UT Rio Grande Valley | October 17, 2021
PHYS-3305 Classical Mechanics | Andreas Hanke

Problem Set 6 – due Friday, October 22 by 12:00 PM midnight

The Problem Set has **5 questions** on **3 pages**, with a total maximum credit of **30 points**.

Please turn in well-organized, clearly written solutions (no scrap work).

Problem 1) Line integral of a vector function [6 points]

Consider the vector function $\vec{\mathbf{F}}(x,y,z) = \begin{pmatrix} 2y \\ 3yz \\ xz^2 \end{pmatrix}$.

a) Calculate $\vec{\nabla} \times \mathbf{F}$. Is $\vec{\mathbf{F}}$ conservative?

b) Calculate the line integrals $\int_{C_{ab}} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}})$ from $\vec{\mathbf{r}}_a = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to $\vec{\mathbf{r}}_b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ along the curves

1. C_{ab} = straight line from $\vec{\mathbf{r}}_a$ to $\vec{\mathbf{r}}_b$.

   Hint: Use the parameterization $\vec{\mathbf{r}}(t) = t\,\vec{\mathbf{r}}_b$ with curve parameter $t: 0 \to 1$.

2. C_{ab} = sequence of straight lines $\vec{\mathbf{r}}_a \xrightarrow{A} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{C} \vec{\mathbf{r}}_b$.

   Hint: Use $\int_{C_{ab}} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \int_A d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}) + \int_B d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}) + \int_C d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}})$

   and use the following parameterizations for the straight lines A, B, C:

   $$\vec{\mathbf{r}}_A(t) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \text{ with } t: 0 \to 1; \quad \vec{\mathbf{r}}_B(t) = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix} \text{ with } t: 0 \to 1; \quad \vec{\mathbf{r}}_C(t) = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \text{ with } t: 0 \to 1.$$

3. C_{ab} : $\vec{\mathbf{r}}(t) = \begin{pmatrix} t \\ t^2 \\ t^4 \end{pmatrix}$ with $t: 0 \to 1$.

Problem 2) Curl of a central force field [6 points]

A central force field for a single particle has the general form $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = F(r)\,\hat{\mathbf{r}}$ where $F(r)$ is a scalar function of the magnitude $r = |\vec{\mathbf{r}}|$ and $\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r}$ is the unit vector of $\vec{\mathbf{r}}$.

Question: Show that for a central force field: $\vec{\nabla} \times \vec{\mathbf{F}} = 0$.

Hint: Write $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = g(r)\,\vec{\mathbf{r}}$ with $g(r) := \frac{F(r)}{r}$. Use Problem 5c of the previous Problem Set 5 to show $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\nabla} \times (g\vec{\mathbf{r}}) = (\vec{\nabla} g) \times \vec{\mathbf{r}} + g\vec{\nabla} \times \vec{\mathbf{r}}$ (where $\vec{\mathbf{F}}$ in Problem 5c of PS 5 is equal to $\vec{\mathbf{r}}$ here). Now use Problems 4c and 5b of the previous Problem Set 5.

Problem 3) Line integral of a central force field [6 points]

a) Consider a time-dependent position vector $\vec{\mathbf{r}}(t)$ with magnitude $r(t) = |\vec{\mathbf{r}}(t)|$ and unit vector $\hat{\mathbf{r}}(t)$.

Show: $\frac{d}{dt} r = \frac{d\vec{\mathbf{r}}}{dt} \cdot \hat{\mathbf{r}}$

Hint: Use Problem 4e of the previous Problem Set 5 with the choice $f(\vec{\mathbf{r}}) = r$.

b) Consider a central force field $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = F(r)\,\hat{\mathbf{r}}$ as in Problem 2 and the line integral

$\int_{C_{ab}} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \int_{C_{ab}} d\vec{\mathbf{r}} \cdot \hat{\mathbf{r}}\, F(r)$ for an arbitrary curve C_{ab} between given points $\vec{\mathbf{r}}_a$ and $\vec{\mathbf{r}}_b$.

Show: $\int_{C_{ab}} d\vec{\mathbf{r}} \cdot \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \int_{r_a}^{r_b} dr\, F(r)$ where $r_a = |\vec{\mathbf{r}}_a|$ and $r_b = |\vec{\mathbf{r}}_b|$. Hint: Use the result of part a)

Is $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ conservative? Is your conclusion consistent with the result of Problem 2?

Problem 4) Kinetic energy of a rolling hollow spherical shell [6 points]

Consider a hollow spherical shell (e.g. a basketball or a soccer ball) of radius R and mass M rolling on a surface without slipping with linear speed $v = \left| \frac{d}{dt} \vec{\mathbf{R}}_{CM}(t) \right|$, where $\vec{\mathbf{R}}_{CM}(t)$ is the position of the mass center (CM) = sphere center. As shown in class the total kinetic energy of the rolling spherical shell is

$$T = T_{CM} + T_{rot} = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

where I is the moment of inertia and ω is the angular speed of the rotating spherical shell.

Question: Find T in terms of M and v.

Hint: The moment of inertia of a hollow spherical shell of radius R and mass M about the diameter is given by $I = \frac{2}{3} M R^2$. Use the no-slip condition to relate ω to v.

Problem 5) Energy of a two-dimensional harmonic oscillator [6 points]

Reconsider Problem 1 of Problem Set 4 "Two-dimensional harmonic oscillator: elliptical orbits". Consider a general elliptical orbit, as given in Eq. (3) of this problem.

a) Find the kinetic energy $T(t) = \frac{1}{2} m \left[v(t) \right]^2$.

b) Find the potential energy $U(t) = \frac{1}{2} k \left[r(t) \right]^2$ (where $r(t) = \left| \vec{\mathbf{r}}(t) \right|$).

c) Find the total energy $E = T(t) + U(t)$, and show that E is constant.

d) Now consider the special case where $b = a$, i.e., a circular orbit of radius a.
Show that in this case both T and U are constant, and $T = U$.