Research Statement

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Broadly, my mathematical research interests lie in the field of commutative algebra. Put simply, commutative algebra is the study of basic arithmetic operations such as addition and multiplication. We refer to a set R that contains 0 as a *ring* whenever addition, subtraction, and multiplication can be performed on any pair of objects in R such that the resulting sum, difference, or product belongs to R and the familiar distributive laws of addition and multiplication hold. Given that the product of two elements r and s of R satisfies $r \cdot s = s \cdot r$, we say that R is *commutative*; if there exists a unique element 1 in R such that $1 \cdot r = r = r \cdot 1$ holds for every element r of R, we say that R is *unital*.

Even though their structure is completely determined by the operations of addition and multiplication, commutative unital rings enjoy a wealth of interesting and beautiful properties. Essential to our understanding is the introduction of additional algebraic structures associated to the ring. Particularly, we study the *ideals* and *modules* associated to a ring. Essentially, an ideal I of a commutative ring R is a collection of ring elements such that (a.) any two elements of I can be added together to produce a element of I and (b.) any element of the ideal can be multiplied by an arbitrary ring element to produce an element of I. On the other hand, a module M over a commutative ring R is a collection of elements (not necessarily belonging to R) such that any pair of elements can be added and multiplication can be performed on any pair (r, m) such that r lies in R and m lies in M.

Polynomials give rise to a familiar class of commutative rings, e.g., the ring $\mathbb{Q}[x]$ of polynomials with rational coefficients. Polynomials behave intuitively; however, if we allow ourselves only to work with the variables x^s for some positive integers s, things become much more interesting. For instance, the polynomial $f(x) = 3x^8 - x^5 + x^2 + 1$ makes sense in $\mathbb{Q}[x]$, but if we restrict ourselves to polynomials in the variables x^3 and x^7, then $f(x)$ is no longer feasible. Explicitly, the only powers of x appearing in such a polynomial must be of the form $3a + 7b$ for some non-negative integers a and b. Consequently, polynomials can be used to model more sophisticated problems in additive number theory — namely, the so-called "membership problem" (or Frobenius problem) of a *numerical semigroup*.

Even more, we may associate to any finite simple graph G on n vertices a squarefree monomial ideal $I(G)$ of the n-variate polynomial ring over a field. By this correspondence, we may deduce information about G via the properties of $I(G)$ and vice-versa. Further, one can show that the ideal $I(G)$ is the *Stanley-Reisner ideal* associated to the *independence complex* of the graph G. Ultimately, these identifications forge a strong relationship — known as Stanley-Reisner theory — between the study of polynomial rings and combinatorics that has been fruitfully implemented for decades.

Understanding the behavior of objects such as polynomials in these various settings forms the basis of my current research, on which I elaborate in the following sections.

1 Numerical Semigroups and Numerical Semigroup Rings

Univariate polynomial rings over fields exhibit well-studied and well-understood properties; however, if we restrict our attention to polynomials in the variables x^s for some positive integers s, it is immediately more complicated to determine whether a polynomial in x can be written as a polynomial in the x^ss. For instance, the polynomial $x^{12} - 17x^7 + 3x^4 + 2x^3$ can be written as a polynomial in x^3 and x^4 because $12 = 3 \cdot 4$ and $7 = 3 + 4$ implies that $x^{12} = (x^4)^3 = (x^3)^4$ and $x^7 = x^3 \cdot x^4$; however, the polynomial $x^5 - x^2 + x$ cannot be written as a polynomial in x^3 and x^4.

Considering that multiplication of the monomials x^a and x^b corresponds to addition of the exponents $a + b$, understanding this "polynomial membership problem" reduces to understanding the "additive membership problem" on the exponents of the monomials that generate the so-called *monomial subring*. Based on this observation, it is fruitful to study the *numerical semigroup* S consisting of the non-negative integer powers of the monomials x^s to which we restrict our consideration.

