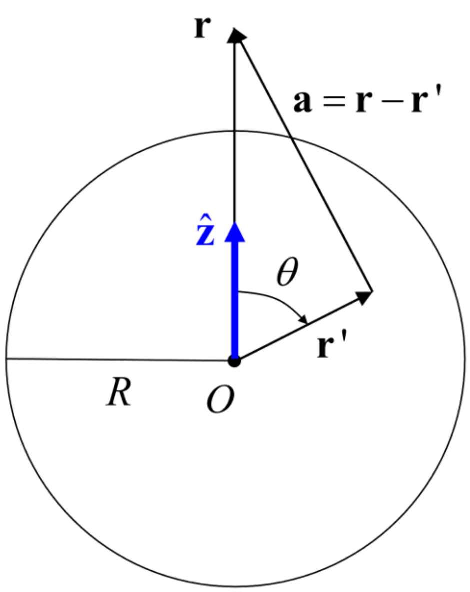
Problem Set 11 – Solutions 11/27/21

Problem 1) Gravitational potential of a sphere

Following the Instructions:

Since the sphere is homogeneous we can without restriction place the point $\mathbf{r}$ on the z - axis as shown in the figure.



Using spherical polar coordinates, we obtain

$$\begin{aligned} V(\mathbf{r}) &= -G\rho_0 \int_{\text{sphere}} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= -G\rho_0 \int_0^R dr' r'^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\phi \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= -G\rho_0 2\pi \int_0^R dr' r'^2 \int_0^\pi d\theta \sin(\theta) \frac{1}{a} \end{aligned} \tag{1}$$

where a is the magnitude of the vector $\mathbf{a}=\mathbf{r}-\mathbf{r}'$ shown in the figure:

$$a = |\mathbf{a}| = |\mathbf{r}-\mathbf{r}'| = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')} = \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2} = \sqrt{r^2 - 2rr'\cos(\theta) + r'^2}\ .$$

To calculate the integral in (1) we perform the variable substitution $\theta \to a$:

$$\frac{da}{d\theta} = \frac{d}{d\theta}\sqrt{r^2 - 2rr'\cos(\theta) + r'^2} = \frac{1}{2}\frac{1}{\sqrt{r^2 - 2rr'\cos(\theta) + r'^2}}2rr'\sin(\theta) = \frac{1}{a}rr'\sin(\theta)$$

$$\Rightarrow \; da = \frac{1}{a}rr'\sin(\theta)d\theta \; . \tag{2}$$

Substituting (2) in (1) gives for $r > R$

$$V(r) = -G\rho_0 2\pi \frac{1}{r}\int_0^R dr'r'\int_{r-r'}^{r+r'} da \; . \tag{3}$$

The integration limits for the integral over a follow because if θ varies from 0 to π then $a = |\mathbf{r} - \mathbf{r}'|$ varies from $r - r'$ to $r + r'$ if $r > R$. Calculating (3) gives, using $\rho_0 V = M$:

$$\begin{aligned} V(r) &= -G\rho_0 2\pi \frac{1}{r}\int_0^R dr'r'\underbrace{\left[r + r' - (r - r')\right]}_{2r'} \\ &= -G\rho_0 4\pi \frac{1}{r}\int_0^R dr'r'^2 \\ &= -G\rho_0 \underbrace{\frac{4\pi}{3}R^3}_{V}\frac{1}{r} \\ &= -G\frac{M}{r} \quad \text{for } r > R \; . \end{aligned}$$

Problem 2) Inertia tensor for mass points at the corners of a cube

Taylor, Chapter 10, Problem 10.22 (page 411) "A rigid body comprises 8 equal ... "

For part a), consider Taylor, Figure 10.5 (page 382), and consider a rotation of the cube about the z - axis through the origin O. For part b), consider a rotation of the cube parallel to the z - axis through the center of the cube.

10.22 ⋆⋆ **(a)** From (10.37), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\left(\sum y_\alpha^2 + \sum z_\alpha^2\right)$. In the first sum in the last expression, four of the points lie in the plane $y = 0$, while the other four have $y_\alpha = a$; thus this first sum is $4a^2$. The same applies to the second sum, and we conclude that $I_{xx} = 8ma^2$. The other two diagonal elements are clearly the same. Similarly, from (10.38), $I_{xy} = -m\sum x_\alpha y_\alpha$. In this sum, four of the points lie in the plane $x = 0$, and of the remaining four points, two lie in the plane $y = 0$. This leaves two points, both with $x_\alpha = y_\alpha = a$. Thus $I_{xy} = -2ma^2$. All the remaining off-diagonal elements are the same, and the inertia tensor is as shown on the left below.

$$\mathbf{I}(\text{part a}) = ma^2\begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{I}(\text{part b}) = ma^2\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) As in part (a), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\left(\sum y_\alpha^2 + \sum z_\alpha^2\right)$, but now all eight terms in both sums are the same and equal to $(a/2)^2$. Therefore $I_{xx} = 4ma^2 = I_{yy} = I_{zz}$. Because the body has reflection symmetry in all three coordinate planes, all of the off-diagonal elements are zero, and the inertia tensor is as shown above right.

