<u>Linear Advection Analysis</u> 5/28-29

$u_t - u_x = 0 \quad , \; x \in [0,1] \; , \; u_0(x) = \sin(2\pi x)$

a) $u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

<u>Accuracy:</u>

$$L_k(x,t) = \frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) - \frac{1}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

T.S. Expansion $\longrightarrow$ $= \frac{1}{\Delta t}\left(\cancel{u^n(x_j)} + \Delta t\, u^n(x_j)_t + \frac{1}{2}\Delta t^2 u^n(x_j)_{tt} + \dots - \cancel{u^n(x_j)}\right)$

$\longrightarrow - \frac{1}{\Delta x}\left(\cancel{u^n(x_j)} - \left(\cancel{u^n(x_j)} - \Delta x\, u^n(x_j)_x + \frac{1}{2}\Delta x^2 u^n(x_j)_{xx} + \dots\right)\right)$

$$L_k(x,t) = \underbrace{u^n(x_j)_t - u^n(x_j)_x}_{=0} + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

$$(\mathcal{O}(\Delta x) > \mathcal{O}(\Delta t))$$

$$\boxed{L_k(x,t) = \mathcal{O}(\Delta x) = 1^{st} \text{ Order Accurate Method}}$$

<u>Stability:</u> assume $u_j^n = \alpha^n e^{ik(x_j)}$

$\hookrightarrow \alpha^{n+1} \cancel{e^{ik(x_j)}} = \cancel{\alpha^n} \cancel{e^{ik(x_j)}} + \frac{\Delta t}{\Delta x}\left(\cancel{\alpha^n} \cancel{e^{ik(x_j)}} - \cancel{\alpha^n} e^{ik(x_j) - \Delta x}\right)$

$\alpha = 1 + \frac{\Delta t}{\Delta x}\left(1 - \underbrace{e^{-ik\Delta x}}\right) \Rightarrow 1 + \frac{\Delta t}{\Delta x}\left(1 - \left(\cos(k\Delta x) - i\sin(k\Delta x)\right)\right)$

$|\alpha|^2 = \left(1 + \frac{\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\right)^2 + \left(\frac{\Delta t}{\Delta x}\sin(k\Delta x)\right)^2 = 1 + \frac{2\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right) + \frac{\Delta t^2}{\Delta x^2}\left(1 - 2\cos(k\Delta x) + \cos^2 \ldots\right)$ 1 $\sin$

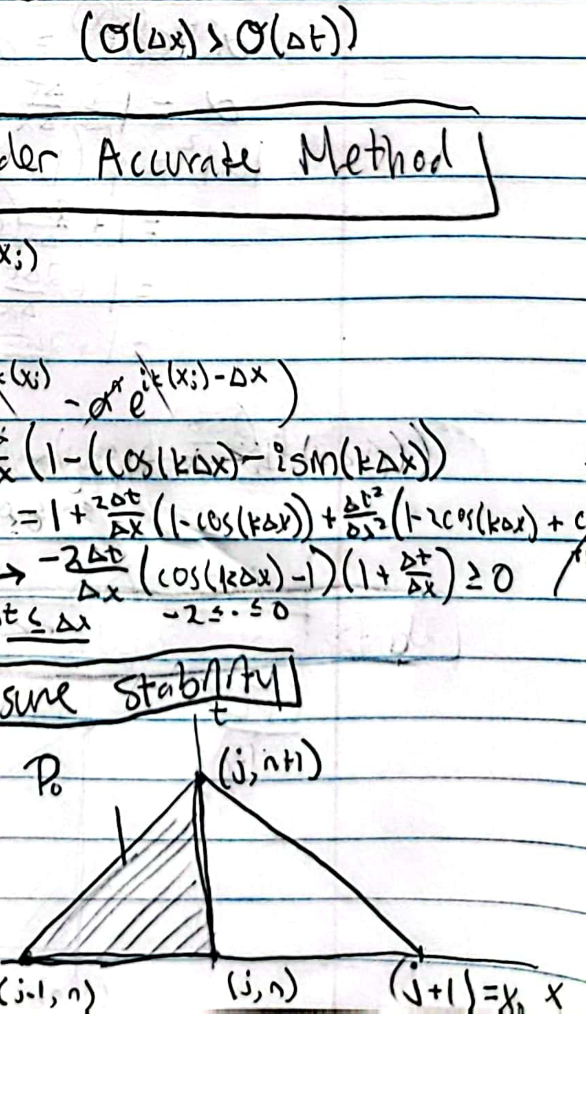
$= \frac{2\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\left(1 + \frac{\Delta t}{\Delta x}\right) + 1 \le 1 \rightarrow \frac{-2\Delta t}{\Delta x}\left(\cos(k\Delta x) - 1\right)\left(1 + \frac{\Delta t}{\Delta x}\right) \ge 0$

$-2 \le \cdot \le 0$

$\left(4\Delta t / \cancel{\Delta x}\right)\left(1 + \frac{\Delta t}{\Delta x}\right) \ge 0 \rightarrow \underline{\Delta t \le \Delta x}$

