Problem Set 5 - Solutions

Problem 1

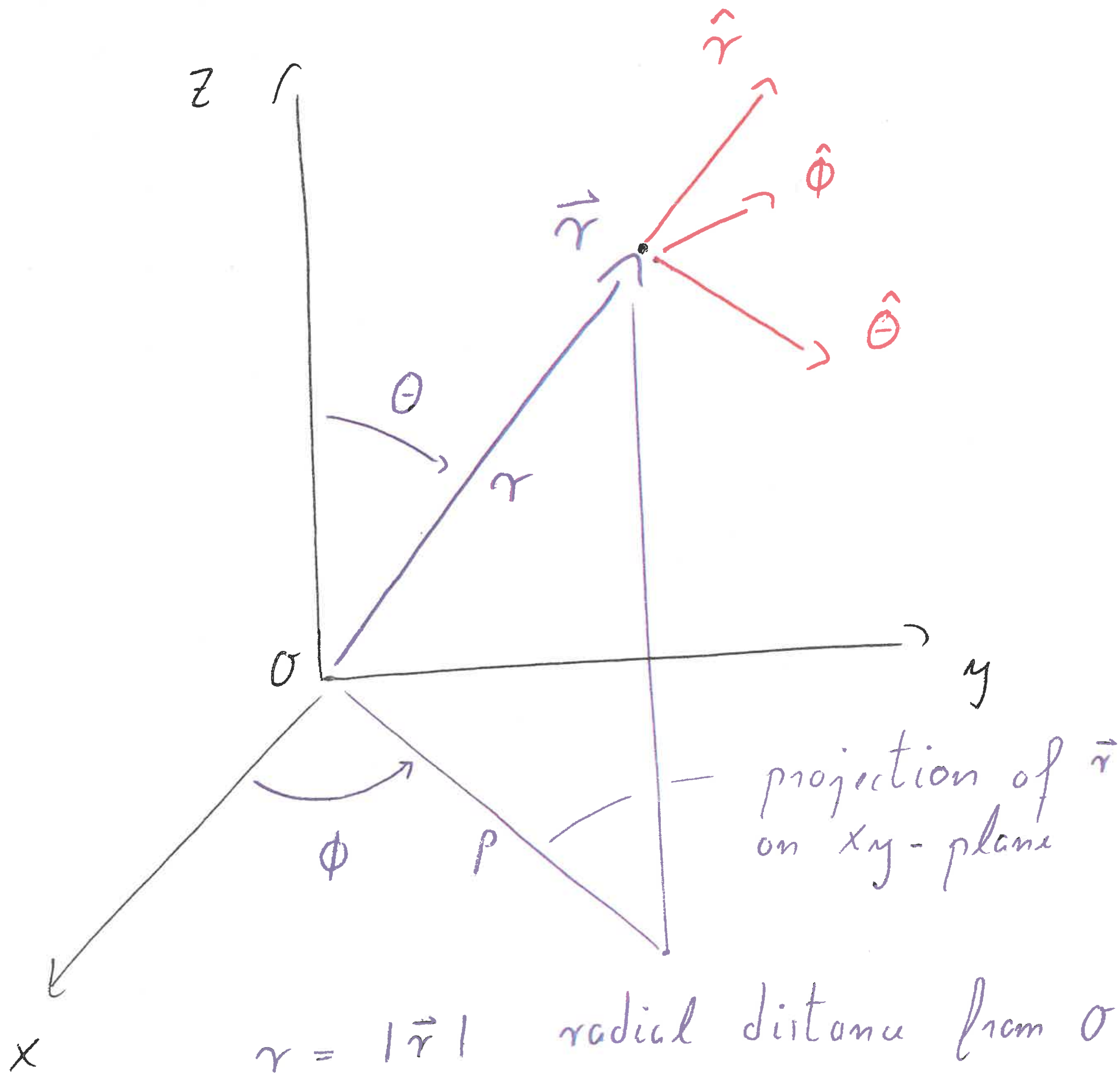
a) $$\frac{\partial}{\partial \vartheta}\hat{r} = \cos\vartheta \cos\phi\, \hat{x} + \cos\vartheta \sin\phi\, \hat{y} - \sin\vartheta\, \hat{z}$$
$$= \hat{\theta}$$

