Problem Set 8 – Solutions 11/07/21

<u>Problem 1</u> Yoyo (1)

String is wrapped around yoyo with radius $R \;\Rightarrow\; v = \dot{x} = \omega R \;\Rightarrow\; \omega = \dfrac{\dot{x}}{R}$ (1)

(Note: (1) corresponds to no-slip condition of rolling disk with radius R)

Total kinetic energy of yoyo, using (1) and $I = \dfrac{1}{2} mR^2$:

$$\begin{aligned} T &= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2} mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2} m\dot{x}^2 + \frac{1}{4} m\dot{x}^2 = \frac{3}{4} m\dot{x}^2 \end{aligned} \tag{2}$$

Lagrange function for adapted coordinate x :

$$\mathcal{L}(x, \dot{x}) = T - U = \frac{3}{4} m\dot{x}^2 + mgx \tag{3}$$

Euler-Lagrange equation: $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} = \dfrac{\partial \mathcal{L}}{\partial x} \;\Rightarrow\; \dfrac{3}{2} m\ddot{x} = mg \;\Rightarrow\; \ddot{x} = \dfrac{2}{3} g$ (4)

Downward acceleration is more slowly than free-fall acceleration because some of the gravitational force is used to accelerate the rotation of the yoyo.

Problem 2 Yoyo (2)

a) As stated in the hint in the question, the Lagrange function $\mathcal{L}(z, \dot{z}, \dot{\phi})$ of the yoyo <u>without</u> the constraint is given by (keeping in mind that z is directed vertically downward)

$$\mathcal{L}(z, \dot{z}, \dot{\phi}) = T - U = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} I \dot{\phi}^2 + mgz \quad . \tag{1}$$

b) The constraint is given by the fact that the yoyo is connected to the string, which itself is suspended at time-dependent height $z_0(t)$. This constraint can be expressed in the following way. Assume that the string has constant total length L. One portion of the string is wrapped around the yoyo of radius R. The length of this portion is $(-\phi R)$ (see figure in question). The remaining portion of the string is equal to the vertical distance $z - z_0$ between the yoyo and the suspension point z_0. Thus

$$-\phi R + z - z_0(t) = L \quad . \tag{2}$$

Eq. (2) corresponds to a constraint that couples the variables z and ϕ. The constraint is time-dependent because of the time-dependence of $z_0(t)$. The corresponding constraint function is

$$c(z, \phi; t) = z - z_0(t) - \phi R - L = 0 \quad . \tag{3}$$

c) z - equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} - \frac{\partial \mathcal{L}}{\partial z} = \lambda \frac{\partial c}{\partial z}$ (4)

with $\mathcal{L}(z, \dot{z}, \dot{\phi})$ from (1) and $c(z, \phi; t)$ from (3). This gives: $m\ddot{z} - mg = \lambda$. (5)

ϕ - equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = \lambda \frac{\partial c}{\partial \phi}$. Using (1) and (3) this gives: $I\ddot{\phi} = -\lambda R$. (6)

$I\ddot{\phi} = \Gamma$ where Γ is the torque $\Rightarrow \lambda = -\frac{\Gamma}{R}$

d) Combining (5) and (6) to eliminate λ gives: $m\ddot{z} - mg = -\frac{I}{R}\ddot{\phi}$ (7)

Using the constraint in (2) to express ϕ in terms of z :

(2) $\Rightarrow \phi = \frac{1}{R}\left(z - z_0 - L\right) \Rightarrow \ddot{\phi} = \frac{1}{R}\left(\ddot{z} - \ddot{z}_0\right)$ (8)

Inserting (8) in (7) $\Rightarrow m\ddot{z} - mg = -\frac{I}{R^2}\left(\ddot{z} - \ddot{z}_0\right)$. (9)

Using $I = \frac{1}{2}mR^2 \Rightarrow m\ddot{z} - mg = -\frac{1}{2}m\left(\ddot{z} - \ddot{z}_0\right) = -\frac{1}{2}m\ddot{z} + \frac{1}{2}m\ddot{z}_0$

$\Rightarrow \ddot{z} = \frac{1}{3}\ddot{z}_0 + \frac{2}{3}g$ (10)

e) Without external vertical driving, i.e., $z_0 = 0$, (10) reproduces the solution (4) of Problem 1.

Remark (not required in homework):

General solution of (10):

$$z(t) = \frac{1}{3}z_0(t) + \frac{1}{3}g\,t^2 + a\,t + b \quad \text{with constants } a,\, b \;.$$

For example, $z_0(t) = A\sin\left(\omega_0 t\right)$ sinusoidal driving with frequency ω_0 and amplitude A,

and assuming $a = b = 0$:

$$z(t) = \frac{A}{3}\sin\left(\omega_0 t\right) + \frac{1}{3}g\,t^2$$

= overall downward acceleration $2g/3$ with sinusoidal modulation of amplitude $A/3$.

