DERIVED FUNCTORS

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1. δ-functors

According to Alexander Grothendieck, in his 1957 Tohoku article "Sur quelques points d'algèbre homologique", the correct framework to talk about derived functors is via δ-functors.

Definition 1.1. A homological (respectively cohomological) δ-functor between abelian categories $\mathscr{A}$ and $\mathscr{B}$ is a collection of additive functors $T_n : \mathscr{A} \to \mathscr{B}$ (respectively $T^n : \mathscr{A} \to \mathscr{B}$) for $n \geq 0$ with morphisms

$$\delta_n : T_n(C) \to T_{n-1}(A)$$

(respectively

$$\delta_n : T^n(C) \to T^{n+1}(A) \ )$$

associated to each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$. These data must satisfy two conditions:

(1) Associated to each short exact sequence $0 \to A \to B \to C \to 0$. there exists a long exact sequence

$$...T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A)...$$

(respectively

$$...T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} T^{n+1}(A)... \quad).$$

(2) For all morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\zeta"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd},$$

the δ morphisms give rise to commutative diagrams

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta"] \arrow[d, "T_n(\zeta)"] & T_{n-1}(A) \arrow[d, "T_{n-1}(\varphi)"] \\ T_n(C') \arrow[r, "\delta"] & T_{n-1}(A') \end{tikzcd}$$

(respectively

$$\begin{tikzcd} T^n(C) \arrow[r, "\delta"] \arrow[d, "T^n(\zeta)"] & T^{n+1}(A) \arrow[d, "T^{n+1}(\varphi)"] \\ T^n(C') \arrow[r, "\delta"] & T^{n+1}(A') \end{tikzcd} \quad).$$

Remark 1.2. By definition, the functor $T_0 : \mathscr{A} \to \mathscr{B}$ is right exact while T^0 is left exact.

Definition 1.3. A morphism $S \to T$ of δ-functors is a system of natural transformations $S_n \to T_n$ (respectively $S^n \to T^n$) which commutes with δ. This means that there exists a commutative diagram in the shape of a ladder between the long exact sequences of S and T associated to each short exact sequence.

A homological δ-functor T is universal if, given a homological δ-funtor S and a natural transformation $f_0 : S_0 \to T_0$, there exists a unique morphism of δ-functors $\{f_n : S_n \to T_n\}_{n\geq 0}$ which extends f_0. Similarly, a cohomological δ-functor T is universal if, given a cohomological δ-funtor S and a natural transformation $f^0 : S^0 \to T^0$, there exists a unique morphism of δ-functors $\{f^n : S^n \to T^n\}_{n\geq 0}$ which extends f^0.

Remark 1.4. Universal δ-functors are unique up to isomorphism due to their universal property. We shall soon see examples of universal δ-functors, namely the homology (respectively cohomology) functor of a chain complex (respectively cochain complex), and derived functors when they exist.

2. The Long Exact (Co)Homology Sequence

Definition 2.1. A sequence of chain complexes (respectively cochain complexes)

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is exact in the abelian category $\mathbf{Ch}(\mathscr{A})$ of chain complexes of objects of $\mathscr{A}$ if and only if each sequence

$$0 \to A_n \to B_n \to C_n \to 0$$

is exact in $\mathscr{A}$.

Theorem 2.2. *Let $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ be an exact sequence of chain complexes. Then there is a natural morphism $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ called the connecting morphism, such that the sequence*

$$.... \xrightarrow{g_*} H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \xrightarrow{f_*} ...$$

is exact.

Similarly, let $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ be an exact sequence of cochain complexes. Then there is a natural morphism $\partial : H^n(C^\bullet) \to H^{n+1}(A_\bullet)$ called the connecting morphism, such that the sequence

$$.... \xrightarrow{g^*} H^{n-1}(C^\bullet) \xrightarrow{\partial} H^n(A^\bullet) \xrightarrow{f^*} H^n(B^\bullet) \xrightarrow{g^*} H^n(C^\bullet) \xrightarrow{\partial} H^{n+1}(A^\bullet) \xrightarrow{f^*} ...$$

is exact.

Proof. See [1] Ch IV.2 pg 121-122. □

Remark 2.3. The long exact sequence in Theorem 2.2 is called the *long exact sequence in (co)homology* induced by the short exact sequence of (co)chain complexes. The (co)homology functor $H_\bullet : \mathbf{Ch}_{\geq 0} \to \mathscr{A}$ (respectively $H^\bullet : \mathbf{Ch}_{\geq 0} \to \mathscr{A}$) gives a universal (co)homological δ-functor.

3. Left Derived Funtors

Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor between abelian categories. If category $\mathscr{A}$ has enough projectives, one can construct the *left derived functors* L_iF of F for $i \geq 0$ in the following way. Let A be an object of $\mathscr{A}$ and let $P_\bullet \to A$ be a projective resolution of A. One defines

$$L_iF(A) := H_i(F(P_\bullet)).$$

Remark 3.1. Since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, one always has $L_0(A) \cong F(A)$.

