Roth's Theorem on 3-Term Arithmetic Progressions

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Outline

1. The Question: 3-Term APs
2. Attempts and Intuition
3. Proof Sketch (The 5 Ideas)
4. The 5 Ideas, Explained
5. Beyond Roth’s Theorem

Notation

- $\{1, 2, \ldots, N\}$ and $[N]$ are used interchangeably.
- C and c refer to generic large and small constants respectively. In different places, they may refer to different constants. For example, $(cx^2 - Cx - C)/(Cx + C) \geq cx$ for $x \geq C$.
- Some equations, although not explicitly stated, only apply to large N.
- Blue items are additional technical pointers.

The Question: 3-Term APs

$0 < \alpha \leq 1$

$A \subseteq \{1, 2, \ldots, N\}$, $|A| = \alpha N$, s.t. A has no 3-term APs. How big can $|A|$ be?

$$\alpha_{\max}(N) := \max_{A \subseteq [N]} \{\alpha\} \tag{1}$$

When $N \to +\infty$, does $\alpha_{\max} \to 0$?

Intuition

Arithmetic progressions have rich structure:

- AP-indexed subsequences of APs are also APs.
- The common difference may be viewed as the "period" of occurence.
- (a, b, c) is a 3-term AP iff $a + c = 2b$.

$\{1, 2, \ldots, N\}$ may be replaced with any N-term AP.

Intuition

Suppose the structure of interest was instead (a, b, c) where $a + b = c$. Then $\alpha_{\text{max}} \approx \frac{1}{2}$ for all N. (Exercise)

This tells us that:

- The rich structure of APs and 3-term APs is **very important.**
- Techniques related to number theory might be required.

Intuition

Considering consecutive triples, $\alpha \leq \frac{2}{3} + \frac{2}{N}$. This settles $N = 1, 2, 3, 4, 5, 6$.

For $N = 7$, consider the middle element. $|A|_{\text{max}} = 4$.

For $N = 8$ and $N = 9$, $|A|_{\text{max}} = 4$ and $|A|_{\text{max}} = 5$ respectively.

3×3 grid pattern for $N = 9$ may improve the $\frac{2}{3}$ bound (consider $N = 27$); possible approach to consider a $3 \times 3 \times \cdots \times 3$ lattice. (Hales-Jewett)

SPOILER ALERT

If you wish to try the problem yourself, stop here.

The Answer: Roth's Theorem

(Roth, 1953) We may upper-bound α_{max} in terms of N as follows:

$$\alpha_{\text{max}} \leq \frac{C^{\;568000}}{(\log \log N)^{1/5}} \tag{2}$$

In particular, $\alpha_{\text{max}} \to 0$ when $N \to +\infty$.

Proof Sketch

Note: this is not Roth's original proof, but it is chosen for its elegance of integrating key ideas.

Density-Increment: $\alpha' = \alpha + c\alpha^6$

Indicator Functions

$f := 1_A - \alpha 1_{[N]}$

$AP_3 := \mathbb{E}_{x,d} g_1(x) g_2(x+d) g_3(x+2d)$

Cauchy-Schwarz and U^2-norm

$|AP_3(g_1, g_2, g_3)| \leq \|g_i\|_{U^2}$

$\|f\|_{U^2} \geq c\alpha^3$

Discrete Fourier Analysis

$\exists \theta \in [0, 1]$ s.t.

$|\sum f(x) e(\theta x)| \geq c\alpha^6 N$

Hardy-Littlewood Circle Method

$\exists P' \subseteq [N], |P'| \geq N^{1/3}$, s.t.

$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Idea 1: Density Increment Strategy

Density-Increment: $\alpha' = \alpha + c\alpha^6$

Indicator Functions	Cauchy-Schwarz and U^2-norm	Discrete Fourier Analysis	Hardy-Littlewood Circle Method
$f := 1_A - \alpha 1_{[N]}$ $AP_3 := \mathbb{E}_{x,d} g_1(x) g_2(x+d) g_3(x+2d)$	$\lvert AP_3(g_1, g_2, g_3)\rvert \le \lVert g_i \rVert_{U^2}$ $\lVert f \rVert_{U^2} \ge c\alpha^3$	$\exists \theta \in [0,1]$ s.t. $\lvert \sum f(x) e(\theta x)\rvert \ge c\alpha^6 N$	$\exists P' \subseteq [N], \lvert P'\rvert \ge N^{1/3}$, s.t. $\sum_{x \in P'} f(x) \ge c\alpha^6 \lvert P'\rvert$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \le \frac{C}{(\log \log N)^{1/5}}}$$

Idea 1: Density-Increment Strategy

$3E19$ 18

Suppose $A \subseteq [N]$, $|A| = \alpha N$, and A has no 3-term APs. If $N \geq C\alpha^{-C}$, then:

There is a sub-AP $P' \subseteq [N]$ of length $|P'| \geq N^{1/3}$ with $|A \cap P'| = \alpha' |P'|$ s.t.

$$\alpha' \geq \alpha + c\alpha^6 \tag{3}$$

$1/5E5$

Idea 1: Density-Increment Strategy

$3E19$ 18

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There is a sub-AP $P' \subseteq [N]$ of length $|P'| \geq N^{1/3}$ with $|A \cap P'| = \alpha'|P'|$ s.t.

$$\alpha' \geq \alpha + c\alpha^6 \tag{4}$$

$1/5E5$

Proof: Density-Increment Strategy $\Rightarrow$ (2)

Suppose the Density-Increment statement is true.

