Groups Acting on Trees

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Contents

1 Introduction

The purpose of this essay is to investigate the structure of free groups, free products, and more generally, amalgamations of groups, and graphs of groups. We will use Bass-Serre theory to establish a geometric perspective: understanding these groups and others via actions on (simplicial) trees.

§2 introduces free groups and amalgams. Adopting an algebraic perspective, a structure theorem for amalgams is established: every element is expressible in reduced word form. This concretely captures the idea that an amalgam is the 'gluing' of groups along a common subgroup. One immediate application is that the free product of two finite groups contains a free subgroup of finite index. In favour of developing the geometric viewpoint in subsequent sections, proofs in this section are sketched or omitted.

§3 introduces graphs, trees, and graph actions. The geometric perspective of free groups and amalgams is explored by studying their tree actions, in particular understanding the orbits and stabilisers of the vertices and edges. We establish a correspondence between free groups and free actions on a tree, using the fact that trees are simply connected. We draw another correspondence between amalgams and tree actions without inversion with a segment as fundamental domain, by tracing a path between a segment and its image under action of a group element. We use the correspondences to show that all subgroups of free groups and certain conjugate-avoiding subgroups of amalgams are free.

§4 explores the structure of amalgams D_∞, $\mathrm{SL}_2(\mathbb{Z})$, and the trefoil knot group $\mathcal{T}$. We use Tits' ping-pong lemma to show that certain automorphism groups are free products, by establishing conditions that imply a nested inclusion of images upon action of the generators. These examples provide a concrete backdrop for the results established thus far regarding free products and amalgams.

§5 introduces direct limits and graphs of groups as a generalisation of amalgams. We introduce the HNN extension as the 'gluing' of two isomorphic subgroups of a group via conjugation by a new element. Building upon HNN extensions, we ultimately establish a correspondence between graphs of groups (G, Y) and tree actions $G \curvearrowright X$ without inversion. We also obtain a structure theorem for such groups, allowing us to view them as a generalisation of fundamental groups of graphs. The theory has several deep consequences: we prove Kurosh's theorem concerning certain subgroups of amalgams, show existence of an infinite, finitely generated simple group, and also show existence of a group with exactly 2 conjugacy classes.

§6 studies general automorphisms and automorphism groups of trees, and establishes geometric properties that enable us to show that a group cannot be a non-trivial amalgam. Crucially, we understand that a tree automorphism either fixes point(s) or acts via translation along a line. We then extend this structure to tree actions of nilpotent groups. The theory enables us to show that the following groups are not amalgams:

1. Triangle reflection groups W_{lmn} for $l, m, n \geq 2$.
2. $\mathrm{Aut}(\mathbb{F}_n)$ for $n \geq 3$.
3. $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, and its subgroups of finite index.

Finally, §7 provides a terse conclusion and suggests areas of further study.

2 Amalgams

In this section, we define free groups, free products, and more generally, an amalgamation of groups. We establish a structure theorem that allows us to understand and work concretely with elements of amalgams. We also lay out basic properties.

2.1 Free groups and amalgams

Definition 2.1. Group G is a **free group** if there exists a generating subset $S \subset G$ such that for any $n \in \mathbb{N}$, $s_1, \dots, s_n \in S, \epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ such that $(s_i, \epsilon_i) \neq (s_{i+1}, -\epsilon_{i+1})$ for all i,

$$s_1^{\epsilon_1} \dots s_n^{\epsilon_n} \neq 1.$$

Such an S (not necessarily unique) is called a **free basis** of G; write $G = \mathbb{F}_S$.

Theorem 2.2 (structure theorem for free groups)**.** *Let S be a set.*

(a) The free group $\mathbb{F}_S$ is the group G along with injection $S \hookrightarrow G$ with the following universal property: for all function $f : S \to H$ into a group H, there exists unique homomorphism $h : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[dr, dashed, "\exists! h"] & \\ S \arrow[ur, hook] \arrow[rr, "f"] & & H \end{tikzcd}$$

Furthermore, G is unique up to isomorphism.

*(b) Each element $g \in \mathbb{F}_S$ has a unique **reduced word**[1] expression:*

$$g = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$$

where $n \in \mathbb{N}$, $s_1, \dots, s_n \in S, \epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ such that $(s_i, \epsilon_i) \neq (s_{i+1}, -\epsilon_{i+1})$ for all i.

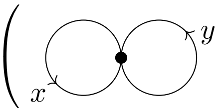
(c) $\mathbb{F}_S$ may be identified with the fundamental group of a bouquet of circles indexed by S.

Proof omitted. □

Intuitively, $G = \mathbb{F}_S$ is the largest possible group generated by elements of S; imposing additional relations would result in a quotient of $\mathbb{F}_S$. Therefore, every group is a quotient of a free group.

Example 2.3. The first two free groups are $\mathbb{F}_1 \cong \mathbb{Z}$ and

$$\mathbb{F}_2 = \mathbb{F}_{\{x,y\}} = \langle x, y \mid \ \rangle = \pi_1 \left(\quad \right).$$



More generally, we may define an amalgam of groups:

Definition 2.4. Let A and $\{G_i\}_{i \in I}$ be groups, and $\iota_i : A \hookrightarrow G_i$ injective group homomorphisms. The **amalgam** $G = *_A\{G_i\}$, accompanied with homomorphisms $f_i : G_i \to G$ agreeing on A, is the group G with the following universal property: for any group H and homomorphisms $h_i : G_i \to H$ agreeing on A, there exists unique homomorphism $h : G \to H$ such that the following diagram commutes for all $i \in I$:

[1]The reduced word of $g \in \mathbb{F}_S$ may be obtained by first writing it as a word in S, then exhaustively removing sub-words of the form ss^{-1} or $s^{-1}s$ for $s \in S$. The final word is independent of the order of 'reductions' performed.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, hook, "\iota_i"'] & & G \arrow[dr, dashed, "\exists! h"] & \\ & G_i \arrow[ur, "f_i"] \arrow[rr, "h_i"] & & H \end{tikzcd}$$

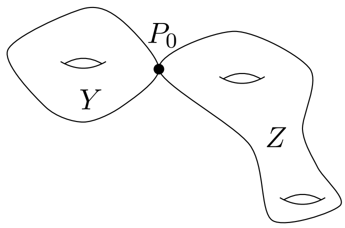
In the case of trivial A, we denote by $*\{G_i\}$ the **free product** of the groups G_i. In the case $|I| = 2$, we denote the amalgam by $G = G_1 *_A G_2$.

Lemma 2.5. *The amalgam in Definition 2.4 exists, and is unique up to unique isomorphism.*

Proof sketch. The amalgam $G = *_A\{G_i\}$ may be constructed as follows: consider the quotient of the free group on $\bigsqcup_i G_i$ by relations due to group operations of the G_i ($xyz^{-1} : x, y, z \in G_i$) and the ι_i ($xy^{-1} : y = \iota_i(x)$). Uniqueness is due to the universal property. $\square$

Example 2.6. The free group $G = \mathbb{F}_S$ is the free product of copies of $\mathbb{Z}$ indexed by S.

Example 2.7. Let Y, Z be path-connected subsets of a topological space X intersecting at a single point P_0.



Then the fundamental group of $Y \cup Z$ is a free product

$$\pi_1(Y \cup Z, P_0) = \pi_1(Y, P_0) * \pi_1(Z, P_0).$$

One may observe similarities between the universal property of the free product and that of the direct product. We will see in §5 that free products and amalgams are a special case of the direct limit[2] of a collection of a groups and homomorphisms, just as direct products are a special case of the inverse limit.

2.2 The structure theorem

Similar to Theorem 2.2(b), we have a structure theorem for the amalgam of groups:

Theorem 2.8 (structure theorem for amalgams)**.** *Let $G = *_A\{G_i\}$ be an amalagam of groups G_i indexed by I. For each $i \in I$, let S_i be a choice of right coset representatives of $A\backslash G_i$, with $1 \in S_i$. Then the maps $f_i : G_i \to G$ are injective, and each $g \in G$ has a unique **reduced word** expression:*

$$g = as_1s_2 \dots s_n$$

for $n \in \mathbb{N}$, $a \in A$, $i_1, \dots, i_n \in I$ with $i_m \neq i_{m+1}$ for all m, and $s_m \in S_{i_m} \setminus \{1\}$ for all m.

Proof sketch. Let G act on the set of reduced words as follows:

1. Let X be the set of reduced words $(a; s_1, \dots, s_n)$ for $n \in \mathbb{N}$, $a \in A$, $i_1, \dots, i_n \in I$ with $i_m \neq i_{m+1}$ for all m, and $s_m \in S_{i_m} \setminus \{1\}$ for all m.

[2]Limits/colimits are notions in category theory, applicable to other abstract structures such as rings, modules, and vector spaces. The direct limit is a special case of the colimit.

2. Define actions $G_i \curvearrowright X$ by left multiplication, and note the common agreement on $A \curvearrowright X$.
3. By the universal property of amalgams, this defines action $G \curvearrowright X$.
4. The action defines a map $\alpha : G \to X$ defined by $g \mapsto g \cdot e$, where e is the empty word. One may verify that this map is the two-sided inverse of the natural map $\beta : X \to G$.
5. The maps α and β identify each $g \in G$ with its unique reduced word.

The uniqueness of reduced word expressions also show the maps f_i are injective. □

Remark 2.9. Even though the structure theorem involves choice of coset representatives, one may alternatively formulate it as a representative-free result, e.g. by defining reduced words as quotients of the action of A^{n-1} on $G_{i_1} \times \ldots G_{i_n}$ via $(a_1, \ldots, a_{n-1}) \cdot (g_1, \ldots, g_n) = (g_1 a_1^{-1}, a_1 g_2 a_2^{-1}, \ldots, a_{n-1} g_n)$. It follows that each $g \in G$ has a unique **type** $i = (i_1, \ldots, i_n)$ and **length** $l(g) = n \in \mathbb{N}$ with respect to the amalgam.

The importance of the structure theorem is that it confirms our geometric intuition of what it means to take a free product or amalgam: to 'glue' the constituent groups G_i together via common subgroup A. We not only retain the presence of each group G_i in the amalgam, but also understand the exact extent to which different G_i's interact.

In §5, we will apply the structure theorem to show that certain groups defined by generators and relations are non-trivial, e.g. by showing that they may be obtained from successive amalgams. We will also define the HNN extension, which intuitively amounts to 'gluing' two isomorphic subgroups of a single group G together, via conjugation by a new element outside the group.

2.3 Auxiliary results

There is the notion of rank of a free group:

Proposition 2.10. *If $G = \mathbb{F}_S$ is a free group, its **rank** $r(G) = |S|$ is well-defined, and is equal to the minimal number of generators of G.*

Proof sketch. Upon abelianisation and tensoring with $\mathbb{Q}$, the result follows from the uniqueness of dimension of vector spaces. □

The following result shows that 'most' elements of the amalgam are of infinite order:

Lemma 2.11. *Let $G = *_A\{G_i\}$ be an amalgam, and $g \in G$ an element of type $i = (i_1, \ldots, i_{l(g)})$.*

(a) $l(g) = 0 \iff g \in A$.

(b) $l(g) \leq 1 \iff g \in G_i$ for some $i \in I$.

*(c) Suppose $i_1 \neq i_{l(g)}$; say that g is **cyclically reduced**.*

(i) $l(g^m) = |m| \cdot l(g)$ for all $m \in \mathbb{Z}$.

(ii) g has infinite order.

(d) Each element is conjugate to a cyclically reduced element or an element of some G_i.

(e) Every element $g \in G$ of finite order is contained in a conjugate of some G_i.

Proof sketch. (a)–(d) follow from the structure theorem. (e) follows from (c) and (d). □

The amalgam preserves certain properties of its constituent groups:

Lemma 2.12. *Let $G = *_A\{G_i\}$ be an amalgam.*

(a) If the G_i are countable, then so is G.

(b) If the G_i are torsion-free, then so is G.

(c) If $B \leq A$ and $H_i \leq G_i$ are subgroups such that $B = H_i \cap A$ for each i, then $*_B\{H_i\}$ injects naturally into $*_A\{G_i\}$.

Proof sketch. (a) follows from the structure theorem. (b) follows from Lemma 2.11(e). (c) follows from the structure theorem by extending coset representatives of $B\backslash H_i$ to $A\backslash G_i$. $\square$

In the special case where the amalgamated subgroup commutes with the constituent groups, the amalgam descends to a free product of the quotient groups:

Lemma 2.13. *Let $G = *_A\{G_i\}$ be an amalgam where A commutes with all G_i. Then*

$$G/A \cong (*\{G_i/A\}) \,.$$

Proof sketch. The result follows from applying the structure theorem to $*_A\{G_i\}$ and $*\{G_i/A\}$. $\square$

There is also a result concerning the 'commutator subgroup' of free products:

Proposition 2.14.

(a) Let A, B be groups. The kernel of projection $A * B \to A \times B$ is freely generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ for $a \in A \setminus \{1\}, b \in B \setminus \{1\}$.

(b) The free product of two finite groups contains a free subgroup of finite index.

Proof sketch. (a) follows from the structure theorem by inductively writing $g = [a_1, b_1]^{\epsilon_1} \ldots [a_n, b_n]^{\epsilon_n}$ in reduced word form; for instance, if $\epsilon_1 = \epsilon_n = 1$, then $g = a_1^{-1}b_1^{-1} \ldots a_n b_n$ and has length $\geq n+3$.

(b) follows from (a). $\square$

3 Graphs and Trees

In this section, we define graphs and trees; the geometric perspective is emphasised. We introduce the notion of a group action on a graph. We establish a correspondence between amalgams and group actions on trees with a segment as fundamental domain.

3.1 Graphs and geometry

Definition 3.1. A **graph** $X = (V, E)$ consists of a vertex set V and edge multiset E of vertex-pairs (possibly including self-loops). Its **realisation** $\mathrm{real}(X)$ is the associated topological space formed by an appropriate quotient of $E \times [0, 1]$: each edge is a unit interval, and endpoints corresponding to the same vertex are identified.

A graph is **simple** if the edge set contains no self-loops, and each vertex pair occurs at most once as an edge.

A **tree** is a connected simple graph $X = (V, E)$ with no cycles.

Throughout this essay, we will use other standard graph theory terminology such as subgraph, walk, path, cycle, connectedness, degree, morphism, and automorphism. We will also consider graphs with additional information, such as edge orientation, vertex/edge labellings, or colourings.

It is useful to reconcile the discrete picture of a graph $X = (V, E)$ with its geometric realisation $\mathrm{real}(X)$. This is summarised in the following topology-inspired definitions and the subsequent lemma:

Definition 3.2. Let $X = (V, E)$ be a graph, and $P_0 \in V$ a vertex. The **fundamental group** of X with respect to **base point** P_0, denoted $\pi_1(X, P_0)$, is the group formed by closed walks starting and ending at P_0 without backtracking, with the group operation being walk concatenation. The **universal covering tree** $\tilde{X}$ is the simple graph with vertex set $\tilde{V}$ consisting of walks starting at P_0 without backtracking, with adjacency between any pair of walks differing by a single edge.

Remark 3.3. One may show that $\pi_1(X, P_0) \cong \pi_1(X, Q_0)$ for any two base points P_0 and Q_0, via a non-canonical isomorphism (conjugation by a walk from P_0 to Q_0). It is thus possible to refer to the fundamental group abstractly as $\pi_1(X)$. Similarly, $\tilde{X}$ is independent of the base point P_0.

Lemma 3.4. *Let $X = (V, E)$ and X' be connected graphs.*

(a) $\mathrm{real}(X)$ *is compact* $\iff$ X *is finite.*

(b) $\mathrm{real}(X)$ *is locally compact* $\iff$ X *is locally finite.*

(c) Suppose X is locally finite. Then X is infinite $\iff$ *X contains an infinite path.*

(d) $\mathrm{real}(X)$ *is a metric space: the distance between two points is the minimum length of a path between the points, where each edge is of unit length and inherits the metric from* $[0, 1]$.

(e) $\mathrm{real}(X)$ *is a (path-connected) geodesic metric space.*

(f) $\pi_1(X) \cong \pi_1(\mathrm{real}(X))$ *and is a free group.*

(g) $\tilde{X}$ *is a tree.*

(h) $\mathrm{real}(\tilde{X})$ *is the (geometric) universal cover of* $\mathrm{real}(X)$.

(i) A morphism $s : X \to X'$ lifts to a continuous map $\hat{s} : \mathrm{real}(X) \to \mathrm{real}(X')$.

(j) An automorphism $s \in \mathrm{Aut}(X)$ lifts to a unique isometry $s \in \mathrm{Aut}(\mathrm{real}(X))$.

(k) *X admits a maximal subtree Y containing all vertices of X.*

(l) *If $\Gamma = \{\Gamma_i\}$ is a collection of disjoint subtrees of X, one may define the* ***contraction*** *X/Γ to be the graph obtained by contracting each Γ_i to a single vertex. Then*

   (i) $\mathrm{real}(X/\Gamma)$ *is the quotient space of* $\mathrm{real}(X)$ *by identifying each subspace* $\mathrm{real}(\Gamma_i)$ *to a point.*

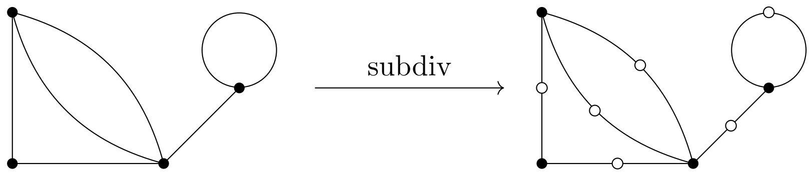
   (ii) *The canonical projection* $\mathrm{real}(X) \to \mathrm{real}(X/\Gamma)$ *is a homotopy equivalence.*

(m) $\mathrm{real}(X)$ *is homotopically equivalent to a bouquet of circles.*

Proof sketch. Proofs of (a)–(j) are omitted. (k) is a consequence of Zorn's lemma, or may be constructed via an inverse system (see Lemma 3.9(c)). The intuition for (l) is that contraction of subtrees do not remove or add cycles. (m) follows from contraction of a maximal subtree. □

Finally, we make an observation that allows one to convert general graphs to simple graphs while preserving the geometry:

Definition 3.5. Let $X = (V, E)$ be a graph. Its **subdivision** $X' = \mathrm{subdiv}(X)$ is the graph with vertex set $V' = V \sqcup E$ and edge set E' consisting of vertex-edge pairs of X where the vertex is an endpoint of the edge.