Remark 3.2. The objects $L_iF(A)$ of $\mathscr{B}$ are well defined up to natural isomorphism, i.e. if $Q_\bullet \to A$ is another projective resolution, then there exists a canonical isomorphism $L_iF(A) = H_i(F(P_\bullet)) \xrightarrow{\cong} H_i(F(Q_\bullet))$, which follows from the comparison theorem for projective resolutions (see [1] Ch IV.5 Prop 5.1) such that there exists a morphism of chain complexes $f : P_\bullet \to Q_\bullet$ which lifts the identity id_A and any other morphism of chain complexes $f' : P_\bullet \to Q_\bullet$ which also lifts the identity id_A is homotopic to the induced morphism $f_* : H_i(F(P_\bullet)) \to H_i(F(Q_\bullet))$ of f in homology. Vice versa, there exists a morphism of chain complexes $g : Q_\bullet \to P_\bullet$ which lifts the identity id_A , and it follows that the induced morphisms in homology f_* and g_* are isomorphisms.

Remark 3.3. As a consequence of remark 3.2, if A is projective, then $L_iF(A) = 0$ for all $i > 0$.

Theorem 3.4. *Suppose that the category $\mathscr{A}$ has enough projectives. Then for all right exact functors $F : \mathscr{A} \to \mathscr{B}$, the left derived functors $L_\bullet F$ form a universal δ-functor.*

Proof. The proof goes beyond the scope of these notes, and we refer to [3] Theorem 2.46 and 2.4.7 p45-49. The first condition which needs to be satisfied to form a homological δ-functor however can be proved using Theorem 2.2 applied to $0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$ and the fact that the additive functor F takes a split exact sequence (obtained by projectives P''_n) to a a split exact sequence. $\square$

4. Right Derived Functors

Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories. If $\mathscr{A}$ has enough injectives, then one can construct the *right derived functors* R^iF of F for $i \geq 0$. For an object A of $\mathscr{A}$, one takes an injective resolution $A \to I^\bullet$ and one defines

$$R^iF(A) := H^i(F(I^\bullet)).$$

As before, since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, one always have $R^0F(A) \cong F(A)$. Since F is a left exact, $F^{op} : \mathscr{A}^{op} \to \mathscr{B}^{op}$ is right exact and moreover $\mathscr{A}^{op}$ has enough projectives. One can hence construct the left derived functors L_iF^{op} of F^{op}. The complex $I^\bullet$ becomes a projective resolution of A in $\mathscr{A}^{op}$ and hence,

$$R^iF(A) = (L_iF^{op})^{op}(A).$$

Hence all the results obtained for left derived functors can be applied to right derived functors. In particular, the objects $R^iF(A)$ are independent of the choice of injective resolution, $R^\bullet F$ is a universal cohomological δ-functor and $R^iF(I) = 0$ for all $i > 0$ and all injective object I.

5. Ext

Lemma 5.1. *The functor $F : R$-**mod** $\to$ **Ab** defined by $F(B) = Hom_R(A, B)$ is left exact.*

Proof. Let $0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$ be a short exact sequence, hence we want to show that the induce sequence

$$0 \to \mathrm{Hom}_R(A, B') \xrightarrow{f} \mathrm{Hom}_R(A, B) \xrightarrow{g} \mathrm{Hom}_R(A, B'')$$

is exact. One has $g_* \circ f_* = 0$, hence $\ker(g_*) \supset \mathrm{im}(f_*)$.

Conversely, if $\varphi \in \ker(g_*)$, i.e. for all $a \in A, g \circ \varphi(a) = 0, \varphi(a) \in \ker(g) = \mathrm{im}(f)$, hence there exists a unique $b' \in B'$ such that $f(b') = \varphi(a)$. The map $a \mapsto b'$ is well-defined morphism ψ, and hence we have $\varphi = f_*(\psi)$.

It remains to prove that f_* is injective: if $f_*(\psi) = f \circ \psi = 0$ for some $\psi : A \to B'$, one has that the inclusion of $\psi(a)$ in B is null for all $a \in A$. But this implies that $\psi(a) = 0$, hence $\psi = 0$. $\square$

Definition 5.2. (Ext Functors) For all fixed R-module A, the functor $F(B) = \mathrm{Hom}_R(A, B)$ is left exact. Its right derived functors are called Ext:

$$\mathrm{Ext}^i_R(A, B) = R^i\mathrm{Hom}_R(A, -)(B).$$

In particular, $\mathrm{Ext}^0_R(A, B) = \mathrm{Hom}_R(A, B)$.

Proposition 5.3. *The following are equivalent for an R-module B:*

(1) B is injective.
(2) $Hom_R(-, B)$ is an exact functor.
(3) $Ext^i_R(A, B) = 0$ for all $i \geq 0$ and for every R-module A.
(4) $Ext^1_R(A, B) = 0$ for every R-module A.

Proof. (1) $\iff$ (2) The property of being injective is the possibility of lifting a morphism $\alpha : A' \to B$ to $\beta : A \to B$ for a short exact sequence $0 \to A' \to A \to A'' \to 0$. This is equivalent to the functor $A \mapsto \mathrm{Hom}_R(A, B)$ which is left exact, being also right exact. Since the previous argument shows the surjectivity of the last arrow to the right

$$0 \to \mathrm{Hom}_R(A'', B) \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A', B),$$

if B was injective, which implies that the functor $A \mapsto \mathrm{Hom}_R(A, B)$ is exact.

