UT Rio Grande Valley November 27, 2021
PHYS-3305 Classical Mechanics Andreas Hanke

Problem Set 12 – due Friday, December 3 by 12:00 PM midnight

The Problem Set has **3 questions** on **2 pages** with a total maximum credit of **30 points**.

Please turn in well-organized, clearly written solutions (no scrap work).

This Problem Set is relatively short since the class ends on Wednesday, December 1. However, the deadline is Friday as usual.

The final exam will be a take-home exam on **Wednesday, December 8** (following the UTRGV Final Exams Schedule for Fall 2021). The final exam will be distributed online via blackboard at 10 AM on December 8 and due by 12:00 PM midnight of the same day.

Problem 1) Rotating body [10 points]

Note regarding notation: In the lecture notes, the body-fixed frame is denoted $\{\hat{\mathbf{e}}'_i(t)\}$ using primes to distinguish it from the lab-fixed frame $\{\hat{\mathbf{e}}_i(t)\}$. In this problem we omit the primes for the body-fixed frame to simplify notation.

Consider a rotating rigid body with inertia tensor I_{ij} in a body-fixed frame $\{\hat{\mathbf{e}}_i(t)\}$ centered at the mass center CM. Assume that the angular velocity vector $\vec{\boldsymbol{\omega}}$ is given by

$$\vec{\boldsymbol{\omega}}(t) = \sum_{i=1}^{3} \omega_i \,\hat{\mathbf{e}}_i(t)$$

with constant (time-independent) components ω_i in the rotating frame $\{\hat{\mathbf{e}}_i(t)\}$.

a) The angular momentum can be expanded as $\vec{\mathbf{L}} = \sum_{i=1}^{3} L_i \,\hat{\mathbf{e}}_i$. Write down the relation between the components L_i and ω_i in the frame $\{\hat{\mathbf{e}}_i\}$. Are the components L_i time-dependent?

Hint: Use that the components I_{ij} of the inertia tensor in the body-fixed frame $\{\hat{\mathbf{e}}_i\}$ are time-independent.

b) Express the time derivative $\frac{d}{dt}\vec{\mathbf{L}}$ in terms of $\vec{\mathbf{L}}$ and $\vec{\boldsymbol{\omega}}$. Under which condition is $\frac{d}{dt}\vec{\mathbf{L}} = 0$?

Hint: Use the relation $\frac{d}{dt}\hat{\mathbf{e}}_i = \vec{\boldsymbol{\omega}} \times \hat{\mathbf{e}}_i$ for the rotating frame $\{\hat{\mathbf{e}}_i(t)\}$.

Problem 2) Two carts connected by springs [10 points]

Taylor, Chapter 11, Problem 11.5 (page 448) "(a) Find the normal frequencies ... "

Hint: Set $k_3 = 0$ in Figure 11.1 and use the results in Section 11.1 to find the mass matrix M and spring matrix K for $k_3 = 0$. The two normal frequencies ω_1, ω_2 follow from $\det\left(K - \omega^2 M\right) = 0$. The associated eigenmodes $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ follow from $\left(K - \omega^2 M\right)\vec{\mathbf{v}} = 0$ by setting $\omega = \omega_1$ and $\omega = \omega_2$, respectively.

Problem 3) Double pendulum [10 points]

Consider two identical pendulums (each of length L and mass m) that are joined by a massless spring (force constant k) as shown below. The masses are subject to the downward gravitational force with Earth's acceleration constant g. The positions of the pendulums are specified by the angles ϕ_1 and ϕ_2 as shown. The equilibrium length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical.

a) Find the Lagrange function $\mathcal{L}\left(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\right) = T - U$ in harmonic approximation, i.e., ignoring terms of higher than second order in ϕ_1, ϕ_2.

Solution: $\mathcal{L}\left(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\right) = \frac{1}{2} m L^2 \left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) - \frac{1}{2} m g L \left(\phi_1^2 + \phi_2^2\right) - \frac{1}{2} k L^2 \left(\phi_2 - \phi_1\right)^2$.

b) Find and describe the normal modes of the coupled pendulums.

Instruction: Use the Euler Lagrange equations for ϕ_1 and ϕ_2 to find the mass matrix M and spring matrix K. Use a similar strategy as in Problem 2 to find the two normal frequencies ω_1, ω_2 and associated eigenmodes $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$.