 1.11.9. Explain the difficulty in using integration by parts with $u = x^3$ and $dv = e^{x^2}\,dx$ to compute the antiderivative of $x^3 e^{x^2}$. Group the terms differently, and try again successfully.

Observe that in two of the above examples, we were required to use integration by parts multiple times in order to find the antiderivatives of the given functions. Generally, if we wish to evaluate the antiderivative of the product of a function $f(x)$ and a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$, we may use a shorthand version of integration by parts known as the **tabular method**.

Theorem 1.11.10 (Tabular Method for Integration)**.** *Given any function $f(x)$ whose antiderivatives are known and any polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with nonzero a_n, we have that*

$$\int p(x)f(x)\,dx = \sum_{k=0}^{n} (-1)^k p^{(k)}(x) I^{k+1} f(x),$$

where $p^{(k)}(x)$ denotes the kth derivative of $p(x)$ and $I^k f(x)$ denotes the k-fold antiderivative of $f(x)$.

Proof. Observe that the nth derivative of $p(x)$ is given by $p^{(n)}(x) = a_n n!$ so that $p^{(n+1)}(x) = 0$. By the method of Integration by Parts with $u = p(x)$ and $dv = f(x)\,dx$, we have that

$$\int p(x)f(x)\,dx = p(x)F(x) - \int p'(x)F(x)\,dx$$

for some real function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$. By hypothesis, the antiderivative of $F(x)$ is known, hence we may use integration by parts with $u = p'(x)$ and $dv = F(x)\,dx$ to find that

$$\int p'(x)F(x)\,dx = p'(x)I^2 f(x) - \int p''(x)I^2 f(x)\,dx,$$

where $I^2 f(x)$ denotes the antiderivative of $F(x)$, i.e., $\frac{d}{dx}I^2 f(x) = F(x)$ so that $\frac{d^2}{dx^2}I^2 f(x) = f(x)$. Combined with the above displayed equation, we have that

$$\begin{aligned}\int p(x)f(x)\,dx &= p(x)F(x) - \left(p'(x)I^2 f(x) - \int p''(x)I^2 f(x)\,dx\right) \\ &= p(x)F(x) - p'(x)I^2 f(x) + \int p''(x)I^2 f(x)\,dx.\end{aligned}$$

Using integration by parts once again with $u = p''(x)$ and $dv = I^2 f(x)\,dx$, we have that

$$\int p''(x)I^2 f(x)\,dx = p''(x)I^3 f(x) - \int p'''(x)I^3 f(x)\,dx.$$

Combined with the above displayed equation, we find that

$$\int p(x)f(x)\,dx = p(x)F(x) - p'(x)I^2 f(x) + p''(x)I^3 f(x) - \int p'''(x)I^3 f(x)\,dx.$$

Continue in this manner until $u = p^{(n)}(x)$. By our opening remarks, we have that $du = p^{(n+1)}(x) = 0$ so that $\int v\,du = 0$. Observing the pattern and using $F(x) = \int f(x)\,dx = I^1 f(x)$, we are done. $\square$

Graphically, we can quite simply implement the tabular method by writing out a table with four columns: the first column consists of the index k; the second column consists of the sign $(-1)^k$; the third column consists of the consecutive derivatives of $p(x)$ up to and including 0; and the fourth column consists of the consecutive antiderivatives $I^{k+1}f(x)$ of $f(x)$. Once we have these, the tabular method guarantees that $\int p(x)f(x)\,dx$ can be found by adding the consecutive products of the kth row of the second and third columns by the $(k+1)$th row of the fourth column.

Example 1.11.11. We will illustrate the tabular method to compute the antiderivative of $x^2\sin(x)$ as in Example 1.11.7. Construct the following table with $p(x) = x^2$ and $f(x) = \sin(x)$.

k	$(-1)^k$	$p^{(k)}(x)$	$I^{k+1}f(x)$
0	$+$	x^2	$\sin(x)$
1	$-$	$2x$	$-\cos(x)$
2	$+$	2	$-\sin(x)$
3	$-$	0	$\cos(x)$

Consequently, we find that $\int x^2\sin(x)\,dx = x^2(-\cos(x)) - 2x(-\sin(x)) + 2\cos(x)$, as desired.

Exercise 1.11.12. Use the tabular method to verify your solution to Example 1.11.8.

Exercise 1.11.13. Use the tabular method to compute the antiderivative of $x^{10}(2x+1)^4$.

1.12 Trigonometric Integrals

Given positive integers (or whole numbers) m and n, we refer to an integral of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

as a **trigonometric integral**: indeed, the integrand is a product of powers of basic trigonometric functions. Quickly, one can glean that u-substitution fails, and integration by parts is hopelessly complicated. Using basic trigonometry, however, we are able to evaluate these integrals by converting them to a form in which we can use the tried-and-true methods of yore. Given a right triangle with hypotenuse of length $h > 0$, base of length a, and height of length o, the Pythagorean Theorem states that $o^2 + a^2 = h^2$. By dividing each term in this equation by h, we have that $\frac{o^2}{h^2} + \frac{a^2}{h^2} = 1$. Using x to represent the angle whose opposite side has length o and whose adjacent side has length a, the Pythagorean Theorem yields the so-called **Pythagorean Identity**

$$\sin^2(x) + \cos^2(x) = 1.$$

Consequently, we may convert any even power of $\cos(x)$ into a power of $1-\sin^2(x)$ (and vice-versa). Considering that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, we have the following stratagem.

Strategy 1.12.1 (Trigonometric Integration, Case I)**.** Consider the case that either m or n is odd.

(a.) Given that m is odd, we may write $m = 2k+1$ for some positive integer k so that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k+1}(x)\cos^n(x)\,dx = \int [\sin^2(x)]^k\cos^n(x)(\sin(x)\,dx).$$

Considering that $\sin^2(x) = 1-\cos^2(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, letting $u = \cos(x)$ yields that

$$\int \sin^m(x)\cos^n(x)\,dx = -\int (1-u^2)^k u^n\,du.$$

Expanding the polynomial $(1-u^2)^k$ and using the Power Rule, we can find the antiderivative.

(b.) Given that n is odd, we may write $n = 2\ell+1$ for some positive integer ℓ so that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{2\ell+1}(x)\,dx = \int \sin^m(x)(\cos^2(x))^\ell(\cos(x)\,dx).$$

Considering that $\cos^2(x) = 1-\sin^2(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$, letting $v = \sin(x)$ yields that

$$\int \sin^m(x)\cos^n(x)\,dx = \int v^m(1-v^2)^\ell\,dv.$$

Expanding the polynomial $(1-v^2)^\ell$ and using the Power Rule, we can find the antiderivative.

Example 1.12.2. Compute the indefinite integral of $\sin^3(x)\cos^2(x)$.

Solution. Observe that $\sin^3(x)\cos^2(x)\,dx = \sin^2(x)\cos^2(x)(\sin(x)\,dx)$. By the Pythagorean Identity, we have that $\sin^2(x) = 1 - \cos^2(x)$, from which it follows that

$$\sin^3(x)\cos^2(x)\,dx = (1 - \cos^2(x))\cos^2(x)(\sin(x)\,dx).$$

Using the substitution $u = \cos(x)$, we have that $du = -\sin(x)\,dx$ so that

$$\sin^3(x)\cos^2(x)\,dx = (1 - u^2)u^2(-du) = (u^2 - 1)u^2\,du = (u^4 - u^2)\,du.$$

Consequently, we find that

$$\int \sin^3(x)\cos^2(x)\,dx = \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C. \qquad \diamond$$

Exercise 1.12.3. Compute the indefinite integral of $\sin^5(x)$.

Unfortunately, this method fails in the case that both m and n are even. Consider the trigonometric integral of $\sin^2(x)\cos^2(x)$. By setting $u = \sin(x)$, we find that $du = \cos(x)\,dx$ so that

$$\sin^2(x)\cos^2(x)\,dx = u^2\cos(x)\,du.$$

But the lingering factor of $\cos(x)$ obstructs our efforts to take the indefinite integral. Likewise, a similar obstruction appears if we attempt to let $u = \cos(x)$. Luckily, we have more trigonometric tools at our disposal. Recall the following **angle addition formulas**.

$$\begin{aligned}\sin(x + y) &= \sin(x)\cos(y) + \sin(y)\cos(x)\\ \cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y).\end{aligned}$$

Using these, we can derive the **double-angle formulas** by plugging in $x = y$.

