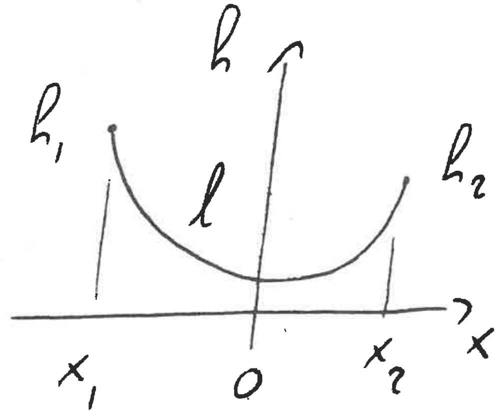
Problem Set 7 - Solutions

Problem 1

a) $U = g\rho \int_0^l ds \, \{h(s) - c\}$



Curve $[x, h(x)]$

with curve parameter x

$$\rightarrow \quad ds = \sqrt{1 + \left(\frac{dh}{dx}\right)^2} \, dx$$

or, using $y(x) = h(x) - c$: $\dot{h} = \dot{y}$, $\cdot = \frac{d}{dx}$

$$\Rightarrow \quad ds = \sqrt{1 + \dot{y}^2} \, dx$$

$$U[y] = g\rho \int_{x_1}^{x_2} dx \sqrt{1+\dot{y}^2} \, y = \int_{x_1}^{x_2} dx \, \mathcal{L}(y, \dot{y})$$

with $\mathcal{L}(y, \dot{y}) = g\rho \sqrt{1+\dot{y}^2} \, y$

b) $\frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = 0$

$\frac{\partial \mathcal{L}}{\partial \dot{y}} = g\rho \frac{\dot{y} y}{\sqrt{}}$, $\frac{\partial \mathcal{L}}{\partial y} = g\rho \sqrt{}$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\dot{y} y}{\sqrt{1+\dot{y}^2}}\right) = \sqrt{1+\dot{y}^2} \qquad \text{(EL)}$$

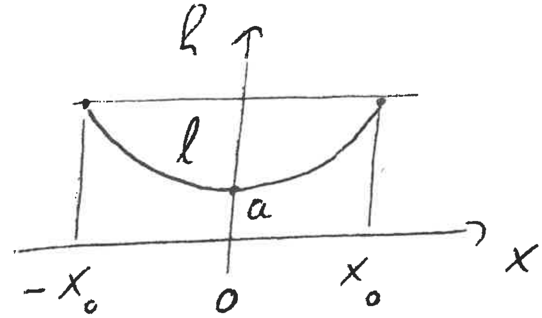
c) ansatz: $y(x) = a \cosh\left(\frac{x-b}{a}\right)$

$\Rightarrow \dot{y} = \sinh\left(\frac{x-b}{a}\right)$

$1+\dot{y}^2 = 1+\sinh^2\left(\frac{x-b}{a}\right) = \cosh^2\left(\frac{x-b}{a}\right)$

EL $\rightarrow$ $\underbrace{\frac{d}{dx}\, a \sinh\left(\frac{x-b}{a}\right)}_{\cosh\left(\frac{x-b}{a}\right)} \overset{!}{=} \cosh\left(\frac{x-b}{a}\right)$ o.K.

d)



$h(x) = y(x) + c = a \cosh\left(\frac{x-b}{a}\right) + c$

$h_1 = h_2$, $h(0) = a$ $\Rightarrow$ $b = c = 0$

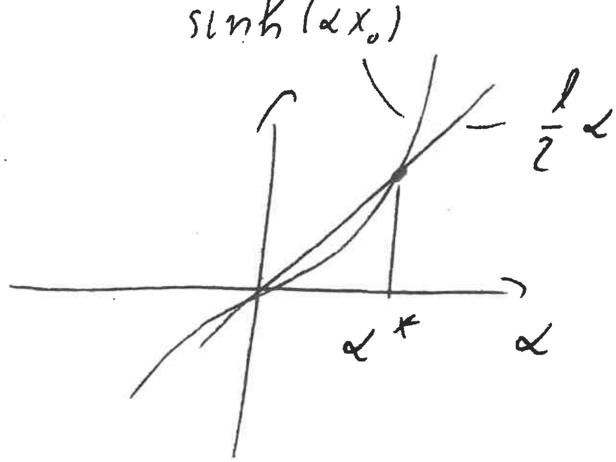
$\Rightarrow h(x) = a \cosh\left(\frac{x}{a}\right)$

$l \overset{!}{=} \int_{-x_0}^{x_0} dx \sqrt{1+\dot{h}^2} = \int_{-x_0}^{x_0} dx \cosh\left(\frac{x}{a}\right) = 2a \sinh\left(\frac{x_0}{a}\right)$

Determine $\alpha := \frac{1}{a}$:

$\frac{l}{2}\alpha \overset{!}{=} \sinh(\alpha x_0)$

$\Rightarrow \alpha$ from intersection of curves $\frac{l}{2}\alpha$, $\sinh(\alpha x_0)$:



Consistency: $\alpha^* > 0$ only exists if $\frac{l}{2} > x_0$

i.e., $l > 2x_0$, o.K. (slope at $\alpha=0$)

For $l = 4x_0$:

$2x_0\alpha \stackrel{!}{=} \sinh(\alpha x_0)$, solve $2y - \sinh(y) = 0$ for $y := \alpha x_0 = \frac{x_0}{a}$

$\Rightarrow$ $\underline{y^* = 2.1773}$ (see Mathematica notebook)

$$\Rightarrow \quad h(x) = \frac{x_0}{y^*}\cosh\left(y^* \frac{x}{x_0}\right)$$

measure h, x in units of x_0:

$$\Rightarrow \quad \hat{h}(\hat{x}) = \frac{1}{y^*}\cosh(y^* \hat{x}) \quad \text{universal curve}$$

$$\hat{h} = \frac{h}{x_0}, \quad \hat{x} := \frac{x}{x_0}, \quad \hat{l} = \int_{-1}^{1} d\hat{x}\sqrt{1+\dot{\hat{h}}^2} = 4$$

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In[1]:= f1[y_] := 2 y

In[2]:= f2[y_] := Sinh[y]

In[3]:= p1 := Plot[f1[y], {y, -1, 3}]

In[4]:= p2 := Plot[f2[y], {y, -1, 3}]

In[5]:= Show[p1, p2]
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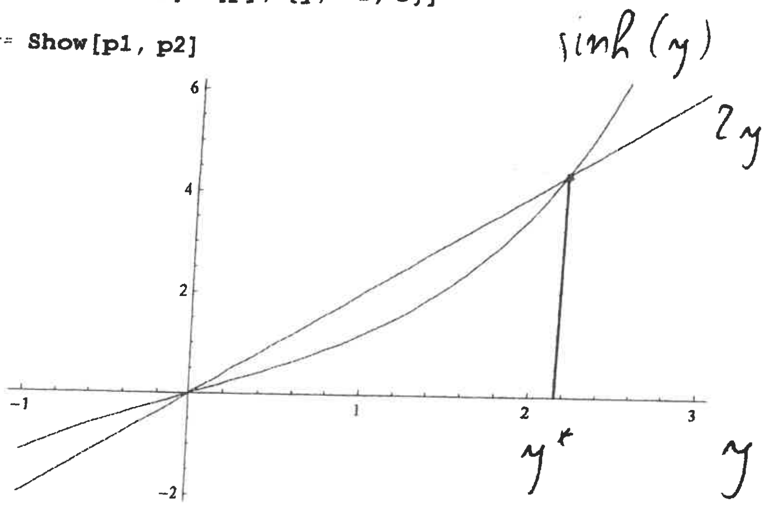
Out[5]=

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In[6]:= ystar := 2.1773

In[7]:= f1[ystar]

Out[7]= 4.3546

In[8]:= f2[ystar]

Out[8]= 4.35455

In[9]:= h[x_] := Cosh[ystar * x] / ystar

In[10]:= Integrate[Sqrt[1 + (h'[x])^2], {x, -1, 1}]
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$= \frac{\ell}{x_0}$

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Out[10]= 3.99996

In[11]:= Plot[h[x], {x, -1, 1}]
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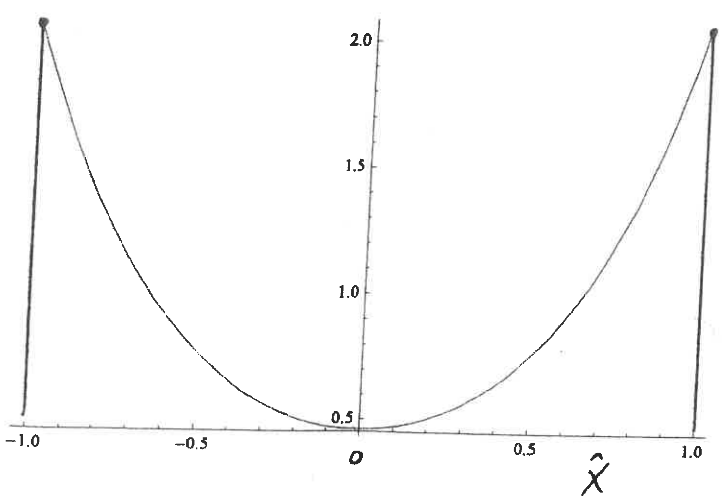
Out[11]=



universal curve

$$\hat{h}(\hat{x}) = \frac{1}{y^*} \cosh(y^* \hat{x})$$

for $\hat{h} = \frac{h}{x_0}$, $\hat{x} = \frac{x}{x_0}$

$$\hat{\ell} = \frac{\ell}{x_0} = 4$$

(all lengths in units of x_0)