b) $$\frac{\partial}{\partial \phi}\hat{r} = -\sin\vartheta \sin\phi\, \hat{x} + \sin\vartheta \cos\phi\, \hat{y}$$
$$= \sin\vartheta\,(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y})$$
$$= \sin\vartheta\, \hat{\phi}$$

c) $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r})$$
$$= \dot{r}\hat{r} + r\frac{d}{dt}\hat{r}$$
$$= \dot{r}\hat{r} + r\left(\underbrace{\frac{\partial \hat{r}}{\partial \vartheta}}_{\hat{\theta},\ a)}\frac{d\vartheta}{dt} + \underbrace{\frac{\partial \hat{r}}{\partial \phi}}_{\sin\vartheta\, \hat{\phi},\ b)}\frac{d\phi}{dt}\right)$$
$$= \dot{r}\hat{r} + r\dot{\vartheta}\hat{\theta} + r\sin\vartheta\, \dot{\phi}\,\hat{\phi}$$

Spherical Polar Coordinates

cp Taylor, pages 134-136 $\qquad (r, \theta, \phi)$

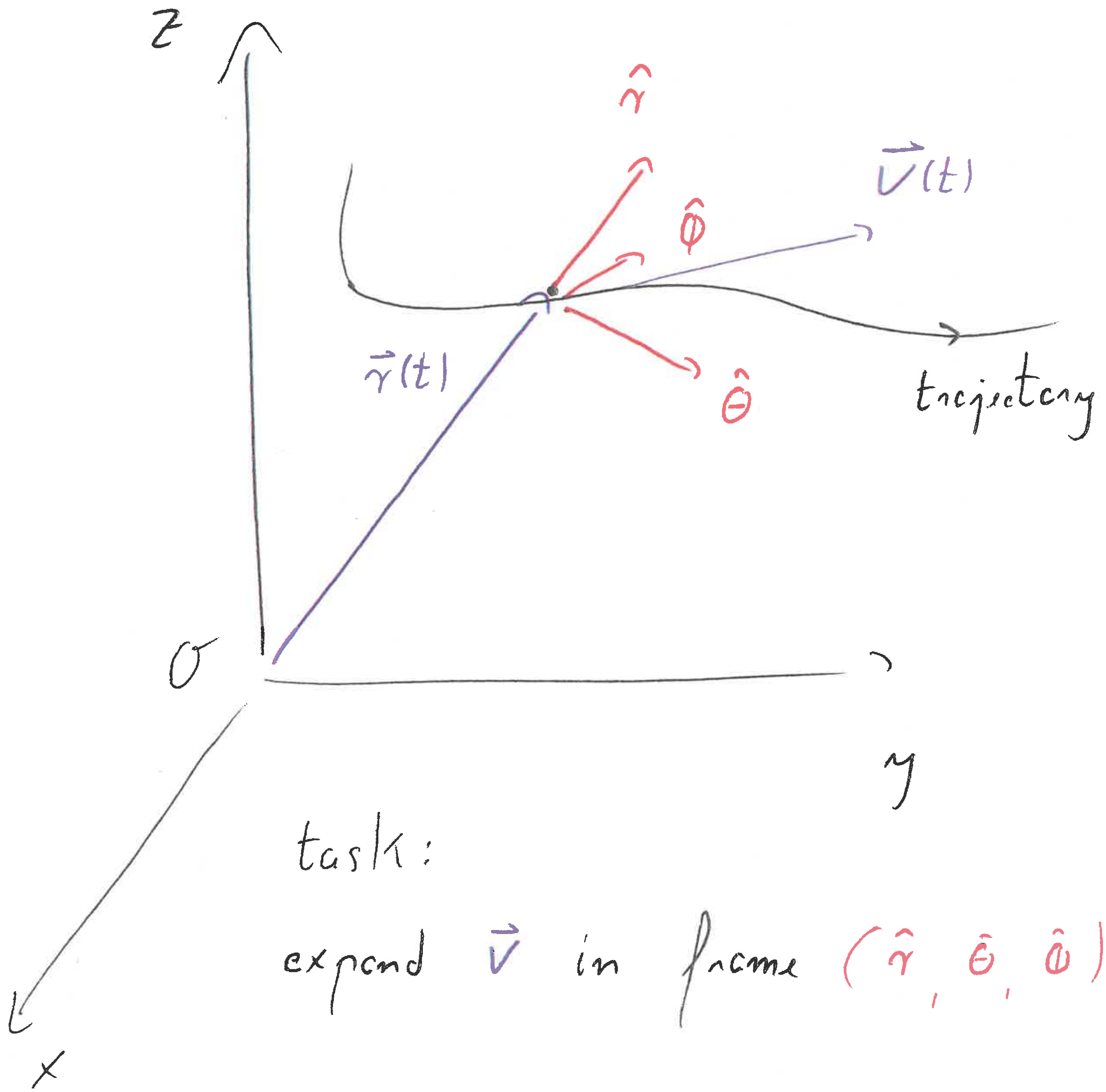


$r = |\vec{r}|$ radial distance from O

$\theta =$ polar angle w/ z-axis

$\phi =$ azimuthal angle

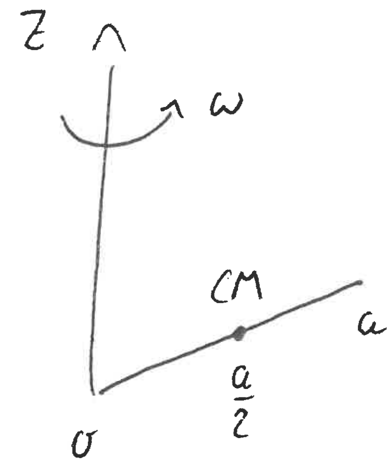
Illustration Problem 1c:



task:

expand $\vec{v}$ in frame $(\hat{r}, \hat{\theta}, \hat{\phi})$

Problem 2



PS 4, Problem 5:

$$I_{CM} = \frac{1}{12} M a^2$$

$$I_{end} = \frac{1}{3} M a^2$$

$$L_z \Big|_O = \omega I_{end} = \omega \frac{1}{3} M a^2$$

$$L_z^{(CM)} = \omega M \left(\frac{a}{2}\right)^2 = \omega \frac{1}{4} M a^2$$

$\hat{=}$ mass point $M \hat{=}$ CM spinning about O at radial distance $\frac{a}{2}$ from O

$$L_z' = \omega I_{CM} = \omega \frac{1}{12} M a^2$$

$\hat{=}$ rod spinning about CM = center of rod

