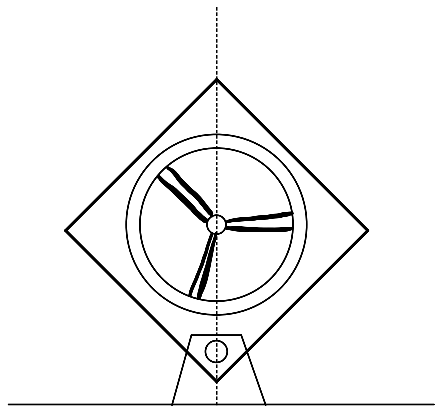
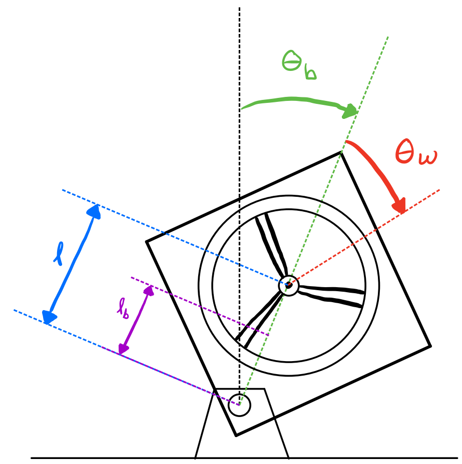
One Dimensional Proto type





θ_b = tilt angle of pendulum body

θ_w = rotational displacement of momentum wheel w/ respect to body

m_b = mass of Body

m_w = mass of wheel

I_b = Moment of inertia of Pendulum body around piviot point

I_w = moment of inertia of wheel & motor rotor around rotation axis of motor

l = distance between the motor axis and piviot point

l_b = distance between the center of mass of pendulum body and piviot point

$m_b = 24g$

$m_w = 30g$

$I_b = 729.37 \times 10^{-2}\ g \cdot mm^2$

$I_w = 122.18 \times 10^{4}\ g \cdot mm^2$

$l = 48.6\ mm$

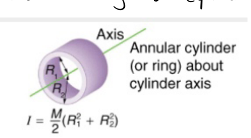
$l_b = 31.5\ mm$

$$\ddot{\theta}_b = \frac{(m_b l_b + m_w l)\, g \sin\theta_b + T_m - C_b\dot{\theta}_b + C_w\dot{\theta}_w}{I_b + m_w l^2}$$

$$\ddot{\theta}_w = \frac{(I_b + I_w + m_w l^2)(T_m - C_w\dot{\theta}_w)}{I_w(I_b + m_w l^2)} - \frac{(m_b l_b + m_w l)\, g \sin\theta_b - C_b\dot{\theta}_b}{(I_b + m_w l^2)}$$

Moment of inertia of wheel & motor rotor around rotation axis of motor

Assume ring is cylinder



$R_1 = 80\ mm$

$R_2 = 100\ mm$

$m_{motor} = 119g$

$m_{wheel} = 30g$

$M = m_{motor} + m_{wheel}$

$M = 149g$

$$I_w = \frac{M}{2}(R_1^2 + R_2^2)$$

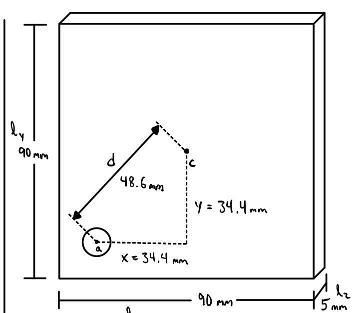
$$= \frac{149g}{2}(80^2 + 100^2)\,mm$$

$$= 122.18 \times 10^4\ g \cdot mm^2$$

$$= (122.18 \times 10^4\ g \cdot mm^2)\left(\frac{1 \times 10^{-9}\ kg \cdot m^2}{1\ g \cdot mm^2}\right)$$

$$= 122.18 \times 10^{-5}\ kg \cdot m^2$$

$$\boxed{I_w = 122.18 \times 10^4\ g \cdot mm^2 \text{ or } 122.18 \times 10^{-5}\ kg \cdot m^2}$$



