Problem Set 6 - Solutions

Problem 1

$$\vec{F}(x,y,z) = 2y\,\hat{x} + 3yz\,\hat{y} + xz^2\,\hat{z}$$

$$\Rightarrow \quad F_x = 2y\,, \quad F_y = 3yz\,, \quad F_z = xz^2$$

a) $$\vec{\nabla}\times\vec{F}\big|_1 = \frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y = 0 - 3y = -3y$$

$$\vec{\nabla}\times\vec{F}\big|_2 = \frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z = 0 - z^2 = -z^2$$

$$\vec{\nabla}\times\vec{F}\big|_3 = \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x = 0 - 2 = -2$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{F} = \underline{-3y\,\hat{x} - z^2\,\hat{y} - 2\,\hat{z}}$$

$\vec{\nabla}\times\vec{F} \neq 0 \Rightarrow$ $\vec{F}$ is <u>not</u> conservative

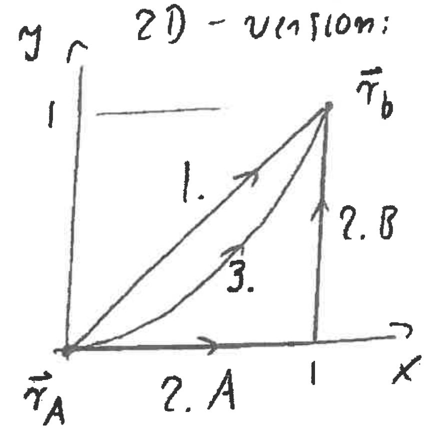
$\Rightarrow \int_{\mathcal{C}_{ab}} d\vec{r}\cdot\vec{F}(\vec{r})$ is path-dependent

as shown in b)

b) $I = \int_{\mathcal{C}_{ab}} d\vec{r} \cdot \vec{F}(\vec{r})$

for curves $\mathcal{C}_{ab}$: $\vec{r}_a \to \vec{r}_b$

$\vec{r}_a = 0$, $\vec{r}_b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



1. $\vec{r}(t) = t\,\vec{r}_b$, $t : 0 \to 1$

Following steps in hint:

$$\frac{d\vec{r}}{dt} = \vec{r}_b$$

$$\vec{F}[\vec{r}(t)] = \begin{pmatrix} 2t \\ 3t^2 \\ t^3 \end{pmatrix}$$

$$\frac{d\vec{r}}{dt} \cdot \vec{F}[\vec{r}(t)] = 2t + 3t^2 + t^3$$

$$\Rightarrow I_1 = \int_0^1 dt\,(2t + 3t^2 + t^3) = \underline{\frac{9}{4}}$$

2. Line A

$$\vec{r}_A(t) = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \quad t : 0 \to 1$$

$$\frac{d\vec{r}_A}{dt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{F}[\vec{r}_A(t)] = 0$$

$$\Rightarrow I_A = 0$$

Line B

$$\vec{r}_B(t) = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix}, \quad t: 0 \to 1$$

$$\frac{d\vec{r}_B}{dt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{F}[\vec{r}_B(t)] = \begin{pmatrix} 2t \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{d\vec{r}_B}{dt} \cdot \vec{F}[\vec{r}_B(t)] = 0$$

$$\Rightarrow \quad I_B = 0$$

Line C

$$\vec{r}_c(t) = \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}, \quad t: 0 \to 1$$

$$\frac{d\vec{r}_c}{dt} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{F}[\vec{r}_c(t)] = \begin{pmatrix} 2 \\ 3t \\ t^2 \end{pmatrix}$$

$$\frac{d\vec{r}_c}{dt} \cdot \vec{F}[\vec{r}_c(t)] = t^2$$

$$\Rightarrow \quad I_C = \int_0^1 dt\, t^2 = \frac{1}{3}$$

$$\Rightarrow \quad I_2 = I_A + I_B + I_C = \underline{\frac{1}{3}}$$

3. $$\bar{r}(t) = \begin{pmatrix} t \\ t^2 \\ t^4 \end{pmatrix}, \quad t : 0 \to 1$$

$$\frac{d\bar{r}}{dt} = \begin{pmatrix} 1 \\ 2t \\ 4t^3 \end{pmatrix}$$

$$\bar{\bar{F}}[\bar{r}(t)] = \begin{pmatrix} 2t^2 \\ 3t^6 \\ t^9 \end{pmatrix}$$

$$\frac{d\bar{r}}{dt} \cdot \bar{\bar{F}}[\bar{r}(t)] = 2t^2 + 6t^7 + 4t^{12}$$

$$I_3 = \int_0^1 dt \, (2t^2 + 6t^7 + 4t^{12}) = \underline{\frac{269}{156}}$$

$$I_1 \neq I_2 \neq I_3$$

$$\Rightarrow I = \int_{\mathcal{C}_{ch}} d\bar{r} \cdot \bar{\bar{F}} \text{ is path-dependent}$$

