3) (10 pts) ANL (Recurrence Relations)

What is the closed form solution to the following recurrence relation? Please use the iteration technique, show all of your work and provide your final answer in Big-Oh notation.

T(1) = 1
T(n) = 2T(n/4) + 1

Iterate the recurrence three times:

$T(n) = 2T\left(\frac{n}{4}\right) + 1$ (one iteration)
$T(n) = 2(2T\left(\frac{n}{16}\right) + 1) + 1$
$T(n) = 4T\left(\frac{n}{16}\right) + 3$ (two iterations)
$T(n) = 4(2T\left(\frac{n}{64}\right) + 1) + 3$
$T(n) = 8T\left(\frac{n}{64}\right) + 7$ (three iterations)

Now, let's make a guess as to the form of the recurrence after iterating k times based on the first three iterations:

$$T(n) = 2^k T\left(\frac{n}{4^k}\right) + (2^k - 1)$$

Since we know T(1), we want to plug in the value of k for which $\frac{n}{4^k} = 1$, in for k. Solving, we find that $n = 4^k$. Taking the square root of both sides, we find $\sqrt{n} = \sqrt{4^k} = \sqrt{2^{2k}} = (2^{2k})^{\frac{1}{2}} = 2^k$. Substituting for both 4^k and 2^k, in the right hand of the recurrence, we get:

$$T(n) = \sqrt{n}T\left(\frac{4^k}{4^k}\right) + \left(\sqrt{n} - 1\right) = \sqrt{n}T(1) + \left(\sqrt{n} - 1\right) = \sqrt{n} + \sqrt{n} - 1 \in \boldsymbol{O}(\sqrt{\boldsymbol{n}})$$

Grading: 1 pt for first iteration
1 pt for second iteration
2 pts for third iteration
2 pts for general form guess
2 pts to plug in n = 4^k into general form (or equivalent)
2 pts to substitute and get to the final answer.