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ \cos(2x) &= \cos^2(x) - \sin^2(x)\end{aligned}$$

Considering that $\sin^2(x) + \cos^2(x) = 1$, we may simplify these identities as follows.

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) = [1 - \sin^2(x)] - \sin^2(x) = 1 - 2\sin^2(x)\\ \cos(2x) &= \cos^2(x) - \sin^2(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1\end{aligned}$$

By solving for $\sin^2(x)$ and $\cos^2(x)$ above, we obtain the **power-reduction formulas**.

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \qquad\qquad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

One way to memorize the distinction is to "remember your sign" when using sine. Or as my former student Ronald Heminway so eloquently put it, we may use the mnemonic device "sinus minus."

Strategy 1.12.4 (Trigonometric Integration, Case II)**.** Consider the case that neither of the integers m and n is odd. Put another way, consider the case that both of the integers m and n are even.

(a.) Given that $m = n = 2k$ for some positive integer k, we have that

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k}(x)\cos^{2k}(x)\,dx = \int [\sin(x)\cos(x)]^{2k}\,dx.$$

Using the double-angle formula $\sin(2x) = 2\sin(x)\cos(x)$, we have that

$$[\sin(x)\cos(x)]^{2k} = \left[\frac{\sin(2x)}{2}\right]^{2k} = \frac{[\sin^2(2x)]^k}{4^k}.$$

Using the power-reduction formula $\sin^2(2x) = \frac{1}{2}[1 - \cos(4x)]$, we can then obtain a polynomial in $\cos(4x)$. Continue using the power-reduction formula for cosine to obtain a linear combination of $\cos(4x)$, $\cos(8x)$, $\cos(16x)$, etc. Each of these has an elementary antiderivative.

(b.) Given that $m = 2i$ and $n = 2j$ for some distinct positive integers i and j, use the power-reduction formulas repeatedly to express $\sin^{2i}(x)\cos^{2j}(x) = [\sin^2(x)]^i[\cos^2(x)]^j$ as a linear combination of $\cos(2x)$, $\cos(4x)$, $\cos(8x)$, etc. Each of these has an elementary antiderivative.

Example 1.12.5. Compute the indefinite integral of $\sin^2(x)\cos^2(x)$.

Solution. By the double-angle formula, we have that

$$\sin^2(x)\cos^2(x)\,dx = [\sin(x)\cos(x)]^2\,dx = \left(\frac{1}{2}\sin(2x)\right)^2 dx = \frac{1}{4}\sin^2(2x)\,dx.$$

Using the power-reduction formula, we find that

$$\sin^2(x)\cos^2(x)\,dx = \frac{1}{4}\sin^2(2x)\,dx = \frac{1}{4}\cdot\frac{1}{2}[1 - \cos(4x)]\,dx$$

has an elementary antiderivative. Consequently, we conclude that

$$\int \sin^2(x)\cos^2(x)\,dx = \frac{1}{8}\int [1 - \cos(4x)]\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + C. \qquad \diamond$$

Exercise 1.12.6. Compute the indefinite integral of $\cos^4(x)$.

Using the Pythagorean Identity $\sin^2(x) + \cos^2(x) = 1$, we can obtain another identity

$$\tan^2(x) + 1 = \sec^2(x)$$

by dividing each term by $\cos^2(x)$ and recalling that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$. Consequently, we can adapt our stratagem from Trigonometric Integration, Case I to evaluate integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

Crucially, toward achieving this end, we must observe the following facts.

1.) By the Quotient Rule, we have that

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos^2(x) - (-\sin(x))(\sin(x))}{\cos^2(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \sec^2(x).$$

2.) By the Chain Rule, we have that

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}[\cos(x)]^{-1} = -[\cos(x)]^{-2}[-\sin(x)] = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x).$$

3.) Using the substitution $u = \cos(x)$ with $du = -\sin(x)\,dx$, we have that

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

4.) Using the substitution $u = \sec(x) + \tan(x)$ with $du = [\sec(x)\tan(x) + \sec^2(x)]\,dx$, we have

$$\int \frac{\sec(x)[\sec(x) + \tan(x)]}{\sec(x) + \tan(x)}\,dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\tan(x) + \sec(x)}\,dx = \ln|\sec(x) + \tan(x)| + C.$$

Strategy 1.12.7 (Trigonometric Integration, Case III)**.** Consider the case that $n \geq 2$ is an even integer. Explicitly, assume that $n = 2k$ for some positive integer k, from which it follows that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^m(x)\sec^{2k}(x)\,dx = \int \tan^m(x)[\sec^2(x)]^{k-1}(\sec^2(x)\,dx).$$

Considering that $\sec^2(x) = 1 + \tan^2(x)$ and $\frac{d}{dx}\tan(x) = \sec^2(x)$, letting $u = \tan(x)$ yields that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^m(x)(1 + \tan^2(x))^{k-1}(\sec^2(x)\,dx) = \int u^m(1 + u^2)^{k-1}\,du.$$

Expanding the polynomial $(1 + u^2)^{k-1}$ and using the Power Rule, we can compute the integral.

Exercise 1.12.8. Compute the indefinite integral of $\tan^2(x)\sec^2(x)$.

Exercise 1.12.9. Compute the indefinite integral of $\tan^5(x)\sec^4(x)$.

Strategy 1.12.10 (Trigonometric Integration, Case IV)**.** Consider the case that $m \geq 1$ is odd and $n \geq 1$. Explicitly, assume that $m = 2\ell + 1$ for some positive integer ℓ, from which it follows that

$$\int \tan^m(x)\sec^n(x)\,dx = \int \tan^{2\ell+1}(x)\sec^n(x)\,dx = \int [\tan^2(x)]^\ell \sec^{n-1}(x)(\sec(x)\tan(x)\,dx).$$

Considering that $\tan^2(x) = \sec^2(x) - 1$ and $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, letting $v = \sec(x)$ yields that

$$\int \tan^m(x)\sec^n(x)\,dx = \int [\sec^2(x) - 1]^\ell \sec^{n-1}(x)(\sec(x)\tan(x)\,dx) = \int (v^2 - 1)^\ell v^{n-1}\,dv.$$

Expanding the polynomial $(v^2 - 1)^{\ell-1}$ and using the Power Rule, we can compute the integral.

Exercise 1.12.11. Compute the indefinite integral of $\tan(x)\sec^2(x)$.

Exercise 1.12.12. Compute the indefinite integral of $\tan^3(x)\sec^3(x)$.

Unfortunately, it is difficult to compute the indefinite integral of the function $\tan^m(x)\sec^n(x)$ when $m \geq 2$ is an even integer and $n \geq 1$ is an odd integer; however, in this case, it is possible to use integration by parts and the Pythagorean Identity to transform the integrand into one that falls into either Trigonometric Integration, Case III or Trigonometric Integration, Case IV as follows.

Example 1.12.13. Consider the trigonometric function $\tan^2(x)\sec(x)$. Observe that if $u = \tan(x)$ and $dv = \sec(x)\tan(x)\,dx$, then by the method of Integration by Parts, we have that

$$\int \tan^2(x)\sec(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx.$$

We are now in a position to compute the indefinite integral by evaluating the indefinite integral of $\sec^3(x)$. By the Pythagorean Identity $1 + \tan^2(x) = \sec^2(x)$, we have that

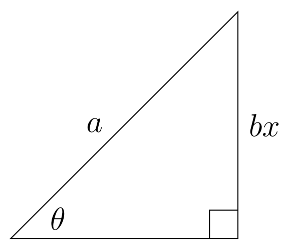
$$\int \sec^3(x)\,dx = \int \sec(x)[\tan^2(x) + 1]\,dx = \int \tan^2(x)\sec(x)\,dx + \int \sec(x)\,dx.$$

By plugging this back into our above displayed equation and rearranging, it follows that

$$\int \tan^2(x)\sec(x) = \frac{1}{2}\left[\sec(x)\tan(x) - \int \sec(x)\,dx\right] = \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

1.13 Trigonometric Substitution

Beyond their extensive applications in geometry and physics, the trigonometric functions yield a very powerful substitution method for integration. Consider the following right triangle.



