UT Rio Grande Valley November 20, 2021
PHYS-3305 Classical Mechanics Andreas Hanke

Problem Set 11 – due Friday, November 26 by 12:00 PM midnight

The Problem Set has **3 questions** on **3 pages** with a total maximum credit of **30 points**.

Please turn in well-organized, clearly written solutions (no scrap work).

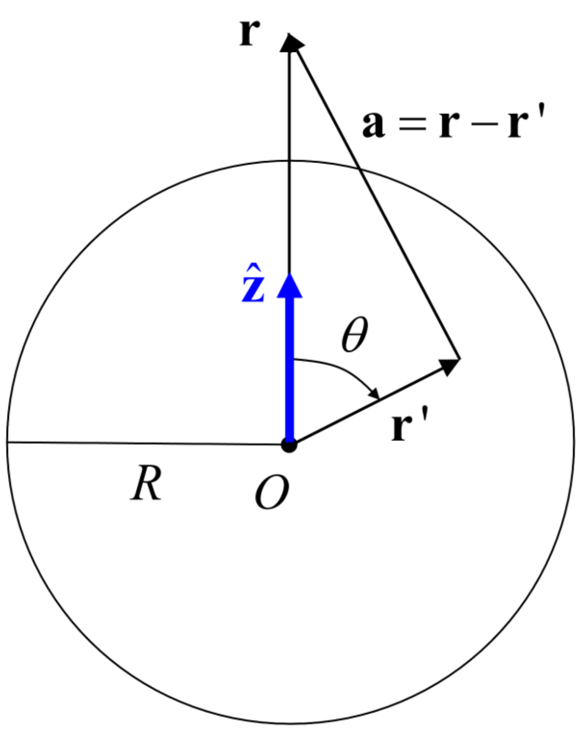
Problem 1) Gravitational potential of a sphere [10 points]

Consider a sphere with radius R and homogeneous mass distribution $\rho(\mathbf{r}')=\begin{cases}\rho_0\ , & r'<R\\ 0\ , & r'>R\end{cases}$

where $r'=|\mathbf{r}'|$ and $\rho_0=\dfrac{M}{V}$. Here M is the total mass of the sphere and $V=\dfrac{4\pi}{3}R^3$ the volume.

Find the gravitational potential $V(\mathbf{r})=-G\rho_0\int_{\text{sphere}} d^3r'\,\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$ for a point $\mathbf{r}$ <u>outside</u> of the sphere, i.e., $r>R$.

Instruction: Since the sphere is homogeneous, we can without restriction place the point $\mathbf{r}$ on the z - axis as shown in the figure.



Using spherical polar coordinates, we obtain

$$V(\mathbf{r}) = -G\rho_0 \int_0^R dr' r'^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\phi \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -G\rho_0 2\pi \int_0^R dr' r'^2 \int_0^\pi d\theta \sin(\theta) \frac{1}{a} \ . \qquad (1)$$

For the second equality we performed the ϕ - integration resulting in the factor 2π (using that the integrand is independent of ϕ) and a is the magnitude of the vector $\mathbf{a} = \mathbf{r} - \mathbf{r}'$ shown in the figure: $a = |\mathbf{a}| = |\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 - 2rr'\cos(\theta) + r'^2}$ (show this). To calculate the integral in (1) we perform the variable substitution $\theta \to a$. Calculate $\frac{da}{d\theta}$ for fixed r, r' to show that

$$da = \frac{1}{a} r r' \sin(\theta) d\theta \ . \qquad (2)$$

Use (2) in (1) to show that for $r > R$ (argue why the integration limits are correct):

$$V(r) = -G\rho_0 2\pi \frac{1}{r} \int_0^R dr' r' \int_{r-r'}^{r+r'} da \ . \qquad (3)$$

Complete the calculation to show that $V(r) = -G\frac{M}{r}$.

Problem 2) Inertia tensor for mass points at the corners of a cube [10 points]

Taylor, Chapter 10, Problem 10.22 (page 411) "A rigid body comprises 8 equal ... "

(continued next page)

Problem 3) Principal axis transformation of the inertia tensor [10 points]

Suppose that in a certain body-fixed coordinate system $C = \left[O, \{ \hat{\mathbf{e}}_i \} \right]$ the inertia tensor of a rigid body is given by the symmetric matrix

$$I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} .$$

a) Find the eigenvalues λ_α and normalized eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$ of I (i.e. $I\hat{\mathbf{v}}^{(\alpha)} = \lambda_\alpha \hat{\mathbf{v}}^{(\alpha)}$ and $\left|\hat{\mathbf{v}}^{(\alpha)}\right| = 1$ for $\alpha = 1,2,3$).

b) Show that different eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$ of I are orthogonal to each other.

c) Find the matrix R such that $I' \equiv R I R^T$ is a diagonal matrix.
Hint: The line vectors of R (and therefore the column vectors of R^T) are the eigenvectors $\hat{\mathbf{v}}^{(\alpha)}$.

d) Show that the matrix R is orthogonal (i.e., R is a rotation matrix).

e) The diagonal matrix I' can be interpreted as the inertia tensor of the rigid body in a rotated coordinate system $C = \left[O, \{ \hat{\mathbf{e}}'_i \} \right]$. The basis vectors $\{ \hat{\mathbf{e}}'_i \}$ in which I' is diagonal correspond to the principal axes of inertia of the body. Find the relation between $\{ \hat{\mathbf{e}}_i \}$ and $\{ \hat{\mathbf{e}}'_i \}$.