UT Rio Grande Valley October 9, 2021
PHYS-3305 Classical Mechanics Andreas Hanke

Problem Set 5 – due Friday, October 15 by 12:00 PM midnight

The Problem Set has **5 questions** on **4 pages**, with a total maximum credit of **30 points**. Please turn in well-organized, clearly written solutions (no scrap work). Questions 3 is taken from the textbook.

Problem 1) Velocity vector in spherical polar coordinates [6 points]

Read the section "Spherical Polar Coordinates" in Chapter 4, pages 134, 135 of the textbook, including Figure 4.17 on page 136. (You don't need to read the rest of the text on page 136.) From Figure 4.17 one may deduce the following expressions of the unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\phi}$ as functions of the angles θ, ϕ (make sure you understand this):

$$\hat{\mathbf{r}} = \sin(\theta)\cos(\phi)\,\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\,\hat{\mathbf{y}} + \cos(\theta)\,\hat{\mathbf{z}} \tag{1}$$

$$\hat{\theta} = \cos(\theta)\cos(\phi)\,\hat{\mathbf{x}} + \cos(\theta)\sin(\phi)\,\hat{\mathbf{y}} - \sin(\theta)\,\hat{\mathbf{z}} \tag{2}$$

$$\hat{\phi} = -\sin(\phi)\,\hat{\mathbf{x}} + \cos(\phi)\,\hat{\mathbf{y}} \tag{3}$$

a) Show: $\frac{\partial}{\partial\theta}\hat{\mathbf{r}} = \hat{\theta}$ (where $\frac{\partial}{\partial\theta}$ denotes the partial derivative with respect to θ at fixed ϕ)

b) Show: $\frac{\partial}{\partial\phi}\hat{\mathbf{r}} = \sin(\theta)\,\hat{\phi}$

c) Consider a general time-dependent position vector $\vec{\mathbf{r}}(t) = r(t)\,\hat{\mathbf{r}}(t)$.

Show: $\vec{\mathbf{v}} = \frac{d}{dt}\vec{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta} + r\sin(\theta)\dot{\phi}\,\hat{\phi}$.

Hint: Use the product rule. To find $\frac{d}{dt}\hat{\mathbf{r}}(t)$, consider $\hat{\mathbf{r}}(t) = \hat{\mathbf{r}}\left[\theta(t),\phi(t)\right]$ and use the chain rule for partial derivatives: $\frac{d\hat{\mathbf{r}}}{dt} = \frac{\partial\hat{\mathbf{r}}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{\mathbf{r}}}{\partial\phi}\frac{d\phi}{dt}$. Use a) and b) to express $\frac{\partial\hat{\mathbf{r}}}{\partial\theta}$, $\frac{\partial\hat{\mathbf{r}}}{\partial\phi}$ in terms of $\hat{\theta}$ and $\hat{\phi}$, respectively.

Problem 2) Angular momentum of a spinning rod: Separation of CM motion [6 points]

Consider a thin rod of length a and mass M spinning with angular speed ω about the end, as in Problem 5b of Problem Set 4 (here we use the letter a for the rod length to avoid confusion with the angular momentum L). Assume that the end of the rod coincides with the origin O of the lab frame, and take the z - axis as rotation axis. Then the z - component of the angular momentum, $\vec{\mathbf{L}}$, of the spinning rod w.r.t. O is $L_z = \omega I_{\text{end}}$ with $I_{\text{end}} = \frac{1}{3} M L^2$.

Show: $L_z = L_z^{(\text{CM})} + L_z^{'}$.

Here $L_z^{(\text{CM})}$ is the angular momentum of the CM of the rod w.r.t. O (corresponding to a fictitious mass M located at CM = midpoint of the rod), and $L_z^{'}$ is the angular momentum of the rod w.r.t. CM, as in Problem 5a of Problem Set 4 (notation as in class).

Problem 3) Solid disk rolling down an incline [6 points]

Textbook, Problem 3.35 (page 103) "Consider a uniform solid disk ..."