Problem 3) Principal axis transformation of the inertia tensor

a) Find the eigenvalues λ_α and normalized eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$ of $I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

Find eigenvalues λ_α of I :

$$\det\left(I - \lambda\mathbf{1}\right) = \det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{pmatrix} \overset{!}{=} 0$$

$$(2-\lambda)^2(4-\lambda)+1+1-(2-\lambda)-(2-\lambda)-(4-\lambda)=0 \quad \text{combine numbers in blue}$$

$$(2-\lambda)^2(4-\lambda)-(2-\lambda)-(2-\lambda)-(2-\lambda)=0$$

$$(2-\lambda)^2(4-\lambda)-3(2-\lambda)=0$$

$$(2-\lambda)\left[(2-\lambda)(4-\lambda)-3\right]=0$$

$$(2-\lambda)\left(\lambda^2-6\lambda+5\right)=0$$

Use quadratic equation for $\lambda^2-6\lambda+5=0 \;\Rightarrow\; \lambda = \frac{1}{2}\left(6 \pm \sqrt{36-4\cdot 5}\right) = \frac{1}{2}(6\pm 4) = \begin{cases} 5 \\ 1 \end{cases}$

$$\Rightarrow\; \lambda^2-6\lambda+5 = (\lambda-5)(\lambda-1)$$

$$\Rightarrow\; \det\left(I-\lambda\mathbf{1}\right)=(2-\lambda)(\lambda-5)(\lambda-1) \overset{!}{=} 0$$

$\Rightarrow$ Eigenvalues (in increasing order): $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 5$

Find associated normalized eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$ of I :

i) Eigenvector $\hat{\mathbf{v}}^{(1)}$ for $\lambda_1 = 1$:

$$I\vec{\mathbf{v}} = \lambda_1\vec{\mathbf{v}} \;\Rightarrow\; \left(I-\lambda_1\mathbf{1}\right)\vec{\mathbf{v}} = 0 \;\Rightarrow\; \begin{pmatrix} 2-\lambda_1 & 1 & 1 \\ 1 & 2-\lambda_1 & 1 \\ 1 & 1 & 4-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

Using $\lambda_1 = 1 \;\Rightarrow\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \;\Rightarrow\; \begin{matrix} v_1+v_2+v_3=0 \\ v_1+v_2+v_3=0 \\ v_1+v_2+3v_3=0 \end{matrix}$ (first 2 equations are identical)

$$\Rightarrow\ \vec{\mathbf{v}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = c\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$ where c is a constant used to normalize $\hat{\mathbf{v}}^{(1)}$ so that $\left|\hat{\mathbf{v}}^{(1)}\right| = 1$

$$\Rightarrow\ c = \frac{1}{\sqrt{2}}\ \ \Rightarrow\ \hat{\mathbf{v}}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\ \text{ for } \lambda_1 = 1$$

ii) Use similar procedure to find normalized eigenvectors $\hat{\mathbf{v}}^{(2)}$ for $\lambda_2 = 2$ and $\hat{\mathbf{v}}^{(3)}$ for $\lambda_3 = 5$:

$$\Rightarrow\ \hat{\mathbf{v}}^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\ \text{ for } \lambda_2 = 2\ \ ,\qquad \hat{\mathbf{v}}^{(3)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\ \text{ for } \lambda_3 = 5$$

b) Show that different eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$ of I are orthogonal to each other.

To show: $\hat{\mathbf{v}}^{(\alpha)} \cdot \hat{\mathbf{v}}^{(\beta)} = \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$ for all $\alpha, \beta = 1, 2, 3$

E.g. $$\hat{\mathbf{v}}^{(1)} \cdot \hat{\mathbf{v}}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(+1-1+0\right) = 0$$ etc.

