Compactifications of Metrics on Heterotic Total Spaces

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String theory concerns itself with understanding the fundamental interactions of physics. This considers the compactifications of extra dimensions as well as our current 4-dimensional understanding (referred to as lower dimensions). Much research has shown that different choices for the extra dimensions of space influcence effective lower dimensional equations of Einstein, Maxwell, and Klein-Gordon. To deeply understanding the implications of the extra geometry on the lower dimensional physics, certain metrics and Gauge potentials must be specified for the extra space. Current approaches use machine learning and other numerical methods to approximate Calabi-Yau metrics then use these results to aproximate the Gauge potentials. However, these methods are often sluggish and lead to unwanted error propagation. Therefore, our work has us considering a total metric space that specify forms for our metric and Gauge potentials of interest. Calculating desired information using these techniques reduces error propogation and allows a more streamlined approach to determine the scalar field ϕ and a gauge field strength H.

I. INTRODUCTION

Einstein's General Relativity and Quantum Field Theory are the most fundamental and successful theories in modern physics. For example, gravity is the curvature of spacetime due to massive objects and energy, while QFT describes the weak, strong, and E&M forces in the Standard Model. However, for the past century, many questions in theoretical physics have been attempting to unify the frameworks provided by quantum field theory and general relativity. String theory is one of many proposed schemes.

A relatively large assumption needs to be made to use string theory; it only works when there are more than the three spatial and one-time dimensions proposed by Einstein. In conjunction with this question, string theorists also ask: what extra dimensions are allowed, how does their shape change the 4D physics, and which geometries agree with the already known physics? Unfortunately, according to previous research[5], there exist half a billion relevant manifolds that do not have accessible definable metric spaces. To address this issue, many string theorists have been turning to numerical metrics derived from techniques in machine learning. However, the current process, which will be discussed more in Section V, is time-consuming and introduces heavy error propagation. This is because the extra dimensions and the observed dimensions are treated separately. However, our work aims to treat both the extra and observed dimensions simultaneously in the machine learning algorithm.

II. PRELIMINARY MATHEMATHICS

This section briefly discusses the relevant mathematical techniques needed to understand the research project's breadth fully.

A. Manifolds and Metrics

Modern theoretical physics is heavily rooted in understanding the notion of a manifold. According to Carrol[1], manifolds correspond to a space that may be curved and have a complicated topology, but in local regions look like $\mathbb{R}^n$ pieces that are smoothly sewed together.

Metrics are essential mathematical objects describing the geometry of the manifolds discussed in General Relativity. The metric tensor computes path lengths and similarly determines the "shortest distance" between two points on any non-trivial manifold. For example, the metrics on $\mathbb{R}^3$ and the $\mathbb{R}^2$ polar Euclidean metric can be seen below

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{1}$$

The manifolds that will be considered in this work are Kähler manifolds, which admit a complex metric of $U(N)$ holonomy and a closed Kähler form. The consequences of this choice of geometry will be discussed in Section II D.

B. Gauge Theories

From Lessa's notes on gauge theories[2], gauge appears when there exist more degrees of freedom than exist in the physics itself. This freedom manifests as degrees of freedom from the Lagrangian of a scalar field theory. Theories that leave the Lagrangian invariant under gauge transformations are considered gauge theories. The most straightforward example to think of is classical E&M, where the vector and scalar potentials are invariant under the transformation