Lemma 3.6. *Let $X = (V, E)$ be a graph, and $X' = \mathrm{subdiv}(X)$.*

(a) *X' is bipartite; in particular, it has no self-loops.*

(b) $\mathrm{real}(X')$ *is geometrically equivalent to* $\mathrm{real}(X)$*, up to a metric scale factor of* 2.

(c) $X'' = \mathrm{subdiv}(X')$ *is simple.*

Proof sketch. The subdivision may be viewed as adding a new vertex at the midpoint of every existing edge. □

3.2 Trees

In this subsection, we establish properties about trees. We begin with a geometric characterisation:

Lemma 3.7. *Let $X = (V, E)$ be a connected graph. The following are equivalent:*

(a) *X is a tree.*

(b) $\tilde{X} = X$.

(c) *Every pair of vertices $P, Q \in V$ are joined by a unique path P—Q (a* ***geodesic****) in X.*

(d) *Every pair of points in* $\mathrm{real}(X)$ *is joined by a unique injective path.*

(e) $\pi_1(X)$ *is trivial.*

(f) $\mathrm{real}(X)$ *is contractible.*

Proof omitted. ☐

We also establish results about subtrees:

Lemma 3.8. *Let X be a tree.*

(a) Connected subgraphs of X are subtrees.

(b) $Y \subset X$ is a subtree $\iff$ for any $P, Q \in V(Y)$, the geodesic P—Q is contained in Y.

(c) If $Y \subset X$ is a subgraph or subset of vertices, the ***span*** *of Y is the union of all geodesics P—Q for $P, Q \in V(Y)$. It is a subtree of X, and of equal diameter to Y.*

(d) The intersection of any family of subtrees is empty or a subtree.

(e) If a finite family of subtrees is such that any pair has non-empty intersection, then the entire family has non-empty intersection.

(f) If Y, Z are disjoint subtrees, one may define the ***distance*** *$d(Y, Z)$ as the minimum-length path between a vertex of Y and a vertex of Z. Then*

(i) The definition agrees with the usual distance function on vertices.

(ii) $d(P, Q) = d(Y, Z)$ for exactly one pair of vertices $(P, Q) \in V(Y) \times V(Z)$.

(iii) The geodesic P—Q is edge-disjoint from Y and Z.

(iv) P—Q is the intersection of all subtrees with non-empty intersection with both Y and Z.

We refer to P—Q as the ***geodesic*** *between the two subtrees Y, Z.*

(g) If $f : Y \to X$ is a locally injective morphism from a connected graph Y to tree X, then f is injective. In particular, Y is a tree.

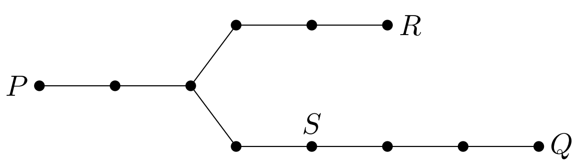
Proof omitted. ☐

Miscellaneous properties of trees:

Lemma 3.9. *Let $X = (V, E)$ be a tree.*

(a) Let $P \in V$. Then $\deg(P) = 1 \iff X \setminus P$ is a tree $\iff X \setminus P$ is connected. Such vertices are ***leaves*** *of the tree X.*

(b) If $P, Q, R \in V$ and $S \in V(Q$—$R)$, then $d(P, S) \le \max\{d(P, Q), d(P, R)\}$.



(c) Let $P_0 \in V$ and $X_n = \{Q \in V(X) : d(P_0, Q) = n\}$. The tree may be ***rooted*** *at P_0, and viewed as a (possibly finite)* ***inverse system*** *$\cdots \to X_n \to \cdots \to X_2 \to X_1 \to X_0 = \{P_0\}$.*

(d) If X is finite, or more generally of finite diameter, then

(i) If X is finite, then $|E(X)| = |V(X)| - 1$.

(ii) X has at least one leaf.

(iii) If the diameter of X is even (resp. odd), then X has a vertex (resp. edge) invariant under all automorphisms.

(e) If $\Gamma = \{\Gamma_i\}$ is a collection of disjoint subtrees of X, the contraction X/Γ is a subtree.

Proof sketch. (a) follows from the more general fact that removing vertex P from a tree splits it into a forest of $\deg(P)$ subtrees. (b) follows from the 'tripod picture' above. (c) follows from identifying each vertex Q with the geodesic Q—P_0. (d) follows from the fact that the set of leaves is stable under automorphisms, and removing all leaves of a finite tree X reduces its diameter by 2. (e) is a special case of Lemma 3.4(l). □

Finally, we setup the notion of 'points at infinity' of a tree:

Definition 3.10. Let X be a tree with vertex $P_0 \in V(X)$. The **ends** of X, denoted $\partial_\infty X$, are the infinite paths in X starting from base point P_0. The notion is independent of the base point P_0, and may be naturally identified with the topological ends of the end-compactification of real(X).

3.3 Cayley graphs

A useful perspective to understanding groups by its generators is to examine its Cayley graph:

Definition 3.11. Let G be a group and $S \subset G$. Its **Cayley graph** $\Gamma(G, S)$ is the directed graph with vertex set G and directed edges $g \to gs$ for all $(g, s) \in G \times S$.

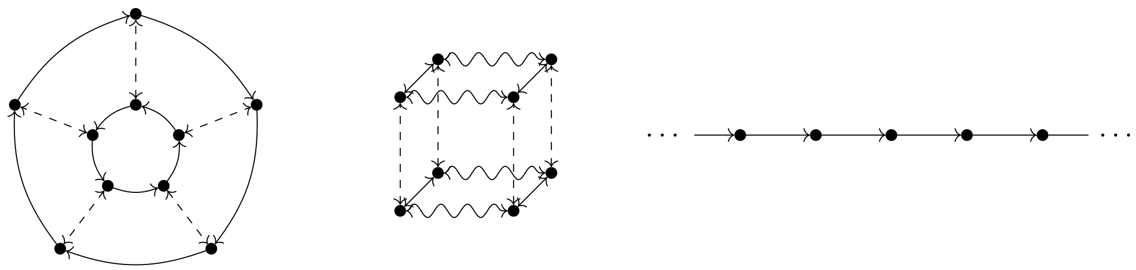


Figure 1: Cayley graphs of D_{10} (left), $C_2 \times C_2 \times C_2$ (middle), and $\mathbb{Z}$ (right) with the usual generators.

Lemma 3.12. *Let G be a group, and $S \subset G$ with Cayley graph $\Gamma = \Gamma(G, S)$.*

(a) Γ is connected $\iff$ S generates G.

(b) Γ is free of self-loops $\iff$ $1 \notin S$.

(c) The underlying undirected graph of Γ is simple $\iff$ $S \cap S^{-1} = \emptyset$.

(d) Γ is a tree $\iff$ $G = \mathbb{F}_S$ is a free group with basis S.

Proof omitted. □

3.4 Group actions on graphs

Definition 3.13. A **graph action** of group G on graph X, denoted $G \curvearrowright X$, is a group homomorphism $G \to \mathrm{Aut}(X)$. One may naturally define orbits and stabilisers of vertices and edges, as well as the **quotient graph** $G \backslash X$.

A **tree action** is a graph action $G \curvearrowright X$ on a tree X.

The action is **without inversion** if there is an orientation of X preserved by G.

The action is **free** if it is without inversion, and all vertex stabilisers are trivial.

If $S \subset G$, the set of **common S-fixed points** is denoted by $X^S = \{P \in V(X) : sP = P \forall s \in S\}$.

A **G-stable subgraph** is a subgraph $Y \subset X$ whose vertex and edge sets are stable under G-action. The action $G \curvearrowright X$ induces an action $G \curvearrowright Y$.

Example 3.14. Let X be a graph with universal covering tree $\tilde{X}$. For any base point $P_0 \in V(X)$, the fundamental group $G = \pi_1(X, P_0)$ acts freely on $\tilde{X}$ via left walk concatenation. The quotient graph is $G \backslash \tilde{X} = X$.

Example 3.15. Group G acts freely on any of its Cayley graphs $\Gamma(G, S)$ via left multiplication.

Example 3.16. Let X be the star tree with vertex set $\{\star\} \sqcup G$ and an edge between $\star$ and each $g \in G$. Then G acts without inversion on X by fixing $\star$ and acting on the other vertices via left multiplication. If G is non-trivial, then $X^G = \{\star\}$.

Example 3.17. Let $H_0 \leq H_1 \leq H_2 \leq \dots$ be a (possibly infinite) chain of subgroups of G such that $\bigcup_i H_i = G$. The associated **coset tree** X is the tree corresponding to the inverse system

$$G/H_0 \to G/H_1 \to G/H_2 \to \dots \to G/H_n \xrightarrow{gH_n \mapsto gH_{n+1}} G/H_{n+1} \to \dots$$

In other words, the vertex set is $\bigsqcup_i G/H_i$, with edges between gH_n and gH_{n+1} for all $g \in G, n \in \mathbb{N}$. (Note: the tree in Example 3.15 is a special case of the coset tree of the chain $1 \leq G$.)

There is a natural action of G on the coset tree without inversion, via left multiplication: $g \cdot (aH_n) = gaH_n$ for all $g \in G, aH_n \in G/H_n$. The following lemma describes properties of this action:

Lemma 3.18. *For the coset tree action as described in Example 3.16,*

(a) Each vertex $P = gH_n \in V(X)$ has stabliliser $Stab_G P = gH_n g^{-1}$.

(b) If the chain of subgroups is non-stationary (i.e. $H_n \neq G$ for all n), then $X^G = \emptyset$.

Proof omitted. □

We now establish more general results about graph actions. First, we show that a graph action $G \curvearrowright X$ is geometrically equivalent to one in which X is simple, and the action is without inversion:

Lemma 3.19. *Let $G \curvearrowright X$ be a graph action.*

(a) Let $X' = \operatorname{subdiv}(X)$. The action $G \curvearrowright X$ induces action $G \curvearrowright X'$.

(b) Let $X'' = \operatorname{subdiv}(X')$. The action $G \curvearrowright X$ induces action $G \curvearrowright X''$ without inversion.

Proof sketch. The result follows from Lemma 3.6 and the fact that the action preserves the bipartite structure of X'. □

Lemma 3.20. *Let X be a connected graph, and $G \curvearrowright X$ without inversion.*

(a) $G \backslash X$ is connected.

(b) Every subtree of $G \backslash X$ lifts to a subtree of X.

*(c) There exists a **tree of representatives** of $X \bmod G$: a subtree of X containing one element of each vertex orbit.*

Proof sketch. (a) follows from the fact that walks are preserved under the quotient map. (b) follows from Zorn's lemma: for a subtree T of X, consider a maximal subtree of X injecting into T. (c) follows from lifting a spanning tree of $G \setminus X$. □

Definition 3.21. Let X be a connected graph, and $G \curvearrowright X$ without inversion. A **fundamental domain** of $X \bmod G$ is a subgraph Y of X such that $Y \to G\backslash X$ is a graph isomorphism.

Lemma 3.22. *Let $G \curvearrowright X$ be a tree action without inversion. A fundamental domain of the action exists $\iff$ $G\backslash X$ is a tree.*

Proof. By Lemma 3.8(a), the fundamental domain, if it exists, is a subtree of X. Conversely, by Lemma 3.20(b), if $G\backslash X$ is a tree, then it lifts to a subtree of X. □

Lemma 3.23. *Let $G \curvearrowright X$ be a tree action without inversion, and $S \subset G$.*

(a) X^S is a subtree of X.

(b) $X^S = X^{\langle S\rangle} = \bigcap_{s\in S} X^s$, where $\langle S\rangle$ is the subgroup of G generated by elements of S.

(c) Let $K \triangleleft G$ be a normal subgroup with $X^K \neq \emptyset$. Then $G \curvearrowright X$ induces an action of the quotient group $G/K \curvearrowright X^K$.

Proof. (a) follows from Lemma 3.8(b) and the observation $P, Q \in X^S \implies P{-}Q \in X^S$. (b) is clear. (c) follows from the fact that for all $g \in G$ and $k \in K$, $gkg^{-1} \in H$ fixes X^K, thus k fixes gX^K. Thus $gX^K \subset X^K$ for all $g \in G$, i.e. X^K is stable under G. □

3.5 Free group $\equiv$ free action on tree

In this subsection, we establish a correspondence between free groups and free group actions on a tree.

Proposition 3.24. *Let $G \curvearrowright X$ be a free tree action. Choose an orientation of edges of X preserved by G, and a tree of representatives T of $X \bmod G$.*

(a) Let S be the set of elements $g \in G \setminus \{1\}$ such that there is a directed edge from a vertex of T to a vertex of gT. Then S is a free basis for G.

(b) Let Γ be the orbit of T (a collection of disjoint subtrees). Then $X/\Gamma \cong \Gamma(G,S)$ corresponds to its Cayley graph.

(c) If $|E(G\backslash X)|$ is finite, then $|S| = |E(G\backslash X)| - |E(T)|$.

Proof omitted; see Example 3.26. □

Corollary 3.25 (correspondence for free groups)**.**

(a) A group G is free $\iff$ G acts freely on a tree.

(b) (Schreier) Every subgroup of a free group is free.

(c) (Schreier index formula) If $H \leq G$ are free groups with finite index $n = |G : H|$, then

$$r(H) - 1 = n(r(G) - 1).$$

Proof. (a) follows from Lemma 3.12(d) and Proposition 3.24(b), noting by Lemma 3.9(e) that X/Γ is a tree. (b) follows from (a) and the fact that any subgroup action of a free action is free. (c) follows from Proposition 3.24(c) and degree counting: $|V(T)| = |G : H| = n$, $|E(T)| = n-1$, $|S| = r(H)$, and $|E(G\backslash X)| = |V(T)|\cdot r(G) = n \cdot r(G)$. (Note: in the Cayley graph of G with respect to its free generators, each vertex has exactly $r(G)$ outgoing edges.) □

Example 3.26. The free group $G = \mathbb{F}_2 = \langle x, y \mid \ \rangle$ acts freely on its (oriented) Cayley graph $X = \Gamma(G, \{x, y\})$, a 4-regular tree.

Let $H \leq \mathbb{F}_2$ be the subgroup of index 4 determined by the kernel of projection $\mathbb{F}_2 \to C_2 \times C_2$. Then H acts freely on X (by subgroup action). Selecting a tree of representatives T with $|V(T)| = |G : H| = 4$, one notes that there are 5 incoming and 5 outgoing edges between T and others in its orbit. With the notation of Proposition 3.24, $S = \{x^2, yxy^{-1}x^{-1}, y^2, xyxy^{-1}, xy^2x^{-1}\}$ and $H = \mathbb{F}_S = \mathbb{F}_5$.

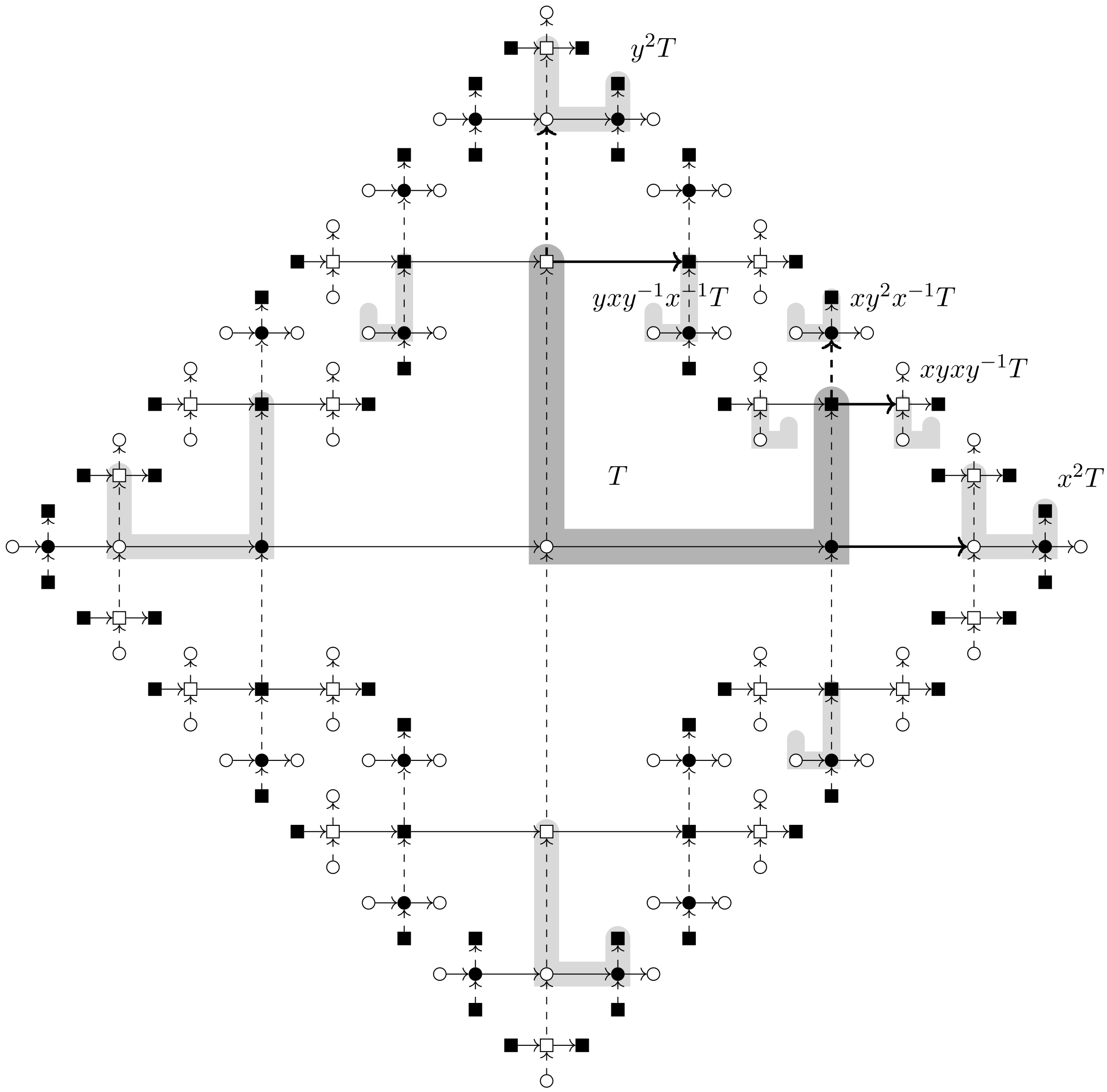


Figure 2: The action of subgroup H on the Cayley graph of free group $G = \mathbb{F}_2$ in Example 3.26. The directed edges correspond to x (solid) and y (dashed). The vertices are denoted by one of four symbols, corresponding to their equivalence classes under projection $G \to C_2 \times C_2$. Tree of representatives T is darkly shaded. Adjacent copies hT ($h \in H$) are lightly shaded. The 5 copies hT ($h \in H$) for which some edge emerges from T and ends in hT are labelled, with the respective edges bolded; these correspond to the free generators of H.