By the Pythagorean Theorem, the side adjacent to the interior angle θ has length $\sqrt{a^2 - b^2x^2}$ so that $a\cos(\theta) = \sqrt{a^2 - b^2x^2}$. Observe that $bx = a\sin(\theta)$ so that $b\,dx = a\cos(\theta)\,d\theta$, and we have that

$$\begin{aligned}
\int \sqrt{a^2 - b^2x^2}\,dx &= \int a\cos(\theta)\left(\frac{a}{b}\cos(\theta)\,d\theta\right) \\
&= \frac{a^2}{b}\int \cos^2(\theta)\,d\theta \\
&= \frac{a^2}{2b}\int [1 + \cos(2\theta)]\,d\theta && \text{(power-reduction formula)} \\
&= \frac{a^2}{2b}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{a^2}{2b}[\theta + \sin(\theta)\cos(\theta)] + C && \text{(double-angle formula)} \\
&= \frac{a^2}{2b}\left[\arcsin\left(\frac{bx}{a}\right) + \frac{bx}{a^2}\sqrt{a^2 - b^2x^2}\right] + C,
\end{aligned}$$

where the last equality comes from the substitution $bx = \sin(\theta)$ and the above triangle.

Strategy 1.13.1 (Trigonometric Substitution, Case I)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{a^2 - b^2x^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ by making the substitution $bx = a\sin(\theta)$ so that $b\,dx = a\cos(\theta)\,d\theta$.

Example 1.13.2. Use a trigonometric substitution to compute the indefinite integral of $x^2\sqrt{1-x^2}$.

Solution. Considering that this function has a factor of $\sqrt{1-x^2}$, we may make the trigonometric substitution $x = \sin(\theta)$ so that $dx = \cos(\theta)\,d\theta$. Observe that $x^2 = \sin^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$. Consequently, we find that

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2(\theta)\cos(\theta)(\cos(\theta)\,d\theta) = \int \sin^2(\theta)\cos^2(\theta)\,d\theta.$$

By Example 1.12.5 above and the double-angle formulas, we have that

$$\begin{aligned}\int \sin^2(\theta)\cos^2(\theta)\,d\theta &= \frac{1}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right] + C \\ &= \frac{1}{8}\left[\theta - \frac{1}{2}\sin(2\theta)\cos(2\theta)\right] + C \\ &= \frac{1}{8}(\theta - \sin(\theta)\cos(\theta)[\cos^2(\theta) - \sin^2(\theta)]) + C \\ &= \frac{1}{8}[\theta - \sin(\theta)\cos^3(\theta) + \sin^3(\theta)\cos(\theta)] + C.\end{aligned}$$

Using the substitution $x = \sin(\theta)$ and the fact that $\sqrt{1-x^2} = \cos(\theta)$, we conclude that

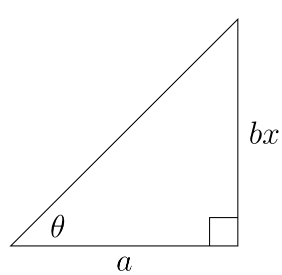
$$\int x^2\sqrt{1-x^2}\,dx = \frac{1}{8}\left[\arcsin(x) - x(1-x^2)^{3/2} + x^3\sqrt{1-x^2}\right] + C. \qquad \diamond$$

Exercise 1.13.3. Use a trigonometric substitution to compute the indefinite integral of $\frac{x}{\sqrt{1-x^2}}$.

Exercise 1.13.4. Use a trigonometric substitution to compute the indefinite integral of $x^5\sqrt{1-9x^2}$.

Exercise 1.13.5. Use a trigonometric substitution to compute the indefinite integral of $\frac{x^2}{\sqrt{9-x^2}}$.

Certainly, it is possible to consider other possibilities for our initial right triangle. Explicitly, suppose that the altitude and base of a right triangle are given as follows.



By the Pythagorean Theorem, the hypotenuse of the above right triangle has length $\sqrt{a^2+b^2x^2}$. Observe that $bx = a\tan(\theta)$ so that $b\,dx = a\sec^2(\theta)\,d\theta$. Consequently, we have that

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2+b^2x^2}} &= \int \frac{\frac{a}{b}\sec^2(\theta)\,d\theta}{\sqrt{a^2+a^2\tan^2(\theta)}} \\
&= \frac{a}{b}\int \frac{\sec^2(\theta)\,d\theta}{\sqrt{a^2(1+\tan^2(\theta))}} \\
&= \frac{a}{b}\int \frac{\sec^2(\theta)\,d\theta}{\sqrt{a^2\sec^2(\theta)}} && \text{(Pythagorean Identity)} \\
&= \frac{1}{b}\int \sec(\theta)\,d\theta && (a>0 \text{ and } \sec(\theta)>0) \\
&= \frac{1}{b}\ln|\sec(\theta)+\tan(\theta)|+C \\
&= \frac{1}{b}\ln\left|\frac{\sqrt{a^2+b^2x^2}+bx}{a}\right|+C,
\end{aligned}$$

where the last equality comes from the substitution $bx = a\tan(\theta)$ and the above triangle.

Strategy 1.13.6 (Trigonometric Substitution, Case II)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{a^2+b^2x^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ by making the substitution $bx = a\tan(\theta)$ so that $b\,dx = a\sec^2(\theta)\,d\theta$.

Example 1.13.7. Use a trigonometric substitution to compute the indefinite integral of $x^3\sqrt{1+x^2}$.

Solution. Considering that this function has a factor of $\sqrt{1+x^2}$, we may make the trigonometric substitution $x=\tan(\theta)$ with $dx = \sec^2(\theta)\,d\theta$. Observe that $x^2=\tan^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{1+x^2}=\sqrt{1+\tan^2(\theta)}=\sqrt{\sec^2(\theta)}=\sec(\theta)$. Consequently, we find that

$$\int x^3\sqrt{1+x^2}\,dx = \int \tan^3(\theta)\sec(\theta)(\sec^2(\theta)\,d\theta) = \int \tan^3(\theta)\sec^3(\theta)\,d\theta.$$

We are now in a position to evaluate a trigonometric integral. By the technique outlined in Trigonometric Integration, Case IV, we may borrow a factor of $\tan(\theta)$ and a factor of $\sec(\theta)$ and use the Pythagorean Identity $\tan^2(\theta)=\sec^2(\theta)-1$ to simplify the integrand $\tan^3(\theta)\sec^3(\theta)\,d\theta$ as follows.

$$\int \tan^3(\theta)\sec^3(\theta)\,d\theta = \int (\sec^2(\theta)-1)\sec^2(\theta)(\sec(\theta)\tan(\theta)\,d\theta)$$

We now employ the substitution $u=\sec(\theta)$ with $du=\sec(\theta)\tan(\theta)\,d\theta$ to obtain the following.

$$\int (\sec^2(\theta)-1)\sec^2(\theta)(\sec(\theta)\tan(\theta)\,d\theta) = \int (u^2-1)u^2\,du = \int (u^4-u^2)\,du = \frac{1}{5}u^5-\frac{1}{3}u^3+C.$$

Considering that $u = \sec(\theta) = \sqrt{1+x^2}$, it follows that

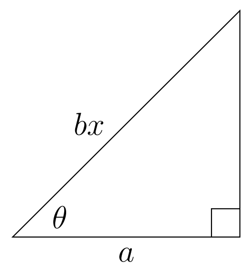
$$\int x^3\sqrt{1+x^2}\,dx = \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C. \qquad \diamond$$

Exercise 1.13.8. Use a trigonometric substitution to compute the indefinite integral of $(x^2+1)^{-3/2}$.

Exercise 1.13.9. Use a trigonometric substitution to compute the indefinite integral of $x^2(x^2+9)^{3/2}$.

Exercise 1.13.10. Use a trigonometric substitution to compute the indefinite integral of $x^5\sqrt{4+x^2}$.

Last, consider the following right triangle in which the base and hypotenuse are given.



