Serre's Condition and Cohen-Macaulayness

Dylan C. Beck

28 October 2021

Contents

1 Conventions and Prerequisites

We will assume throughout this expository note that R is a commutative unital ring with additive identity 0_R and multiplicative identity 1_R. Recall that an **ideal** I of R is a subgroup of R that is closed under multiplication by elements of R, i.e., we have that $ri \in I$ for every element $r \in R$ and $i \in I$. We say that a proper ideal P of R is **prime** if and only if the quotient ring $R/P = \{r + P \mid r \in R\}$ is a **domain**. We say that a proper ideal M of R is **maximal** if and only if R/M is a **field**. By convention and for convenience, we make the following definitions, as well.

Definition 1.1. We denote by $\operatorname{Spec}(R)$ the collection of prime ideals of R, i.e.,

$$\operatorname{Spec}(R) = \{P \subseteq R \mid P \text{ is a prime ideal of } R\}.$$

Occasionally, we will write $\operatorname{MaxSpec}(R) = \{M \subseteq R \mid M \text{ is a maximal ideal of R }\}$. We refer to $\operatorname{Spec}(R)$ as the **spectrum** of R and to $\operatorname{MaxSpec}(R)$ as the **maximal spectrum** of R.

Example 1.2. Let $\mathbb{Z}$ denote the ring of integers. We have that $\operatorname{Spec}(\mathbb{Z}) = \{p\mathbb{Z} \mid p \text{ is prime}\} \cup \{0\}$ because $\mathbb{Z}$ is a Euclidean domain and $\operatorname{MaxSpec}(\mathbb{Z}) = \operatorname{Spec}(\mathbb{Z}) \setminus \{0\}$.

Definition 1.3. We say that R is **Noetherian** if any of the following equivalent conditions hold.

(i.) Every ascending chain of ideals of R stabilizes. Explicitly, for every sequence of inclusions of ideals $I_1 \subseteq I_2 \subseteq \cdots$, there exists an integer $n \gg 0$ such that $I_k = I_n$ for all integers $k \geq n$.

(ii.) Every nonempty collection of ideals has a maximal element with respect to inclusion.

(iii.) Every ideal I of R is finitely generated. Explicitly, there exist elements $x_1, \dots, x_n \in I$ such that for every element $x \in I$, we have that $x = r_1 x_1 + \cdots + r_n x_n$ for some elements $r_1, \dots, r_n \in R$.

Example 1.4. Let k be a field. Observe that the ideals of k are $\{0_k\}$ and k: indeed, the ideals of k (or any ring) are in one-to-one correspondence with the kernels of the unital ring homomorphisms $k \to S$ as S ranges over all commutative unital rings. Every nonzero element of k is a unit, so any unital ring homomorphism $\varphi : k \to S$ must be injective or identically zero, i.e., $\ker \varphi = \{0_k\}$ or $\ker \varphi = k$. Both of these are finitely generated ideals, as k is generated as an ideal by 1_k (as with any ring). Consequently, any field k is Noetherian by Definition 1.3(iii.).

Definition 1.5. We say that R is **local** if R admits a *unique* maximal ideal $\mathfrak{m}$. For emphasis, we write $(R, \mathfrak{m}, k)$ to denote the local ring R with unique maximal ideal $\mathfrak{m}$ and **residue field** $k = R/\mathfrak{m}$.

Proposition 1.6. Let R be a commutative unital ring. The following conditions are equivalent.

(1.) R is local with unique maximal ideal $\mathfrak{m}$.

(2.) For every element $r \in R$, either r or $1_R + r$ is a unit.

Example 1.7. Given a field k and indeterminate x, consider the quotient ring $S = k[x]/(x^2)$. We denote by $\bar{x}$ the class of x modulo (x^2). By the Correspondence Theorem, the ideals of S are in bijection with the ideals of $k[x]$ that contain (x^2) via the map that sends an ideal I of $k[x]$ to the ideal $I/(x^2)$ of S. Considering that $k[x]$ is a principal ideal domain, the ideals of S are (0_S), $(\bar{x})$, and S, corresponding to the ideals (x^2), (x), and $k[x]$, respectively. Of these, $(\bar{x})$ is maximal by the Third Isomorphism Theorem. Consequently, $(S, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m} = (\bar{x})$.

2 Krull Dimension and Height

One of the most important invariants of a commutative unital Noetherian ring is its dimension.

Definition 2.1. We define the **(Krull) dimension** of R to be the extended natural number

$$\dim(R) = \sup\{n \mid P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n \text{ and } P_0, P_1, \dots, P_n \in \operatorname{Spec}(R)\},$$

i.e., $\dim(R)$ is the supremum of the lengths of strictly descending chains of prime ideals of R.

Example 2.2. Let k be a field. We have already seen in Example 1.4 that k is a Noetherian ring with $\operatorname{Spec}(k) = \{0_k\} = \operatorname{MaxSpec}(k)$. (By an abuse of notation, we will often use 0_k to denote both the zero element and the zero ideal of k.) Consequently, we have that $\dim(k) = 0$: indeed, 0_k is the only prime ideal of k, hence the only strictly descending chain of prime ideals of k is 0_k of length 0.

Example 2.3. By Example 1.2, we have that $\operatorname{Spec}(\mathbb{Z}) = \{p\mathbb{Z} \mid p \text{ is a prime}\} \cup \{0\}$. Consequently, every strictly descending chain of prime ideals of $\mathbb{Z}$ is of the form $p\mathbb{Z} \supsetneq \{0\}$ for some prime p. (We assume implicitly that a prime p is nonzero.) We conclude that $\dim(\mathbb{Z}) = 1$.

On the other hand, we note that $\mathbb{Z}$ is a principal ideal domain, hence every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer $n \geq 0$. By the Fundamental Theorem of Arithmetic, we have that $n = p_1^{e_1} \cdots p_k^{e_k}$ for some distinct primes $p_1, \dots, p_n$ and integers $e_1, \dots, e_k \geq 0$, so any ascending chain of ideals beginning with $n\mathbb{Z}$ eventually stabilizes in $\mathbb{Z}$. By Definition 1.3(i.), $\mathbb{Z}$ is Noetherian.

Proposition 2.4. A principal ideal domain has (Krull) dimension at most one.

Proof. Every nonzero prime ideal of a principal ideal domain is maximal. Consequently, every maximal strictly descending chain of prime ideals consists of a nonzero prime (maximal) ideal and the zero ideal. We conclude that the (Krull) dimension of a PID is at most one. □

Corollary 2.5. Let k be a field. We have that $\dim(k[x]) = 1$.

One can show moreover that the n-variate polynomial ring over a field k has dimension n.

Proposition 2.6. Let k be a field. We have that $\dim(k[x_1, \dots, x_n]) = n$.

Essentially, the idea is to proceed by induction: the base case has already been established by Corollary 2.5; however, even in this case, the proof is beyond the scope of this expository note.

dGenerally, the following result holds for polynomial rings over Noetherian rings.

Proposition 2.7. Let R be a Noetherian ring. We have that $\dim(R[x_1, \dots, x_n]) = \dim(R) + n$.

Caution: there exist Noetherian rings of infinite (Krull) dimension; see this Math StackExchange post. On the other hand, there exist commutative unital rings of finite (Krull) dimension that are not Noetherian; see this MathOverflow post. Both of these examples are quite involved, which illustrates that such rings are more pathological than ubiquitous. Even more, we will soon see that every Noetherian local ring has finite (Krull) dimension (cf. Corollary 2.12).

Computing the dimension of an arbitrary commutative unital ring can be computationally burdensome. Our immediate aim is therefore to introduce several concepts and facts that can be used to simplify this procedure. We begin by describing the dimension of R in a different way.

Definition 2.8. We define the **height** of a prime ideal P of R to be the extended natural number

$$\operatorname{ht}(P) = \sup\{n \mid P \supseteq P_1 \supsetneq \cdots \supsetneq P_{n-1} \supsetneq P_n \text{ and } P_1, \dots, P_{n-1}, P_n \in \operatorname{Spec}(R)\},$$

i.e., $\operatorname{ht}(P)$ is the supremum of the lengths of strictly descending chains of prime ideals contained in P. Given an arbitrary ideal I of R, we define $\operatorname{ht}(I) = \inf\{\operatorname{ht}(P) \mid P \supseteq I \text{ and } P \in \operatorname{Spec}(R)\}$.

Proposition 2.9. We have that $\dim(R) = \sup\{\operatorname{ht}(M) \mid M \in \operatorname{MaxSpec}(R)\}$. Put another way, the (Krull) dimension of R is the supremum of the heights of the maximal ideals of R.

Proof. Every strictly descending chain of prime ideals begins with (or can be extended to a strictly descending chain of prime ideals that begins with) a maximal ideal because every maximal ideal is prime and every (prime) ideal is contained in a maximal ideal. Consequently, every maximal strictly descending chain of prime ideals begins with a maximal ideal, and the inequality $\geq$ holds. Conversely, every strictly descending chain of prime ideals contained in a maximal ideal M gives rise to a strictly descending chain of prime ideals of R, and the inequality $\leq$ holds. □

Caution: there exist commutative unital rings in which two maximal ideals have different heights. In fact, there exist Hilbert domains with this property; see this paper by Roberts.

Example 2.10. By Proposition 2.9, for a local ring $(R, \mathfrak{m})$, we have that $\dim(R) = \operatorname{ht}(\mathfrak{m})$. Particularly, for any prime ideal P of R, we have that $\dim(R_P) = \operatorname{ht}(P)$, where

$$R_P = \left\{\frac{r}{s} : r \in R,\, s \in R \setminus P, \text{ and } \frac{r}{s} = \frac{r'}{s'} \iff \exists t \in R \setminus P \text{ s.t. } t(rs' - r's) = 0_R\right\} = S^{-1}R$$

is the localization of R at the multiplicatively closed set $S = R \setminus P$.

Our next proposition shows that height is a well-behaved invariant.

