Dr. Nakao Research Journal

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1 RAIL for Cylindrical Coordinates

If we consider the 2D spatial heat equation

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{\mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\mathbf{r} \frac{\partial \mathbf{U}}{\partial r} \right) + \frac{1}{\mathbf{r^2}} \left(\frac{\partial^2 \mathbf{U}}{\partial \theta^2} \right) + \left(\frac{\partial^2 \mathbf{U}}{\partial z^2} \right) \tag{1}$$

where we assume azimuthal symmetry

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{\mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\mathbf{r} \frac{\partial \mathbf{U}}{\partial r} \right) + \left(\frac{\partial^2 \mathbf{U}}{\partial z^2} \right) \tag{2}$$

$$\frac{d\mathbf{U}}{dt} = \frac{1}{\mathbf{r}} \mathbf{U}_{rr} + \mathbf{U}_{zz} \tag{3}$$

To solve this diffusive heat equation, we presume the solution assumes a low-rank structure that can be factored into time-dependent, orthogonal spatial basis vectors $\mathbf{V}_r$, $\mathbf{V}_z$ and a diagonal weighting vector $\mathbf{S}$ via truncated singular value decomposition (SVD) such that

$$\mathbf{U}(t^n) = \mathbf{V}^r(t^n)\mathbf{S}(t^n)(\mathbf{V}^z(t^n))^T \tag{4}$$

where $\mathbf{V}^r$ and $\mathbf{V}^z$ are orthogonal basis matrices for the r, and z subspaces, respectively, and $\mathbf{S}$ is a diagonal weighting vector. However, it is important to note that because we are working in cylindrical coordinates, functions $f(x)$, $g(x)$ are only orthogonal in the L^2 norm if

$$< f, g >_r := \int_0^L f(x)g(x) r dr = \begin{cases} 0, & f(x) \neq g(x) \\ 1, & f(x) = g(x) \end{cases} \tag{5}$$

because the determinant of the linear transformation matrix from Cartesian to cylindrical coordinates equals the radius r. Therefore, when we define the orthogonal basis matrix $\mathbf{V}^r$ as orthogonal only if

$$(\mathbf{V}^r)^T (\mathbf{r} \star \mathbf{V}^r_*) = I \tag{6}$$

where the $\star$ operator denotes the Hasmaard product between the orthogonal basis matrix and the vector of r-values.

We represent the second partial derivatives $\mathbf{U}_{rr}$, $\mathbf{U}_{zz}$ using Laplacians $\mathbf{D}_{rr}$, $\mathbf{D}_{zz}$ that act on the discretized basis vectors to yield high-order approximations of the second partial derivatives.

We can represent the spatially discretized solution $\mathbf{U}^n \approx \mathbf{U}(t^n)$ and at each time-step, we separately update $\mathbf{V}^r$, S, and $\mathbf{V}^z$ to yield the complete solution $\mathbf{U}^{n+1}$ at time t^{n+1}.

$$\mathbf{U}^{n+1} = \mathbf{V}^{r,n+1}\mathbf{S}^{n+1}(\mathbf{V}^{z,n+1})^T \tag{7}$$

If we update the solution using time-step size Δt, we can achieve a first-order temporal approximation via Backward Euler method

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \left(\frac{1}{\mathbf{r}} \mathbf{D}_{rr}\mathbf{U}^{n+1} + \mathbf{U}^{n+1}\mathbf{D}_{zz}^T \right) \tag{8}$$

We use the dynamic low-rank (DLR) framework to separate equation (8) into its basis vector components, isolating and solving for $\mathbf{V}_r$, $\mathbf{V}_z$, and S separately. We project $\mathbf{U}^n$ onto the low-rank basis subspaces of $\mathbf{V}^r$ and $\mathbf{V}^z$ using first-order approximations $\mathbf{V}_*^{r,n+1} \approx \mathbf{V}^{r,n+1}$, and $\mathbf{V}_*^{z,n+1} \approx V^{z,n+1}$.

