Problem Set 9 - Solutions

Problem 1

$$\mathcal{L} = \frac{1}{2} m_1 \left(\frac{d\bar{r}_1}{dt}\right)^2 + \frac{1}{2} m_2 \left(\frac{d\bar{r}_2}{dt}\right)^2 - U(|\bar{r}_1 - \bar{r}_2|)$$

a) $\bar{r}_1' = \bar{r}_1 + s\hat{x}$, $\bar{r}_2' = \bar{r}_2 + s\hat{x}$

translation in x-direction

$$1 \bullet \xrightarrow{s} \bullet\, 1'$$

$$2 \bullet \xrightarrow{s} \bullet\, 2'$$

$\mathcal{L}$ invariant b/c

$|\bar{r}_1' - \bar{r}_2'| = |\bar{r}_1 - \bar{r}_2|$ in U

$\to$ system is <u>translationally invariant</u>

b) $$C_x = \sum_{i=1}^{6} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \underbrace{\frac{dq_i'}{ds}\bigg|_{s=0}}_{= \begin{cases} 1, & q_i' = x_1', x_2' \\ 0, & \text{else} \end{cases}} = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} + \frac{\partial \mathcal{L}}{\partial \dot{x}_2}$$

$$= m_1 \dot{x}_1 + m_2 \dot{x}_2 = p_{1,x} + p_{2,x} = P_x$$

analogously: P_y, P_z are conserved

$\Rightarrow$ <u>total momentum</u> $\vec{p} = m_1 \dot{\bar{r}}_1 + m_2 \dot{\bar{r}}_2$ conserved

Problem 2   Taylor, Problem 8.1

see lecture notes

Problem 3   Taylor, Problem 8.5

$$\mathcal{L} = T - U = \frac{1}{2} M \dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - U(r)$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \mu \dot{x}_i \,, \qquad \vec{r} = (x, y, z)$$

$$\Rightarrow \vec{p} = \mu \dot{\vec{r}}$$

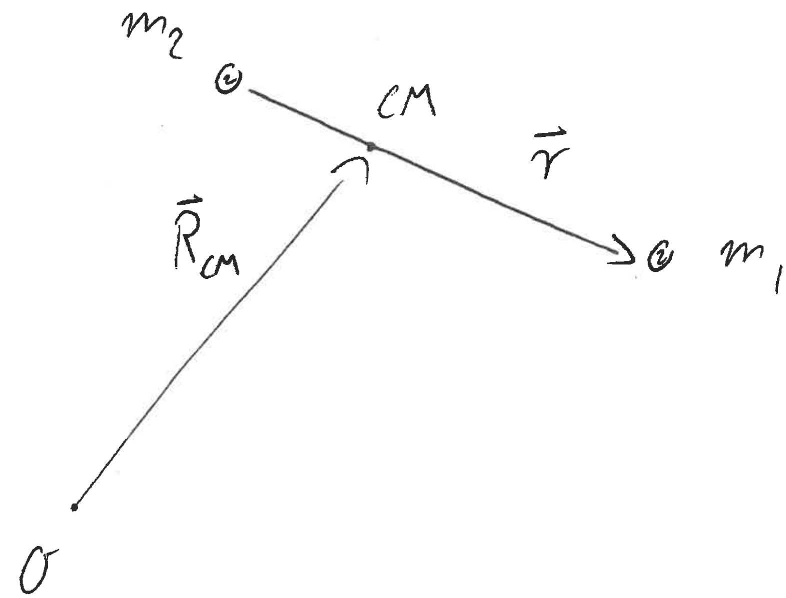
In CM frame (see lecture notes)

$$\vec{p}_1^{\,\prime} = m_1 \dot{\vec{r}}_1^{\,\prime} = \underbrace{m_1 \frac{m_2}{M}}_{\mu} \dot{\vec{r}} = \mu \dot{\vec{r}} = \vec{p}$$

$$\vec{p}_2^{\,\prime} = m_2 \dot{\vec{r}}_2^{\,\prime} = - \underbrace{m_2 \frac{m_1}{M}}_{\mu} \dot{\vec{r}} = - \mu \dot{\vec{r}} = - \vec{p}$$

$(\vec{p}_1^{\,\prime} + \vec{p}_2^{\,\prime} = 0$ by definition of CM$)$

Problem 4 Taylor, Problem 8.8



$$\mathcal{L}(\cancel{\vec{R}_{CM}}, \vec{r}, \dot{\vec{R}}_{CM}, \dot{\vec{r}}) = T - U$$

$$= \frac{1}{2} M \dot{\vec{R}}_{CM}^{\,2} + \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - \frac{1}{2} k r^2$$

1. $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{R}_{CM,i}} = \dfrac{\partial \mathcal{L}}{\partial R_{CM,i}} = 0$ b/c $\vec{R}_{CM}$ cyclic

$\Rightarrow M \ddot{\vec{R}}_{CM} = 0$, $\vec{P} = M \dot{\vec{R}}_{CM}$ conserved

2. $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{x}_i} = \dfrac{\partial \mathcal{L}}{\partial x_i} \Rightarrow \mu \ddot{\vec{r}} = -k\vec{r}$

2D harmonic oscillator with $\omega = \sqrt{\dfrac{k}{\mu}}$