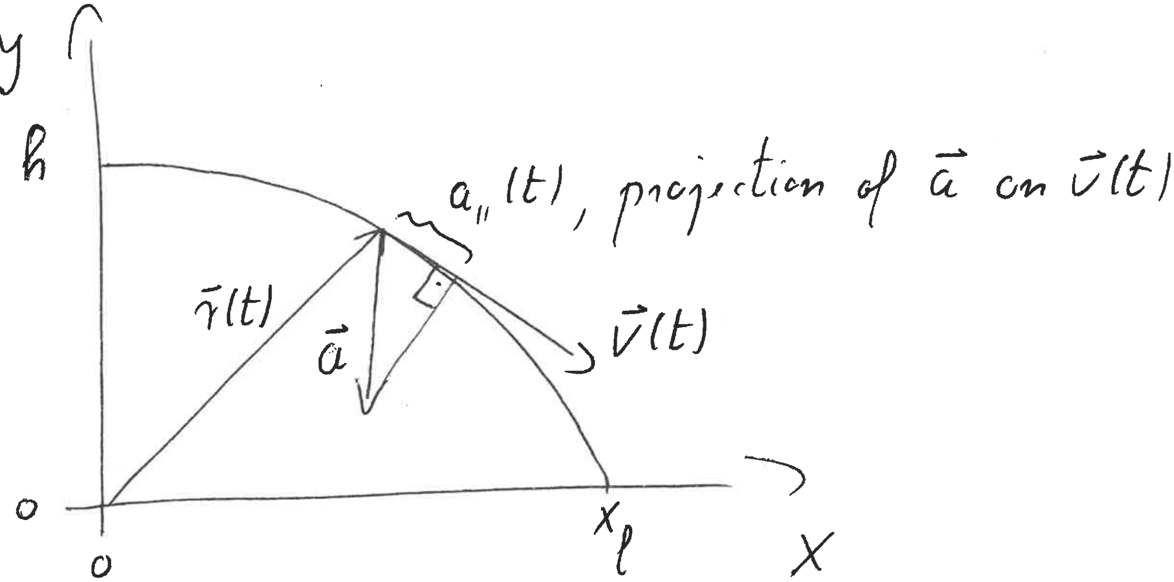
Problem Set 2 - Solutions

Problem 1

a)



b) $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} = wt\,\hat{x} + (h - \frac{1}{2}gt^2)\,\hat{y}$

$\Rightarrow y(t) = h - \frac{1}{2}gt^2$

$y(t_l) = h - \frac{1}{2}gt_l^2 = 0 \Rightarrow t_l = \sqrt{\frac{2h}{g}}$

c) $\vec{v} = \frac{d}{dt}\vec{r} = w\hat{x} - gt\hat{y}$

$v = |\vec{v}| = \sqrt{w^2 + g^2t^2}$ (1), $\hat{v} = \frac{\vec{v}}{v}$

d) $\vec{a} = \frac{d}{dt}\vec{v} = -g\hat{y} =: \vec{g} = \text{const}$

e) $\frac{dv}{dt} \overset{(1)}{=} \frac{1}{2}\frac{1}{\sqrt{\ }}g^2 2t = \frac{g^2 t}{v}$; $a_{\parallel} = \vec{a}\cdot\hat{v} = \frac{g^2 t}{v}$

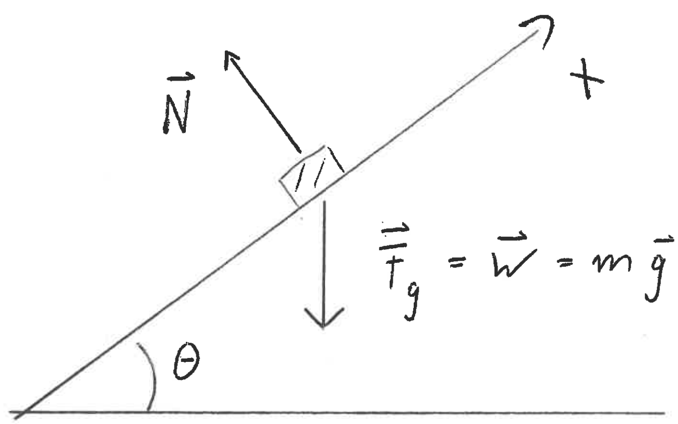
$\rightarrow \frac{dv}{dt} = a_{\parallel}$, $a_{\parallel}$ speeds up the projectile