By the Pythagorean Theorem, the side opposite the interior angle θ has length $\sqrt{b^2x^2-a^2}$. Observe that $bx = a\sec(\theta)$ so that $b\,dx = a\sec(\theta)\tan(\theta)\,d\theta$. Consequently, we have that

$$\begin{aligned}
\int \frac{dx}{b^2x^2-a^2} &= \int \frac{\frac{a}{b}\sec(\theta)\tan(\theta)\,d\theta}{a^2\tan^2(\theta)} \\
&= \frac{1}{ab}\int \frac{\sec(\theta)\,d\theta}{\tan(\theta)} \\
&= \frac{1}{ab}\int \csc(\theta)\,d\theta \\
&= -\frac{1}{ab}\ln|\csc(\theta)+\cot(\theta)| + C \\
&= -\frac{1}{ab}\ln\left|\frac{bx+a}{\sqrt{b^2x^2-a^2}}\right| + C,
\end{aligned}$$

where the last equality comes from the substitution $bx = a\sec(\theta)$ and the above triangle.

Strategy 1.13.11 (Trigonometric Substitution, Case III)**.** Given a function $f(x)$ that can be written as $g(x)\sqrt{b^2x^2-a^2}$ for some nonzero real numbers a and b and some function $g(x)$, we may attempt to compute $\int f(x)\,dx$ via the substitution $bx = a\sec(\theta)$ so that $b\,dx = a\sec(\theta)\tan(\theta)\,d\theta$.

Example 1.13.12. Use a trigonometric substitution to compute the indefinite integral of $x^3\sqrt{x^2-1}$.

Solution. Considering that this function has a factor of $\sqrt{x^2-1}$, we may make the substitution $x = \sec(\theta)$ so that $dx = \sec(\theta)\tan(\theta)\,d\theta$. Observe that $x^2 = \sec^2(\theta)$ so that by the Pythagorean Identity, we have that $\sqrt{x^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)} = \tan(\theta)$. Consequently, we find that

$$\int x^3\sqrt{x^2-1}\,dx = \int \sec^3(\theta)\tan(\theta)(\sec(\theta)\tan(\theta)\,d\theta) = \int \tan^2(\theta)\sec^4(\theta)\,d\theta.$$

Observe that we may use the substitution $u = \tan(\theta)$ with $du = \sec^2(\theta)\,d\theta$ to obtain

$$\begin{aligned}\int \tan^2(\theta)\sec^4(\theta)\,d\theta &= \int \tan^2(\theta)\sec^2(\theta)(\sec^2(\theta)\,d\theta) \\ &= \int \tan^2(\theta)(1+\tan^2(\theta))(\sec^2(\theta)\,d\theta) && \text{(Pythagorean Identity)} \\ &= \int u^2(1+u^2)\,du \\ &= \int (u^2+u^4)\,du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C.\end{aligned}$$

Considering that $u = \tan(\theta) = \sqrt{x^2-1}$, it follows that

$$\int x^3\sqrt{x^2-1}\,dx = \frac{1}{3}(x^2-1)^{3/2} + \frac{1}{5}(x^2-1)^{5/2} + C. \qquad \diamond$$

Exercise 1.13.13. Use a trigonometric substitution to compute $\int (x^2-4)^{-3/2}\,dx$.

Exercise 1.13.14. Use a trigonometric substitution to compute $\int \sqrt{4x^2-9}\,dx$.

Exercise 1.13.15. Use a trigonometric substitution to compute $\int x^5\sqrt{x^2-16}\,dx$.

1.14 Partial Fraction Decomposition

We have thus far discussed several satisfactory techniques for integrating power functions, algebraic functions, exponential functions, logarithmic functions, trigonometric functions, and their products; however, we have not yet uniformly dealt with the problem of integrating **rational functions**. By definition, a rational function is a quotient of two polynomial expressions, e.g., the rational functions

$$\frac{1}{x^2+2x} \text{ and } \frac{x-2}{x-5} \text{ and } \frac{x^3-1}{x^2+1}.$$

We say that a rational function is **proper** if and only if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Of the displayed rational functions above, only the first is a proper rational function. By performing **polynomial long division**, we may convert any **improper** rational function into a linear combination of proper rational functions. Explicitly, we have that $x - 2 = (x - 5) + 3$ so that dividing each side by $x - 5$ yields that

$$\frac{x-2}{x-5} = 1 + \frac{3}{x-2}.$$

We may subsequently compute the antiderivative of this rational function by elementary methods.

$$\int \frac{x-2}{x-5}\,dx = \int \left(1 + \frac{3}{x-2}\right) dx = \int 1\,dx + \int \frac{3}{x-2}\,dx = x + 3\ln|x-2| + C$$

Likewise, by polynomial long division, we find that $x^3 - 1 = x(x^2+1) - (x+1)$ so that the improper rational function can be written as the following linear combination of proper rational functions.

$$\frac{x^3-1}{x^2+1} = x - \frac{x+1}{x^2+1} = x - \frac{x}{x^2+1} - \frac{1}{x^2+1}$$

Once again, the antiderivative of this rational function can be found with relative ease.

$$\int \frac{x^3-1}{x^2+1}\,dx = \int x\,dx - \int \frac{x}{x^2+1}\,dx - \frac{1}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) - \arctan(x) + C$$

Unfortunately, the antiderivative of the proper rational function $(x^2+2x)^{-1}$ cannot be obtained by any technique we have discussed so far; however, it is possible to integrate this function by noticing (quite cleverly) that it can be written as a difference of proper rational functions as follows.

$$\int \frac{1}{x^2+2x}\,dx = \int \left(\frac{1}{2x} - \frac{1}{2(x+2)}\right) dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx = \frac{1}{2}(\ln|x| - \ln|x+2|) + C$$

Essentially, the content of this observation is the method of **partial fraction decomposition**.

Before we delve into the method of partial fraction decomposition, we must continue to recall some important notions from college algebra. We say that a polynomial is **irreducible** if it cannot be written as a product of two polynomials of strictly lesser degree. Consequently, a linear polynomial $ax + b$ is irreducible; it can be shown that a quadratic polynomial is irreducible if and only if it has no roots. By the Quadratic Equation, the roots of a real quadratic polynomial $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that ax^2+bx+c is irreducible if and only if $b^2-4ac < 0$. We refer to the real number b^2-4ac as the **discriminant** of the quadratic: if this quantity is negative, the quadratic has only imaginary roots. One of the most useful (and nontrivial) facts about real polynomials is that the only irreducible polynomials with real coefficients are linear or quadratic. Put another way, it turns out that every real polynomial factors as a product of linear and irreducible quadratic polynomials.

Theorem 1.14.1 (Partial Fraction Decomposition Theorem)**.**

(a.) (Distinct Linear Factors) *Given any real numbers a, b, c, and d such that a and c are nonzero and $ax + b$ and $cx + d$ are distinct, there exist nonzero real numbers A and B such that*

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}.$$

(b.) (Powers of Distinct Linear Factors) *Given any real numbers a, b, c, and d such that a and c are nonzero and $ax+b$ and $cx+d$ are distinct and any pair of positive integers m and n, there exist real numbers $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ not all of which are zero such that*

$$\frac{1}{(ax+b)^m(cx+d)^n} = \sum_{i=1}^{m} \frac{A_i}{(ax+b)^i} + \sum_{j=1}^{n} \frac{B_j}{(cx+d)^j}.$$

(c.) (Linear and Irreducible Quadratic Factors) *Given any real numbers a, b, c, d, and e such that a and c are nonzero and $d^2 - 4ce < 0$, there exist real numbers A, B, C not all zero such that*

$$\frac{1}{(ax+b)(cx^2+dx+e)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+dx+e}.$$

(d.) (Distinct Irreducible Quadratic Factors) *Given any real numbers a, b, c, d, e, and f such that a and d are nonzero, ax^2+bx+c and dx^2+ex+f are distinct, $b^2-4ac < 0$, and $e^2-4df < 0$, there exist real numbers A, B, C, and D not all of which are zero such that*

$$\frac{1}{(ax^2+bx+c)(dx^2+ex+f)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{dx^2+ex+f}.$$

(e.) (Powers of Distinct Irreducible Quadratic Factors) *Given any pair of positive integers m and n and any real numbers a, b, c, d, e, and f such that a and d are nonzero, $b^2 - 4ac < 0$, $e^2 - 4df < 0$, and ax^2+bx+c and dx^2+ex+f are distinct, there exist real numbers $A_1, \ldots, A_m, B_1, \ldots, B_m, C_1, \ldots, C_n$, and $D_1, D_2, \ldots, D_n$ not all of which are zero such that*

$$\frac{1}{(ax^2+bx+c)^m(dx^2+ex+f)^n} = \sum_{i=1}^{m} \frac{A_ix+B_i}{(ax^2+bx+c)^i} + \sum_{j=1}^{n} \frac{C_jx+D_j}{(dx^2+ex+f)^j}.$$

Even more, these are all of the possible cases of proper rational functions with numerator 1.

