Preliminary

- An Example
- The Axiomatic Method
- Properties of Axiomatic Systems
- Euclid’s Axiomatic Geometry and It’s Limitations

An Example

Suppose we have the following rules (axioms) about students and classes in a university.

A1 There are exactly three students.

A2 For every pair of students, there is exactly one class in which they are enrolled.

A3 Not all of the students belong to the same class.

A4 Two separate classes share at least one student in common.

Consequences of the Axioms

A1 There are exactly three students.

A2 For every pair of students, there is exactly one class in which they are enrolled.

A3 Not all of the students belong to the same class.

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Thm 1.1. Two separate classes share one and only one student in common.

Thm 1.2. There are exactly three classes in our system.

Thm 1.3. Each class has exactly two students.

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Consequences of the Axioms

A1 There are exactly three points.

A2 For every pair of points, there is exactly one line in which they belong.

A3 Not all of the points belong to the same line.

A4 Two separate lines share at least one point in common.

Thm 1.1. Two separate lines share one and only one point in common.

Thm 1.2. There are exactly three lines in our system.

Thm 1.3. Each line has exactly two points.

The Axiomatic Method

Axiomatic System

An axiomatic system consists of four components listed below:

1. Undefined Terms
2. Axioms (or Postulates)
3. Defined Terms
4. Theorems

<u>Example</u>: Student-Class system

Undefined terms: Student, Class

Student-Class System

A1 There are exactly three students.

A2 For every pair of students, there is exactly one class in which they are enrolled.

A3 Not all of the students belong to the same class.

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Thm 1.1. Two separate classes share one and only one student in common.

Thm 1.2. There are exactly three classes in our system.

Thm 1.3. Each class has exactly two students.

Model of an Axiomatic System

A **model** is an interpretation of the undefined terms of an axiomatic system such that all of the axioms are true statements in this new interpretation.

Example

Three-point geometry as a model of the Student-Class system in the previous slide.

Exercise 1

Consider a system where we have children in a classroom choosing different flavors of ice cream. Suppose we have the following axioms:

A1 There are exactly five flavors of ice cream: vanilla, chocolate, strawberry, cookie dough, and bubble gum.

A2 Given any two different flavors, there is exactly one child who likes these two flavors.

A3 Every child likes exactly two different flavors among the five.

Q. Give a model of the system with using points and lines. Use the model to answer the following questions.

- How many children are there in this classroom?
- Show that any pair of children likes at most one common flavor.
- Show that for each flavor there are exactly four children who like that flavor.

PROPERTIES OF AXIOMATIC SYSTEMS

- Consistency
- Independence
- Completeness

Consistency

An axiomatic system is **consistent** if no two statements (these could be two axioms, an axiom and theorem, or two theorems) contradict each other.

Non-Example

A1 There are exactly three points.

A2 There are exactly two points.

Relative consistency

Let system *A* is embedded in another system *B*. If we know that system *B* is consistent, then system *A* must itself be consistent.

Independence

An individual axiom in an axiomatic system is called **independent** if it cannot be proved from the other axioms.

<u>Example</u>

A1 There are exactly three points.

A2 Two distinct points belong to one and only one line.

A3 Not all of the points belong to the same line.

A4 Two separate lines have at least one point in common.

A5 A line has exactly two points.

Independence (cont'd)

An individual axiom in an axiomatic system is called **independent** if it cannot be proved from the other axioms.

Q. How can we show that an axiom is independent of the other axioms in an axiomatic system?

>> An axiom X is independent in a system, if we could find two different models of the system including Axiom *X* and its negation, respectively.

Independence (cont'd)

An individual axiom in an axiomatic system is called **independent** if it cannot be proved from the other axioms.

<u>Example</u>

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Completeness

An axiomatic system is called **complete** if it is impossible to add a new consistent and independent axiom to the system. (The new axiom can use only defined and undefined terms of the original system.)

In other words, every statement involving defined and undefined terms is provably true or false in the system.

<u>Example</u>

Any set of three axioms among A1 –A4 is **NOT** complete.