Problem 2

Cp. hints:

$$s(x) \overset{(3)}{=} \frac{n_1}{c}\sqrt{x^2+h_1^2} + \frac{n_2}{c}\sqrt{(x_2-x)^2+h_2^2}$$

(cp. Fig. 6.9, p. 231)

$$\Rightarrow \frac{ds}{dx} = \frac{n_1}{c}\frac{x}{\sqrt{}} - \frac{n_2}{c}\frac{x_2-x}{\sqrt{}}$$

$$= \frac{n_1}{c}\sin(\vartheta_1) - \frac{n_2}{c}\sin(\vartheta_2)$$

$$\left.\frac{ds}{dx}\right|_{\tilde{x}} = 0 \Rightarrow n_1 \sin(\vartheta_1) = n_2 \sin(\vartheta_2)$$

Snell's law of refraction

Problem 3

a) $U(r) = \frac{1}{2} k r^2$

$\vec{F} = -\vec{\nabla} U(r) = -U'(r)\,\hat{r}$ (cp. PS5, P4c)

$= -k r \hat{r} = -k\vec{r}$

$\hat{=}$ 2D harmonic oscillator

b) Lagrangian using coordinates x, y:

$\mathcal{L} = T - U$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (x^2 + y^2)$

$= \mathcal{L}(x, y, \dot{x}, \dot{y})$

$x: \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow m\ddot{x} = -kx \qquad (1)$

$y: \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y} \Rightarrow m\ddot{y} = -ky \qquad (2)$

cp PS 4, Problem 1:

(1), (2) are independent (decoupled) harmonic oscillator equations in x and y directions, with angular frequency $\Omega = \sqrt{\frac{k}{m}}$;

superposition gives <u>elliptical orbits</u> in xy-plane.

Problem 4

a) $\vec{r} = r\hat{r} \Rightarrow \frac{d}{dt}\vec{r} = \dot{r}\hat{r} + r\underbrace{\frac{d}{dt}\hat{r}}_{\dot{\phi}\hat{\phi}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$

$\Rightarrow \bar{L} = \bar{r}\times\bar{p} = m\bar{r}\times\frac{d}{dt}\bar{r} = m r\hat{r}\times(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$

$= \underbrace{m r^2\dot{\phi}}_{L_z}\hat{z}$ using $\hat{r}\times\hat{r} = 0$, $\hat{r}\times\hat{\phi} = \hat{z}$

$\Rightarrow L_z = m r^2\dot{\phi}$

b) $\vec{F} = -\bar{\nabla}U(r) = -U'(r)\hat{r}$ (cp. PS5, P4c)

central force b/c $\parallel$ to $\bar{r}$

c) $\mathcal{L} = T - U = \frac{1}{2}m\left(\frac{d\bar{r}}{dt}\right)^2 - U(r)$

$\overset{a)}{=} \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$

$= \mathcal{L}(r, \cancel{\phi}, \dot{r}, \dot{\phi})$

i.e., $\mathcal{L}$ is independent of ϕ

d) ϕ - equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{b/c } \mathcal{L} \text{ is independent of } \phi.$$

$$p_\phi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} \quad \underline{\text{conjugate momentum}} \text{ for } \phi.$$

$$\Rightarrow \frac{d}{dt} p_\phi = 0 \,, \quad p_\phi \text{ is conserved}$$

a) $\Rightarrow$ $p_\phi = L_z$, momentum conjugate to ϕ is angular momentum L_z !

r - equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{dt} m\dot{r} = m r \dot{\phi}^2 - U'(r)$$

$$m\ddot{r} = -U'(r) + m r \dot{\phi}^2 \qquad (1)$$

One may express the r.h.s. of (1) in terms of the (constant) angular momentum $L_z = m r^2 \dot\varphi$:

$$\dot\varphi^2 = \frac{L_z^2}{m^2 r^4}$$

$$\Rightarrow \quad m\ddot r = -U'(r) + \frac{L_z^2}{m r^3} =: -\frac{d}{dr} U_{eff}(r)$$

with $U_{eff}(r) := U(r) + \frac{L_z^2}{2m}\frac{1}{r^2}$

$\Rightarrow$ motion of $r(t) = |\vec r(t)|$ corresponds to 1-dimensional motion in the <u>effective potential</u> $U_{eff}(r)$;

repulsive term $\frac{L_z^2}{2m}\frac{1}{r^2}$ in $U_{eff}(r)$ is called <u>centrifugal barrier</u>, preventing the particle from approaching the origin $r=0$:

e.g. $U(r) \sim -\frac{1}{r}$ (gravitation)