Problem Set 12 - Solutions

Problem 1

a) $\vec{L} = I\vec{\omega} \Rightarrow L_i = \sum_{j=1}^{3} I_{ij} \underbrace{\omega_j}_{\text{const}} = \text{const}$

in frame $\{\hat{e}_i(t)\}$

b) $\vec{L}(t) = \sum_{i=1}^{3} \underbrace{L_i}_{\text{const, due to a)}} \hat{e}_i(t)$

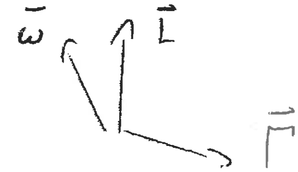
$$\Rightarrow \frac{d}{dt}\vec{L} = \sum_{i=1}^{3} L_i \underbrace{\frac{d}{dt}\hat{e}_i}_{\vec{\omega}\times\hat{e}_i} = \vec{\omega} \times \underbrace{\sum_{i=1}^{3} L_i \hat{e}_i}_{\vec{L}} = \vec{\omega}\times\vec{L}$$

$$\Rightarrow \frac{d}{dt}\vec{L} = 0 \text{ if } \vec{\omega} \parallel \vec{L}$$

note: if $\vec{\omega}$ not $\parallel \vec{L} \Rightarrow \frac{d}{dt}\vec{L} = \vec{\omega}\times\vec{L} = \vec{\Gamma}$

where external torque $\vec{\Gamma} \perp$ to $\vec{\omega}, \vec{L}$

= special type of precession where ω_i, L_i fixed in body frame $\{\hat{e}_i\}$



Problem 2) Two carts connected by springs

11.5 ⋆⋆ **(a)** The quickest way to find the equation of motion for the system of Fig.11.15 is to set $k_3 = 0$ in Fig.11.1. With $m_1 = m_2$ and $k_1 = k_2$ as well, the mass and spring-constant matrices are given by Eq.(11.5) as

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}.$$

If we define $\omega_o = \sqrt{k/m}$ and hence $k = m\omega_o^2$, the characteristic equation becomes

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2(\omega^4 - 3\omega_o^2\omega^2 + \omega_o^4) = 0,$$

so the normal frequencies are given by $\omega^2 = \omega_o^2(3 \pm \sqrt{5})/2$.

(b) If we substitute $\omega = \omega_1 = \omega_o\sqrt{(3-\sqrt{5})/2}$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ yields $a_2 = a_1(1+\sqrt{5})/2 = 1.62a_1$. Thus the first normal mode has the form

$$x_1 = A\cos(\omega_1 t - \delta) \qquad \text{and} \qquad x_2 = 1.62A\cos(\omega_1 t - \delta).$$

where A and δ are arbitrary constants. In the first mode the two carts oscillate in phase, the second one with the larger amplitude. Similarly, in the second mode, we find $a_2 = a_1(1-\sqrt{5})/2 = -0.62a_1$ and so

$$x_1 = A\cos(\omega_2 t - \delta) \qquad \text{and} \qquad x_2 = -0.62A\cos(\omega_2 t - \delta).$$

where (in general) A and δ are different constants. In this mode the two carts move 180° out of phase, and cart 2 has the smaller amplitude.

Problem 3) Double pendulum

11.14 ⋆⋆ **(a)** The kinetic energy is $T = \frac{1}{2}mL^2(\dot{\phi}_1^2 + \dot{\phi}_2^2)$. The gravitational potential energy of either pendulum has the form $mgL(1 - \cos\phi) \approx \frac{1}{2}mgL\phi^2$, and the spring's PE is $\frac{1}{2}kx^2 \approx \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$. Putting these together,

$$\mathcal{L} = \tfrac{1}{2}mL^2(\dot{\phi}_1^2 + \dot{\phi}_2^2) - \tfrac{1}{2}mgL(\phi_1^2 + \phi_2^2) - \tfrac{1}{2}kL^2(\phi_2 - \phi_1)^2$$

from which we get the Lagrange equations:

$$\begin{aligned} \ddot{\phi}_1 &= -\omega_o{}^2\phi_1 + (k/m)(\phi_2 - \phi_1) \\ \ddot{\phi}_2 &= -\omega_o{}^2\phi_2 - (k/m)(\phi_2 - \phi_1) \end{aligned}$$

where I have divided through by mL^2 and introduced the natural frequency for either pendulum (without the spring) given by $\omega_o{}^2 = g/L$.

(b) From the equations of motion, you can write down the "mass matrix" $\mathbf{M}$ and "spring matrix" $\mathbf{K}$, and thence the matrix

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} \omega_o{}^2 + k/m - \omega^2 & -k/m \\ -k/m & \omega_o{}^2 + k/m - \omega^2 \end{bmatrix}.$$

The determinant of this matrix is $(\omega_o{}^2 - \omega^2)(\omega_o{}^2 + 2k/m - \omega^2)$, and the two normal frequencies are

$$\omega_1 = \omega_o \qquad \text{and} \qquad \omega_2 = \sqrt{\omega_o{}^2 + 2k/m}.$$

The corresponding motions are found by solving the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ with ω set equal to ω_1 and ω_2 in turn. For the first mode, this gives the eigenvector $\mathbf{a} = A(1, 1)$ (actually a 2×1 column, of course). This means that in the first mode the two pendulums oscillate in unison (in phase with equal amplitudes). In this mode the spring is unstretched, its presence is irrelevant, and the frequency is just the natural frequency for a single pendulum.

For the second mode, $\mathbf{a} = A(1, -1)$, and the two pendulums oscillate with equal amplitudes but exactly out of phase. Notice that, in either mode, the two pendulums behave just like the two carts of Section 11.2.