$3E19$ 18

Let $N_0 = N$, $P_0 = [N]$, $A_0 = A$, and $\alpha_0 = \alpha$. Now for each $i \in \mathbb{N}$, if $N_i \geq C\alpha_i{}^{-C}$:

- There is some sub-AP $P_{i+1} \subseteq P_i$ of length $N_{i+1} \geq N_i{}^{1/3}$ such that,
- Setting $A_{i+1} = A_{i+1} \cap P_{i+1}$ and $|A_{i+1}| = \alpha_{i+1} N_{i+1}$,

  $1/5E5$
- One has $a_{i+1} \geq \alpha_i + c\alpha_i{}^6$.

Proof: Density-Increment Strategy $\Rightarrow$ (2)

The α_i's are increasing rapidly:

- Starting from $\alpha_0 = \alpha$, it would take $\leq \underset{1/5E5}{\frac{1}{c\alpha^5}} = \overset{5E5}{C\alpha^{-5}}$ iterations for it to double to 2α.
- The next double to 4α would be reached in $\leq \overset{5E5}{C(2\alpha)^{-5}}$ more iterations.
- The k-th double to $2^k\alpha$ would require $\leq \overset{5E5}{C\alpha^{-5}2^{-5k}}$ iterations.

However, the α_i's are also bounded above by 1.

Proof: Density-Increment Strategy $\Rightarrow$ (2)

Therefore, the total number of iterations is at most
$i_{\max} = \underset{5E5}{C\alpha^{-5}}(1 + 2^{-5} + 2^{-10} + \dots) \leq \underset{5.17E5}{C\alpha^{-5}}$.

Since the iterations must stop, therefore $N_{i_{\max}} \leq \overset{3E19}{C\alpha_{i_{\max}}}{}^{-C} \leq \overset{18}{C\alpha^{-C}}$.

Since $N_{i_{\max}} \geq N^{(1/3)^{i_{\max}}} \geq N^{(1/3)^{C\alpha^{-5}}}$, thus $\overset{5.17E5}{N^{(1/3)^{C\alpha^{-5}}}} \leq \overset{3E19}{C\alpha}{}^{\overset{18}{-C}}$.

Taking double-log, $\log\log N - \overset{5.17E5}{(\log 3)C\alpha^{-5}} \leq \overset{45}{\log}(\overset{18}{C - C}\log\alpha)$.

Proof: Density-Increment Strategy $\Rightarrow$ (2)

Note that $\log(\overset{45}{C} - \overset{18}{C}\log\alpha) \le \overset{4}{C}\alpha^{-5}$.

Therefore $\log\log N \le \overset{5.68E5}{C}\alpha^{-5}$.

Rearranging, $\alpha \le \dfrac{\overset{5.68E5}{C}}{(\log\log N)^{1/5}}$. $\square$

Idea 2: Indicator Functions

Density-Increment: $\alpha' = \alpha + c\alpha^6$

Indicator Functions

$f := 1_A - \alpha 1_{[N]}$

$AP_3 := \mathbb{E}_{x,d}\, g_1(x) g_2(x+d) g_3(x+2d)$

Cauchy-Schwarz and U^2-norm

$|AP_3(g_1, g_2, g_3)| \leq \|g_i\|_{U^2}$

$\|f\|_{U^2} \geq c\alpha^3$

Discrete Fourier Analysis

$\exists \theta \in [0,1]$ s.t.

$|\sum f(x) e(\theta x)| \geq c\alpha^6 N$

Hardy-Littlewood Circle Method

$\exists P' \subseteq [N], |P'| \geq N^{1/3}$, s.t.

$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$\Downarrow$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Idea 2: Indicator Functions

We want to a function to count the number of 3-term APs in A.

Define indicator 1_A as $1_A(x) := \{1 \text{ if } x \in A, 0 \text{ if } x \notin A\}$.

N' is chosen this way to prevent counting of wrap-around APs.

Consider the finite abelian group $G = \mathbb{Z}/N'\mathbb{Z}$ where $N' = 2N + 1$.

$|A|$ arises from the trivial $d = 0$ solutions.

Then $\mathbb{E}_{x,d \in G} 1_A(x) 1_A(x+d) 1_A(x+2d) = (|A| + 2 \cdot \#\text{ 3-term APs in } A)/N'^2$.