Let k be a field. Consider the monomial subring $k[x^s \mid s \in S]$ consisting of all polynomials in the monomials x^s for some subset $S \subseteq \mathbb{Z}_{\geq 0}$ of the non-negative integers that contains 0, is closed under addition, and satisfies $\mathbb{Z}_{\geq 0} \setminus S$ is finite. We refer to the set S as a *numerical semigroup* and to the commutative unital ring $k[S] = [x^s \mid s \in S]$ as its corresponding *numerical semigroup ring*. Crucially, observe that x^a belongs to $k[S]$ if and only if the integer a belongs to S. Observe that the *Frobenius number* $\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S)$ is well-defined. We define the *pseudo-Frobenius numbers* of S as

$$\mathrm{PF}(S) = \{n \in \mathbb{Z}_{\geq 0} \setminus S \mid n + s \in S \text{ for all nonzero elements } s \in S\}.$$

One can show that there exist relatively prime integers $a_1 < \cdots < a_n \in S$ such that S consists precisely of all $\mathbb{Z}_{\geq 0}$-linear combinations of these elements $a_1, \ldots, a_n$, i.e., we have that $S = \mathbb{Z}_{\geq 0}\langle a_1, \ldots, a_n \rangle$. Further, one can take the generators $a_1, \ldots, a_n$ to be minimal among all sets of generators of S. We refer to the minimum number of generators of S as the *embedding dimension* $\mu(S)$; the least positive integer belonging to S is its *multiplicity* $e(S)$. We note that $\mu(S) \leq e(S)$ must always hold.

On their own, numerical semigroups have been studied at length by the likes of Fröberg (cf. [Fr94]), García-Sánchez (cf. [AsGS, DelG-SR-P, G-SR99, G-SR09]), and Herzog (cf. [He69, HeHiSta21, HeKumSta]) because the numerical semigroup rings constitute an interesting class of one-dimensional Cohen-Macaulay rings whose properties are intimately connected with the properties of the corresponding numerical semigroup. For instance, one can show that the embedding dimension and (Hilbert-Samuel) multiplicity of $k[S]$ and S are equal. Kunz showed that the numerical semigroup ring $k[S]$ is Gorenstein if and only if the numerical semigroup S is symmetric (cf. [Kun]). Further, all numerical semigroups of embedding dimension two are Gorenstein. In a landmark result [He69, Theorem 4.2.1] of his 1969 thesis, Herzog showed that a numerical semigroup of embedding dimension three is Gorenstein if and only if it is a complete intersection. Later, Fröberg demonstrated that the Cohen-Macaulay type of $k[S]$ is equal to the cardinality of the set of pseudo-Frobenius elements of S (cf. [Fr94]).

Originally motivated by an interesting new class of numerical semigroups — called *far-flung Gorenstein* — introduced by Herzog-Kumashiro-Stamate in [HeKumSta], in joint work [BeDao] with Hailong Dao, we define the *difference-Frobenius* numbers of a numerical semigroup S as

$$\mathrm{DF}(S) = \{\mathrm{F}(S) - x \mid x \in \mathrm{PF}(S)\},$$

where $\mathrm{F}(S)$ is the Frobenius number of S and $\mathrm{PF}(S)$ is the set of pseudo-Frobenius numbers of S. We say that S is *divisive* whenever 1 is a difference-Frobenius number. By definition, a divisive numerical semigroup is not symmetric. We find that a far-flung Gorenstein numerical semigroup is divisive, but the converse does not hold in general. It turns out that the divisive numerical semigroups that have certain simple or desirable properties can sometimes be described elegantly. Particularly, we determine completely those numerical semigroups generated by intervals that are divisive, and we show that a numerical semigroup of *minimal multiplicity* (i.e., *maximal embedding dimension*) is divisive if and only if its largest and second-largest generators differ by 1. We exhibit a class of "sparse" numerical semigroups with embedding dimension four that are divisive. Ultimately, we ask the following.