(continued next page)

Problem 4) Gradient of a scalar function [6 points]

For a scalar function $f(x,y,z)$ the *gradient* of f, denoted $\vec{\nabla} f$, is the vector defined as

$$\vec{\nabla} f := \frac{\partial f}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\,\hat{\mathbf{e}}_i \ . \tag{1}$$

In what follows, $\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = \sum_{i=1}^{3} x_i \hat{\mathbf{e}}_i$ is a vector with magnitude $r = |\vec{\mathbf{r}}| = \sqrt{x^2 + y^2 + z^2}$.

a) Find the gradient $\vec{\nabla} f$ of the scalar function $f(x,y,z) = x^2 + 2xy + xz^3$.

b) Show: $\vec{\nabla} r = \hat{\mathbf{r}}$.

c) For a general scalar function $g(r)$, show: $\vec{\nabla} g(r) = g'(r)\,\hat{\mathbf{r}}$ (where $g'(r) = \frac{dg}{dr}$)

Hint: The notation $g(r)$ means that the scalar function g depends on x, y, z only in terms of the magnitude $r = \sqrt{x^2 + y^2 + z^2}$. Use the product rule and part b).

d) Find $\vec{\nabla}\left(\frac{1}{r}\right)$.

Hint: Use the result of part c) with $g(r) = \frac{1}{r}$.

e) Let $\vec{\mathbf{r}}(t)$ be time-dependent. Show: $\frac{d}{dt} f(x,y,z) = \vec{\nabla} f \cdot \frac{d\vec{\mathbf{r}}}{dt}$.

Hint: Use the chain rule for partial derivatives, $\frac{d}{dt} f\left[x(t), y(t), z(t)\right] = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$,

and write the r.h.s. as scalar product of the vectors $\vec{\nabla} f$ and $\frac{d\vec{\mathbf{r}}}{dt}$.

Problem 5) Curl of a vector function [6 points]

For a vector function $\vec{\mathbf{F}}(x,y,z) = F_x(x,y,z)\,\hat{\mathbf{x}} + F_y(x,y,z)\,\hat{\mathbf{y}} + F_z(x,y,z)\,\hat{\mathbf{z}} = \sum_{i=1}^{3} F_i(x,y,z)\,\hat{\mathbf{e}}_i$ the *curl* of $\vec{\mathbf{F}}$, denoted $\vec{\nabla}\times\vec{\mathbf{F}}$, is the vector defined as

$$\begin{aligned}\vec{\nabla}\times\vec{\mathbf{F}} &:= \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{\mathbf{z}} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\,\frac{\partial}{\partial x_j}F_k\,\hat{\mathbf{e}}_i\end{aligned} \tag{1}$$

where ε_{ijk} is the Levi-Civita symbol (antisymmetric symbol) introduced earlier in class.

In what follows, $f(x,y,z)$ is a scalar function, $\vec{\mathbf{F}}(x,y,z)$ is a vector function,

and $\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = \sum_{i=1}^{3} x_i\,\hat{\mathbf{e}}_i$ is a vector with magnitude $r = |\vec{\mathbf{r}}| = \sqrt{x^2+y^2+z^2}$.

a) Sketch the vector function $\vec{\mathbf{F}}(x,y,z) := -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ in the xy - plane, and find the curl $\vec{\nabla}\times\vec{\mathbf{F}}$.

b) Find $\vec{\nabla}\times\vec{\mathbf{r}}$ (i.e., the curl of the vector function $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}$)

c) Show: $\vec{\nabla}\times\left(f\,\vec{\mathbf{F}}\right) = \left(\vec{\nabla}f\right)\times\vec{\mathbf{F}} + f\,\vec{\nabla}\times\vec{\mathbf{F}}$ where $\vec{\nabla}f$ is the gradient of f.

d) Show: $\vec{\nabla}\times\left(\vec{\nabla}f\right) = 0$ for any scalar function $f(x,y,z)$.

Hint: It is helpful for this type of calculations (saving many lines of writing) to use summation symbols $\sum$ to write sums, and to write the curl using ε_{ijk} as in the 2nd line of Eq. (1).