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1. Consider this classification problem, mapping points (x_1, x_2) to class labels ± 1:

$$(-1, 0) \to -1, \quad (0, 0) \to -1, \quad (1, 1) \to +1, \quad (2, 1) \to +1.$$

Implement a SVM classifier for it. Show that the classifier has Lagrange multipliers $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [0, 1, 1, 0]$ (i.e., that λ_1 and λ_4 are slack), and that the equation of the optimal separating boundary between the two classes is $x_1 + x_2 = 1$.

2. Show that the location of the decision boundary for a 1-d optimal Bayesian classifier with equal class priors and $P(x|+1) = G(x; \mu_1, \sigma_1)$, $P(x|-1) = G(x; \mu_2, \sigma_2)$ (i.e., Gaussians) is given by:

$$x = \frac{\mu_2\sigma_1^2 - \mu_1\sigma_2^2}{\sigma_1^2 - \sigma_2^2} \pm \frac{\sqrt{\sigma_1^2\sigma_2^2\left[(\mu_1 - \mu_2)^2 + 2(\sigma_1^2 - \sigma_2^2)\ln\left(\frac{\sigma_1}{\sigma_2}\right)\right]}}{\sigma_1^2 - \sigma_2^2}.$$

If $\sigma_1 = \sigma_2$, show that this expression simplifies to

$$x = \frac{\mu_1 + \mu_2}{2}$$

i.e., the midpoint between the means.

3. Show that the probability of a false negative classification in the optimal Bayesian classifier discussed in the lectures (i.e., the 2-d classifier where $\mu_{+1} = (1, 1)$, $\mu_{-1} = (0, 0)$ and $\sigma = 0.5$) is 0.0786.

You may find the following definitions useful.

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}, \qquad \text{erfc}(x) = \tfrac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.$$

4. A 2-class optimal Bayesian classifier with Gaussian likelihoods is trained using the following dataset of 2-d points.

$$\text{Class 1:} \quad \tfrac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 & 5 & 3 \\ 5 & 3 & 3 & 1 \end{bmatrix}, \qquad \text{Class 2:} \quad \tfrac{1}{\sqrt{2}} \begin{bmatrix} -1 & -3 & 3 & 1 \\ 3 & 1 & -1 & -3 \end{bmatrix}.$$

Find the equation of the decision surface for this classifier and show that it is parabolic.