Example 1.14.2. Use the Partial Fraction Decomposition Theorem to compute $\displaystyle\int \frac{1}{x^2-5x-6}\,dx$.

Solution. Observe that $x^2-5x-6 = (x-6)(x+1)$ is a factorization of x^2-5x-6 into distinct linear factors, hence the method of partial fraction decomposition yields that

$$\frac{1}{x^2-5x-6} = \frac{A}{x-6} + \frac{B}{x+1}.$$

Clearing denominators and using the fact that $(x-6)(x+1) = x^2-5x-6$, we find that

$$1 = A(x+1) + B(x-6).$$

By setting $x = 6$, we find that $1 = 7A$ so that $A = \frac{1}{7}$. By setting $x = -1$, we find that $1 = -7B$ so that $B = -\frac{1}{7}$. Consequently, the method of partial fraction decomposition reveals that

$$\frac{1}{x^2-5x-6} = \frac{\frac{1}{7}}{x-6} + \frac{-\frac{1}{7}}{x+1}.$$

We may therefore return to compute our indefinite integral with elementary techniques.

$$\int \frac{1}{x^2-5x-6}\,dx = \frac{1}{7}\int \frac{1}{x-6}\,dx - \frac{1}{7}\int \frac{1}{x+1}\,dx = \frac{1}{7}\ln|x-6| - \frac{1}{7}\ln|x+1| + C. \qquad \diamond$$

Example 1.14.3. Use the method of partial fraction decomposition to compute $\int (x^4-1)^{-1}\,dx$.

Solution. Observe that $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$ is a factorization of x^4-1 into distinct linear and quadratic factors. Considering that $0-4(1)(1)=-4<0$, it follows that x^2+1 is irreducible. Using the method of partial fraction decomposition, it follows that

$$\frac{1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}.$$

Clearing denominators and using the fact that $(x-1)(x+1)=x^2-1$, we find that

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1).$$

Considering that this identity holds for all x, it follows that $4A=1$ by plugging in $x=1$, $-4B=1$ by plugging in $x=-1$, and $A-B-D=1$ by plugging in $x=0$. We find immediately that

$$A=\frac{1}{4},\ B=-\frac{1}{4},\ \text{and } D=A-B-1=\frac{1}{2}-1=-\frac{1}{2}.$$

Expanding the polynomial on the right in the second-to-last displayed equation, we find that

$$0x^3+1=1=(A+B+C)x^3+\text{some polynomial of degree at most two.}$$

Comparing coefficients gives that $A+B+C=0$ so that $C=0$. We conclude that

$$\frac{1}{x^4-1}=\frac{\frac{1}{4}}{x-1}-\frac{\frac{1}{4}}{x+1}-\frac{\frac{1}{2}}{x^2+1},$$

from which it follows that

$$\int \frac{1}{x^4+1}\,dx=\frac{1}{4}\int\frac{1}{x-1}\,dx-\frac{1}{4}\int\frac{1}{x+1}-\frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan(x)+C. \qquad \diamond$$

Caution: it is not necessarily always possible to eliminate variables by plugging in carefully chosen values $x=a$ when implementing the method of partial fraction decomposition. Ultimately, it is in fact best to use the **method of undetermined coefficients**, as outlined in our next example.

Example 1.14.4. Use the Partial Fraction Decomposition Theorem to compute $\displaystyle\int \frac{2x+1}{x^4+2x^2+1}\,dx$.

Solution. Observe that $x^4+4x^2+3=(x^2+1)(x^2+3)$ is a factorization of x^4+4x^2+3 into distinct irreducible factors. Using the method of partial fraction decomposition, we have that

$$\frac{2x+1}{(x^2+1)(x^2+3)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}.$$

Clearing denominators, we find that

$$2x+1=(Ax+B)(x^2+3)+(Cx+D)(x^2+1).$$

Considering that $x^2 + 1$ and $x^2 + 3$ are irreducible, we cannot eliminate either of these quadratic factors by substituting $x = a$ for any real number a. Consequently, we must compare coefficients. Expanding the right-hand side in the second-to-last displayed equation, we find that

$$2x + 1 = (A + C)x^3 + (B + D)x^2 + (3A + C)x + 3B + D,$$

from which we obtain the following linear system of equations.

$$\begin{aligned} A + C &= 0 \qquad & 3A + C &= 2 \\ B + D &= 0 \qquad & 3B + D &= 1 \end{aligned}$$

We have therefore that $A = -C$ and $B = -D$ so that $2 = -3C + C = -2C$ and $1 = -3D + D = -2D$. We conclude that $A = 1$, $B = \frac{1}{2}$, $C = -1$, and $D = -\frac{1}{2}$, from which it follows that

$$\begin{aligned} \int \frac{2x+1}{x^4+4x^2+3}\,dx &= \int \left(\frac{x+\frac{1}{2}}{x^2+1} - \frac{x+\frac{1}{2}}{x^2+3} \right) dx \\ &= \frac{1}{2}\int \frac{2x+1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x+1}{x^2+3}\,dx \\ &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - \frac{1}{2}\int \frac{1}{x^2+3}\,dx \\ &= \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\arctan(x) - \frac{1}{2}\ln|x^2+3| - \frac{1}{2\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C, \end{aligned}$$

where the last integral is determined by $x^2 + 3 = 3\left[\left(\frac{x}{\sqrt{3}}\right)^2 + 1\right]$ and the substitution $u = \frac{x}{\sqrt{3}}$. ◇

Example 1.14.5. Use the Partial Fraction Decomposition Theorem to compute $\displaystyle\int \frac{1}{x^2-1}\,dx$.

Exercise 1.14.6. Use the method of partial fraction decomposition to compute $\displaystyle\int \frac{2x+3}{x^3-2x^2+4x-8}\,dx$.

Observe that the method of partial fraction decomposition applies to proper rational functions; however, by polynomial long division, every rational function induces a proper rational function.

Example 1.14.7. Use polynomial long division to express the following rational function as the sum of a polynomial and a proper rational function; then, compute its indefinite integrals.

$$f(x) = \frac{x^3+1}{x^2+x+1}$$

Solution. We proceed by polynomial long division. Our task is to sequentially eliminate the largest power of x in each polynomial that appears as the **dividend** in the long division.

1.) Our dividend is $x^3 + 1$, and our **divisor** is $x^2 + x + 1$. Observe that

$$(x^3 + 1) - x(x^2 + x + 1) = (x^3 + 1) - (x^3 + x^2 + x) = -x^2 - x + 1.$$

2.) Our dividend is now $-x^2 - x + 1$, and our divisor is $x^2 + x + 1$. Observe that

$$(-x^2 - x + 1) - (-1)(x^2 + x + 1) = (-x^2 - x + 1) + (x^2 + x + 1) = 2.$$

3.) Our dividend of 2 has lesser degree than $x^2 + x + 1$, so the division terminates.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad x - 1 \\ \hline x^2 + x + 1 & x^3 \quad\quad\quad\quad\quad + 1 \\ & -x^3 - x^2 - x \\ \hline & \quad\quad -x^2 - x + 1 \\ & \quad\quad\;\; x^2 + x + 1 \\ \hline & \quad\quad\quad\quad\quad\quad 2 \end{array}$$

Ultimately, we find that $x^3 + 1 = (x - 1)(x^2 + x + 1) + 2$ so that

$$\frac{x^3 + 1}{x^2 + x + 1} = x - 1 + \frac{2}{x^2 + x + 1}.$$

Considering that $1^2 - 4(1)(1) = -3 < 0$, it follows that $x^2 + x + 1$ is an irreducible quadratic polynomial, hence the method of partial fraction decomposition fails to improve the situation here; rather, we may revert to the method of **completing the square** to find that

$$x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

By setting $u = x + \frac{1}{2}$, we find that $du = dx$ so that

$$\int \frac{2}{x^2 + x + 1}\,dx = 2\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = 2\int \frac{1}{u^2 + \frac{3}{4}}\,du = \frac{8}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}u\right)^2 + 1}\,du.$$

One can perform a substitution $t = \frac{2}{\sqrt{3}}u$ with $dt = \frac{2}{\sqrt{3}}\,du$ or simply recognize this integral as

$$\frac{8}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}u\right)^2 + 1}\,du = \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}u\right) + C = \frac{4}{\sqrt{3}}\arctan\left(\frac{2x + 1}{\sqrt{3}}\right) + C.$$

Ultimately, we conclude that the function has the following general antiderivative.

$$\int \frac{x^3 + 1}{x^2 + x + 1}\,dx = \int \left(x - 1 + \frac{2}{x^2 + x + 1}\right)dx = \frac{1}{2}x^2 - x + \frac{4}{\sqrt{3}}\arctan\left(\frac{2x + 1}{\sqrt{3}}\right) + C \qquad \diamond$$

Example 1.14.8. Use polynomial long division to express the following rational functions as the sum of a polynomial and a proper rational function; then, compute their indefinite integrals.