The long exact sequence of the right derived functor shows that (2) $\iff$ (3) and similarly, it also shows (2) $\iff$ (4). $\square$

Proposition 5.4. *The following are equivalent for an R-module A:*

(1) A is projective.

(2) *$Hom_R(A, -)$ is an exact functor.*
(3) *$Ext^i_R(A, B) = 0$ for all $i \geq 0$ and for every R-module B.*
(4) *$Ext^1_R(A, B) = 0$ for every R-module B.*

Proof. The proof is similar to Prop 5.3. □

Remark 5.5. One can also consider derived functors of *contravariant* functors. Let F for instance be a contravariant functor which is left exact from $\mathscr{A}$ to $\mathscr{B}$. F then can be identified with a covariant functor, left exact from $\mathscr{A}^{op}$ to $\mathscr{B}$. Hence, if $\mathscr{A}$ has enough projectives (i.e. dually if $\mathscr{A}^{op}$ has enough injectives), one can then define the right derived functor $R^\bullet F(A)$ to be the cohomology of $F(P_\bullet)$, where $P_\bullet$ is a projective resolution of A. This is again a universal cohomological δ-functor with $R^0F(A) = F(A)$ and $R^iF(P) = 0$ for all $i > 0$ and all projectives P.

For instance, for all fixed R-module B, the functor $G(A) = \mathrm{Hom}_R(A, B)$ is contravariant and left exact. Hence we have the right derived functors $R^\bullet G(A)$. It can be shown (see [3] Theorem 2.7.6 pg 63) that these are exactly the right derived functors $\mathrm{Ext}^\bullet_R(A, B)$ defined above, and hence we have

$$R^\bullet \mathrm{Hom}_R(-, B)(A) \cong R^\bullet \mathrm{Hom}_R(A, -)(B) = \mathrm{Ext}^\bullet_R(A, B).$$

Lemma 5.6. *For all $n \geq 0$ and every ring R, one has*
(a) *$Ext^n_R(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha Ext^n_R(A_\alpha, B)$.*
(b) *$Ext^n_R(A, \prod_\beta B_\beta) \cong \prod_\beta Ext^n_R(A, B_\beta)$.*

Proof. Let $P_\alpha \to A_\alpha$ be projective resolutions of A_α. Hence $\oplus_\alpha P_\alpha \to \oplus_\alpha A_\alpha$ is a projective resolution of the direct sum since the direct sum of projective modules is projective. Similarly, if $B_\beta \to Q_\beta$ are injective resolutions of B_β, then $\prod_\beta B_\beta \to \prod_\beta Q_\beta$ is an injective resolution of the product since the direct product of injectives is injective. The statement in the lemma can be deduced directly from $\prod_\alpha \mathrm{Hom}_R(A_\alpha, B) = \mathrm{Hom}_R(\oplus_\alpha A_\alpha, B)$ and from $\mathrm{Hom}_R(A, \prod_\beta B_\beta) = \prod_\beta \mathrm{Hom}_R(A, B_\beta)$, which can be shown by Theorem B.19 in the appendix[1], and from commutativity in taking cohomology of the complex with the product. □

Lemma 5.7. *$Ext^n_{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$ and all abelian groups A and B.*

Proof. One embeds B in an injective abelian group I^0. The quotient I^1 is again divisible i.e. $\forall n \neq 0, n \in \mathbb{Z}$ and $\forall a \in I^1$, $\exists\ b \in I^1$ such that $a = nb$, hence injective. The groups $\mathrm{Ext}^\bullet_{\mathbb{Z}}(A, B)$ are calculated from the taking the cohomology of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, I^0) \to \mathrm{Hom}_{\mathbb{Z}}(A, I^1) \to 0$$

[1]We shall see in the next section, $\mathrm{Hom}_{\mathbb{Z}}(A, -) : \mathbf{Ab} \to R\text{-}\mathbf{mod}$ is right adjoint to the functor $A \otimes_R - :$ $\mathbf{mod}\text{-}R \to \mathbf{Ab}$ and this is true more generally for the pair $\mathrm{Hom}_R(A, -)$ for arbitrary ring R and $A \otimes_R -$, and right adjoint functors preserve products.

□

Example 5.8. One can calculate explicitly the Ext groups for $A = \mathbb{Z}/p\mathbb{Z}$. Indeed, the following short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p\times} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

can be taken as a projective resolution (in fact free) of $\mathbb{Z}/p\mathbb{Z}$. Applying the left exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, B)$ give the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \xrightarrow{(p\times)^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B).$$

One can identify $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \cong B$ such that $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A, B)$ is the cohomology of the complex

$$0 \to B \xrightarrow{\times p} B \to 0.$$

One hence obtains that

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) =: {}_pB := \{b \in B \mid pb = 0\},$$

the abelian p-torsion subgroup of B, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = B/pB$ and from the complex, $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) = 0$ for all $i > 1$.

Remark 5.9. Note that for $B = \mathbb{Z}/q\mathbb{Z}$, the group ${}_pB$ can be identified with $\mathbb{Z}/\gcd(p, q)\mathbb{Z}$.

