Alternate Solution to #1

Instead of summing each number in the original problem individually (1 + 2 + 1 + 3 + 1 + 2 + 1, which we decomposed into 4 x 1 + 2 x 2 + 1 x 3), we can view the problem differently and count how many times each bit gets flipped. The least significant bit gets flipped every time, or $2^n - 1$ times. The second least significant bit gets flipped slightly less than half of that, exactly $2^{n-1} - 1$ times. (To see this, note that we flip this bit every other time, with the flip occurring on the second of each pair. Formally, we flip this bit $\left\lfloor \frac{2^n-1}{2} \right\rfloor = 2^{n-1} - 1$ times. More generally, the kth least significant bit gets flipped exactly $\left\lfloor \frac{2^n-1}{2^{k-1}} \right\rfloor = 2^{n-k+1} - 1$ times. Thus, we can add up the total number of bit flips by adding the number of times each individual bit itself gets flipped, giving us the following summation to evaluate:

$$\sum_{k=1}^{n} 2^{n-k+1} - 1$$

$$\sum_{k=1}^{n} (2^{n-k+1} - 1) = \left(\sum_{k=1}^{n} 2^{n-k+1}\right) - (\sum_{k=1}^{n} 1)$$

$$= \left(\sum_{k=1}^{n} 2^k\right) - n$$

$$= 2(2^n - 1) - n$$

$$= 2^{n+1} - 2 - n$$

$$= 2^{n+1} - n - 2$$

Yet a third way to view this problem is to let T(n) be the answer to the question for an n-bit counter. Using the observation above where we note that the least significant bit flips every time ($2^n - 1$), notice that the remaining n – 1 bits are essentially playing the role of an n – 1 bit-counter. (Basically, the n – 1 most significant bits stay frozen every other step and then just count regularly on the even numbered steps. This means that $T(n) = T(n-1) + 2^n - 1$. The solution to this recurrence is the summation above.

Grading: 4 pts for setting up summation or recurrence relation which corresponds to the answer to the question.

6 pts for evaluating the derived summation or recurrence relation.

Give partial credit as you see fit for both parts.

If initial summation is incorrect but that incorrect sum is evaluated correctly, give a maximum of four points out of six for evaluating the summation, depending on complexity of it.