As the action is free, all vertex and edge stabilisers are trivial. The quotient graph $G\backslash X$ is homotopic to a bouquet of 5 circles; this can be seen by contracting T in $G\backslash X$. The generators S trace out 5 loops of $G\backslash X$; these generate its fundamental group, as the projection $G\backslash X \to (G\backslash X)/T$ is a homotopy equivalence (see Lemma 3.4(l)).

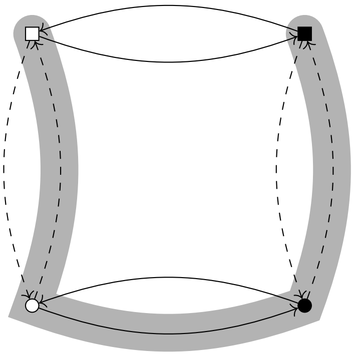


Figure 3: The quotient graph $H\backslash X$ in Example 3.26, with tree of representatives T shaded.

3.6 Amalgam $\equiv$ ($G\backslash X$ = segment)

In this subsection, we establish a correspondence between amalgams and tree actions without inversion, with a segment as fundamental domain.

Definition 3.27. Let $G = G_1 *_A G_2$ be an amalgam. The **associated tree action** $G \curvearrowright X$ of the amalgam is defined as follows: let X be the bipartite graph with vertex set $(G/G_1) \sqcup (G/G_2)$ and edge set G/A, where each edge gA adjoins vertices gG_1 and gG_2. There is a natural action $G \curvearrowright X$ without inversion, via left multiplication on the cosets.

Lemma 3.28. *With reference to Definition 3.27,*

(a) The action is edge-transitive, i.e. $G\backslash X$ is a segment.

(b) Let $P = G_1, Q = G_2$, and $y = A$. Then $y = PQ$ is a fundamental domain of $X \bmod G$, with vertex and edge stabilisers $Stab_G(P) = G_1, Stab_G(Q) = G_2$, and $Stab_G(y) = A$ respectively.

(c) X is indeed a tree.

Proof. (a) and (b) are clear. (c) follows from the structure theorem of amalgams: suppose otherwise, that there is a cycle in X with vertices $g_0G_1, g_1G_2, g_2G_1, \ldots, g_{n-1}G_2, g_nG_1 = g_0G_1$ and edges $g_1A, \ldots, g_nA$. Taking indices mod n, $g_i = g_{i-1}s_i$ for s_i alternating between G_1 and G_2. The fact that the cycle does not backtrack implies that the s_i in fact alternate between $G_1 \setminus A$ and $G_2 \setminus A$. Multiplying all relations gives $s_1s_2 \ldots s_n = 1$, contradicting the structure theorem. □

Theorem 3.29. *Let X be a connected graph and $G \curvearrowright X$ without inversion, such that $G\backslash X$ is a single segment. Let y be an edge between two vertices P, Q (this is a fundamental domain). Then*

(a) The edge stabiliser G_y is a subgroup of the vertex stabilisers G_P and G_Q.

(b) X is connected $\iff$ G is generated by $G_P \cup G_Q$.

(c) X is cycle-free $\iff$ the map $G_P *_{G_y} G_Q \to G$ induced by inclusion maps is an injection.

(d) X is a tree $\iff$ $G = G_P *_{G_y} G_Q$.

Proof sketch. (a) is clear. (b) and (c) follow from the structure theorem of amalgams, and the following observations:

- Let $PyQy_1R_1y_2R_2 \ldots y_nR_n$ be a path starting from y. There exists $g_1 \in G_Q$ sending $y \mapsto y_1$; there exists $g_2 \in G_P \setminus A$ sending $y \mapsto g_1^{-1}y_2$; there exists $g_3 \in G_Q \setminus A$ sending $y \mapsto (g_1g_2)^{-1}y_3$; iterating, there exists $g = g_1 \ldots g_n \in G$ sending $y \mapsto y_n$, where the g_i alternate between $G_P \setminus A$ and $G_Q \setminus A$.

- Conversely, given $g = g_1 \dots g_n$ with $a \in A$ and the g_i alternating between $G_P \setminus A$ and $G_Q \setminus A$, one may construct the corresponding walk $PyQy_1R_1y_2R_2 \dots y_nR_n$ without backtracking.

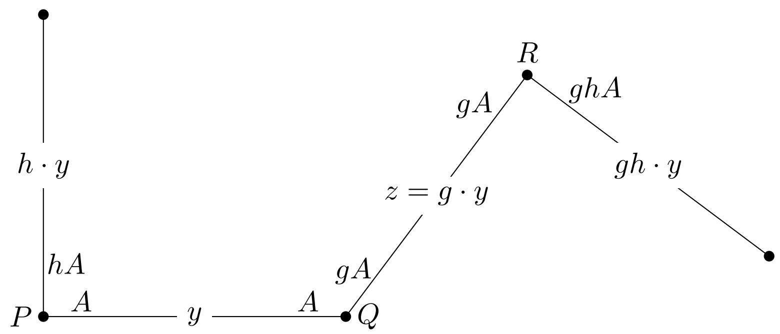
(d) follows from (b) and (c). $\square$

The intuition behind Definition 3.27 and Theorem 3.29 is as follows: information about the vertex and edge stabilisers G_P, G_Q, and G_y already determine the local structure of the graph about edge $y = PQ$ in a rigid manner. Since G_P fixes P and A fixes edge y, thus the edges adjoining vertex P may be identified with left cosets G_P/A, with edge y identified with the trivial coset A, such that the action of G_Q permutes the edges of P via left multiplication on the cosets. Similarly, edges adjoining vertex Q may be identified with left cosets G_Q/A.

Each additional edge should now also have the same local structure. For instance, if $z = QR \neq y$ is another edge adjoining Q, then one may pick representative $g \in G_Q \setminus A$ such that $g \cdot y = z = QR$. Then R and z have stabilisers $\mathrm{Stab}_G(R) = gG_Pg^{-1}$ and $\mathrm{Stab}_G(z) = gAg^{-1}$ respectively, and the edges of R may be identified with left cosets gG_Pg^{-1}/gAg^{-1}. This is related to Definition 3.27 via

$$\left(gG_Pg^{-1} \curvearrowright gG_Pg^{-1}/gAg^{-1}\right) \equiv \left(gG_Pg^{-1} \curvearrowright gG_P/A\right)$$

so we may instead label the edges adjoining R with cosets $gG_P/A \subset G/A$.



For each $h \in G_P$, the element $gh \in G$ must send $y \mapsto gh \cdot y = ghg^{-1} \cdot gy = ghg^{-1} \cdot z$, which is evaluated by left multiplication on the cosets labelling the edges adjoining R. By 'iterative path construction' starting from initial segment $y = PQ$, the structure of tree X and the action $G \curvearrowright X$ is uniquely determined.

We are finally able to state the main result of this subsection:

Corollary 3.30 (correspondence for amalgams)**.** *Let $G = G_1 *_A G_2$ be an amalgam. There is tree action $G \curvearrowright X$ without inversion, unique up to isomorphism, with fundamental domain a segment $y = PQ$ with vertex and edge stabilisers $Stab_G(P) = G_1, Stab_G(Q) = G_2$, and $Stab_G(y) = A$.*

Proof sketch. This is a direct consequence of Lemma 3.28 and Theorem 3.29; one may verify that they are indeed converse results. $\square$

We conclude the section with an application of this correspondence.

Proposition 3.31. *Let H be a subgroup of amalgam $G = G_1 *_A G_2$ such that $H \setminus \{1\}$ does not meet any conjugate of G_1 or G_2. Then H is a free group.*

Proof. By Corollary 3.30, the associated tree action $G \curvearrowright X$ of the amalgam has vertex stabilisers being conjugates of G_1 or G_2. The condition is thus equivalent to the subgroup action $H \curvearrowright X$ being free. The result then follows from Corollary 3.25(a). $\square$

4 Explicit Amalgams

In this section, we explore three explicit examples of amalgams: D_∞, $\mathrm{PSL}_2(\mathbb{Z})$, and the trefoil knot group $\mathcal{T}$. We understand them via their natural actions on geometric spaces, as well as on trees.

4.1 $D_\infty = C_2 * C_2$

The simplest non-trivial amalgam is the free product $G = C_2 * C_2$. Denoting the generators by a and b, the elements of G are of the form $(ba)^k$, $a(ba)^k$, $(ba)^k b$, or $(ab)^k$ for $k \in \mathbb{Z}_{\geq 0}$.

To relate this to the infinite dihedral group D_∞, the $\mathbb{Z}$-preserving geometric automorphisms of $\mathbb{R}$, we identify the generators a, b with the reflections $\tau_a : x \mapsto -x$ and $\tau_b : x \mapsto 1 - x$ respectively.

Lemma 4.1. *G injects into D_∞ via $a \mapsto \tau_a, b \mapsto \tau_b$.*

Proof. It suffices to show that $g_1 g_2 \dots g_n \neq 1$ for any $g_1, \dots, g_n \in \{\tau_a, \tau_b\}$ alternating between τ_a and τ_b. Let $X_a = \{x \in \mathbb{R} : x < 0\}$ and $X_b = \{x \in \mathbb{R} : x > \frac{1}{2}\}$; note these are disjoint. Note that

$$\tau_a X_b = \{x < -\frac{1}{2}\} \subsetneq X_a \quad \text{and} \quad \tau_b X_a = \{x > 1\} \subsetneq X_b.$$

Thus in the case $g = g_1 g_2 \dots g_n = \tau_a \tau_b \dots \tau_a$,

$$gX_b = \tau_a \dots \tau_b \tau_a X_b \subsetneq \tau_a \dots \tau_b X_a \subsetneq \cdots \subsetneq \tau_a X_b \subsetneq X_a,$$

so $g \neq 1$. The other cases are similar, since the inclusions are strict and X_a, X_b are disjoint. □

This argument is generalised here:

Lemma 4.2 (Tits' ping-pong lemma)**.** *Let group G act on a set X, $H_1, \dots, H_k \leq G$ be subgroups ($k \geq 2$), at least one of which has order greater than 2, and $X_1, \dots, X_k \subset X$ such that the following property holds: for all $\alpha, \beta \in \{1, \dots, k\}$ with $i \neq j$, and for all $h \in H_\alpha \setminus \{1\}$, $hX_\beta \subset X_\alpha$.*

Then $\langle H_1, \dots, H_k \rangle = H_1 * \cdots * H_k$.

Proof. It suffices to show that $g = g_1 g_2 \dots g_n \neq 1$ for any $g_i \in H_{\alpha_i} \setminus \{1\}$, where $(\alpha_1, \dots, \alpha_n)$ is a sequence in $\{1, \dots, k\}$ with no two consecutive indices equal. Note for all $\beta \in \{1, \dots, k\} \setminus \{\alpha_n\}$,

$$gX_\beta = g_1 \dots g_{n-1} g_n X_\beta \subset g_1 \dots g_{n-1} X_{\alpha_n} \subset \cdots \subset g_1 X_{\alpha_2} \subset X_{\alpha_1}.$$

If $k > 2$ or $\alpha_1 = \alpha_n$, one may pick $\beta \neq \alpha_1, \alpha_n$ to conclude $g \neq 1$ (since $X_\beta \cap X_{\alpha_1} = \emptyset$).

Otherwise, we have $k = 2, \alpha_1 \neq \alpha_n$, and WLOG $\alpha_1 = 1, \alpha_n = 2$, and $|H_1| > 2$. Pick $h \in H_1 \setminus \{1, g_1\}$ and apply the previous case to show $h^{-1} g h = (h^{-1} g_1) g_2 \dots g_n h \neq 1$. Thus $g \neq 1$. □

It remains to show $G = D_\infty$.

Proposition 4.3. *$D_\infty = C_2 * C_2$.*

Proof. By Lemma 4.1, it suffices to show D_∞ is generated by τ_a and τ_b.

Let $g \in D_\infty$. Applying translations $\tau_a \tau_b : x \mapsto x - 1$ and $\tau_b \tau_a : x \mapsto x + 1$, there exists $h \in \langle \tau_a, \tau_b \rangle g$ such that $h(0) = 0$. Thus $h \in \{1, \tau_a\}$, and $g \in \langle \tau_a, \tau_b \rangle$. □

A useful geometric perspective is to view the real line $\mathbb{R}$ as a line tree X (with vertex set $\mathbb{Z}$ and edges between consecutive integers). One may readily identify $\mathrm{Aut}(X) = D_\infty$ as the full symmetry group. The action is vertex-transitive, edge-transitive, and flag-simply transitive.

To remove inversion, consider the action of $G = D_\infty$ on the subdivided tree $X' = \text{subdiv}(X)$ with vertex set $\frac{1}{2}\mathbb{Z}$ and edges between consecutive half-integers. The action $G \curvearrowright X'$ is now edge-simply transitive; in particular, the action is without inversion, and the quotient $G\backslash X'$ is a single segment. If y is the edge between vertices $P = 0$ and $Q = \frac{1}{2}$, then $\text{Stab}_G(P) = \langle \tau_a \rangle \cong C_2, \text{Stab}_G(Q) = \langle \tau_b \rangle \cong C_2$, and $\text{Stab}_G(y) = 1$. This agrees with the correspondence established in Corollary 3.30.

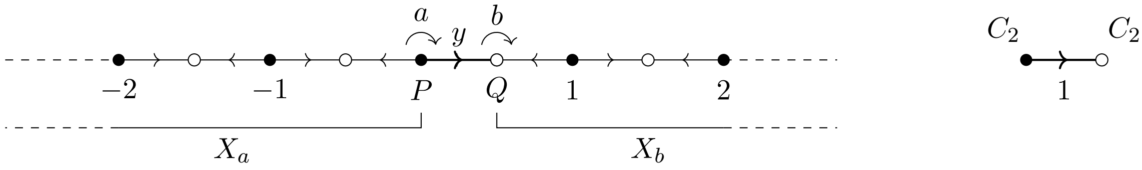


Figure 4: (left) The action of $D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$ on $\mathbb{R}$. Elements a, b correspond to reflections about $P = 0$ and $Q = \frac{1}{2}$ respectively. Sets X_a, X_b as in the proof of Lemma 4.1 are depicted. The bolded segment $y = PQ$ is a fundamental domain of the action. (right) The quotient graph $D_\infty \backslash X'$, labelled with vertex and edge stabilisers of the fundamental domain.

4.2 $\text{PSL}_2(\mathbb{Z}) = C_2 * C_3$

The next simplest non-trivial amalgam is the free product $G = C_2 * C_3$. In this subsection, we identify it with the discrete matrix group $\text{PSL}_2(\mathbb{Z})$.

Let $a = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ be elements of $\text{PSL}_2(\mathbb{Z})$ of orders $2, 3$ respectively.

Lemma 4.4. $\langle a, b \rangle = \langle a \rangle * \langle b \rangle$.

Proof 1. Identify $\text{PSL}_2(\mathbb{Z})$ with the lattice-preserving geometric automorphisms of $\mathbb{R}^2/\{\pm 1\}$. Set $X_a = \{(x, y) : xy > 0\}$ and $X_b = \{(x, y) : xy < 0\}$. One may verify that $aX_b = X_a$ and $bX_a, b^{-1}X_a \subset X_b$. The result follows from the ping-pong lemma (Lemma 4.2). $\square$

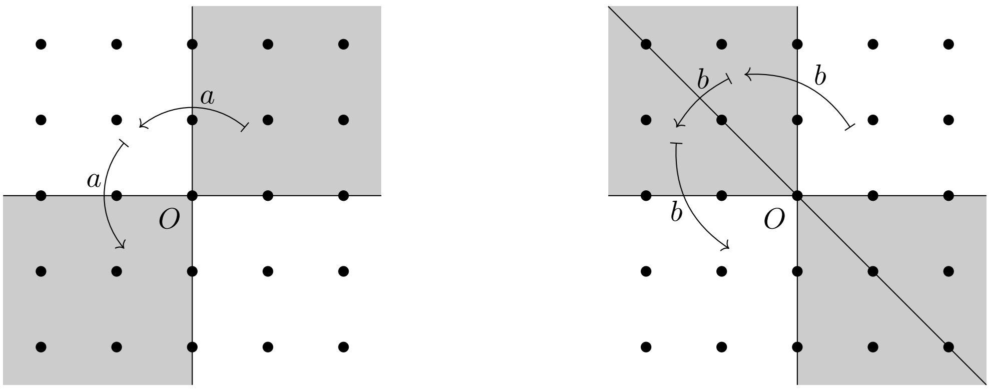


Figure 5: (left) The action of a on $\mathbb{R}^2/\{\pm 1\}$ via $\frac{\pi}{2}$-rotation. The region X_a is shaded. (right) The action of b on $\mathbb{R}^2/\{\pm 1\}$. The region X_b is shaded.

