MA172: Calculus II

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Chapter 1

Differentiation and Integration

Broadly speaking, differential calculus is the study of instantaneous change. Early on in a first calculus course, students learn that the derivative of a function at a point measures the slope of the line tangent at that point; the slope of the tangent line at a point is simply limit of the slopes of the secant lines passing through the specified point, and these slopes measure the average rate of change of the function. Consequently, the derivative measures the instantaneous change of a function. Bearing this in mind, calculus is immediately applicable in a wide range of fields — from physics and engineering to biology, chemistry, and medicine. Conversely, it is the aim of integral calculus to quantify change over time given the instantaneous rate of change. Combined, differential and integral calculus constitute an indispensable tool in many applied sciences today.

1.1 Limits and Continuity

Calculus is the study of change in functions. Essentially, a **function** is simply a rule that assigns to each input x one and only one output $y = f(x)$. Often, in this course, we will simply consider **real functions**, i.e., functions that are defined such that their inputs and outputs are **real numbers**. We are unwittingly very familiar with real numbers: the real numbers $\mathbb{R}$ include zero, all positive and negative whole numbers, all positive and negative rational numbers (or fractions), all positive and negative square roots of positive rational numbers, and transcendental numbers like π and e.

We will use the notation $f : \mathbb{R} \to \mathbb{R}$ to express that f is a function whose **domain** is the real numbers $\mathbb{R}$ and whose **codomain** is the real numbers $\mathbb{R}$. Explicitly, the domain of a function is the set of all possible inputs of a function, and the codomain of a function is the set of all possible outputs of the function. Even more, the collection of all possible outputs of a function is the **range** of the function. We will adopt the **set-builder** notation for the domain and range of a function f.

$D_f = \{x \in \mathbb{R} \mid f(x) \text{ is a real number}\}$ consists of real numbers x such that $f(x)$ is a real number.

$R_f = \{f(x) \in \mathbb{R} \mid x \in D_f\}$ consists of real numbers $f(x)$ such that x lies in the domain of f.

Example 1.1.1. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. By definition, this function outputs the real number x that is input. We refer to this as the **identity function** on the real numbers. Consequently, the domain of f is $D_f = \mathbb{R}$ because the output of any real number is a real number, and the range of f is $R_f = \mathbb{R}$ because every real number is the output of itself.

Caution: the domain of a real function might not be all real numbers; the range of a real function might not be all real numbers, either, as our next pair of examples illustrate.

Example 1.1.2. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. By definition, this function outputs the square x^2 of the real number x that is input. Certainly, the square of any real number is a real number, hence the domain of f is $D_f = \mathbb{R}$; on the other hand, the only real numbers that are the square of another real number are the non-negative real numbers. Explicitly, for any real number x, the real number $f(x) = x^2$ is a non-negative real number, i.e., we have that $x^2 \geq 0$. Consequently, the codomain of f is $\mathbb{R}$, but the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Example 1.1.3. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. By definition, this function outputs the square root $\sqrt{x}$ of the real number x that is input. We cannot take the square root of a negative real number, hence the domain of f consists of all non-negative real numbers, i.e., we have that $D_f = \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$; on the other hand, every non-negative real number can be realized as the square root of a non-negative real number. Explicitly, for any non-negative real number y, the real number y^2 satisfies that $y = \sqrt{y^2} = f(y^2)$. Consequently, the codomain of f is $\mathbb{R}$, but once again, the range of f is $R_f = \mathbb{R}_{\geq 0} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Generally, the restrictions on the domain of a real function consist of the following situations.

(a.) We cannot divide by zero.

(b.) We cannot take the even root of a negative real number.

(c.) We cannot take the logarithm of a non-positive real number.

Occasionally, it is necessary to split the domain or the range of a function into distinct chunks of the real number line. By the above rule, the domain of the real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ consists of all nonzero real numbers. Consequently, we can certainly realize the domain of f as $D_f = \{x \in \mathbb{R} \mid x \neq 0\}$, but it is sometimes more convenient to describe this set using the **union** symbol $\cup$. Put simply, the union symbol $\cup$ functions as the logical connective "or." Clearly, a nonzero real number is either positive or negative, hence we can partition the domain of f into those real numbers that are positive and those real numbers that are negative. We achieve this with the union symbol as $D_f = \{x \in \mathbb{R} \mid x > 0\} \cup \{x \in \mathbb{R} \mid x < 0\}$. Even more, we learn in college algebra (or earlier) that the set of real numbers x satisfying the **inequalities** $x > 0$ and $x < 0$ can be described respectively using the **open intervals** $(0, \infty)$ and $(-\infty, 0)$. Consequently, in **interval notation**, the domain of the real function $f(x) = x^{-1}$ is given by $D_f = (-\infty, 0) \cup (0, \infty)$.

Exercise 1.1.4. Compute the domain and range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Exercise 1.1.5. Compute the domain and range of the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^{-3}$.

Exercise 1.1.6. Compute the domain and range of the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \frac{1}{\ln(x)}$.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ whose domain is D_f. Given any real number a in D_f, we say that the **limit** of $f(x)$ as x approaches a is the quantity L (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$. Put another way, the quantity L can be made arbitrarily close to the value of $f(x)$ by taking x to be sufficiently close in value to a. Conveniently, if the quantity L exists, then we write $L = \lim_{x \to a} f(x)$.

Example 1.1.7. Let us compute the limit of $f(x) = x^2$ as x approaches $a = 1$ using the definition. Computing the limit is essentially like playing a game of limbo: we are handed a real number $\varepsilon > 0$ (the limbo bar), and our challenge is to find a real number $\delta > 0$ such that $|x^2 - 1| < \varepsilon$ whenever we assume that $|x - 1| < \delta$. Of course, we are at liberty to take δ as small as necessary to ensure that $|x^2 - 1| < \varepsilon$. We may therefore assume that $0 < \delta \leq 1$. Considering that $x^2 - 1 = (x - 1)(x + 1)$, if we assume that $|x - 1| < \delta \leq 1$, then we must have that $0 < x < 2$, from which it follows that $|x + 1| \leq |x| + 1 = x + 1 < 3$ by the Triangle Inequality. Consequently, we have that

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x - 1||x + 1| < 3\delta.$$

Last, if we wish to have that $|x^2 - 1| < \varepsilon$, then we should choose δ to be the minimum of 1 and $\frac{\varepsilon}{3}$.

One-sided limits can be defined analogously to the limit above: the **left-hand limit** of $f(x)$ as x approaches a is the quantity L^- (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $-\delta < x - a < 0$ implies that $|f(x) - L^-| < \varepsilon$. Likewise, the **right-hand limit** of $f(x)$ as x approaches a is the quantity L^+ (if it exists) such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $0 < x - a < \delta$ implies that $|f(x) - L^+| < \varepsilon$.

$L^- = \lim_{x \to a^-} f(x)$ is the symbolic way to express the left-hand limit of $f(x)$ as x approaches a.

$L^+ = \lim_{x \to a^+} f(x)$ is the symbolic way to express the right-hand limit of $f(x)$ as x approaches a.

Ultimately, the two-sided limit exists if and only if the left- and right-hand limits exist and are equal; thus, the two-sided limit is equal to the common value of the left- and right-hand limits.

$$L^- = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = L^+$$

Graphically, it is possible to compute the two-sided limit L of some functions $f(x)$ as x approaches a by tracing one's finger along the graph of $f(x)$ from the left- and right-hand sides.

