Compactifications of Metrics on Heterotic Total Spaces

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What’s the *deal* with string theory?

- Success of Einstein’s gravity (General Relativity)
- Success of Quantum Mechanics (QFT)
  - These two don’t really play together!

How do we combine our successes into one big success?

String theory attempts to do this by combining Einstein’s gravity with quantum field theory!

Only one problem…. 10 dimensions!

Toolkit 1: The Metric

Metrics give a notion of distance on spaces, encapsulating all information about the space

You already know one! Metric on R^3

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Can be much more complicated....

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Can get much information from metric, including curvature, torsion, etc.

Toolkit 2: Gauge Fields

QFT says that there is a field corresponding to each fundamental particle (or really, vice versa!)

Force-carrying particles are "gauge fields" in the SM

Gauge fields have symmetries that leave Lagrangian unchanged, another example of something you already know!

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Left unchanged by...

$$\mathbf{A} \to \mathbf{A} + \nabla f$$

$$V \to V - \frac{\partial f}{\partial t}$$

Other Examples...

$$S = \int \bar{\psi}\left(i\hbar c \gamma^\mu \partial_\mu - mc^2\right)\psi \, \mathrm{d}^4 x$$

Invariant under

$$\psi \to e^{i\theta}\psi$$

$$-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2$$

Invariant under

$$\mathbf{A}^a_\mu \to \mathbf{A}^a_\mu + \partial_\mu \epsilon^a(x) + C_{abc}\epsilon^b(x)\mathbf{A}^c_\mu$$

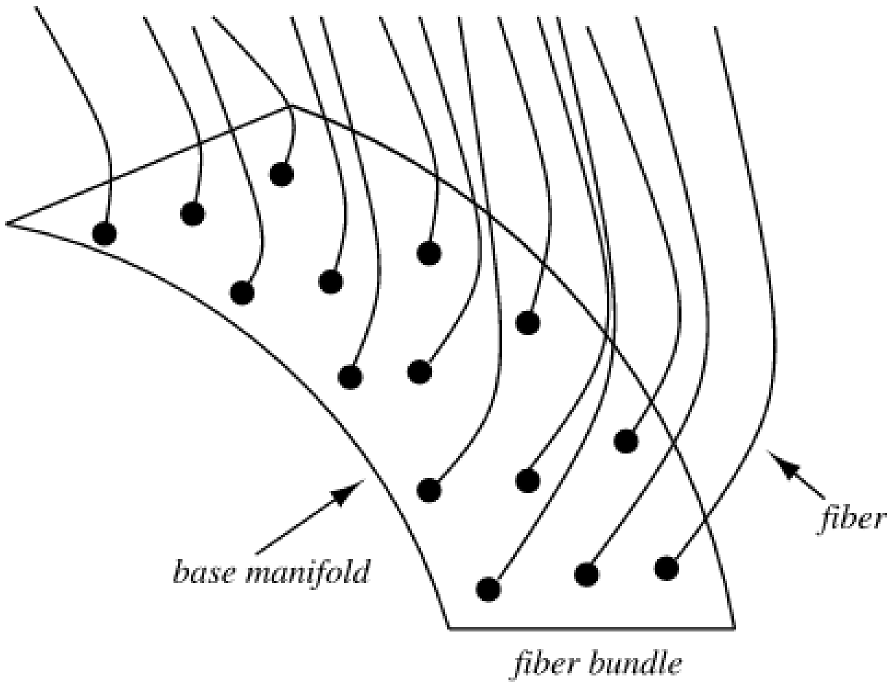
Toolkit 3: Fiber Bundles

Constructed from a base space X and a space F called the "fiber"

Total space called V, or $\pi: V \to X$

For open sets $U \subset X$, looks locally like $U \times F$

Can think of fields as taking values in the fiber, e.g. scalar fields taking values in fiber of R



Toolkit 4: Einstein Field Equations

"Spacetime tells matter how to move, matter tells spacetime how to curve"

Einstein imagined that gravity was effect of curvature of spacetime $\longrightarrow$ General theory of Relativity

Starting with an action built of the metric, variation (Euler Lagrange equations) gives equations of motion

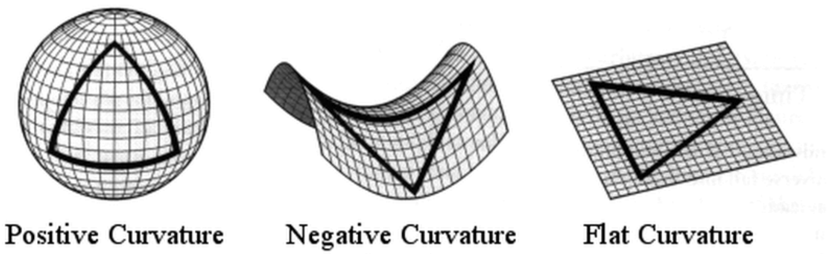
$$S = \int \mathrm{d}^4x \sqrt{-g}\, R \xrightarrow{\text{varying}} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

where

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\nu\mu} - \partial_\nu \Gamma^\sigma_{\sigma\mu} + \Gamma^\sigma_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