But, the system including A1 – A4 is complete. (why?)

Exercise 2

Consider an axiomatic system that consists of elements in a set *S* and a set *P* of pairings of elements (a, b) that satisfy the following axioms:

A1 If (a, b) is in *P*, then (b, a) is not in *P*.

A2 If (a, b) is in *P* and (b, c) is in *P*, then (a, c) is in *P*.

Given two models of the system, answer the questions below.

M1: $S = \{1, 2, 3, 4\}$, $P = \{(1, 2), (2, 3), (1, 3)\}$

M2: Let *S* be the set of real numbers and let *P* consist of all pairs (x, y) where $x < y$.

Find another independent axiom A3 of the system, which is true in M1, but not in M2. Use this result to argue that the original system is not complete.

Exercise 2 (cont'd)

An example of such independent axiom

A3 There is an element *x* in *S* such that (y, x) is not in *P* for any y in *S*.

Then, **A3** is true in **M1** since there is 4 in *S* with no pairs in *P* containing. But, every element in *S* of **M2**, namely, every real number has infinitely many smaller numbers in *S*. Thus, **M1** is a model of the augmented system containing A1, A2, A3, where as **M2** is not. It concludes that the given system with A1 and A2 is not complete.

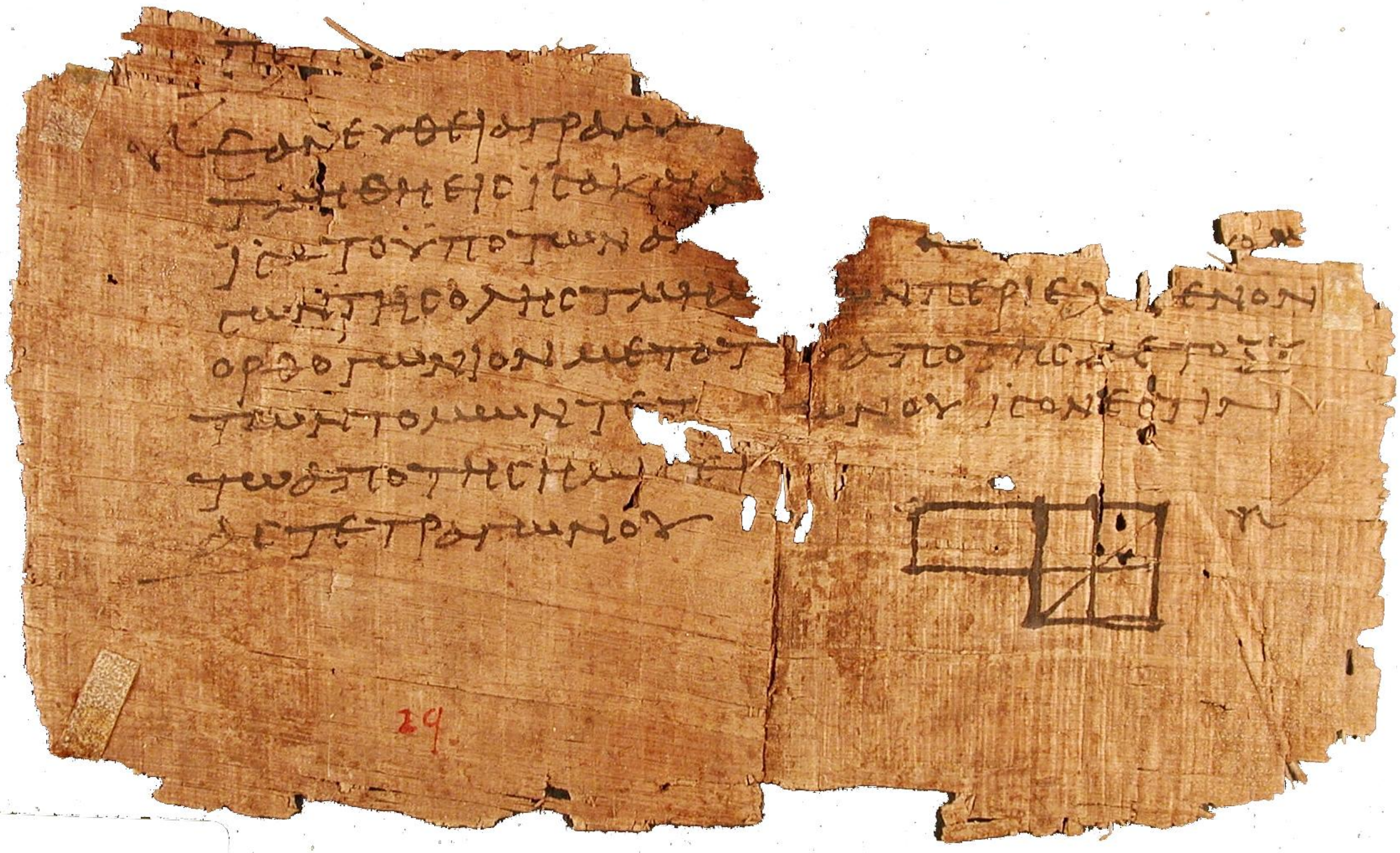
