Vieta's Formulas

Vieta's Formulas, otherwise called Viète's Laws, are a set of equations relating the roots and the coefficients of polynomials.

Introduction

Vieta's Formulas were discovered by the French mathematician François Viète. Vieta's Formulas can be used to relate the sum and product of the roots of a polynomial to its coefficients. The simplest application of this is with quadratics. If we have a quadratic $x^2+ax+b=0$ with solutions p and q, then we know that we can factor it as:

$$x^2+ax+b=(x-p)(x-q)$$

(Note that the first term is x^2, not ax^2.) Using the distributive property to expand the right side we now have

$$x^2+ax+b=x^2-(p+q)x+pq$$

Vieta's Formulas are often used when finding the sum and products of the roots of a quadratic in the form ax^2+bx+c with roots r_1 and r_2. They state that:

$$r_1+r_2=-\frac{b}{a}$$

and

$$r_1\cdot r_2=\frac{c}{a}.$$

We know that two polynomials are equal if and only if their coefficients are equal, so $x^2+ax+b=x^2-(p+q)x+pq$ means that $a=-(p+q)$ and $b=pq$. In other words, the product of the roots is equal to the constant term, and the sum of the roots is the opposite of the coefficient of the x term.

A similar set of relations for cubics can be found by expanding $x^3+ax^2+bx+c=(x-p)(x-q)(x-r)$.

We can state Vieta's formulas more rigorously and generally. Let $P(x)$ be a polynomial of degree n, so $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where the coefficient of x^i is a_i and $a_n\neq 0$. As a consequence of the Fundamental Theorem of Algebra, we can also write $P(x)=a_n(x-r_1)(x-r_2)\cdots(x-r_n)$, where r_i are the roots of $P(x)$. We thus have that

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=a_n(x-r_1)(x-r_2)\cdots(x-r_n).$$

Expanding out the right-hand side gives us

$$a_nx^n-a_n(r_1+r_2+\cdots+r_n)x^{n-1}+a_n(r_1r_2+r_1r_3+\cdots+r_{n-1}r_n)x^{n-2}+\cdots+(-1)^na_nr_1r_2\cdots r_n.$$

The coefficient of x^k in this expression will be the $(n-k)$-th elementary symmetric sum of the r_i.

We now have two different expressions for $P(x)$. These must be equal. However, the only way for two polynomials to be equal for all values of x is for each of their corresponding coefficients to be equal. So, starting with the coefficient of x^n, we see that

$$\begin{aligned} a_n &= a_n \\ a_{n-1} &= -a_n(r_1+r_2+\cdots+r_n) \\ a_{n-2} &= a_n(r_1r_2+r_1r_3+\cdots+r_{n-1}r_n) \\ &\vdots \\ a_0 &= (-1)^na_nr_1r_2\cdots r_n \end{aligned}$$

More commonly, these are written with the roots on one side and the a_i on the other (this can be arrived at by dividing both sides of all the equations by a_n).

If we denote σ_k as the k-th elementary symmetric sum, then we can write those formulas more compactly as $\sigma_k=(-1)^k\cdot\frac{a_{n-k}}{a_n}$, for $1\le k\le n$. Also, $-b/a=p+q, c/a=p\cdot q$.

Proving Vieta's Formula

Basic proof: This has already been proved earlier, but I will explain it more. If we have $x^2 + ax + b = (x-p)(x-q)$, the roots are p and q. Now expanding the left side, we get: $x^2 + ax + b = x^2 - qx - px + pq$. Factor out an x on the right hand side and we get: $x^2 + ax + b = x^2 - x(p+q) + pq$ Looking at the two sides, we can quickly see that the coefficient a is equal to $-(p+q)$. $p+q$ is the actual sum of roots, however. Therefore, it makes sense that $p+q = \frac{-b}{a}$. The same proof can be given for $pq = \frac{c}{a}$.

Note: If you do not understand why we must divide by a, try rewriting the original equation as $ax^2 + bx + c = (x-p)(x-q)$ SuperJJ

General Form

For a polynomial of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ with roots $r_1, r_2, r_3, \ldots r_n$, Vieta's formulas state that:

$$\begin{aligned}
s_1 &= r_1 + r_2 + r_3 + \cdots + r_n &&= -\frac{a_{n-1}}{a_n} \\
s_2 &= r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-2}r_{n-1} &&= \frac{a_{n-2}}{a_n} \\
s_3 &= r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &&= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
s_n &= r_1r_2r_3\cdots r_n &&= (-1)^n \frac{a_0}{a_n}.
\end{aligned}$$

These formulas are widely used in competitions, and it is best to remember that when the n roots are taken in groups of k (i.e. $r_1 + r_2 + r_3 \ldots + r_n$ is taken in groups of 1 and $r_1 r_2 r_3 \ldots r_n$ is taken in groups of n), this is equivalent to $(-1)^k \frac{a_{n-k}}{a_n}$.

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