Proposition 2.11. Let I and J be ideals of R.

(1.) If $I \subseteq J$, then $\operatorname{ht}(I) \leq \operatorname{ht}(J)$.

(2.) We have that $\operatorname{ht}(I) = \operatorname{ht}(\sqrt{I})$, where $\sqrt{I}$ is the **radical** of I, i.e.,

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some integer } n \geq 1\}.$$

(3.) We have that $\operatorname{ht}(I) + \dim(R/I) \leq \dim(R)$.

(4.) If R is a domain that is a finitely generated algebra over a field, then

$$\operatorname{ht}(I) + \dim(R/I) = \dim(R).$$

(5.) (Krull's Height Theorem) If I is finitely generated by at least n generators, then $\operatorname{ht}(I) \leq n$.

Proof. (1.) Observe that any prime ideal P such that $P \supseteq J$ satisfies $P \supseteq I$, hence any prime ideal that satisfies $\operatorname{ht}(J) = \operatorname{ht}(P)$ must satisfy $\operatorname{ht}(I) \leq \operatorname{ht}(P) = \operatorname{ht}(J)$.

(2.) Observe that a prime ideal P satisfies $P \supseteq I$ if and only if it satisfies $P \supseteq \sqrt{I}$. One direction is clear in view of the fact that $I \subseteq \sqrt{I}$. Conversely, if $P \supseteq I$, then for any element $r \in \sqrt{I}$, we have that $r^n \in I$ implies that $r^n \in P$ so that $r \in P$ by the primality of P, i.e., $P \supseteq \sqrt{I}$.

(3.) Let P be a prime ideal of R such that $\operatorname{ht}(I) = \operatorname{ht}(P)$. If $\operatorname{ht}(P)$ is infinite, then we obtain an infinite strictly descending chain of prime ideals $P \supseteq P_1 \supseteq \cdots$, hence $\dim(R)$ is infinite. Otherwise, we obtain a strictly descending chain of prime ideals $P \supsetneq P_1 \supsetneq \cdots \supsetneq P_{n-1} \supsetneq P_n$. On the other hand, every strictly descending chain of prime ideals of R/I corresponds to a strictly descending chain of prime ideals of R such that the smallest (with respect to inclusion) prime ideal contains I. By construction, the longest among these ends with P, so we obtain a strictly descending chain of prime ideals $Q_m \supsetneq \cdots Q_1 \supsetneq P \supsetneq P_1 \cdots \supsetneq P_n$ of R. By definition, we have that

$$\operatorname{ht}(I) + \dim(R/I) = n + m \leq \dim(R).$$

We omit the proofs of (4.) and (5.) for sake of simplicity. □

Corollary 2.12. Every Noetherian local ring has finite (Krull) dimension.

Proof. Let $(R, \mathfrak{m})$ be a Noetherian local ring. By Example 2.10, we have that $\dim(R) = \operatorname{ht}(\mathfrak{m})$. Considering that R is Noetherian, we have that $\mathfrak{m}$ is finitely generated by Definition 1.3(iii.), hence $\operatorname{ht}(\mathfrak{m})$ is finite by Proposition 2.11(5.) (Krull's Height Theorem). □

Corollary 2.13. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let $\mu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, where $\mathfrak{m}/\mathfrak{m}^2$ is viewed as a k-vector space.

(1.) We have that $\mu(\mathfrak{m})$ is the minimum number of generators of $\mathfrak{m}$.

(2.) We have that $\dim(R) \leq \mu(\mathfrak{m})$.

Proof. Observe that (1.) holds by Nakayama's Lemma; (2.) holds by Krull's Height Theorem. □

On its own, the invariant $\mu(\mathfrak{m})$ of a Noetherian local ring $(R, \mathfrak{m})$ is of critical importance.

Definition 2.14. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with residue field $k = R/\mathfrak{m}$. We refer to the invariant $\mu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ as the **embedding dimension** of R.

Definition 2.15. We say that a Noetherian local ring $(R, \mathfrak{m})$ is **regular** if $\dim(R) = \mu(\mathfrak{m})$.

Example 2.16. By Example 2.2, every field is a regular local ring of dimension 0: the zero ideal is its unique maximal ideal. Conversely, every regular local ring of dimension 0 is a field.

By the end of this expository note, we will have the tools to prove the following.

Proposition 2.17. [BH, Proposition 2.2.3] Every regular local ring is a domain.

We direct our attention now to the height-zero prime ideals of a ring.

Definition 2.18. We refer to a prime ideal P of R such that $\operatorname{ht}(P) = 0$ as a **minimal prime** of R. Equivalently, P is a minimal prime of R if and only if for any prime ideal Q of R such that $Q \subseteq P$, we have that $P = Q$. Equivalently, P is a minimal prime of R if and only if $\dim(R_P) = 0$. Occasionally, it will be convenient for us to denote $\operatorname{MinSpec}(R) = \{P \subseteq R \mid P \text{ is a minimal prime ideal of } R\}$.

Proposition 2.19. Let R be a commutative Noetherian ring. We have that $|\operatorname{MinSpec}(R)| < \infty$. Put another way, there are only finitely many minimal prime ideals of R.

Proof. (Daniel Katz) We will prove moreover that there are only finitely many minimal prime ideals of the quotient ring R/I for any ideal I of R. Our claims holds by taking I to be the zero ideal.

Observe that the primes in $\operatorname{MinSpec}(R/I)$ are precisely the primes P of R such that $I \subseteq P$ is a maximal chain of prime ideals containing I. On the contrary, we will assume that $\operatorname{MinSpec}(R/I)$ is infinite, i.e., there exist infinitely many primes $\{P_i\}_{i=1}^\infty$ such that $I \subseteq P_i$ for each integer $i \geq 1$. Observe that the collection $\mathcal{J} = \{J \subseteq R \,|\, J \text{ is an ideal of } R \text{ and } J \subseteq P_i \text{ for infinitely many } i\}$ is nonempty by assumption, hence there exists a maximal element P of $\mathcal{J}$ by hypothesis that R is Noetherian. We claim that P is prime, hence we have reached a contradiction. Explicitly, if P is prime, then we must have that $P \supseteq P_i$ for all integers $i \geq 1$ by the minimality of P_i. But also, we must have that $P \subseteq P_i$ for all integers $i \geq 1$ by definition of P — a contradiction.

On the contrary, if P were not prime, then there would exist elements $a, b \in R \setminus P$ with $ab \in P$. Consequently, we have that $P \subsetneq P + aR$ and $P \subsetneq P + bR$, hence by the maximality of P, neither $P + aR$ nor $P + bR$ is contained in infinitely many of the P_i. Observe that $(P + aR)(P + bR) \subseteq P$, hence $(P + aR)(P + bR)$ is contained in infinitely many of the P_i so that either $P + aR$ or $P + bR$ is contained in infinitely many of the P_i — a contradiction. Ultimately, P is prime. $\square$

3 Regular Sequences and Associated Primes

One of our main objectives throughout this expository note is to establish the property of Proposition 2.11(4.) for a more general class of commutative unital Noetherian rings. To do this, we must relate the topological invariant of (Krull) dimension with some homological invariant.

We will assume throughout this section that M is an R-module. By definition, $(M, +)$ is an abelian group such that there is a map $\cdot : R \times M \to M$ that sends $(r, m) \mapsto r \cdot m$ satisfying

(1.) $r \cdot (m + n) = r \cdot m + r \cdot n$,

(2.) $(r + s) \cdot m = r \cdot m + s \cdot m$,

(3.) $r \cdot (s \cdot m) = (rs) \cdot m$, and

(4.) $1_R \cdot m = m$

for all elements $r, s \in R$ and $m, n \in M$. Clearly, R is an R-module. We will reserve the notation 0 for the zero element of M. Often, it will be convenient to write $r \cdot m$ as rm with the understanding that r is an element of R that is acting on the element m of M via the specified action.

Definition 3.1. We say that an element $x \in R$ is **M-regular** whenever

(i.) $xm = 0$ implies that $m = 0$ and

(ii.) $xM \neq M$.

In case x only satisfies condition (i.), we say that x is **weakly M-regular**. One might also see some authors refer to such an element as a **non-zero divisor** of M. Under this naming convention, an element $x \in R$ that does not satisfy condition (i.) of Definition 3.1 is called a **zero divisor** of M.

Remark 3.2. We note that condition (ii.) of Definition 3.1 is a provision to prevent the "degenerate" case. Particularly, if $M = 0$, then $xm = 0$ implies that $m = 0$ trivially, hence every element of R is M-regular for the zero module. On the other hand, every unit u of a ring satisfies $uR = R$, so we would like to restrict our attention to non-units acting on nonzero modules.

We will soon focus explicitly on the case that $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module. Given that $x \in \mathfrak{m}$ satisfies $xM = M$, it would follow by Nakayama's Lemma that $M = 0$, hence condition (i.) would be satisfied trivially. On the other hand, if $M \neq 0$, then $xM \neq M$ for any element $x \in \mathfrak{m}$ by the contrapositive of Nakayama's Lemma. Consequently, condition (ii.) in Definition 3.1 is satisfied by any element of $\mathfrak{m}$ (i.e., any non-unit of R).

Example 3.3. Every nonzero non-unit of $\mathbb{Z}$ is $\mathbb{Z}$-regular because $\mathbb{Z}$ is a domain that is not a field. In fact, this is the case with any domain that is not a field. On the other hand, for any nonzero element n of $\mathbb{Z}$, we have that $n\mathbb{Q} = \mathbb{Q}$, hence a nonzero integer is only weakly $\mathbb{Q}$-regular.

Definition 3.4. We say that a sequence $(x_1, \ldots, x_n) \in R$ is an **M-regular sequence** whenever

(i.) x_1 is an M-regular element of R and

(ii.) x_{i+1} is an $M/(x_1, \ldots, x_i)M$-regular element of R for each integer $1 \leq i \leq n-1$.

