September 25, 2021
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Problem Set 4 – due Friday, October 1 by 12:00 PM midnight

The Problem Set has **5 questions** on **3 pages**, with a total maximum credit of **30 points**.
Please turn in well-organized, clearly written solutions (no scrap work). Questions 2 and 3 are taken from the textbook.

Problem 1) Two-dimensional harmonic oscillator: elliptical orbits [10 points]

Consider a two-dimensional harmonic oscillator with mass m and spring constant k in the xy - plane. We treat this system using Cartesian coordinates. The position vector of the mass is $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ and the harmonic restoring force on the mass is $\vec{\mathbf{F}} = -k\vec{\mathbf{r}}$.

a) Using Newton's 2nd law, $\vec{\mathbf{F}} = m\dfrac{d^2}{dt^2}\vec{\mathbf{r}}$, derive explicitly the following (independent) differential equations (DEs) for the components $x(t)$ and $y(t)$:

$$\ddot{x} = -\Omega^2 x \ , \quad \ddot{y} = -\Omega^2 y \ , \qquad (1)$$

where $\Omega = \sqrt{k/m}$.

Note: We here use the notation Ω instead of ω to avoid confusion, because Ω is different from the time-dependent angular velocity $\omega(t) = \dfrac{d}{dt}\phi(t)$ where $\phi(t)$ is the azimuthal angle of the elliptical orbit (see figure next page). $\omega = \dfrac{d}{dt}\phi = \sqrt{k/m} = \text{const.}$ only holds for the special case of a spherical orbit (with $a = b = R$ in figure next page, cp. PS 2, P5).

b) Show that the DEs (1) are solved by

$$x(t) = a\cos(\Omega t) \ , \quad y(t) = b\sin(\Omega t) \qquad (2)$$

where $a > 0$, $b > 0$ are constant amplitudes (without loss of generality we can assume $a \geq b$).

c) The components $x(t)$, $y(t)$ in Eq. (2) correspond to the following orbit in the xy - plane:

$$\vec{\mathbf{r}}(t) = a\cos(\Omega t)\hat{\mathbf{x}} + b\sin(\Omega t)\hat{\mathbf{y}} \quad . \tag{3}$$

Show that the orbit is an ellipse with long axis a and short axis b (see figure below).

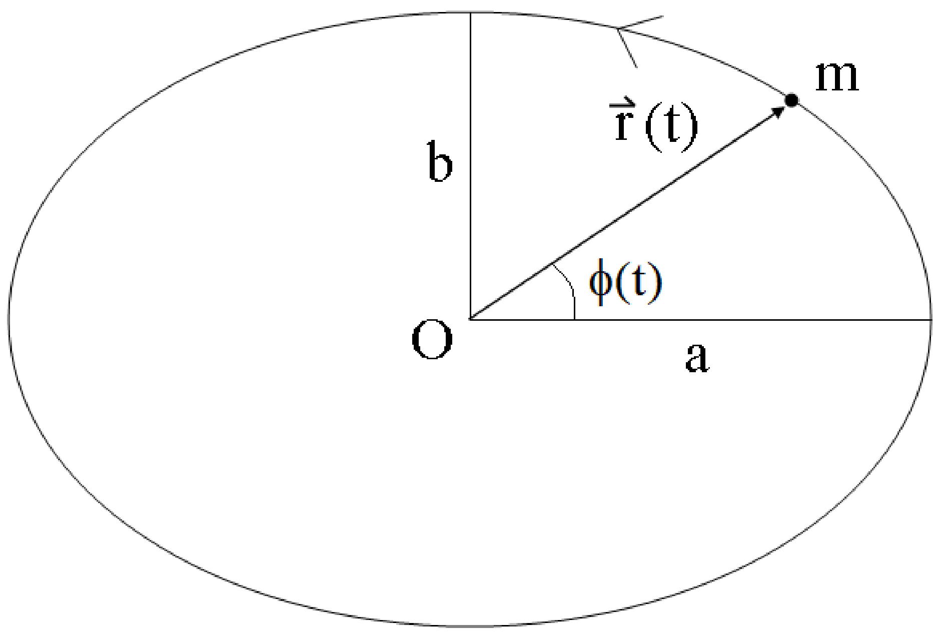
Hint: Show that $x(t)$, $y(t)$ fulfill the equation of an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

