UT Rio Grande Valley — December 8, 2021

PHYS-3305 Classical Mechanics — Andreas Hanke

Final Exam – due today by 12:00 PM midnight

The exam has **4 problems** on **4 pages**. The maximum credit of the exam is **320 points**.
Please submit your exam as a single file (not multiple files) using blackboard,
Course Materials / Final Exam.

Problem 1) Sphere rolling on track [80 points]

A solid sphere of mass $M = 1\text{kg}$ with gravitational potential energy $U(y) = Mgy$ rolls without slipping on the track shown below, starting from rest at point A. Here y is the height of the bottom of the sphere over ground and $g = 9.8\,\frac{\text{m}}{\text{s}^2}$ is Earth's acceleration of gravity.

1. Assume that the sphere is rolling with linear speed v.

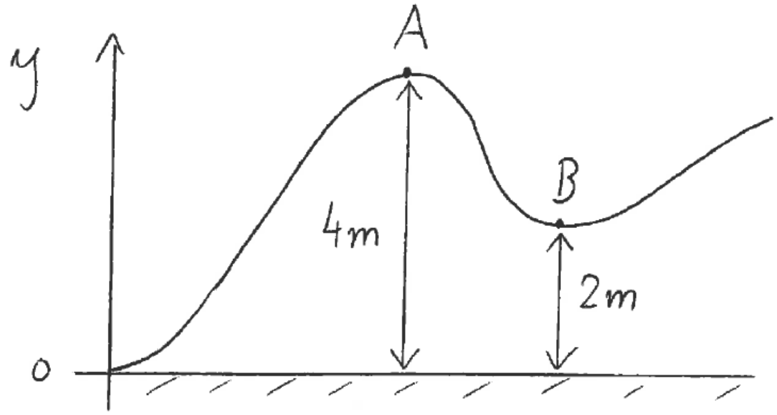
   Show that the total kinetic energy of the sphere is $T = \frac{7}{10} M v^2$.

   Hint: The moment of inertia of a solid sphere of mass M and radius R is $I = \frac{2}{5} M R^2$.

   Use the no-slip condition $\omega = \frac{v}{R}$ where ω is the angular velocity of the rolling sphere.

2. Find the total mechanical energy, E, in units of joules (J), of the sphere.

3. Find the linear speed, v, of the sphere at point B.



Problem 2) Line integral of a vector function [80 points]

Consider the vector function $\vec{\mathbf{F}}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$.

1. Calculate $\vec{\nabla}\times\mathbf{F}$. Is $\vec{\mathbf{F}}$ conservative?

2. Consider the scalar function $U(x,y,z) = -xyz$. Show: $\vec{\mathbf{F}} = -\vec{\nabla}U$.

3. Use the result from part 2 to find the line integral $\int_C d\vec{\mathbf{r}}\cdot\vec{\mathbf{F}}(\vec{\mathbf{r}})$ along an arbitrary curve

   $C: O \to \vec{P}$ from the origin $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to an arbitrary point $\vec{P} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ (see figure below).

   Hint: It is not required to calculate the line integral. You may directly express $\int_C d\vec{\mathbf{r}}\cdot\vec{\mathbf{F}}(\vec{\mathbf{r}})$

   in terms of U by interpreting $\vec{\mathbf{F}}$ as a force and U as the associated potential energy.

Problem 3) Kepler problem [80 points]

Consider a planet of mass m with position vector $\vec{\mathbf{r}}(t) = r(t)\,\hat{\mathbf{r}}(t)$ moving in the xy - plane. The planet is subject to the attractive gravitational force towards the star at the origin O of the xy - plane. The potential energy of the planet is $U(r) = -\frac{k}{r}$ with a constant $k > 0$.

1. Find the angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ of the planet using polar coordinates.

   Hint: Use $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$, $\vec{\mathbf{p}} = m\frac{d}{dt}\vec{\mathbf{r}}$, $\frac{d}{dt}\hat{\mathbf{r}} = \dot{\phi}\hat{\phi}$, and $\hat{\mathbf{r}} \times \hat{\phi} = \hat{\mathbf{z}}$. Result: $\vec{\mathbf{L}} = L_z \hat{\mathbf{z}}$ with $L_z = mr^2\dot{\phi}$.

2. Find the force $\vec{\mathbf{F}} = -\vec{\nabla}U(r)$ on the planet.

3. Find the Lagrange function $\mathcal{L}(r, \dot{r}, \dot{\phi}) = T - U$ for the two coordinates r, ϕ.

4. Find the Euler-Lagrange (EL) equation for ϕ (the ϕ - equation) for the Lagrange function found in part 3. Show that the ϕ - equation implies that L_z from part 1 is conserved.

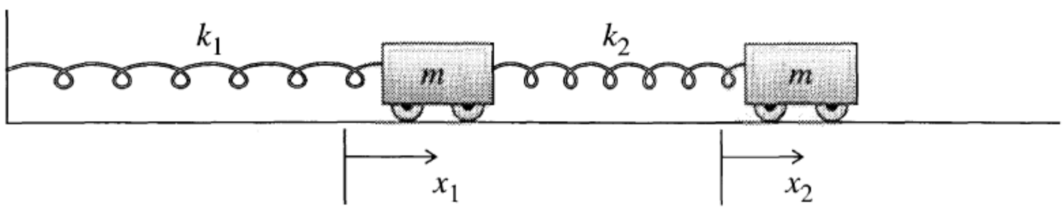
5. Find the EL equation for r (the r - equation) for the Lagrange function found in part 3. Show that the r - equation can be brought into the form $m\ddot{r} = -\frac{d}{dr}U_{eff}(r)$ with the effective potential $U_{eff}(r) = -\frac{k}{r} + \frac{L_z^2}{2m}\frac{1}{r^2}$ with L_z from part 1.

6. Qualitatively sketch the effective potential $U_{eff}(r)$ found in part 5 as a function of r.

7. Consider the special case of a <u>circular</u> orbit with constant radius R. The radius R corresponds to the minimum of $U_{eff}(r)$ found in part 5. Use the condition $\left.\frac{d}{dr}U_{eff}(r)\right|_{r=R} = 0$ to find the (constant) angular velocity $\omega = \dot{\phi}$ of the circular orbit.

Problem 4) Two carts connected by springs [80 points]

Two carts with equal mass m can move on a horizontal track. The left cart is attached to a fixed wall by a spring with force constant k_1 and the two carts are attached to each other by a spring with force constant k_2. Assume that $k_1 = 3k_2/2$ by writing $k_1 = 3k$ and $k_2 = 2k$ (with the same constant k for both carts). The displacements from the equilibrium positions of the two carts are x_1 and x_2, respectively (see figure below).



1. Find the Lagrange function $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ of the system.

2. Bring the Lagrange function to the form $\mathcal{L} = \frac{1}{2}\dot{\vec{x}} \cdot M \dot{\vec{x}} - \frac{1}{2}\vec{x} \cdot K \vec{x}$ with the vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and identify the mass matrix M and spring matrix K.

   Result: $M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $K = \begin{pmatrix} 5k & -2k \\ -2k & 2k \end{pmatrix}$.

3. Find the two eigenfrequencies ω_1, ω_2 from $\det(K - \omega^2 M) = 0$.

4. Find the associated eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ from $(K - \omega^2 M)\vec{\mathbf{v}} = 0$ with $\omega = \omega_1$ and $\omega = \omega_2$, respectively. Describe the normal modes of the system associated with these eigenvectors.