Proof 2. Identify $\text{PSL}_2(\mathbb{Z})$ with isometries of the hyperbolic half-plane $\mathcal{H}$ via $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$. Under this identification, a corresponds to π-rotation about point $P = i \in \mathcal{H}$, while b corresponds to $\frac{2\pi}{3}$-rotation about point $Q = e^{2\pi i/3} \in \mathcal{H}$. The result then follows from the ping-pong lemma with $X_a = \{z \in \mathcal{H} : \text{Re}(z) > 0\}$ and $X_b = \{z \in \mathcal{H} : \text{Re}(z) < 0\}$. $\square$

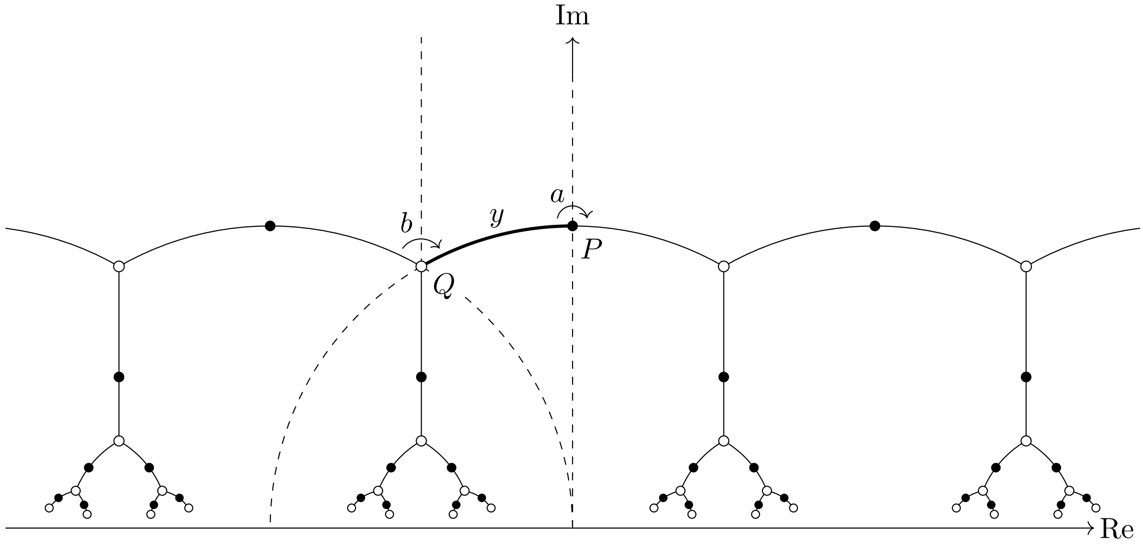


Figure 6: The action of $a, b \in \mathrm{PSL}_2(\mathbb{Z})$ on the hyperbolic half-plane $\mathcal{H}$. Region $X_a = \{\mathrm{Re}(z) > 0\}$ is contained within a $\frac{2\pi}{3}$-sector about Q, so bX_a and $b^{-1}X_a$ are contained within the left half $X_b = \{\mathrm{Re}(z) < 0\}$. The image of hyperbolic segment $y = PQ$ under action of $\langle a, b\rangle$ generates a $(2,3)$-biregular tree X embedded in $\mathcal{H}$.

Proposition 4.5. $\mathrm{PSL}_2(\mathbb{Z}) = C_2 * C_3$.

Proof 1. $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ via fractional linear transformations $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \frac{p}{q} \mapsto \frac{\alpha p + \beta}{\gamma q + \delta}$.

Construct the **Farey graph** with vertices $\mathbb{Q}_\infty$ and edges between fractions $\frac{p}{q}$ and $\frac{r}{s}$ with $|ps - qr| = 1$. Viewing $\mathbb{Q}_\infty$ as points on $\partial\mathcal{H}$, one observes that the Farey graph dissects $\mathcal{H}$ into ideal triangles. Thus $\mathrm{SL}_2(\mathbb{Z})$ acts on the **Farey tree**, the face-edge incidence tree of the Farey graph.

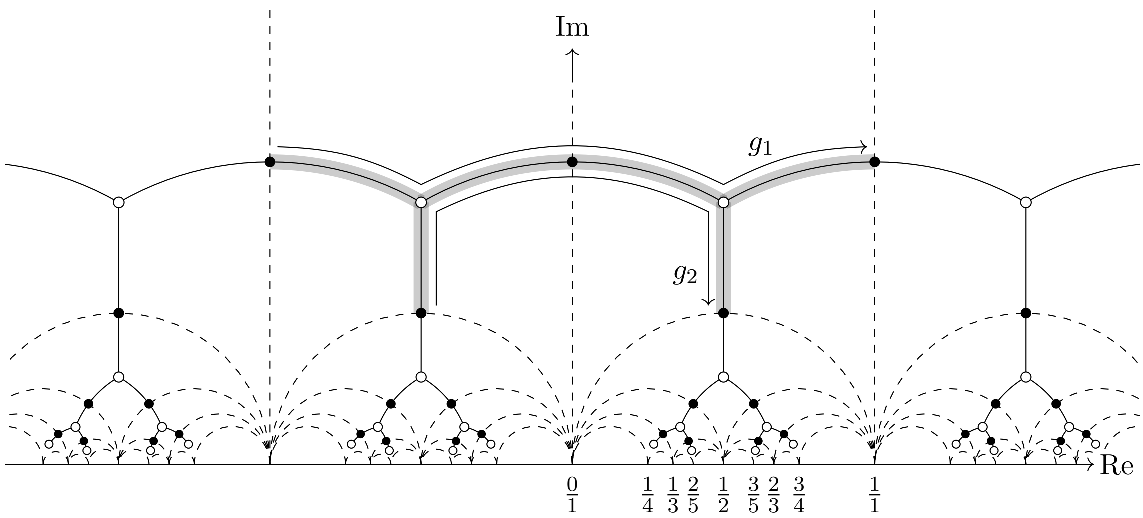


Figure 7: The Farey tree (solid) and Farey graph (dotted) embedded in the hyperbolic half-plane $\mathcal{H}$. Note the agreement of the Farey tree with the tree in Figure 6 (see Lemma 4.4, Proof 2). A fundamental domain of the subgroup action $H = \langle g_1, g_2\rangle$ (see Proposition 4.8) is shaded.

The result follows from Corollary 3.30, noting that the edge between 0 and ∞ has stabiliser $\langle a\rangle$, and the ideal triangle with vertices $0, 1, \infty$ has stabiliser $\langle b\rangle$. $\square$

Proof 2. By Lemma 4.4, it suffices to show $\mathrm{PSL}_2(\mathbb{Z})$ is generated by a and b.

For any $g = \pm\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$, note that $bg, b^{-1}g, bag, b^{-1}ag$ are the elements

$$\pm\begin{pmatrix}-\gamma & -\delta\\ \alpha+\gamma & \beta+\delta\end{pmatrix}, \pm\begin{pmatrix}\alpha+\gamma & \beta+\delta\\ -\alpha & -\beta\end{pmatrix}, \pm\begin{pmatrix}-\gamma & -\delta\\ \alpha-\gamma & \beta-\delta\end{pmatrix}, \pm\begin{pmatrix}\alpha-\gamma & \beta-\delta\\ \gamma & \delta\end{pmatrix}$$

respectively. If $g \notin \langle a\rangle$, then by a routine case check on the relative sizes and signs of $\alpha, \beta, \gamma, \delta$, one of the above elements has a strictly lower value of $|\alpha| + |\beta| + |\gamma| + |\delta|$. Thus an element of the right coset $\langle a, b\rangle g$ that minimises $|\alpha| + |\beta| + |\gamma| + |\delta|$ must belong to $\langle a\rangle$; in particular, $g \in \langle a, b\rangle$. □

Remark 4.6. The extremised quantity $|\alpha|+|\beta|+|\gamma|+|\delta|$ of Proposition 4.5, Proof 2 is not unique; the argument may be adapted to other quantities such as $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ or $\operatorname{Im}\left(\frac{\alpha i+\beta}{\gamma i+\delta}\right)$.

Corollary 4.7. $\mathrm{SL}_2(\mathbb{Z}) = C_4 *_{C_2} C_6$.

Proof. The result follows from Lemma 2.13, noting that $Z(\mathrm{SL}_2(\mathbb{Z})) = \{\pm 1\}$. Alternatively, $G = \mathrm{SL}_2(\mathbb{Z})$ also acts on the tree in Lemma 4.4, Proof 2, with fundamental domain a segment $y = PQ$, and vertex and edge stabilisers $\mathrm{Stab}_G(P) \cong C_4, \mathrm{Stab}_G(Q) \cong C_6$, and $\mathrm{Stab}_G(y) \cong C_2$; the result follows from Corollary 3.30. □

Proposition 4.8. *The subgroup H of* $\mathrm{PSL}_2(\mathbb{Z})$ *generated by* $\pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ *and* $\pm\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ *is a free subgroup of index* 6.

Proof 1. Note that $H = \langle g_1, g_2\rangle$ where $g_1 = abab = \pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $g_2 = baba = \pm\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$. One may verify that

- g_1, g_2 have infinite order.
- Any non-trivial reduced word in g_1, g_2 corresponds to a non-trivial reduced word in a, b.
- If $g \in \mathrm{PSL}_2(\mathbb{Z})$ is written as a reduced word in a, b, one may right-multiply by g_1, g_1^{-1}, g_2, or g_2^{-1} to reduce $l(g)$, until $g \in \{e, a, b, b^{-1}, ab, ab^{-1}\}$.
- H is contained in the kernel of projection $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, with $|\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})| = 6$.

Thus $\langle g_1, g_2\rangle$ is a free subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of index precisely 6. □

Proof 2. $H = \langle g_1, g_2\rangle$ acts without inversion on the Farey tree X, with the fundamental domain being a tree with 6 edges (shaded in Figure 7). Thus $|\mathrm{PSL}_2(\mathbb{Z}) : H| = 6$.

One may apply the ping-pong lemma to g_1, g_2 with sets

$$X_1 = \{z : |\mathrm{Re}(z)| > 1\} \quad\text{and}\quad X_2 = \left\{z : \left|z - \frac{1}{2}\right| < \frac{1}{2} \text{ or } \left|z + \frac{1}{2}\right| < \frac{1}{2}\right\}$$

to show that $H = \langle g_1\rangle * \langle g_2\rangle \cong \mathbb{F}_2$. □

Corollary 4.9. *The subgroup $\hat{H}$ of* $\mathrm{SL}_2(\mathbb{Z})$ *generated by* $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ *and* $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ *is a free subgroup of index* 12.

Proof. By Proposition 4.8, it suffices to show that the projection $\hat{H} \to \mathrm{PSL}_2(\mathbb{Z})$ has trivial kernel. This follows from the structure theorem for free groups and the fact that $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ have infinite order. □

4.3 Trefoil knot group $\mathcal{T} = \mathbb{Z} *_{2\mathbb{Z} \cong 3\mathbb{Z}} \mathbb{Z}$

Unlike the previous two examples, the trefoil knot group is not a free product. To understand it better, we employ two standalone results of subsequent sections: Theorem 5.5 regarding 'gluing' geometric fundamental groups, and Corollary 6.8 concerning automorphisms of a tree without fixed points. One may wish to skip ahead to these results before returning to this subsection.

Definition 4.10. The **trefoil knot group** is the amalgam

$$\mathcal{T} = \mathbb{Z} *_{2\mathbb{Z} \cong 3\mathbb{Z}} \mathbb{Z} = \langle a, b \mid a^2 = b^3 \rangle.$$

Remark 4.11. An alternative presentation of $\mathcal{T}$ is

$$\mathcal{T} = \langle x, y \mid xyx = yxy \rangle.$$

The presentations are equal by identifying $a = xyx, b = yx$ and $x = ab^{-1}, y = b^2 a^{-1}$.

Proposition 4.12. $\mathcal{T}/Z(\mathcal{T}) = \mathcal{T}/\langle a^2 \rangle \cong \mathrm{PSL}_2(\mathbb{Z})$.

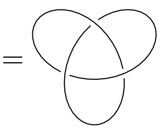
Proof. This follows from Lemma 2.13, noting that $Z(\mathcal{T}) = \langle a^2 \rangle$ is precisely the amalgamated subgroup, and Proposition 4.5. □

Proposition 4.13. $\mathcal{T}$ *is not a free product.*

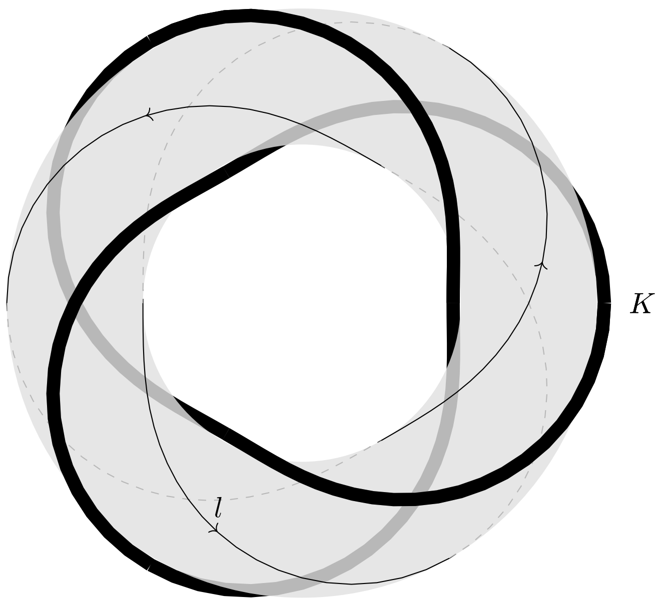
Proof. Suppose otherwise that $\mathcal{T}$ is a free product. By Corollary 3.25, there is a tree X upon which $\mathcal{T}$ acts freely.

Since all stabilisers are trivial, thus by Corollary 6.8, the generators $a, b \in \mathcal{T}$ act hyperbolically on X via non-trivial translations along lines l_a, l_b of X respectively. Consequently, $a^2 = b^3$ is a non-trivial translation along the same line $l_a = l_b = l$.

Finally, note $g = [a, b] = a^{-1}b^{-1}ab \neq 1$ fixes l, a contradiction. Thus $\mathcal{T}$ is not a free product. □

Proposition 4.14. $\mathcal{T} = \pi_1\left(\mathbb{R}^3 \setminus K\right)$, *where* $K =$  *is the trefoil knot.*

Proof. Consider the trefoil knot K as a tube resting on the surface of a torus Ψ:



Let Φ denote the closure of the complement $(\mathbb{R}^3 \setminus K) \setminus \Psi$. Then Ψ and Φ are each homotopic to a loop, and $\Psi \cap \Phi$ is a band, with $\pi_1(\Psi \cap \Phi)$ generated by a loop l that corresponds to two loops around Ψ and three loops around Φ. By Theorem 5.5,

$$\pi_1(\Psi \cup \Phi) = \varinjlim \left(\begin{tikzcd} \pi_1(\Psi \cap \Phi) \arrow[r] \arrow[dr] & \pi_1(\Psi) \\ & \pi_1(\Phi) \end{tikzcd} \right) = \varinjlim \left(\begin{tikzcd} \mathbb{Z} \arrow[r, "x \mapsto 2x"] \arrow[dr, "x \mapsto 3x"'] & \mathbb{Z} \\ & \mathbb{Z} \end{tikzcd} \right) = \mathbb{Z} *_{2\mathbb{Z} \cong 3\mathbb{Z}} \mathbb{Z}$$

which is the desired equality. □

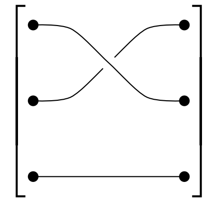
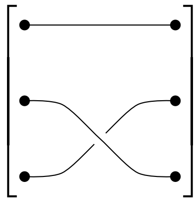
Corollary 4.15. *The trefoil knot K is not ambient-isotopic to the unknot in $\mathbb{R}^3$.*

Proof. The result follows from $\pi_1(\mathbb{R}^3 \setminus \{\text{unknot}\}) = \mathbb{Z} \neq \mathcal{T}$ (because $\mathcal{T}$ is non-abelian, say). □

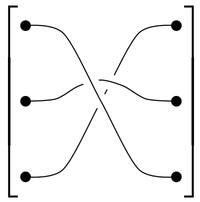
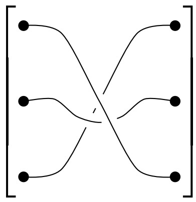
Proposition 4.16. *$\mathcal{T}$ is isomorphic to the* ***braid group***

$$B_3 = \pi_1(\{\text{3-element subsets of the plane}\}).$$

Proof sketch. With the presentation in Remark 4.11, we may identify generators x, y of $\mathcal{T}$ with the Reidemeister moves

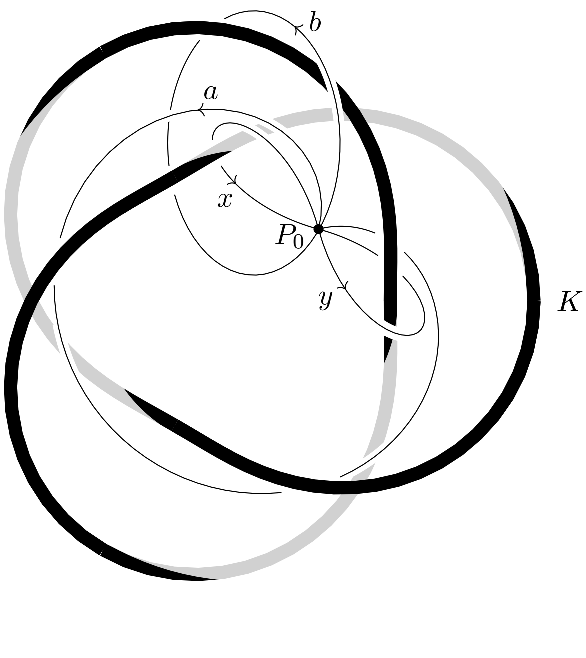
$x =$  $, y =$ 

which generate the braid group B_3, and agree on the 'rearrangement of crossings'

$xyx =$  $=$  $= yxy.$

Conversely, any braid which cannot be trivialised by 'rearranging crossings' and 'straightening strings' (corresponding to $xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1$) is non-trivial. □

Remark 4.17. The generators x, y and a, b may be represented by the following homotopy classes in $\mathcal{T} = \pi_1(\mathbb{R}^3 \setminus K, P_0)$:



5 Direct Limits and Graphs of Groups

In this section, we investigate direct limits of groups as a generalisation of amalgams. We use it to construct trees of groups, HNN extensions, and more generally, graphs of groups. We ultimately establish a correspondence between graphs of groups (G, Y) and tree actions $G \curvearrowright X$ without inversion.

5.1 Direct limits

Definition 5.1. Let $(I, \leq)$ be a directed set and $\{G_i\}_{i \in I}$ a collection of groups indexed by I, along with homomorphisms $\alpha_{ij} : G_i \to G_j$ for all $i \leq j$, such that $\alpha_{ii} = \mathrm{id}_{G_i}$ and $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ for all $i \leq j \leq k$. The **limit** $G = \varinjlim\{G_i\}$, accompanied with homomorphisms $f_i : G_i \to G$ commuting with the α_{ij}'s, is the group G such that for any group H and homomorphisms $h_i : G_i \to H$ commuting with the α_{ij}'s, there exists unique homomorphism $h : G \to H$ such that the following diagram commutes for all $i \leq j$:

$$\begin{tikzcd}
G_i \arrow[rr, "f_i"] \arrow[dr, "\alpha_{ij}"'] \arrow[drrr, "h_i"] & & G \arrow[dr, dashed, "\exists! h"] & \\
& G_j \arrow[ur, "f_j"] \arrow[rr, "h_j"] & & H
\end{tikzcd}$$

Lemma 5.2. *The direct limit in Definition 5.1 exists, and is unique up to unique isomorphism.*

Proof sketch. Similar to Lemma 2.5, the direct limit G may be constructed as follows: consider the quotient of the free group on $\bigsqcup_i G_i$ by relations due to group operations of the G_i ($xyz^{-1} : x, y, z \in G_i$) and the α_{ij} ($xy^{-1} : y = \alpha_{ij}(x)$). Uniqueness is due to the universal property. □

Remark 5.3. $G = \varinjlim\{G_i\}$ is generated by the images $f_i(G_i)$ of the constituent groups.