$$\begin{aligned} \mathbf{K}^{n+1} &:= \mathbf{U}^{n+1}\mathbf{V}_*^{z,n+1} = \mathbf{V}^{r,n+1}\mathbf{S}^{n+1}(\mathbf{V}^{z,n+1})^T\mathbf{V}_*^{z,n+1} = \mathbf{V}^{r,n+1}\mathbf{S}^{n+1} \\ \mathbf{L}^{n+1} &:= (\mathbf{U}^{n+1})^T \left(\mathbf{r} \star \mathbf{V}_*^{r,n+1}\right) = \mathbf{V}^{z,n+1}(\mathbf{S}^{n+1})^T(\mathbf{V}^{r,n+1})^T \left(\mathbf{r} \star \mathbf{V}_*^{r,n+1}\right) = \mathbf{V}^{z,n+1}(\mathbf{S}^{n+1})^T \end{aligned} \tag{9}$$

We project equation (8) onto the orthogonal vector $\mathbf{V}_*^{z,n+1}$, to achieve

$$\mathbf{K}^{n+1} = \mathbf{K}^n + \Delta t \left(\frac{1}{\mathbf{r}} \mathbf{D}_{rr}\mathbf{K}^{n+1} + \mathbf{K}^{n+1} \left(\mathbf{D}_{zz}\mathbf{V}^{z,n+1}\right)^T \mathbf{V}^{z,n+1} \right) \tag{10}$$

We can rearrange this equation to yield a Sylvester equation solveable for $\mathbf{K}^{n+1}$.

$$\left(\mathbf{I} - \Delta t \frac{1}{\mathbf{r}} \mathbf{D}_{rr}\right)\mathbf{K}^{n+1} - \mathbf{K}^{n+1}\left(\Delta t \left(\mathbf{D}_{zz}\mathbf{V}^{z,n+1}\right)^T \mathbf{V}^{z,n+1}\right) = \mathbf{K}^n \tag{11}$$

Similarly, we can project equation (8) onto the orthogonal vector product $\mathbf{V}_*^{r,n+1}\mathbf{r}$ to yield

$$\left(\mathbf{I} - \Delta t \mathbf{D}_{zz}\right)\mathbf{L}^{n+1} - \mathbf{L}^{n+1}\left(\Delta t \left(\mathbf{D}_{rr}\mathbf{V}^{r,n+1}\right)^T \left(\mathbf{r} \star \mathbf{V}^{r,n+1}\right)\right) = \mathbf{L}^n \tag{12}$$

Note that equation (12) differs from equation (11) in that the it results from the projection of equation (8) onto $\left(\mathbf{r} \star \mathbf{V}_*^{r,n+1}\right)$ because we want to ensure that the r-subspace basis $\mathbf{V}^{\mathbf{r},\mathbf{n+1}}$ is projected out due to the definition of weighted subspace orthogonality (5).

We solve equations (11) and (12) to get $\mathbf{K}^{n+1}$ and $\mathbf{L}^{n+1}$, respectively. We then compute reduced QR-factorization on $\mathbf{K}^{n+1}$ and $\mathbf{L}^{n+1}$ to obtain updated orthogonal basis vectors $\mathbf{V}_{\ddagger}^{r,n+1}$, and $\mathbf{V}_{\ddagger}^{z,n+1}$, respectively. The reduced QR-factorizations are defined via $\mathbf{K}^{n+1} = \mathbf{QR} := \mathbf{V}_{\ddagger}^{r,n+1}\mathbf{R}$ and $\mathbf{L}^{n+1} = \mathbf{QR} := \mathbf{V}_{\ddagger}^{z,n+1}\mathbf{R}$. However, to ensure $\mathbf{V}^{r,n+1}$ remains orthogonal in the weighted r-susbspace, we compute the $\mathbf{K}^{n+1}$ reduced QR-factorization via

$$\texttt{QR-factorization}(\sqrt{\mathbf{r}} \star \mathbf{K}^{n+1}) \to \frac{\mathbf{Q}}{\sqrt{\mathbf{r}}} = \mathbf{V}_{\ddagger}^{r,n+1} \tag{13}$$