$$\Rightarrow L_z^{(CM)} + L_z' = \omega \left(\frac{1}{4} M a^2 + \frac{1}{12} M a^2 \right)$$

$$= \omega \frac{1}{3} M a^2 = L_z \Big|_O$$

Problem 3

3.35 ** Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at angle γ to the horizontal. The instantaneous point of contact between the disk and the incline is called P. **(a)** Draw a free-body diagram, showing all forces on the disk. **(b)** Find the linear acceleration $\dot{v}$ of the disk by applying the result $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ for rotation about P. (Remember that $L = I\omega$ and the moment of inertia for rotation about a point on the circumference is $\frac{3}{2}MR^2$. The condition that the disk not slip is that $v = R\omega$ and hence $\dot{v} = R\dot{\omega}$.) **(c)** Derive the same result by applying $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ to the rotation about the CM. (In this case you will find there is an extra unknown, the force of friction. You can eliminate this by applying Newton's second law to the motion of the CM. The moment of inertia for rotation about the CM is $\frac{1}{2}MR^2$.)

see next page

Problem 3

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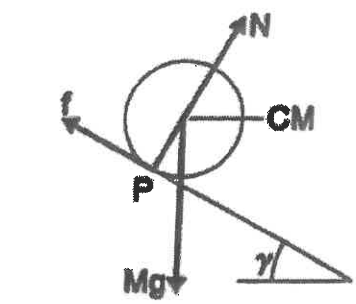
(b) The condition $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ applied about P becomes $I_P\dot{\omega} = MgR\sin\gamma$, whence $\dot{v} = \frac{2}{3}g\sin\gamma$.