1.37 ⋆ **(a)** The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. If we choose axes with x measured up the slope, y along the outward normal, and z horizontally across the slope, then $\mathbf{N} = (0, N, 0)$ and $\mathbf{g} = (-g\sin\theta, -g\cos\theta, 0)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = -mg\sin\theta \\ m\ddot{y} = N - mg\cos\theta \\ m\ddot{z} = 0 \end{cases}$$

Since $\dot{z} = 0$ initially, it remains so and hence $z = 0$ for all t. The normal force adjusts itself so that $\ddot{y} = 0$, and $y = 0$ for all t. Finally, $\ddot{x} = -g\sin\theta$, which can be integrated twice to give $x = v_o t - \frac{1}{2}gt^2\sin\theta$.

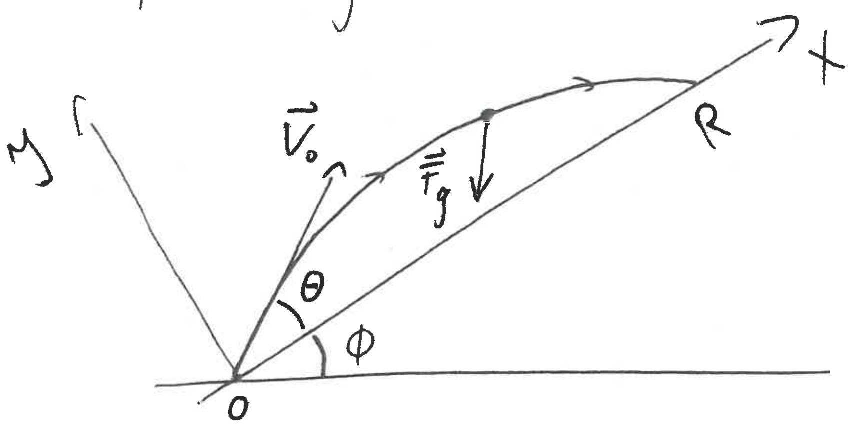
(b) Solving for the times when $x = 0$, we find that $t = 0$ (at launch) or $t = 2v_o/(g\sin\theta)$ (the answer of interest).



(a)

1.39 ** $x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$, $y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$, $z = 0$. When the ball returns to the plane, y is 0, which implies that $t = 2v_o\sin\theta/(g\cos\phi)$. Substituting this time into x and using a couple of trig identities yields the claimed answer for the range R. To find the maximum range, differentiate R with respect to θ and set the derivative equal to zero. This gives $\theta = (\pi - 2\phi)/4$, and substitution into R (plus another trig identity) yields the claimed value of R_{max}. (b)

more explicitly:



force on ball: $\vec{F}_g = m\vec{g} = -mg\sin(\phi)\,\hat{x} - mg\cos(\phi)\,\hat{y}$

(N2), (1.35):

$\cancel{m}\ddot{x} = F_x = -\cancel{m}g\sin(\phi)$, $\cancel{m}\ddot{y} = F_y = -\cancel{m}g\cos(\phi)$

initial velocity: $\vec{v}_o = v_o\cos(\theta)\,\hat{x} + v_o\sin(\theta)\,\hat{y}$

$$\Rightarrow \quad x(t) = v_o\cos(\theta)\cdot t - \frac{1}{2}g\sin(\phi)\,t^2 \qquad (1)$$

$$y(t) = v_o\sin(\theta)\cdot t - \frac{1}{2}g\cos(\phi)\,t^2 \qquad (2)$$

$\Rightarrow$ (a)