Also show that $\left(\hat{\mathbf{v}}^{(1)}, \hat{\mathbf{v}}^{(2)}, \hat{\mathbf{v}}^{(3)}\right)$ is a right-handed system (and not a left-handed system):

To show $\hat{\mathbf{v}}^{(1)} \times \hat{\mathbf{v}}^{(2)} = \hat{\mathbf{v}}^{(3)}$:

$$\hat{\mathbf{v}}^{(1)} \times \hat{\mathbf{v}}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \hat{\mathbf{v}}^{(3)}$$

c) Find the matrix R such that $I' \equiv R I R^T$ is a diagonal matrix.

$$R := \begin{pmatrix} - & \hat{\mathbf{v}}^{(1)} & - \\ - & \hat{\mathbf{v}}^{(2)} & - \\ - & \hat{\mathbf{v}}^{(3)} & - \end{pmatrix}$$ ($\hat{\mathbf{v}}^{(\alpha)}$ as line vectors) $$\Rightarrow\ R^T = \begin{pmatrix} | & | & | \\ \hat{\mathbf{v}}^{(1)} & \hat{\mathbf{v}}^{(2)} & \hat{\mathbf{v}}^{(3)} \\ | & | & | \end{pmatrix}$$ ($\hat{\mathbf{v}}^{(\alpha)}$ as column vectors)

$$\Rightarrow \; I R^T = I \begin{pmatrix} | & | & | \\ \hat{\mathbf{v}}^{(1)} & \hat{\mathbf{v}}^{(2)} & \hat{\mathbf{v}}^{(3)} \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \lambda_1 \hat{\mathbf{v}}^{(1)} & \lambda_2 \hat{\mathbf{v}}^{(2)} & \lambda_3 \hat{\mathbf{v}}^{(3)} \\ | & | & | \end{pmatrix}$$

using matrix multiplication rule and that $\hat{\mathbf{v}}^{(\alpha)}$ are eigenvectors of I with eigenvalues λ_α

$$\Rightarrow \; I' = R I R^T = \begin{pmatrix} - & \hat{\mathbf{v}}^{(1)} & - \\ - & \hat{\mathbf{v}}^{(2)} & - \\ - & \hat{\mathbf{v}}^{(3)} & - \end{pmatrix} \begin{pmatrix} | & | & | \\ \lambda_1 \hat{\mathbf{v}}^{(1)} & \lambda_2 \hat{\mathbf{v}}^{(2)} & \lambda_3 \hat{\mathbf{v}}^{(3)} \\ | & | & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ b/c } \hat{\mathbf{v}}^{(\alpha)} \cdot \hat{\mathbf{v}}^{(\beta)} = \delta_{\alpha\beta}$$

d) Show that the matrix R is orthogonal (i.e., R is a rotation matrix).

To show: $R R^T = \mathbf{1}$

$$R R^T = \begin{pmatrix} - & \hat{\mathbf{v}}^{(1)} & - \\ - & \hat{\mathbf{v}}^{(2)} & - \\ - & \hat{\mathbf{v}}^{(3)} & - \end{pmatrix} \begin{pmatrix} | & | & | \\ \hat{\mathbf{v}}^{(1)} & \hat{\mathbf{v}}^{(2)} & \hat{\mathbf{v}}^{(3)} \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} \text{ because } \hat{\mathbf{v}}^{(\alpha)} \cdot \hat{\mathbf{v}}^{(\beta)} = \delta_{\alpha\beta}$$

e) The diagonal matrix I' can be interpreted as the inertia tensor of the rigid body in a rotated coordinate system $C = \left[O, \{\hat{\mathbf{e}}'_\alpha\}\right]$. The basis vectors $\{\hat{\mathbf{e}}'_\alpha\}$ in which I' is diagonal correspond to the principal axes of inertia of the body. Find the relation between $\{\hat{\mathbf{e}}_\alpha\}$ and $\{\hat{\mathbf{e}}'_\alpha\}$.

Definition of rotation matrix R: $\hat{\mathbf{e}}'_\alpha = \sum_{\beta=1}^{3} R_{\alpha\beta} \hat{\mathbf{e}}_\beta \Rightarrow R_{\alpha\beta} = \hat{\mathbf{e}}'_\alpha \cdot \hat{\mathbf{e}}_\beta$ see lecture notes

On the other hand, definition of rotation matrix R in part c) with $\hat{\mathbf{v}}^{(\alpha)}$ as line vectors:

$\Rightarrow R_{\alpha\beta} = v_\beta^{(\alpha)} = \hat{\mathbf{v}}^{(\alpha)} \cdot \hat{\mathbf{e}}_\beta$ = component β of $\hat{\mathbf{v}}^{(\alpha)}$ in frame $\{\hat{\mathbf{e}}_\alpha\}$

$\Rightarrow \hat{\mathbf{v}}^{(\alpha)} = \hat{\mathbf{e}}'_\alpha = \sum_{\beta=1}^{3} R_{\alpha\beta} \hat{\mathbf{e}}_\beta$: rotated basis vectors $\hat{\mathbf{e}}'_\alpha$ are the eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$

= principal axes of I