(a.) $\dfrac{x^3 + 1}{x^2 + x + 1}$ (b.) $\dfrac{x^4 - x^2 + 1}{x^2 - 1}$ (c.) $\dfrac{x^5 - 4x^4 + 9x^2 - 6}{x^3 + x^2 - x - 1}$

1.15 Improper Integration

Our interest in integrals so far has been to find the net area bounded by the curve $f(x)$. Because of this, we have restricted ourselves to closed and bounded intervals of the form $[a, b]$. Often, we are interested in how a mathematical model behaves in the long-run, i.e., as x grows arbitrarily large (or approaches $\pm\infty$). Under this framework, we develop the concept of the improper integral.

Given a function $f(x)$ that is integrable over the closed region $[a, b]$ for every real number $b > a$, the **improper integral** of $f(x)$ over the interval $[a, \infty)$ is defined (if it exists) as

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

By the Fundamental Theorem of Calculus, Part I, for any antiderivative $F(x)$ of $f(x)$, we have that

$$\lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}[F(b) - F(a)].$$

One can analogously define the improper integral of $f(x)$ over the interval $(-\infty, b]$ as

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

whenever $f(x)$ is integrable over the closed and bounded interval $[a, b]$ for every real numbers $a < b$. Even more, the doubly improper integral of $f(x)$ over $(-\infty, \infty)$ is defined as

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{b\to\infty}\left(\lim_{a\to-\infty}\int_a^b f(x)\,dx\right) = \lim_{a\to-\infty}\left(\lim_{b\to\infty}\int_a^b f(x)\,dx\right)$$

whenever $f(x)$ is integral over the closed and bounded interval $[a, b]$ for all real numbers a and b.

Exercise 1.15.1. Compute the improper integral $\int_1^\infty x^{-2}\,dx$.

Exercise 1.15.2. Compute the improper integral $\int_{-\infty}^1 e^x\,dx$.

Exercise 1.15.3. Compute the improper integral $\int_0^\infty xe^{-x}\,dx$.

Exercise 1.15.4. Compute the improper integral $\int_{-\infty}^\infty (1 + x^2)^{-1}\,dx$.

Exercise 1.15.5. Compute the improper integral $\int_{-\infty}^\infty xe^{-x^2}\,dx$.

Each of the above functions admits horizontal asymptotes, hence the improper integrals we computed were all finite, and the ends of our computations justified the means.

One can also consider the improper integral of a function with a vertical asymptote. Given that $f(x)$ is continuous on the half-open interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$, we have that

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx = \lim_{t\to b^-}[F(t) - F(a)]$$

for any antiderivative $F(x)$ of $f(x)$ (if this limit exists). One can analogously define the improper integral of $f(x)$ over the half-open interval $(a, b]$ whenever $\lim_{x\to a^+} f(x) = \pm\infty$ (provided it exists) as

$$\int_a^b f(x)\,dx = \lim_{u\to a^+}\int_u^b f(x)\,dx = \lim_{u\to a^+}[F(b) - F(u)].$$

Even if the integrand $f(x)$ is unbounded as $x > a$ approaches a and as $x < b$ approaches b, it is still possible to define the doubly improper integral of $f(x)$ over (a, b) as

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \left(\lim_{u \to a^+} \int_u^t f(x)\,dx \right) = \lim_{u \to a^+} \left(\lim_{t \to b^-} \int_u^t f(x)\,dx \right)$$

provided that $f(x)$ is integrable over the closed interval $[u, t]$ for all real numbers $a < u < t < b$.

Exercise 1.15.6. Compute the improper integral $\int_0^1 (x-1)^{-1}\,dx$.

Exercise 1.15.7. Compute the improper integral $\int_0^1 x^{-1/2}\,dx$.

Exercise 1.15.8. Compute the improper integral $\int_{-1}^1 x^{-2/3}\,dx$.

Conventionally, we say that an improper integral **converges** whenever the limit of definition exists, and we say that it **diverges** if the limit does not exist. Even if we cannot explicitly compute an improper integral, the **Comparison Theorem** allows us to say whether it converges or diverges.

Theorem 1.15.9 (Comparison Theorem for Improper Integrals)**.** *Consider any pair of continuous functions $f(x)$ and $g(x)$ such that $f(x) \geq g(x) \geq 0$ for all real numbers $x \geq a$.*

(a.) *If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.*

(b.) *If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.*

One can make analogous statements for the improper integrals $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^b g(x)\,dx$, doubly improper integrals, and improper integrals of a function with a vertical asymptote.

Exercise 1.15.10. Determine if the improper integral $\int_0^\infty xe^x\,dx$ converges.

Exercise 1.15.11. Determine if the improper integral $\int_0^\infty x^{-2}\sin^2(x)\,dx$ converges.

Chapter 2

Integration: Applications and Modeling

2.1 The Area Between Curves

Our introduction to the notion of integration already gave us an interpretation of the definite integral $\int_a^b f(x)\,dx$ as the net area bounded by the graph of the curve $f(x)$ and the x-axis. Consequently, there are myriad benefits of using a definite integral to capture information about real-life observations: the Net Change Theorem states that if $f'(x)$ is the derivative of an integrable function $f(x)$, then the definite integral $\int_a^b f'(x)\,dx = f(b) - f(a)$ measures the net change of $f(x)$ over the closed interval $[a, b]$. For instance, if $f'(t)$ is the velocity of a particle observed from time $t = a$ to time $t = b$, then the definite integral $f(b) - f(a) \int_a^b f'(t)\,dt$ is the net displacement of the particle (i.e., the net distance traveled by the particle) during the time frame in which we observed it.

Crucially, we can view the x-axis of the Cartesian plane as the curve $y = g(x) = 0$, hence if $f(x)$ satisfies that $f(x) \geq g(x) = 0$ (i.e., $f(x)$ is non-negative) for all real numbers x such that $a \leq x \leq b$, then the definite integral $\int_a^b f(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$ measures the area between the curves $f(x)$ and $g(x)$. Generalizing this notion gives us a way to measure the area between any two curves $f(x)$ and $g(x)$ satisfying $f(x) \geq g(x)$ for all real numbers x such that $a \leq x \leq b$.

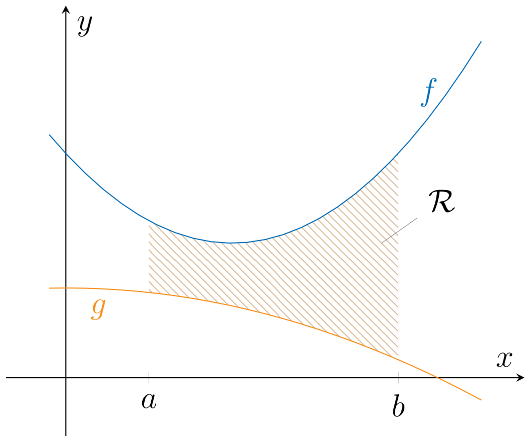
Formula 2.1.1 (Area Formula for a Region Bounded by Two Curves)**.** *Consider any pair of functions $f(x)$ and $g(x)$ satisfying that $f(x) \geq g(x)$ for all real numbers x such that $a \leq x \leq b$. Provided that $f(x)$ and $g(x)$ are both integrable on $[a, b]$, the curves $f(x)$, $g(x)$, $x = a$, and $x = b$ bound a region $\mathcal{R}$ in the Cartesian plane of finite area. Explicitly, the area of this region is given by*

$$\text{area}(\mathcal{R}) = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Basically, the proof of this formula boils down to the fact if $f(x)$ and $g(x)$ are both integrable functions, then for any choice of partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of the interval $[a, b]$ and any choice of sample points x_i^*, the limit that defines the integral of $f(x) - g(x)$ over $[a, b]$ exists.

$$\int_a^b [f(x) - g(x)]\,dx = \lim_{n\to\infty} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)]\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n [f(x_i^*)]\,\Delta x_i - \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\,\Delta x_i$$

We will repeatedly see this kind of rationale applied throughout this chapter of the lecture notes: if we want to compute area, volume, moment of inertia, work, etc., we can approximate by a Riemann sum and take a limit to reduce the error of our approximation to zero.