Each 3-term AP is double-counted by $\pm d$.

Idea 2: Indicator Functions

We therefore consider the linear indicator function
$AP_3(g_1, g_2, g_3) := \mathbb{E}_{x,d \in G} g_1(x) g_2(x+d) g_3(x+2d)$.

This term, $\approx \alpha/4N$, is small.

Since A has no 3-term APs, thus $AP_3(1_A, 1_A, 1_A) = |A|/N'^2 = \alpha N/(2N+1)^2$.

We compare A to a random subset of $[N]$ of size αN.

$AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]}) = \alpha^3 AP_3(1_{[N]}, 1_{[N]}, 1_{[N]})$.

Idea 2: Indicator Functions

Note that $[N]$ has $\geq \lfloor N/3 \rfloor^2$ 3-term APs formed by $x, d = 1, 2, \ldots, \lfloor N/3 \rfloor$.

Thus $AP_3(1_{[N]}, 1_{[N]}, 1_{[N]}) \geq \overset{\approx 1/18}{(N + 2\lfloor N/3 \rfloor^2)/(2N+1)^2} \geq \overset{1/19}{c}$ for $N \geq \overset{50}{C}$.

So for $N \geq \overset{50}{C}$, $AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]}) \geq \overset{1/19}{c}\alpha^3$.

Idea 2: Indicator Functions

We now compare $AP_3(1_A, 1_A, 1_A) \leq \frac{\alpha}{4N}$ to $AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]}) \geq c\alpha^3$.

Define function $f := 1_A - \alpha 1_{[N]}$. f is *balanced* because $\mathbb{E}_{x \in G} f(x) = 0$.

Expanding linearly, $AP_3(1_A, 1_A, 1_A) = AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, \alpha 1_{[N]})$
$+AP_3(f, \alpha 1_{[N]}, \alpha 1_{[N]}) + AP_3(\alpha 1_{[N]}, f, \alpha 1_{[N]}) + AP_3(\alpha 1_{[N]}, \alpha 1_{[N]}, f)$
$+AP_3(f, f, \alpha 1_{[N]}) + AP_3(f, \alpha 1_{[N]}, f) + AP_3(f, f, \alpha 1_{[N]}) + AP_3(f, f, f)$.

Since f is *balanced*, the 7 terms of the form $AP_3(f, *, *)$, $AP_3(*, f, *)$, and $AP_3(*, *, f)$ should be small; however, the equation above says otherwise.

Idea 2: Indicator Functions

Explicitly, we have that $|AP_3(g_1, g_2, g_3)| \geq \overset{1/19}{\frac{1}{7}}(c\alpha^3 - \frac{\alpha}{4N})$ for 1-bounded functions g_1, g_2, g_3 among which $g_i = f$ for some $i \in \{1, 2, 3\}$.

For $N \geq \overset{190}{C}\alpha^{-\overset{2}{C}}$, $\overset{1/19}{\frac{1}{7}}(c\alpha^3 - \frac{\alpha}{4N}) \geq \overset{1/140}{c}\alpha^3$.

To make sense of the AP_3 indicator function, we will relate $|AP_3(g_1, g_2, g_3)|$ to the "norm"s of g_1, g_2, and g_3. We hope that this will give us a lower bound for the "norm" of f, which in turn gives us information about A.

Idea 3: Cauchy-Schwarz Inequality

Density-Increment: $\alpha' = \alpha + c\alpha^6$

Indicator Functions

$f := 1_A - \alpha 1_{[N]}$

$AP_3 := \mathbb{E}_{x,d} g_1(x) g_2(x+d) g_3(x+2d)$

Cauchy-Schwarz and U^2-norm

$|AP_3(g_1, g_2, g_3)| \leq \|g_i\|_{U^2}$

$\|f\|_{U^2} \geq c\alpha^3$

Discrete Fourier Analysis

$\exists \theta \in [0,1]$ s.t.

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Hardy-Littlewood Circle Method

$\exists P' \subseteq [N], |P'| \geq N^{1/3}$, s.t.

$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Idea 3: Cauchy-Schwarz Inequality

To bound AP_3, we will attempt to use the Cauchy-Schwarz Inequality. We first rephrase the statement with the use of expected value: for any functions $\beta, \gamma : X \to \mathbb{C}$,

$$|(\mathbb{E}_{x\in X}\beta(x)\overline{\gamma(x)})|^2 \leq \mathbb{E}_{x\in X}|\beta(x)|^2\mathbb{E}_{x'\in X}|\gamma(x')|^2 \tag{5}$$

Note that by setting $\gamma := \overline{\gamma}$, we may remove the conjugation on the left.

If β is 1-bounded, then $|\mathbb{E}_{x\in X}\beta(x)\gamma(x)|^2 \leq \mathbb{E}_{x\in X}|\gamma(x)|^2$.