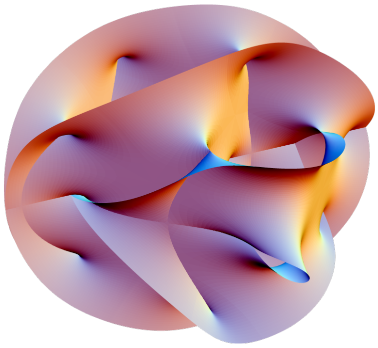

FIG. 1: 3D shadow of a Calabi Yau manifold, a certain class of Kähler manifolds

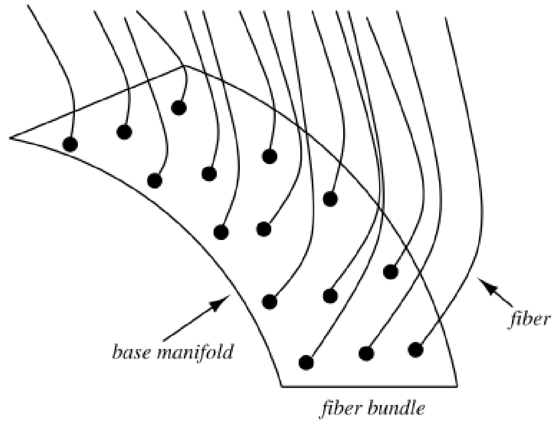

FIG. 2: Visual representation of a fiber bundle. Image taken from the Wolfram Alpha page on Fiber Bundles.

$$V' = V - \frac{\partial f}{\partial t} \tag{2}$$
$$\vec{A}' = \vec{A} + \nabla f \tag{3}$$

The gauge theories of interest in our research are non-abelian gauge theories in which the transformation of the gauge fields is represented by

$$A^a_\mu \to A^a_\mu + \partial_\mu \epsilon^a + C_{abc}\epsilon^b A^c_\mu \tag{4}$$

where C_{abc} are the structure constants (a consequence of the non-vanishing commutator of the gauge fields); these gauge theories are more interesting because the observed Standard Model is the non-abelian theory $SU(3)\times SU(2)\times U(1)$.

C. Fiber Bundles

The symmetries of physical laws can be encoded in an extra geometry called a fiber bundle. For intuition, at every point in a base space, another geometry exists to track needed information. For example, mathematicians encode our 3-dimensional space with an interval of real numbers at each point to track temperature. In the case of this work, the fiber bundles are helpful geometric tools for encoding the notion of the gauge fields discussed previously.

D. Einstein Field Equations

The Einstein Field Equations are derived from varying the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} R \tag{5}$$

where the variation is with respect to the metric. From this, we get Ricci Tensor and Christoffel symbol definitions

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\nu\mu} - \partial_\nu \Gamma^\sigma_{\sigma\mu} + \Gamma^\sigma_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{6}$$
$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{7}$$

From Section II A, we discussed the choice of Kähler metric making our work easier. This manifests in the fact that a distinction can be drawn between holomorphic (μ, ν) and anti-holomorphic $(\bar{\mu}, \bar{\nu})$ coordinates. Kähler metrics provide an interesting constraint for the nonzero components of the metric

$$g_{ab} = g_{\bar{a}\bar{b}} = 0 \quad g_{\bar{a}b} \neq 0 \tag{8}$$

Since only the off-diagonal components of the metric are nonzero, the definitions for the Ricci tensor and Christofell symbols will simplify for the Kähler metrics.[4]

$$R_{\bar{a}b} = -\partial_c \Gamma^{\bar{a}}_{\bar{b}\bar{a}} \tag{9}$$
$$\Gamma^a_{bc} = g^{a\bar{d}}\partial_b g_{c\bar{d}} \quad \Gamma^{\bar{c}}_{\bar{b}\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{\bar{c}d} \tag{10}$$

III. 5D KALUZA-KLEIN THEORY

5-dimensional Kaluza-Klein theory was one of the first attempts at unifying gravity and electromagnetism by considering a 5-dimensional Einstein gravity defined by the geometry

$$X_{tot} = \mathbb{R}^{1,3} \times S^1$$

$\mathbb{R}^{1,3}$ refers to the standard 4-dimensional Einstein space in GR, and the S^1 is a circle fiber defined at every point in space. A 5-dimensional metric ansatz can be defined by

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{11}$$

where $g_{\mu\nu}$ is the base metric and ϕ is the coordinate associated with the S^1 fibers. Kaluza and Klein theorized that a 5-dimensional Einstein field equation could reduce to the observed 4-dimensional Einstein equations and Maxwell's equations with this ansatz. The information in this section was taken from the work by Bailin and Love[3].

IV. WORK PERFORMED

Our work generalizes the Kaluza-Klein theory discussed in the previous section. The interest in the original theory stems from the idea that the $U(1)$ abelian gauge symmetry could be manifestly reproduced by considering a $U(1)$ fiber symmetry on a base metric. Furthermore, it was a compact way of carrying all of the metric and gauge information of the base space and the fiber.