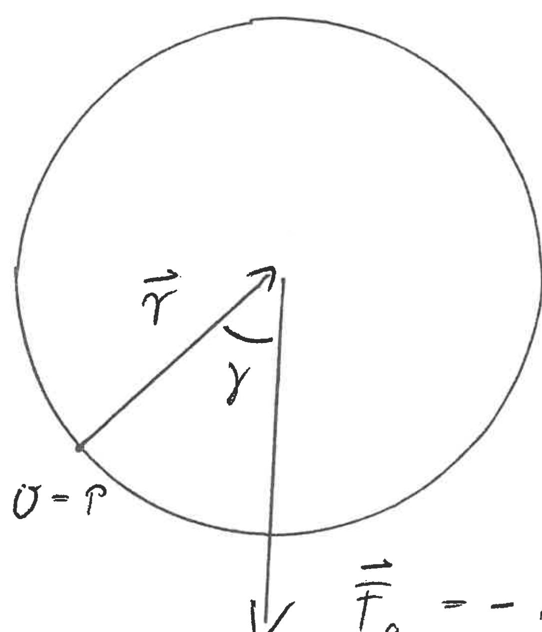
(c) The same condition applied about the CM gives $I_{\text{cm}}\dot{\omega} = fR$. To eliminate the unknown frictional force f, we must use Newton's second law, $M\dot{v} = Mg\sin\gamma - f$. Eliminating f, we get the same answer as before.

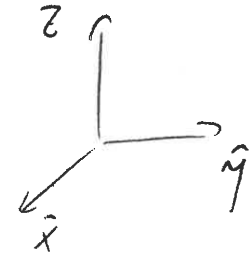
(a)



more explicitly:

b)





$\vec{r} = R\,\hat{r}$

$\vec{F}_g = -Mg\,\hat{z}$

$$\vec{F}^{ext} = \vec{F}_g = -Mg\,\hat{z}$$

$$\Rightarrow \vec{\Gamma}^{ext}\Big|_O = \vec{r}\times\vec{F}_g = -MgR\,\underbrace{\hat{r}\times\hat{z}}_{\sin(\gamma)\,\hat{x}}$$

out of page

$$\Rightarrow \left|\frac{d}{dt}\vec{L}\right| = I_P\,\dot{\omega} = MgR\sin(\gamma) \quad (1)$$

about $O = P$

$$I_P = \frac{3}{2} M R^2$$

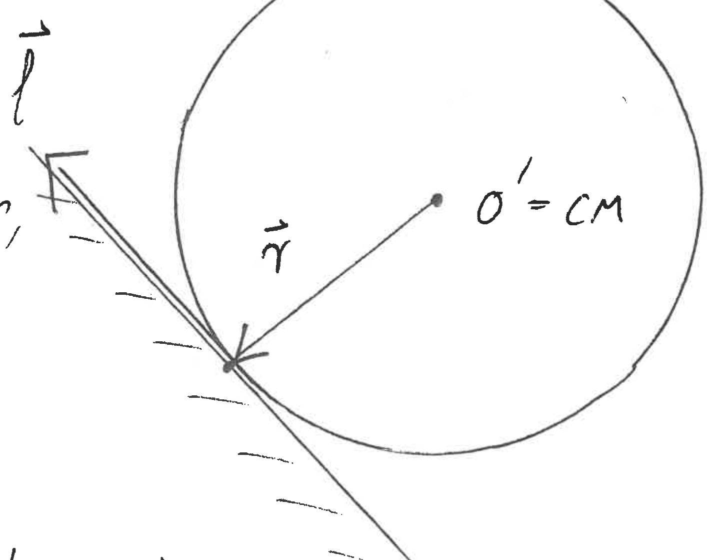
$$\dot{\omega} = \frac{d}{dt}\omega = \frac{d}{dt}\left(\frac{v}{R}\right) = \frac{1}{R}\dot{v}$$

$$(1) \Rightarrow \frac{3}{2}\cancel{M}\cancel{R^2}\frac{1}{\cancel{R}}\dot{v} = \cancel{M} g \cancel{R} \sin(\gamma)$$

$$\Rightarrow \underline{\dot{v} = \frac{2}{3} g \sin(\gamma)}$$

c)