Other examples

- *S* and *P* are finite sets: Cardinality
- There exists an element in *S* that is not included in any pair in *P*: Existence of incomparable element in *S*.
- Two separate pairs in *P* share at least one element common in *S*.

Note: Less terms (undefined and/or defined) used in the axioms, more simplicity and clarity obtained in the system.

Euclid’s Elements

- The oldest geometry textbook (written in 300 BC)
- 13-volume work by Greek scholar Euclid (ca. 325-265 BC)



Euclid’s Elements (cont’d)

In the first volume of his book, Euclid starts with a few definitions and statements, and build up :

- Definitions

  e.g. “A point is that which has no part.”

  e.g., “A line is breadthless length.”

- Five geometric statements (called Postulates)

  e.g., “To draw a straight line from any point to any point.”

- Five common notions

  e.g., “Things that are equal to the same thing are also equal to one another.”

Euclid's Postulates

Euclid I To draw a straight line from any point to any point.

Euclid II To produce a finite straight line continuously in a straight line.

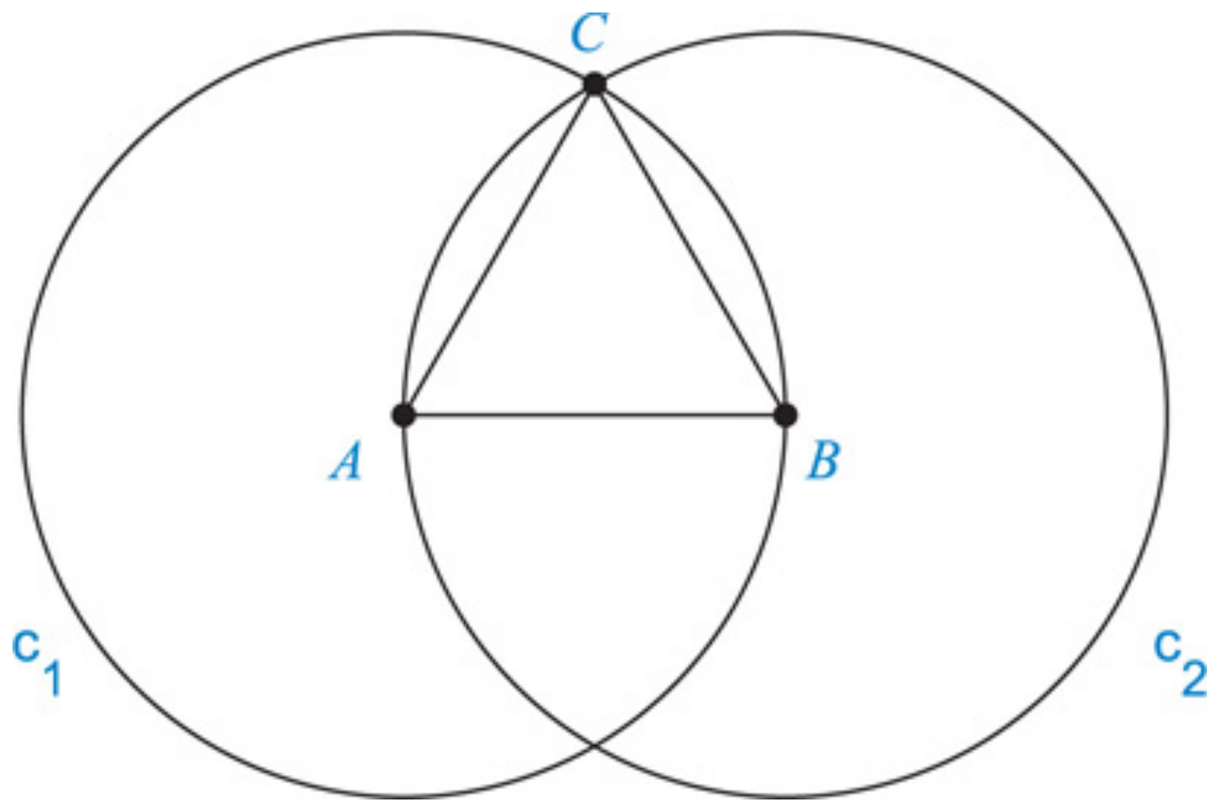
Euclid III To describe a circle with any center and distance (i.e., radius).