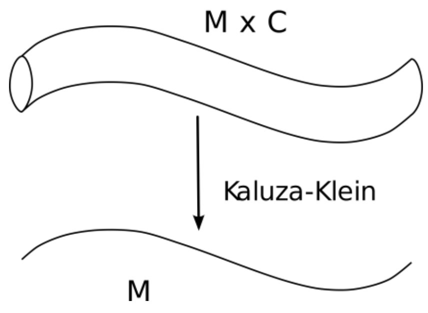
Very coupled PDEs!



Kaluza-Klein Theory

A first attempt at unifying GR with other theories (EM)

Supposed that there is extra (fifth) dimension curled up like a circle



$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}$$

$$A_\mu = (\phi, A_x, A_y, A_z)$$

What if you transform coords?

$$x'^\mu = x^\mu$$
$$\theta' = \theta - f(x^\mu)$$

$$\longrightarrow \quad A_\mu \to A_\mu + \partial_\mu f$$

Seem familiar?

Symmetry of circle reproduces E&M! U(1) Symmetry

$$\Box\phi = \tfrac{1}{4}\phi^3 F^{\mu\nu} F_{\mu\nu} \ , \quad \frac{1}{2} g^{\mu\sigma} \nabla_\mu \left(\phi^3 F_{\rho\sigma}\right) = 0 \quad \longrightarrow \quad \nabla^\mu F_{\mu\nu} = 0$$

Where does our work begin?

Let's expand our use of the Kaluza-Klein Theory to consider more than just the theory of E&M.

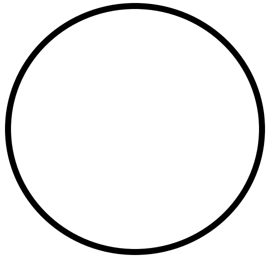
Abelian Gauge Theory [U(1)]
(before only worried about the photon)

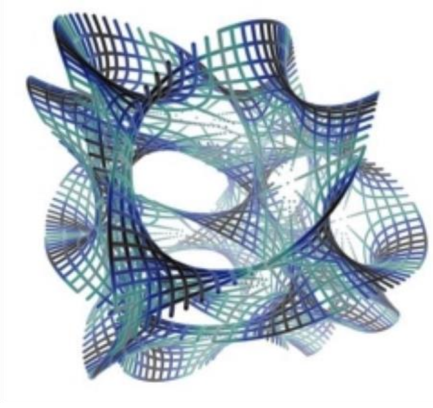
$$A_\mu \rightarrow A_\mu + \partial_\mu f$$

Non-Abelian Gauge Theory [SU(3) x SU(2) x U(1)]
(now need photon, weak bosons, gluons, and hopefully a graviton)

$$\mathbf{A}_\mu^a \rightarrow \mathbf{A}_\mu^a + \partial_\mu \epsilon^a(x) + C_{abc}\epsilon^b(x)\mathbf{A}_\mu^c$$

Changing the base and fiber manifolds of interest





This is the 3D shadow of a Calabi-Yau manifold to replace the base

Higher Dimensional KK metric ansatz

$$\mathring{g}_{\bar{A}B} = \begin{pmatrix} g_{\bar{\mu}\nu} + \tilde{g}_{\bar{m}n} B^{\bar{m}}_{\bar{\mu}} B^{n}_{\nu} & \tilde{g}_{\bar{m}n} B^{\bar{m}}_{\bar{\mu}} \\ \tilde{g}_{\bar{m}n} B^{n}_{\nu} & \tilde{g}_{\bar{m}n} \end{pmatrix}$$

Note the indices with the bars.

$$B^{\bar{m}}_{\bar{\mu}} = \xi^{\bar{m}}_{a}(y) A^{a}_{\bar{\mu}}(x)$$

Killing vectors

Gauge Fields

See Balin and Love's Kaluza-Kelin theories to see more about different metric ansatz

What to do with this ansatz?

Plug it into Einstein's Field Equations!