Question 1.1. Other than numerical semigroups of minimal multiplicity and numerical semigroups generated by intervals, for what classes of "well-behaved" numerical semigroups is it tractable to describe the circumstances under which such a numerical semigroup is divisive?

Despite their subtlety, these questions are well-suited for interested undergraduate and graduate students in mathematics. One immediate direction regarding Question 1.1 is to investigate *pinched* *discrete interval* numerical semigroups, which are derived from numerical semigroups generated by intervals by removing some elements. Recently, the author has classified all pinched discrete interval numerical semigroups of the form $\langle n, n+1, 2n-2, 2n-1 \rangle$ as symmetric, divisive, or neither.

Let I be a subset of $\mathbb{Z}$. We say that I is a *relative ideal* of S if $I \supseteq S + I = \{s + i \mid s \in S \text{ and } i \in I\}$ and $s + I \subseteq S$ for some element $s \in S$. Put another way, I must be closed with respect to addition by elements of S, and there must exist a smallest integer of I. We recall that a numerical semigroup S with maximal relative ideal $\mathfrak{M} = S \setminus \{0\}$ and canonical relative ideal $\Omega = \{-n \mid n \in \mathbb{Z} \setminus S\}$ is said to be *nearly Gorenstein* if $\mathfrak{M} \subseteq \Omega + (S - \Omega)$ and *almost Gorenstein* if $\mathfrak{M} + \Omega = \mathfrak{M}$, where $S - \Omega = \{m \in \mathbb{Z} \mid m + \Omega \subseteq S\}$ is the relative ideal of differences of S and Ω. Observe that any almost Gorenstein numerical semigroup is nearly Gorenstein; however, it is known that the converse does not hold. Recent results from [MosStr] and [HeHiSta19, HeHiSta21] suggest an interesting connection between these classes of numerical semigroups and the trace ideal of the canonical module of the numerical semigroup ring $k[\![S]\!]$. Based on this, Dao and the author define the *blow-up* of a numerical semigroup S with respect to a proper ideal $I \subseteq S$ generated by $x_1 < \cdots < x_n$ as

$$B_S(I) = S + \mathbb{Z}_{\geq 0}\langle x_i - x_1 \mid 1 \leq i \leq n \rangle.$$

We say that a numerical semigroup S has the *Gorenstein canonical blow-up* (GCB) property whenever its canonical blow-up $B_S(\Omega)$ is symmetric. Our main theorem of [BeDao] demonstrates that many interesting classes of numerical semigroups exhibit the GCB property, e.g., Arf, far-flung Gorenstein, divisive, and multiplicity ≤ 3 numerical semigroups are GCB. Even more, if S has minimal multiplicity, then the GCB property is equivalent to the condition that S is almost Gorenstein.

2 Properties of Stable and Trace Ideals of Curve Singularities

Commutative algebraists classify the "largeness" of a commutative unital local ring R primarily through two invariants: *depth* and *Krull dimension*, denoted by $\operatorname{depth}(R)$ and $\dim(R)$, respectively. Briefly put, depth measures "homological largeness," and Krull dimension measures "topological largeness." If these two invariants coincide, i.e., $\operatorname{depth}(R) = \dim(R)$, we say that R is *Cohen-Macaulay*. Vast amounts of research have been dedicated to the study of Cohen-Macaulay rings (cf. [BrHe]).

Gorenstein rings form a strict subclass of Cohen-Macaulay rings that have been studied extensively for decades (cf. [BrHe], [Fox], [HeKun], or [LeuWie]). Crucially, a Gorenstein local ring R admits a *canonical module* ω_R that is "structurally equivalent" to R, i.e., we have that $\omega_R \cong R$ as R-modules. Even more, a canonical module of a Noetherian local ring $(R, \mathfrak{m})$ behaves well with respect to the topological process of $\mathfrak{m}$-*adic completion*. Particularly, if we denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R, we have that $\omega_{\widehat{R}} \cong \widehat{\omega_R}$ is a canonical module of the Noetherian local ring $(\widehat{R}, \mathfrak{m}\widehat{R})$. We say that R is *analytically unramified* if $\widehat{R}$ is *reduced*, i.e., every power of a nonzero element of $\widehat{R}$ is nonzero.

