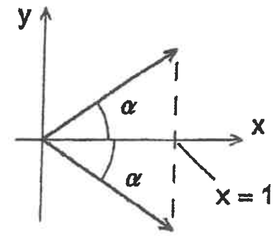
Problem Set 1 - Solutions

1.6 ⋆ $\mathbf{b}\cdot\mathbf{c} = 1 - s^2$, which is zero if and only if $s = \pm 1$. The vectors $\mathbf{b}$ and $\mathbf{c}$ make equal angles, α, above and below (or below and above) the x axis. The angle between them can be 90° only if $\alpha = 45°$.



1.8 ⋆ (a) Starting from the definition (1.7), we see that

$$\mathbf{r}\cdot(\mathbf{u}+\mathbf{v}) = \sum r_i(u_i + v_i) = \sum (r_i u_i + r_i v_i) = \sum r_i u_i + \sum r_i v_i = \mathbf{r}\cdot\mathbf{u} + \mathbf{r}\cdot\mathbf{v}$$

where the second equality follows from the distributive property of ordinary numbers. third equality is just a rearrangement of the six terms of the sum, and the last is just definition (1.7) of the two scalar products.

(b) Starting again from (1.7), we find

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d}{dt}\sum r_i s_i = \sum \left(r_i \frac{ds_i}{dt} + \frac{dr_i}{dt} s_i \right) = \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{s}.$$

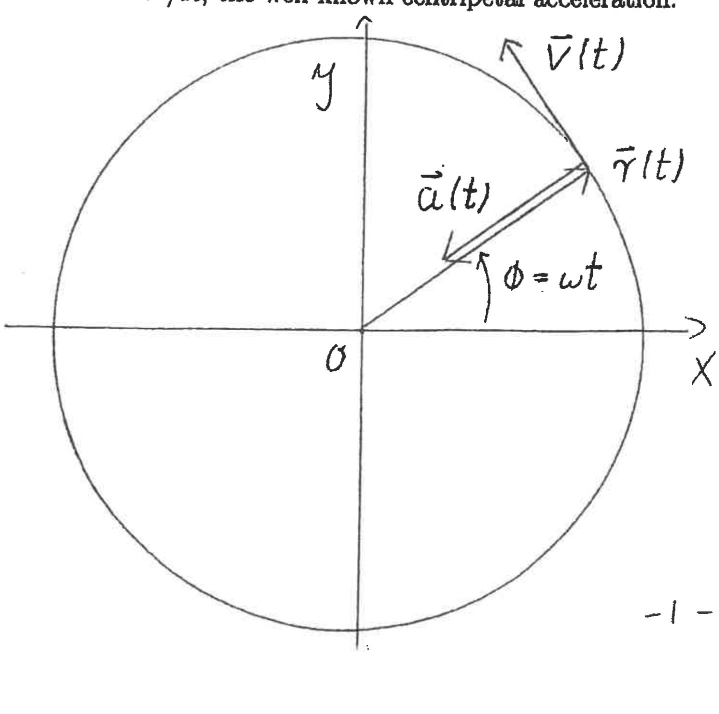
1.10 ⋆ The particle's polar angle is $\phi = \omega t$, so $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$ or

$$\mathbf{r} = \hat{\mathbf{x}}\, R\cos(\omega t) + \hat{\mathbf{y}}\, R\sin(\omega t).$$

Differentiating, we find that $\dot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega R\sin(\omega t) + \hat{\mathbf{y}}\,\omega R\cos(\omega t)$ and then (1) $\quad \dot{} \;\hat{=}\; \frac{d}{dt}$

$$\ddot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega^2 R\cos(\omega t) - \hat{\mathbf{y}}\,\omega^2 R\sin(\omega t) = -\omega^2\mathbf{r} = -\omega^2 R\hat{\mathbf{r}}. \quad (2)$$

That is, the acceleration is antiparallel to the radius vector and has magnitude $a = \omega^2 R = v^2/R$, the well known centripetal acceleration.



$$V = |\dot{\vec{r}}|$$

$$\overset{(1)}{=} \sqrt{(\omega R)^2 \underbrace{(\sin^2\omega t + \cos^2\omega t)}_{=1}}$$

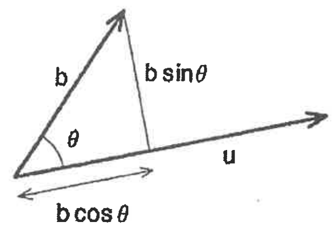
$$= \omega R \quad (3)$$

$$\Rightarrow a = |\ddot{\vec{r}}|$$

$$\overset{(2)}{=} \sqrt{(\omega^2 R)^2 (\cos^2\omega t + \sin^2\omega t)}$$

$$= \omega^2 R \overset{(3)}{=} \frac{V^2}{R}$$

1.13 ⋆ If θ is the angle between $\mathbf{b}$ and the unit vector $\mathbf{u}$, then $\mathbf{u}\cdot\mathbf{b} = b\cos\theta$ is the component of $\mathbf{b}$ in the direction of $\mathbf{u}$. Similarly, $|\mathbf{u}\times\mathbf{b}| = b\sin\theta$ is the component of $\mathbf{b}$ perpendicular to $\mathbf{u}$. Thus the result $b^2 = (\mathbf{u}\cdot\mathbf{b})^2 + (\mathbf{u}\times\mathbf{b})^2$ is simply a statement of Pythagoras' theorem.