Idea 3: Cauchy-Schwarz Inequality

We now apply Cauchy-Schwarz to bound $AP_3(g_1, g_2, g_3)$ in terms of g_1 alone; to do this, we will use the fact that g_2 and g_3 are 1-bounded.

Note that $AP_3(g_1, g_2, g_3) = \mathbb{E}_{x,y\in G} g_1(2x - y)g_2(x)g_3(y)$.

There is some freeplay in parameterising 3-term APs.

Since g_2 is 1-bounded, we let $\gamma(x) := \mathbb{E}_{y\in G} g_1(2x - y)g_3(y)$. Then,

$$|AP_3(g_1, g_2, g_3)|^2 \leq \mathbb{E}_{x\in G}|\mathbb{E}_{y\in G} g_1(2x - y)g_3(y)|^2 \tag{6}$$

which we may rewrite as $\mathbb{E}_{x\in G}\mathbb{E}_{y\in G} g_1(2x - y)g_3(y)\mathbb{E}_{y'\in G}\overline{g_1(2x - y')g_3(y')}$.

Idea 3: Cauchy-Schwarz Inequality

We apply Cauchy-Schwarz once again: $g_3(y)\overline{g_3(y')}$ is 1-bounded, thus

$$|\mathbb{E}_{y,y',x\in G}g_1(2x-y)\overline{g_1(2x-y')}g_3(y)\overline{g_3(y')}|^2 \leq \mathbb{E}_{y,y'\in G}|\mathbb{E}_{x\in G}g_1(2x-y)\overline{g_1(2x-y')}|^2$$

which we rewrite as $\mathbb{E}_{x,x',y,y'\in G}g_1(2x-y)\overline{g_1(2x-y')g_1(2x'-y)}g_1(2x'-y')$,

and reparameterise as $\mathbb{E}_{x,d_1,d_2\in G}g_1(x)\overline{g_1(x+d_1)g_1(x+d_2)}g_1(x+d_1+d_2)$.

This is solely in terms of g_1, and is non-negative! "norm"-worthy.

Idea 3: Cauchy-Schwarz Inequality

Define the Gowers U^2-norm on function $F : G \to \mathbb{C}$ as

$$\|F\|_{U^2} := \sqrt[4]{\mathbb{E}_{x,d_1,d_2 \in G} F(x)\overline{F(x+d_1)F(x+d_2)}F(x+d_1+d_2)} \tag{7}$$

As its norm-related properties will not be used, we postpone the proof that it is in fact a norm to the end of the slides.

The two applications of Cauchy-Schwarz show $|AP_3(g_1, g_2, g_3)|^4 \leq \|g_1\|_{U^2}^4$.

Thus $|AP_3(g_1, g_2, g_3)| \leq \|g_1\|_{U^2}$.

Idea 3: Cauchy-Schwarz Inequality

Similarly, we may also show $|AP_3(g_1, g_2, g_3)| \leq \|g_2\|_{U^2}, \|g_3\|_{U^2}$.

We know that $|AP_3(g_1, g_2, g_3)| \geq \overset{1/140}{c}\alpha^3$ for $N \geq \overset{190}{C}\alpha^{-\overset{2}{C}}$ and 1-bounded functions g_1, g_2, g_3 among which $g_i = f$ for some $i \in \{1, 2, 3\}$.

Thus for $N \geq \overset{190}{C}\alpha^{-\overset{2}{C}}$, $\|f\|_{U^2} \geq \overset{1/140}{c}\alpha^3$.

We will now make sense of $\|f\|_{U^2}$ using Discrete Fourier Analysis.

Idea 4: Discrete Fourier Analysis

$$\boxed{\text{Density-Increment: } \alpha' = \alpha + c\alpha^6}$$

Indicator Functions

$f := 1_A - \alpha 1_{[N]}$
$AP_3 := \mathbb{E}_{x,d} g_1(x) g_2(x + d) g_3(x + 2d)$

Cauchy-Schwarz and U^2-norm

$|AP_3(g_1, g_2, g_3)| \leq \|g_i\|_{U^2}$
$\|f\|_{U^2} \geq c\alpha^3$

Discrete Fourier Analysis

$\exists \theta \in [0, 1]$ s.t.
$|\sum f(x) e(\theta x)| \geq c\alpha^6 N$

Hardy-Littlewood Circle Method

$\exists P' \subseteq [N], |P'| \geq N^{1/3}$,
s.t.
$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Idea 4: Discrete Fourier Analysis

A trained eye would recognise that the U^2-norm can be treated with Fourier Analysis. We first establish the basic tools required.