We assume throughout this section that $(R, \mathfrak{m}, k)$ is an analytically unramified one-dimensional Cohen-Macaulay local ring with infinite residue field k, total ring of fractions $Q(R)$, and integral closure $\overline{R}$. Under these conditions, it is known that R possesses a canonical ideal $\omega_R \subseteq R$ (cf. [HeKun]) and every $\mathfrak{m}$-primary ideal of R has a principal reduction (cf [HuSw, Proposition 8.3.7]). Based on recent work of Herzog, Stamate, et al. (cf. [HeHiSta19, HeHiSta21, HeKumSta]), Hailong Dao and I have sought in [BeDao] to understand the role that the canonical ideal ω_R plays in determining the structure of R. Generally, our work aims to answer questions about when the canonical blow-up

$$B(\omega_R) = \bigcup_{n \geq 0} (\omega_R^n : \omega_R^n) = \{\alpha \in Q(R) \mid \alpha\omega_R^n \subseteq \omega_R^n\}$$

is Gorenstein, almost Gorenstein, or nearly Gorenstein and to determine when $B(\omega_R) = \overline{R}$.

Using the properties of stable ideal theory outlined in the seminal work of Lipman [Lip] in conjunction with techniques related to trace ideals described in recent work of Dao [Dao21], Dao-Lindo [DaoLin], Dao-Maitra-Sridhar [DaoMaSr], and Kobayashi-Takahashi [KoTa], we have determined that the properties of $B(\omega_R)$ are intimately related to the behavior of sufficiently large powers of ω_R and their trace ideals. Particularly, the Gorensteinness of $B(\omega_R)$ is equivalent to any of the following.

(1.) There exists an $n \gg 0$ such that $\omega_R^n \cong (\omega_R^n)^\vee$, where $-^\vee$ denotes the canonical dual of $-$.

(2.) There exists an $n \gg 0$ such that ω_R^n is self-dual.

(3.) There exists an $n \gg 0$ such that $\operatorname{tr}(\omega_R^n)$ is stable.

Further, we show that any Arf ring has the GCB property. Using the dual of the canonical blow-up $b(\omega_R) = (R : B(\omega_R))$, we exhibit equivalent properties for R to be almost Gorenstein. For instance, if $\mathfrak{m}$ is ω_R-Ulrich or $\mathfrak{m} \subseteq b(\omega_R)$, then R is almost Gorenstein; the converses hold, as well.

Extensive effort has been made jointly by Dao and the author toward the case that $R_S = k[\![S]\!]$ is the numerical semigroup ring corresponding to the numerical semigroup S. Observe that a numerical semigroup ring is a complete one-dimensional Noetherian local domain and hence Cohen-Macaulay. Further, by a result of Nagata in [Na], the integral closure of a numerical semigroup ring is module-finite, hence the integral closure of R_S is a module-finite birational extension of R_S. Using the methods described in the last two paragraphs of the previous section, there is an elegant description of the canonical blow-up of R_S as an R_S-algebra whose generators are simply the difference-Frobenius numbers. Consequently, the properties of the canonical blow-up of R_S are intimately related to the properties of the canonical blow-up of S. We show that the divisive numerical semigroup rings are a natural generalization of the far-flung Gorenstein numerical semigroup rings of Herzog-Kumashiro-Stamate.