Example 5.10. One can now calculate more generally the Ext groups for A, a finitely generated abelian group: Since $\mathbb{Z}$ is free, hence projective, one has $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/, B) = 0$. By the structural theorem of finitely generated abelian groups, $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/p_n\mathbb{Z}$, and since $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A_1 \oplus A_2, B) \cong \mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A_1, B) \oplus \mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A_2, B)$, combined with similar calculations of example 5.8, one can calculate the $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A, B)$ groups. For a more general abelian group A which is not finitely generated, the calculations turn out to be more difficult (see in the context of the Theorem of Stein-Serre [1] Thm 6.1 pg 107).

Example 5.11. One can also reason in the second variable of Ext taking the left exact covariant functor $\mathrm{Hom}_{\mathbb{Z}}(A, -)$. So suppose $B = \mathbb{Z}$ and let $A^* := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the *Pontryagin dual* of the abelian group A. Using the injective resolution

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

we calculate $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}(A, \mathbb{Z})$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(A, -)$, we have

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Once again, the Ext groups are the cohomology groups of the following complex:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \to 0$$

If A is a torsion group, then $\mathrm{Ext}^0_{\mathbb{Z}}(A,B) = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(A,B) = A^*$.

6. Tor

Definition 6.1. (Tor Functors) For all fixed left R-module B, the functor $T(A) := A \otimes_R B$ from **mod**-R to **Ab** is right exact. Its left derived functors are called Tor:

$$Tor_i^R(A,B) := L_i T(A).$$

Remark 6.2. In particular, $Tor_0^R(A,B) = A \otimes_R B$. Recall that these abelian groups were calculated by choosing a projective resolution $P_\bullet \to A$ and then taking homology groups of the complexe $P_\bullet \otimes_R B$. In particular, if A is a proiective R-module, then $Tor_i^R(A,B) = 0$ for all $i \geq 0$.

Lemma 6.3. *For all right R-module A, the functor $A \otimes_R - : R\text{-}\boldsymbol{mod} \to \boldsymbol{Ab}$ is left adjoint to the functor $Hom_{\mathbb{Z}}(A,-) : \boldsymbol{Ab} \to R\text{-}\boldsymbol{mod}$.*

Proof. The left R-module structure of $\mathrm{Hom}_{\mathbb{Z}}(A,-)$ is induced by the right R-module structure of A. One needs to show that for all abelian group G, one has the following natural isomorphism:

$$\tau : \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R B, G) \xrightarrow{\cong} \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(A,G)).$$

Given $\varphi : A \otimes_R B \to G$, one defines $\tau(\varphi)$ by

$$\tau(\varphi)(b)(a) := \varphi(a \otimes b).$$

Conversely, given $\psi : B \to \mathrm{Hom}_{\mathbb{Z}}(A,G)$, one defines $\widetilde{\tau}(\psi)$ by

$$\widetilde{\tau}(\psi)(a \otimes b) := \psi(b)(a).$$

It is straightforward to show that τ and $\widetilde{\tau}$ are well-defined and natural transformations which are inverses to each other. $\square$

Proposition 6.4. *(a) Let $\{B_i\}_{i \in I}$ be a family of left R-modules and A a right R-module. One has a natural isomorphism*

$$A \otimes_R (\oplus_{i \in I} B_i) \cong \otimes_{i \in I} A \otimes_R B_i.$$

(b) Let $0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$ be a short exact sequence of left R-modules. Then for all right R-module A, the sequence

$$A \otimes_R B' \xrightarrow{f_*} A \otimes_R B \xrightarrow{g_*} A \otimes_R B'' \to 0$$

is exact.

Proof. (a) By Theorem B.20 in the appendix, a left adjoint functor preserves coproducts and cokernels.

(b) By Theorem B.21, a left adjoint functor is right exact. □

Definition 6.5. A left R-module B is called *flat* if for all short exact sequence $0 \to A' \to A \to A'' \to 0$, the induced sequence

$$0 \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0$$

is exact.

Proposition 6.6. *Every projective R-module is flat.*

Proof. Every projective module is a direct summand of a free one. Since $-\otimes_R B$ preserves direct sums, it suffices to show that free modules are flat. By the same argument, it suffice to show that the R-module R is flat. This is true since $A \otimes_R R \cong A$. □

Proposition 6.7. *The following are equivalent for a left R-module B:*

(1) B is flat.
(2) $Tor_i^R(A, B) = 0$ for all $i \geq 0$ and for every right R-module A.
(3) $Tor_1^R(A, B) = 0$ for every right R-module A.