$\hookrightarrow \therefore$ $\boxed{\Delta t \le \Delta x \text{ to ensure stability}}$

<u>CFL:</u> $\frac{dx}{dt} = -1 \rightarrow x_0 = x + t$



$x_0 = j+1$ characteristic $\notin D_0$

$\hookrightarrow$ $\boxed{\text{CFL-condition not met}}$

b) $u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}(u_{j+1}^n - u_j^n)$

Accuracy: $L_k(x,t) = \frac{1}{\Delta t}(u_j^{n+1} - u_j^n) - \frac{1}{\Delta x}(u_{j+1}^n - u_j^n)$

$$= \frac{1}{\Delta t}\left(\cancel{u^n(x_j)} + \Delta t\, u^n(x_j)_t + \tfrac{1}{2}\Delta t^2 u^n(x_j)_{tt} + \dots - \cancel{u^n(x_j)}\right)$$
$$- \frac{1}{\Delta x}\left(\cancel{u^n(x_j)} + \Delta x\, u^n(x_j)_x + \tfrac{1}{2}\Delta x^2 u^n(x_j)_{xx} + \dots - \cancel{u^n(x_j)}\right)$$

$$L_k(x,t) = \underbrace{\left(u^n(x_j)_t - u^n(x_j)_x\right)}_{0} + O(\Delta t) + O(\Delta x)$$

$$\boxed{= O(\Delta x) = 1^{st}\text{-Order Method}}$$

Stability: assume $u_j^n = \alpha^n e^{ik(x_j)}$

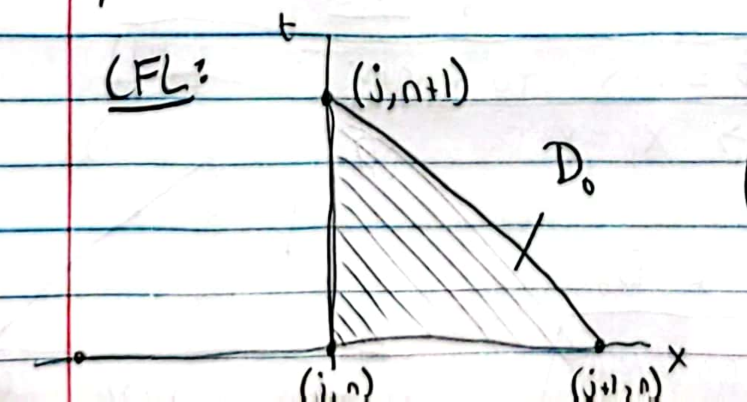
$$\alpha^{n+1} \cancel{e^{ik(x_j)}} = \cancel{\alpha^n e^{ik(x_j)}} + \frac{\Delta t}{\Delta x}\left(\alpha^n e^{ik(x_j+\Delta x)} - \alpha^n e^{ik(x_j)}\right)$$
$$\alpha = 1 + \frac{\Delta t}{\Delta x}\left(e^{ik\Delta x} - 1\right) = 1 + \frac{\Delta t}{\Delta x}\left(\cos(k\Delta x) + i\sin(k\Delta x) - 1\right)$$
$$|\alpha|^2 = \sqrt{\left(1 + \frac{\Delta t}{\Delta x}(\cos(k\Delta x) - 1)\right)^2 + \frac{\Delta t}{\Delta x}\sin(k\Delta x)^2}^{\,2}$$
$$= 1 + \frac{2\Delta t}{\Delta x}(\cos(k\Delta x) - 1) + \frac{\Delta t^2}{\Delta x^2}\left(\cos^2(k\Delta x) - 2\cos(k\Delta x) + \cancel{1} + \sin^2(k\Delta x)\right)$$
$$= 1 + \frac{2\Delta t}{\Delta x}(\cos(k\Delta x) - 1) + \frac{\Delta t^2}{\Delta x^2}\left(2 - 2\cos(k\Delta x)\right)$$
$$= 1 + \frac{2\Delta t}{\Delta x}(\cos(k\Delta x) - 1) - \frac{2\Delta t^2}{\Delta x^2}(\cos(k\Delta x) - 1) \le 1$$

$\hookrightarrow \left(\frac{2\Delta t}{\Delta x}\right)(\cos(k\Delta x) - 1)\left(1 - \frac{\Delta t}{\Delta x}\right) \le 0$ $\Big\}$ $\left(-\cancel{\frac{4\Delta t}{\Delta x}}\right)\left(1 - \frac{\Delta t}{\Delta x}\right) \le 0$

$-2 \le \cdot \le 0$ $\quad 1 - \frac{\Delta t}{\Delta x} \ge 0 \rightarrow \boxed{\Delta t \le \Delta x}$

$$\boxed{\Delta t \le \Delta x \text{ to maintain stability}}$$

CFL:



$\frac{dx}{dt} = -1 \rightarrow x_0 = x + t$

$\hookrightarrow$ characteristics $\in D_0$

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