

### 3) (10 pts) ANL (Recurrence Relations)

Using the iteration technique, determine a closed-form solution for the following recurrence relation in terms of  $n$ . Note: Your answer should be **EXACT** and not a Big-Oh bound.

$$T(0) = 1$$
$$T(n) = 4T(n-1) + 2^n, \text{ for integers } n > 0$$

Using the iteration technique for three iterations, we get:

$$T(n) = 4T(n-1) + 2^n \quad \text{Grading: 1 pt}$$

$$T(n) = 4(4T(n-2) + 2^{n-1}) + 2^n$$

$$T(n) = 16T(n-2) + (2^2)2^{n-1} + 2^n$$

$$T(n) = 16T(n-2) + [2^{n+1} + 2^n] \quad \text{Grading: 2 pts}$$

$$T(n) = 16(4T(n-3) + 2^{n-2}) + [2^{n+1} + 2^n]$$

$$T(n) = 64T(n-3) + (2^4)2^{n-2} + [2^{n+1} + 2^n]$$

$$T(n) = 64T(n-3) + [2^{n+2} + 2^{n+1} + 2^n] \quad \text{Grading: 2 pts}$$

From here we see that after  $k$  iterations, the recurrence is:

$$T(n) = 4^k T(n-k) + \sum_{i=0}^{k-1} 2^{n+i} \quad \text{Grading: 2 pts}$$

Since we know  $T(0)$ , let's plug in  $k = n$  into the formula above to yield:

$$T(n) = 4^n T(n-n) + \sum_{i=0}^{n-1} 2^{n+i} \quad \text{Grading: 1 pt}$$

$$T(n) = 4^n T(0) + \sum_{i=0}^{n-1} 2^n 2^i$$

$$T(n) = 2^{2n} + 2^n \sum_{i=0}^{n-1} 2^i$$

$$T(n) = 2^n (2^n) + 2^n \sum_{i=0}^{n-1} 2^i$$

$$T(n) = 2^n [2^n + \sum_{i=0}^{n-1} 2^i]$$

$$T(n) = 2^n [\sum_{i=0}^n 2^i]$$

$$T(n) = 2^n \left( \frac{2^{n+1}-1}{2-1} \right)$$

$$T(n) = 2^n (2^{n+1} - 1) \quad \text{Grading: 2 pts}$$

**Grading Note:** Please accept any reasonable final form (there are quite a few that are reasonably simplified). For the last two points, award 1 pt if there is some algebra but not quite enough to get to a reasonable final form.