For function $f : G = \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}$, define its transform $\hat{f} : \mathbb{Z}/N'\mathbb{Z} \to \mathbb{C}$ as

$$\hat{f}(r) := \mathbb{E}_{x \in G} f(x) \exp\left(-2\pi i \frac{rx}{N'}\right) \tag{8}$$

To simplify notation, we define $e(x) := \exp(2\pi i x)$, which has period 1. Then

$$\hat{f}(r) := \mathbb{E}_{x \in G} f(x) e\left(-\frac{rx}{N'}\right) \tag{9}$$

Idea 4: Discrete Fourier Analysis

The elegance of the Fourier transform shows in the inverse transform:

$$f(x) = \sum_{r \in G} \hat{f}(r) e(\frac{rx}{N'}) \tag{10}$$

To see why this is true:

$$\sum_{r \in G} \hat{f}(r) e(\frac{rx}{N'}) = \sum_{r \in G} \mathbb{E}_{y \in G} f(y) e(-\frac{ry}{N'}) e(\frac{rx}{N'}) = \sum_{y \in G} f(y) \mathbb{E}_{r \in G} e(\frac{r(x-y)}{N'})$$

Neat trick to swap $\sum$ with $\mathbb{E}$ over the variables y and r.

Idea 4: Discrete Fourier Analysis

The critical observation is

$$\mathbb{E}_{r\in G}e\left(\frac{r(x-y)}{N'}\right) = \{1 \text{ if } x = y, 0 \text{ if } x \neq y\} = \delta_{xy}$$

which we immediately apply to achieve

$$\sum_{r\in G}\hat{f}(r)e\left(\frac{rx}{N'}\right) = \sum_{y\in G}f(y)\mathbb{E}_{r\in G}e\left(\frac{r(x-y)}{N'}\right) = \sum_{y\in G}f(y)\delta_{xy} = f(x)$$

as desired.

The fact that the inverse transform involves a summation ($\sum$) instead of an expected value ($\mathbb{E}$) is important.

Idea 4: Discrete Fourier Analysis

Here are other facts required: Note that $\mathbb{E}$ is applied to the original function while $\sum$ is applied to the transform.

1. Define the norms $\|f\|_2 = (\mathbb{E}_{x\in G}|f(x)|^2)^{1/2}$ and $\|\hat{f}\|_2 = (\sum_{r\in G}|\hat{f}(r)|^2)^{1/2}$. Then $\|f\|_2 = \|\hat{f}\|_2$. Again, we omit the proof of the norm-related properties.
2. Define the *convolution* of two functions as $f * g(x) := \mathbb{E}_{y\in G} f(y)g(x-y)$. Then this operation is commutative, and $\widehat{(f * g)}(r) = \hat{f}(r)\hat{g}(r)$.
3. Define $\|\hat{f}\|_4 = (\sum_{r\in G}|\hat{f}(r)|^4)^{1/4}$. Then $\|f\|_{U^2} = \|\hat{f}\|_4$.

In particular, Statement 3 is relevant to interpreting the U^2-norm.

Idea 4: Discrete Fourier Analysis

We shall prove Statements 1, 2, and 3 in sequence. For Statement 1,

$$\|f\|_2^2 = \mathbb{E}_{x\in G}|f(x)|^2 = \mathbb{E}_{x\in G} f(x)\overline{f(x)} = \mathbb{E}_{x\in G} \sum_{r,r'\in G} \hat{f}(r)e(\frac{rx}{N'})\overline{\hat{f}(r')e(\frac{r'x}{N'})} = \dots$$

Reshuffling,

$$\dots = \sum_{r,r'\in G} \hat{f}(r)\overline{\hat{f}(r')}\mathbb{E}_{x\in G} e(\frac{(r-r')x}{N'}) = \sum_{r,r'\in G} \hat{f}(r)\overline{\hat{f}(r')}\delta_{rr'} = \sum_{r\in G} |\hat{f}(r)|^2 = \|\hat{f}\|_2^2$$

Idea 4: Discrete Fourier Analysis

For Statement 2,

$$\widehat{(f * g)}(r) = \mathbb{E}_{x \in G}(f * g)(x)e(-\frac{rx}{N'}) = \mathbb{E}_{x,y \in G}f(y)g(x-y)e(-\frac{rx}{N'}) = \ldots$$

Reparameterising x as $x' = x - y$,

$$\cdots = \mathbb{E}_{x',y \in G}f(y)g(x')e(-\frac{r(y+x')}{N'}) = \mathbb{E}_{x',y \in G}f(x')e(-\frac{rx'}{N'})f(y)e(-\frac{ry}{N'}) = \hat{f}(r)\hat{g}(r)$$

Idea 4: Discrete Fourier Analysis

For Statement 3, we reparameterise the U^2-norm:

$$\|f\|_{U^2}^4 = \mathbb{E}_{x\in G}\mathbb{E}_{y,y'\in G} f(y)f(x-y)\overline{f(y')f(x-y')} = \mathbb{E}_{x\in G}(f*f)(x)\overline{(f*f)(x)} = \|f*f\|_2^2$$