Example 1.1.8. Let us graphically compute the limit of $f(x) = x^2$ as x approaches $a = 1$. Using the graph of $f(x) = x^2$, we find that the limit is 1. Particularly, if we trace the graph with our left pointer finger, moving from left to right toward the point $x = 1$, our finger stops at $y = f(1) = 1$. Likewise, if we trace the graph with our right pointer finger moving from right to left toward $x = 1$, our finger stops at $y = f(1) = 1$. Put in the language of calculus, we have that $L^- = 1 = L^+$.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a real number a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Explicitly, we require three things to be true of the function $f(x)$ in this case.

1.) We must have that f is defined at the real number a, i.e., $f(a)$ must be in the range of f.

2.) We must have that $\lim_{x \to a^-} f(x) = f(a)$, i.e., the left-hand limit of f at a must be $f(a)$.

3.) We must have that $\lim_{x \to a^+} f(x) = f(a)$, i.e., the right-hand limit of f at a must be $f(a)$.

Consequently, if any of these criteria is violated, then the function f cannot be continuous at a.

Example 1.1.9. One of the easiest ways to detect that a function is not continuous at a real number a is to observe that the function is not defined at a. Explicitly, the function $f(x) = \frac{1}{x}$ is not continuous at $a = 0$ because the domain of f excludes $a = 0$ (i.e., we cannot divide by zero).

Example 1.1.10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ that is defined **piecewise** as follows.

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and} \\ -1 & \text{if } x < 0 \end{cases}$$

Graphically, if we trace our fingers along f from the left-hand side, when we arrive at $a = 0$ from the left-hand side, we find that the limiting value here is -1; however, if we trace our fingers along f from the right-hand side, when we arrive at $a = 0$ from the right-hand side, we find that the limiting value here is 1. Consequently, the function $f(x)$ is not continuous at $a = 0$.

Example 1.1.11. Let us prove by definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous for all real numbers a. Observe that f is defined piecewise as follows.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0 \end{cases}$$

Consequently, it suffices to show that $g(x) = x$ and $h(x) = -x$ are everywhere continuous. Given real numbers $\varepsilon_1, \varepsilon_2 > 0$, we must find real numbers $\delta_1, \delta_2 > 0$ such that $|x - a| < \varepsilon_1$ whenever $|x - a| < \delta_1$ and $|-x - (-a)| < \varepsilon_2$ whenever $|x - a| < \delta_2$. Considering that the absolute value is multiplicative, we have that $|-x - (-a)| = |-x + a| = |-(x - a)| = |x - a|$, we may simply take the real numbers $\delta_1 = \varepsilon_1$ and $\delta_2 = \varepsilon_2$. We conclude that $g(x) = x$ and $h(x) = -x$ are continuous for all real numbers a so that $f(x) = |x|$ is continuous for all nonzero real numbers by the piecewise definition of $f(x)$ prescribed above. We are done as soon as we show that

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x|.$$

By continuity of the functions $g(x)$ and $h(x)$ and by definition of $|x|$, the left-hand limit is given by $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} h(x) = h(0) = 0$, and the right-hand limit is $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} g(x) = g(0) = 0$.

Generally, continuity can be defined as a property of a function on any **subset** of its domain, i.e., on any collection of real numbers that lie in the domain. Often, we will consider functions that are continuous on their entire domain, but it is possible that a function is not continuous at some point in its domain. We say that a function f is **discontinuous** at a real number a if f is not continuous at the real number a. By the above three criteria, we can classify these **discontinuities**.

- We say that f has a **removable discontinuity** at a real number a if a is not in the domain of f but the left- and right-hand limits of f at a exist and are equal, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
- We say that f has a **jump discontinuity** at a real number a if both of the left- and right-hand limits of f at a exist but are not equal, i.e., $\lim_{x \to a^-} f(x) = L^- \neq L^+ = \lim_{x \to a^+} f(x)$.
- We say that f has an **essential discontinuity** at a real number a if either the left- or the right-hand limit of f at a does not exist, i.e., either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist.

Often, if a function f is continuous for every real number in its domain D_f, we will say that the function is **continuous** on its domain. Explicitly, if the domain of a function f is all real numbers and f is continuous on its domain, then we will say that f is **everywhere continuous**. Graphically, we may detect that a function is continuous if we can draw it without lifting our pencil.

Example 1.1.12. We can graph $|x|$ without lifting our pencil, hence it is everywhere continuous.

Example 1.1.13. We cannot graph x^{-2} without lifting our pencil at $x = 0$, hence x^{-2} is not continuous at $a = 0$. On the other hand, for all real numbers a other than $a = 0$, we can graph this function without lifting our pencil, hence x^{-2} is continuous on its domain $(-\infty, 0) \cup (0, \infty)$.

Continuous functions abound: **polynomial** functions such as $x^3 - 2x^2 + x - 7$ and **exponential** functions such as e^x are defined for all real numbers and are everywhere continuous. Likewise, the **trigonometric** functions $\sin(x)$ and $\cos(x)$ are defined for all real numbers and are everywhere continuous. **Logarithmic** functions such as $\ln(x)$ and $\log(x)$ and **algebraic** functions such as $\sqrt{x}$ and $x^{3/2}$ are defined for all positive real numbers and are continuous on their domains. Further, addition, subtraction, multiplication, division, composition, and any finite combination of these operations on continuous functions result in functions that are typically continuous on their domains.

1.2 Differentiation and L'Hôpital's Rule

Given any real numbers a and $h > 0$ and any real function $f(x)$ such that $f(a)$ and $f(a + h)$ are defined, consider the closed interval $[a, a + h]$ consisting of all real numbers x with $a \leq x \leq a + h$. We define the **secant line** of $f(x)$ over this interval as the line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$. Observe that the slope of the secant line is given by the **difference quotient**

$$Q_a(h) = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

By taking the limit of $Q_a(h)$ as h approaches 0, we obtain the **derivative** of $f(x)$ at a

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Of course, this limit might not exist; however, if it does, we interpret it geometrically as the slope of the line tangent to $f(x)$ at the point $(a, f(a))$. Given that the quantity $f'(a)$ exists, we say that $f(x)$ is **differentiable** at a. One fundamental interpretation of the derivative in the context of a function that measures something physical (e.g., velocity) is as the instantaneous rate of change.

Example 1.2.1. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^3$.

Example 1.2.2. Use the limit definition of the derivative to compute $g'(x)$ for $g(x) = \frac{1}{x}$.

Example 1.2.3. Use the limit definition of the derivative to compute $h'(x)$ for $h(x) = \sqrt{x}$.

One of the most important properties of differentiable real functions is the following.

Proposition 1.2.4. *If a real function f is differentiable at a real number a, then f is continuous at a. Explicitly, a function that is differentiable at a point in its domain is necessarily continuous there. Conversely, there exists a function that is continuous but not differentiable on its domain.*

Proof. We will assume that f is differentiable at a real number a. Consequently, the limit

$$f'(a) = \lim_{h \to 0} Q_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Using the substitution $x = a + h$, we have that $h = x - a$. Crucially, under this substitution, the limit of any function $g(h)$ as h approaches 0 is equal to the limit of the function $g(x-a)$ as x approaches a. (Verify this by definition of the limit.) Consequently, the following identity holds.

$$f'(a) = \lim_{x \to a} Q_a(x-a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Considering that $x - a$ is a polynomial function, it is continuous at a, and we conclude that

$$\lim_{x \to a}(x - a) = a - a = 0.$$

Using the fact that the limit of a product is the product of limits (when both limits exist),

$$0 = f'(a) \cdot \lim_{x \to a}(x-a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot \lim_{x \to a}(x-a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot x - a = \lim_{x \to a}[f(x) - f(a)]$$

yields the result that $\lim_{x \to a} f(x) = \lim_{x \to a}[f(a) + f(x) - f(a)] = \lim_{x \to a} f(a) + \lim_{x \to a}[f(x) - f(a)] = f(a)$.