static friction, keeps disk from slipping



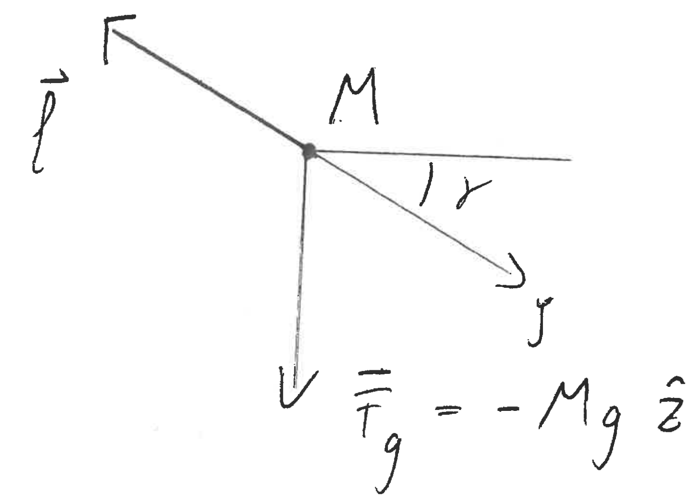
now: $\vec{F}^{ext} = \vec{f}$

$$\vec{\Gamma}^{ext}\Big|_{CM} = \vec{r} \times \vec{f} = -R f \hat{x}$$

$$\Rightarrow \left|\frac{d}{dt}\vec{L}\right| = I_{cm}\dot{\omega} = R f \qquad (2)$$

about $O' = CM$

how to eliminate f ?



Newton's 2nd law for motion of mass M in direction y:

$$F_y = M\dot{v} \,;$$

$$F_y = Mg\sin(\gamma) - f \,, \quad \text{see figure}$$

$$\overset{(2)}{=} Mg\sin(\gamma) - \underbrace{\frac{1}{R} I_{cm}\,\dot{\omega}}_{\frac{1}{R}\frac{1}{2}MR^2\frac{\dot{v}}{R} = \frac{1}{2}M\dot{v}}$$

$$\Rightarrow \cancel{M}\dot{v} = \cancel{M}g\sin(\gamma) - \frac{1}{2}\cancel{M}\dot{v} \quad \Big| + \frac{1}{2}\dot{v}$$

$$\frac{3}{2}\dot{v} = g\sin(\gamma) \Rightarrow \underline{\dot{v} = \frac{2}{3}g\sin(\gamma)}, \text{ as b)}$$

Problem 4

a) $$\vec{\nabla} f = \vec{\nabla}(x^2 + 2xy + xz^3)$$

$$= (2x + 2y + z^3)\,\hat{x} + 2x\,\hat{y} + 3xz^2\,\hat{z}$$

b) $$\vec{\nabla} r = \left(\frac{\partial}{\partial x} r\right)\hat{x} + \left(\frac{\partial}{\partial y} r\right)\hat{y} + \left(\frac{\partial}{\partial z} r\right)\hat{z}$$

$$\frac{\partial}{\partial x} r = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \cancel{\frac{1}{2}}\,\frac{1}{\sqrt{x^2 + y^2 + z^2}}\,\cancel{2}x$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

similarly: $$\frac{\partial}{\partial y} r = \frac{y}{r}, \quad \frac{\partial}{\partial z} r = \frac{z}{r}$$

$$\Rightarrow \vec{\nabla} r = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}$$

$$= \frac{1}{r}\underbrace{(x\hat{x} + y\hat{y} + z\hat{z})}_{\vec{r}}$$

$$= \frac{\vec{r}}{r} = \underline{\hat{r}}$$

c) $\vec{\nabla} g(r) = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$

$$\frac{\partial g}{\partial x} = \frac{dg}{dr}\frac{\partial r}{\partial x} = g'(r)\frac{\partial r}{\partial x}$$

similarly: $\frac{\partial g}{\partial y} = g'(r)\frac{\partial r}{\partial y}$, $\frac{\partial g}{\partial z} = g'(r)\frac{\partial r}{\partial z}$

$$\Rightarrow \vec{\nabla} g(r) = g'(r)\underbrace{\left(\frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z}\right)}_{\vec{\nabla} r,\ \text{as in b)}}$$

$$= g'(r)\vec{\nabla} r$$

$$= \underline{g'(r)\hat{r}}, \quad \text{using b)}$$

d) $\vec{\nabla}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\hat{r}$ using c) w/ $g(r) = \frac{1}{r}$