Euclid IV That all right angles are equal to each other.

Euclid V If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Propositions

E.g., Proposition 1: Construction of equilateral triangle



Propositions

Proposition 1: Construction of equilateral triangle

1. Given segment $\overline{AB}$
2. Given center A and distance equal to the length of $\overline{AB}$, construct circle c_1.
3. Given center B and distance equal to the length of $\overline{BA}$, construct circle c_2.
4. Let C be a point of intersection of circles c_1 and c_2.
5. Construct segments from A to C, from C to B, and from B to A.
6. Since $\overline{AC}$ and $\overline{AB}$ are radii of circle c_1, and $\overline{CB}$ and $\overline{AB}$ are radii of c_2, then $\overline{AC}$ and $\overline{CB}$ are both equal to $\overline{AB}$ and, thus, must be equal to one another.

Euclid’s Work Isn’t Perfect

Proposition 1. Construction of equilateral triangle

- All the steps in the construction are justified in terms of the initial set of five postulates and five common notions, except for the step 4, where the intersection point of the two circles is found.

Postulate 4. That all right angles are equal to each other.

- By introducing the postulate, Euclid wants to assume that segment lengths and angles remain unchanged when moving a geometric figure. (e.g. Side-Angle-Side (SAS) theorem)
- More solid logical foundation of this movement of geometric figures is introduced in the late 1800’s, which is called *transformational geometry.*

Euclid’s Fifth Postulate

“If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.”

- The fifth postulate has been questioned since Euclid introduced it. Even in his work, it is never used until Proposition 29.
- It seems to complicated comparing to other four postulates, and even considered unnecessary in the system.

<u>Q. Can we simplify the postulate?</u>

Statements Equivalent to the Fifth

- **Angle Sum of a Triangle**

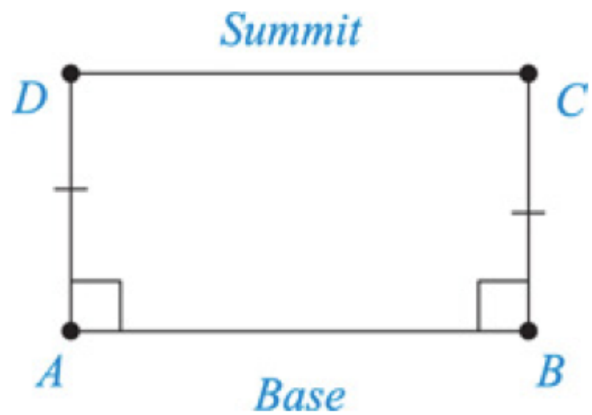
"For any triangle, the sum of three interior angles equals the straight angle."