Conversely, the function $|x|$ is continuous on its domain, but it is not differentiable at $a = 0$: indeed, by Example 1.1.10, the piecewise function $f(x)$ satisfying that $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$ is not continuous because the left- and right-hand limits do not agree at 0. One can readily verify that this function is exactly the derivative of $|x|$, hence the claim holds. □

Computing limits by definition is even more tedious than it looks, but luckily, there are plenty of tools that allow us to compute derivatives of functions without ever touching a limit. Particularly,

- the **Power Rule** says that if $f(x) = x^r$ for some real number r, then $f'(x) = rx^{r-1}$;
- the **Product Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x);$$

- the **Quotient Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g'(x)]^2}; \text{ and}$$

- the **Chain Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f \circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) = [f' \circ g(x)] \cdot g'(x).$$

Computing the limit of a function that is continuous is quite easy: we may simply "plug and chug;" however, there exist functions that are not continuous. Even worse, when evaluating limits, we can encounter situations that result in an **indeterminate form** when the limit is the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Example 1.2.5. Given any real functions $f(x)$ and $g(x)$ that are differentiable for all real numbers x such that $a < x < b$ except possibly $x = c$ for some real number $a \leq c \leq b$, consider the following.

1.) We have that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$.

2.) We have that $g'(x) \neq 0$ for any real number x such that $a < x < b$ and $x \neq c$.

3.) We have that $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists.

Granted that each of the above conditions holds, it follows that $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \lim_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Example 1.2.6. Compute the limit of $f(x) = \dfrac{\ln(x)}{x^3 - 1}$ as x approaches $a = 1$.

Example 1.2.7. Compute the limit of $g(x) = (2x - \pi)\sec(x)$ as x approaches $a = \frac{\pi}{2}$ from the left.

Example 1.2.8. Compute the limit of $h(x) = \dfrac{\sin(x)}{\sin(x) + \tan(x)}$ as x approaches $a = 0$.

Example 1.2.9. If $\frac{d}{dx} \sin(x) = \cos(x)$, compute the limit of $f(x) = \dfrac{\sin(x)}{x}$ as x approaches $a = 0$.

Caution: Unfortunately, the above example is not a valid proof of this limit identity: in fact, this limit identity is needed to prove that $\frac{d}{dx} \sin(x) = \cos(x)$, so in order to prove this identity in a rigorous and non-circular manner, we must use tools from trigonometry and the Squeeze Theorem.

1.3 Implicit Differentiation

Curves in the Cartesian plane can be represented by an equation involving a function of two variables. Explicitly, we are familiar with such curves as $xy = 1$ and $y - x^2 = 0$; they are respectively the functions $y = f(x) = x^{-1}$ and $y = g(x) = x^2$. We refer to the functions $f(x)$ and $g(x)$ as the **explicit** forms of the curves. Unfortunately, it is not possible to write every curve in the Cartesian plane as a function of one variable: curves such as the unit circle $x^2 + y^2 = 1$ or the hyperbola $y^2 - x = 0$ cannot be represented as functions because they fail the **Vertical Line Test**; however, we will see throughout this semester that these curves provide important models in calculus. Curves that do not admit closed-form expressions of the form $y = f(x)$ can be written **implicitly**.

Under certain conditions, it is possible to find a "small enough" region in the Cartesian plane in which an implicit curve can be represented by a function; thus, in this "window," the slope and tangent line of such curves are well-defined. Consequently, we may define the **implicit derivative** by assuming that y is a function of x (on some "small window" in the plane) with derivative $y' = \frac{dy}{dx}$.

Example 1.3.1. Consider the unit circle $x^2 + y^2 = 1$. We will find the slope of the tangent line to the circle at a point (x, y). Considering the variable y as a function of x and using the fact that $y' = \frac{dy}{dx}$, we must invoke the Chain Rule in order to find that

$$0 = \frac{d}{dx} 1 = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy'.$$

Crucially, each time the derivative operator $\frac{d}{dx}$ encounters the variable y, we differentiate y as we would the function $y = f(x)$ that represents y locally. Consequently, if y is nonzero, then

$$\frac{dy}{dx} = y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

Otherwise, the tangent line does not exist if y is zero because this implies that $x = 1$.

Example 1.3.2. Consider the hyperbola $y^2 - x = 0$. We have that

$$0 = \frac{d}{dx}0 = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

so that $\frac{dy}{dx} = y' = (2y)^{-1}$ for all points (x, y) on the hyperbola such that y is nonzero.

1.4 Exponential and Logarithmic Functions

Given any positive real number a, the **exponential** function with **base** a is given by $\exp_a(x) = a^x$. Crucially, the most important exponential function is simply $\exp(x) = e^x$: here, the base is **Euler's number** $e \approx 2.72$. Later, we will concern ourselves with the definition of Euler's number; for now, we need only recall the following properties of exponential functions for any real numbers x and y.

1.) $a^{x+y} = a^x a^y$

2.) $a^{x-y} = a^x a^{-y}$

3.) $a^{xy} = (a^x)^y$

4.) $(ab)^x = a^x b^x$ for any real number $b > 0$

We do not yet have the machinery available to use to prove the following, but it is true that

$$\frac{d}{dx}e^x = e^x.$$

Considering that $e^x > 0$ for all real numbers x, it follows that e^x is a strictly increasing function, hence it passes the **Horizontal Line Test** and must therefore admit an **inverse** function; we refer to this function as the **natural logarithmic** function $\ln(x)$. Put another way, we have that

$$e^{\ln(x)} = x \text{ for all real numbers } x > 0 \text{ and } \ln(e^x) = x \text{ for all real numbers } x.$$

Observe that the range of e^x is $(0, \infty)$, hence the domain of $\ln(x)$ is $(0, \infty)$. Conversely, the domain of e^x is $(-\infty, \infty)$, hence the range of $\ln(x)$ is $(-\infty, \infty)$. We will also simply assert that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

We may also deduce the following properties of logarithmic functions for any real numbers $x, y > 0$.

1.) $\log_a(xy) = \log_a(x) + \log_a(y)$

2.) $\log_a(x^r) = r\log_a(x)$ for all real numbers r

3.) $\log_a(xy^{-1}) = \log_a(x) - \log_a(y)$

4.) $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

Even more, for any real number $a > 0$, the exponential function $\exp_a(x) = a^x$ is differentiable for all real numbers x. Further, observe that $y = a^x$ is strictly positive for all real numbers x, hence the function $\ln(y) = x\ln(a)$ is well-defined. Using the Chain Rule, we find that

$$\frac{1}{y}\cdot y' = \frac{d}{dx}\ln(y) = \frac{d}{dx}[x\ln(a)] = \ln(a)\cdot\frac{d}{dx}x = \ln(a) \text{ and } \frac{d}{dx}a^x = \frac{d}{dx}y = \frac{dy}{dx} = y' = y\ln a = a^x\ln a.$$

By a similar rationale as before, one can define the **logarithmic** function $\log_a(x)$ **base** a for any positive real number a as the function inverse of a^x; its domain is $(0,\infty)$, and its range is $(-\infty,\infty)$.

Example 1.4.1. Compute the derivative of $y = \log_a(x)$ by using the fact that $a^y = x$.