Ball returns to plane:

$$y(t) = 0 \Rightarrow v_0 \sin\theta \, \cancel{t} - \frac{1}{2} g \cos\phi \cdot t^{\cancel{2}} = 0$$

$$\Rightarrow t = \frac{2v_0}{g} \, \frac{\sin\theta}{\cos\phi}$$

In (1):

$$R = x(t) = \frac{2v_0^2}{g} \underbrace{\cos\theta \, \frac{\sin\theta}{\cos\phi}}_{\frac{\sin\theta}{\cos^2\phi} \cos\theta \, \cos\phi}$$

$$- \underbrace{\frac{1}{2} g \, \frac{4v_0^2}{g^2}}_{\frac{2v_0^2}{g}} \underbrace{\sin\phi \, \frac{\sin^2\theta}{\cos^2\phi}}_{\frac{\sin\theta}{\cos^2\phi} \sin\theta \, \sin\phi}$$

$$R = \frac{2v_0^2}{g} \frac{\sin\theta}{\cos^2\phi} \underbrace{(\cos\theta\cos\phi - \sin\theta\sin\phi)}_{\cos(\theta+\phi)}$$

$$= \frac{2v_0^2}{g} \frac{\sin\theta}{\cos^2\phi} \cos(\theta+\phi) \qquad (3)$$

result quoted in question

R_{max} from $\frac{dR}{d\theta} = 0$:

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g\cos^2\phi} \left\{ \underbrace{\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)}_{\cos(2\theta+\phi)} \right\}$$

$$= 0$$

$$\Rightarrow 2\theta + \phi = \frac{\pi}{2} \hat{=} 90^\circ$$

$$2\theta + \phi = \frac{\pi}{2} \qquad (4)$$

$$\Rightarrow \quad \theta + \phi = \frac{\pi}{2} - \theta$$

in (3): $\cos(\theta + \phi) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$$\Rightarrow \quad R_{max} = \frac{2v_0^2}{g} \frac{\sin^2\theta}{\cos^2\phi} \qquad (5)$$

$$(4) \Rightarrow \quad \theta = \frac{1}{2}\left(\frac{\pi}{2} - \phi\right) \qquad (b)$$

$$\Rightarrow \quad \sin^2\theta = \sin^2\left\{\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right\}$$

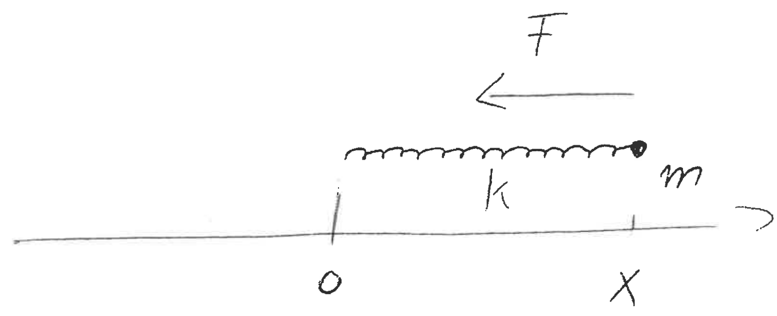
$$= \frac{1}{2}\left\{1 - \underbrace{\cos\left(\frac{\pi}{2} - \phi\right)}_{\sin\phi}\right\} \quad \text{trig. relation}$$

in (5): $\cos^2\phi = 1 - \sin^2\phi = (1 + \sin\phi)(1 - \sin\phi)$

$$\Rightarrow \quad R_{max} = \frac{v_0^2}{g} \frac{\cancel{1 - \sin\phi}}{(1 + \sin\phi)\cancel{(1 - \sin\phi)}}$$

$$= \frac{v_0^2}{g(1 + \sin\phi)} \quad \text{quoted in question}$$

Problem 4



$$F = -kx \quad (1)$$

a) $$F = ma = m\frac{d^2x}{dt^2} \quad (N2)$$

$$\overset{(1)}{\Rightarrow} \quad m\frac{d^2x}{dt^2} = -kx \quad \Big| \cdot \frac{1}{m}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad (2)$$