or, following hint:

$$(\hat{u}\cdot\vec{b})^2 + (\hat{u}\times\vec{b})^2 = |\hat{u}\cdot\vec{b}|^2 + |\hat{u}\times\vec{b}|^2$$

$$= \underbrace{|\hat{u}|^2}_{1}\underbrace{|\vec{b}|^2}_{b^2}\cos^2\theta + |\hat{u}|^2|\vec{b}|^2\sin^2\theta$$

$$= b^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = b^2$$

1.17 ⋆⋆ (a) Let us start with the x component of $\mathbf{r}\times(\mathbf{u}+\mathbf{v})$. From the definition (1.9), we see that

$$[\mathbf{r}\times(\mathbf{u}+\mathbf{v})]_x = r_y(u_z+v_z)-r_z(u_y+v_y) = (r_yu_z-r_zu_y)+(r_yv_z-r_zv_y) = (\mathbf{r}\times\mathbf{u})_x+(\mathbf{r}\times\mathbf{v})_x .$$

Since the y and z components follow in the same way, we conclude that $\mathbf{r}\times(\mathbf{u}+\mathbf{v}) = \mathbf{r}\times\mathbf{u}+\mathbf{r}\times\mathbf{v}$.

(b) Starting again from (1.9), we find for the x component

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s})_x = \frac{d}{dt}(r_ys_z-r_zs_y) = \left(r_y\frac{ds_z}{dt}-r_z\frac{ds_y}{dt}\right)+\left(\frac{dr_y}{dt}s_z-\frac{dr_z}{dt}s_y\right) = \left(\mathbf{r}\times\frac{d\mathbf{s}}{dt}+\frac{d\mathbf{r}}{dt}\times\mathbf{s}\right)_x .$$

This is the x component of the desired identity. Since the y and z components follow in exactly the same way, our proof is complete.

or, following hint:

(b) $$\vec{r}\times\vec{s} = \sum_{ijk}\varepsilon_{ijk}\, r_j\, s_k\, \hat{e}_i$$

$$\Rightarrow \frac{d}{dt}(\vec{r}\times\vec{s}) = \sum_{ijk}\varepsilon_{ijk}(\dot{r}_j s_k + r_j\dot{s}_k)\,\hat{e}_i = \dot{\vec{r}}\times\vec{s} + \vec{r}\times\dot{\vec{s}}$$

$$\left(\dot{} = \frac{d}{dt}\right)$$

1.19 ★★ $\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot(\dot{\mathbf{v}}\times\mathbf{r} + \mathbf{v}\times\dot{\mathbf{r}})$.
The final term $\mathbf{a}\cdot(\mathbf{v}\times\dot{\mathbf{r}})$ is zero because $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{v}\times\mathbf{v} = 0$. The second to last term is $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{r}) = 0$, because $\mathbf{a}\times\mathbf{r}$ is perpendicular to $\mathbf{a}$, so their scalar product is zero. This leaves us with the requested identity.

or, following hint:

$$\bar{a}\cdot(\bar{v}\times\bar{r}) = \sum_{ijk} a_i v_j r_k$$

$$\Rightarrow \frac{d}{dt}[\bar{a}\cdot(\bar{v}\times\bar{r})] = \sum_{ijk}(\dot{a}_i v_j r_k + a_i \dot{v}_j r_k + a_i v_j \dot{r}_k)$$

$$= \dot{\bar{a}}\cdot(\bar{v}\times\bar{r}) + \underbrace{\bar{a}\cdot(\dot{\bar{v}}\times\bar{r})}_{\substack{=\bar{a}\cdot(\bar{a}\times\bar{r}) \\ =0 \text{ b/c } \bar{a}\times\bar{r}\perp\bar{a}}} + \underbrace{\bar{a}\cdot(\bar{v}\times\dot{\bar{r}})}_{=\bar{a}\cdot(\underbrace{\bar{v}\times\bar{v}}_{=0})=0}$$

$$= \underline{\dot{\bar{a}}\cdot(\bar{v}\times\bar{r})}$$

note: if $\bar{a} = \text{const}$, i.e., $\dot{\bar{a}} = 0$:

$$\Rightarrow \frac{d}{dt}[\bar{a}\cdot(\bar{v}\times\bar{r})] = 0$$

i.e., $\bar{a}\cdot(\bar{v}\times\bar{r}) = \text{const.}$
(constant of motion)