1.5 Inverse Trigonometric Functions

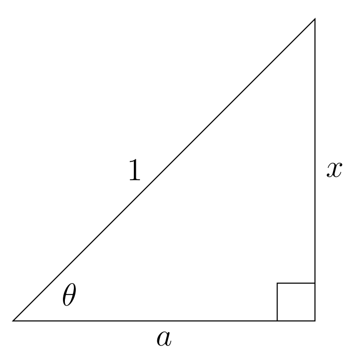
Even though the trigonometric functions like $\sin(x)$, $\cos(x)$, and $\tan(x)$ are **periodic**, we can find a region on the x-axis in which these functions pass the Horizontal Line Test and admit function inverses. Explicitly, the inverse trigonometric functions are denoted as follows.

$$\begin{array}{lll} \arcsin(x) = \sin^{-1}(x) & \text{domain: } [-1,1] & \text{range: } \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \arccos(x) = \cos^{-1}(x) & \text{domain: } [-1,1] & \text{range: } [0,\pi] \\ \arctan(x) = \tan^{-1}(x) & \text{domain: } (-\infty,\infty) & \text{range: } \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{array}$$

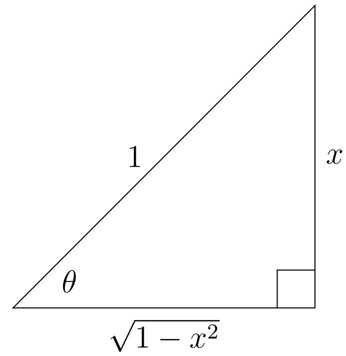
Considering that the input of the sine function is an angle, the output of the arcsine function is an angle. Consequently, if $x = \sin(\theta)$, then it follows by definition that $\theta = \arcsin(x)$ so that

$$\frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx}.$$

Observe that $\sin(\theta)$ is the ratio of the opposite side and the hypotenuse of a right triangle, so we may construct a right triangle whose opposite side has length x and whose hypotenuse has length 1 in order to obtain $\sin(\theta) = x$. Our right triangle therefore has the following form.



By the **Pythagorean Theorem**, we must have that $x^2 + a^2 = 1$ so that $a = \sqrt{1-x^2}$.

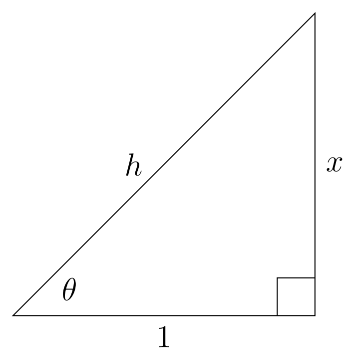


Using the Chain Rule, we can compute $\dfrac{d\theta}{dx}$. Explicitly, we have that

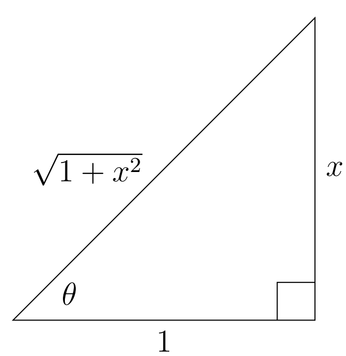
$$\cos(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\sin(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arcsin(x) = \frac{d\theta}{dx} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x^2}}.$$

Example 1.5.1. Use a right triangle involving 1, x, and $\sqrt{1-x^2}$ to compute $\frac{d}{dx}\arccos(x)$.

Using a similar idea as the one we employed to compute the derivative of $\arcsin(x)$ and $\arccos(x)$, we will set up a triangle with $\tan(\theta) = x$. Observe that $\tan(\theta)$ is the ratio of the opposite side and the adjacent side of a right triangle, so we may construct a right triangle whose opposite side has length x and whose adjacent side has length 1 in order to obtain $\tan(\theta) = x$.



Once again, by the Pythagorean Theorem, we find that $h^2 = x^2 + 1^2$ so that $h = \sqrt{1+x^2}$.

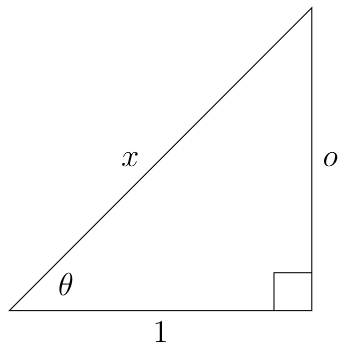


Using the Chain Rule, we can compute $\dfrac{d\theta}{dx}$. Explicitly, we have that

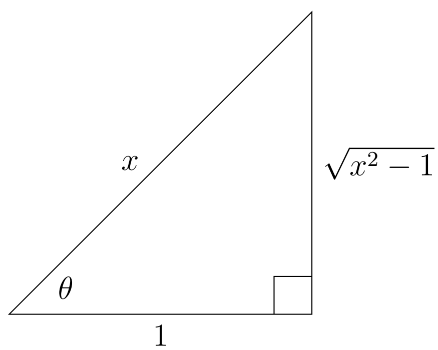
$$\sec^2(\theta) \cdot \frac{d\theta}{dx} = \frac{d}{dx}\tan(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\arctan(x) = \frac{d\theta}{dx} = \cos^2(\theta) = \frac{1}{1+x^2}.$$

Example 1.5.2. Use a right triangle involving 1, x, and $\sqrt{1+x^2}$ to compute $\frac{d}{dx}\operatorname{arccot}(x)$.

Last but not least, we will set up a triangle with $\sec(\theta) = x$. Observe that $\sec(\theta)$ is the ratio of the hypotenuse to the adjacent side of a right triangle, so we obtain the following diagram.



Once again, by the Pythagorean Theorem, we find that $x^2 = o^2 + 1^2$ so that $o = \sqrt{x^2-1}$.



Using the Chain Rule, we can compute $\dfrac{d\theta}{dx}$. Explicitly, we have that

$$\sec(\theta)\tan(\theta)\cdot\frac{d\theta}{dx} = \frac{d}{dx}\sec(\theta) = \frac{d}{dx}x = 1 \text{ so that } \frac{d}{dx}\operatorname{arcsec}(x) = \frac{d\theta}{dx} = \cos(\theta)\cot(\theta) = \frac{1}{x\sqrt{x^2-1}}.$$

Example 1.5.3. Use a right triangle involving 1, x, and $\sqrt{x^2-1}$ to compute $\frac{d}{dx}\operatorname{arccsc}(x)$.

1.6 Antidifferentiation

Considering that a derivative is a rate of change, it is natural in the applied sciences to begin with a rate of change and use it to estimate the net change of a process over time. Explicitly, if we observe that the velocity of a body is given by a function $f(x)$ over some interval of time, then we may seek a function $F(x)$ such that $F'(x) = f(x)$ over this interval of time. Given that such a function $F(x)$ exists and satisfies that $F'(x) = f(x)$, we refer to $F(x)$ as an **antiderivative** of $f(x)$.

Example 1.6.1. Prove that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Example 1.6.2. Prove that the function $G(x) = x\ln(x) - x$ is an antiderivative of $g(x) = \ln(x)$.

Example 1.6.3. Prove that the function $H(x) = xe^x - e^x$ is an antiderivative of $h(x) = xe^x$.

Observe that for any antiderivative $F(x)$ of a function $f(x)$, there exists a family of antiderivatives indexed by the real numbers. Particularly, the function $G(x) = F(x) + C$ is an antiderivative of $f(x)$ for every real number C. Even more, by the **Mean Value Theorem**, every antiderivative of $f(x)$ is of the form $F(x) + C$ for some antiderivative $F(x)$ of $f(x)$ and some real number C. Consequently, we may define the **general antiderivative** or **indefinite integral** of $f(x)$ to be

$$\int f(x)\, dx = F(x) + C$$

for any real number C. By the familiar derivative rules, we obtain

- the **Power Rule**, i.e., $\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C$ for all real numbers $r \neq -1$ and
- the **Chain Rule**, i.e., $\int f'(g(x))g'(x)\, dx = f(g(x)) + C$.