Like before, we say that $(x_1, \ldots, x_n)$ is a **weakly M-regular** sequence if x_1 is weakly M-regular or x_{i+1} is weakly $M/(x_1, \ldots, x_i)M$-regular for some integer $1 \leq i \leq n-1$.

Caution: a permutation of a (weakly) M-regular sequence is not necessarily (weakly) M-regular.

Example 3.5. [BH, Exercise 1.1.3] Consider the polynomial ring $S = k[x, y, z]$ over a field k. Observe that x is an S-regular element because it is a nonzero element of the domain S. Further, we have that $y - xy$ is an $S/(x)$-regular element because it is equal to y modulo x and $S/(x) \cong k[y, z]$. Last, we have that $z - xz$ is an $S/(x, y - xy)$-regular element because $(x, y - xy) = (x, y)$ implies that $S/(x, y - xy) \cong k[z]$ and $z - xz$ is equal to z modulo $(x, y - xy)$. We conclude therefore that $(x, y - xy, z - xz)$ is an S-regular sequence. On the other hand, the sequence $(y - xy, z - xz, x)$ is not S-regular because $(z - xz)y = z(y - xy)$ shows that $z - xz$ is not $S/(y - xy)$-regular.

Our next proposition shows that under mild conditions, a permutation of an M-regular sequence is once again M-regular. We omit the proof, but we note that it follows from Nakayama's Lemma.

Proposition 3.6. [BH, Proposition 1.1.6] Let $(R, \mathfrak{m})$ be Noetherian local ring. Let M be a finitely generated R-module. If $(x_1, \ldots, x_n)$ is an M-regular sequence, then so is any permutation of it.

We have characterized nonzero elements of R whose action on any nonzero element of M results in a nonzero element of M as (weakly) M-regular (or as a non-zero divisor on M). We will now investigate those elements of R whose action on a given nonzero element of M is always zero.

Definition 3.7. Let M be a nonzero R-module. We define the R-**annihilator** of a nonzero element $m \in M$ as $\operatorname{ann}_R(m) = \{r \in R \mid rm = 0\}$. Often, we will refer to this simply as the annihilator of m. We define also the R-**annihilator** of the entire module M as $\operatorname{ann}_R(M) = \bigcap_{m \in M} \operatorname{ann}_R(m)$.

Observe that the annihilator of any nonzero element $m \in M$ is an ideal of R: indeed, if r and s belong to $\operatorname{ann}_R(m)$, then we have that $(r+s)m = rm + sm = 0$ and $(ar)m = a(rm) = a(0) = 0$ for all elements $a \in R$. Consequently, we may consider the case that $\operatorname{ann}_R(m)$ is a prime ideal of R.

Definition 3.8. Let M be a nonzero R-module. We say that a prime ideal P of R is an **associated prime** of M if there exists a nonzero element $m \in M$ such that $P = \operatorname{ann}_R(m)$.

Example 3.9. Let $S = k[x]$ be the univariate polynomial ring over a field k. Let $M = k[x]/(x^2)$. We will denote by $\bar{x}$ the class of x modulo x^2. Observe that $x\bar{x} = \bar{x}^2 = \bar{0}_k$, hence the ideal of S generated by x is contained in the annihilator of $\bar{x}$, i.e., $(x) \subseteq \operatorname{ann}_S(\bar{x})$. But (x) is a maximal ideal of S and $\operatorname{ann}_R(\bar{x})$ is a proper ideal of S, hence we have that $\operatorname{ann}_S(\bar{x}) = (x)$ is an associated prime of M. Observe that (x) is also a minimal prime ideal of S. We will soon see that this is no coincident.

Before we proceed, we should investigate sufficient conditions for the existence of associated primes of a nonzero module. Unfortunately, this requires additional tools that are not immediately relevant to us; instead, we state the following proposition without proof.

Proposition 3.10. Every nonzero module M over a Noetherian ring R admits an associated prime. Further, if M is Noetherian, then M admits only finitely many associated primes.

We denote by $\operatorname{Ass}_R(M)$ the collection of associated primes of a nonzero module M over a Noetherian ring R. By the previous proposition, if M is Noetherian, then $|\operatorname{Ass}_R(M)| < \infty$.

We will now establish a connection between the associated primes of M and M-regular elements.

Proposition 3.11. Let R be Noetherian. Let M be an R-module. The following are equivalent.

(1.) The element $x \in R$ is a zero divisor on M.

(2.) The element $x \in R$ belongs to some associated prime P of M.

Put another way, the collection of zero divisors of M is the union of all associated primes of M.

Proof. Let $x \in R$ be a zero divisor on M. By Proposition 3.10, M admits an associated prime P. If $x = 0_R$, then x belongs to P because every ideal of R contains 0_R. We may assume that x is nonzero. By hypothesis that x is a zero divisor on M, there exists a nonzero element $m \in M$ such that $xm = 0$, hence x belongs to $\operatorname{ann}_R(m)$. Given that $\operatorname{ann}_R(m)$ is prime, our proof is complete. We assume therefore that $\operatorname{ann}_R(m)$ is not prime. By hypothesis that R is Noetherian, the collection

$$\mathscr{A} = \{\operatorname{ann}_R(m') \mid m' \in M, \operatorname{ann}_R(m') \text{ is a proper ideal of } R, \text{ and } \operatorname{ann}_R(m) \subseteq \operatorname{ann}_R(m')\}$$

has a maximal element P because it contains $\operatorname{ann}_R(m)$ by construction. We claim that P is a prime ideal. Consider the case that some elements y and z of R satisfy $yz \in P$ and $z \notin P$. Observe that $P \subseteq \operatorname{ann}_R(ym')$ because every element of P annihilates m' and so must annihilate ym'. On the other hand, we have that $z(ym') = (yz)m' = 0$ by assumption that $yz \in P$, hence z is an element of $\operatorname{ann}_R(ym') \setminus P$. By the maximality of P and the fact that $\operatorname{ann}_R(m) \subseteq \operatorname{ann}_R(ym')$, we must have that $\operatorname{ann}_R(ym') = R$ so that $ym' = 1_R(ym') = 0$ and y annihilates m', i.e., we have that $y \in P$. We conclude that P is an associated prime ideal of M that contains $\operatorname{ann}_R(m)$ and x.

Conversely, if $x \in R$ belongs to some associated prime ideal P of M, then there exists a nonzero element $m \in M$ such that $xm = 0$, hence x is a zero divisor on M. $\square$

Corollary 3.12. Let R be Noetherian. Let M be an R-module. The following are equivalent.

(1.) The element $x \in R$ is M-regular.

(2.) The element $x \in R$ does not belong to any associated prime P of M.

Corollary 3.13. Let R be a Noetherian ring. Let M be an R-module. Let I be an ideal of R that consists of zero divisors of M. There exists an associated prime P of M such that $I \subseteq P$.

Proof. We prove the contrapositive. Given that $I \not\subseteq P$ for all associated primes P of M, there exists an element $x \in I$ such that $x \notin P$ for any associated prime P by the Prime Avoidance Lemma. By Corollary 3.12, we conclude that x is M-regular, i.e., x is not a zero divisor on M. $\square$

One can also view the property that P is an associated prime of M as a homological condition.

Proposition 3.14. Let M be a nonzero R-module. Consider the following conditions.

(1.) P is an associated prime of M.

(2.) M contains an R-submodule that is isomorphic to R/P for some prime ideal P.

(3.) There exists a nonzero R-module homomorphism $\psi : R/P \to M$ for some prime ideal P of R. Put another way, we have that $\operatorname{Hom}_R(R/P, M) \neq 0$ for some prime ideal P of R.

We have that (1.) $\iff$ (2.) $\implies$ (3.). Conversely, if either (a.) P is a maximal ideal of R or (b.) the associated primes of M are the minimal primes of R, then (3.) $\implies$ (1.).

Proof. By definition, if P is an associated prime of M, then there exists a nonzero element $m \in M$ such that $P = \operatorname{ann}_R(m)$. Consider the map $\varphi : R \to M$ defined by $\varphi(r) = rm$. One can easily verify that this is an R-module homomorphism, hence $\varphi(R)$ is an R-submodule of M. By definition, we have that $\ker \varphi = \{r \in R \mid rm = 0\} = \operatorname{ann}_R(m) = P$, and we conclude that $R/P \cong \varphi(R)$.

Conversely, if M contains an R-submodule that is isomorphic to R/P for some prime ideal P of R, then there exists an injective R-module homomorphism $\varphi : R/P \to M$. Consequently, we have that $P = \ker \varphi = \{r + P \mid r\varphi(1_R + P) = 0\} = \operatorname{ann}_R(\varphi(1_R + P))$ is an associated prime of M.

If M contains an R-submodule N such that $\varphi : R/P \to N$ is an R-module isomorphism, then the composite map $\psi : R/P \xrightarrow{\varphi} N \xrightarrow{\iota} M$ is a nonzero R-module homomorphism.

Last, we will assume that there exists a nonzero R-module homomorphism $\psi : R/P \to M$ for some prime ideal P of R. Recall that $\psi : R/P \to M$ is an R-module homomorphism if and only if

(a.) ψ is well-defined, i.e., $r + P = 0_R + P$ implies that $\psi(r + P) = 0$ and

(b.) ψ is R-linear, i.e., $\psi(r + P) = r \cdot \psi(1_R + P)$ for all elements $r \in R$.

Combined, these properties say that every nonzero R-linear homomorphism $R/P \to M$ is uniquely determined by the nonzero element $\psi(1_R + P) \in M$ and $\psi(1_R + P)$ must be annihilated by P. Consequently, we find that $P \subseteq \operatorname{ann}_R(\psi(1_R + P))$. Given that (a.) P is a maximal ideal of R, we conclude that $P = \operatorname{ann}_R(\psi(1_R + P))$ is an associated prime of M. On the other hand, if P is not maximal, it follows by Corollary 3.13 that $P \subseteq Q$ for some associated prime Q of M. Given that (b.) the associated primes of M are the minimal primes of R, we conclude that $P = Q$. $\square$

We shall soon discuss the connection between regular sequences contained in the maximal ideal $\mathfrak{m}$ of a Noetherian local ring $(R, \mathfrak{m}, k)$ and the nonzero R-linear maps $k \to M$. Before we are able to state this relationship explicitly, we investigate the deeper interplay between the M-regular elements of R contained in the annihilator of some R-module N and the R-linear maps $N \to M$.