3 Additive Number Theory, Sumsets, and Monomial Subrings

Consider a nonempty (not necessarily finite) subset $S \subseteq \mathbb{Z}_{\geq 0}$. We say that S is *Sidon* if for every pair of non-negative integers $i \leq j$, the sum $s_i + s_j$ of the elements $s_i, s_j \in S$ is unique. Put another way, there do not exist distinct pairs of integers $i \leq j$ and $i' \leq j'$ such that $s_i + s_j = s_{i'} + s_{j'}$ for some elements $s_i, s_j, s_{i'}, s_{j'} \in S$. Originally introduced by Simon Sidon in his study of Fourier series, Sidon sets culled significant interest in the field of additive number theory after a result of Erdös and Turán showed that for every real number $x > 0$, the number of elements of a Sidon set that are $\leq x$ is at most $\sqrt[4]{x} + O(\sqrt[4]{x})$ (cf. [ErdTu]). Even now, it remains an open problem to determine the maximum number of elements not exceeding a given real number $x > 0$ that a Sidon set can contain.

Essentially, the question of Sidon is to determine how "dense" a Sidon set can be if its largest element does not exceed some real number $x > 0$. On the other hand, given a positive integer m, one can ask the question of how "sparse" a set can be such that the n-fold sum of its elements achieves a maximum value of m. Colloquially, this problem is known as the Postage Stamp Problem, as it can be interpreted accordingly: let n and k be positive integers. Given that an envelope affords enough space for n stamps and we have k distinct denominations of stamps available to us, what is the maximum cost of postage m such that any letter of cost $0, 1, \ldots, m$ can be mailed?

Let A be a nonempty subset of non-negative integers. We define the n-fold sum of A as

$$nA = \underbrace{A + A + \cdots + A}_{n \text{ summands}} = \{a_1 + a_2 + \cdots + a_n \mid a_1, a_2, \ldots, a_n \in A\},$$

and we denote by $[m] = \{0, 1, \ldots, m\}$ the discrete interval $[0, m] \cap \mathbb{Z}$. Considered the first to state the Postage Stamp Problem, Rohrbach defined the invariants $m(n, A) = \max\{m : \{0, 1, \ldots, m\} \subseteq nA\}$ and $m(n, k) = \max\{m(n, A) : |A| = k\}$ in his seminal 1937 paper (cf. [Roh]). Even though it is relatively simple to state, it has been shown that the computational complexity of the problem is exponential in both n and k, and there remain many open questions that relate to $m(n,k)$ (cf. [AlBa]).

One particular case of the Postage Stamp Problem centers around the notion of a complete double. Consider any nonempty subset $A \subseteq [a]$. We say that A is a *complete double* of $[a]$ if $2A = [2a]$. Clearly, every discrete interval $[a]$ is a complete double of itself, hence one might naturally ask, "What is the least cardinality of A such that $\mathcal{A}$ is a complete double of a?" Let $\mu(a) = \min\{|A| : 2A = [2a]\}$. Based on a discussion to a post by Hailong Dao on MathOverflow (cf. [Dao19]), I have established bounds for $\mu(a)$ in [Be]. Further, in the same paper, I have defined the *regularity* $\operatorname{reg}(A) = \inf\{n \mid nA = [na]\}$ of A and exhibited necessary and sufficient conditions to guarantee that $\operatorname{reg}(A)$ is finite. Given a subset $\mathcal{A} \subseteq [a]$ that contains $\{0, 1, a-1, a\}$, we define the monomial subring

$$k[x,y]^{(\mathcal{A})} = k[x^i y^{a-i} \mid i \in \mathcal{A}] \subseteq k[x,y].$$

In [Be], I have computed the Hilbert series, (Hilbert-Samuel) multiplicity, and (Castelnuovo-Mumford) regularity of $k[x,y]^{(\mathcal{A})}$. Remarkably, it turns out that $\operatorname{reg}(k[x,y]^{(\mathcal{A})}) = \operatorname{reg}(\mathcal{A})$.

Further questions remain regarding the connection between the cardinality of A and $\operatorname{reg}(A)$.

Question 3.1. Is there an elegant relationship between $|A|$ and $\operatorname{reg}(A)$?

Question 3.2. What are the bounds on $|A|$ such that $\operatorname{reg}(A) = 3$, i.e., A is a complete triple of $[a]$?

Question 3.3. Is there a sharper bound between $\operatorname{reg}(A)$ and $a - 2 = \max\{\operatorname{reg}(A) : \operatorname{reg}(A) < \infty\}$?