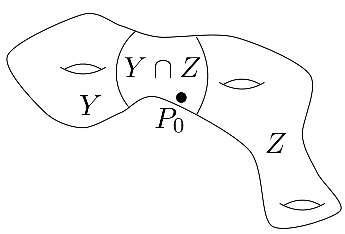
Example 5.4. The amalgam $*_A\{G_i\}$ (see Definition 2.4) is the direct limit of the collection of groups $\{A\} \cup \{G_i\}_{i \in I}$ along with the injective homomorphisms $\iota_i : A \hookrightarrow G_i$. In particular,

$$\varinjlim \left(\begin{tikzcd} A \arrow[r, hook] \arrow[dr, hook] & G_1 \\ & G_2 \end{tikzcd} \right) = G_1 *_A G_2.$$

This also allows us to establish 'associativity' of successive amalgams:

$$\varinjlim \left(\begin{tikzcd} A \arrow[r, hook] \arrow[dr, hook] & G_1 \\ B \arrow[r, hook] \arrow[dr, hook] & G_2 \\ & G_3 \end{tikzcd} \right) = (G_1 *_A G_2) *_B G_3 = G_1 *_A (G_2 *_B G_3).$$

Theorem 5.5 (Seifert-Van Kampen; generalisation of Example 2.7)**.** *Let Y, Z be path-connected open subsets of a topological space X with non-empty, path-connected intersection $Y \cap Z$.*



For any $P_0 \in Y \cap Z$, the fundamental group of $Y \cup Z$ is the direct limit

$$\varinjlim \left(\begin{tikzcd} \pi_1(Y \cap Z, P_0) \arrow[r] \arrow[dr] & \pi_1(Y, P_0) \\ & \pi_1(Z, P_0) \end{tikzcd} \right) = \pi_1(Y \cup Z, P_0).$$

Proof sketch. The result follows from the fact that every closed loop in $Y \cup Z$ from P_0 is homotopic to a finite concatenation of loops from P_0, each one either residing entirely in Y or entirely in Z. □

Example 5.6. The following direct limit is trivial:

$$\varinjlim \left(\begin{tikzcd} C_2 \arrow[r, hook] \arrow[dr] & S_3 \\ & 1 \end{tikzcd} \right) = 1.$$

Example 5.7. Let p be a prime. The direct limit of the following chain

$$\varinjlim \left(\mathbb{Z} \hookrightarrow \mathbb{Z} \hookrightarrow \ldots \hookrightarrow \mathbb{Z} \xhookrightarrow{x \mapsto px} \mathbb{Z} \hookrightarrow \ldots \right) = \mathbb{Z}[1/p]$$

may be identified with the rationals with denominator a power of p. Similarly,

$$\varinjlim \left(1 \hookrightarrow \mathbb{Z}/p\mathbb{Z} \hookrightarrow \ldots \hookrightarrow \mathbb{Z}/p^n\mathbb{Z} \xhookrightarrow{x \mapsto px} \mathbb{Z}/p^{n+1}\mathbb{Z} \hookrightarrow \ldots \right) = \mathbb{Z}[1/p]/\mathbb{Z}.$$

More generally, the limit of any injective chain of groups

$$\varinjlim \left(H_0 \hookrightarrow H_1 \hookrightarrow \ldots \hookrightarrow H_n \hookrightarrow H_{n+1} \hookrightarrow \ldots \right) = \bigcup_{i \in \mathbb{N}} H_i$$

may be identified with a union of the groups in the appropriate sense.

Example 5.8. For $n \geq 2$, the group

$$G_n = \langle x_1, \ldots, x_n \mid x_1 x_2 x_1^{-1} = x_2^2, x_2 x_3 x_2^{-1} = x_3^2, \ldots, x_n x_1 x_n^{-1} = x_1^2 \rangle$$

may be expressed as the direct limit of the following groups and injective homomorphisms:

$$\begin{tikzcd}[column sep=small] A_1 \arrow[dr, hook] \arrow[rrrrrrrd, hook, bend left=10] & & A_2 \arrow[dl, hook] \arrow[dr, hook] & & \ldots \arrow[dl, hook] \arrow[dr, hook] & & A_n \arrow[dl, hook] \arrow[dr, hook] & \\ & B_1 & & B_2 & \ldots & B_{n-1} & & B_n \end{tikzcd}$$

where $A_i = \langle x_i \rangle \cong \mathbb{Z}$ and $B_i = \langle x_i, x_{i+1} \mid x_i x_{i+1} x_i^{-1} = x_{i+1}^2 \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{2}]$. One may verify by a series of tedious computations that G_2 and G_3 are trivial:

- In G_2, $x_2 = x_1 x_2 x_1^{-1} x_2^{-1} = x_1^{-1}$; thus G is abelian, and $x_i = x_i^2 \implies x_i = 1$.
- In G_3, we have $x_3^{65536} x_2^{16} x_1^2 = x_2^{16} x_3 x_1^2 = x_2^{16} x_1^4 x_3 = x_1^4 x_2 x_3 = x_1^4 x_3^2 x_2 = x_3^2 x_1 x_2 = x_3^2 x_2^2 x_1$. Therefore, $x_1 = x_2^{-16} x_3^{-65534} x_2^2$. The relation $x_1 x_2 x_1^{-1} = x_2^2$ becomes $x_3^{65534} x_2 x_3^{-65534} = x_2^2$, which simplifies to $x_3^{-65534} = x_2$. Consequently, $x_1, x_2 \in \langle x_3 \rangle$, so G_3 is abelian and generated by x_3. Finally, the relation $x_2 x_3 x_2^{-1} = x_3^2$ implies $x_3 = 1$, and G_3 is trivial.

The following result, involving elementary number theory and extremal ideas, provides a deeper reason why G_n cannot be finite but non-trivial.

Lemma 5.9. *G_n has no proper normal subgroup of finite index.*

Proof. Let $K \triangleleft G$ be a normal subgroup of finite index; then $Q = G/K$ is finite. Let m_i be the order of x_i in Q. Since $x_{i+1} = x_i^{m_i} x_{i+1} x_i^{-m_i} = (x_{i+1})^{2^{m_i}}$ in Q, thus $m_{i+1} \mid 2^{m_i} - 1$.

For any prime $p \mid m_{i+1}$, it follows that $p \mid 2^{n_i} - 1$. By Fermat's Little Theorem, $p \mid 2^{\gcd(m_i, p-1)} - 1$. Thus $\gcd(n_i, p-1) > 1$, and n_i has a prime factor strictly smaller than p. A contradiction follows from considering the minimal prime factor among the m_i, unless $m_i = 1$ for all i. However, since Q is generated by the x_i, thus $Q = 1$ and $K = G$. □

However, further investigation will reveal that G_n is non-trivial for $n \geq 4$. In particular, we show that G_4 contains a copy of $\mathbb{F}_2$:

Proposition 5.10. *G_n is non-trivial for $n \geq 4$.*

Proof sketch. Build G_4 by successive amalgams:

- Define $C_{123} = \langle x_1, x_2, x_3 \mid x_1 x_2 x_1^{-1} = x_2^2, x_2 x_3 x_2^{-1} = x_3^2 \rangle \cong B_1 *_{A_2} B_2$.
- Since $A_1 \cap A_2 = 1$ in B_1 and $A_3 \cap A_2 = 1$ in B_2, thus by Lemma 2.11(c), the amalgam C_{123} contains a copy of $A_1 * A_3 \cong \mathbb{F}_2$.
- Similarly, the amalgam $C_{341} = B_3 *_{A_4} B_4$ contains a copy of $A_1 * A_3$.
- Thus $G_4 = C_{123} *_{\mathbb{F}_2} C_{341}$, and the non-trivial groups C_{123}, C_{341}, and $\mathbb{F}_2$ inject into G_4.

By similar arguments, G_n is non-trivial for $n \geq 4$. □

Corollary 5.11. *There exists an infinite simple group generated by 4 elements.*

Proof. Take a quotient of G_n ($n \geq 4$) by a maximal proper normal subgroup (which exists by Zorn's lemma, since G_n is finitely generated). □

We often wish to avoid scenarios such as Examples 5.6 and 5.8 ($G_n, n \leq 3$) where certain relations trivialise the direct limit. We circumvent this by building the group from amalgams and applying the structure theorem to guarantee injectivity. This is summarised in the following result:

Proposition 5.12. *Suppose in Definition 5.1 that all α_{ij} are injective, and there exists a cycle-free subset of the α_{ij} that generate the rest of the maps by composition. Then all maps $f_i : A_i \to G$ are injective, and G is generated by the images of the A_i.*

Proof sketch. The statement is true for finite index sets I, by induction: one may construct G from repeated amalgamations (see, for instance, Example 5.4) and applying the structure theorem. The general case may then be obtained by taking a direct limit. □

5.2 Tree of groups $\equiv$ ($G\backslash X =$ tree)

Definition 5.13. A **graph of groups** (G, Y) consists of a connected graph $Y = (V, E)$, a group G_P for each vertex $P \in V$, and a group G_y for each edge $y = PQ \in E$ along with injections $G_y \to G_P$ and $G_y \to G_Q$.

Each edge y is arbitrarily assigned an orientation, with $\bar{y}$ denoting the opposite orientation. For $a \in G_y$, denote by $a^y, a^{\bar{y}}$ the corresponding images of a in G_P, G_Q respectively.

If Y is a tree, we call (G, Y) a **tree of groups**.

One way to obtain graphs of groups is via tree actions:

Definition 5.14. Let $G \curvearrowright X$ be a tree action without inversion, with quotient graph $Y = G\backslash X$. The **associated graph of groups** (G, Y) to the tree action $G \curvearrowright X$ is defined as follows:

- Pick tree of representatives $\hat{T}$ of $X \bmod G$. Identify $\hat{T}$ with its image T in Y under the quotient map $\pi : X \to Y = G\backslash X$. Note T is a spanning tree of Y.
- For each vertex $\hat{P} \in V(\hat{T})$, assign to the corresponding vertex $P \in V(Y)$ the stabiliser $G_P := \mathrm{Stab}_G(\hat{P})$. For each edge $\hat{y} \in E(\hat{T})$, assign to the corresponding edge $y \in E(Y)$ the stabiliser $G_y := \mathrm{Stab}_G(\hat{y})$. The edge stabilisers inject naturally into the vertex stabilisers.
- For each remaining (oriented) edge $y = PQ \in E(Y) \setminus E(T)$, lift it to edge $\hat{y} = \hat{P}\hat{Q}' \in E(X)$ where $\hat{P} \in V(\hat{T}), \hat{Q}' \notin V(\hat{T})$. Assign to edge y the stabiliser $G_y := \mathrm{Stab}_G(\hat{y})$; note this injects naturally into $G_P = \mathrm{Stab}_G(\hat{P})$ and $\mathrm{Stab}_G(\hat{Q}')$. Let $\hat{Q} \in V(\hat{T})$ correspond to vertex Q; pick representative element g^y that sends $\hat{Q} \mapsto \hat{Q}'$. Then $\mathrm{Stab}_G(\hat{Q}') = g^y G_Q (g^y)^{-1} \cong G_Q$, and G_y also injects into G_Q upon conjugation by g^y. Set $g^{\bar{y}} = (g^y)^{-1}$.

Lemma 5.15. *For the graph of groups (G, Y) constructed in Definition 5.14, the vertex groups G_P and elements g^y for each edge $y \in E(Y) \setminus E(T)$ generate the original group G.*

Proof. The lemma follows from connectivity of tree X; the idea is similar to the argument in §3.6.

Let $g \in G$ be an element. Induct on $m = d(\hat{T}, g\hat{T})$. If $m = 0$, then $\hat{T}$ and $g\hat{T}$ share some vertex $P \in V(\hat{T})$, thus $g \in G_P$.

If $m > 0$, let $\hat{y} = \hat{P}\hat{Q}'$ be the first edge along geodesic $\hat{T}$—$g\hat{T}$. Then $\hat{P} \in V(\hat{T}), \hat{Q}' \notin V(\hat{T})$. Let $\hat{Q} \in V(\hat{T})$ and $\hat{Q}'$ project to the same vertex $Q \in V(T)$. Then one of the following holds:

1. $\pi(\hat{y}) \in E(T)$. Then there is some $g_1 \in G_P$ sending edge $\hat{P}\hat{Q} \mapsto \hat{y} = \hat{P}\hat{Q}'$. Note $\hat{Q}' \in g_1\hat{T}$.
2. $\pi(\hat{y}) \in E(Y) \setminus E(T)$. Then $g_1 = g^y$ sends vertex $\hat{Q} \mapsto \hat{Q}'$. Note $\hat{Q}' \in g^y\hat{T}$.

In either case, one may apply induction hypothesis to $h = g_1^{-1}g \in G$, noting that $m = d(\hat{T}, g\hat{T}) > d(g_1\hat{T}, g\hat{T}) = d(\hat{T}, h\hat{T})$. In particular, this algorithm generates an explicit expression $g = g_1 g_2 \ldots g_n$ where each g_i belongs either to a vertex group G_P, or is g^y for some (oriented) $y \in E(Y) \setminus E(T)$. $\square$

Recall that Corollary 3.30 establishes a direct correspondence between amalgam $G = G_1 *_A G_2 = \varinjlim(G, Y)$ and an associated tree action $G \curvearrowright X$. The associated segment of groups is

$$(G, Y) = \left[G_1 \bullet \xrightarrow{\quad A \quad} \bullet\, G_2 \right]$$

We now extend this to a correspondence between a general tree of groups (G, Y) and tree action $G \curvearrowright X$ without inversion, with fundamental domain $Y \cong G\backslash X$ and corresponding vertex and edge stabilisers. The tools and ideas are drawn from §3.6.

Theorem 5.16. *(correspondence for trees of groups) Let (G, Y) be a tree of groups. There is a tree action $G \curvearrowright X$ without inversion, unique up to isomorphism, with associated graph of groups (G, Y). Furthermore, $G = \varinjlim(G, Y)$ may be identified with the direct limit.*

We refer to G and $G \curvearrowright X$ as the ***associated group and tree action*** *to the tree of groups (G, Y).*

Proof. By Proposition 5.12, the vertex and edge groups of (G, Y) inject into G, and generate G. One may thus construct tree X in an analogous manner to Definition 3.25, with vertex and edge sets

$$V(X) = \bigsqcup_{P \in V(Y)} G/G_P, \quad E(X) = \bigsqcup_{y \in E(Y)} G/G_y$$

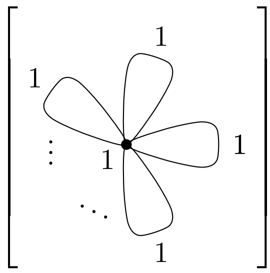
where edge gG_y (for $y = PQ \in E(Y)$) adjoins vertices gG_P and gG_Q. The action $G \curvearrowright X$ is defined naturally by left multiplication. By a similar argument to Lemma 3.28, X is connected (by Lemma 5.15, the vertex groups generate G) and cycle-free (by the structure theorem for amalgams), thus it is a tree. Furthermore, identifying $P \in V(Y)$ with coset G_P, the vertex and edge sets are indeed those in (G, Y). The converse association in Definition 5.14 implies uniqueness. $\square$

Corollary 5.17. *Let (G, Y) be a tree of groups associated with tree action $G \curvearrowright X$.*

1. *If the vertex groups G_P are countable, then so is G.*
2. *If the vertex groups G_P are torsion-free, then so is G.*

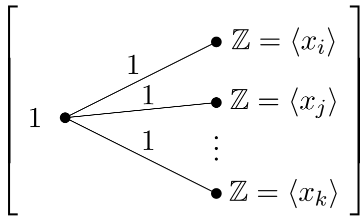
Proof. (a) follows from Lemma 5.15. (b) follows from Lemma 2.11(e) and the fact that for finite Y, $G = \varinjlim(G, Y)$ may be built from successive amalgams (with the case of infinite Y resolved by taking a direct limit). $\square$

Example 5.18. The free action of $G = \mathbb{F}_S$ on its Cayley graph $X = \Gamma(G, S)$ has associated graph of groups (G, Y) being a bouquet of loops indexed by S at a single vertex, with all vertex and edge groups being trivial:

$(G, Y) =$ 

Conversely, $F_S = \pi_1(Y)$ is the fundamental group of a bouquet of loops indexed by S.

Alternatively, consider the tree action $\mathbb{F}_S \curvearrowright X'$ associated with viewing F_S as a free product of copies of $\mathbb{Z}$ indexed by S (see Example 5.4 and Theorem 5.16). Note that X' is the point-line incidence graph of X restricted to 'S-parallel' lines, i.e. lines $g\langle x_i \rangle$ for $g \in G, x_i \in S$. The associated graph of groups is

$(G', Y') =$ 

We also establish a structure theorem, generalising Theorem 2.8:

Definition 5.19. Let (G, Y) be a tree of groups, with associated tree action $G \curvearrowright X$ and $Y \subset X$ identified with a fundamental domain. Fix base point $P_0 \in V(Y)$.

A **word** (c, μ) consists of a closed walk $c = P_0 y_1 P_1 y_2 \dots y_n P_n$ starting and ending at $P_0 = P_n$, and elements $\mu = (g_0, \dots, g_n)$ with $g_i \in G_{P_i}$ for each i. The following concise notation is used:

$$(c, \mu) = g_0 y_1 g_1 y_2 \cdots y_n g_n.$$

The **projection** to G is denoted by

$$|c, \mu| = g_0 g_1 \dots g_n \in G.$$

The **lift** of a word (c, μ) to X is the walk $\hat{c} = \hat{P}_0 \hat{y}_1 \hat{P}_1 \hat{y}_2 \dots \hat{y}_n \hat{P}_n$ beginning at $\hat{P}_0 = P_0$, with $\hat{y}_i = g_0 g_1 \dots g_{i-1} y_i$ and $\hat{P}_i = g_0 g_1 \dots g_{i-1} P_i$ for each $i = 1, \dots, n$. (Note: since g_{i-1} fixes vertex

P_{i-1}, thus edge $\hat{P}_{i-1}\hat{y}_i\hat{P}_i$ is the image of edge $P_{i-1}y_iP_i$ under the action of element $g_0g_1\dots g_{i-1}$. Therefore, $\hat{c}$ is indeed a walk in X.)