(2) contains 2nd derivative $\frac{d^2x}{dt^2}$

$\hat{=}$ 2nd order differential equation (DE) for position variable $x(t)$

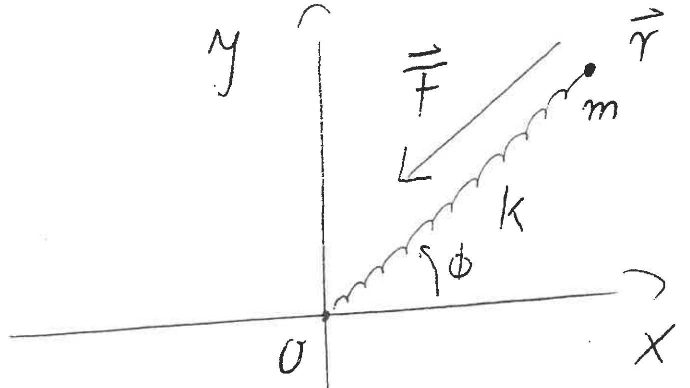
b) Show: $x(t) = a\cos(\omega t) + b\sin(\omega t)$ solves the DE (2):

$$\frac{dx}{dt} = -a\sin(\omega t)\,\omega + b\cos(\omega t)\cdot\omega$$

$$\frac{d^2x}{dt^2} = -a\cos(\omega t)\cdot\omega^2 - b\sin(\omega t)\cdot\omega^2$$

$$= -\omega^2 x(t) \Rightarrow \text{(2) fulfilled, with } \underline{\omega = \sqrt{\tfrac{k}{m}}}$$

Problem 5



$\vec{r}(t) = r(t)\,\hat{r}(t)$

note: $r = |\vec{r}|$ (magnitude)

$\vec{F}(t) = -k\,r(t)\,\hat{r}(t) \qquad (1)$

a) $\vec{a}(t) = \dfrac{d^2}{dt^2}\vec{r}(t)$ in polar coordinates: (1.47), $\cdot \,\hat{=}\, \dfrac{d}{dt}$

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\,\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\phi}$$

$\vec{F} = m\vec{a}$ with $\vec{F}$ from (1):

$$-kr\hat{r} = m(\ddot{r} - r\dot{\phi}^2)\,\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{\phi}$$

comparing components in $\hat{r}$, $\hat{\phi}$:

$$\Rightarrow \quad -kr = m(\ddot{r} - r\dot{\phi}^2) \qquad (2)$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \qquad (3)$$

(2), (3) are two coupled 2nd order DEs for $r(t)$, $\phi(t)$; finding solutions is difficult in general!

b) Ansatz for $r(t)$, $\phi(t)$:

$$r = R = \text{const.}$$
$$\phi(t) = \omega t, \quad \omega = \text{const} \qquad (4)$$

= circular orbit with radius R

$$\Rightarrow \quad \dot{r} = 0, \quad \ddot{r} = 0$$
$$\dot{\phi} = \omega, \quad \ddot{\phi} = 0$$

in (2): $$-kR \overset{!}{=} m(0 - R\omega^2)$$
$$-k\cancel{R} = -m\cancel{R}\omega^2$$
fulfilled if $\underline{\omega = \sqrt{\frac{k}{m}}}$

in (3): $$R \cdot 0 + 2 \cdot 0\,\omega \overset{!}{=} 0$$
$0 = 0$, fulfilled

$\Rightarrow$ (4) is a solution of (2), (3)
with $\omega = \sqrt{\frac{k}{m}}$, same ω as in P4!

explanation:

write $\vec{F} = -k r \hat{r} = -k \vec{r}$

in Cartesian $(x-y)$ coordinates:

$$\vec{F} = -k\vec{r} = -k(x\hat{x} + y\hat{y})$$
$$= -kx\hat{x} - ky\hat{y}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y}$$

comparing components in $\hat{x}, \hat{y}$:

$$\vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{1}{m}\vec{F}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x, \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y \quad (5)$$

(5) are two indendent, one-dimensional DEs for $x(t)$, $y(t)$ as in P4!

$\Rightarrow$ harmonic motions for $x(t)$, $y(t)$ with $\omega = \sqrt{\frac{k}{m}}$ as in P4; for a circular orbit, $x(t)$, $y(t)$ are $90°$ out of phase:

$$x(t) = R\cos(\omega t), \quad y(t) = R\cos(\omega t - \tfrac{\pi}{2}) = R\sin(\omega t)$$