Problem 2

Central force field:

$$\vec{F}(\vec{r}) = F(r)\,\hat{r} = \underbrace{\frac{F(r)}{r}}_{g(r)}\,\vec{r} = g(r)\,\vec{r}$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (g\,\vec{r})$$

$$= (\vec{\nabla} g) \times \vec{r} + g\,\vec{\nabla} \times \vec{r}, \quad \text{PS5, P5c}$$

$$\vec{\nabla} g(r) = g'(r)\,\hat{r}, \quad \text{PS5, P4c}$$

$$\vec{\nabla} \times \vec{r} = 0, \quad \text{PS5, P5b}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{F} = g'(r)\,\underbrace{\hat{r} \times \vec{r}}_{=0 \text{ b/c } \hat{r} \parallel \vec{r}} = 0$$

$$\vec{\nabla} \times \vec{F} = 0$$

$\Rightarrow$ $\vec{F}$ is conservative

$\Rightarrow$ $\int_{\mathcal{C}_{ab}} d\vec{r} \cdot \vec{F}$ is path-independent

$$\left(= \int_{r_a}^{r_b} dr\, F(r), \text{ shown in P3 below} \right)$$

Problem 3

a) PS5, Problem 4e: $\frac{d}{dt} f[\vec{r}(t)] = \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}$

with $f[\vec{r}(t)] = r(t)$:

$$\Rightarrow \frac{d}{dt} r = \underbrace{\vec{\nabla} r}_{= \hat{r}} \cdot \frac{d\vec{r}}{dt} = \hat{r} \cdot \frac{d\vec{r}}{dt} = \underline{\frac{d\vec{r}}{dt} \cdot \hat{r}}$$

b) $$\int_{\ell_{ab}} d\vec{r} \cdot \vec{F}(\vec{r}) = \int_{\ell_{ab}} d\vec{r} \cdot \hat{r}\, F(r)$$

$$= \int_{t_a}^{t_b} dt \underbrace{\frac{d\vec{r}}{dt} \cdot \hat{r}}_{= \frac{dr}{dt},\ \text{part a)}} F(r) \quad \text{def. of line integral}$$

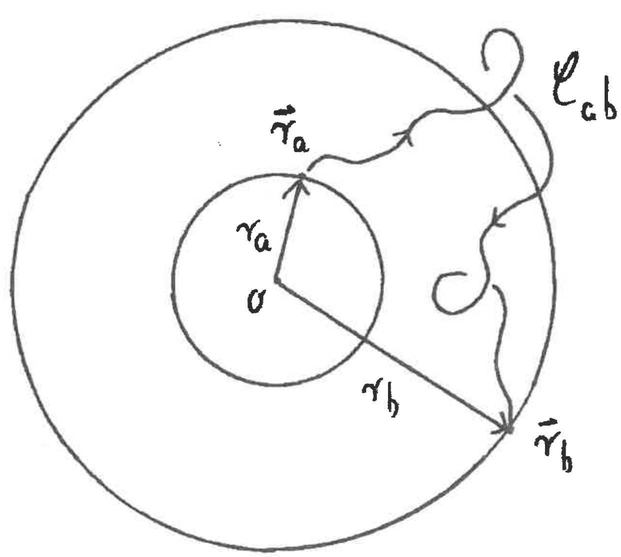
$$= \int_{t_a}^{t_b} dt \frac{dr}{dt} F(r) = \underline{\int_{r_a}^{r_b} dr\, F(r)}$$

$$\Rightarrow \int_{\mathcal{C}_{ab}} d\vec{r} \cdot \vec{F}(\vec{r}) = \int_{r_a}^{r_b} dr\, F(r)$$

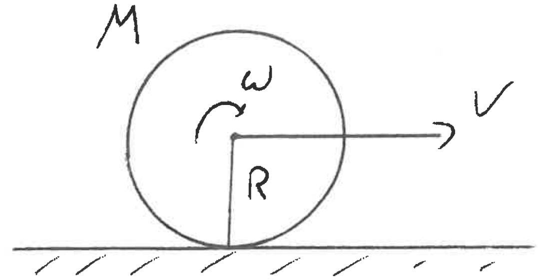
is path-independent b/c depends only on end points $\vec{r}_a, \vec{r}_b$ (by the magnitudes r_a, r_b), but not on shape of $\mathcal{C}_{ab}$

$\Rightarrow \vec{F}(\vec{r}) = F(r)\,\hat{r}$ is conservative

(consistent w/ $\vec{\nabla} \times \vec{F} = 0$ found in P2)



Problem 4



No-slip condition: $v = R\omega \Rightarrow \omega = \frac{v}{R}$

$I = \frac{2}{3} M R^2$ hollow spherical shell about diameter

$$\Rightarrow T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} M v^2 + \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} M \cancel{R^2} \frac{v^2}{\cancel{R^2}}$$

$$= \underbrace{\left(\frac{1}{2} + \frac{1}{3}\right)}_{\frac{5}{6}} M v^2 = \underline{\frac{5}{6} M v^2}$$

Problem 5

Cp. solution of PS4, P1

a) $v(t) = \omega \left(a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t) \right)^{1/2}$

$\Rightarrow T(t) = \frac{1}{2} m\, v(t)^2 = \frac{1}{2} \underbrace{m \omega^2}_{=k} \left(a^2 \sin^2(\omega t) + b^2 \cos^2(\omega t) \right)$

b) $r(t) = \left(a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t) \right)^{1/2}$

$\Rightarrow U(t) = \frac{1}{2} k\, r(t)^2 = \frac{1}{2} k \left(a^2 \cos^2(\omega t) + b^2 \sin^2(\omega t) \right)$

Both $T(t)$, $U(t)$ are <u>not</u> constant, i.e., depend on t; but E is constant:

c) $E = T(t) + U(t)$

$= \frac{1}{2} k \left\{ a^2 \underbrace{(\sin^2 \omega t + \cos^2 \omega t)}_{=1} + b^2 \underbrace{(\cos^2 \omega t + \sin^2 \omega t)}_{=1} \right\}$

$= \frac{1}{2} k \left(a^2 + b^2 \right) = \text{const.}$

d) $b = a$: $\Rightarrow v = \omega a$, $T = \frac{1}{2} \underbrace{m \omega^2}_{=k} a^2 = \frac{1}{2} k a^2$,

$U = \frac{1}{2} k a^2$

$\Rightarrow T = U = \frac{1}{2} E$