d) Find the velocity vector $\vec{\mathbf{v}}(t) = \frac{d}{dt}\vec{\mathbf{r}}(t) = v_x(t)\,\hat{\mathbf{x}} + v_y(t)\,\hat{\mathbf{y}}$, and the speed $v(t) = |\vec{\mathbf{v}}(t)|$.

Indicate in the figure at what points along the ellipse the speed v is largest and smallest.

e) Find the angular momentum $\vec{\ell} = m\vec{\mathbf{r}}(t) \times \vec{\mathbf{v}}(t)$. (Result: $\vec{\ell} = abm\Omega\,\hat{\mathbf{z}}$)

f) The result in e) implies that $\vec{\ell}$ is conserved. Why is this to be expected?



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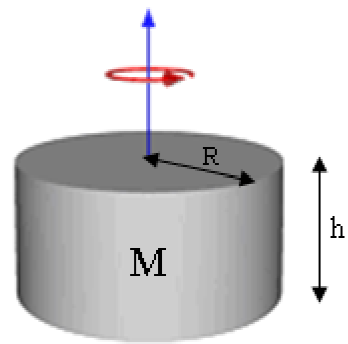
2) Taylor, Problem 3.25 (page 102) "A particle of mass m is moving ..." [5 points]

3) Taylor, Problem 3.30 (page 103) "Consider a rigid body rotating ..." [5 points]

Problem 4) Moment of inertia of a solid cylinder [6 points]



Find the moment of inertia I of a solid cylinder of radius R and mass M spinning around its center axis.

Instruction: Replace the expression in Eq. (3.31), $I = \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2$, by an integral using a uniform mass density $\mu_0 = \frac{M}{V}$ (similarly as we did before in Eq. (3.13)). Show:

$$I = \mu_0 \int_V \left[\rho(\vec{\mathbf{r}})\right]^2 dV \qquad (1)$$

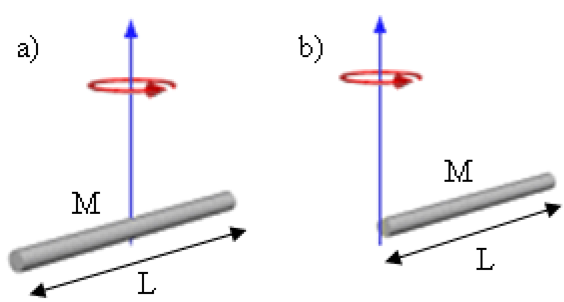
where $\rho(\vec{\mathbf{r}})$ is the radial distance from the rotation *axis* (not from the origin O!) of the point $\vec{\mathbf{r}}$ in the body. In cylindrical polar coordinates the volume element is given by $dV = d\rho\,\rho\,d\phi\,dz$ and Eq. (1) reads for the cylinder: $I = \mu_0 \int_0^R d\rho\,\rho \int_0^{2\pi} d\phi \int_0^h dz\,\rho^2 = \mu_0 2\pi h \int_0^R d\rho\,\rho^3$. Evaluate the integral and use $M = \mu_0 V = \mu_0 \pi R^2 h$. (Result: $I = \frac{1}{2} M R^2$)

Problem 5) Moment of inertia of a rod [4 points]

Find the moment of inertia I of a thin rod of length L and mass M

a) spinning around the center axis;

b) spinning around the end.



Hint: Put the rod on the x-axis and use $I = \mu_0 \int_{\text{rod}} \left[\rho(x)\right]^2 dx$ where $\rho(x)$ is the radial distance from the rotation axis of the point x along the rod. $\mu_0 = M/L$ is the mass *per length* of the rod.