Proposition 3.15. [BH, Proposition 1.2.3] Let M and N be R-modules. The following hold.

(1.) If $\operatorname{ann}_R(N)$ contains an M-regular element, then $\operatorname{Hom}_R(N, M) = 0$.

(2.) Conversely, if R is Noetherian and M and N are finitely generated, then the condition $\operatorname{Hom}_R(N, M) = 0$ implies that $\operatorname{ann}_R(N)$ contains an M-regular element.

Proof. (1.) Consider an R-module homomorphism $\varphi : N \to M$. For every element $n \in N$ and $x \in \operatorname{ann}_R(N)$, we have that $\varphi(xn) = \varphi(0) = 0$. Considering that φ is R-linear and x belongs to R, we have that $0 = \varphi(xn) = x\varphi(n)$. Given that x is M-regular, we have that $\varphi(n) = 0$. But this holds for every element $n \in N$, hence we conclude that φ is the zero map so that $\operatorname{Hom}_R(N, M) = 0$.

(2.) We omit the proof for sake of simplicity; however, we note that it proceeds by the contrapositive, using Corollary 3.13 to obtain an associated prime of M containing $\operatorname{ann}_R(N)$. $\square$

Example 3.16. Let $S = k[x, y]$ be the bivariate polynomial ring over a field k. Let $M = k[x, y]/(x^2)$, and let $N = k[x, y]/(x, y)$. Observe that x and y annihilate N, hence we have that $\operatorname{ann}_R(N) = (x, y)$. On the other hand, the element $y \in \operatorname{ann}_R(N)$ is M-regular. We conclude that $\operatorname{Hom}_R(N, M) = 0$.

Our next proposition is the basis for the proof of the main theorem of the next section.

Proposition 3.17. Given any R-modules M and N and a weakly M-regular sequence $(x_1, \dots, x_n)$ in $\operatorname{ann}_R(N)$, we have that $\operatorname{Hom}_R(N, M/(x_1, \dots, x_n)M) \cong \operatorname{Ext}_R^n(N, M)$.

Proof. We proceed by induction on n. Observe that $\operatorname{Hom}_R(N, M) \cong \operatorname{Ext}_R^0(N, M)$ (because Ext is the right-derived functor of Hom), hence the claim holds for $n = 0$. We will assume inductively that the claim holds for all integers $1 \leq i \leq n-1$. We note that x_i is an $M/(x_1, \dots, x_{i-1})M$-regular element by hypothesis for each integer $1 \leq i \leq n$, hence Proposition 3.15 implies that $\operatorname{Ext}_R^{i-1}(N, M) = 0$ for each integer $1 \leq i \leq n$ by induction. We note that the short exact sequence

$$0 \to M \xrightarrow{x_n \cdot} M \to M/x_nM \to 0$$

induces a long exact sequence of Ext by applying the covariant functor $\operatorname{Hom}_R(N, -)$. But as we observed in the previous paragraph, the lower Ext vanish by induction, hence we obtain

$$0 \to \operatorname{Ext}_R^{n-1}(N, M/x_nM) \xrightarrow{\psi} \operatorname{Ext}_R^n(N, M) \xrightarrow{\varphi} \operatorname{Ext}_R^n(N, M).$$

$\operatorname{Ext}_R^i(N, -)$ preserves multiplication (as it is a right-derived functor) for all indices $i \geq 0$, hence we have that φ is multiplication by x_n. By hypothesis that x_n belongs to $\operatorname{ann}_R(N)$, we find that φ is the zero map. We conclude that ψ is an isomorphism, i.e., $\operatorname{Ext}_R^{n-1}(N, M/x_nM) \cong \operatorname{Ext}_R^n(N, M)$. Using induction in the second equivalence, we obtain the desired result as follows.

$$\begin{aligned} \operatorname{Ext}_R^n(N, M) &\cong \operatorname{Ext}_R^{n-1}(N, M/x_nM) \\ &\cong \operatorname{Hom}_R\left(N, \frac{M/x_nM}{(x_1, \dots, x_{n-1})M/x_nM}\right) \\ &\cong \operatorname{Hom}_R(N, M/(x_1, \dots, x_n)M) \end{aligned}$$

$\square$

4 Depth and the Cohen-Macaulay Condition

We will assume throughout this section that $(R, \mathfrak{m}, k)$ is a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We will also assume that M is a finitely generated R-module. Our next proposition illustrates the nice behavior of R and M in this setting.

Proposition 4.1. The following statements hold.

(1.) R has finite (Krull) dimension. Further, we have that $\dim(R) = \operatorname{ht}(\mathfrak{m})$.

(2.) R admits finitely many associated primes. In particular, R admits an associated prime.

(3.) An element $x \in R$ is R-regular if and only if x does not belong to any associated prime of R.

(4.) M is a Noetherian R-module.

(5.) Every permutation of an M-regular sequence is an M-regular sequence.

(6.) M admits finitely many associated primes. In particular, M admits an associated prime.

(7.) An element $x \in R$ is M-regular if and only if x does not belong to any associated prime of M.

(8.) We have that $\operatorname{Hom}_R(k, M) = 0$ if and only if $\mathfrak{m}$ contains an M-regular element.

(9.) Given any M-regular sequence $(x_1, \dots, x_n) \in \mathfrak{m}$, for all integers $0 \leq i \leq n-1$, we have that

$$\operatorname{Ext}^i_R(k, M) \cong \operatorname{Hom}_R(k, M/(x_1, \dots, x_i)M) = 0.$$

Proof. Observe that property (1.) holds by Corollary 2.12. Property (2.) holds by Proposition 3.10, and property (6.) holds by the same proposition as soon as we establish property (4.). Properties (3.) and (7.) hold by Corollary 3.12. Property (5.) holds by Proposition 3.5. Property (8.) holds by Proposition 3.15. Property (9.) holds by the proof of Proposition 3.17.

One can show that property (4.) is equivalent to the condition that M is finitely generated when R is a Noetherian ring. Explicitly, if M is finitely generated by n elements, then M is isomorphic to a quotient of the Noetherian R-module $R^{\oplus n}$, hence M is Noetherian. Conversely, if M is Noetherian, then M is finitely generated by the analog of the third condition of Definition 1.3. $\square$

By hypothesis that R is Noetherian, every ascending chain of ideals of R eventually stabilizes. Consequently, we can recursively build M-regular sequences of elements in the maximal ideal $\mathfrak{m}$ of R. Observe that if $\mathfrak{m}$ is an associated prime of M, then every element $x \in \mathfrak{m}$ is a zero divisor on M. Conversely, if $\mathfrak{m}$ is not an associated prime of M, then there exists an M-regular element $x_1 \in \mathfrak{m}$. We can subsequently ask if there exists an M/x_1M-regular element $x_2 \in \mathfrak{m}$. Continuing in this way, we obtain an ascending chain of ideals $(x_1) \subseteq (x_1, x_2) \subseteq \cdots$ that must eventually stabilize. One natural question to ask of this is, "How many elements can we possibly fit in an M-regular sequence?" Our immediate task is to answer this question. We introduce the tools to do so next.

Definition 4.2. We say that an M-regular sequence $\underline{x} = (x_1, \dots, x_n)$ is a **maximal M-regular sequence** if $\mathfrak{m}$ consists of zero divisors for $M/\underline{x}M$, i.e., $\mathfrak{m}$ is an associated prime of $M/\underline{x}M$.

Theorem 4.3. (Rees) Every maximal M-regular sequence in $\mathfrak{m}$ consists of the same number of terms. Particularly, this invariant is referred to as the **depth** of M, and it is given by

$$\operatorname{depth}(M) = \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

Proof. Consider a maximal M-regular sequence $\underline{x} = (x_1, \dots, x_n)$ in $\mathfrak{m}$. By definition, each element x_{i+1} is $M/(x_1, \dots, x_i)M$-regular for each integer $0 \le i \le n-1$. Consequently, we have that

$$\operatorname{Ext}_R^i(k, M) \cong \operatorname{Hom}_R(k, M/(x_1, \dots, x_i)M) = 0$$

for each integer $0 \le i \le n-1$ by Proposition 4.1. On the other hand, by hypothesis that $\underline{x}$ is a maximal M-regular sequence in $\mathfrak{m}$, it follows that $\mathfrak{m}$ consists of zero divisors of $M/\underline{x}M$. By Corollary 3.13, we conclude that $\mathfrak{m}$ is an associated prime of $M/\underline{x}M$. By Proposition 3.14, we conclude that $\operatorname{Hom}_R(k, M/\underline{x}M) \neq 0$ so that $\operatorname{Ext}_R^n(k, M) \cong \operatorname{Hom}_R(k, M/\underline{x}M) \neq 0$. $\square$

Corollary 4.4. We have that $\operatorname{depth}(M) = 0$ if and only if $\mathfrak{m}$ is an associated prime of M.

Proof. Observe that $\operatorname{depth}(M) = 0$ if and only if $\operatorname{Ext}_R^0(k, M) \neq 0$ if and only if $\operatorname{Hom}_R(k, M) \neq 0$ if and only if $\mathfrak{m}$ is an associated prime of M by Proposition 3.14. $\square$

Example 4.5. Let k be a field. Let $k[\![x, y]\!]$ denote the ring of bivariate formal power series. Observe that $k[\![x, y]\!]$ is a Noetherian local ring: it is the completion of the Noetherian ring $k[x, y]$ at the homogeneous maximal ideal (x, y). Consider the Noetherian local ring $R = k[\![x, y]\!]/(x^2, xy)$. We claim that $\operatorname{depth} R = 0$. Each of the generators of the maximal ideal $\mathfrak{m} = (\bar{x}, \bar{y})$ is a zero divisor on R, hence we conclude that $\mathfrak{m}$ is an associated prime of R and $\operatorname{depth} R = 0$ by Corollary 4.4.