For the first time this presentation, our assumptions will make our equations look easier than the original

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\nu\mu} - \partial_\nu \Gamma^\sigma_{\sigma\mu} + \Gamma^\sigma_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

$$\downarrow$$

$$R_{\bar{a}b} = -\partial_b \Gamma^{\bar{c}}_{\bar{a}\bar{c}}$$

$$\Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{\bar{c}d}$$

$$\Gamma^{a}_{bc} = g^{a\bar{d}}\partial_{b} g_{c\bar{d}}$$

Equations for complex expressions taken from Section 15.3.3 of Green, Shwarz, Witten: Superstring Theory Volume 2

Using Einstein's Field Equations

Our metric ansatz gives us three different equations for the total dimensional space

$$\mathring{R}_{\bar{\mu}\nu} = R_{\bar{\mu}\nu} - \partial_\nu \left(\tilde{g}^{\bar{m}n} \partial_{\bar{\mu}} \tilde{g}_{\bar{m}n} \right) \longleftarrow$$ should be Einstein's Gravity in base

$$\mathring{R}_{\bar{k}\nu} = -\partial_\nu \left(\tilde{g}^{\bar{m}n} \partial_{\bar{k}} \tilde{g}_{\bar{m}n} \right) \longleftarrow$$ should be Yang-Mills in base (like Maxwell's Eqs.)

$$\mathring{R}_{\bar{k}l} = \tilde{R}_{\bar{k}l} \longleftarrow$$ should be Klein-Gordon in base (like Schrödinger Eq.)

Not quite done...

Still need to figure out what we condition need to be imposed on the total space to get the full equations on the base space!

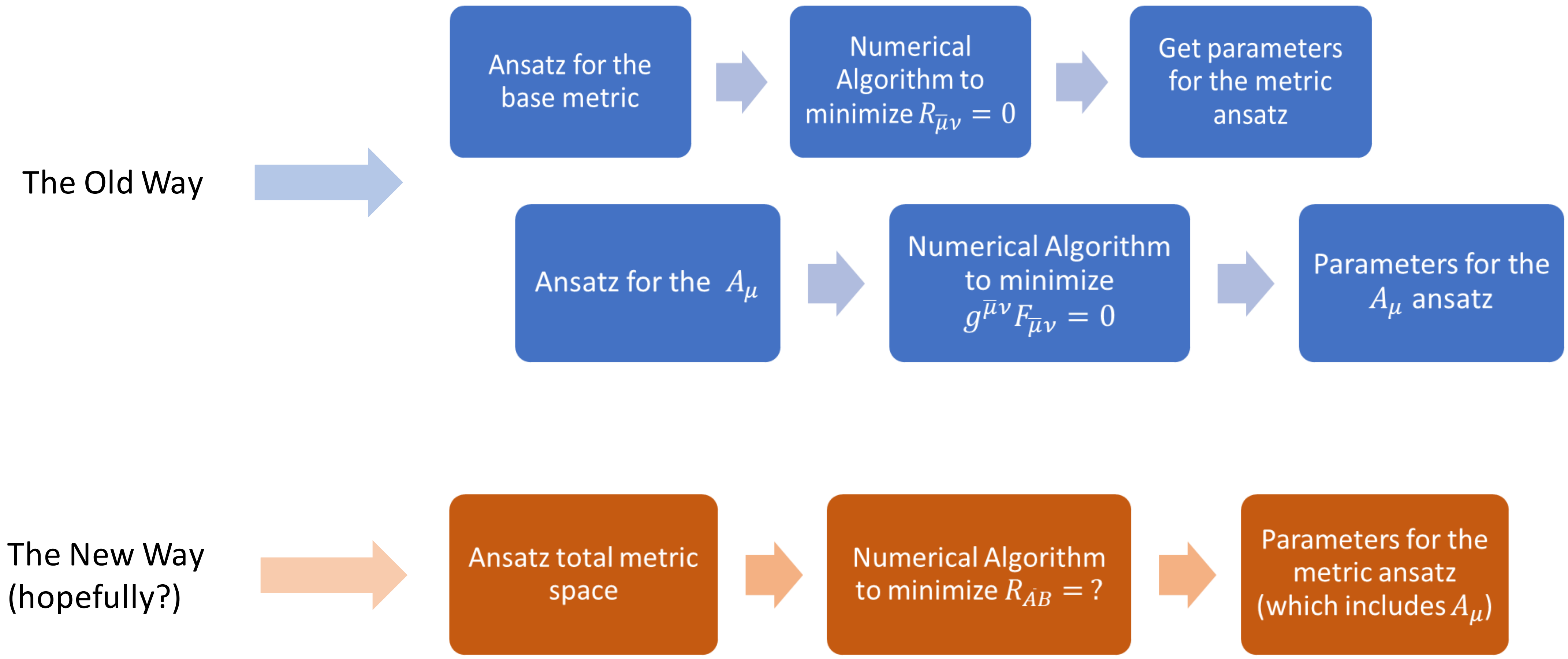
Constant scalar curvature?

Einstein Metric?

$$\mathring{R}_{\bar{\mu}\nu} \propto \mathring{g}_{\bar{\mu}\nu}$$

Where do we go from here?

Numerical Approximations!



Any Questions?