4 Combinatorial Commutative Algebra and Stanley-Reisner Theory

We say that a pair $G = (V, E)$ consisting of a nonempty set V of *vertices* and a nonempty subset $E \subseteq V \times V$ of *edges* is a *finite simple graph* whenever V consists of finitely many elements and E does not contain any *loops* or *multiple edges*. We recall that a loop is a pair of vertices of the form (v, v), and an edge (v, w) is a multiple edge if (w, v) is also an edge. Conventionally, we assume that the n vertices of G are the positive integers 1 through n, i.e., $V = \{1, 2, \ldots, n\}$. We say that $V' \subseteq V$ constitutes an *independent vertex set* if for any two vertices $i, j \in V'$, (i, j) is not an edge of G.

Generalizing the notion of a finite simple graph is that of a *simplicial complex* Δ. We say that Δ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$ if Δ is a nonempty subset of $2^{[n]}$ such that for every pair of subsets $\sigma, \tau \in 2^{[n]}$ with $\tau \subseteq \sigma$, if σ belongs to Δ, then τ belongs to Δ. Put another way, Δ must be closed with respect to the operation of taking subsets. We note that some familiar geometric objects — such as line segments, triangles, and tetrahedra — are simplicial complexes.

Every finite simple graph on n vertices gives rise to a quotient of the polynomial ring in n variables. Explicitly, for a field k, a finite simple graph $G = (V, E)$ on n vertices can be related to the quotient ring $k(G) = k[x_1, \ldots, x_n]/I(G)$ by the squarefree monomial ideal $I(G) = (x_i x_j \mid (i,j) \in E)$. We refer to $I(G)$ as the *edge ideal* of G and to $k(G)$ as the *edge ring* of G. Likewise, every simplicial complex on n vertices gives rise to a quotient of the polynomial ring in n variables. For simplicity, we use the same residue field. Explicitly, we define the *Stanley-Reisner ring* $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$, where

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_k} \mid \{i_1, i_2, \ldots, i_k\} \subseteq 2^{[n]} \setminus \Delta)$$

is the *Stanley-Reisner ideal* of $k[x_1, \ldots, x_n]$ generated by the monomials corresponding to subsets of $2^{[n]}$ that do not belong to Δ. Often, the elements of Δ are referred to as *faces*, hence I_Δ is generated by monomials corresponding to non-faces of Δ. We do not assume that all of the integers of $[n]$ correspond to vertices of Δ, hence it is possible that x_i belongs to I_Δ for some integer $1 \le i \le n$ so that $k[\Delta]$ is a quotient of the polynomial ring in fewer than n elements. We recall that the set Δ_G consisting of the independent vertex sets of G is a simplicial complex, eponymously called the *independence complex* of G. One can show that the edge ideal of G and the Stanley-Reisner ideal of Δ_G are equal, hence we have that $k(G) = k[\Delta_G]$ (cf. [MooRogSa-Wa, Theorem 4.4.9]). Consequently, the so-called Stanley-Reisner theory can be employed to understand properties of the edge ring $k[G]$ and vice-versa.

Recently, Souvik Dey and I introduced in [BeDey] some new invariants $\operatorname{ms}(R)$ and $\operatorname{cs}(R)$ of a Noetherian (standard graded) local ring $(R, \mathfrak{m}, k)$. Explicitly, we have that

$$\operatorname{cs}(R) = \min\{\dim_k(I/\mathfrak{m}I) \mid I \subseteq \mathfrak{m} \text{ is a (homogeneous) ideal of } R \text{ such that } \mathfrak{m}^2 = I^2\} \text{ and}$$
$$\operatorname{ms}(R) = \min\{\dim_k(I/\mathfrak{m}I) \mid I \subseteq \mathfrak{m} \text{ is a (homogeneous) ideal of } R \text{ such that } \mathfrak{m}^2 \subseteq I\}.$$