$l_x = 90\ mm$

$l_y = 90\ mm$

$l_z = 5\ mm$

$m = 24g$

$d = 48.6\ mm$

$$\boxed{I_b = 729.37 \times 10^{-2}\ g \cdot mm^2 \text{ or } 7.2937 \times 10^{-5}\ kg \cdot m^2}$$

$$I_{c,x} = \frac{1}{12} m (l_y^2 + l_z^2)$$

$$= \frac{1}{12}(24g)(90^2 + 5^2)\,mm$$

$$= 162.50 \times 10^2\ g \cdot mm^2$$

$$= (162.50 \times 10^2\ g \cdot mm^2)\left(\frac{1 \times 10^{-9}\ kg \cdot m^2}{1\ g \cdot mm^2}\right)$$

$$= 1.625 \times 10^{-5}\ kg \cdot m^2$$

Parallel axis theorem

$$I_{a,x} = I_{c,x} + m d^2$$

$$= (162.50 \times 10^2\ g \cdot mm^2) + (24g)(48.6mm)^2$$

$$= 729.37 \times 10^{-2}\ g \cdot mm^2$$

$$= (729.37 \times 10^{-2}\ g \cdot mm^2)\left(\frac{1 \times 10^{-9}\ kg \cdot m^2}{1\ g \cdot mm^2}\right)$$

$$= 7.2937 \times 10^{-5}\ kg \cdot m^2$$

C_ω = dynamic friction Coefficent of wheel
C_b = dynamic friction Coefficent of body
T_m = Torque produced by motor
K_m = Torque constant of electric motor
u = current input

C_ω =
C_b =

$K_m = 37.5$ mNm/A or 37.5×10^{-3} Nm/A

$$T_m = K_m u \longrightarrow T_m(t) = K_m u(t) \quad \text{where} \quad T_m(t) - T_f = I_w \frac{d\dot{\theta}_w(t)}{dt}$$

$T_m(t) = T_f$ (When angular velocity $\dot{\theta}_w$ is Constant)

I_w & C_w found from driving momentum wheel driven with different current steps, while pendulum body was rigidly fixed, and the time trace of $\dot{\theta}_w$ was recorded.

Motor Current

A least squares fit with the measurments to :

$$I_w \ddot{\theta}_w(t) = K_m u(t) - C_w \dot{\theta}_w(t)$$

angular acceleration of wheel

angular velocity of wheel

I_b & C_b found after rigidly fixing the momenum wheel with the pendulum body, the whole setup was hung upside down and was made to swing

angular accleration of the body

dynamic friction Coefficent of body

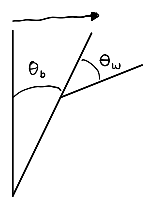
tilt angle of body

A least squares fit with the measurments to :

$$\underbrace{(I_b + I_w + m_w \ell^2)}_{\text{moment of Inertia of Body about center of rotation of the body}} \ddot{\theta}_b(t) = -C_b \dot{\theta}_b(t) + \underbrace{(m_b \ell_b + m_w \ell)\, g \sin(\theta_b(t))}_{\text{Force of body + wheel due to gravity in x-dir}}$$

angular Velocity of the body

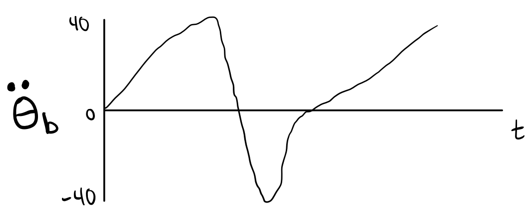
When $\ddot{\theta}_b(t) = 0$ (When angular velocity $\dot{\theta}_b$ is Constant)



$$C_b \dot{\theta}_b(t) = (m_b \ell_b + m_w \ell)\, g \sin(\theta_b(t))$$

$$C_b = \frac{(m_b \ell_b + m_w \ell)\, g \sin(\theta_b(t))}{\dot{\theta}_b(t)}$$

When constant



$$\dot{\theta}_b = \text{Const} \quad \text{when} \quad \frac{d\ddot{\theta}(t)}{dt} = 0$$

Total Kinetic Energy

$$K = K_1 + K_2 = \frac{1}{2}\left(m_B l_B^2 + m_w l^2 + I_B + I_w\right)\dot{\theta}_B^2 + I_w \dot{\theta}_B \dot{\theta}_w + \frac{1}{2} I_w \dot{\theta}_w^2$$