Proof. The proof is analogous to Prop 5.3. □

Example 6.8. On can calculate Tor groups explicitly when for instance $A = \mathbb{Z}/p\mathbb{Z}$. A projective resolution for $\mathbb{Z}/p\mathbb{Z}$ as we saw in example 5.8 is

$$0 \to \mathbb{Z} \xrightarrow{p\times} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Applying the right exact covariant functor $- \otimes B$, on finds the exact sequence

$$\mathbb{Z} \otimes B \xrightarrow{(p\times)_*} \mathbb{Z} \otimes B \to \mathbb{Z}/p\mathbb{Z} \otimes B \to 0$$

such that the abelian groups $\mathrm{Tor}_i^R(\mathbb{Z}/p\mathbb{Z}, B)$ for all abelian group B is the homology of the complex

$$0 \to B \xrightarrow{\times p} B \to 0.$$

One hence obtains that $\mathrm{Tor}_0^R(\mathbb{Z}/p\mathbb{Z}, B) = B/pB$ and $\mathrm{Tor}_1^R(\mathbb{Z}/p\mathbb{Z}, B) = {}_pB$, where we recall that ${}_pB := \{b \in B \mid pb = 0\}$. Moreover $\mathrm{Tor}_i^R(\mathbb{Z}/p\mathbb{Z}, B) = 0$ for all $i \geq 2$.

Proposition 6.9. *For all abelian groups of finite type A and B, one has*

(a) $Tor_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group.
(b) $Tor_i^{\mathbb{Z}}(A, B) = 0$ for all $i \geq 2$.

Proof. This follows from the the additivity of $\mathrm{Tor}_\bullet^{\mathbb{Z}}(A, B)$, from the structural theorem of finitely generated abelian groups, that $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}^m, -) = 0$ since $\mathbb{Z}^m$ is projective and from previous calculations in example 6.8 $\square$

Remark 6.10. More generally, the above is true for all abelian groups and one shows this by using the fact that every abelian group is the inductive limit of its finitely generated subgroups and how $\mathrm{Tor}^{\mathbb{Z}}$ behaves with respect to direct limits. Similar reasoning shows that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is a torsion group for all abelian group B and that $\mathrm{Tor}_i^{\mathbb{Z}}(A, B) = 0$ for all $i \geq 1$ for all torsion-free abelian group A and for all abelian group B.

Remark 6.11. For a *commutative* ring R, the right R-modules can be identified with left R-modules and vice-versa. Furthermore, one has that $\mathrm{Tor}_i^{\mathbb{Z}}(A, B) \cong \mathrm{Tor}_i^{\mathbb{Z}}(B, A)$, where the isomorphism is induced by $A \otimes_R B \cong B \otimes_R A$. One abstract way of proving it is to note that $\mathrm{Tor}_\bullet^{\mathbb{Z}}(A, B)$ and $\mathrm{Tor}_\bullet^{\mathbb{Z}}(B, A)$ are both universal δ-functors on $A \otimes_R B \cong B \otimes_R A$. One concludes with the same previous reasoning.

Corollary 6.12. *For all abelian group, one has*

$$Tor_1^{\mathbb{Z}}(A, -) = 0 \iff A \text{ is torsion free} \iff Tor_1^{\mathbb{Z}}(-, A) = 0$$

Remark 6.13. This result means that on $\mathbb{Z}$, the property of being flat is equivalent to being torsion-free. This result is true in general on any given principal ideal domain.

Appendix A. Categories

Definition A.1. A *category* $\mathscr{C}$ is the datum of a class of objects $\mathrm{ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ for all pairs (A, B) of objects of $\mathscr{C}$, of identity morphisms id_A for all object $A \in \mathrm{ob}(\mathscr{C})$ and a map

$$\circ : \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$

(composition of morphisms) for all triplets (A, B, C) of objects of $\mathscr{C}$. The datum must satisfy (i) associativity, i.e. for $f : A \to B$, $g : B \to C$ and $h : C \to D$, we must have $(h \circ g) \circ f = h \circ (g \circ f)$, and (ii) the axiom of unity, i.e. for all $f : A \to B$, we must have $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A$.

Definition A.2. A morphism $f : B \to C$ in a category $\mathscr{C}$ is called a *monomorphism* if for all distinct morphism $e_1, e_2 : A \to B$, one must have $f \circ e_1 \neq f \circ e_2$. Analogously, an *epimorphism* $g : A \to B$ is a one such that for all distinct morphism $e_1, e_2 : B \to C$, one has $e_1 \circ g \neq e_2 \circ g$.

Remark A.3. For a *monomorphism*, this is equivalent to the possibility of left cancellation, i.e. for all morphisms $a_1, a_2 : A \to B$ such that $f \circ a_1 = f \circ a_2$, this implies that $a_1 = a_2$. Similarly for an *epimorphism*, this translates to the possibility of right cancellation.

Definition A.4. The kernel $A := \ker(f)$ of a morphism $f : B \to C$ in a category $\mathscr{C}$ consist of an object A and a morphism $i : A \to B$ which has the following universal property: For all morphism $e : A' \to B$ in $\mathscr{C}$ such that $f \circ e = 0$, there exists a unique morphism $e' : A' \to A$ such that e factorises as $e = i \circ e'$.

Definition A.5. The cokernel $D := \mathrm{coker}(f)$ of a morphism $f : B \to C$ in a category $\mathscr{C}$ consist of an object D and a morphism $p : C \to D$ which has the following universal property: For all morphism $g : C \to D'$ in $\mathscr{C}$ such that $g \circ f = 0$, there exists a unique morphism $g' : D \to D'$ such that g factorises as $g = g' \circ p$.