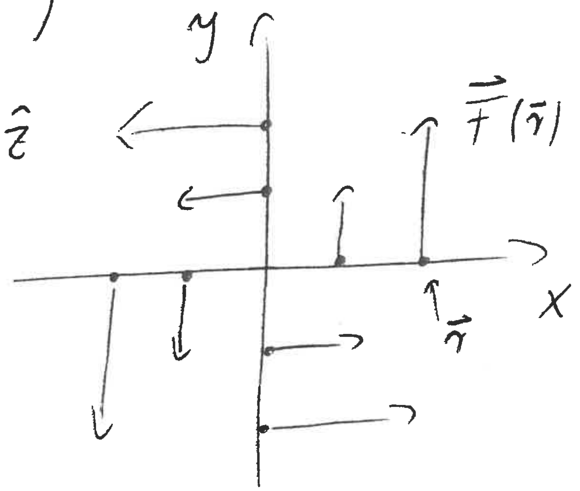
$$= \underline{-\frac{1}{r^2}\hat{r}}$$

e) $\frac{d}{dt} f[x(t), y(t), z(t)] = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt}$

$$= \sum_{i=1}^{3} \vec{\nabla} f\big|_i \frac{d\vec{r}}{dt}\Big|_i = \underline{\vec{\nabla} f \cdot \frac{d\vec{r}}{dt}}$$

Problem 5

a) $\vec{F}(x,y) = -y\hat{x} + x\hat{y}$



$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x\right)\hat{z}$$

$$= (1 - (-1))\hat{z}$$

$$= \underline{2\hat{z}}$$

b) $$\underbrace{\nabla \times \vec{r}\,\Big|_i}_{\text{component } i} = \sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} \underbrace{\vec{r}\,\big|_k}_{x_k}$$

$$= \sum_{j,k} \varepsilon_{ijk} \underbrace{\frac{\partial}{\partial x_j} x_k}_{= \delta_{jk} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}}$$

$$= \sum_{j=1}^{3} \underbrace{\varepsilon_{ijj}}_{=0} = 0$$

combine components $i = 1, 2, 3 \Rightarrow \underline{\nabla \times \vec{r} = 0}$

c) $\vec{\nabla} \times (\rho \vec{F})|_i$ fixed component i

$$= \sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} \underbrace{(\rho \vec{F})_k}_{\rho F_k}$$

$$= \sum_{j,k} \varepsilon_{ijk} \underbrace{\frac{\partial}{\partial x_j} (\rho F_k)}_{\left(\frac{\partial}{\partial x_j} \rho\right) F_k + \rho \frac{\partial}{\partial x_j} F_k}$$

normal product rule

$$= \sum_{j,k} \varepsilon_{ijk} \underbrace{\left(\frac{\partial}{\partial x_j} \rho\right)}_{\vec{\nabla} \rho|_j} F_k + \rho \underbrace{\sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k}_{\vec{\nabla} \times \vec{F}|_i}$$

$$= (\vec{\nabla} \rho) \times \vec{F}|_i + \rho\, \vec{\nabla} \times \vec{F}|_i \quad ; \quad i = 1, 2, 3$$

$$\Rightarrow \quad \vec{\nabla} \times (\rho \vec{F}) = (\vec{\nabla} \rho) \times \vec{F} + \rho\, \vec{\nabla} \times \vec{F}$$

d) $\vec{\nabla} \times (\vec{\nabla} \ell)|_i$ fixed component i

$$= \sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} \underbrace{\vec{\nabla}\ell|_k}_{\frac{\partial}{\partial x_k}\ell}$$

indices j, k switched

$$= \sum_{j,k} \varepsilon_{ijk} \frac{\partial^2 \ell}{\partial x_j \partial x_k}$$

$$= \sum_{\text{pairs } \langle jk \rangle} \left(\varepsilon_{ijk} \frac{\partial^2 \ell}{\partial x_j \partial x_k} + \underbrace{\varepsilon_{ikj}}_{= -\varepsilon_{ijk}} \frac{\partial^2 \ell}{\partial x_k \partial x_j} \right)$$

$$= \sum_{\text{pairs } \langle jk \rangle} \varepsilon_{ijk} \underbrace{\left(\frac{\partial^2 \ell}{\partial x_j \partial x_k} - \frac{\partial^2 \ell}{\partial x_k \partial x_j} \right)}_{=0}$$

$$= 0 \quad , \quad i = 1, 2, 3$$

$$\Rightarrow \underline{\vec{\nabla} \times (\vec{\nabla} \ell) = 0}$$ curl of gradient vanishes!