UT Rio Grande Valley October 23, 2021
PHYS-3305 Classical Mechanics Andreas Hanke

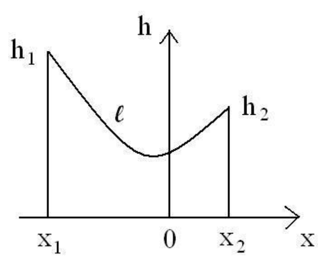
Problem Set 7 – due Friday, October 29 by 12:00 PM midnight

The Problem Set has **4 questions** on **4 pages**, with a total maximum credit of **30 points**.

Please turn in well-organized, clearly written solutions (no scrap work). Questions 2 and 3 are taken from the textbook by Taylor.

Problem 1) Shape of a hanging chain [8 points]

A chain of fixed length ℓ and uniform linear mass density ρ is attached at its ends at heights h_1 and h_2 in the gravitational field of the Earth with gravitational constant g. The horizontal distance between the ends is $x_2 - x_1$. The goal of this problem is to find the curve $h(x)$ of the hanging chain at mechanical equilibrium using calculus of variations.



The equilibrium curve of the chain minimizes the potential energy $U = g\rho\int_0^{\ell} ds\left[h(s) - c\right]$ (1)

where $h(s)$ is the height of the chain over ground as a function of arc length s. The constant c accounts for the freedom to add a constant $(-g\rho\ell c)$ to the potential energy.

a) Substitute $s \to x$ in the integral in (1) and identify the Lagrange function $\mathcal{L}(y, \dot{y})$ for the curve $y(x) := h(x) - c$, with $\dot{y} = \frac{dy}{dx}$, such that

$$U[y] = \int_{x_1}^{x_2} dx\,\mathcal{L}(y, \dot{y}) \quad . \qquad \text{Hint: } ds = \sqrt{1 + \left(\frac{dh}{dx}\right)^2}\, dx \,. \tag{2}$$

b) Interpret $U[y]$ in (2) as an action integral for curves $y(x)$. From the Lagrange function $\mathcal{L}(y,\dot{y})$ found in a) derive the Euler-Lagrange (EL) equation

$$\frac{d}{dx}\left(\frac{y\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = \sqrt{1+\dot{y}^2} \quad . \tag{3}$$

Hint: The EL equation for the Lagrange function $\mathcal{L}(y,\dot{y})$ is given by $\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)-\frac{\partial\mathcal{L}}{\partial y}=0$ where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial\dot{y}}$ denote partial derivatives with respect to the arguments y and $\dot{y}$ of $\mathcal{L}$, respectively, and $\frac{d}{dx}$ is a total derivative.

c) Verify by inserting that the EL equation (3) is solved by the curve

$$y(x)=a\cosh\left(\frac{x-b}{a}\right) \tag{4}$$

where a, b are constants. Note that it is not required to *derive* (4) from the EL equation (3); just verify that (4) is a solution of (3).

d) (4) and the definition $y(x)=h(x)-c$ imply that the general form of the equilibrium height function is

$$h(x)=a\cosh\left(\frac{x-b}{a}\right)+c \quad . \tag{5}$$

The constants a, b, c are determined by the conditions $h(x_1)=h_1$, $h(x_2)=h_2$, and that the chain length is ℓ. Find a, b, c for the symmetric case $h_1=h_2$, $x_1=-x_0$, $x_2=+x_0$, and assuming without restriction $h(0)=a$. While this trivially determines b and c, the determination of a (using the chain length ℓ) leads to a non-algebraic equation for a. Solve this equation numerically for the case $\ell=4x_0$, i.e., the chain is twice as long as the horizontal distance. Plot the resulting height function of the chain.

Problem 2) Snell's law derived from Fermat's principle [8 points]

Read the first three pages of Chapter 6 (pages 215-217).

Do Problem 6.4 (page 231) "A ray of light ...".