We then apply Statements 1 and 2:

$$\|f*f\|_2^2 = \|\widehat{f*f}\|_2^2 = \sum_{r\in G}\widehat{(f*f)}(r)\overline{\widehat{(f*f)}(r)} = \sum_{r\in G}\hat{f}(r)^2\overline{\hat{f}(r)^2} = \sum_{r\in G}|\hat{f}(r)|^4 = \|\hat{f}\|_4^4$$

Therefore $\|f\|_{U^2} = \|\hat{f}\|_4$. $\square$

Idea 4: Discrete Fourier Analysis

For Statement 3, we reparameterise the U^2-norm:

$$\|f\|_{U^2}^4 = \mathbb{E}_{x\in G}\mathbb{E}_{y,y'\in G} f(y)f(x-y)\overline{f(y')f(x-y')} = \mathbb{E}_{x\in G}(f*f)(x)\overline{(f*f)(x)} = \|f*f\|_2^2$$

We then apply Statements 1 and 2:

$$\|f*f\|_2^2 = \|\widehat{f*f}\|_2^2 = \sum_{r\in G} \widehat{(f*f)}(r)\overline{\widehat{(f*f)}(r)} = \sum_{r\in G} \hat{f}(r)^2\overline{\hat{f}(r)^2} = \sum_{r\in G} |\hat{f}(r)|^4 = \|\hat{f}\|_4^4$$

Therefore $\|f\|_{U^2} = \|\hat{f}\|_4$. $\square$

Idea 4: Discrete Fourier Analysis

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Linking back to the problem at hand: for $N \geq C\alpha^{-C}$, $\|\hat{f}\|_4 \geq c\alpha^3$.

Each $\hat{f}(r)$ represents the values of f weighted in a certain periodic fashion. Note that $f(x) = \{\alpha \text{ if } x \in A, \alpha - 1 \text{ if } x \notin [N] \setminus A, 0 \text{ otherwise}\}$. If $|\hat{f}(r)|$ is big for some $r \in G$, this could indicate an unequal density distribution related to some period.

In particular, we shall show that if function $f : G = \mathbb{Z}/N'\mathbb{Z}$ is 1-bounded and $\|\hat{f}\|_4 \geq \delta$, then $|\hat{f}(r)| \geq \delta^2$ for some $r \in G$.

Idea 4: Discrete Fourier Analysis

Since f is 1-bounded, thus $\|\hat{f}\|_2 = \|f\|_2 \leq 1$.

Let $\|\hat{f}\|_\infty = \max_{r \in G} |\hat{f}(r)|$. Then

$$\delta^4 \leq \|\hat{f}\|_4^4 = \sum_{r \in G} |\hat{f}(r)|^4 \leq \|\hat{f}\|_\infty^2 \sum_{r \in G} |\hat{f}(r)|^2 = \|\hat{f}\|_\infty^2 \|\hat{f}\|_2^2 \leq \|\hat{f}\|_\infty^2$$

thus $\|\hat{f}\|_\infty \geq \delta^2$.

Idea 4: Discrete Fourier Analysis

So for $N \geq \overset{190}{C}\alpha^{-\overset{2}{C}}$, there is some $r \in G$ such that $|\hat{f}(r)| \geq (\overset{1/140}{c}\alpha^3)^2 = \overset{1/19600}{c}\alpha^6$.

Let $\theta := -\frac{r}{N'} \pmod 1$. Then we may write

$$|\hat{f}(r)| = |\mathbb{E}_{x\in G} f(x)e(-\frac{rx}{N'})| = \frac{1}{N'}|\sum_{x\in[N]} f(x)e(\theta x)|$$

and therefore ($N' = 2N+1$)

$$|\sum_{x\in[N]} f(x)e(\theta x)| \geq \overset{1/19600}{c}\alpha^6 N' \geq \overset{1/9800}{c}\alpha^6 N$$

Idea 5: Hardy-Littlewood Circle Method

Density-Increment: $\alpha' = \alpha + c\alpha^6$

Indicator Functions

$f := 1_A - \alpha 1_{[N]}$

$AP_3 := \mathbb{E}_{x,d} g_1(x) g_2(x+d) g_3(x+2d)$

Cauchy-Schwarz and U^2-norm

$|AP_3(g_1, g_2, g_3)| \leq \|g_i\|_{U^2}$

$\|f\|_{U^2} \geq c\alpha^3$

Discrete Fourier Analysis

$\exists \theta \in [0,1]$ s.t.

$|\sum f(x) e(\theta x)| \geq c\alpha^6 N$

Hardy-Littlewood Circle Method

$\exists P' \subseteq [N], |P'| \geq N^{1/3}$, s.t.

$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Idea 5: Hardy-Littlewood Circle Method

Note for any subset $S \subseteq [N]$, the value $\sum_{x \in S} f(x)$ tells us about the density of A in S. In particular, if $\sum_{x \in S} f(x) > 0$, then $|A \cap S| > \alpha |S|$, and vice versa.

