Problem Set 4 - Solutions

Problem 1

a) Spring: $\vec{F} = -k\vec{r}$

N2: $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$

$$\Rightarrow m\frac{d^2\vec{r}}{dt^2} = -k\vec{r} \quad (1)$$

using $\vec{r} = x\hat{x} + y\hat{y}$

$(1) \Rightarrow m\ddot{x} = -kx, \quad m\ddot{y} = -ky$

$\dfrac{k}{m} = \Omega^2 \Rightarrow \ddot{x} = -\Omega^2 x, \quad \ddot{y} = -\Omega^2 y$

① ②

b) $x(t) = a\cos(\Omega t)$

$\Rightarrow \dot{x} = -a\Omega\sin(\Omega t)$

$\Rightarrow \ddot{x} = -a\Omega^2\cos(\Omega t) = -\Omega^2 x$

$\Rightarrow$ ① is fulfilled

$y(t) = b\sin(\Omega t)$

$\Rightarrow \dot{y} = b\Omega\cos(\Omega t)$

$\Rightarrow \ddot{y} = -b\Omega^2\sin(\Omega t) = -\Omega^2 y$

$\Rightarrow$ ② is fulfilled

c) $$\frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} = \cos^2(\Omega t) + \sin^2(\Omega t) = 1$$

$$\Rightarrow \quad \vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y}$$

is an elliptical orbit

d) $$\frac{d}{dt}\vec{r} = -a\Omega \sin(\Omega t)\,\hat{x} + b\Omega\cos(\Omega t)\,\hat{y}$$

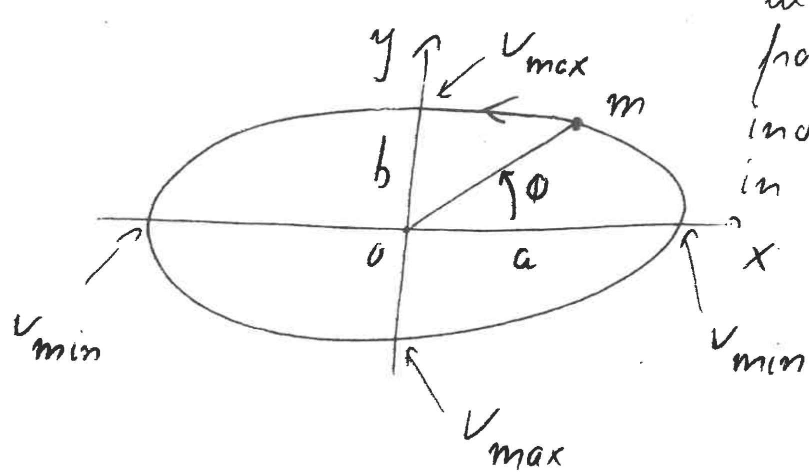
$$v = |\vec{v}| = \left|\frac{d}{dt}\vec{r}\right|$$

$$= \Omega\left\{a^2\sin^2(\Omega t) + b^2\cos^2(\Omega t)\right\}^{1/2}$$

if $a > b$:

v is <u>maximum</u> for $\Omega t = \frac{\pi}{2}, \frac{3\pi}{2}$; $v_{max} = \Omega a$

v is <u>minimum</u> for $\Omega t = 0, \pi$; $v_{min} = \Omega b$



elliptical orbit results from superposition of independent oscillations in x- and y directions

e) $\vec{l} = m\vec{r} \times \vec{v} = m\vec{r} \times \frac{d\vec{r}}{dt}$

d)

$$= m\{a\cos(\Omega t)\hat{x} + b\sin(\Omega t)\hat{y}\}$$

$$\times \{-a\Omega\sin(\Omega t)\hat{x} + b\Omega\cos(\Omega t)\hat{y}\}$$

using $\hat{x}\times\hat{x} = 0$, $\hat{y}\times\hat{y} = 0$

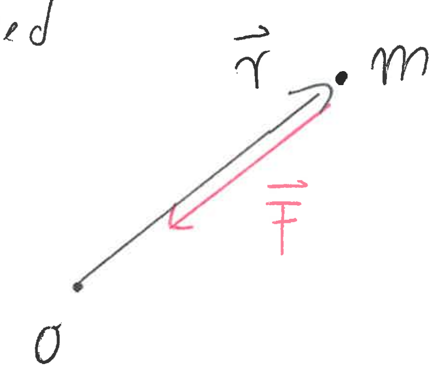
$$= m\{ab\Omega\cos^2(\Omega t)\underbrace{\hat{x}\times\hat{y}}_{\hat{z}} - ab\Omega\sin^2(\Omega t)\underbrace{\hat{y}\times\hat{x}}_{-\hat{z}}\}$$

$$= mab\Omega\underbrace{\{\cos^2(\Omega t) + \sin^2(\Omega t)\}}_{=1}\hat{z}$$

$$= mab\Omega\hat{z}$$ conserved $\left(\Omega = \sqrt{\frac{k}{m}} = \text{const}\right)$

f) $\vec{F} = -k\vec{r}$ is central force

$\Rightarrow \vec{l}$ conserved



note:

here $\Omega \neq \dot{\phi} = \frac{d}{dt} \phi$!