Hint/Instruction: You don't need to show that, on the actual path followed, Q lies in the same vertical plane as P_1 and P_2. Instead, assume from the start that all points shown in Figure 6.9 are in the xy - plane. Thus $Q=(x,0,0)$, i.e., the z - component of Q is zero. Fermat's principle states that the actual path of a light ray between two points P_1 and P_2 is the path for which the time of travel is minimum. Let's define the action functional $S[C]$ for a given, arbitrary path $C: P_1 \to P_2$ from P_1 to P_2 as the time of travel of the light ray along this path:

$$S[C] := \text{(time of travel)} = \frac{1}{c} \int_{C:P_1\to P_2} n(\vec{\mathbf{r}})\,ds \qquad (1)$$

where c is the speed of light in vacuum, $n(\vec{\mathbf{r}})$ is the refractive index of the medium at point $\vec{\mathbf{r}}$ along the path, and s is the length coordinate along the path. The light ray chooses the path $\tilde{C}$ for which $S[C]$ is minimum.

In the present problem, the refractive index n is constant in the media above and below the interface shown in Figure 6.9. Thus Eq. (1) simplifies to

$$S[C] = \frac{n_1}{c} \int_{C_1:P_1\to Q} ds + \frac{n_2}{c} \int_{C_2:Q\to P_2} ds = \frac{n_1}{c}\left(\text{length of } C_1\right) + \frac{n_2}{c}\left(\text{length of } C_2\right) \qquad (2)$$

where C_1, C_2 are the sections of C above and below the interface, respectively. Clearly, the shortest path between two given points is a straight line. Using this, we can rewrite $S[C]$ as a <u>function</u> $s(x)$ (not functional) of the position of point $Q=(x,0,0)$:

$$S[C] = s(x) := \frac{n_1}{c}\left(\text{length of straight line } P_1 \to Q\right) + \frac{n_2}{c}\left(\text{length of straight line } Q \to P_2\right) \qquad (3)$$

Express the lengths of the straight lines in Eq. (3) in terms of h_1, h_2, x using Pythagoras' theorem and derive Snell's law from the condition that $s(x)$ is minimum, i.e., $\left.\frac{ds}{dx}\right|_{x=\tilde{x}} = 0$.

Problem 3) Two-dimensional harmonic oscillator: Lagrange equations [6 points]

Taylor, Problem 7.3 (page 281) "Consider a mass m moving in two dimensions ..."

a) Show that the potential energy $U(x, y) = \frac{1}{2} k r^2$ given in the question results in the force $\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$ corresponding to a two-dimensional harmonic oscillator (as in PS 4, Problem 1).

Hint: Use the result of Problem 4c of Problem Set 5.

b) Answer the questions in Taylor, Problem 7.3.

Problem 4) Particle in two dimensions subject to a central force [8 points]

Work out Example 7.2 in Taylor, pages 242, 243.

Consider a particle of mass m moving in the xy - plane subject to a potential energy U which depends on the position vector $\vec{\mathbf{r}}$ only in terms of the magnitude $r = |\vec{\mathbf{r}}|$, i.e., $U = U(r)$.

a) Find the angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ using cylindrical polar coordinates (r, ϕ, z).

Hint: Use $\vec{\mathbf{r}}(t) = r(t)\,\hat{\mathbf{r}}(t)$, $\vec{\mathbf{p}}(t) = m\frac{d}{dt}\vec{\mathbf{r}}(t)$, and $\hat{\mathbf{r}}(t) \times \hat{\phi}(t) = \hat{\mathbf{z}}$.

Result: $\vec{\mathbf{L}} = L_z\hat{\mathbf{z}}$ with $L_z = mr^2\dot{\phi}$.

b) Find the force, $\vec{\mathbf{F}} = -\vec{\nabla}U(r)$, and show that $\vec{\mathbf{F}}$ is a central force.

Hint: Use the result of Problem 4c of Problem Set 5.

c) Find the Lagrange function $\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = T - U$ for the two coordinates r, ϕ.

d) Find the Euler-Lagrange (EL) equations for r (the r - equation) and ϕ (the ϕ - equation) for the Lagrange function found in c). Show that the ϕ - equation implies $\frac{d}{dt}L_z = 0$, with $L_z = mr^2\dot{\phi}$ from part a. (Thus showing that L_z is conserved, as expected for a central force.)

Hint for c), d): Follow the calculations shown in Example 7.2, page 242, assuming that U only depends on r but not on ϕ.