Further, indefinite integration is **linear**: for all real functions $f(x)$ and $g(x)$, we have

- the **Multiples Rule** $\int kf(x)\, dx = k\big(\int f(x)\, dx\big)$ for all real numbers k and
- the **Sum Rule** $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

Example 1.6.4. Compute the indefinite integral of $f(x) = x^{-1}$.

Example 1.6.5. Compute the indefinite integral of $g(x) = 2xe^{x^2}$.

Example 1.6.6. Compute the indefinite integral of $h(x) = 2\sin(x)\cos(x)$.

Circling back to the opening remarks of this section, we will assume that the velocity of a body over an interval of time is a continuous function $v(t)$. Even more, suppose that we note the position $s(t)$ of the particle at time $t = 0$, i.e., the quantity $s(0)$ is known. Considering that $s'(t) = v(t)$, it follows that $s(t)$ must differ from $\int v(t)\, dt$ by a constant C that depends on the quantity $s(0)$. We refer to this scenario as an **initial value problem** of the **differential equation** $s'(t) = v(t)$.

Example 1.6.7. Consider the velocity function $v(t) = 3t^2 - 4t + 2$ of a body whose position $s(t)$ at time $t = 0$ is given by $s(0) = 7$. Observe that $s(t) = \int v(t)\, dt = \int (3t^2 - 4t + 2)\, dt$, hence it suffices to find an antiderivative of $3t^2 - 4t + 2$. One can readily verify that $t^3 - 2t^2 + 2t$ is an antiderivative of $v(t)$, hence we have that $s(t) = t^3 - 2t^2 + 2t + C$. By plugging in our initial position value of $s(0) = 7$, we find that $7 = s(0) = C$ so that $s(t) = t^3 - 2t^2 + 2t + 7$.

Example 1.6.8. Consider tossing a ball upward with an initial velocity of 48 feet per second and constant acceleration of -32 feet per second from the edge of a cliff of height 432 feet. Compute the maximum height of the ball; then, find the time it takes for the ball to reach the ground.

1.7 Computing Area Bounded by a Curve of One Variable

Continuing in the theme of extrapolating data from intermittent observations, suppose that we observe the velocity $v(t)$ of a particle over a period of time $0 \leq t \leq 25$, taking care to mark down the velocity of the particle every five seconds. Consider along these lines the following table.

t	0	5	10	15	20	25
$v(t)$	25	31	35	43	47	46

We can roughly approximate the total distance traveled by the body for $0 \leq t \leq 25$ by assuming (incorrectly) that the body maintains a constant velocity each time we see it. Computing the total distance travelled by the particle during our observation amounts to finding the **displacement** of the body over each time interval and adding these quantities together. Explicitly, we have that

$$\text{total distance traveled} = 25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 + 47 \cdot 5 + 46 \cdot 5 = 1135.$$

Certainly, we can improve this estimation by taking more measurements: even recording one more observation will give us a better understanding of the behavior of the particle over the specified interval of time. Better yet, the more observations we record, the more accurate our understanding of the total distance traveled; however, this also requires adding more numbers together. Consequently, it will be convenient to develop notation to take sums of arbitrarily large quantities of data.

Let us assume for the moment that we have a collection of n real numbers $a_1, a_2, \dots, a_n$ for some positive integer n. Certainly, the sum of these real numbers can be realized as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

We refer to this as **sigma notation**: indeed, the Greek letter sigma Σ is used as a mnemonic device for "sum"; the subscript $i = 1$ denotes the **index of summation** and informs us of the first term a_1 in our collection of data; and the superscript n tells us that the sum terminates with the last term a_n in our collection of data. We refer to the real number a_i as the ith **summand** for each integer $1 \leq i \leq n$; the entire sum $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called a **finite sum**.

Often, we will consider finite sums whose ith summand can be conveniently expressed in **closed-form**. Explicitly, this means that there exists a function $f(x)$ such that $a_i = f(i)$.

Example 1.7.1. Consider the finite sum $1 + 2 + 3 + \cdots + 10$ of the first ten positive integers. Observe that the ith summand is simply the positive integer i, hence we have that $a_i = i$ and

$$1 + 2 + 3 + \cdots + 10 = \sum_{i=1}^{10} i.$$

Crucially, we point out another way to **index** the given sum — namely, we have that

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = 0 + 1 + 2 + 3 + \cdots + 10 = \sum_{i=0}^{10} i.$$

Often, if a sum involves a summand of zero, we will simply omit it (unless it is more convenient to include it). We could have also written this sum in a third way as follows.

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + \cdots + 10 = (1 + 2 + 3 + \cdots + 20) - (11 + 12 + 13 + \cdots + 20) = \sum_{i=1}^{20} i - \sum_{i=11}^{20} i.$$

Example 1.7.2. Consider the finite sum $1 + 4 + 9 + \cdots + 100$ of squares of the first ten positive integers in which the ith summand is simply the positive integer i^2. We have that $a_i = i^2$ and

$$1 + 4 + 9 + \cdots + 100 = \sum_{i=1}^{10} i^2.$$

Example 1.7.3. Express the finite sum $1^3 + 2^3 + 3^3 + \cdots + 1000^3$ of cubes of the first 1000 positive integers in summation notation, identifying the closed-form expression for the ith summand a_i.

Quite importantly, finite sums admit a convenient arithmetic of their own.

Proposition 1.7.4 (Properties of Finite Sums)**.** *Given any positive integer n and any real numbers $a_1, \dots, a_n, b_1, \dots, b_n$, and C, the following identities hold.*

(i.) (Empty Sum Law) *We have that $\sum_{i=n}^m a_i = 0$ for all integers $m < n$.*

(ii.) (Constant Sum Formula) *We have that $\sum_{i=m}^n C = C(n - m + 1)$ for all integers $m \leq n$.*

(iii.) (Linearity of a Finite Sum I) *We have that $\sum_{i=1}^n Ca_i = C(\sum_{i=1}^n a_i)$.*

(iv.) (Linearity of a Finite Sum II) *We have that $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$.*

One can easily prove the above formulas by expanding and comparing the expressions on both sides of the equation. We will not endeavor to prove the following identities because these details are beyond the scope of this course; however, they will be indispensable in what follows.

Proposition 1.7.5. *Consider any positive integer n.*

(i.) *We have that* $\displaystyle\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

(ii.) *We have that* $\displaystyle\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

(iii.) *We have that* $\displaystyle\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Going back to our example of tracking a particle over a period of time, if we know the velocity $v(t)$ of the particle at any time $0 \leq t \leq 25$, then we can approximate the total distance traveled by the particle by recording the velocity a positive integer n times and computing the total displacement of the particle over each interval of time. Explicitly, if we observe the particle for some real numbers $0 = t_0 < t_1 < \cdots < t_n = 25$ and we assume that the particle has constant velocity $v(t_i)$ for each integer $0 \leq i \leq n$, then the total distance traveled by the particle between time t_{i-1} and time t_i is given by the real number $\Delta t_i = t_i - t_{i-1}$ and the total displacement of the particle on this closed interval $[t_{i-1}, t_i]$ is $v(t_i)\,\Delta t_i$ (rate $\times$ time). Consequently, in sigma notation, we have that

$$\text{total distance traveled} = v(t_1)\,\Delta t_1 + v(t_2)\,\Delta t_2 + \cdots + v(t_n)\,\Delta t_n = \sum_{i=1}^n v(t_i)\,\Delta t_i.$$

By viewing the points $(t_i, v(t_i))$ as lying on the graph of the velocity curve $v(t)$, we may recognize $\sum_{i=1}^n v(t_i)\,\Delta t_i$ as an approximation of the area between the curve $v(t)$ and the t-axis, i.e., the area bounded by the curve $v(t)$ of one variable. We will now generalize this idea.