We therefore wish to consider sub-APs $P \subseteq [N]$ of which for $x \in P$, the values of $e(\theta x)$ are nearly identical.

This is in fact true when the common difference d is such that $\|\theta d\|_{\mathbb{R}/\mathbb{Z}}$ is small, and the length of the sub-AP is not too large ($|P| \cdot \|\theta d\|_{\mathbb{R}/\mathbb{Z}}^2 \ll 1$).

Idea 5: Hardy-Littlewood Circle Method

We first show that we can pick d such that $\|\theta d\|_{\mathbb{R}/\mathbb{Z}}$ is small.

Let $\delta > 0$; then we can pick $d \in \{1, 2, \ldots, \lfloor 1/\delta \rfloor\}$ such that $\|\theta d\|_{\mathbb{R}/\mathbb{Z}} \leq \delta$.

To show this, apply Pigeonhole Principle to $0, \theta, 2\theta, \ldots, \lfloor 1/\delta \rfloor \theta \pmod 1$ on $\mathbb{R}/\mathbb{Z}$. There must be at least one pair $a\theta, b\theta$ of distance $\|a\theta - b\theta\|_{\mathbb{R}/\mathbb{Z}} \leq \delta$.

Set $d = |a - b|$; then $\|\theta d\|_{\mathbb{R}/\mathbb{Z}} \leq \delta$ as desired.

Idea 5: Hardy-Littlewood Circle Method

Now we consider dividing $[N]$ into sub-APs of common difference d.

There would be d sub-APs, each of length $\geq \lfloor N/d \rfloor \geq \lfloor \delta N \rfloor$.

We wish to further subdivide these APs such that the values of $e(\theta x)$ are nearly identical, so that

$$|\sum_{x \in P} f(x)e(\theta x)| \approx |\sum_{x \in P} f(x)|$$

to sufficient proximity.

Idea 5: Hardy-Littlewood Circle Method

Let $P = a_0 - \lfloor \frac{|P|}{2} \rfloor d, \ldots, a_0 - d, a_0, a_0 + d, \ldots, a_0 + (\lceil \frac{|P|}{2} \rceil - 1)d$.
Then $|\sum_{x \in P} f(x)| = |\sum_{x \in P} f(x) e(\theta a_0)|$. So

$$|\sum_{x \in P} f(x) e(\theta x)| = \left| \sum_{x \in P} f(x) e(\theta a_0) + \sum_{x \in P} f(x)(e(\theta x) - e(\theta a_0)) \right|$$

Let $\Delta_P = \left| |\sum_{x \in P} f(x) e(\theta x)| - |\sum_{x \in P} f(x)| \right|$. By Triangle Inequality,

$$\Delta_P \leq \left| \sum_{x \in P} f(x)(e(\theta x) - e(\theta a_0)) \right| \leq \sum_{x \in P} |e(\theta x) - e(\theta a_0)|$$

f is 1-bounded

Idea 5: Hardy-Littlewood Circle Method

We can evaluate and bound the RHS.

Note that $|e(x) - e(y)| = 2\sin\left(\pi\|x-y\|_{\mathbb{R}/\mathbb{Z}}\right) \leq 2\pi\|x-y\|_{\mathbb{R}/\mathbb{Z}}$. Therefore

$$\sum_{x \in P} |e(\theta x) - e(\theta a_0)| \leq 2\pi \sum_{x \in P} \|\theta x - \theta a_0\|_{\mathbb{R}/\mathbb{Z}} = 2\pi \sum_{y \in Q} \|\theta d y\|_{\mathbb{R}/\mathbb{Z}}$$

where $Q = \lfloor \frac{|P|}{2} \rfloor, \ldots, -2, -1, 0, 1, 2, \ldots, \lceil \frac{|P|}{2} \rceil - 1$.

Idea 5: Hardy-Littlewood Circle Method

Since $\|\theta d\|_{\mathbb{R}/\mathbb{Z}} \leq \delta$, thus

$$2\pi \sum_{y \in Q} \|\theta d y\|_{\mathbb{R}/\mathbb{Z}} \leq 2\pi \sum_{y \in Q} |y| \cdot \|\theta d\|_{\mathbb{R}/\mathbb{Z}} \leq 2\pi\delta \sum_{y \in Q} |y|^2 \leq 2\pi\delta \frac{|P|^2}{4} = \frac{\pi}{2}\delta |P|^2$$

and ultimately $\Delta_P \leq \frac{\pi}{2}\delta|P|^2$.