Our next proposition illustrates that depth behaves well with respect to short exact sequences.

Proposition 4.6. The **Depth Lemma** asserts that for any short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated R-modules, the following inequalities hold.

(1.) $\operatorname{depth}(L) \ge \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$

(2.) $\operatorname{depth}(M) \ge \min\{\operatorname{depth}(L), \operatorname{depth}(N)\}$

(3.) $\operatorname{depth}(N) \ge \min\{\operatorname{depth}(L) - 1, \operatorname{depth}(M)\}$

Further, if $\operatorname{depth}(M) \ge \operatorname{depth}(N) + 1$, then we have that $\operatorname{depth}(L) = \operatorname{depth}(N) + 1$.

Proof. Consider a short exact sequence $0 \to L \to M \to N \to 0$ of finitely generated modules over a local ring $(R, \mathfrak{m}, k)$. We have that $\operatorname{depth}(L) = \min\{i \mid \operatorname{Ext}_R^i(k, L) \neq 0\}$, hence we may apply $\operatorname{Hom}_R(k, -)$ to our short exact sequence to obtain a long exact sequence

$$0 \to \operatorname{Hom}_R(k, L) \to \operatorname{Hom}_R(k, M) \to \operatorname{Hom}_R(k, N)$$

$$\to \operatorname{Ext}_R^1(k, L) \to \operatorname{Ext}_R^1(k, M) \to \operatorname{Ext}_R^1(k, N) \to \cdots.$$

(i.) Given that $\operatorname{depth}(L) = d$, we have that $\operatorname{Ext}_R^d(k, L) \neq 0$ and $\operatorname{Ext}_R^i(k, L) = 0$ for all integers $0 \le i \le d-2$. Consequently, there are R-module isomorphisms $\operatorname{Ext}_R^i(k, M) \cong \operatorname{Ext}_R^i(k, N)$ for all integers $0 \le i \le d-1$, and the rest of our long exact sequence can be written as

$$0 \to \operatorname{Ext}_R^{d-1}(k, M) \to \operatorname{Ext}_R^{d-1}(k, N) \to \operatorname{Ext}_R^d(k, L) \to \operatorname{Ext}_R^d(k, M) \to \operatorname{Ext}_R^d(k, N) \to \cdots.$$

We claim that $\operatorname{depth}(L) \ge \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$. On the contrary, we will assume that $\operatorname{depth}(M) \ge \operatorname{depth}(L) + 1$ and $\operatorname{depth}(N) \ge \operatorname{depth}(L)$. But this implies that $\operatorname{Ext}_R^{d-1}(k, M) = \operatorname{Ext}_R^d(k, M) = 0$ and $\operatorname{Ext}_R^d(k, L) \cong \operatorname{Ext}_R^{d-1}(k, N) = 0$ — a contradiction. We conclude that $\operatorname{depth}(L) \ge \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$. We note that the other assertions are proved in a similar way by analyzing the situation for $\operatorname{depth}(M)$ and $\operatorname{depth}(N)$. $\square$

Even more, depth behaves well with respect to taking quotients by regular sequences.

Proposition 4.7. Let $\underline{x} = (x_1, \dots, x_n)$ be an M-regular sequence. We have that

$$\operatorname{depth}(M/\underline{x}M) = \operatorname{depth}(M) - n.$$

Proof. By the proof of Proposition 3.17, we have that $\operatorname{Ext}_R^i(k, M) \cong \operatorname{Ext}_R^{i-n}(k, M/\underline{x}M)$ for all integers $i \geq n$. By hypothesis, we have that $\operatorname{depth}(M) \geq n$, hence we conclude that

$$\begin{aligned}
\operatorname{depth}(M) - n &= \inf\{i \geq 0 \mid \operatorname{Ext}_R^i(k, M) \neq 0\} - n \\
&= \inf\{i - n \geq 0 \mid \operatorname{Ext}_R^i(k, M) \neq 0\} \\
&= \inf\{i - n \geq 0 \mid \operatorname{Ext}_R^{i-n}(k, M/\underline{x}M) \neq 0\} \\
&= \operatorname{depth}(M/\underline{x}M),
\end{aligned}$$

where the first and last equalities hold by Theorem 4.3 and the third holds by the isomorphism. $\square$

Unlike with taking quotients, localizing at a prime ideal can sometimes increase depth.

Proposition 4.8. Let P be a prime ideal of R. We have that

(1.) $\operatorname{depth}(M) \leq \dim(R/P)$ if P is an associated prime of M and

(2.) $\operatorname{depth}(M) \leq \dim(R/P) + \operatorname{depth}(M_P)$.

Proof. (1.) We proceed by induction on $\operatorname{depth}(M)$. Given that $\operatorname{depth}(M) = 0$, the claim holds trivially. Given that $\operatorname{depth}(M) = 1$, by Proposition 4.4, $\mathfrak{m}$ is not an associated prime of M, hence for any associated prime P of M, we have that $\mathfrak{m} \supsetneq P$ so that $\dim(R/P) \geq 1$, and the claim holds. Consider the case that $\operatorname{depth}(M) \geq 2$. By definition, there exists an M-regular element $x \in \mathfrak{m}$. Given an associated prime P of M, we have that $P = \operatorname{ann}_R(m)$ for some nonzero element $m \in M$, hence the collection $\mathscr{C} = \{\operatorname{ann}_R(m) \mid m \in M \text{ is nonzero and } \operatorname{ann}_R(m) \subseteq P\}$ is nonempty. By Proposition 4.1(4.), M is Noetherian, hence there exists a maximal element of $\mathscr{C}$, i.e., a maximal ideal $\operatorname{ann}_R(a)$ that is annihilated by P. On the contrary, if a belonged to xM, then there would exist a nonzero element $b \in M$ such that $a = xb$. Observe that P annihilates a, hence P annihilates xb, so P must annihilate b because x is M-regular. Consequently, we would find that $\operatorname{ann}_R(b) \subseteq P$ $\operatorname{ann}_R(a) \subsetneq \operatorname{ann}_R(b)$ — a contradiction. We conclude that a does not belong to xM, hence P annihilates $a + xM$ so that P consists of zero divisors of M/xM. By Corollary 3.13, P belongs to some associated prime Q of M/xM. We claim that $P \subsetneq Q$, from which it follows that

$$\dim(R/P) - 1 \geq \dim(R/Q) \geq \operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$$

by induction, and we conclude that $\operatorname{depth}(M) \leq \dim(R/P)$. Observe that $x \notin P$ by hypothesis that P annihilates m and x is M-regular, hence x belongs to $R \setminus P$ so that $(M/xM)_P = 0$ (cf. Example 2.10). On the other hand, as Q is an associated prime of M/xM, there exists a nonzero element $m' + xM \in M/xM$ such that $Q = \operatorname{ann}_R(m' + xM) = \{r \in R \mid rm' \in xM\}$. Consequently, for every element $s \in R \setminus Q$, we have that $sm' \notin xM$ so that $(M/xM)_Q \neq 0$. We conclude that $P \subsetneq Q$.

(2.) By convention, if $M_P = 0$, then $\operatorname{depth}(M_P)$ is infinite, and the claim holds. Our proof is also complete if $\operatorname{depth}(M) \leq \dim(R/P)$. We may assume therefore that $\operatorname{depth}(M) > \dim(R/P)$ and M_P is nonzero. Consequently, by (1.), P is not an associated prime of M, hence P cannot belong to any associated prime of M. By Corollary 3.13, there exists an M-regular element $x \in P$. By Proposition 4.7, we have that $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$ and $\operatorname{depth}(M_P/xM_P) = \operatorname{depth}(M_P) - 1$. By induction on $\operatorname{depth}(M)$, we conclude that $\operatorname{depth}(M) \leq \dim(R/P) + \operatorname{depth}(M_P)$. $\square$

Observe that the depth of a module measures its "homological bigness." On the other hand, the (Krull) dimension of a module measures its "topological bigness." Our immediate aim is to compare the two invariants. Before we do so, we need to demonstrate that depth and dimension behave well with respect to taking the quotient by a regular sequence (known colloquially as "cutting down").

Definition 4.9. We define the **(Krull) dimension** of a module as $\dim(M) = \dim(R/\operatorname{ann}_R(M))$.

Proposition 4.10. Let $\underline{x} = (x_1, \dots, x_n)$ be an M-regular sequence. We have that

$$\dim(M/\underline{x}M) = \dim(M) - n.$$

Proof. We omit the proof of this fact because it is beyond the scope of this note. $\square$

Proposition 4.11. We have that $\operatorname{depth}(M) \leq \dim(M)$.

Proof. By Theorem 4.3, it follows that $\operatorname{depth}(M)$ is equal to the number of terms of any maximal M-regular sequence. Observe that for any maximal M-regular sequence $\underline{x} = (x_1, \dots, x_n)$ in $\mathfrak{m}$, we have that $\dim(M/\underline{x}M) = \dim(M) - n$ by Proposition 4.10. By Definition 4.9, we have that $\dim(M/\underline{x}M) = \dim(R/\operatorname{ann}_R(M/\underline{x}M)) \geq 0$ so that $\operatorname{depth}(M) = n \leq \dim(M)$. $\square$

Our next example illustrates that this inequality may be strict.

Example 4.12. Let k be a field. Consider the Noetherian local ring $R = k[\![x, y]\!]/(x^2, xy)$ of Example 4.5. We claim that $\dim R = 1$. Observe that $\operatorname{ht}(x^2, xy) = \operatorname{ht}(x, xy) = \operatorname{ht}(x) = 1$ in $k[\![x, y]\!]$, hence $\dim R \leq \dim k[\![x, y]\!] - \operatorname{ht}(x^2, xy) = 2 - 1 = 1$ by Proposition 2.11(4.). On the other hand, $(\bar{x}, \bar{y}) \supsetneq (\bar{x})$ is a strictly descending chain of prime ideals in R so that $\dim R = 1 > 0 = \operatorname{depth}(R)$.

