Foundations of Euclidean Geometry

- Introduction to FEG
- Part 1: Primitive Concepts and Postulates

Introduction to FEG

- In the first module of the course, we will develop an axiomatic system for Euclidean geometry. To begin with, we will be given a handful of undefined terms, definitions, and postulates (a.k.a. axioms).

- Given this initial system, we will investigate logical consequences of the postulates with respect to what we know about the nature of Euclidean geometry to the best we can derive from the given.

- In each of the nine parts, we will develop key concepts in geometry and explore possible models of the system at-the-moment by adding postulates and definitions to the system, until we will eventually shape it into what we call Euclidean geometry.

Introduction to FEG (cont'd)

- In the process of this development, students are expected to be active participants in constructing the knowledge, not just passive receivers.

- The major learning activity is to complete each part of the under-developed document, Foundations of Euclidean Geometry. Due to its purpose, this document is intentionally composed underdeveloped, and need to be completed by students. It is highly recommended to keep student work in organized form to supplement any missing parts.

- The student-generated components will include some definitions, theorems with proofs, and responses to exercise problems. At the end of each part of the notes, you have a list of tasks for weekly assignments. Note that weekly lecture notes, slides, and lectures will be uploaded in Blackboard.

Part 1. Primitive Concepts and Postulates

In Part 1, we will discuss the primitive concepts and postulates that introduce the notion of incidence geometry.

Incidence geometry is a theory of relationships between geometric figures in the system. For instance, existence and/or uniqueness of intersections between figures and inclusion/exclusion of figures will be discussed.

Learning objectives in Part 1 include to

- Explain the primitive concepts and postulates for foundations of Euclidean Geometry,
- Examine geometric models or non-models of the given axiomatic systems with Postulate 1-6,
- Identify theorems that describe existence and intersections of geometric figures (a.k.a. incidence geometry) using models, and
- Produce mathematical proofs of theorems using undefined terms, definitions, and Postulates 1-6

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- Postulate 1-6
  Each postulate describes relationships between the basic geometric figures. Can you give names to the postulates? It will help you to create your own meanings of the postulates.

Postulates 1-6

P1 L is a set of subsets of S. P is a set of subsets of S.

P2 Given any two distinct points, there is exactly one line containing them.

P3 Given any three distinct noncollinear points, there is exactly one plane containing them.

P4 If two distinct points lie in a plane, then the line containing them lies in the plane.

P5 If two distinct planes intersect, then their intersection is a line.

P6 Every line contains at least two distinct points. Every plane contains at least three distinct noncollinear points. Space contains at least four distinct noncollinear and noncoplanar points.

Exercise 1

In Problem 1 to 4, consider the following interpretation of the sets S, L, P, where S contains exactly four points A, B, C, D, and the lines are the sets with exactly two distinct points, and the planes are the sets with exactly three distinct points.

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- Using the diagram, verify that Postulate 1-6 hold in this model.

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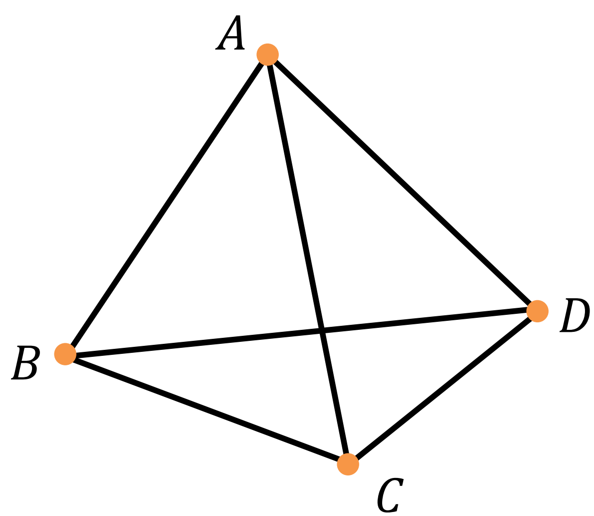

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We can write a statement to answer the question:

"For any two distinct lines, there is at most one intersecting point."

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Enough to show that any two distinct lines cannot have more than two intersecting points. Let's prove it by contradiction.

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Let's say arbitrary two distinct line *l* and *m* have two distinct intersecting points. Then, P2 says there exists exactly one line passing thru the points. It contradicts that *l* and *m* are distinct lines.

Exercise 1

Problem 3-4

- Answer the following questions using your diagram.

  (b) What are possible intersections of a line and a plane not containing the line?

  (c) For given a line and a point not on the line, how many planes are there containing the line and the point?

  (d) What is the union of two distinct lines intersecting each other?

- Write statements describing your responses to each question in the previous problem. Provide proofs of the statements for your Week 2 assignment.

Exercise 1

Problem 5-6

In Problem 5 and 6, consider the following interpretation of the sets S, L, P, where S contains exactly five points A, B, C, D, E, and the lines are the sets with exactly two distinct points, and the planes are the sets with exactly three distinct points.

- Draw a diagram of this interpretation on the white board in this room.
- Using the diagram, verify that Postulate 1-6 hold in this model.

Exercise 1

Problem 7-8

In Problem 7 and 8, consider a modified system obtained by replacing the last statement of Postulate 6 with the following:

"Space contains at least four distinct noncoplanar points."

- Is the interpretation in Problem 1-4 a model for the modified system?
- Consider the following interpretation of the sets S, L, P, where S = $\{A, B, C, D\}$, L = $\{\{A, B, C, D\}\}$, and P = $\emptyset$. Is this interpretation a model for the modified system? Can you draw a figure that illustrates this model?

Week 2 Class Discussion: Part 1

Give your responses to the following reflective questions in Blackboard discussion forum named, Week 2: Class Discussion.

Q1. Do you think the four-point model is a unique model of the system (P1-6) with finite set S? Consider the followings: (a) Do you think you can give a different interpretation of the sets L and P for given $S = \{A, B, C, D\}$? (b) Do you think you can give a model of the system that has exactly five points?

Q2. Do you think the system consisting of P1-P6 (and the terms) is complete? Explain why.

Part 1: Summary

- We introduced primitive concepts and postulates for foundations of Euclidean Geometry.

- The four-point model provides a particular way to interpret the undefined terms of the system. We can show that this model satisfies all six the postulates in the system. Of course, there could be different models of the system (what else could be?)

- Creation of theorems can be done by observing some patterns of figures representing a model of the system and proving conjectures. A proven conjecture in the system is a theorem. We proved and will prove in your assignments some describing existence and intersections of geometric figures (a.k.a. incidence geometry).