Idea 5: Hardy-Littlewood Circle Method

We now return to the problem at hand. We partition $[N]$ into sub-APs $P_1, P_2, \ldots, P_k$, having common difference d; then for $N \geq C\alpha^{-C}$, 2

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1/9800

$$c\alpha^6 N \leq \left|\sum_{x\in[N]} f(x)e(\theta x)\right| \leq \sum_{j=1}^{k}\left|\sum_{x\in P_j} f(x)e(\theta x)\right| \leq \sum_{j=1}^{k}\left(\left|\sum_{x\in P_j} f(x)\right| + \Delta_{P_j}\right)$$

Therefore

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$$\sum_{j=1}^{k}\left|\sum_{x\in P_j} f(x)\right| \geq c\alpha^6 N - \sum_{j=1}^{k}\Delta_{P_j} \geq c\alpha^6 N - \sum_{j=1}^{k}\frac{\pi}{2}\delta|P_j|^2$$

Idea 5: Hardy-Littlewood Circle Method

$1/1.54E6$

We now carefully set our variables for success. Let $\delta = c\alpha^6 N^{-1/3}$ and d as defined previously.

Now $[N]$ may be partitioned into d sub-APs, each of common difference d and length $\geq \lfloor \delta N \rfloor \geq \lfloor c\alpha^6 N^{2/3} \rfloor$. Taking away $\lceil N^{1/3} \rceil$ at a time, the only cases left to consider is when $\lfloor N^{1/3} \rfloor$ or $\lfloor N^{1/3} \rfloor - 1$ remains.

$1/1.54E6$

Since $\lfloor c\alpha^6 N^{2/3} \rfloor \geq 2\lceil N^{1/3} \rceil$, thus the APs may be chopped finer into sub-APs of length $\geq N^{1/3}$ and $\leq 2N^{1/3}$. Since $N^{1/3} \geq 2$, we may split the remainder between two of the $\lceil N^{1/3} \rceil$ blocks.

true if $N \geq (3E19)\alpha^{-18}$

Idea 5: Hardy-Littlewood Circle Method

Linking back,

$$\sum_{j=1}^{k} \frac{\pi}{2}\delta|P_j|^2 \leq \sum_{j=1}^{k} \frac{\pi}{2}(\overset{1/1.54E6}{c}\alpha^6 N^{-1/3})(2N^{1/3})|P_j| = \overset{1/4.9E5}{c}\alpha^6 \sum_{j=1}^{k} |P_j| = \overset{1/4.9E5}{c}\alpha^6 N$$

Therefore

$$\sum_{j=1}^{k} \Big|\sum_{x\in P_j} f(x)\Big| \geq \overset{1/9800}{c}\alpha^6 N - \overset{1/4.9E5}{c}\alpha^6 N = \overset{1/1E4}{c}\alpha^6 N$$

Idea 5: Hardy-Littlewood Circle Method

Finally, we apply the fact that f is balanced:

$$\sum_{j=1}^{k} \sum_{x \in P_j} f(x) = \sum_{x \in [N]} f(x) = 0$$

to get

$$\sum_{j=1}^{k} \left(\left| \sum_{x \in P_j} f(x) \right| + \sum_{x \in P_j} f(x) \right) \overset{1/1E4}{\geq} c\alpha^6 N \overset{1/1E4}{=} c\alpha^6 \sum_{j=1}^{k} |P_j|$$

Idea 5: Hardy-Littlewood Circle Method

Ultimately, there is some $j \in \{1, 2, \ldots, k\}$ such that

$$\sum_{x \in P_j} f(x) \geq \overset{1/1E4}{\frac{1}{2} c\alpha^6} |P_j| = \overset{1/5E5}{c\alpha^6} |P_j|$$

Now P_j has length $\geq N^{1/3}$, and $|A \cap P_j| \geq (\alpha + \overset{1/5E5}{c\alpha^6})|P_j|$.

This proves the Density-Increment under the condition $N \geq \overset{3E19}{C}\alpha^{-\overset{18}{C}}$. $\square$

Proof Summary

Density-Increment: $\alpha' = \alpha + c\alpha^6$

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$\sum_{x \in P'} f(x) \geq c\alpha^6 |P'|$

$$\Downarrow$$

$$\boxed{\alpha_{\max} \leq \frac{C}{(\log \log N)^{1/5}}}$$

Alternative Approaches

Apart from the approach detailed in these slides (Probabilistic Method + Fourier Analysis), it is worth exploring others:

- (Hales-Jewett) In a $3 \times 3 \times \cdots \times 3$ hypercube with $\alpha 3^n$ selected cells, there is a combinatorial line of n selected cells. (Ergodic Theory)

- (Corners) In an $N \times N$ grid of αN^2 selected cells, there is a "corner" triplet $\{(x, y), (x + h, y), (x, y + h)\}$. (Graph Theory)

Extensions

- (Szemeredi) Consider k-term APs.
- Consider multidimensional APs.
- (Leibman, Bergelson) Consider polynomial progressions.
- (Erdös) Strengthen the density condition to large sets, $\sum \frac{1}{a_i} \to +\infty$.