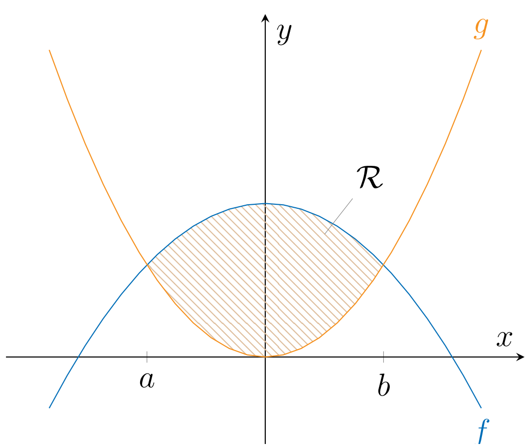
Example 2.1.2. Consider the functions $f(x) = x^2 + 1$ and $g(x) = 2x$. Observe that $f(x) = g(x)$ if and only if $x^2 + 1 = 2x$ if and only if $x^2 - 2x + 1 = 0$ if and only if $(x-1)^2 = 0$ if and only if $x = 1$. By the same rationale, we have that $f(x) \geq g(x)$ for all real numbers $x \geq 1$ because $f(x) - g(x) = (x-1)^2 \geq 0$ for all real numbers $x \geq 1$. We can therefore consider the region $\mathcal{R}$ bounded by the curves $f(x) = x^2 + 1$, $g(x) = 2x$, and $x = 4$; the area of this region is given by

$$\text{area}(\mathcal{R}) = \int_1^4 [f(x) - g(x)]\, dx = \int_1^4 (x-1)^2\, dx = \left[\frac{(x-1)^3}{3}\right]_1^4 = 9.$$

Example 2.1.3. Consider the functions $f(x) = e^{2x}$ and $g(x) = e^x$. Observe that $f(x) = g(x)$ if and only if $e^{2x} = e^x$ if and only if $e^{2x} - e^x = 0$ if and only if $e^x(e^x - 1) = 0$ if and only if $e^x - 1 = 0$ if and only if $e^x = 1$ if and only if $x = 0$. Even more, we have that $e^{2x} \geq e^x$ for all real numbers $x \geq 0$ because $e^{2x} - e^x = e^x(e^x - 1)$ and $e^x - 1 \geq 0$ for all real numbers $x \geq 0$. Consequently, the region $\mathcal{R}$ bounded by the curves $f(x) = e^{2x}$, $g(x) = e^x$, and $x = \ln(10)$ has area given by

$$\text{area}(\mathcal{R}) = \int_0^{\ln(10} [f(x) - g(x)]\, dx = \int_0^{10} (e^{2x} - e^x)\, dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^{\ln(10)} = \frac{81}{2}.$$

Often, we will be interested in the region bounded by two curves $f(x)$ and $g(x)$ satisfying that $f(a) = g(a)$, $f(b) = g(b)$, and $f(x) \geq g(x)$ for all real numbers x such that $a \leq x \leq b$.



Below, we outline the basic strategy to find the area bounded by curves $f(x)$ and $g(x)$ such that $f(a) = g(a)$, $f(b) = g(b)$, and $f(x) \geq g(x)$ for all real numbers x such that $a \leq x \leq b$.

Algorithm 2.1.4 (Determining the Region Bounded by Two Curves)**.** Complete the following steps to determine the area bounded by the graphs of some curves $f_1(x)$, $f_2(x)$, $x = a$, and $x = b$.

(i.) Provided that the equation $f_1(x) = f_2(x)$ is readily solved by algebraic methods, one can find the intersection point(s) of $f_1(x)$ and $f_2(x)$ by solving this equation in terms of x.

(ii.) By plugging in points, we can determine whether if $f_1(x) \geq f_2(x)$ or $f_1(x) \leq f_2(x)$ on the interval. Label the larger function as $f(x)$, and label the smaller function as $g(x)$.

(iii.) By the Area Formula for a Region Bounded by Two Curves, the area of the region $\mathcal{R}$ bounded by the curves can be found using $x = a$, $x = b$, and the intersection points of $f_1(x)$ and $f_2(x)$.

(iv.) Conversely, if the equation $f_1(x) = f_2(x)$ is difficult to solve, then choose several x-values so that the value of the function $f_1(x)$ is known (or can easily approximate and accurately plotted on a graph). Use the rule of thumb that if f is a polynomial of degree n, it is best to choose $n+1$ different x-values to plot $f(x)$; use at least four points for other functions.

(v.) Plot the corresponding points $(x, f_1(x))$, and use these to sketch the graph of $f_1(x)$.

(vi.) Repeat the second and third steps of the algorithm for the function $f_2(x)$.

(vii.) Label the top function as $f(x)$ and the bottom function as $g(x)$ based on the graph.

(viii.) Use the Area Formula for a Region Bounded by Two Curves to compute the area of $\mathcal{R}$.

Example 2.1.5. Compute the area of the region $\mathcal{R}$ bounded by $y = -x^2 + 4$ and $y = x^2 - 4$.

Solution. We must first determine the intersection points of the curves $y = -x^2+4$ and $y = x^2-4$. Consequently, we solve the equation $-x^2 + 4 = x^2 - 4$. We find that $2x^2 = 8$ so that $x^2 = 4$ and $x = -2$ or $x = 2$. Even more, the inequality $-x^2+4 \geq x^2-4$ holds for all real numbers x such that $-2 \leq x \leq 2$ because $-0^2 + 4 = 4 > -4 = 0^2 - 4$. We conclude therefore that

$$\text{area}(\mathcal{R}) = \int_{-2}^{2} [(-x^2 + 4) - (x^2 - 4)]\, dx = \int_{-2}^{2} (-2x^2 + 8)\, dx = \left[-\frac{2}{3}x^3 + 8x\right]_{-2}^{2} = \frac{64}{3}. \qquad \diamond$$

Example 2.1.6. Compute the area of the region $\mathcal{R}$ bounded by the curves $y = 2x+1$, $y = 2x-4$, $x = -1$, and $x = 2$; then, use geometry to verify that this area is correct. Last, discuss what would happen if the curves $x = -1$ and $x = 2$ were not given in the problem statement.

Solution. Considering that $5 = (2x+1) - (2x-4) > 0$, it follows that $2x+1 > 2x-4$ for all real numbers x. (One can visualize these curves as parallel lines of slope 2.) Consequently, we find that

$$\text{area}(\mathcal{R}) = \int_{-1}^{2} [(2x+1) - (2x-4)]\, dx = \int_{-1}^{2} 5\, dx = 5[2 - (-1)] = 15.$$

We can verify this geometrically: indeed, the region $\mathcal{R}$ is the union of two right triangles with base of length $3 = 2 - (-1)$ and height $5 = 2(2) + 1$, hence the area is $2(1/2)(3)(5) = 15$.

Last, observe that if the curves $x = -1$ and $x = 2$ were not given in the problem statement, then the area bounded by the curves $y = 2x+1$ and $y = 2x-4$ is infinite! $\diamond$

Exercise 2.1.7. Compute the area of the region $\mathcal{R}$ bounded by the curves $y = \sqrt{x}$ and $y = x^2$.

Generally, we say that a region $\mathcal{R}$ in the Cartesian plane is **vertically simple** if there exist curves $y = f_1(x)$, $y = f_2(x)$, $x = a$, and $x = b$ such that every point (x, y) in the region $\mathcal{R}$ satisfies that $a \leq x \leq b$ and $f_1(x) \leq y \leq f_2(x)$. Graphically, the curve $y = f_1(x)$ can be viewed as the "bottom" of the region $\mathcal{R}$, and the curve $y = f_2(x)$ can be viewed as the "top" of $\mathcal{R}$. Commonly, we will refer to $f_1(x)$ as y_{bottom} and $f_2(x)$ as y_{top}. Below is a reformulation of the above area formula.