We note that Examples 3.9 and 4.5 are exemplary of a more general phenomenon.

Proposition 4.13. Every minimal prime of R is an associated prime of R.

Proof. Observe that a minimal prime ideal P of R must have $\operatorname{ht}(P) = 0$, hence we have that $\operatorname{depth}(R_P) \leq \dim(R_P) = \operatorname{ht}(P) = 0$. By Corollary 4.4, we have that PR_P is an associated prime of R_P, hence there exists an element r/s of R_P such that $PR_P = \operatorname{ann}_{R_P}(r/s)$. Using properties of localization, we conclude that $P = \operatorname{ann}_R(r)$ (cf. [G13/14, Proposition 6.7] for details). $\square$

We have seen in Proposition 4.11 that M is at least as "topologically large" as it is "homologically large." Consequently, it is worth investigating when these two notions of size agree.

Definition 4.14. Given that $\operatorname{depth}(M) = \dim(M)$, we say that M is **Cohen-Macaulay**. We say that the Noetherian local ring R is Cohen-Macaulay if it is Cohen-Macaulay as an R-module.

Example 4.15. Let k be a field. Let $S = k[\![x, y]\!]$ denote the bivariate ring of formal power series. Observe that (x, y) is an S-regular sequence, hence we have that $0 = \dim(S/(x, y)) = \dim(S) - 2$ by Proposition 4.10. On the other hand, we have that $2 \leq \operatorname{depth}(S) \leq \dim(S) = 2$ by Theorem 4.3 and Proposition 4.11. We conclude that $k[\![x, y]\!]$ is Cohen-Macaulay. Considering that $\mathfrak{m}$ is minimally generated by (x, y), we find that $\mu(\mathfrak{m}) = 2$, hence $k[\![x, y]\!]$ is a regular local ring by Definition 2.15. We will soon show that this is not a coincidence (cf. Corollary 5.7 for details).

Our next proposition illustrates that Cohen-Macaulay rings behave well with respect to "cutting down" by an R-regular sequence. Quite importantly, this allows us to reduce to the 0-dimensional case by taking the quotient of a Cohen-Macaulay ring by a maximal R-regular sequence.

Proposition 4.16. Let $\underline{x} = (x_1, \dots, x_n)$ be an R-regular sequence. We have that R is Cohen-Macaulay if and only if $R/\underline{x}R$ is Cohen-Macaulay.

Proof. By Proposition 4.7, we have that $\operatorname{depth}(R/\underline{x}R) = \operatorname{depth}(R) - n$. By Proposition 4.10, we have that $\dim(R/\underline{x}R) = \dim(R) - n$. Consequently, we have that $\dim(R) = \operatorname{depth}(R)$ if and only if $\dim(R) - n = \operatorname{depth}(R) - n$ if and only if $\dim(R/\underline{x}R) = \operatorname{depth}(R/\underline{x}R)$. □

Our next proposition illustrates that the ideals of Cohen-Macaulay local rings exhibit behavior similar to the ideals of a domain that is a finitely generated algebra over a field. Particularly, Proposition 2.11(4.) holds for the ideals of a Cohen-Macaulay local ring.

Proposition 4.17. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d.

(1.) For each prime ideal P of R, we have that R_P is Cohen-Macaulay.

(2.) For each prime ideal P of R, we have that

$$\operatorname{ht}(P) + \dim(R/P) = \dim(R).$$

Consequently, for any ideal I of R, we have that

$$\operatorname{ht}(I) + \dim(R/I) = \dim(R).$$

(3.) We have that $\operatorname{Ass}_R(R) = \operatorname{MinSpec}(R) = \{P \in \operatorname{Spec}(R) \mid \dim(R/P) = \dim(R)\}$.

Proof. (1.) We proceed by induction on the dimension d of R. Observe that if $d = 0$, every prime ideal of R has $\dim(R_P) = \operatorname{ht}(P) = 0$, and the claim holds by Proposition 4.11. We will assume the claim holds for $d - 1$. Consider a strictly descending chain of prime ideals

$$\mathfrak{m} \supsetneq P_1 \supsetneq \cdots \supsetneq P_{n-1} \supsetneq P_n = P$$

of maximum length n. Observe that $\dim(R/P_1) = 1$. Certainly, the inequality $\geq$ holds by the Correspondence Theorem. On the other hand, if it were a strict inequality $>$, then we would obtain a longer strictly descending chain of prime ideals of R — a contradiction. On the other hand, we have that $\dim(R_{P_1}) \leq d - 1$ because $\mathfrak{m}$ can be appended to any strictly descending chain of prime ideals contained in P_1. By Proposition 4.8, we find that

$$\operatorname{depth}(R_{P_1}) \geq \operatorname{depth}(R) - \dim(R/P_1) = \operatorname{depth}(R) - 1 = d - 1 \geq \dim(R_{P_1})$$

by hypothesis that R is Cohen-Macaulay. By a similar rationale (or induction on the length n), we find that $\operatorname{depth}(R_P) \geq \dim(R_P)$, and our claim holds by induction.

(2.) By part (1.), R_P is Cohen-Macaulay, from which it follows that $\dim(R_P) = \operatorname{depth}(R_P)$. By Proposition 2.11(3.), the inequality $\leq$ holds. Conversely, by Proposition 4.8, we have that $\operatorname{ht}(P) + \dim(R/P) = \dim(R_P) + \dim(R/P) = \operatorname{depth}(R_P) + \dim(R/P) \geq \operatorname{depth}(R) = \dim(R)$.

(3.) By Proposition 4.13, the inclusion $\supseteq$ holds. Conversely, if P is an associated prime of R, then we have that $\operatorname{ht}(P) = \dim(R_P) = \operatorname{depth}(R_P) = 0$ by Corollary 4.4, hence P is a minimal prime of R. Given any minimal prime P of R, we have that $\dim(R) = \dim(R/P) + \operatorname{ht}(P) = \dim(R/P)$. □

5 Systems of Parameters and Regular Local Rings

Every ideal of a Noetherian local ring that is generated by a regular sequence can be extended to an ideal whose radical is equal to the maximal ideal. One of our main objectives in this section is to establish that for a Cohen-Macaulay local ring, the converse holds. We will assume throughout that $(R, \mathfrak{m}, k)$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$, and $\dim(R) = d$.

Definition 5.1. We say that a collection of elements $x_1, \dots, x_d \in \mathfrak{m}$ is a **system of parameters** (or s.o.p.) whenever there exists an integer $n \gg 0$ such that the ideal $I = (x_1, \dots, x_d)$ satisfies $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$ (or equivalently, if $\sqrt{I} = \mathfrak{m}$, i.e., I is $\mathfrak{m}$-primary). We refer to an ideal of R that is generated by a system of parameters as a **parameter ideal**. Given that the elements $x_1, \dots, x_d$ are R-regular, moreover, we say that $(x_1, \dots, x_d)$ is a **regular system of parameters**.

Proposition 5.2. If I is a parameter ideal of R, then $\mu(I) = \dim_k(I/\mathfrak{m}I) \geq \dim(R) = d$.

Proof. Observe that $d = \dim(R) = \operatorname{ht}(\mathfrak{m}) = \operatorname{ht}(\sqrt{I}) = \operatorname{ht}(I) \leq \mu(I)$ by Krull's Height Theorem. $\square$

Equivalently, the quotient of R by a parameter ideal I is **Artinian**, i.e., $\dim(R/I) = 0$.

Proposition 5.3. The following conditions are equivalent.

(1.) There exist elements $x_1, \dots, x_d \in \mathfrak{m}$ such that $I = (x_1, \dots, x_d)$ satisfies $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$.

(2.) There exist elements $x_1, \dots, x_d \in \mathfrak{m}$ such that $I = (x_1, \dots, x_d)$ satisfies $\dim(R/I) = 0$.

Proof. We will assume first that condition (1.) holds. Consider a prime ideal P of R that contains I. Observe that $\mathfrak{m}^n \subseteq I \subseteq P$ implies that $\mathfrak{m} \subseteq P$, from which we conclude that $P = \mathfrak{m}$. Put another way, we have that $\operatorname{Spec}(R/I) = \{\mathfrak{m}/I\}$ so that $\dim(R/I) = 0$, as desired.

Conversely, suppose that condition (2.) holds. Each of the generators of I belongs to $\mathfrak{m}$, hence we have that $I \subseteq \mathfrak{m}$. On the other hand, if there were another prime ideal P of R such that $I \subseteq P \subsetneq \mathfrak{m}$, then we would obtain a strictly descending chain of ideals $\mathfrak{m}/I \supsetneq P/I$ of R/I of length 1 — a contradiction. We conclude that $\mathfrak{m}$ is the only prime ideal of R lying over I, hence we have that $\sqrt{I} = \mathfrak{m}$. Considering that R is Noetherian, this is equivalent to $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$. $\square$

Our next proposition illustrates that the quotient of a ring by an ideal generated by elements belonging to a system of parameters behaves similarly to the quotient of a ring by a regular sequence.