Remark A.6. Every kernel $i : A \to B$ is a monomorphism and two kernels of the same morphism are isomorphic. Analogously, every cokernel $p : C \to D$ is an epimorphism and two cokernels of the same morphism are isomorphic. A given morphism in a category does not necessarily have a kernel or a cokernel.

Definition A.7. A category $\mathscr{C}$ is *additive* if for all pairs (A, B) of objects of $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group with 0 and such that the categorical composition is distributive with respect to addition in the abelian group. If f, g, g', h are composable morphisms, then distributivity is as follows:

$$f \circ (g + g') = f \circ g + f \circ g'$$

and

$$(g + g') \circ h = g \circ h + g' \circ h.$$

Furthermore, an additive category must have a zero object 0 and a product $A \times B$ for every pair of objects.

Definition A.8. An *abelien* category is an additive category $\mathscr{A}$ which satisfies the following three properties:

(AB1) Every morphism in $\mathscr{A}$ has a kernel and a cokernel.
(AB2) Every monomorphism in $\mathscr{A}$ is the kernel of its cokernel.
(AB3) Every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

Remark A.9. Hence in an abelian category, a monophism is equivalent to the being the kernel of a morphism. Similarly, an epimorphism is equivalent to being a cokernel. For a unitary associative ring R, the category of left $R-$modules, $R-$**mod** is an abelian category and in some sense the only example. The Freyd-Mitchell Theorem (1964) states that every *small* abelian category (i.e. an abelian category such that its class of objects is a set) embeds into a category $R-$**mod** for some ring R (see [3] pg 25). By choosing a "universe", one can ensure that our categories of interest are small (see [2] pg 12, Remark on "large" categories).

Appendix B. Functors

Definition B.1. A (covariant) functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object $C \in \mathscr{C}$ an object $F(C) = FC = F_C \in \mathscr{D}$ and to each morphism $f : C_1 \to C_2$ in $\mathscr{C}$ a morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathscr{D}$. A functor must send identities to identities: $F(\mathrm{id}_{\mathscr{C}}) = \mathrm{id}_{F(\mathscr{C})}$ and must send composition to composition: $F(f \circ g) = F(f) \circ F(g)$.

Remark B.2. A contravariant functor on the other hand sends objects of $\mathscr{C}$ to $\mathscr{D}$ but reverses the arrows i.e. for a morphism $f : C_1 \to C_2$ in $\mathscr{C}$, it assigns a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathscr{D}^{op}$, the opposite category of $\mathscr{D}$ obtained by formally reversing the direction of all its morphisms while retaining their original composition law.

Definition B.3. Let $F, G : \mathscr{C} \to \mathscr{D}$ be two funtors from $\mathscr{C}$ to $\mathscr{D}$. A *natural transformation* $\eta : F \Rightarrow G$ assigns to each object $C \in \mathscr{C}$ a morphism $\eta_C : F(C) \to G(C)$ in $\mathscr{D}$ such that for each morphism $f : C \to C'$ in $\mathscr{C}$, one has a commutative diagram

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\eta_C"] & F(C') \arrow[d, "\eta_{C'}"] \\ G(C) \arrow[r, "G(f)"] & G(C') \end{tikzcd}$$

Remark B.4. In particular, when each η_C is an isomorphism, then η is called a *natural isomorphism* and it is denoted by $\eta : F \cong G$.

Definition B.5. A functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories is *additive* if it preserves finite biproducts, i.e. F maps zero to zero ($F(0) \simeq 0 \in \mathscr{B}$), and given two objects $A, A' \in \mathscr{A}$, there is an isomorphism $F(A \oplus A') \cong F(A) \oplus F(A')$ which respect inclusion and projection maps of the direct sum.

Definition B.6. A functor $F : \mathscr{C} \to \mathscr{D}$ is *fully faithful* if and only if for every object $X, Y \in \mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(F(X), F(Y))$ is a bijection.

We now make a small detour to a fundamental result in category theory: Yoneda lemma and Yoneda embedding. One of its many applications, used here in these notes, is to prove the right and left exactness of adjoint functors, which will be defined shortly.

Remark B.7. Let $\mathscr{C}$ be a category. Then every object X of $\mathscr{C}$ defines a contravariant functor

$$h_X : \mathscr{C}^{op} \to \mathfrak{Sets}$$
$$A \mapsto \mathrm{Hom}_{\mathscr{C}}(A, X)$$

Let $F : \mathscr{C}^{op} \to \mathfrak{Sets}$ be a contravariant functor. If $\tau : h_X \Rightarrow F$ is a natural transformation, then one can consider the morphism $\tau_X : h_X(X) = \mathrm{Hom}_{\mathscr{C}}(X, X) \to F(X)$.

Remark B.8. Here $\mathfrak{Sets}$ is the category of sets where the class of objects consists of sets (note that the class itself is not a set since the notion of a set of all sets leads to contradictions).

Lemma B.9. *(Yoneda lemma) Let $\mathscr{C}$ be a category, $X \in \mathscr{C}$ and $F : \mathscr{C}^{op} \to \mathfrak{Sets}$ be a contravariant functor. The map*

$$Hom(h_X, F) \cong F(X), \qquad \tau \mapsto \tau_X(id_X)$$

is bijective and functorial in X. Moreover this isomorphism is natural in X and F when both sides are regarded as functors from $\mathscr{C} \times \mathfrak{Sets}^{\mathscr{C}^{op}}$ to $\mathfrak{Sets}$.

