

## 3) (10 pts) ANL (Summations and Recurrence Relations)

Using the iteration technique, find a tight Big-Oh bound for the recurrence relation defined below:

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2, \text{ for } n > 1$$

$$T(1) = 1$$

Hint: You may use the fact that  $\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = 4$  and that  $3^{\log_2 n} = n^{\log_2 3}$ , and that  $\log_2 3 < 2$ .

Iterate the given recurrence two more times:

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2$$

$$T(n) = 3\left(3T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right) + n^2$$

$$T(n) = 9T\left(\frac{n}{4}\right) + \frac{3n^2}{4} + n^2$$

$$T(n) = 9T\left(\frac{n}{4}\right) + n^2\left(1 + \frac{3}{4}\right)$$

$$T(n) = 9\left(3T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right) + n^2\left(1 + \frac{3}{4}\right)$$

$$T(n) = 27T\left(\frac{n}{8}\right) + \frac{9n^2}{16} + n^2\left(1 + \frac{3}{4}\right)$$

$$T(n) = 27T\left(\frac{n}{8}\right) + n^2\left(1 + \frac{3}{4} + \frac{9}{16}\right)$$

In general, after the  $k^{\text{th}}$  iteration, we get the recurrence

$$T(n) = 3^k T\left(\frac{n}{2^k}\right) + n^2 \left(\sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i\right)$$

To solve the recurrence, find  $k$  such that  $\frac{n}{2^k} = 1$ . This occurs when  $n = 2^k$  and  $k = \log_2 n$ . Plug into the equation above for this value of  $k$  to get:

$$T(n) = 3^{\log_2 n} T(1) + n^2 \left(\sum_{i=0}^{k-1} \left(\frac{3}{4}\right)^i\right) \leq 3^{\log_2 n} + n^2 \left(\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i\right) = n^{\log_2 3} + 4n^2 = O(n^2)$$

**Grading:** 1 pt for copying recurrence, 1 pt for getting 2<sup>nd</sup> iteration, 2 pts for getting third iteration (in any form), 3 pts for  $k^{\text{th}}$  iteration, 1 pt for what to plug in form, 2 pts to complete the problem. (Be somewhat generous as this is probably the hardest problem on the exam.)

# Computer Science Foundation Exam

January 13, 2018

## Section II B

### ALGORITHMS AND ANALYSIS TOOLS

### **SOLUTION**

**NO books, notes, or calculators may be used,  
and you must work entirely on your own.**

Question #	Max Pts	Category	Passing	Score
1	10	DSN	7	
2	5	ALG	3	
3	10	DSN	7	
TOTAL	25		17	

**You must do all 3 problems in this section of the exam.**

**Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.**