Problem 7.35 (page 287) "Figure 7.16 is a bird's-eye view ... "

As pointed out in the lecture notes, the kinetic energy of a bead is

$$T = \frac{1}{2} m \left(\frac{d\vec{\mathbf{r}}}{dt} \cdot \frac{d\vec{\mathbf{r}}}{dt} \right)$$

where $\vec{\mathbf{r}}(t)$ is the time-dependent position vector relative to a <u>stationary</u> origin O.

Choose the origin O to be stationary point A shown in Figure 7.16 in the question.

(The origin O cannot be chosen as the center C of the hoop because C is moving!)

Thus the position vector $\vec{\mathbf{r}}$ relative to $O = A$ can be decomposed as $\vec{\mathbf{r}} = \vec{\mathbf{c}} + \vec{\mathbf{r}}'$ where $\vec{\mathbf{c}}$ is the vector from A to C (the center of the hoop) and $\vec{\mathbf{r}}'$ is the vector from C to $\vec{\mathbf{r}}$.

Thus:

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d\vec{\mathbf{c}}}{dt} + \frac{d\vec{\mathbf{r}}'}{dt} \quad . \tag{1}$$

The center C of the hoop is turning about A with angular velocity ω. So, the velocity of C is $\vec{\mathbf{v}}_C = \frac{d\vec{\mathbf{c}}}{dt} = R\omega\hat{\mathbf{v}}_C$ where R is the radius of the hoop (= half the diameter AB).

For *constant* angle ϕ, the bead is rotating about C with angular velocity ω (following the rotation of the hoop). Thus, for non-constant angle ϕ the bead is rotating about C with total angular velocity $\omega + \dot{\phi}$. The velocity of the vector $\vec{\mathbf{r}}'$ from C to $\vec{\mathbf{r}}$ is thus

$$\vec{\mathbf{v}}' = \frac{d\vec{\mathbf{r}}'}{dt} = R\left(\omega + \dot{\phi}\right)\hat{\mathbf{v}}'.$$

The angle between $\vec{\mathbf{v}}_C$ and $\vec{\mathbf{v}}'$ is equal to ϕ (see figure below).

Thus: $\hat{\mathbf{v}}_C \cdot \hat{\mathbf{v}}' = \cos(\phi)$.

Using the above results for $\vec{\mathbf{v}}_C$ and $\vec{\mathbf{v}}\,'$ one finds for the kinetic energy of the bead:

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\frac{d\vec{\mathbf{r}}}{dt}\cdot\frac{d\vec{\mathbf{r}}}{dt}\right) = \frac{1}{2}m\left(\frac{d\vec{\mathbf{c}}}{dt}+\frac{d\vec{\mathbf{r}}\,'}{dt}\right)\cdot\left(\frac{d\vec{\mathbf{c}}}{dt}+\frac{d\vec{\mathbf{r}}\,'}{dt}\right) \\
&= \frac{1}{2}m\left(\vec{\mathbf{v}}_C+\vec{\mathbf{v}}\,'\right)\cdot\left(\vec{\mathbf{v}}_C+\vec{\mathbf{v}}\,'\right) = \frac{1}{2}m\left(v_C^2+v'^2+2\vec{\mathbf{v}}_C\cdot\vec{\mathbf{v}}\,'\right) \\
&= \frac{1}{2}m\left[R^2\omega^2+R^2\left(\omega+\dot{\phi}\right)^2+2R^2\omega\left(\omega+\dot{\phi}\right)\cos(\phi)\right] \\
&= \frac{1}{2}mR^2\left[\omega^2+\left(\omega+\dot{\phi}\right)^2+2\omega\left(\omega+\dot{\phi}\right)\cos(\phi)\right]
\end{aligned}
\qquad (2)
$$

Since there is no potential energy, the Lagrange function for the adapted coordinate ϕ is given by the last line in (2):

$$
\mathcal{L}\left(\phi,\dot{\phi}\right)=T=\frac{1}{2}mR^2\left[\omega^2+\left(\omega+\dot{\phi}\right)^2+2\omega\left(\omega+\dot{\phi}\right)\cos(\phi)\right] \qquad (3)
$$