Consider any real function $f(x)$ that is continuous on a closed and bounded interval $[a, b]$. Choose any positive integer n; then, choose n real numbers $a = x_0 < x_1 < \cdots < x_n = b$. Consider the closed and bounded intervals $[x_{i-1}, x_i]$ for each integer $1 \leq i \leq n$. We refer to the collection $\mathcal{P}$ of such closed and bounded intervals as a **partition** of $[a, b]$, and we denote by $\Delta x_i = x_i - x_{i-1}$ the length of the interval $[x_{i-1}, x_i]$. Choosing **sample points** x_i^* such that $x_{i-1} \leq x_i^* \leq x_i$ yields a so-called **tagged partition** $(\mathcal{P}, x_i^*)$ consisting of closed and bounded intervals and sample points within them. We associate to each tagged partition a **Riemann sum** (or **Riemann approximation**)

$$\sum_{i=1}^{n} f(x_i^*)\, \Delta x_i = f(x_1^*)\, \Delta x_1 + f(x_2^*)\, \Delta x_2 + \cdots + f(x_n^*)\, \Delta x_n.$$

Geometrically, we may realize $f(x_i^*)$ as the height of a rectangle with base Δx_i, hence the above Riemann sum provides an approximation of the **net area** bounded by the curve $f(x)$ over the closed interval $[a, b]$. Common tagged partitions are formed by taking x_i^* to be the left- or right-**endpoint** or the **midpoint** of $[x_{i-1}, x_i]$. Each of these tagged partitions uses $n + 1$ **equally-spaced** points $a = x_0 < x_1 < \cdots < x_n = b$; the common length of each interval $[x_{i-1}, x_i]$ is Δx. Considering that

$$b - a = \text{length}([a, b]) = \sum_{i=1}^{n} \Delta x = n \Delta x$$

by the second part of Proposition 1.7.4, we conclude that $\Delta x = \dfrac{b - a}{n}$.

- We denote by $\mathcal{L}_n$ the **left-endpoint Riemann approximation** with $\ell_i = a + (i - 1)\, \Delta x$.
- We denote by $\mathcal{R}_n$ the **right-endpoint Riemann approximation** with $r_i = a + i\Delta x$.
- We denote by $\mathcal{M}_n$ the **midpoint Riemann approximation** with $m_i = a + \big(i - \frac{1}{2}\big)\Delta x$.

Example 1.7.6. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $f(x) = x$ on the closed and bounded interval $[0, 4]$ using four equally-spaced points. By recognizing that $a = 0$ and $b = 4$, the common length of each interval of the partition is

$$\Delta x = \frac{4 - 0}{4} = \frac{4}{4} = 1.$$

Consequently, the left-endpoint approximation satisfies that $\ell_i = 0 + (i - 1)1 = i - 1$; the right-endpoint approximation satisfies that $r_i = 0 + i = i$; and the midpoint approximation satisfies that $m_i = 0 + \big(i - \frac{1}{2}\big)1 = i - \frac{1}{2}$ for each integer $1 \leq i \leq 4$. We conclude therefore that the following hold.

$$\mathcal{L}_4 = \sum_{i=1}^{4} f(\ell_i)\, \Delta x = \sum_{i=1}^{4} \ell_i = \sum_{i=1}^{4} (i - 1) = \sum_{i=1}^{4} i - \sum_{i=1}^{4} 1 = \frac{4(4 + 1)}{2} - 4 = 6$$

$$\mathcal{R}_4 = \sum_{i=1}^{4} f(r_i)\, \Delta x = \sum_{i=1}^{4} r_i = \sum_{i=1}^{4} i = \frac{4(4 + 1)}{2} = 10$$

$$\mathcal{M}_4 = \sum_{i=1}^{4} f(m_i)\, \Delta x = \sum_{i=1}^{4} \left(i - \frac{1}{2}\right) = \sum_{i=1}^{4} i - \sum_{i=1}^{4} \frac{1}{2} = 10 - \frac{1}{2}(4) = 8$$

Example 1.7.7. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $g(x) = x^2$ on the closed and bounded interval $[0, 1]$ using five equally-spaced points. Like before, the first step is to see that $a = 0$ and $b = 1$ so that the common length of each interval is

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}.$$

By the above, the left-endpoint approximation uses the sample points $\ell_i = 0 + (i-1)\,\Delta x = \frac{i-1}{5}$; the right-endpoint approximation uses the sample points $r_i = 0 + i\Delta x = \frac{i}{5}$; and the midpoint approximation uses the sample points $m_i = 0 + \bigl(i - \frac{1}{2}\bigr)\Delta x = \frac{2i-1}{10}$. We conclude that

$$\mathcal{L}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{\ell_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^2 = \frac{1}{5}\left(0 + \frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right) = \frac{30}{75} = \frac{6}{25}$$

$$\mathcal{R}_5 = \sum_{i=1}^{5} g(\ell_i)\,\Delta x = \sum_{i=1}^{5} \frac{r_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{i}{5}\right)^2 = \frac{1}{5}\left(\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25} + \frac{25}{25}\right) = \frac{55}{75} = \frac{11}{25}$$

$$\mathcal{M}_5 = \sum_{i=1}^{5} g(m_i)\,\Delta x = \sum_{i=1}^{5} \frac{m_i^2}{5} = \frac{1}{5}\sum_{i=1}^{5}\left(\frac{2i-1}{10}\right)^2 = \frac{1}{5}\left(\frac{1}{100} + \frac{9}{100} + \frac{25}{100} + \frac{49}{100} + \frac{81}{100}\right) = \frac{33}{100}$$

Example 1.7.8. Compute the left- and right-endpoint and midpoint Riemann approximations of the curve $h(x) = x^3$ on the closed and bounded interval $[0, 2]$ using eight equally-spaced points.

By allowing the number of sample points to grow arbitrarily large, the error of approximating the area bounded by a curve of one variable by a Riemann sum shrinks to zero, hence we define

$$\text{area bounded by the curve } f(x) \text{ on the closed and bounded interval } [a, b] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i,$$

where x_i^* are sample points of a partition $\mathcal{P}$ of $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$.

Example 1.7.9. Let us apply the definition to compute the area bounded by the curve $f(x) = x^2$ on the closed and bounded interval $[0, 1]$. Crucially, the above definition does not depend on the sample points x_i^* or the partition $\mathcal{P}$ of $[0, 1]$, so we may select these to make things as convenient as possible. Given any choice of equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$, we have that $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. We may choose the right-endpoint approximation so that $x_i^* = \frac{i}{n}$ and

$$\mathcal{R}_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

by the second part of Proposition 1.7.5. By taking the limit as $n \to \infty$, we conclude that

$$\text{area bounded by } x^2 \text{ on } [0, 1] = \lim_{n\to\infty} \mathcal{R}_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}.$$

1.8 Definite Integration

Given any real function $f(x)$ and any real numbers a and b, consider any collection of points $(x_n, f(x_n))$ on the graph of $f(x)$ with $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ for each integer $1 \leq i \leq n$. Each of the closed and bounded intervals $[x_{i-1}, x_i]$ gives rise to a partition $\mathcal{P}$ of the closed interval $[a, b]$, and we may choose sample points x_i^* for each integer $1 \leq i \leq n$ such that $x_{i-1} \leq x_i^* \leq x_i$ and $x_1^* < x_2^* < \cdots < x_n^*$. Crucially, we are not assuming here that the points $x_0, x_1, \dots, x_n$ are equally-spaced, hence we may denote $\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\}$. We define

$$\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\, \Delta x_i$$

as the **definite integral** of $f(x)$ on the closed and bounded interval $[a, b]$. Provided that the above limit exists, we say that $f(x)$ is **integrable** on $[a, b]$. We refer to the function $f(x)$ in this case as the **integrand**; the real numbers a and b are the **limits of integration**. By our work in the previous section, we may interpret the definite integral $\int_a^b f(x)\, dx$ as the net area bounded by $f(x)$: indeed, $\sum_{i=1}^n f(x_i^*)\, \Delta x_i$ is a Riemann sum representing rectangles of height $f(x_i^*)$ and width Δx_i.

