Problem Set 3 - Solutions

Problem 1

a) $\vec{F}_g = -mg\hat{y} = -mg(\sin\phi\,\hat{r} + \cos\phi\,\hat{\phi})$

$= -mg\sin\phi\,\hat{r} - mg\cos\phi\,\hat{\phi}$

$\vec{F}_n = -F_n\,\hat{r}$

$\Rightarrow \vec{F} = \vec{F}_g + \vec{F}_n = (-mg\sin\phi - F_n)\,\hat{r}$

$- mg\cos\phi\,\hat{\phi}$

$\Rightarrow F_r = -mg\sin\phi - F_n, \quad F_\phi = -mg\cos\phi \quad (1)$

b) Eq. (1.48) with $r = R = \text{const}, \ \dot{r} = \ddot{r} = 0$:

$F_r = -mR\dot{\phi}^2, \quad F_\phi = mR\ddot{\phi} \quad (2)$

c) component F_r of (1), (2)

$\Rightarrow -mg\sin\phi - F_n = \underbrace{-mR\dot{\phi}^2}_{\text{centripetal force}}$

If $F_n = 0$ at top of loop $(\theta = 90°)$:

$$\Rightarrow \quad -mg\underbrace{\sin(90°)}_{=1} - \underbrace{\cancel{F_n}}_{=0} = -mR\underbrace{\dot{\theta}^2}_{\left(\frac{v}{R}\right)^2} \quad (v = \dot{\theta}R)$$

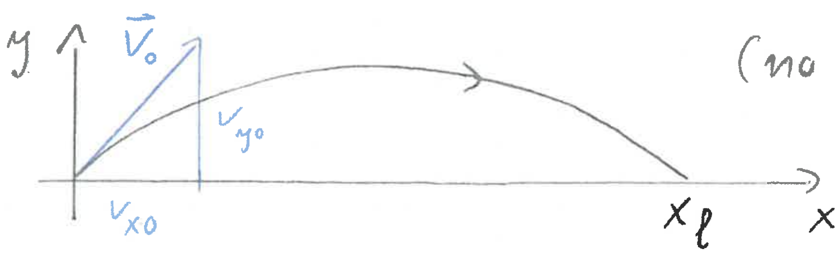
$$\Rightarrow \quad -\cancel{m}g = -\cancel{m}R\left(\frac{v}{R}\right)^2$$

$$\Rightarrow \quad v = v_{min} = \underline{\sqrt{gR}}$$

This is the minimum speed of the car to stay on the track b/c $\underline{F_n = 0}$ means that the car is <u>losing contact with the track</u>

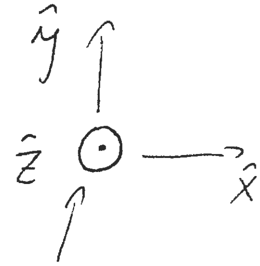
Problem 2 Range of projectile

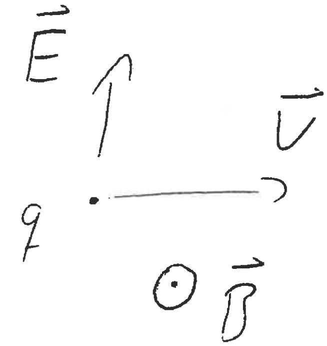
2.15 ★ Since the only force is the projectile's weight $m\mathbf{g}$, Newton's second law implies that $\ddot{\mathbf{r}} = \mathbf{g}$ and its two components can be integrated twice to give the well-known results $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$ (if we take $x_o = y_o = 0$). At landing, $y = 0$, which gives the time of flight as $t = 2v_{yo}/g$. The range is just the value of x at this time, namely $x_\ell = v_{xo}t = 2v_{xo}v_{yo}/g$.



(no air resistance)

Problem 3





= out of paper plane

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B} \quad , \quad \vec{E} = E\hat{y}$$

$$\vec{v}\times\vec{B} = (v\hat{x})\times(B\hat{z}) = vB\,\underbrace{\hat{x}\times\hat{z}}_{=-\hat{y},\ \text{see below}} = -vB\hat{y}$$

$$\Rightarrow \quad \vec{F} = q\,(\underbrace{E - vB}_{=0})\,\hat{y} \stackrel{!}{=} 0$$

$$\Rightarrow \quad \underline{v = \frac{E}{B}}$$

Application: velocity selector (Wien filter); google it!

products of unit vectors:

$\hat{x}\cdot\hat{x} = 1$, $\hat{x}\cdot\hat{y} = 0$, etc

in short: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} := \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$

for all $i, j = 1, 2, 3$

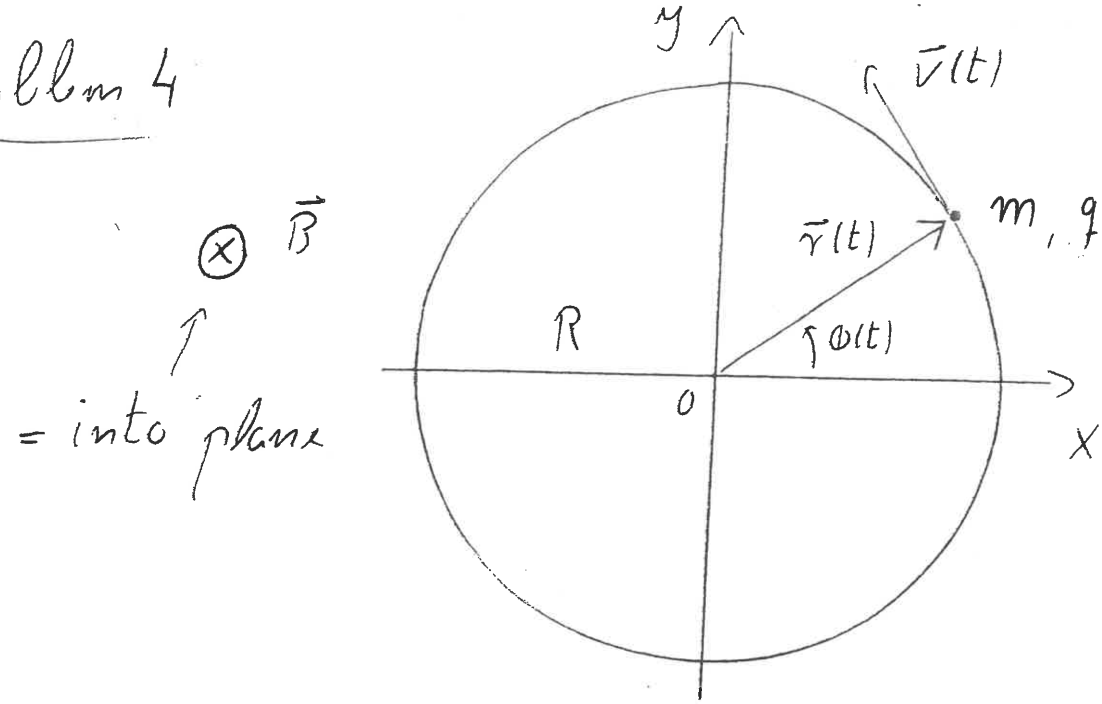
$\hat{x}\times\hat{y} = \hat{z}$, $\hat{y}\times\hat{z} = \hat{x}$,

$\hat{x}\times\hat{x} = 0$, etc.

in short:

$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\,\hat{e}_k$

Problem 4



a) $\hat{r} = \cos(\phi)\,\hat{x} + \sin(\phi)\,\hat{y}$

with $\phi = \phi(t) = \omega t$

$$\Rightarrow \frac{d}{dt}\hat{r} = \underbrace{[-\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}]}_{\hat{\phi}}\underbrace{\frac{d\phi}{dt}}_{\omega}$$

$$= \omega\,\hat{\phi} \qquad (1)$$

$$\frac{d}{dt}\hat{\phi} = \underbrace{[-\cos(\phi)\,\hat{x} - \sin(\phi)\,\hat{y}]}_{-\hat{r}}\underbrace{\frac{d\phi}{dt}}_{\omega}$$

$$= -\omega\,\hat{r} \qquad (2)$$

b) $\vec{r} = R\hat{r}$

$$\vec{v} = \frac{d}{dt}\vec{r} = R\frac{d}{dt}\hat{r} \overset{(1)}{=} R\omega\hat{\phi}$$

$$\vec{a} = \frac{d}{dt}\vec{v} = R\omega\frac{d}{dt}\hat{\phi} \overset{(2)}{=} -R\omega^2\hat{r} \qquad (3)$$

c) $$\hat{\phi}\times\hat{z} = [-\sin(\phi)\hat{x} + \cos(\phi)\hat{y}]\times\hat{z}$$

$$= -\sin(\phi)\underbrace{\hat{x}\times\hat{z}}_{-\hat{y}} + \cos(\phi)\underbrace{\hat{y}\times\hat{z}}_{\hat{x}}$$

$$= \cos(\phi)\hat{x} + \sin(\phi)\hat{y} = \hat{r} \qquad (4)$$

$$\Rightarrow \quad \vec{F} = q\,\vec{v}\times\vec{B} = q\,R\omega\,\hat{\phi}\times\underbrace{\vec{B}}_{= -B\hat{z}}$$

$$= -q\,R\omega B\,\underbrace{\hat{\phi}\times\hat{z}}_{=\hat{r}\text{, using (4)}}$$

$$= -q\,R\omega B\,\hat{r} \quad \text{= directed to } O$$

(5) (centripetal force)

d) $\bar{F} = m\bar{a}$

$$\Rightarrow \quad -qR\omega B\hat{r} = -mR\omega^2\hat{r} \quad \text{using (3), (5)}$$

$$\Rightarrow \quad q\cancel{R}\cancel{\omega}B = m\cancel{R}\omega^{\cancel{2}}$$

$$\Rightarrow \quad \omega = \frac{qB}{m}$$ cyclotron frequency, independent of R!