Proof. Let $\tau : h_X \Rightarrow F$ be a natural transformation, then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X, X) & \xrightarrow{\tau_X} & F(X) \\ \downarrow{\scriptstyle -\circ u} & & \downarrow{\scriptstyle F(u)} \\ \mathrm{Hom}_{\mathscr{C}}(A, X) & \xrightarrow{\tau_A} & F(A) \end{array}$$

for every object $A \in \mathscr{C}$ and every morphism $u \in \mathrm{Hom}_{\mathscr{C}}(A, X)$. In particular, since $\tau_A(u) = \tau_A(id_X \circ u) = F(u)(\tau_X(id_X))$, τ is entirely determined by $\tau_X(id_X)$. Hence it suffices to show that if $x \in F(X)$, the collection of maps $\tau_A : \mathrm{Hom}_{\mathscr{C}}(A, X) \to F(A)$ defined by $\tau_A(u) = F(u)(x)$ for $u \in \mathrm{Hom}_{\mathscr{C}}(A, X)$ is a natural transformation. If $v : A \to B$ is a morphism in $\mathscr{C}$, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(B, X) & \xrightarrow{\tau_B} & F(B) \\ \big\downarrow{\scriptstyle -\circ v} & & \big\downarrow{\scriptstyle F(v)} \\ \mathrm{Hom}_{\mathscr{C}}(A, X) & \xrightarrow{\tau_A} & F(A) \end{array}$$

since for $u \in \mathrm{Hom}_{\mathscr{C}}(B, X)$, one has that $(F(v) \circ \tau_B)(u) = (F(v) \circ F(u))(x) = F(u \circ v)(x)$ and $(\tau_A \circ (- \circ v))(u) = \tau_A(u \circ v) = F(u \circ v)(x)$. □

Remark B.10. $\mathfrak{Sets}^{\mathscr{C}^{op}}$ or sometimes $[\mathscr{C}^{op}, \mathfrak{Sets}]$ denotes the category of functors (or functor category) from $\mathscr{C}^{op}$ to $\mathfrak{Sets}$, where $\mathscr{C}^{op}$ is a small category.

Corollary B.11. *(Yoneda embedding) Let (C) be a category. The functor*

$$h : \mathscr{C} \to \mathfrak{Sets}^{\mathscr{C}^{op}}$$
$$X \mapsto h_X$$

is fully faithful.

Proof. If $u : X \to Y$ is a morphism in $\mathscr{C}$, then its associated natural transformation $h(u) : h_X \Rightarrow h_Y$ is given by, for every object $A \in \mathscr{C}$,

$$h(u)_A : \mathrm{Hom}_{\mathscr{C}}(A, X) \to \mathrm{Hom}_{\mathscr{C}}(A, Y)$$
$$v \mapsto u \circ v$$

By Yoneda lemma, $\mathrm{Hom}(h_X, h_Y) \cong h_Y(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y), \tau \mapsto \tau_X(\mathrm{id}_X)$ is a bijection and hence the image of $h(u)$ under this bijection is $h(u)_X(\mathrm{id}_X) = u \circ \mathrm{id}_X = u$. □

Remark B.12. If $\mathscr{A}$ is an additive category, then $\mathscr{A}$ embeds via the functor $h : A \mapsto h_A = \mathrm{Hom}_{\mathscr{A}}(-, A)$ into the abelian category $\mathbf{Ab}^{\mathscr{A}^{op}}$, the category of functors from $\mathscr{A}^{op} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups.

Remark B.13. A closely related definition is that of a *representable* functor: a functor $F : \mathscr{C}^{op} \to \mathfrak{Sets}$ is *representable* if and only if it is isomorphic to h_X for some object $X \in \mathscr{C}$. Note that such an X is unique up to isomorphism.

Lemma B.14. *Let $\mathscr{A}$ be an additive category. The Yoneda embedding reflects exactness, i.e. a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathscr{A}$ is exact if for every object $M \in \mathscr{A}$, we have the following exact sequence*

$$Hom_{\mathscr{A}}(M, A) \xrightarrow{\alpha_*} Hom_{\mathscr{A}}(M, B) \xrightarrow{\beta_*} Hom_{\mathscr{A}}(M, C).$$

Proof. Letting $M = A$, we see that $\beta \circ \alpha = \beta_* \circ \alpha_*(id_A) = 0$, hence $Im(\alpha) \subset ker(\beta)$. Letting $M = ker(\beta)$, we see that the monomorphism $\iota : ker(\beta) \to B$ satisfies $\beta_*(\iota) = \beta \circ \iota = 0$. By exactness, there exists a morphism $\gamma \in \mathrm{Hom}_{\mathscr{A}}(M, A)$ such that $\iota = \alpha_*(\gamma) = \alpha \circ \gamma$, which implies that $ker(\beta) = Im(\iota) \subset Im(\alpha)$. □