Example 1.8.1. Express the following as the definite integral of a function on the interval $[1, 8]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2}\, \Delta x_i$$

Considering that we do not know the partition $\mathcal{P}$ or the sample points x_i^*, there is not much we can do other than recognize the function $f(x)$. Comparing the limit with the definition above, we recognize that $f(x) = \sqrt{2x + x^2}$ so that the limit in question is $\int_1^8 \sqrt{2x + x^2}\, dx$.

Example 1.8.2. Express the following as the definite integral of a function on the interval $[0, \pi]$.

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n x_i^* \sin(x_i^*)\, \Delta x_i$$

Often, it is most simple to work with a **regular partition** $\mathcal{P}$, i.e., a partition of $[a, b]$ with $n+1$ equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$ such that $\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x = \frac{b-a}{n}$. Under this identification, we have that $\Delta x_1 = x_1 - x_0$ so that $x_1 = x_0 + \Delta x_1 = a + \Delta x$, from which it follows that $x_2 = x_1 + \Delta x_2 = (a + \Delta x) + \Delta x = a + 2\Delta x$ and $x_i = a + i\Delta x$ for each integer $1 \leq i \leq n$. Choosing our sample points such that $x_i^* = x_i = a + i\Delta x$ and using the fact that

$$\|\mathcal{P}\| = \max\{\Delta x_i \mid 1 \leq i \leq n\} = \Delta x = \frac{b-a}{n}$$

approaches zero if and only if n approaches ∞, we conclude that

$$\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^n f(x_i^*)\, \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^n \frac{f(a + i\Delta x)(b-a)}{n}.$$

Example 1.8.3. Express the following as the definite integral of a function on a closed interval.

$$\lim_{n \to \infty} \sum_{i=1}^n \cos\left(-\pi + i\frac{2\pi}{n}\right)\left(\frac{2\pi}{n}\right)$$

Considering that $\Delta x = \frac{2\pi}{n} = \frac{b-a}{n}$ and $a = -\pi$, we must have that $b = a + n\Delta x = -\pi + 2\pi = \pi$. Even more, the function in question is $\cos(x)$, hence the limit describes the quantity $\int_{-\pi}^{\pi} \cos(x)\,dx$.

Before we endeavor to compute any definite integrals by the limit definition provided above, it is conceptually important to note that the definite integral can be computed by hand in some cases without appealing to any limits. Explicitly, for any real numbers c and d, we have that $\int_a^b (cx+d)\,dx$ represents the net area bounded by the line $cx+d$ and the coordinate axes. Consequently, this area can be computed geometrically as a linear combination of areas of triangles and rectangles.

Example 1.8.4. Compute the definite integral $\int_{-2}^{3} (3x-2)\,dx$ using geometry.

Example 1.8.5. Compute the definite integral $\int_{-3}^{2} (5-2x)\,dx$ using geometry.

Likewise, for any function of the form $y = f(x) = \sqrt{r^2 - x^2}$, it follows that $x^2 + y^2 = r^2$ yields a circle of radius r, hence we can determine an integral of the form $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$.

Example 1.8.6. Compute the definite integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ using geometry.

Often, we will deal with definite integrals that cannot be computed by geometry; for now, if we encounter this situation, we can sometimes use the limit definition of the definite integral.

Example 1.8.7. Compute the definite integral $\int_0^1 x^2\,dx$ as the limit of a Riemann sum as the number of terms approaches infinity. Considering that $a = 0$ and $b = 1$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so that $a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$. Consequently, it follows that

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}.$$

Example 1.8.8. Compute the definite integral $\int_0^3 (x^3 - 6x)\,dx$ as the limit of a Riemann sum as the number of terms approaches infinity. Considering that $a = 0$ and $b = 3$, we have that

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so that $a + i\Delta x = 0 + \frac{3i}{n} = \frac{3i}{n}$. Consequently, it follows that

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{3}{n^2} \sum_{i=1}^{n} \left(\frac{27i^3}{n^2} - 18i\right).$$

Granted that the limit of each of these Riemann sums exists, the limit of their difference is given by the difference of their limits, hence it suffices to compute these limits separately.

$$\lim_{n\to\infty} \frac{3}{n^2} \sum_{i=1}^{n} \frac{81i^3}{n^2} = \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{81}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{81}{4}$$

$$\lim_{n\to\infty} \frac{3}{n^2} \sum_{i=1}^{n} 18i = \lim_{n\to\infty} \frac{54}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{54}{n^2} \cdot \frac{n(n+1)}{2} = \frac{54}{2} = \frac{108}{4}$$

Consequently, we have that $\int_0^3 (x^3 - 6x)\,dx = \frac{81}{4} - \frac{108}{4} = -\frac{27}{4}$.

Based on the definition of the definite integrals and the summation properties outlined in the previous section, we can extrapolate the following properties of definite integrals.

Proposition 1.8.9 (Properties of Definite Integrals)**.** *Given any real function $f(x)$ that is integrable on a closed and bounded interval $[a, b]$, the following properties hold for $\int_a^b f(x)\,dx$.*

(i.) (Empty Integral Law) $\int_a^a f(x)\,dx = 0$

(ii.) (Reversing the Limits of Integration) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii.) (Additivity of Adjacent Intervals) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all real numbers* c

(iv.) (Constant Integral Formula) $\int_a^b k\,dx = k(b-a)$ *for all real numbers* k

(v.) (Linearity of a Definite Integral I) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ *for all real numbers* k

(vi.) (Linearity of a Definite Integral II) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 1.8.10. We will compute the definite integral $\int_0^1 (3x^2+4)\,dx$ by appealing to Example 1.8.7 and Proposition 1.8.9. Considering that $\int_0^1 4\,dx = 4(1-0) = 4$, we have that

$$\int_0^1 (3x^2+4)\,dx = \int_0^1 3x^2\,dx + \int_0^1 4\,dx = 3\int_0^1 x^2\,dx + 4 = 3\left(\frac{1}{3}\right) + 4 = 5.$$

Example 1.8.11. Consider any pair of functions $f(x)$ and $g(x)$ such that $\int_{-1}^1 f(x)\,dx = 2$ and $\int_{-1}^1 g(x)\,dx = -1$. We will compute $\int_{-1}^1 [3f(x) - g(x)]\,dx$ by appealing to Proposition 1.8.9.

$$\begin{aligned}\int_{-1}^1 [3f(x)-g(x)]\,dx &= \int_{-1}^1 (3f(x) + [-g(x)])\,dx \\ &= \int_{-1}^1 3f(x)\,dx + \int_{-1}^1 [-g(x)]\,dx \\ &= 3\int_{-1}^1 f(x)\,dx - \int_{-1}^1 g(x)\,dx = 3(2) - (-1) = 7.\end{aligned}$$

Example 1.8.12. Consider any function $f(x)$ such that $\int_0^4 f(x)\,dx = 1$, $\int_{-2}^3 f(x)\,dx = 3$, and $\int_{-2}^0 f(x)\,dx = 5$. We will compute $\int_3^4 f(x)\,dx$ by appealing to Proposition 1.8.9.

$$\begin{aligned}\int_3^4 f(x)\,dx &= \int_3^{-2} f(x)\,dx + \int_{-2}^4 f(x)\,dx \\ &= -\int_{-2}^3 f(x)\,dx + \int_{-2}^4 f(x)\,dx \\ &= -\int_{-2}^3 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_0^4 f(x)\,dx = -3 + 5 + 1 = 3.\end{aligned}$$

1.9 The Fundamental Theorem of Calculus

Calculus can be divided into two topics — differentiation and integration — that are connected by the Fundamental Theorem of Calculus. Essentially, the Fundamental Theorem of Calculus says that differentiation and integration are inverse operations: if $f(x)$ is continuous on an open interval, then $f(x)$ admits an antiderivative by the definite integral, and conversely, the definite integral of $f(x)$ over a closed interval measures the **net change** of any antiderivative over that interval.