Proposition 5.4. If $x_1, \dots, x_i \in \mathfrak{m}$ belong to a system of parameters for R, then

$$\dim(R/(x_1, \dots, x_i)) = d - i.$$

Proof. We proceed by induction on i. We assume first that x_1 belongs to a system of parameters. By definition, there exist elements $y_2, \dots, y_d \in \mathfrak{m}$ such that $I = (x_1, y_2, \dots, y_d)$ is a parameter ideal. Let $I' = (y_2, \dots, y_d)$, $R' = R/x_1R$, and $\dim(R') = d'$. Observe that $R/I \cong R'/I'$, from which it follows that $\dim(R'/I') = \dim(R/I) = 0$ by Proposition 5.3. We conclude that I' is a parameter ideal of R', hence by Proposition 5.2, we must have that $d - 1 \geq \mu(I') \geq \dim(R') = \dim(R/x_1R)$. Conversely, if the images of $z_1, \dots, z_{d'} \in R$ generate a parameter ideal of R', then $x_1, z_1, \dots, z_{d'}$ generate a parameter ideal of R. By the same rationale as before, we have that $d' + 1 \geq \dim(R)$ so that $\dim(R/x_1R) \geq d - 1$. We assume now that the claim holds for $i - 1$. Let $x_1, \dots, x_i$ belong to a system of parameters of R. Let $I' = (x_2, \dots, x_i)$, and let $R' = R/x_1R$. By induction, we have that $\dim(R'/I') = \dim(R') - (i - 1) = (d - 1) - (i - 1) = d - i$, and our proof is complete. $\square$

We establish one of the main results of this section.

Proposition 5.5. The following conditions are equivalent.

(1.) Every system of parameters of R is an R-regular sequence.

(2.) There exists a system of parameters of R that is an R-regular sequence.

(3.) R is Cohen-Macaulay.

Proof. Clearly, condition (1.) implies condition (2.). On the other hand, if there exists a system of parameters of R that is an R-regular sequence, then we must have that $\operatorname{depth}(R) \geq \dim(R)$. By Proposition 4.11, we conclude that R is Cohen-Macaulay, hence condition (2.) implies condition (3.). Last, we will assume that R is Cohen-Macaulay. We proceed by induction on the dimension d of R. We may assume that the claim holds for $d-1$ because the case $d=0$ is vacuously true. Consider a system of parameters $x_1, \dots, x_d \in \mathfrak{m}$. Observe that x_1 cannot belong to any minimal prime P of R; otherwise, we would have that $d-1 = \dim(R/x_1R) \geq \dim(R/P) = \dim(R) = d$ by Propositions 4.17 and 5.4 — a contradiction. Consequently, x_1 does not belong to any associated prime of R by Proposition 4.17. We conclude by Corollary 3.12 that x_1 is R-regular. By induction, we conclude that $(\bar{x}_2, \dots, \bar{x}_d)$ is an R/x_1R-regular sequence, hence $(x_1, \dots, x_d)$ is an R-regular sequence. Considering that this holds for any system of parameters of R, we are done. $\square$

Recall that by Definition 2.15, a regular local ring $(R, \mathfrak{m}, k)$ is a Noetherian local ring for which $\dim(R) = \mu(\mathfrak{m}) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Consequently, the maximal ideal of a regular local ring is generated by a system of parameters. We show moreover that it is generated by an R-regular sequence.

Proposition 5.6. If $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m}$ is generated by an R-regular sequence.

Proof. We proceed by induction on $d = \dim(R)$. Let $x_1 \in \mathfrak{m}$ be any minimal generator of $\mathfrak{m}$. By Proposition 2.17, the regular local ring R is a domain, so x_1 is a non-zero divisor of R. Because x_1 belongs to $\mathfrak{m}$, it is a non-unit, hence x_1R does not equal R and x_1 is R-regular. We conclude that $\mathfrak{m} = x_1R$ is generated by an R-regular sequence. We will assume therefore that the claim holds for $d-1$. Let $x_1, \dots, x_d$ be a minimal system of generators of $\mathfrak{m}$. By definition, $x_1, \dots, x_d$ is a system of parameters for $\mathfrak{m}$, hence by Proposition 5.4, we have that

$$\dim(\bar{R}) = \dim(R/x_1R) = d-1 = \mu(\bar{x}_2, \dots, \bar{x}_d) = \mu(\bar{\mathfrak{m}}).$$

Consequently, $(\bar{R}, \bar{\mathfrak{m}})$ is a regular local ring of dimension $d-1$. By induction, $(\bar{x}_2, \dots, \bar{x}_d)$ is a $\bar{R}$-regular sequence. But x_1 is R-regular, hence $(x_1, \dots, x_d)$ is an R-regular sequence. $\square$

Corollary 5.7. Every regular local ring is Cohen-Macaulay; the converse is not true.

Proof. By Proposition 5.6, the unique maximal ideal of a regular local ring is generated by a regular sequence; such a Noetherian local ring is Cohen-Macaulay by Proposition 5.5.

Conversely, consider the Noetherian local ring $S = k[\![x,y]\!]/(x^2,y^2)$. Let $\bar{x}$ and $\bar{y}$ denote the class of x and y modulo (x^2,y^2). Observe that S has dimension 0, hence S is a Cohen-Macaulay local ring. Explicitly, the prime ideals of S correspond to prime ideals of $k[\![x,y]\!]$ that contain (x^2,y^2). But any such prime ideal must contain both x and y, hence the only prime ideal of S is $(\bar{x}, \bar{y})$. On the other hand, the maximal ideal of S is exactly $\bar{\mathfrak{m}} = (\bar{x}, \bar{y})$ with $\mu(\bar{\mathfrak{m}}) = 2 > 0 = \dim(S)$. $\square$

By Proposition 5.4, the dimension of a Noetherian local ring modulo a subset S of a system of parameters drops by $|S|$. By the proof of Proposition 5.6, the quotient of a regular local ring by a minimal generator of the maximal ideal is a regular local ring. Our next proposition illustrates that this property holds for any ideal generated by a subset of a regular system of parameters.

Proposition 5.8. [BH, Proposition 2.2.4] Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension d. Let I be a proper ideal of R. The following statements are equivalent.

(1.) R/I is a regular local ring.

(2.) I is generated by a subset of a regular system of parameters.

Proof. Given that I is generated by a subset $\{x_1, \dots, x_k\}$ of a (regular) system of parameters of R, it follows that $\dim(R/I) = d - k = \mu(\mathfrak{m}/I)$, hence R/I is a regular local ring.

Conversely, suppose that R/I is a regular local ring. By Proposition 2.17, it follows that I is a prime ideal of R. Further, we have that $\mu(\mathfrak{m}/I) = \dim(R/I) = d'$. Observe that $(\mathfrak{m}/I)^2 = (\mathfrak{m}^2+I)/I$, hence we have that $\mu(\mathfrak{m}/I) = \dim_k(\mathfrak{m}/(\mathfrak{m}^2+I))$. Consider the short exact sequence of k-vector spaces

$$0 \to \frac{I}{\mathfrak{m}^2 \cap I} \xrightarrow{\varphi} \frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{\psi} \frac{\mathfrak{m}}{\mathfrak{m}^2 + I} \to 0$$

determined by $\varphi(x+\mathfrak{m}^2\cap I) = x+\mathfrak{m}^2$ and $\psi(x+\mathfrak{m}^2) = x+\mathfrak{m}^2+I$. By the Rank-Nullity Theorem, we have that $\dim_k(\mathfrak{m}/(\mathfrak{m}^2+I)) + \dim_k(I/(\mathfrak{m}^2 \cap I)) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \mu(\mathfrak{m}) = d$, from which it follows that $\dim_k(I/(\mathfrak{m}^2 \cap I)) = d - \dim_k(\mathfrak{m}/(\mathfrak{m}^2+I)) = d - d'$. Consequently, by Nakayama's Lemma, we obtain elements $x_1, \dots, x_{d-d'}$ of I that belong to a minimal generating set of $\mathfrak{m}$. By hypothesis that $(R, \mathfrak{m})$ is a regular local ring, it follows that $x_1, \dots, x_{d-d'}$ belong to a regular system of parameters, hence we find that $\dim(R/(x_1, \dots, x_{d-d'})) = d - (d - d') = d'$ by Proposition 5.4. On the other hand, we have that $\mu(\mathfrak{m}/(x_1, \dots, x_{d-d'}) = d'$, hence we have that $R/(x_1, \dots, x_{x-d'})$ is a regular local ring. Particularly, $(x_1, \dots, x_{d-d'})$ is a prime ideal of R that is contained in the prime ideal I of R and satisfies $\dim(R/(x_1, \dots, x_{d-d'}) = \dim(R/I)$. We conclude by the Correspondence Theorem that $I = (x_1, \dots, x_{d-d'})$ so that I is generated by a subset of a regular system of parameters. $\square$

Regular local rings are in some sense the "best behaved" class of Noetherian local rings. By Corollary 5.7, every regular local ring is Cohen-Macaulay, but there exist Cohen-Macaulay local rings that are not regular. Consequently, one might naturally wonder "how far" a Cohen-Macaulay local ring is from being regular. We refer the reader to Examples 7.1 and 7.2 for details.

6 Serre's Condition (S_i)

Brilliantly, the French mathematician Jean-Pierre Serre recognized that the depth of a finitely generated module over a Noetherian ring controls many of its nice properties.

Definition 6.1. We say that a finitely generated module M over a Noetherian ring R satisfies **Serre's Condition** (S_i) if for all prime ideals P of R, we have that $\operatorname{depth}(M_P) \geq \inf\{i, \operatorname{ht}(P)\}$.

For instance, the following observations can be made immediately.

Proposition 6.2. If R satisfies Serre's Condition (S_1), then $\operatorname{Ass}_R(R) = \operatorname{MinSpec}(R)$. Put another way, every associated prime of R is a minimal prime of R.

Proof. By Proposition 4.13, the containment $\supseteq$ holds. Conversely, let P be an associated prime of R. On the contrary, assume that P is not a minimal prime, i.e., $\dim(R_P) = \operatorname{ht}(P) \geq 1$. By hypothesis that R is (S_1), we have that $0 = \operatorname{depth}(R_P) \geq \inf\{1, \dim(R_P)\} = 1$ — a contradiction. $\square$

Proposition 6.3. If M satisfies Serre's Condition (S_n), then M_P is Cohen-Macaulay for all prime ideals P with $\operatorname{depth}(M_P) \leq n-1$. Conversely, if M_P is Cohen-Macaulay for all prime ideals P with $\operatorname{depth}(M_P) \leq n-1$, then $\operatorname{depth}(M_P) \geq \inf\{n, \dim(M_P)\}$. Particularly, if R_P is Cohen-Macaulay for all primes P with $\operatorname{depth}(R_P) \leq n-1$, then R satisfies (S_n).