Definition B.15. A pair of functors $L : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ are called *adjoint functors* if there exists a bijection for all objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$

$$\tau = \tau_{A,B} : \mathrm{Hom}_{\mathscr{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(A, R(B))$$

natural in A and B, i.e. for ever morphism $f : A \to A'$ in $\mathscr{A}$ and $g : B \to B'$ in $\mathscr{B}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\tau_{A',B}"] & \mathrm{Hom}_{\mathscr{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau_{A,B}"] & \mathrm{Hom}_{\mathscr{B}}(L(A), B') \arrow[d, "\tau_{A,B'}"] \\
\mathrm{Hom}_{\mathscr{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathscr{A}}(A, R(B)) \arrow[r, "Rg_*"] & \mathrm{Hom}_{\mathscr{A}}(A, R(B'))
\end{tikzcd}$$

Remark B.16. We have the two functors $\mathrm{Hom}_{\mathscr{B}}(L(-), -)$ and $\mathrm{Hom}_{\mathscr{A}}(-, R(-))$ from $\mathscr{A}^{op} \times \mathscr{B}$ to $\mathfrak{Sets}$ and τ, which is a natural isomorphism between them. We say that L is left adjoint to R, and that R is right adjoint to L.

Example B.17. Many examples exists of adjoint functors. A left adjoint functor is often a universal construction which invert a forgetful functor. For example, let k be a field and $L : \mathfrak{Sets} \to \mathbf{Vect}$ the functor which assign to each set X the $k-$vector space with basis indexed by X. Hence, $L(X)$ is the set of linear combinations of elements of X with coefficients in k. Then L is left adjoint to the forgetful functor $R : \mathbf{Vect} \to \mathfrak{Sets}$, which assign to each $k-$vector space its underlying set.

Remark B.18. There is also a definition via counit-unit adjunction where a pair of $(L, R) : \mathscr{A} \to \mathscr{B}$ defines (i) a natural transformations $\eta : \mathrm{id}_A \Rightarrow R \circ L$, the "unit of the adjunction" such that the right adjoint of $f : L(A) \to B$ is $R(f) \circ \eta_A : A \to R(B)$, and (ii) a natural transformations $\epsilon : L \circ R \to \mathrm{id}_B$, the "counit of the adjunction" such that the left adjoint of $g : A \to R(B)$ is $\epsilon_B \circ L(g) : L(A) \to B$. Furthermore, the following compositions give the identity

$$L(A) \xrightarrow{L(\eta)} L \circ R \circ L(A) \xrightarrow{\epsilon L} L(A) \text{ and } R(B) \xrightarrow{\eta R} R \circ L \circ R(B) \xrightarrow{R(\epsilon)} R(B)$$

(see [2] pg 82 Theorem 1 and pg 83 Theorem 2).

Theorem B.19. *If the functor $R : \mathscr{D} \to \mathscr{C}$ has a left adjoint functor, then R preserves products, pull-backs and kernels.*

Proof. See [1] Theorem II.7.7 pg 68. □

Theorem B.20. *(Dual) If the functor $L : \mathscr{C} \to \mathscr{D}$ has a right adjoint functor, then L preserves coproducts, pushouts and cokernels.*

Theorem B.21. *Let $L : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ be a pair of additive adjoint functors between abelian categories. Then L is right exact and R is left exact.*

Proof. Let $0 \to B' \to B \to B'' \to 0$ be a short exact sequence in $\mathscr{B}$. By the naturality of the bijection τ of the adjunction, one has a commutative diagram for all object $A \in \mathscr{A}$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathscr{B}}(L(A), B') \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{B}}(L(A), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathscr{B}}(L(A), B'') \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A, R(B')) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A, R(B)) \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(A, R(B''))
\end{tikzcd}$$

The first row is exact since $\mathrm{Hom}_{\mathscr{B}}(L(A), -)$ is left exact, hence the second row is exact for all $A \in \mathscr{A}$. By Yoneda lemma B.14, it follows that

$$0 \to R(B') \to R(B) \to R(B'')$$

is exact. As a consequence, every right adjoint functor R is left exact. In particular, the right adjoint functor $L^{op} : \mathscr{A}^{op} \to \mathscr{B}^{op}$ is left exact, i.e. L is right exact.

□

Appendix C. Suggested Exercises

The following exercises have been kindly suggested by Dylan C. Beck:

Exercises:

(1) Let R be a commutative ring and I, J ideals. Show that
$$\mathrm{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}.$$

(2) Let R be a commutative ring and M be an R-module. Show that
$$\mathrm{Tor}_1^R(M, RxR) = 0 \iff x \text{ is a non-zero divisor.}$$

(3) (Koszul complex) Let R be a commutative ring and M be an R-module, and let $I = < x_1, ..., x_n >$ be a regular sequence. Show that
$$\mathrm{Tor}_n^R(R/I, M) \cong \{m \in M \mid Im = 0\}.$$

Moreover, show that
$$\begin{aligned}\mathrm{Tor}_1^R(R/I, R/J) &\cong \frac{J : I}{J} \\ &\cong \mathrm{Hom}_R(R/I, R/J)\end{aligned}$$

where $J : I = \{r \in R \mid rI \subseteq J\}$.

References

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[3] Charles Weibel, *An introduction to homological algebra* Cambridge Studies in advanced mathematics **38**, Cambridge University Press 1994