Theorem 1.9.1 (Fundamental Theorem of Calculus, Part I)**.** *Given any real function $f(x)$ that is integrable with a continuous antiderivative $F(x)$ on a closed interval $[a, b]$, we have that*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Even more, this quantity measures the net area bounded by the curve $f(x)$ from $x = a$ to $x = b$.

Proof. Observe that the quantity $F(b)-F(a)$ measures the net change of $F(x)$ on the closed interval $[a, b]$. Given any collection of n real numbers $a = x_0 < x_1 < \cdots < x_n = b$, we have that

$$F(b) - F(a) = F(b) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(a)$$

by adding and subtracting $F(x_i)$ for each integer $1 \le i \le n-1$. Grouping each consecutive pair of differences and using the fact that $a = x_0$ and $b = x_n$, it follows that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

By the Mean Value Theorem applied to $F(x)$, for each integer $1 \le i \le n$, there exists a real number x_i^* such that $x_{i-1} \le x_i^* \le x_i$ and $F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1})$. By assumption that $F(x)$ is an antiderivative of $f(x)$ on the closed interval $[a, b]$, we have that $F'(x) = f(x)$, hence we can rewrite each of these equations as $F(x_i) - F(x_{i-1}) = f(x_i^*)\,\Delta x_i$ for the quantity $\Delta x_i = x_i - x_{i-1}$. Going back to our above displayed equation with this new identity, we have that

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

By taking the limit as n approaches ∞ on both sides, we conclude the desired result that

$$F(b) - F(a) = \lim_{n\to\infty} [F(b) - F(a)] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_a^b f(x)\,dx. \qquad \square$$

Consequently, if $v(t)$ measures the velocity of a particle over time, then the (definite) integral of $v(t)$ over $[a, b]$ measures the total distance travelled by the particle from time $t = a$ to time $t = b$.

Example 1.9.2. Compute the net area bounded by the curve $f(x) = x^3$ from $x = -1$ to $x = 1$.

Example 1.9.3. Compute the net area bounded by the curve $g(x) = \sin(x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

Example 1.9.4. Compute the net area bounded by the curve $h(x) = \frac{1}{x}$ from $x = 1$ to $x = e$.

Remark 1.9.5. Like we previously mentioned, if $F(x)$ is an antiderivative of a real function $f(x)$ on a closed interval $[a, b]$, the Mean Value Theorem implies that every antiderivative of $f(x)$ over $[a, b]$ is of the form $F(x) + C$ for some real number C. Consequently, the choice of antiderivative of $f(x)$ does not matter when it comes to computing the definite integral of $f(x)$ on $[a, b]$:

$$\int_a^b f(x)\,dx = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

holds for all real numbers C by the Fundamental Theorem of Calculus, Part I.

One other way to interpret the first part of the Fundamental Theorem of Calculus is as follows.

Corollary 1.9.6 (Net Change Theorem)**.** *Given any differentiable function $f(x)$ on an open interval (a, b) such that $f(a)$ and $f(b)$ are defined, we have that*

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

Put another way, the net change of $f(x)$ over the closed interval $[a, b]$ is $\int_a^b f'(x)\,dx$.

Example 1.9.7. Consider a leaky water heater that loses $2 + 5t$ gallons of water per hour for each hour after 7 AM. Compute the total amount of water leaked between the time of 9 AM and 12 PM.

Example 1.9.8. Consider any medication that disperses into a patient's bloodstream at a rate of $50 - 2\sqrt{t}$ milligrams per hour from the time it is administered. Compute the amount of medication dispersed into a patient's bloodstream one hour after it is administered. Given that one full dose is 50 milligrams, what percentage of the dose reaches the patient's bloodstream in an hour?

Example 1.9.9. Consider any particle that moves with velocity $t^3 - 10t^2 + 24t$ meters per second after initial observation at time $t = 0$. Compute the total displacement of and the total distance travelled by the particle from time $t = 0$ to time $t = 6$; then, compare the values.

Conversely, the second part of the Fundamental Theorem of Calculus states that every continuous function on a closed interval $[a, b]$ admits an antiderivative in the form of a definite integral.

Theorem 1.9.10 (Fundamental Theorem of Calculus, Part II)**.** *Given any real function $f(x)$ that is continuous on a closed interval $[a, b]$, for all real numbers $a < x < b$, we have that*

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Proof. Considering that $f(x)$ is continuous on $[a, b]$, it is integrable on $[a, b]$, hence we may define

$$F(x) = \int_a^x f(t)\,dt$$

for all real numbers $a \leq x \leq b$. We must demonstrate that for all real numbers $a < x < b$, the limit

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}$$

exists. By the second and third parts of Proposition 1.8.9, it follows that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

By the **Mean Value Theorem for Definite Integrals**, there exists a real number c (depending upon h) such that $x < c < x+h$ and $\int_x^{x+h} f(t)\,dt = f(c)[(x+h)-x] = f(c)h$ so that

$$f(c) = \frac{F(x+h) - F(x)}{h}.$$

Considering that $f(x)$ is continuous on the closed interval $[a, b]$, it follows that

$$f\Big(\lim_{h\to 0} c\Big) = \lim_{h\to 0} f(c) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = F'(x),$$

hence it suffices to compute the limit of c as h approaches 0. By the Squeeze Theorem, we have

$$x = \lim_{h\to 0} x \leq \lim_{h\to 0} c \leq \lim_{h\to 0} (x+h) = x$$

so that $\lim_{h\to 0} c = x$ and $F'(x) = f(x)$ for all real numbers $a < x < b$, as desired. $\square$

Example 1.9.11. Compute the derivative of $\int_0^x \sin(t)\,dt$ for any real number $x > 0$.

Example 1.9.12. Compute the derivative of $\int_{-1}^x e^t\,dx$ for any real number $x > -1$.

Example 1.9.13. Compute the derivative of $\int_1^x \ln(t)\,dt$ for any real number $x > 1$.

Example 1.9.14. Given any differentiable real functions $f(x)$, $g(x)$, and $h(x)$, use the Fundamental Theorem of Calculus, Part II and the Chain Rule for derivatives to prove that

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f'(h(x))h'(x) - f'(g(x))g'(x).$$

Example 1.9.15. Compute the derivative of $\int_0^{x^2} \sin(\cos(t))\,dt$ for any real number $x > 0$.

Example 1.9.16. Compute the derivative of $\int_{\ln(x)}^{10} \sqrt{t^2+1}\,dt$ for any real number $0 < x < e^{10}$.

Example 1.9.17. Compute the derivative of $\int_{x^3}^{x^2} \sqrt{t}\,dt$ for any real number $0 < x < 1$.