Particularly, we sought to understand these invariants in the case of the edge ring $k(G)$ of a finite simple graph G and a fixed field k. Conventionally, we write $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ for $\operatorname{ms}(k(G))$ and $\operatorname{cs}(k(G))$, respectively. All though we can compute $\operatorname{ms}(R)$ and $\operatorname{cs}(R)$ in some cases, it turns out that these invariants behave in a surprisingly subtle manner in the setting of edge rings of finite simple graphs. For instance, we have proved that $\operatorname{ms}(K_n) = 1$, but we have only been able to show that $\operatorname{cs}(K_n) \geq \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$ for the complete graph K_n on n vertices. On the other hand, we have that $\operatorname{ms}(S_n) = n - 1$ and $\operatorname{cs}(S_n) = n$ for the star graph S_n on n vertices. Consequently, we believe that $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ are both determined in some way by the "connectivity" of G; however, despite extensive efforts, pinning down exactly what that means has proven intractable so far.

It is worth mentioning that we have shown that $\dim(R) \leq \operatorname{ms}(R) \leq \operatorname{cs}(R) \leq \mu(\mathfrak{m})$ holds for any Noetherian local ring $(R, \mathfrak{m})$, hence we have that $\operatorname{ms}(G) \geq \alpha(G)$, where $\alpha(G)$ denotes the *independence number* of G. Combining results of Fröberg (cf. [Fr90, Theorem 1]) and Eisenbud-Huneke-Ulrich (cf. [EisHuUl, Corollary 5.2]), we have shown in the case that k is infinite that equality holds whenever G^c is chordal; otherwise, we have shown that $\operatorname{ms}(k(G)) \leq n - \operatorname{mcn}(G) + 3$, where we define

$$\operatorname{mcn}(G) = \min\{i \geq 4 \mid C_i \text{ is an induced subgraph of } G^c\}.$$

If G^c is chordal (i.e., if C_i is not an induced subgraph of G^c for any integer $i \geq 4$), then we declare that $\operatorname{mcn}(G) = n - \alpha(G) + 3$. Our future work will concern the following.

Question 4.1. Can we compute $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ for a large class of familiar graphs? Bounds have been given for (a.) the complete graph K_n, (b.) the path graph P_n, (c.) the cycle graph C_n, and (d.) the wheel graph W_n, but in each case, one (or both) of $\operatorname{ms}(G)$ or $\operatorname{cs}(G)$ has not been found.

Question 4.2. Do we have that $\operatorname{ms}(k(G)) \leq \#\{\text{maximum cliques of } G^c\} + \dim k(G) - 1$?

Question 4.3. If G can be covered by two cliques, do we have that $\operatorname{ms}(k(G)) \leq 2$?

Let I be a quadratic squarefree monomial ideal of $k[x_1, \dots, x_n]$. Let G_I be the graph on the vertex set $[n] = \{1, \dots, n\}$ whose non-edges are precisely the monomials of I. Recently, in conversation with Grigoriy Blekherman, it has been communicated to Souvik Dey and the author that there is a connection between $\operatorname{ms}(k[x_1, \dots, x_n]/I)$ and the *generic completion rank* of the graph G_I (cf. [BlSi]). Based on this discussion, we believe that $\operatorname{ms}(G)$ could be identified with a more mysterious graph invariant called the *maximum likelihood threshold*. Particularly, we ask the following.

Question 4.4. Let $I(G)$ be the edge ideal of a finite simple graph $G = (V, E)$. Let G^c denote the complement graph of G, i.e., the finite simple graph on the vertex set V with an edge (i, j) if and only if (i, j) is not an edge of G. Do we have that $\operatorname{ms}(G) = \operatorname{mlt}(G^c)$?

Given that G^c has no edges (i.e., G is complete) or G^c has no cycles (i.e., G^c is a tree), we can prove explicitly that $\operatorname{ms}(G) = \operatorname{mlt}(G^c)$, the latter of which have been computed in [GroSul].

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