- **Playfair's postulate**

"Given a line and a point not on the line, it is possible to construct one and only one line through the given point parallel to the line."

- **Saccheri Quadrilateral**

"The summit angles in a Saccheri Quadrilateral are right angles", where Saccheri quadrilaterals are defined quadrilaterals having two congruent sides perpendicular to a third side.



Euclid’s Fifth Postulate

Q. Is the fifth postulate necessary in the system? In other words, is it independent?

- A popular method of attack was to assume the logical opposite of Euclid’s fifth postulate and try to prove this new statement false, or find a contradiction to an already accepted result.
- Amazingly, no one could prove that the negation of the fifth postulate was false or produced a contradiction. This counters to the dominant belief that Euclidean geometry was the only consistent geometry possible.

Congruent Triangles

- **How did Euclid view the triangle congruence?**

Euclid *moves* points and segments so as to overlay one triangle on top of the other and thus prove the result. However, there is no axiomatic basis for such transformations in Euclid's original set of five postulates.

- **Triangle congruence as axiom**

Most modern treatments of Euclidean geometry assume SAS congruence as an axiom (e.g., Hilbert, Birkhoff). Then, they showed SAS derives other congruence conditions including ASA, AAS, SSS.

Measurements

Based on the concept of congruence, Euclid seemed to assume the concept of measurement. See the the following definitions.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight standing on the other is called a perpendicular to that on which it stands.

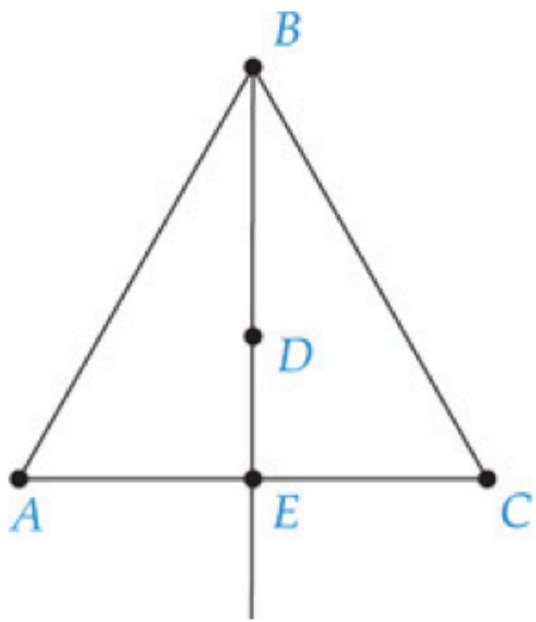
11. An obtuse angle is an angle greater than a right angle.

12. An acute angle is an angle less than a right angle.

Proposition 5 of Book I

Theorem 2.14. *(Prop. 5 of Book I) In an isosceles triangle, the two base angles are congruent.*

Proof: Let triangle ΔABC have sides $\overline{AB}$ and $\overline{BC}$ congruent, as shown in Figure 2.9. Let $\overrightarrow{BD}$ be the bisector of $\angle ABC$, with D inside the triangle. Let $\overrightarrow{BD}$ intersect $\overline{AC}$ at E. Then, by SAS we have triangles ΔABE and ΔCBE congruent and thus $\angle EAB \cong \angle ECB$ and we're done. □



Pasch's Axiom

"Let A, B, C be three non-collinear points and let l be a line that does not pass through A, B, or C. If l passes through side , it must pass through either a point on $\overline{AC}$ or a point on $\overline{BC}$, but not both."

- **Separation property**

A concept that a line separates a plane into two disjoint parts. Based on this concept, we see the triangle separating the plane into two regions: inside and outside.

In a Nutshell,

This brief critical review of Euclid's Elements leads us to identify some limitations. The following concepts will be discussed in our development of foundations for Euclidean geometry.

- Between-ness
- Congruence & Measurement
- Congruent triangles
- Separation