Proof. Given that M satisfies Serre's Condition (S_n), we have that $\operatorname{depth}(M_P) \geq \inf\{n, \dim(R_P)\}$ for all prime ideals P of R. Particularly, for any prime ideal P with $\operatorname{depth}(M_P) \leq n-1$, we must have that $\operatorname{depth}(M_P) \geq \dim(R_P) \geq \dim(M_P) \geq \operatorname{depth}(M_P)$ so that M_P is Cohen-Macaulay.

On the contrary, if it were the case that $\dim(R_P) \geq \operatorname{depth}(M_P) + 1$, then we would have that $\inf\{n, \dim(R_P)\} = n$ so that $n - 1 \geq \operatorname{depth}(M_P) \geq n = \inf\{n, \dim(R_P)\}$ — a contradiction.

Conversely, if M_P is Cohen-Macaulay for all prime ideals P with $\operatorname{depth}(M_P) \leq n - 1$, then we have that $\operatorname{depth}(M_P) = \dim(M_P)$ for all prime ideals P with $\operatorname{depth}(M_P) \leq n - 1$. Considering that a prime ideal P of R satisfies either $\operatorname{depth}(M_P) \leq n - 1$ or $\operatorname{depth}(M_P) \geq n$, we conclude that $\operatorname{depth}(M_P) \geq \inf\{n, \dim(M_P)\}$ for all prime ideals P of R. □

Combined with another criterion originally introduced by Krull, one can say even more about the structure of a finitely generated module over a Noetherian ring — especially when the module is the ring itself. We point the reader to the next section for a brief exposition on this idea.

7 Further Considerations for Future Talks

Based on comments made throughout this note, we list several possible avenues for future talks.

Example 7.1. We have seen that Noetherian local ring $(R, \mathfrak{m}, k)$ is Cohen-Macaulay if and only if $\operatorname{depth}(R) = \dim(R)$, where $\operatorname{depth}(R)$ is the length of a maximal R-regular sequence in $\mathfrak{m}$, i.e.,

$$\operatorname{depth}(R) = \sup\{i \mid \operatorname{Ext}^i_R(k, R) \neq 0\}.$$

Considering that $\operatorname{Ext}^i_R(k, R)$ is a k-vector space, one can find $r(R) = \dim_k \operatorname{Ext}^{\operatorname{depth}(R)}_R(k, R)$. We refer to the invariant $r(R)$ as the **(Cohen-Macaulay) type** of R. Generally, this invariant can be defined for any nonzero finitely generated R-module. We refer to a Cohen-Macaulay local ring of type one as a **Gorenstein** local ring. Consequently, a Gorenstein ring is Cohen-Macaulay, but the converse is not true. For instance, the Noetherian local ring $S = k[\![x, y]\!]/(x^2, xy, y^2)$ is dimension 0 and hence Cohen-Macaulay; however, S is spanned by $\{\bar{1}, \bar{x}, \bar{y}\}$ as a k-vector space and $\operatorname{Hom}_S(k, S)$ is spanned by $\{\bar{x}, \bar{y}\}$ as a k-vector space, hence we have that $r(S) = \dim_k \operatorname{Hom}_R(k, R) = 2 > 1$.

Currently, Gorenstein local rings are a very hot topic of research in commutative algebra. Particularly, much effort is being made to classify families of Cohen-Macaulay rings that are not Gorenstein, e.g., almost Gorenstein rings and nearly Gorenstein rings. On their own, there is much to learn and say about Gorenstein rings. For instance, a Gorenstein ring is isomorphic to its **canonical module**. By a result of Foxby, a Noetherian local ring is Gorenstein if and only if it admits a module that has both finite projective dimension and finite injective dimension.

Example 7.2. Let R be a commutative unital ring. Let I be a proper ideal of R. We can impose a topology τ on R with respect to I — called the ***I*-adic topology** — by declaring that a subset U of R is open if and only if for every element $x \in U$, there exists an integer $n \gg 0$ such that $x + I^n \subseteq U$. Consequently, any ring R can be viewed as a topological ring. On its own, the I-adic topology of a ring and the properties of R as a topological ring can be investigated.

But in the case that $(R, \mathfrak{m})$ is a local ring with unique maximal ideal $\mathfrak{m}$, the completion $\widehat{R}$ of R with respect to the $\mathfrak{m}$-adic topology is a complete local ring. One can establish that the completion of a Noetherian ring is Noetherian, and the completion of a Noetherian local ring with respect to the $\mathfrak{m}$-adic topology is a Noetherian local ring. For example, if $S = k[x_1, \dots, x_n]$ is the n-variate polynomial ring over a field k with unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, then its $\mathfrak{m}$-adic completion is the n-variate formal power series ring $k[\![x_1, \dots, x_n]\!]$. We illustrate the idea for the case that $n = 1$. Given a sequence of polynomials $f_n(x) = \sum_{k=0}^n a_n x^n$, observe that their limit

$$f(x) = \lim_{n \to \infty} f_n(x) = \sum_{k=0}^{\infty} a_k x^k$$

is a formal power series. One can prove that the sum and product of formal power series is a formal power series, hence the set of formal power series in x constitutes a ring $k[\![x]\!]$. Observe that $k[\![x]\!]$ contains $k[x]$, as any polynomial can be realized as a formal power series whose coefficients are nonzero for only finitely many terms. On the other hand, one can realize $k[\![x]\!]$ as the (x)-adic completion of $k[x]$. Generally, we say that a sequence $(r_n)_{n\geq 0}$ of elements of R is **Cauchy** in the I-adic topology if for all integers $k \geq 0$, there exists an integer $N \gg 0$ such that for every pair of integers $m, n \geq N$, we have that $r_m - r_n \in I^k$. Consequently, the Cauchy sequences of polynomials of $k[x]$ with respect to the (x)-adic topology are precisely those sequences of polynomials whose coefficients eventually agree. We may complete $k[x]$ with respect to the (x)-adic topology by adding all of the limits of Cauchy sequences, i.e., all of the formal power series in x.

One of the most powerful theorems in commutative algebra, the **Cohen Structure Theorem** describes the structure of a complete Noetherian local ring $(R, \mathfrak{m}, k)$ in three parts.

(1.) R is the homomorphic image of a complete Noetherian regular local ring.

(2.) Given that R is regular and contains a field, we have that $R \cong k[\![x_1, \dots, x_n]\!]/I$ for some non-negative integer n and some ideal I of R.

(3.) Given that R is regular and of mixed characteristic, if R is unramified, then R is uniquely determined by its residue field k and its dimension $\dim R$.

Example 7.3. We say that an R-module M is **torsion-free** if every R-regular element is M-regular, i.e., if $x \in R$ is R-regular, then $xm = 0$ implies that $m = 0$. One can show that a finitely generated module over a Cohen-Macaulay ring of positive dimension satisfies Serre's Condition (S_1) if and only if it is torsion-free. Essentially, this is because M is torsion-free if and only if $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(R)$. Beyond this, one can show that torsion-free modules exhibit many nice properties.

Example 7.4. Recall that a finitely generated module M over a Noetherian local ring R is Cohen-Macaulay if and only if $\operatorname{depth}(M) = \dim(M)$ and $\dim(M) = \dim(R/\operatorname{ann}_R(M)) \leq \dim(R)$. We say that an R-module M is **maximal Cohen-Macaulay** if $\operatorname{depth}(M) = \dim(R)$. It turns out that maximal Cohen-Macaulay modules are related to torsion-free modules. Particularly, maximal Cohen-Macaulay modules are always torsion-free; the converse holds if R is a domain or $\dim(R) = 1$. If $(R, \mathfrak{m})$ is reduced and $\dim(R) = 1$, then M is torsion-free if and only if $\mathfrak{m} \notin \operatorname{Ass}_R(M)$.

Example 7.5. Every finitely generated R-module M has a free resolution

$$F_\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

We refer to the kernel of the ith free module F_i as the ith **syzygy** module. Over Cohen-Macaulay local rings, ith syzygies satisfies Serre's Condition (S_i). Generally, they satisfy the property that

$$\operatorname{depth}(M) \geq \inf\{i, \operatorname{depth}(R)\}$$

by the Depth Lemma (Proposition 4.6). On their own, syzygy modules are of interest and import.

Given any element $m \in M$, we define the "evaluation at m" map $\operatorname{ev}_m : \operatorname{Hom}_R(M, R) \to R$ by declaring that $\operatorname{ev}_m(\varphi) = \varphi(m)$. We say that an R-module is **reflexive** if the canonical map $\psi : M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R)$ defined by $\psi(m) = \operatorname{ev}_m$ is a bijection. One can show that a finitely generated reflexive module over a Noetherian ring is always a second syzygy. Further, a module is torsion-free if and only if ψ is injective, hence a reflexive module is always torsion-free.

Example 7.6. Let R be a Noetherian ring. One can says that R satisfies **Serre's Condition** (R_i) if R_P is a regular local ring for all prime ideals P of R with $\mathrm{ht}(P) \leq i$. Colloquially, if R satisfies (R_i), then R is said to be **regular in codimension** i. Combined with Serre's Condition (S_i), one can exhibit many nice properties of R given that R satisfies (R_i) and (S_j). For instance, if R satisfies (R_0) and (S_1), then R is reduced. If R satisfies (R_1) and (S_2), then R is normal, i.e., R is reduced and integrally closed in its total ring of fractions $Q(R) = \{r/s \mid r, s \in R \text{ and } s \text{ is a non-zero divisor of } R\}$.

References

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[G13/14] A. Gathmann, *Commutative Algebra*, Class Notes TU Kaiserslautern 2013/14.