Formula 2.1.8 (Area Formula for a Vertically Simple Region). *Given a vertically simple region $\mathcal{R}$ bounded by the curves $y_{\text{top}} = f_2(x)$, $y_{\text{bottom}} = f_1(x)$, $x = a$, and $x = b$, we have that*

$$\operatorname{area}(\mathcal{R}) = \int_a^b (y_{\text{top}} - y_{\text{bottom}})\, dx = \int_a^b [f_2(x) - f_1(x)]\, dx.$$

Our regions have been thus far vertically simple, hence we have been able to compute their areas using the above formula. Unfortunately, there exist regions that are not vertically simple.

Exercise 2.1.9. Prove that the region $\mathcal{R}$ bounded by the curves $y = x$, $y = -x$, and $y = -2$ is not vertically simple; then, express $\mathcal{R}$ as the union of two vertically simple regions $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, and find the area of $\mathcal{R}$ by using the fact that $\operatorname{area}(\mathcal{R}_1 \cup \mathcal{R}_2) = \operatorname{area}(\mathcal{R}_1) + \operatorname{area}(\mathcal{R}_2)$.

Exercise 2.1.9 exhibits a region $\mathcal{R}$ that is not vertically simple; however, if we tilt our head to the side, then we would see a vertically simple region. Explicitly, we say that the region $\mathcal{R}$ is **horizontally simple** if there exist curves $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$ such that every point (x, y) in the region $\mathcal{R}$ satisfies that $g_1(y) \leq x \leq g_2(y)$ and $c \leq y \leq d$. Graphically, the curve $x = g_1(y)$ is the "left" function, and the curve $x = g_2(y)$ is the "right" function. Like before, we will refer to $g_1(y)$ as x_{left} and $g_2(y)$ as x_{right}, so we have the following area formula.

Formula 2.1.10 (Area Formula for a Horizontally Simple Region). *Given a horizontally simple region $\mathcal{R}$ bounded by the curves $x_{\text{right}} = g_2(y)$ and $x_{\text{left}} = g_1(y)$, $y = c$, and $y = d$, we have that*

$$\operatorname{area}(\mathcal{R}) = \int_c^d (x_{\text{right}} - x_{\text{left}})\, dy = \int_c^d [g_2(y) - g_1(y)]\, dy.$$

Example 2.1.11. Prove that the region $\mathcal{R}$ of Example 2.1.9 is horizontally simple by exhibiting well-defined curves $x_{\text{left}} = g_1(y)$, $x_{\text{right}} = g_2(y)$, $y = c$, and $y = d$; then, compute the area of $\mathcal{R}$.

Solution. Observe that the curves $x = g_1(y) = y$ and $x = g_2(y) = -y$ intersect at $y = 0$. Even more, for all real numbers y such that $-2 \leq y \leq 0$, we have that $0 \leq -y \leq 2$ so that $g_1(y) \leq x \leq g_2(y)$ for all real numbers $-2 \leq y \leq 0$. We conclude that the region $\mathcal{R}$ is horizontally simple with $x_{\text{left}} = y$ and $x_{\text{right}} = -y$. By the Area Formula for a Horizontally Simple Region, we conclude that

$$\operatorname{area}(\mathcal{R}) = \int_{-2}^0 (x_{\text{right}} - x_{\text{left}})\, dy = \int_{-2}^0 (-y - y)\, dy = \int_{-2}^0 -2y\, dy = \left[-y^2\right]_{-2}^0 = 4. \qquad \diamond$$

Example 2.1.12. Compute the area of the region $\mathcal{R}$ bounded by $x = \sqrt{1 - y^2}$ and $x = 0$.

Solution. Observe that $\sqrt{1 - y^2} = 0$ if and only if $1 - y^2 = 0$ if and only if $y^2 = 1$ if and only if $y = -1$ or $y = 1$. Even more, for all real numbers y such that $-1 \leq y \leq 1$, we have that

$\sqrt{1-y^2} \geq 0$, hence the region $\mathcal{R}$ is horizontally simple with $x_{\text{left}} = 0$ and $x_{\text{right}} = \sqrt{1-y^2}$. By the Area Formula for a Horizontally Simple Region, we conclude that

$$\text{area}(\mathcal{R}) = \int_{-1}^{1} (x_{\text{right}} - x_{\text{left}})\, dy = \int_{-1}^{1} \sqrt{1-y^2}\, dy = \frac{\pi}{2}.$$

Explicitly, the integral can be evaluated by elementary geometry since $\mathcal{R}$ is half of the unit circle. ◇

Unfortunately, it is also possible for a region to be neither vertically nor horizontally simple.

Example 2.1.13. Prove that the region $\mathcal{R}$ bounded by the curves $y = x-2$, $y = 2-x$, $y = -x+2$, and $y = -x-2$ is neither vertically nor horizontally simple; then, compute the area of $\mathcal{R}$.

Solution. Graphing the region, we find that $\mathcal{R}$ is not vertically simple: indeed, for all real numbers x such that $-2 \leq x \leq 0$, we have that $-x-2 \leq y \leq x+2$; however, for all real numbers x such that $0 \leq x \leq 2$, we have that $x-2 \leq y \leq -x+2$. Consequently, the top and bottom curves of $\mathcal{R}$ are not well-defined. By the symmetry of $\mathcal{R}$, it is not horizontally simple, either.

On the bright side, as our above analysis illustrates, we can represent $\mathcal{R}$ as the union of two vertically simple regions: indeed, we have that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ for the vertically simple regions

$$\mathcal{R}_1 = \{(x,y) \mid -2 \leq x \leq 0 \text{ and } -x-2 \leq y \leq x+2\} \text{ and}$$
$$\mathcal{R}_2 = \{(x,y) \mid 0 \leq x \leq 2 \text{ and } x-2 \leq y \leq -x+2\}.$$

Consequently, in view of the fact that $\text{area}(\mathcal{R}) = \text{area}(\mathcal{R}_1) + \text{area}(\mathcal{R}_2)$, we find that

$$\text{area}(\mathcal{R}) = \int_{-2}^{0} [(x+2) - (-x-2)]\, dx + \int_{0}^{2} [(-x+2) - (x-2)]\, dx$$

$$= \int_{-2}^{0} (2x+4)\, dx + \int_{0}^{2} (-2x+4)\, dx = \left[x^2+4x\right]_{-2}^{0} + \left[-x^2+4x\right]_{0}^{2} = 8. \quad ◇$$

Our above exposition completely determines how to compute the area of a region as soon as we can identify it as vertically or horizontally simple; however, there remains some nuance to these types of problems. Our previous example establishes the existence of regions that are neither vertically nor horizontally simple, so the question remains as to how we deal with these. One strategy is to break up such a region into subregions that are either vertically or horizontally simple. (Later, in Calculus III, we will learn the change of variables method that will make this issue more manageable.)

On the other hand, it is also completely possible that we are handed a region that is both vertically and horizontally simple, and the description of the region as vertically simple renders the integral infeasible to compute. Our best bet in this case is to check the description of the region as horizontally simple and hope that the integrand works out to be nicer in this lens.

Example 2.1.14. Compute $-\int_0^1 \ln(x)\, dx$ by viewing it as the area of some region $\mathcal{R}$.

Solution. Considering that $\ln(x) \leq 0$ for all real numbers x such that $0 < x \leq 1$, it follows that $-\int_0^1 \ln x\, dx$ is the area of the region $\mathcal{R}$ bounded by the curves $y = 0$, $y = \ln(x)$, $x = 0$, and $x = 1$. Consequently, we may view $\mathcal{R}$ as the horizontally simple region bounded by the curves $x = 0$, $x = e^y$, and $y = 0$. By the Area Formula for a Horizontally Simple Region, we conclude that

$$\int_0^1 \ln(x)\, dx = \text{area}(\mathcal{R}) = \int_{-\infty}^{0} e^y\, dy = \lim_{a \to -\infty} \int_a^0 e^y\, dy = \lim_{a \to -\infty} [e^y]_a^0 = \lim_{a \to -\infty} (1 - e^a) = 1. \quad ◇$$