For our work of interest, Kaluza-Klein techniques are a first attempt at creating a compact metric ansatz to apply the computational techniques of interest. We needed to generalize the 5-dimensional Kaluza-Klein theory to reproduce the non-abelian gauge fields on interest potentially. To do this, we can consider a Calabi-Yau base metric with a $\mathbb{CP}^n$ fibration (also represented by a Kähler metric). With this in mind, we attempted a metric anstaz[3] for this new total space

$$\mathring{g}_{\bar{A}B} = \begin{pmatrix} g_{\bar{\mu}\nu}(x) + \tilde{g}_{\bar{m}n}(x,y) B^{\bar{m}}_{\bar{\mu}} B^n_\nu & \tilde{g}_{\bar{m}n}(x,y) B^{\bar{m}}_{\bar{\mu}} \\ \tilde{g}_{\bar{m}n}(x,y) B^n_\nu & \tilde{g}_{\bar{m}n}(x,y) \end{pmatrix}$$

where x are base coordinates and y are fiber coordinates and $B^{\bar{m}}_{\bar{\mu}} = \xi^{\bar{m}}_a(y) A^a_{\bar{\mu}}(x)$. The different $\xi^n_a(y)$ are the Killing vectors (vector fields related to the symmetries of the metrics spaces) associated with the fibration. In the case of $\mathbb{CP}^n$, the Killing vectors are defined to be holomorphic i.e.

$$\partial_l \xi^{\bar{m}}_a = 0 \quad \partial_{\bar{l}} \xi^m_a = 0 \tag{12}$$

Kaluza-Klein theory can be attempted with this metric ansatz to derive physics equations in the base space of interest (in this case, the Calabi-Yau). Using the Kähler definitions of the Ricci tensor and Christoffel symbols in Section II D, we have that

$$\mathring{R}_{\bar{\mu}\nu} = R_{\bar{\mu}\nu} - \partial_\nu \left(\tilde{g}^{\bar{m}n} \partial_{\bar{\mu}} \tilde{g}_{\bar{m}n} \right) \tag{13}$$

$$\mathring{R}_{\bar{k}\nu} = -\partial_\nu \left(\tilde{g}^{\bar{m}n} \partial_{\bar{k}} \tilde{g}_{\bar{m}n} \right) \tag{14}$$

$$\mathring{R}_{\bar{k}l} = \tilde{R}_{\bar{k}l} \tag{15}$$

where Equation 13 should resemble Einstein gravity in the base, Equation 14 resembles Yang-Mills (a generalization of Maxwell's), and Equation 15 resembles Klein-Gordon (a generalization of Schrödinger).

V. FUTURE WORK

We have just shown in the previous section that we can write the physics that would be observed in the base metric. However, our work has yet to imply any additional conditions on the total space besides its Kählerity, which we are afraid might be too constrictive when reducing the effective base physics. Future will involve addressing the implications of imposing Kählerity of the total as well as imposing constraints. For example, since we are assuming the fiber metrics are $\mathbb{CP}^n$, the fiber metric should have constant scalar curvature. With this in mind, the total space should have curvature. The two options of consideration are $\mathring{R}_{\bar{A}B} = \Lambda$ (constant scalar curvature) or $\mathring{R}_{\bar{A}B} \propto \mathring{g}_{\bar{A}B}$ (Einstein metric). Conditions on the total space are required for the machine learning algorithm; the flow of the machine learning algorithm can be seen in the figure below.

FIG. 3: Visual representation of the numerical algorithm. Image taken from the Wolfram Alpha page on Fiber Bundles.

The algorithm uses machine learning to minimize a particular constraint to find parameters for a particular metric ansatz. In our case, the metric ansatz will be the one described in Section IV and the constraints are those that still need to be figured out.

VI. CONCLUSION

In this work, we have successfully derived effective equations of motion in a Calabi-Yau base fibered with a particular Kähler metric. These techniques will address the possibility of considering total space metrics in different string theories instead of treating the base and fiber spaces separately. Furthermore, using this framework for our machine learning algorithms will drastically decrease the computational power and error propagation in the current techniques.

VII. ACKNOWLEDGMENTS

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