Final Exam - Solutions

Problem 1) Sphere rolling on track

1. Total kinetic energy of rolling sphere: $T = T_{cm} + T_{rot} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ with $\omega = \frac{v}{R}$

$$\Rightarrow T = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2 = \frac{7}{10}Mv^2$$

2. Sphere at rest at A $\Rightarrow T_A = 0 \Rightarrow E = U_A = U\left(y_A\right) = Mgy_A = 1\,\text{kg} \times 9.8\frac{\text{m}}{\text{s}^2} \times 4\,\text{m} = 39.2\text{ J}$

3. At B: $E = T_B + U_B = \frac{7}{10}Mv_B^2 + Mgy_B \overset{a)}{\Rightarrow} Mgy_A = \frac{7}{10}Mv_B^2 + Mgy_B \Rightarrow \frac{7}{10}v_B^2 = g\left(y_A - y_B\right)$

$$\Rightarrow v_B = \sqrt{\frac{10}{7}g\left(y_A - y_B\right)} = \sqrt{\frac{10}{7} \times 9.8\frac{\text{m}}{\text{s}^2} \times 2\,\text{m}} = 5.29\,\frac{\text{m}}{\text{s}}$$

Problem 2) Line integral of a vector function

$$\vec{\mathbf{F}}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

1. $\left(\vec{\nabla}\times\vec{\mathbf{F}}\right)_x = \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} = x - x = 0$

$$\left(\vec{\nabla}\times\vec{\mathbf{F}}\right)_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = y - y = 0$$

$$\left(\vec{\nabla}\times\vec{\mathbf{F}}\right)_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = z - z = 0$$

$\Rightarrow\ \vec{\nabla}\times\vec{\mathbf{F}} = 0\ \Rightarrow\ \vec{\mathbf{F}}$ is conservative

2. $U(x,y,z) = -xyz$

$$\Rightarrow\ -\vec{\nabla}U = -\left(\frac{\partial U}{\partial x}\hat{\mathbf{x}} + \frac{\partial U}{\partial y}\hat{\mathbf{y}} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}\right) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \vec{\mathbf{F}}$$

$\Rightarrow\ U$ is potential energy for conservative force $\vec{\mathbf{F}}$

3. $\vec{\mathbf{F}} = -\vec{\nabla}U\ \Rightarrow\ \displaystyle\int_C d\vec{\mathbf{r}}\cdot\vec{\mathbf{F}}(\vec{\mathbf{r}}) = -\left[U(\vec{P}) - U(0)\right] = XYZ - 0 = XYZ$

for every curve $C : O \to \vec{P}$ from $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to $\vec{P} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ (line integral is path-independent)

Problem 3) Kepler Problem

1. $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \left(r\,\hat{\mathbf{r}}\right) \times m\,\frac{d}{dt}\vec{\mathbf{r}} \;=\; rm\,\hat{\mathbf{r}} \times \left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\right) \;=\; mr^2\dot{\phi}\,\hat{\mathbf{r}} \times \phi \;=\; mr^2\dot{\phi}\,\hat{\mathbf{z}}$

   $\Rightarrow\; \vec{\mathbf{L}} = L_z\,\hat{\mathbf{z}}$ with $L_z = mr^2\dot{\phi}$

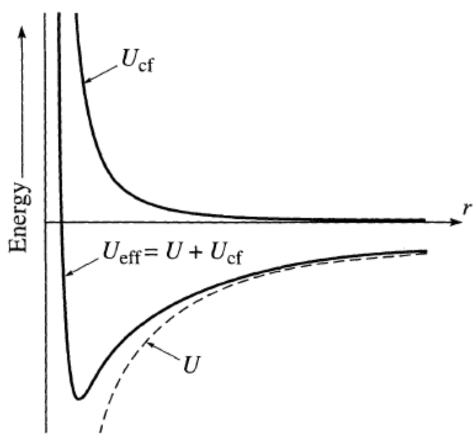
2. $\vec{\mathbf{F}} = -\vec{\nabla}U\left(r\right) = -U'\left(r\right)\hat{\mathbf{r}} = -\frac{k}{r^2}\hat{\mathbf{r}}$ using $U\left(r\right) = -\frac{k}{r}$

3. $T \;=\; \frac{1}{2}m\left(\frac{d}{dt}\vec{\mathbf{r}}\right)\cdot\left(\frac{d}{dt}\vec{\mathbf{r}}\right) \;=\; \frac{1}{2}m\left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\right)\cdot\left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\right) \;=\; \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$

   $\Rightarrow\; \mathcal{L}\left(r,\dot{r},\dot{\phi}\right) = T - U = \;\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \;+\; \frac{k}{r}$

4. $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0$ because $\mathcal{L}$ does not depend on ϕ $\Rightarrow$ $\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} = L_z$ is conserved

5. $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \;\Rightarrow\; m\ddot{r} = mr\dot{\phi}^2 - \frac{k}{r^2} \;=\; \frac{L_z^2}{m}\frac{1}{r^3} - \frac{k}{r^2} \;=\; -\frac{d}{dr}U_{eff}\left(r\right)$ with $U_{eff}\left(r\right) = \frac{L_z^2}{2m}\frac{1}{r^2} - \frac{k}{r}$

6. 

7. For fixed L_z the minimum of $U_{eff}\left(r\right)$ corresponds to a circular orbit with constant radius R

   and $L_z = mR^2\dot{\phi} = mR^2\omega$.

   At minimum: $\left.\frac{d}{dr}U_{eff}\left(r\right)\right|_{r=R} = 0 \;\Rightarrow\; \frac{L_z^2}{m}\frac{1}{R^3} - \frac{k}{R^2} = 0 \;\Rightarrow\; k \;=\; \frac{L_z^2}{mR} \;=\; \frac{\left(mR^2\omega\right)^2}{mR} \;=\; mR^3\omega^2$

   $\Rightarrow\; \omega = \sqrt{\frac{k}{mR^3}}$

Problem 4) Two carts connected by springs

1. $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right)$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 \left(x_1 - x_2 \right)^2 = \frac{1}{2} (3k) x_1^2 + \frac{1}{2} (2k) \left(x_1^2 - 2 x_1 x_2 + x_2^2 \right) = \frac{1}{2} k \left(5x_1^2 - 4x_1 x_2 + 2x_2^2 \right)$$

$$\Rightarrow \mathcal{L}\left(x_1, x_2, \dot{x}_1, \dot{x}_2 \right) = T - U = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{1}{2} k \left(5x_1^2 - 4x_1 x_2 + 2x_2^2 \right)$$

2. $\mathcal{L} = \frac{1}{2} \dot{\vec{x}} \cdot M \dot{\vec{x}} - \frac{1}{2} \vec{x} \cdot K \vec{x}$ with $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $K = \begin{pmatrix} 5k & -2k \\ -2k & 2k \end{pmatrix}$.

3. $$(\mathbf{K} - \omega^2 \mathbf{M}) = \begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix}$$

with determinant $\det(\mathbf{K} - \omega^2 \mathbf{M}) = (m\omega^2 - k)(m\omega^2 - 6k)$. Thus the two normal frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{6k/m}$.

4. The motion in each normal mode is determined by the vector $\mathbf{a}$ satisfying the eigenvector equation $(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this gives $a_2 = 2a_1$, so the two carts oscillate in phase, with the second cart's amplitude equal to twice that of the first. If $\omega = \omega_2$ then $a_2 = -a_1/2$, so the two carts oscillate exactly out of phase, with the second cart's amplitude equal to half that of the first.