Euler-Lagrange equation: $\dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{\phi}}=\dfrac{\partial\mathcal{L}}{\partial\phi}$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\frac{1}{2}mR^2\left[2\left(\omega+\dot{\phi}\right)+2\omega\cos(\phi)\right]$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\frac{1}{2}mR^2\left[2\ddot{\phi}-2\omega\sin(\phi)\dot{\phi}\right]$$

$$\frac{\partial\mathcal{L}}{\partial\phi}=-mR^2\omega\left(\omega+\dot{\phi}\right)\sin(\phi)$$

$$\Rightarrow\ \frac{1}{2}mR^2\left[2\ddot{\phi}-2\omega\sin(\phi)\dot{\phi}\right]=-mR^2\omega\left(\omega+\dot{\phi}\right)\sin(\phi)$$

$$2\ddot{\phi}-2\omega\sin(\phi)\dot{\phi}=-2\omega\left(\omega+\dot{\phi}\right)\sin(\phi)$$

$$\ddot{\phi}-\omega\sin(\phi)\dot{\phi}=-\omega^2\sin(\phi)-\omega\dot{\phi}\sin(\phi)$$ terms shown in blue cancel

$$\ddot{\phi}=-\omega^2\sin(\phi)$$

The last equation is exactly the equation of a simple pendulum with g/l replaced by ω^2 oscillating about the point $\phi=0$ (that is, B); compare Taylor, Section 7.2, Eq. (7.33).

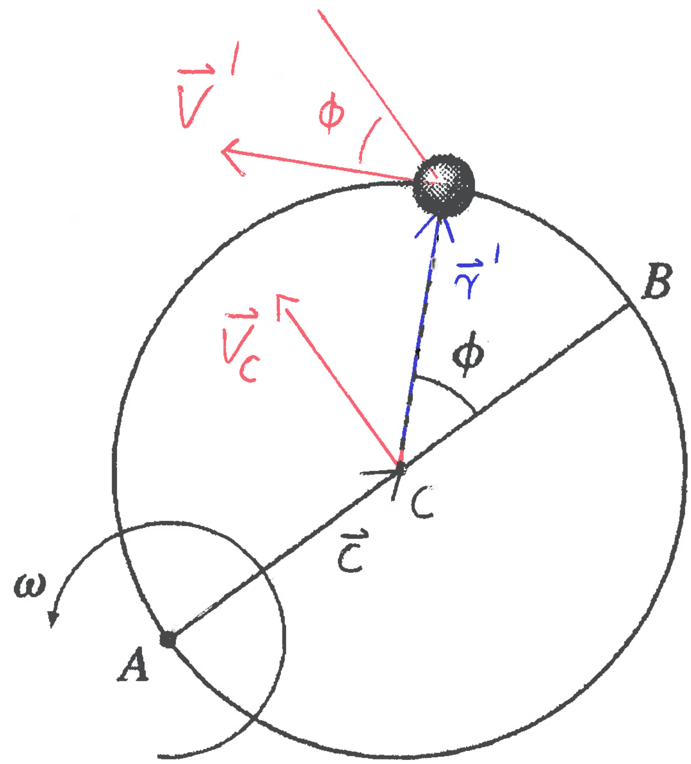

Figure 7.16   Problem 7.35

7.21 $\star$ If we use two-dimensional polar coordinates, the bead's velocity is $\mathbf{v} = (\dot{r}, r\dot{\phi}) = (\dot{r}, r\omega)$, where ω is the fixed angular velocity with which the rod is forced to rotate. Thus $\mathcal{L} = T - U = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$. ($U$ is a constant, which we may as well take to be zero.) The Lagrange equation is $\ddot{r} = \omega^2 r$, the general solution of which is $r(t) = Ae^{\omega t} + Be^{-\omega t}$. If $r(0) = \dot{r}(0) = 0$, then $A = B = 0$ and the bead stays put; that is, $r = 0$ is an equilibrium point (though unstable, as we'll see). If $r(0) = r_\text{o} \neq 0$, but $\dot{r}(0) = 0$, then $A = B = r_\text{o}/2$ and

$$r(t) = \tfrac{1}{2}r_\text{o}(e^{\omega t} + e^{-\omega t}) \to \tfrac{1}{2}r_\text{o}e^{\omega t}$$

as $t \to \infty$. As seen in the rotating frame of the rod, there is an ouward "centrifugal force" $m\omega^2 r$. This causes the bead to accelerate outward, and as r increases, the acceleration increases in proportion — hence the exponential growth of r.

7.23 $\star$ The small cart's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x} + \dot{X})^2 = \frac{1}{2}m(\dot{x} - A\omega\sin\omega t)^2$, and $U = \frac{1}{2}kx^2$. Thus $\partial\mathcal{L}/\partial\dot{x} = m(\dot{x} - A\omega\sin\omega t)$ and Lagrange's equation reads

$$-kx = m\ddot{x} - mA\omega^2\cos\omega t \qquad \text{or} \qquad \ddot{x} + \omega_\text{o}^2 x = B\cos\omega t$$

where I have replaced k/m by ω_o^2 and renamed $A\omega^2$ as B.