Potential Energy

$$P = \bar{m} g \left(\cos(\theta_B) - 1\right)$$

Lagarian Function

$$L = K - P$$

$$L = \frac{1}{2}\left(m_B l_B^2 + m_w l^2 + I_B + I_w\right)\dot{\theta}_B^2 + I_w \dot{\theta}_B \dot{\theta}_w + \frac{1}{2} I_w \dot{\theta}_w^2 - \bar{m} g \left(\cos(\theta_B) - 1\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_B} = \left(m_B l_B^2 + m_w l^2 + I_B + I_w\right)\dot{\theta}_B^2 + I_B \dot{\theta}_B$$

$$\frac{\partial L}{\partial \theta_B} = \bar{m} g \sin(\theta_B)$$

$$\frac{\partial L}{\partial \theta_B} = I_B \dot{\theta}_B + I_w \dot{\theta}_w$$

$$\frac{\partial L}{\partial \theta_B} = 0$$

$$\left(m_B l_B^2 + m_w l^2 + I_B + I_w\right)\dot{\theta}_B^2 + I_B \dot{\theta}_B - \bar{m} g \sin(\theta_B) = 0$$

$$I_B \dot{\theta}_B + I_w \dot{\theta}_w = \tau$$

$$D(q)\ddot{q} + g(q) = u$$

$$u = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

$$D(q) = \begin{bmatrix} \left(m_B l_B^2 + m_w l^2 + I_B + I_w\right) & I_w \\ I_w & I_w \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$g(q) = \begin{bmatrix} -mg\sin(\theta_B) \\ 0 \end{bmatrix}$$

EOM also:

$$d_{11}\ddot{\theta}_B + d_{12}\ddot{\theta}_w + g(\theta_B) = 0$$

$$d_{21}\ddot{\theta}_B + d_{22}\ddot{\theta}_w = \tau$$

Total Energy of Reaction Wheel

$$E = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} + \bar{m} g \left(\cos(\theta_B) - 1\right)$$

$$\dot{E} = \dot{q}^T D(q)\ddot{q} - \bar{m}g\sin(\theta_B)\dot{\theta}_B = \dot{\theta}_w \tau$$

Linearization of System

$$\ddot{\theta}_B = \frac{d_{22}}{\det(D)}\bar{m}g\sin(\theta_B) - \frac{d_{12}}{\det(D)}\tau$$

$$\ddot{\theta}_w = \frac{-d_{21}}{\det(D)}\bar{m}g\sin(\theta_B) + \frac{d_{11}}{\det(D)}\tau$$

$$\det(D) = d_{11}d_{22} - d_{21}d_{12} = \left(m_B l_B^2 + m_w l_w^2 + I_B\right)I_w$$

Vector State

$$\begin{array}{cccccccc} & & & A & x & + & B & \tau \\ X = & \begin{bmatrix}\theta_B\\ \dot{\theta}_B\\ \theta_w\\ \dot{\theta}_w\end{bmatrix} & = & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{d_{22}}{\det(D)}\bar{m}g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-d_{21}}{\det(D)}\bar{m}g & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix}\theta_B\\ \dot{\theta}_B\\ \theta_w\\ \dot{\theta}_w\end{bmatrix} & + & \begin{bmatrix} 0 \\ \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \end{bmatrix} & \tau \end{array}$$

$$= Ax + B\tau$$

$$B = \begin{bmatrix} 0 \\ \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \end{bmatrix} \qquad AB = \begin{bmatrix} \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \\ 0 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 0 \\ \frac{-d_{22}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \\ \frac{d_{21}d_{12}}{(\det(D))^2}\bar{m}g \end{bmatrix} \qquad A^3B = \begin{bmatrix} \frac{-d_{22}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \\ \frac{d_{21}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \end{bmatrix}$$

$$\det(B \mid AB \mid A^2B \mid A^3B) = \frac{d_{12}^2 m^2 g^2}{(\det(D))^4} = \frac{\bar{m}^2 g^2}{(m_B l_B + m_w l^2 + I_B)^4 I_w^2}$$

Full State feedback Controller $\tau = -K^T X$