The **length** of a word is $l(c,\mu) = n$.

A word (c,μ) is **reduced** if either $n = 0$ and $g_0 \neq 1$, or $n \geq 1$ and $g_i \in G_{P_i} \setminus G_{y_i}$ for all vertices P_i for which the walk backtracks (i.e. for each $i = 0,\dots,n$ for which $y_{i+1} = \bar{y}_i$).

Proposition 5.20. *With reference to Definition 5.19,*

(a) Any $g \in G$ is the projection of some word (c,μ).

(b) Any word (c,μ) may be shortened to a reduced word with the same projection to G.

(c) If (c,μ) is a reduced word of length n and projection $g = |c,\mu|$, then $d(P_0, gP_0) = n$, and c is the projection of geodesic P_0—gP_0 to $Y = G\backslash X$.

(d) $|c,\mu| \neq 1$ for any non-trivial reduced word (c,μ).

(e) Let $c^{-1} = P_n\bar{y}_nP_{n-1}\dots\bar{y}_1P_1\bar{y}_0P_0$ and $\mu^{-1} = (g_n^{-1},\dots,g_1^{-1},g_0^{-1})$. Then $\left|c^{-1},\mu^{-1}\right| = |c,\mu|^{-1}$.

(f) (c,μ) is reduced $\iff$ (c^{-1},μ^{-1}) is reduced.

(g) Every $g \in G$ has a unique reduced word expression (with respect to base point P_0).

Proof. (a): by Lemma 5.15, G is generated by vertex groups G_P. One may thus write $g = h_1\dots h_m$ for elements $h_j \in G_{Q_j}$, and pick a closed walk $c = P_0y_1P_1y_1\dots y_nP_n$ passing through points $P_0, Q_1, \dots, Q_m, P_0$ in order. Let $P_{k_j} = Q_j$ for indices $k_1 < \cdots < k_m$; one then sets μ accordingly by $g_{k_j} = h_j$, and the remaining $g_i = 1$.

(b): Let $c = \dots y_{i-1}P_{i-1}y_iP_iy_{i+1}P_{i+1}y_{i+2}\dots$ be a closed walk with backtracking $y_{i+1} = \bar{y}_i$, and $g_i \in G_{y_i}$. Then $g_i \in G_{y_i} \leq G_{P_{i-1}}$, so we may shorten the length of walk c by 2, sending $c \mapsto c' = \dots y_{i-1}P_{i-1}y_{i+2}\dots$ and $\mu = (\dots, g_{i-1}, g_i, g_{i+1}, \dots) \mapsto \mu' = (\dots, g_{i-1}g_ig_{i+1}, \dots)$. We may exhaustively perform this reduction until a reduced word is obtained; this shows the existence of a reduced word expression for every $g \in G$ (where $g = 1$ corresponds to the empty word).

(c) follows from the fact that the condition of word (c,μ) being reduced implies that the lift $\hat{c}$ to X is a walk without backtracking, and thus traces out the geodesic P_0—gP_0 of length n.

(d) follows from (c), with the case $l(c,\mu) = 0$ addressed by the fact that G_{P_0} injects into G.

(e) and (f) follow by definition.

(g): let (c,μ) and (c',μ') be two reduced words projecting to the same $g \in G$. By (c), both words lift to the geodesic P_0—gP_0 in X; consequently, walks $c = c'$ are both equal to the projection of geodesic P_0—gP_0 to $Y = G\backslash X$.

Now consider concatenating (c^{-1},μ^{-1}) with (c',μ'); the resulting word is

$$(c^{-1}c', \mu^{-1}\mu') = g_n^{-1}\bar{y}_n\dots g_1^{-1}\bar{y}_1(g_0^{-1}g_0')y_1g_1'\dots y_ng_n'$$

which by (e), projects to $1 \in G$. However, by (f), the only possible reduction to $(c^{-1}c', \mu^{-1}\mu')$ is at the midpoint of the walk. By (d), it is possible to repeatedly reduce $(c^{-1}c', \mu^{-1}\mu')$ at its midpoint until the empty word is obtained. Reversing the process, we then have $g_n = g_n', \dots, g_1' = g_1, g_0 = g_0$. Thus $(c,\mu) = (c',\mu')$, and the unique reduced word expression is indeed unique. □

Corollary 5.21 (structure theorem for trees of groups)**.** *Let (G,Y) be a tree of groups, and $F(G,Y)$ the group generated by vertex groups G_P and (oriented) elements $y \in E(Y)$, subject to relations*

$$\bar{y} = y^{-1} \quad \text{and} \quad ya^y\bar{y} = a^{\bar{y}} \quad \forall y \in E(Y), a \in G_y.$$

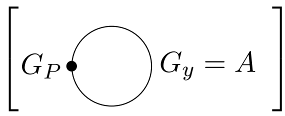
Let $\pi_1(G,Y,P_0)$ be the subgroup of $F(G,Y)$ consisting of words (c,μ). Then

$$G \cong \pi_1(G,Y,P_0).$$

Proof. This follows from Proposition 5.20(g), noting that the projection $\pi_1(G, Y, P_0) \to G$ is surjective and has trivial kernel. $\square$

5.3 HNN extension $\equiv$ ($G\backslash X$ = loop)

We now investigate tree actions $G \curvearrowright X$ without inversion, with associated graph of groups (G, Y) being a loop:

$(G, Y) =$ 

with two injections $A \hookrightarrow G_P$. Identifying $A \leq G_P$ with one of its images in G_P, the other injection corresponds to an injective homomorphism $\theta : A \to A' \leq G$.

According to Definition 5.14, this relates to the original action $G \curvearrowright X$ as follows: there is edge $y = PQ$ of the original tree X, whose vertex and edge stabilisers are $\mathrm{Stab}_G(P) = G_P$, $\mathrm{Stab}_G(y) = A \leq G_P$, and $\mathrm{Stab}_G(Q) = g^y G_P (g^y)^{-1}$ for some element $g^y \in G$ sending $P \mapsto Q$. In particular, the inclusion $\mathrm{Stab}_G(y) \leq \mathrm{Stab}_G(Q)$ implies

$$\theta(a) = (g^y)^{-1} a g^y \quad \forall a \in A.$$

In other words, the subgroups $A, A' = \theta(A) \leq G_P$ are 'glued together' via conjugation by an element g^y of the parent group G.

Similar to Theorem 5.16, given a loop of groups $(G, Y) = (G_P, \theta)$ (i.e. vertex group G_P, subgroup $A \leq G_P$, and injective homomorphism $\theta : A \to A' \leq G_P$), we wish to construct a group action $G \curvearrowright X$ without inversion on a tree X, whose corresponding graph of groups is the loop (G, Y).

Definition 5.22. Let $A \leq G$ be groups and $\theta : A \to G$ be an injective homomorphism. The **HNN extension**, denoted $G*_\theta$, is defined as follows: let $\tilde{G}$ be the direct limit of the following groups and injective maps:

$$\begin{tikzcd}
& A_{[n-1]} \arrow[dl, hook, "\mathrm{id}"'] \arrow[dr, hook, "\theta"] & & A_{[n]} \arrow[dl, hook, "\mathrm{id}"'] \arrow[dr, hook, "\theta"] & & A_{[n+1]} \arrow[dl, hook, "\mathrm{id}"'] \arrow[dr, hook, "\theta"] & \\
\cdots & & B_{[n]} & & B_{[n+1]} & & \cdots
\end{tikzcd}$$

where the $A_{[n]}$ and $B_{[n]}$ are copies of A and G respectively. Then $G*_\theta = \mathbb{Z} \ltimes \tilde{G}$ is a semidirect product, where $\mathbb{Z} = \langle t \rangle$ and $t : \tilde{G} \to \tilde{G}$ is the 'left translation' sending $A_{[n]} \to A_{[n-1]}, B_{[n]} \to B_{[n-1]}$.

Equivalently, if G has presentation $\langle S \mid \mathcal{R} \rangle$, then

$$G*_\theta = \langle S, t \mid \mathcal{R}, tat^{-1} = \theta(a) \quad \forall a \in A \rangle$$

where t is a symbol not in S.

Lemma 5.23 (Higman-Neumann-Neumann)**.** *Let $A \leq G$ be groups and $\theta : A \to G$ be an injective homomorphism.*

(a) $G*_\theta$ contains G and an element $t \in G*_\theta$ such that $\theta(a) = tat^{-1}$ for all $a \in A$.

(b) If G is countable, then so is $G*_\theta$.

(c) If G is torsion-free, then so is $G*_\theta$.

Proof. (a): by Proposition 5.12, the groups $A_{[n]}$ and $G_{[n]}$ inject into $\tilde{G}$, which injects into $G*_\theta$. The result follows by identifying A with $A_{[0]}$ and G with $B_{[0]}$.

(b) follows from Lemma 5.15. (c) follows from Corollary 5.17 applied to $\tilde{G}$. $\square$

Theorem 5.24. *(correspondence for HNN extensions) Let $(G, Y) = (G_P, \theta)$ be a loop of groups. There is tree action $G \curvearrowright X$ without inversion, unique up to isomorphism, with associated graph of groups (G, Y). Furthermore, $G = (G_P)*_\theta$ may be identified with the HNN extension.*

We refer to G and $G \curvearrowright X$ as the ***associated group and tree action*** *to the loop of groups (G, Y).*

Proof. By Theorem 5.16, $\tilde{G}$ is associated with an action $\tilde{G} \curvearrowright X$ the tree of groups

$$(\tilde{G}, \tilde{Y}) = \left[\ \cdots \rule{3em}{0.4pt}\underset{B_{[n-1]}}{\bullet}\overset{A_{[n-1]}}{\rule{5em}{0.4pt}}\underset{B_{[n]}}{\bullet}\overset{A_{[n]}}{\rule{5em}{0.4pt}}\underset{B_{[n+1]}}{\bullet}\overset{A_{[n+1]}}{\rule{5em}{0.4pt}}\underset{B_{[n+2]}}{\bullet}\rule{3em}{0.4pt}\cdots\ \right]$$

where the line $\tilde{Y}$ is a fundamental domain. Let vertex $P_{[k]} \in V(X)$ have stabiliser $B_{[k]}$, and $y_{[k]} = P_{[k]}P_{[k+1]} \in E(\tilde{X})$ have stabiliser $A_{[k]}$. Denote by $g_{[k]}$ the element of $B_{[k]}$ corresponding to $g \in G_P$.

The translational symmetry of $(\tilde{G}, \tilde{Y})$ induces translational symmetry of X; in particular, there is automorphism $t \in \mathrm{Aut}(X)$ performing left-translation by one segment along $\tilde{Y}$, defined in one of the following equivalent ways:

1. By Lemma 5.15, any vertex $Q \in V(X)$ is expressible as $Q = (g_1)_{[k_1]} \dots (g_n)_{[k_n]} P_{[m]}$. Then
$$tQ = (g_1)_{[k_1-1]} \dots (g_n)_{[k_n-1]} P_{[m-1]}.$$
2. With reference to the proof of Theorem 5.16, t acts on vertices and edges of X (represented by left cosets of $\tilde{G}$) by $gB_{[n]} \mapsto t(g)B_{[n-1]}$ and $gA_{[n]} \mapsto t(g)A_{[n-1]}$ respectively.

In particular, automorphism $t \in \mathrm{Aut}(\tilde{X})$ satisfies
$$tg_{[k]}t^{-1} = g_{[k-1]} \in \mathrm{Aut}(\tilde{X}) \quad \forall g \in G_P, k \in \mathbb{Z}.$$

Therefore, the action $\tilde{G} \curvearrowright X$ extends to vertex-transitive action $G \curvearrowright X$. Since $\langle t \rangle$ acts freely on $\tilde{Y}$, thus $\mathrm{Stab}_G(P_{[k]}) = \mathrm{Stab}_{\tilde{G}}(P_{[k]}) = (G_P)_{[k]}$ and $\mathrm{Stab}_G(y_{[k]}) = \mathrm{Stab}_{\tilde{G}}(y_{[k]}) = (G_y)_{[k]}$. It follows that $G \curvearrowright X$ is indeed associated with the loop of groups (G, Y). The converse association in Definition 5.14 implies uniqueness. □

The following result is one application of the HNN extension:

Corollary 5.25.

(a) Every group G may be embedded in a group K such that all elements of the same order are conjugate.

(b) There exists a countable torsion-free group with exactly 2 conjugacy classes.

Proof. (a): For elements $g, h \in G \setminus \{1\}$ of the same order, define isomorphism $\theta : \langle g \rangle \mapsto \langle h \rangle$ sending $g \mapsto h$, and the HNN extension $G_{g,h} = G*_\theta$ containing G, and in which g, h are conjugate.

Let $G' = *_G\{G_{g,h}\}$ be the amalgam of all such HNN extensions. By the structure theorem for amalgams, G and each $G_{g,h}$ inject into G'. Consequently, every pair of elements of G of the same order are conjugate in G'.

Iteratively construct a chain of groups $G = G_0 \leq G_1 \leq G_2 \leq \dots$ where every pair of elements of G_n of the same order are conjugate in G_{n+1}. The result then follows from taking a direct limit (see Example 5.7).

(b) follows from (a) applied to the non-trivial countable torsion-free group $\mathbb{Z}$, noting by Lemmas 2.12 and 5.23 that the successive HNN extensions, amalgams, and direct limits remain countable and torsion-free. □

5.4 Graph of groups $\equiv$ ($G\backslash X$ = graph)

We finally generalise Theorems 5.16 and 5.24 to establish the general correspondence between graph of groups (G, Y) and tree actions $G \curvearrowright X$ without inversion.

Theorem 5.26 (correspondence for graphs of groups)**.** *Let (G, Y, T) be a graph of groups (G, Y) with spanning tree T. There is a tree action $G \curvearrowright X$ without inversion, unique up to isomorphism, with associated graph of groups (G, Y) (as in Definition 5.14, with tree of representatives T).*

We refer to G and $G \curvearrowright X$ as the ***associated group and tree action*** *to graph of groups (G, Y).*

Proof. We first extend (G, Y) to a tree of groups $(\tilde{G}, \tilde{Y})$, where $\tilde{Y}$ is the universal covering tree of Y (see Definition 3.2), with vertex and edge groups lifted accordingly. Identifying $T \subset \tilde{Y}$ with a subtree and lifting each edge of $E(Y) \setminus E(T)$ to an edge of $\tilde{Y}$ adjacent to the lift of T, one may then index vertices, edges, vertex groups, and edge groups of $\tilde{Y}$ by

$$P_{[k]}, y_{[k]}, (G_P)_{[k]}, (G_y)_{[k]}, \quad P \in V(Y), y \in E(Y), k \in \pi_1(Y, T)$$

respectively, where $\pi_1(Y, T)$ is the fundamental group of Y with spanning tree T contracted to a point. By Theorem 5.16, $(\tilde{G}, \tilde{Y})$ is associated with a tree action $\tilde{G} \curvearrowright X$.

There is natural action $\pi_1(Y, T) \to \mathrm{Aut}(\tilde{G})$, defined as follows: each $k \in \pi_1(Y, T)$ corresponds to an automorphism $\tilde{G} \to \tilde{G}$ induced by maps $(G_P)_{[l]} \to (G_P)_{[kl]}, g_{[l]} \mapsto g_{[kl]}$.

Defining $G = \pi_1(Y, T) \ltimes \tilde{G}$, the action $\tilde{G} \curvearrowright X$ induces action $G \curvearrowright X$, defined in one of the following equivalent ways:

1. Any vertex $Q \in V(X)$ is expressible as $Q = (g_1)_{[l_1]} \dots (g_n)_{[l]} P_{[m]}$. Then for $k \in \pi_1(Y, T)$,
$$kQ = (g_1)_{[kl_1]} \dots (g_n)_{[kl]} P_{[km]}.$$
2. With reference to the proof of Theorem 5.16, each $k \in \pi_1(Y, T)$ acts on vertices and edges of X (represented by left cosets of $\tilde{G}$) by $g(G_P)_{[l]} \mapsto k(g)(G_P)_{[kl]}$ and $g(G_y)_{[l]} \mapsto k(g)(G_y)_{[kl]}$ respectively.

Since $\pi_1(Y, T)$ acts freely on $\tilde{Y}$, thus $\mathrm{Stab}_G(P_{[k]}) = \mathrm{Stab}_{\tilde{G}}(P_{[k]}) = B_{[k]}$ and $\mathrm{Stab}_G(y_k) = \mathrm{Stab}_{\tilde{G}}(y_k) = A_{[k]}$. It follows that $G \curvearrowright X$ is indeed associated with the loop of groups (G, Y). The converse association in Definition 5.14 implies uniqueness. □

Remark 5.27. We summarise the tools used to prove Theorem 5.26:

- We first establish the result for a segment of groups (Corollary 3.30) using the structure theorem of amalgams $G = G_1 *_A G_2$. Importantly, we show that the edge and vertex groups inject into, and generate G. The tree X may be constructed using cosets, or alternatively by 'iterative path construction' beginning from an initial segment.
- Next, we establish the result for a tree of groups (Theorem 5.16) by taking successive amalgams (and taking the direct limit in the case where Y is infinite).
- We then establish the result for a loop of groups (Theorem 5.24) using HNN extensions.
- Finally, we establish the result for a general graph of groups, generalising the definitions and results concerning HNN extensions.

One may alternatively prove Theorem 3.25 by taking successive amalgams and HNN extensions (and taking the direct limit in the case where Y is infinite).

Theorem 5.28 (structure theorem for graphs of groups)**.** *Let (G, Y) be a graph of groups.*

(a) Let $P_0 \in V(Y)$ be a base point. Define $F(G, Y)$ and $\pi_1(G, Y, P_0)$ as in Corollary 5.21. Then

$$G \cong \pi_1(G, Y, P_0).$$

(b) Let $T \subset Y$ be a spanning tree. Denote by $\pi_1(G, Y, T)$ the quotient of $F(G, Y)$ by the normal subgroup generated by elements $y \in E(T)$. Then

$$G \cong \pi_1(G, Y, T).$$

Proof sketch. (a): applying Corollary 5.21 to tree of groups $(\tilde{G}, \tilde{Y})$ shows $\tilde{G} \cong \pi_1(\tilde{G}, \tilde{Y}, P_0)$, where spanning tree T is identified with subtree $T_{[1]} \subset X$. The isomorphism is then given by identifying

$$g = (\tilde{c}, \tilde{\mu})k = (g_0)_{[1]}(y_1)_{[k_1]}(g_1)_{[l_1]} \dots (y_n)_{[k_n]}(g_n)_{[l_n]}k \in G = \pi_1(G, Y) \ltimes \tilde{G}$$

with

$$g = (c, \mu) = g_0 y_1 g_1 \dots y_n g_n k \in \pi_1(G, Y, P_0)$$

where $k \in \pi_1(G, Y) \cong \pi_1(G, P_0)$ corresponds to a closed walk in Y beginning and ending at P_0. The converse identification may be obtained as follows: lift closed walk c in Y to walk $\hat{c}$ of $\tilde{Y}$ starting at P_0, then complete the walk back to P_0 to obtain $\tilde{c}$. Let $k \in \pi_1(G, Y)$ be the element such that kP_0 is the endpoint of $\hat{c}$.