i.e. $\phi(t) \neq \Omega t$

reason:

$$\tan \phi = \frac{y}{x} = \frac{b \sin(\Omega t)}{a \cos(\Omega t)} = \frac{b}{a} \tan(\Omega t)$$

$\Rightarrow$ $\phi \neq \Omega t$ unless $\frac{b}{a} = 1$ (circle)

$l = |\vec{l}| = m r^2 \dot{\phi} = \text{const.}$ still holds!

b/c $l = m a b \Omega$:

$\Rightarrow$ $\dot{\phi} = \frac{a b}{r^2} \Omega$ not const. except for circle

for circle: $a = b = r = \text{const} \Rightarrow \dot{\phi} = \Omega$

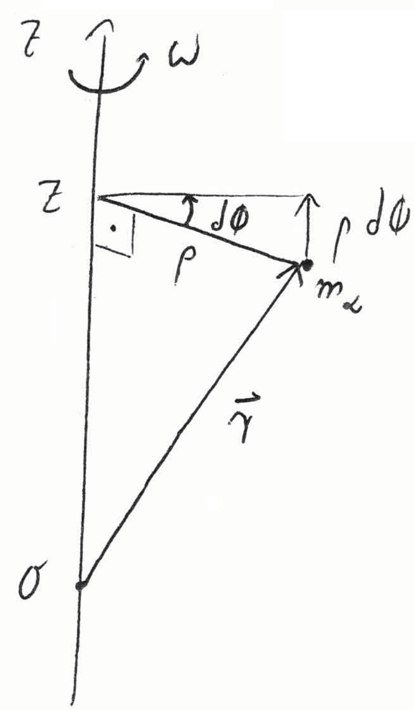
3.25 ⋆ The net force on the particle is just the tension of the string, which is necessarily directed toward the hole in the table at O. Therefore the angular momentum $\boldsymbol{\ell}$ about O is constant. When the particle is travelling in a circle of radius r, the vertical component of $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$ is $\ell_z = rp = rmv = rm(r\omega) = mr^2\omega$. Therefore, the quantity $r^2\omega$ is constant and $r^2\omega = r_o^2\omega_o$; whence $\omega = (r_o/r)^2\omega_o$.

3.30 ⋆⋆ **(a)** If a particle is a distance ρ from the axis of rotation and the body turns through an angle $d\phi$, then the particle moves a distance $\rho d\phi$ in the tangential (ϕ) direction. Dividing by dt we conclude that the particle's speed is $v = \rho d\phi/dt = \rho\omega$ in the ϕ direction. That is, $\mathbf{v} = \rho\omega\hat{\boldsymbol{\phi}}$.

(b) The particle's position is $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so its angular momentum is $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = (\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) \times m\rho\omega\hat{\boldsymbol{\phi}} = m\rho^2\omega\hat{\mathbf{z}} - mz\rho\omega\hat{\boldsymbol{\rho}}$. Therefore its z component is $\ell_z = m\rho^2\omega$.

(c) The total angular momentum has

$$L_z = \sum_{\alpha=1}^{N} \ell_{\alpha z} = \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2 \omega = I\omega \quad \text{where} \quad I = \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2.$$



Problem 4

$$I = \rho \int_V \rho^2(\vec{r})\, dV$$

$$= \rho \int_0^R d\rho\, \rho \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{h} dz}_{h}\, \rho^2$$

$$= \rho\, 2\pi h \int_0^R d\rho\, \rho^3 = \rho\, 2\pi h\, \frac{1}{4} R^4 = \rho\, \frac{\pi}{2} h R^4$$

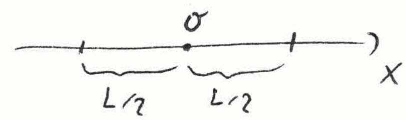
$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 h}$$

$$\Rightarrow \quad I = \frac{M}{\cancel{\pi} R^2 \cancel{h}} \frac{\cancel{\pi}}{2} \cancel{h} R^4 = \underline{\frac{1}{2} M R^2}$$

Problem 5

a) $$I_c = \rho\, 2 \int_0^{L/2} x^2 dx = \rho\, 2\, \frac{1}{3}\left(\frac{L}{2}\right)^3 = \rho\, \frac{L^3}{12}$$

$$= \frac{M}{L}\, \frac{L^3}{12} = \underline{\frac{1}{12} M L^2}$$



b) $$I_{end} = \rho \int_0^L x^2 dx = \rho\, \frac{1}{3} L^3 = \underline{\frac{1}{3} M L^2}$$