# Foundations of Euclidean Geometry

- Part 2: Distance and Ruler Postulate

# Part 2. Distance and Ruler Postulate

In Part 2, we will take an approach to introduce the notion of distance in Euclidean geometry by relying on the real numbers. In this approach, every line in the system corresponds the set of all real numbers. This correspondence will provide properties of distance that we expect for Euclidean geometry.

**Learning objectives in Part 2 include**

- Explain how the concept of distance is established in Euclidean geometry by using the property of the real numbers and the ruler postulate (Postulate 8), and
- Produce mathematical proofs of given theorems using undefined terms, definitions, and Postulates 1-8.

# Postulate 7: Introduction of Distance

$$\mathbb{E}^3(S, L, P, d)$$

- Undefined terms: $S, L, P, d$

**Postulate 7**
$d$ is a function such that $d\colon\ S \times S \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers.

**Definition 4**
Distance between two points: $d(A, B)$

# Coordinate Systems

**Definition 5. Coordinate System**

Let $f\colon l \leftrightarrow \mathbb{R}$ be a one-to-one correspondence between a line $l$ and the set of real numbers. If for all points $A, B$ of $l$ we have

$$AB = |f(A) - f(B)|,$$

then $f$ is called a **coordinate system** for $l$. For each point $A$ of $l$, the real number $x = f(A)$ is called the **coordinate** of $A$.

# Ruler Postulate

**Postulate 8 (The Ruler Postulate)**

Every line has a coordinate system.

Note that $f$ is not defined on the entire set of points, but a line. Postulate 8 says that, for each line, there exists a coordinate system that maps the line to the real numbers.

# Exercise 2

Read Theorem 5-8 and give their proofs.

<u>Theorem 5</u> For every pair of points $A, B$ in $S$,
1. $d(A, B) \geq 0$
2. $d(A, B) = 0$ if and only if $A = B$
3. $d(A, B) = d(B, A)$

<u>Theorem 6</u> If $f$ is a coordinate system for a line $l$ and $g(A) = -f(A)$ for each point $A$ of $l$, then $g$ is a coordinate system for $l$.

<u>Theorem 7</u> If $f$ is a coordinate system for a line $l$ and if $a$ is any real number such that for each $A$ in $l$, $g(A) = f(A) + a$, then $g$ is a coordinate system for $l$.

<u>Theorem 8</u> (The Ruler Placement Theorem) Let $l$ be a line and let $A$ and $B$ be any two distinct points of $l$. Then $l$ has a coordinate system in which the coordinate of $A$ is 0 and the coordinate of $B$ is positive.

# Theorem 6 and 7

Theorem 6 and 7 require the surjectivity of coordinate systems

"For any real number $x$, there exists a point $A$ in $l$ such that $x = g(A)$."

Using the given definition of $g$ in each theorem, show how to find such $A$.

<u>Example (Theorem 6)</u>

Since we have a coordinate system $f$ of $l$, we can find a point $A$ in $l$ such that $f(A) = -x$. By the definition of $g$, $x = -f(A) = g(A)$.

# Theorem 8

In your proof of Theorem 8, clearly indicate where and how you are using Postulate 8, Theorem 7 and 6.

<u>Example(Postulate 8)</u>

"Postulate 8 gives us an arbitrary coordinate system $f$ of $l$ containing points $A$ and $B$. Let's say $f(A) = x$ and $f(B) = y$."

<u>Note</u>: Show how you construct the coordinate system that satisfies Theorem 8 using Theorem 7 and 6.

- Using $f$ and Theorem 7, how can you define a coordinate system $g$ of $l$ such that $g(A) = 0$?
- Using $g$ and Theorem 6, how can you define a coordinate system $h$ of $l$ that satisfies the conclusion of Theorem 8?

# Exercise 2.5

Consider the following interpretation of the undefined term $d$ and expand the four-point model from Exercise 1 Problem 1-4.

$$d(A, A) = 0 \text{ for all point } A \text{ in } S$$
$$d(A, B) = 1 \text{ for all distinct points } A, B \text{ in S}$$

With the existing interpretations of $S, L, P$ in the four-point model, does this interpretation of $d$ satisfy Postulate 7 and 8? Explain why.

# Exercise 2.6

Consider the following modification of the definition of coordinate system $f$. Let $f$ be a one-to-one correspondence between a line and another set of numbers such as $\mathbb{Z}$ (the set of all integers) or $\mathbb{Q}$ (the set of all rational numbers). Do you think Theorem 5-8 still hold in this modified system? Based on your proofs in the previous problems, justify your answer.

# Week 2 Class Discussion: Part 2

Give your responses to the following reflective questions in Blackboard discussion forum named, Week 2: Class Discussion.

Q3. What properties of the real numbers were used to prove Theorem 5-8? Give some examples as much as you can identify.

# Part 2: Summary

- Postulate 7 provides the existence of a function $d$ (called distance) for each line in the system that assigns a real number to a pair of two points on the line.
- The distance function $d$ in Postulate 7 does not allow us to find a "meaning" useful to describe what we want it to be in our Euclidean geometry.
- The definition of coordinate systems and Postulate 8 are intentionally introduced in the system to create such "meaning" by corresponding each line in the system to the real number line.
- The consequences can be shown in Theorems 5-8 and later in the following parts of the lecture note.