This identification provides isomorphism $G \cong \pi_1(G, Y, P_0)$. (b) follows similarly. □

Corollary 5.29. *Let (G, Y) be a graph of groups with all edge groups being trivial. Then*

$$G \cong \left(*\{G_P\}_{P \in V(Y)}\right) * \pi_1(Y, T).$$

Proof. This follows from Theorem 5.28(b), noting that $G = \pi_1(G, Y, T)$ is generated by the G_P and elements $y \in E(Y) \setminus T$ with no additional relations. □

5.5 Applications

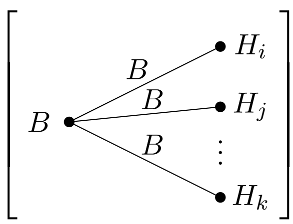
We conclude this section with applications of the general correspondence. In particular, we generalise several results in §3.

Proposition 5.30 (generalisation of Lemma 2.11(e))**.** *Let $G = *_A G_i$ be an amalgam, and $H \leq G$ be a subgroup with $\max_{h \in H} l(h) < \infty$; say that H is* ***bounded****. Then H is contained in a conjugate of some G_i.*

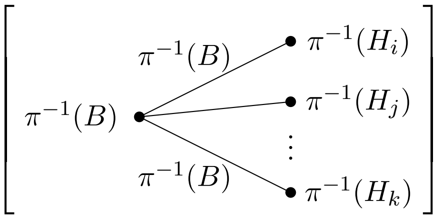
Proof. Let $G \curvearrowright X$ be the associated tree action, with fundamental domain Y. The condition that H is bounded is equivalent to the G-orbit of Y being bounded in diameter. The result then follows from Lemma 3.9(d) applied to the finite-diameter subtree $Z \subset X$ spanned by images of Y under G-action (see Lemma 3.8(c)), noting that Z is stable under G-action. □

Proposition 5.31 (generalisation of Lemma 2.13)**.** *Let $\pi : G \to H$ be a surjective homomorphism. If $H = *_B\{H_i\}$, then $G = *_{\pi^{-1}(B)}\{\pi^{-1}(H_i)\}$.*

Proof. Let $H \curvearrowright X$ with fundamental domain Y and tree of graphs

$(H, Y) =$ 

This induces action $G \curvearrowright X$ with the same fundamental domain Y, but with vertex and edge stabilisers

$(G,Y) =$ 

By Theorem 5.16, this is the amalgam $G = *_{\pi^{-1}(B)}\{\pi^{-1}(H_i)\}$. $\square$

Proposition 5.32. *Let H be a subgroup of the group G associated with the graph of groups (G,Y) such that $H \setminus \{1\}$ does not meet any conjugate of the vertex groups G_P. Then H is a free group.*

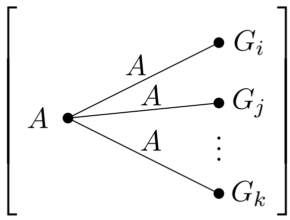
Proof. The argument is identical to Proposition 3.31: The associated tree action $G \curvearrowright X$ of the amalgam has vertex stabilisers being conjugates of the vertex groups. The condition is thus equivalent to the subgroup action $H \curvearrowright X$ being free; the result then follows from Corollary 3.25(a). $\square$

Theorem 5.33 (Kurosh)**.** *Let $G = *_A G_i$ be an amalgam, and $H \leq G$ a subgroup such that $H \setminus \{1\}$ doesn't intersect any conjugate of A. For each $i \in I$, let subset $X_i \subset G/G_i$ be a system of coset representatives for $H\backslash G/G_i$, and let $H_{i,x} = H \cap xG_ix^{-1}$ for each $x \in X_i$. Then*

$$H \cong (*\{H_{i,x}\}_{i \in I, x \in X_i}) * F$$

is the free product of the intersections of H with conjugates of G_i, and a free group F.

Proof. The amalgam $G = *_A G_i$ is associated with the tree of graphs

$(G,Y) =$ 

and tree action $G \curvearrowright X$ without inversion. The condition that $H \setminus \{1\}$ doesn't intersect any conjugate of A is equivalent to the subgroup action $H \curvearrowright X$ having no fixed edges. Therefore, the graph of groups (H,Z) associated with $H \curvearrowright X$ has trivial edge groups.

Let $T \subset Z$ be a spanning tree, identified with a subtree of X. By Corollary 5.29,

$$H \cong \big(*\{H_P\}_{P \in V(Z)}\big) * F$$

where $F = \pi_1(Y,T)$ is a free group.

By the proof of Theorem 5.16, vertices of X may be identified with cosets gG_i. The orbit of one such coset under H is the double coset HgG_i. Thus if $V(Z) = \{xG_i : i \in I, x \in X_i\}$, then the X_i indeed form a system of coset representatives for $H\backslash G/G_i$. The vertex stabilisers of $P = xG_i \in V(Z)$ are then given by

$$H_P = \mathrm{Stab}_H(P) = H \cap \mathrm{Stab}_G(P) = H \cap xG_ix^{-1} = H_{i,x}$$

which concludes the result. $\square$

Example 5.34 ($\mathrm{PSL}_2(\mathbb{Z})$ revisited)**.** With reference to §4.2, let $H \leq \mathrm{PSL}_2(\mathbb{Z})$ be the subgroup generated by elements $c = bab^{-1} = \pm\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ and $d = ab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note c corresponds to π-rotation about point $R = \frac{-1+i}{2}$, while d corresponds to right-translation by 1 unit in $\mathcal{H}$.

Applying Tits' ping-pong lemma to $X_c = \{z : \left|z + \frac{1}{2}\right| < \frac{1}{2}\}$ and $X_d = \{z : \left|\mathrm{Re}(z) + \frac{1}{2}\right| > \frac{1}{2}\}$ shows $H = \langle c \rangle * \langle d \rangle \cong C_2 * \mathbb{Z}$. This agrees with the conclusion of Kurosh's theorem or Corollary 5.29: considering the graph of groups (H, Y) associated with $H \curvearrowright X$, note $H_R = \langle c \rangle$ with all other vertex and edge groups being trivial, and $\pi_1(H\backslash X) \cong \mathbb{Z}$.

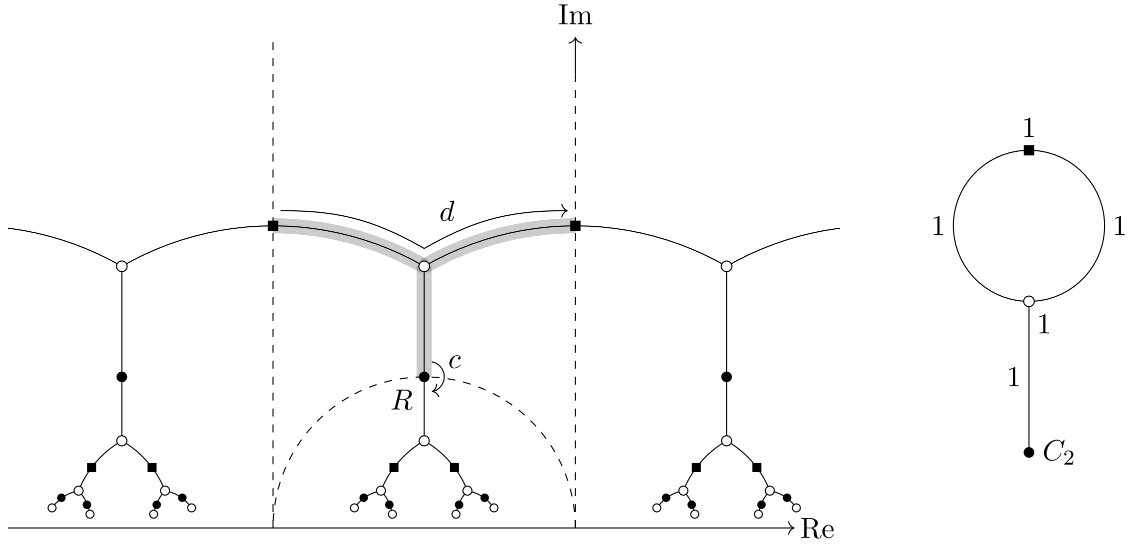


Figure 8: (left) The subgroup action of $H \in \mathrm{PSL}_2(\mathbb{Z})$ on the tree X embedded in hyperbolic half-plane $\mathcal{H}$ (see §4.2). A lift of edges of $Y = H\backslash X$ to a subtree of X is shaded. (right) The graph of groups (H, Y) associated with $H \curvearrowright X$.

Let $K \triangleleft H$ be the kernel of projection $H = C_2 * \mathbb{Z} \to \mathbb{Z}$. Note K is generated by π-rotations $c_n = d^n c d^{-n}$ for $n \in \mathbb{Z}$. Applying Tits' ping-pong lemma to $X_n = \{z : \left|z - n + \frac{1}{2}\right| < \frac{1}{2}\}$ shows

$$K \cong *\{\langle c_n \rangle\}_{n \in \mathbb{Z}} = \cdots * C_2 * C_2 * \ldots$$

The same conclusion follows from Kurosh's theorem or Corollary 5.29: the $\mathbb{Z}$-translates of the shaded subtree in Figure 8 form a fundamental domain of $X \bmod K$. The associated tree of groups (K, Z) to action $K \curvearrowright X$ has vertex groups C_2 indexed by $\mathbb{Z}$, with all other vertex/edge groups trivial.

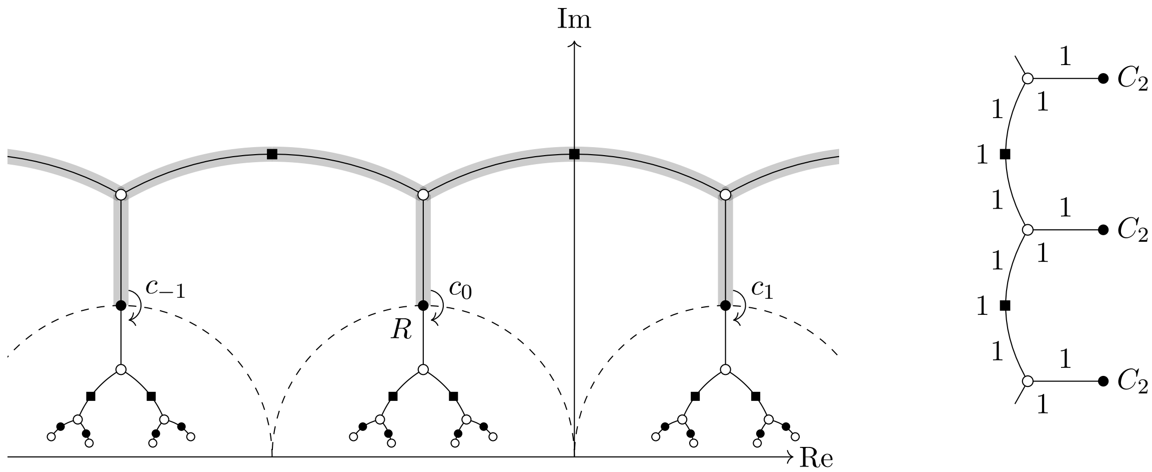


Figure 9: (left) The subgroup action $K \curvearrowright X$, with a fundamental domain shaded. (right) The associated tree of groups (K, Z).

6 Non-amalgams

In this section, we exploit the geometric correspondence established in §5 to establish properties that guarantee that a group cannot possibly be an amalgam. In particular, we will show that the following groups are not amalgams:

1. Triangle reflection groups W_{lmn} for $l, m, n \geq 2$.
2. $\mathrm{Aut}(\mathbb{F}_n)$ for $n \geq 3$.
3. $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, and its subgroups of finite index.

6.1 The (FA) property

The key result that relates non-realisability of a group as an amalgam to its geometric properties is the following:

Theorem 6.1. *Let G be a countable group. The following are equivalent:*

(a) Any action $G \curvearrowright X$ without inversion on a tree X has $X^G \neq \emptyset$.

(b) The following three conditions are satisfied:

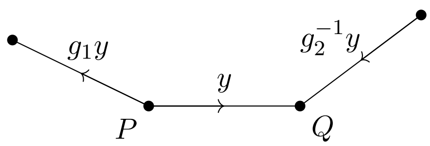
(b.i) G is not a non-trivial amalgam;

(b.ii) G has no quotient isomorphic to $\mathbb{Z}$;

(b.iii) G is finitely generated.

A group G satisfying these conditions is said to have ***property (FA)****.*

Proof. (a) $\implies$ (b.i): if $G = G_1 *_A G_2$ is a non-trivial amalgam, let $G \curvearrowright X$ be the associated action on tree X. Let $y = PQ \in E(X)$ with vertex and edge stabilisers $\mathrm{Stab}_G(P) = G_1, \mathrm{Stab}_G(Q) = G_2$, and $\mathrm{Stab}_G(y) = A$ (see Corollary 3.30). If $g_1 \in G_1 \setminus A$ and $g_2 \in G_2 \setminus A$, then $g_2^{-1}y$ and $g_1 y$ are distinct and coherent; by Corollary 6.8, $g_1 g_2$ acts hyperbolically on X. Thus $X^G = \emptyset$.



(a) $\implies$ (b.ii): if G has a quotient isomorphic to $\mathbb{Z}$, then it can act by translations of the line $\mathbb{Z}$ with no fixed points.

(a) $\implies$ (b.iii): if G is not finitely generated, then it is the union of an infinite non-stationary chain $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq \dots$ of finitely generated subgroups. Then the action of G on the coset tree (see Example 3.17) has no fixed points.

(b) $\implies$ (a): $\pi_1(G\backslash X)$ is a free group isomorphic to a quotient of G; by (b.ii), $\pi_1(G\backslash X)$ is trivial and $G\backslash X$ is a tree. Let T be a tree of representatives of $X \mod G$; by (prev lemma), $G = \varinjlim(G, T)$. by (b.iii), there exists finite subtree T' of T such that $G = \varinjlim(G, T')$.

If T' is non-trivial, then G is an amalgam (consider a leaf of T'), contradicting (b.i). Therefore T' is a single point, and G has a fixed point. ☐

Remark 6.2. When G is uncountable, the theorem remains true with (b.iii) replaced by

(b.iii') G is not the union of a strictly increasing infinite sequence of subgroups.

Why is property (FA) of interest? Similar to Example 3.16, given any action $G \curvearrowright X$ on a tree X, one may extend it to an action $G \curvearrowright \hat{X}$ on a larger tree $\hat{X}$ containing subtree X, under which X is stable under G. Therefore, the structure of G is mainly determined by 'minimal G-tree actions', i.e. actions on a tree X with no proper G-stable subtree. The groups with property (FA) are those for which the only minimal G-tree is a single vertex.

Theorem 6.1 also provides a means to showing a group G is not an amalgam: it suffices to show any G-action on a tree has a fixed point.

Corollary 6.3. *Any group G with property (FA) cannot arise from a non-trivial amalgam, HNN construction, or more generally, from a graph of groups.*

Proof. Suppose action $G \curvearrowright X$ on a tree X arises from a non-trivial graph of groups (G, Y). If Y is a tree, then G is indeed an amalgam. If Y contains a cycle, Theorem 5.20 explicitly guarantees elements of G that act hyperbolically on X. □

Lemma 6.4. *Let G be a group.*

(a) If G is finite, then it has property (FA).

(b) If $K \triangleleft G$ is a normal subgroup and both K and G/K have property (FA), then G has property (FA).

(c) If $H \leq G$ is a subgroup of finite index and H has property (FA), then G has property (FA).

(d) If G has property (FA), every quotient of G has property (FA).

(e) If G has property (FA) and is contained in an amalgam $G_1 *_A G_2$, then G is contained in a conjugate of G_1 or G_2.

Proof. (a) follows from the action on $G_1 *_A G_2$ on a tree (name) has vertex stabilisers being conjugates of G_1 and G_2. (b) follows from Lemma 3.23(c), noting that $G \curvearrowright X$ on a tree X induces action $G/K \curvearrowright X^H$. (c) follows from (a) and (b) by noting that the intersection of conjugates of H is a normal subgroup of G of finite index. (d) follows from lifting the action of a quotient to G to a G-action. (e) follows from the fact that the associated action of the amalgam on a tree has vertex stabilisers being conjugates of G_1 or G_2 (see Corollary 3.30). □

6.2 Automorphisms of a tree

We characterise the behaviour of the two types[3] of automorphisms (without inversion) of a tree:

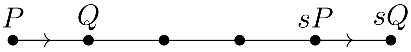
- those with fixed points (**elliptic** automorphisms), and
- those without fixed points (**hyperbolic** automorphisms).

In this subsection, $s \in \mathrm{Aut}(X)$ is an automorphism without inversion of a tree X.

Definition 6.5. Let $y = PQ \in E(X)$ be an edge. Exactly one of the three statements are true:

1. $sy = y$, i.e. edge y is **fixed** by s.
2. $d(P, sP) = d(Q, sQ) \neq 0$, i.e. edges y, sy are distinct and **coherent**.

[3]In general, non-trivial automorphisms of geodesic hyperbolic spaces may be categorised into three types: elliptic (fixes some point), parabolic (fixes a point at infinity), and hyperbolic (fixes two points at infinity). In this subsection, we show that tree actions have no parabolic elements.



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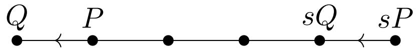
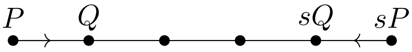


3. $d(P, sP) - d(Q, sQ) = \pm 2$, i.e. edges y, sy are **incoherent**.



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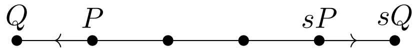


Proposition 6.6 (Tits)**.** *Suppose s acts hyperbolically. Let $m = \inf_{P \in V(X)} d(P, sP) > 0$ and $T = \{P \in V(X) : d(P, sP) = m\}$. Then:*

(a) T is the vertex set of a line of X.

(b) s induces a translation of T of amplitude m.

(c) For any $Q \in V(X)$, $d(Q, sQ) = m + 2 \cdot d(Q, T)$.

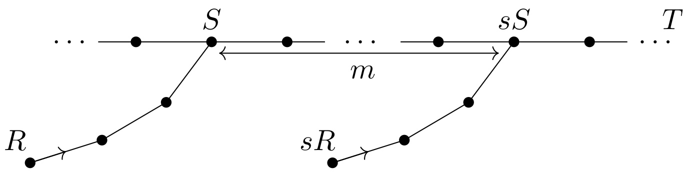
(d) Edges y and sy are coherent $\iff$ y is an edge of T.

(e) Every subtree of X stable under s and s^{-1} contains T.

Proof. Let $P \in T$, and $y = PQ$ be the first edge of geodesic P—sP. By minimality of $d(P, sP)$, sQ does not lie on the geodesic P—sP. Consequently, geodesics

$$\ldots\text{—}s^{-2}P\text{—}s^{-1}P\text{—}P\text{—}sP\text{—}s^2P\text{—}\ldots$$

form a line in X contained in T, under which s acts by a translation of amplitude m. For any other vertex Q not on this line, the geodesic R—S between R and the line will also be 'translated' along the line. This shows that T is exactly the line, and (a)–(d) are true.



(e) follows from the fact that midpoints of all geodesics P—sP (for $P \in V(X)$) trace out line T. $\square$

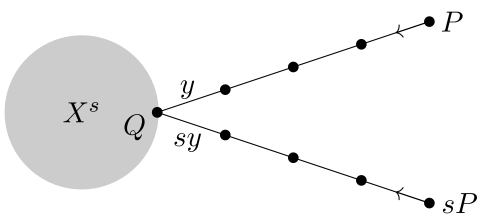
Proposition 6.7. *Suppose s acts elliptically, and $P \in V(X) \setminus X^s$ is a non-fixed vertex.*

(a) Let Q be the vertex in X^s such that P—Q is the geodesic from P to X^s. Then Q is the midpoint of geodesic P—sP.

(b) Geodesic $P - sP$ is of even length, and its midpoint is the unique fixed point along the geodesic.

(c) For all edges $y \in E(X)$, either y is fixed by s, or y and sy are incoherent.

Proof. Let y be the final edge along geodesic P—Q adjoining Q. Since $y \notin X^s$, thus $sy \neq y$, and the geodesic P—Q and its image sP—Q under s meet only at P.



(a)–(c) follow from this observation. ☐

Corollary 6.8. *The following are equivalent:*

(a) s acts hyperbolically.

(b) There is an edge $y \in E(X)$ such that y and sy are distinct and coherent;

(c) There is a line in X stable under s, under which s performs a non-trivial translation.

Furthermore, such an automorphism s necessarily has infinite order.

Proof. The equivalences follow from Propositions 6.6 and 6.7. ☐

Lemma 6.9. *Let $G \curvearrowright X$ be a group action on a tree without inversion.*

(a) Suppose G is generated by three subgroups A, B, C, such that $X^{\langle A,B\rangle}, X^{\langle A,C\rangle}, X^{\langle B,C\rangle} \neq \emptyset$. Then $X^G \neq \emptyset$.

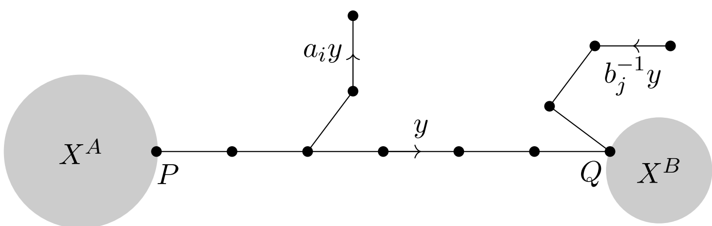
(b) Suppose $a, b \in G$ and a, b, ab each have fixed points. Then $X^{\langle a,b\rangle} \neq \emptyset$.

(c) Suppose G is finitely generated by $a_1, \ldots, a_n$ such that each a_i and $a_i a_j$ have fixed points. Then $X^G \neq \emptyset$.

(d) Suppose G is generated by elements a_i and b_j. Let A and B be the subgroups generated by the a_i and b_j respectively. Suppose $X^A, X^B \neq \emptyset$, and each $a_i b_j$ has fixed points. Then $X^G \neq \emptyset$.

Proof. (a) follows from Lemmas 3.8(d) and 3.23(a).

(b) is a special case of (c), which follows by induction given (d).



(d): suppose otherwise that $X^A \cap X^B = \emptyset$. Let y be an edge along the geodesic P—Q between X^A and X^B. Then y is not fixed by some a_i and some b_j. Consequently, edges $a_i y$ and $b_j^{-1} y$ are disjoint and coherent, implying $a_i b_j$ is hyperbolic, a contradiction. ☐

Lemma 6.10. *Let G be a finitely generated nilpotent group acting on tree X. Exactly one of the following is true:*

(a) G has a fixed point.

(b) There exists a line in X stable under G, under which G acts non-trivially by translations.

In either case, the commutator subgroup $[G, G]$ has a fixed point in X.

Proof. Let $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ be a central series of G, i.e. each quotient G_n/G_{n-1} is abelian. Since G is finitely generated, thus each quotient G_n/G_{n-1} is of finite rank. Refining the central series, one may assume each G_n/G_{n-1} is cyclic.

One now proceeds by induction on n. If $K = G_{n-1}$ has a fixed point, one may apply Corollary 6.8 to the action of the generator of G/K on the subtree X^K.

Otherwise, by induction hypothesis, there is a K-stable line $T \subset X$ upon which G acts non-trivially by translations. Since $K \triangleleft G$, thus for all $g \in T$, gT is also a K-stable path under K; therefore $gT = T$ for all $g \in G$, and T is also stable under G. The action $G \curvearrowright T$ corresponds to homomorphism $G \to \mathrm{Aut}(T)$, whose image is isomorphic to either $\mathbb{Z}$ or D_∞. Since D_∞ is not nilpotent, thus it must be the former, and G acts on T by translations. □

Remark 6.11. Lemma 6.10 shows that the only possible minimal G-trees of finitely generated nilpotent groups are a point and a line. This is not true in general: for instance, the free action of $G = \mathbb{F}_2$ on its Cayley graph (a 4-regular tree) has no proper G-stable subtree.

Corollary 6.12. *Let G be a finitely generated nilpotent group acting on tree X. If G is generated by elements which have fixed points, then $X^G \neq \emptyset$.*

Proof. Suppose otherwise that $X^G = \emptyset$. By Lemma 6.9, there is a line T stable under G, under which G acts non-trivially by translations. If G is generated by elements a_i which have fixed points, then each a_i must fix T (otherwise it is hyperbolic). Thus G fixes T, a contradiction. □

6.3 Explicit non-amalgams

Definition 6.13. Let $l, m, n \geq 2$ be integers. Define the **triangle reflection group**[4] as

$$W_{lmn} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^l = (xz)^m = (yz)^n = 1 \rangle.$$

Geometrically, W_{lmn} is the group generated by reflections about the sides of a triangle with vertex angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$ in spherical, Euclidean, or hyperbolic space (depending on whether $\frac{1}{l} + \frac{1}{m} + \frac{1}{n}$ is > 1, $= 1$, or < 1 respectively).

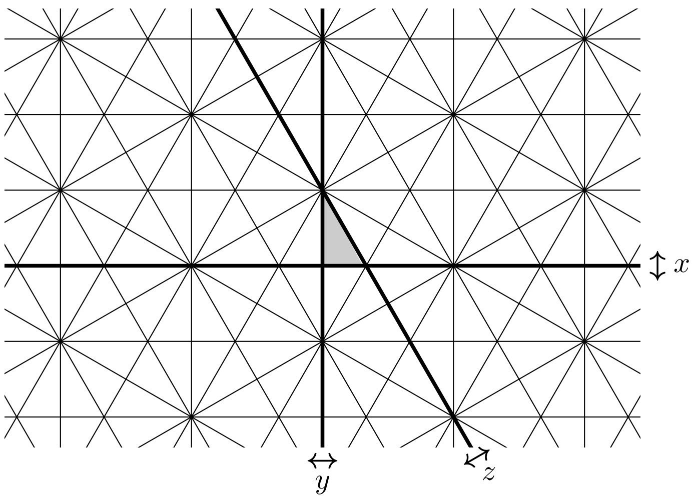


Figure 10: The group W_{236} depicted by its action on the Euclidean plane via reflections about sides of a 30°-60°-90° Euclidean triangle. The axes of generating reflections are bolded and labelled. The shaded triangle is a fundamental domain of the action of W_{236} on the plane; its images generate a tessellation.

Proposition 6.14. *For $l, m, n \geq 2$, W_{lmn} has property (FA).*

Proof. Note $\langle x, y \rangle = D_{2l} \leq G$ is finite, thus has property (FA) (see Lemma 6.4(a)). Similarly, $\langle x, z \rangle$ and $\langle y, z \rangle$ are finite and have property (FA). Let W_{lmn} act on a tree X without inversion. Since $X^x \cap X^y = X^{\langle x,y \rangle} \neq \emptyset$ and similarly $X^{\langle x,z \rangle}, X^{\langle y,z \rangle} \neq \emptyset$. By Lemma 6.9(a), $X^{W_{lmn}} \neq \emptyset$. □

[4]This belongs to the larger class of Coxeter groups.

Proposition 6.15. *For $n \geq 3$, $\mathrm{Aut}(\mathbb{F}_n)$ has property (FA).*

Proof. Let $x_1, \dots, x_n$ be a free basis for $\mathbb{F}_n$. Each automorphism $g \in \mathrm{Aut}(\mathbb{F}_n)$ sends $(x_1, \dots, x_n)$ to a another free basis. We first establish generators for $\mathrm{Aut}(\mathbb{F}_n)$:

- Identify S_n with the subgroup of **permutation automorphisms**: each permutation $\sigma \in A$ sends $x_i \mapsto x_{\sigma(i)}$.
- Identify $\{\pm 1\}^n \cong (C_2)^n$ with the subgroup of **inversion automorphisms**: each $\alpha \in B$ where $\alpha : \{1, \dots, n\} \to \{\pm 1\}$ sends $x_i \mapsto x_i^{\alpha(i)}$.
- Let $\rho \in \mathrm{Aut}(\mathbb{F}_n)$ send $x_1 \mapsto x_1 x_2$ and fix all other x_i.

It is known that $\mathrm{Aut}(\mathbb{F}_n)$ is generated by permutation and inversion automorphisms along with ρ. Following the same idea as in the proof of Proposition 6.14, we first strive for a smaller set of generators with finite order. This is achieved by the following:

- Let involution θ send $x_1 \mapsto x_1 x_2$, $x_2 \mapsto x_2^{-1}$, and fix all other x_i.
- Let involution η send $x_1 \mapsto x_2^{-1}$, $x_2 \mapsto x_1^{-1}$, and fix all other x_i.
- Let involution τ send $x_1 \mapsto x_1^{-1}$, swap $x_2 \leftrightarrow x_3$, and fix all other x_i.
- Let $\Omega = S_{n-2} \ltimes \{\pm 1\}^{n-2} \leq S_n \ltimes \{\pm 1\}^n$ be the finite automorphism subgroup generated by permutations and inversions of $x_3, \dots, x_n$. Note that these commute with θ and η.

We first show $\mathrm{Aut}(\mathbb{F}_n)$ is generated by Ω and the involutions θ, η, τ:

- Ω contains the x_3-inversion ι that sends $x_3 \mapsto x_3^{-1}$ and fixes all other x_i.
- $\eta\tau$ sends $x_1 \mapsto x_2, x_2 \mapsto x_3, x_3 \mapsto x_1^{-1}$, and fixes all other x_i.
- Conjugating ι by $\eta\tau$ gives the x_1- and x_2-inversions.
- Composing τ with an x_1-inversion gives the swap $x_2 \leftrightarrow x_3$ (fixing all other x_i); composing $\eta\tau$ with an x_1-inversion gives the 3-cycle $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_1$ (fixing all other x_i).
- Therefore, Ω, η, τ generate all inversion and permutation automorphisms.
- θ is ρ composed with a x_2-involution. Therefore, $\mathrm{Aut}(\mathbb{F}_n) = \langle \Omega, \theta, \eta, \tau \rangle$.

Next, let $A = \langle \Omega, \eta \rangle \cong \Omega \times C_2$. We show that the groups $\langle A, \theta \rangle$, $\langle A, \tau \rangle$, and $\langle \theta, \tau \rangle$ are finite:

- $\langle \theta, \eta \rangle \cong D_6$, thus $\langle A, \theta \rangle \cong \Omega \times D_6$ is finite.
- $\langle A, \tau \rangle = S_n \ltimes \{\pm 1\}^n$ is finite.
- $\langle \theta, \tau \rangle \cong D_8$ is finite.

The conclusion then follows from Lemmas 6.4(a) and 6.9(a). □

Proposition 6.16. $\mathrm{SL}_3(\mathbb{Z})$ *has property (FA).*

Proof 1. It is known that $G = \mathrm{SL}_3(\mathbb{Z})$ is generated by the following 6 elements:

$$z_1, \dots, z_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let G act on a tree X. Note (taking indices modulo 6) that

1. z_i commutes with z_{i-1} and z_{i+1}.
2. $[z_{i-1}, z_{i+1}] = z_i$ or z_i^{-1}.
3. $\langle z_{i-1}, z_{i+1} \rangle$ is nilpotent; by Lemma 6.10, z_i has a fixed point.
4. By Corollary 6.12, $X^{\langle z_{i-1}, z_{i+1} \rangle} \neq \emptyset$.
5. z_1, z_3, z_5 generate $\mathrm{SL}_3(\mathbb{Z})$; by Lemma 6.8(c), $X^G \neq \emptyset$.

Consequently, $\mathrm{SL}_3(\mathbb{Z})$ has property (FA). □

Proof 2. Note that every automorphism of free group $\mathbb{F}_n$ induces an automorphism of its abelianisation $\mathbb{Z}^n$. Therefore, $\mathrm{SL}_3(\mathbb{Z})$ is a quotient of $\mathrm{Aut}(\mathbb{F}_3)$, which has property (FA) (see Proposition 6.15). The result then follows from Lemma 6.3(d). □

Remark 6.17. Proposition 6.16, Proof 2 also shows $G = \mathrm{Aut}(\mathbb{F}_2)$ is a non-trivial amalgam: if $\mathrm{PSL}_2(\mathbb{Z}) = G/K = C_2 * C_3$ for $K \triangleleft G$, then by Proposition 5.24, $G = (C_2K) *_K (C_3K)$.

Corollary 6.18. *Every subgroup of* $\mathrm{SL}_3(\mathbb{Z})$ *of finite index has property (FA).*

Proof sketch. Repeat the argument of Proposition 6.16, Proof 1 to the subgroup H_N generated by elements z_i^N to show that H_N has property (FA). If $H \leq \mathrm{SL}_3(\mathbb{Z})$ is a subgroup of finite index, then for some large N, H_N is a subgroup of H of finite index. The corollary follows from Lemma 6.3(c). □

Remark 6.19. The proofs of Proposition 6.16 and Corollary 6.18 generalise to show that $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, and its subgroups of finite index, have property (FA).

Corollary 6.20. *The following groups are not amalgams, HNN constructions, or more generally arising from a non-trivial graph of groups:*

1. W_{lmn} *for* $l, m, n \geq 2$.
2. $\mathrm{Aut}(\mathbb{F}_n)$ *for* $n \geq 3$.
3. $\mathrm{SL}_n(\mathbb{Z})$ *for* $n \geq 3$, *and its subgroups of finite index.*

Proof. This follows from Propositions 6.14, 6.15, 6.16, Remark 6.19, and Corollary 6.3. □

7 Conclusion

Through the lens of tree actions, we gain a geometric understanding of free groups, amalgams, and more generally, groups arising from a graph of groups (G, Y). There is still much to explore; we conclude this essay by briefly describing several areas of further study:

1. Further investigation into the structure of free groups (see §3.5) achieves the following result:

Theorem 7.1 (Howson)**.** *Let H_1, H_2 be finitely generated subgroups of a free group G. Then $H_1 \cap H_2$ is finitely generated, with $r(H_1 \cap H_2)$ uniformly bounded in terms of $r(H_1)$ and $r(H_2)$.*

2. A tree action $G \curvearrowright X$ may be extended to an action on its ends $G \curvearrowright \partial_\infty X$ (see Definition 3.10). In addition, one may define the ends of a finitely generated group G as the ends of Cayley graph $\Gamma(G, S)$, for S a finite set of generators; remarkably, the definition is independent of choice of generators S. The following result may be shown:

Theorem 7.2 (Stallings)**.** *Let G be a finitely generated group.*

(a) G has more than one end $\iff$ it is an amalgam or HNN extension over a finite subgroup.

(b) G either has $0, 1, 2,$ or infinitely many ends.

3. One may study general torus knot groups $\mathcal{T}_{p,q} = \langle a, b \mid a^p = b^q \rangle$ (see §4.3) and general fundamental groups of geometric spaces (see, for instance, Theorem 5.5).

4. One may study general SL_2 groups. For instance, $\mathrm{SL}_2(\mathbb{Q}_p)$ (or more generally, over a local field) has a natural action on lattices, which descends to a tree action on lattices under equivalence. The following result may be shown:

Theorem 7.3 (Ihara)**.**

(a) $\mathrm{SL}_2(\mathbb{Q}_p) = \mathrm{SL}_2(\mathbb{Z}_p) *_A \mathrm{SL}_2(\mathbb{Z}_p)$ for an appropriate subgroup A.

(b) Every torsion-free subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ is free.

5. One may study actions $G \curvearrowright X$ on wider classes of geodesic metric spaces X, such as $\mathbb{R}$-trees or hyperbolic spaces. The following results enable one to have a notion of geometry of groups (up to quasi-isometry):

Lemma 7.4. *Let G be a finitely generated group. For a finite generator set S, let d_S be the metric on G inherited by Cayley graph $\Gamma(G, S)$. Then any two such metrics $d_S, d_{S'}$ are quasi-isometric, i.e. there are constants $c, C > 0$ such that*

$$c \cdot d_S(g_1, g_2) \leq d_{S'}(g_1, g_2) \leq C \cdot d_S(g_1, g_2) \quad \forall g_1, g_2 \in G.$$

Theorem 7.5 (Švarc-Milnor)**.** *If G acts properly, cocompactly, and by isometries on a proper geodesic metric space X, then G is quasi-isometric to X.*

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This mathematics essay was written over the summer of 2023. I thank Prof. Henry Wilton for his advice, support, and feedback.