

## E-150 PROJECT 4 (100 POINTS)

### MODELING AND SIMULATION OF 3D FOOD PRINTING

Due: Thursday, October 31st, 3:30 pm

In this project, we will model the extrusion of a two-phase food slurry consisting of small particles suspended in a binder. Such slurries can have very high viscosities, making them challenging to print through small nozzles if the process is not controlled carefully. In this project, you will use a quasi-1D flow model to assess the impact of different processing parameters on the pressure gradients and nozzle wear rates that the equipment will experience while printing with different process parameters.

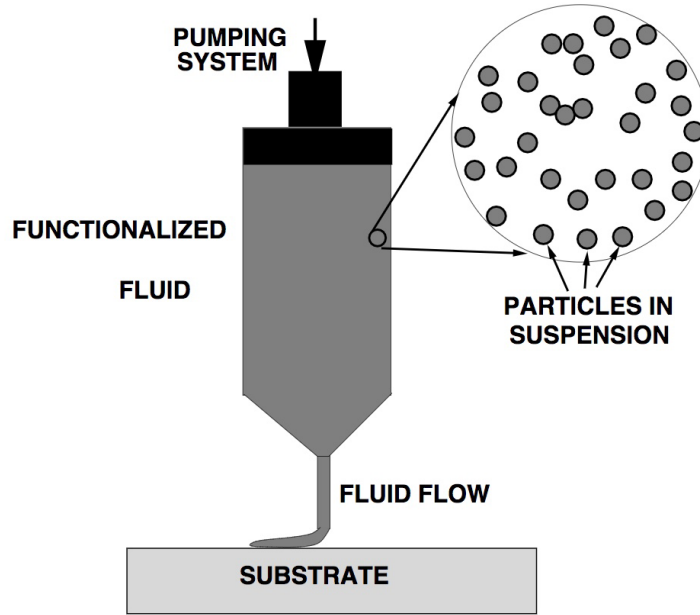


Figure 1: Schematic of printing

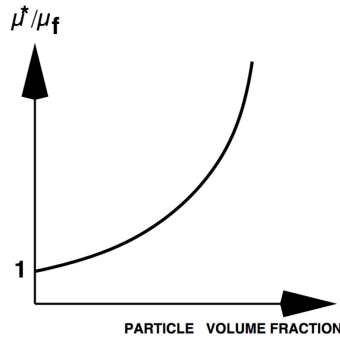


Figure 2: Relationship between particle volume fraction and effective viscosity.

## EXTRUSION & PRESSURE GRADIENTS

### Channel Flow:

The objective of the first part of this analysis is to derive a model for the pressure gradient required to maintain a given flow rate in a channel, as a function of the volume fraction of particles present in the slurry, and the temperature of the slurry. Consider an idealized channel with a circular cross-section of area  $A = \pi R^2$ , with a velocity profile of the form:

$$v(r) = v_{\max} \left(1 - \left(\frac{r}{R}\right)^q\right), \quad (1)$$

where  $v_{\max}$  is the center line velocity and  $r$  is the radial coordinate from the center line of the channel. For fully developed laminar flow,  $q = 2$ . Higher  $q$  values correspond to increasingly turbulent flow. The shear stress in the fluid is given by

$$\tau(r) = \mu^* \frac{\partial v}{\partial r} = -\frac{\mu^* v_{\max} q}{R} \left(\frac{r}{R}\right)^{q-1}, \quad (2)$$

where  $\mu^*$  is the effective viscosity of the slurry. The total flow rate is treated as a known constant:

$$Q = \int_A v dA = Q_o. \quad (3)$$

The centerline velocity can be rewritten in terms of the flow rate as follows:

$$v_{\max} = \frac{Q_o(q+2)}{Aq} = \frac{Q_o(q+2)}{\pi R^2 q}. \quad (4)$$

Plugging this expression for the velocity into the wall stress equation yields:

$$\tau_w = -\tau(r=R) = \frac{\mu^* v_{\max} q}{R} = \frac{\mu^* Q_o(q+2)}{\pi R^3}. \quad (5)$$

### Pressure Gradients:

In the remaining analysis, we will assume that the fluid flow is steady, that the particles are not elongated and that they are mixed uniformly throughout the slurry.<sup>1</sup> The previous expressions allow us to correlate the flow rate of a particle-laden fluid to the shear stress the fluid exerts on the walls of a nozzle. From this shear stress, we can determine the pressure gradient required to maintain steady axial flow<sup>2</sup>:

$$(-(P + \Delta P) + P)\pi R^2 - \tau_w 2\pi R \Delta x = 0, \quad (6)$$

where  $\Delta x$  is a differential length along the axis of the pipe.

$$-\Delta P = \mu^* \frac{Q_o(q+2)}{\pi^2 R^5} 2\pi R \Delta x = \frac{2\mu^* Q_o(q+2)\Delta x}{\pi R^4}, \quad (7)$$

where we used the expression for  $v_{\max}$  and where the effective viscosity is a function of the volume fraction of particles,  $\mu^* = \mu^*(v_p)$ . An explicit relation for  $\mu^*(v_p)$  will be given shortly. Solving for the pressure gradient yields

$$-\frac{\Delta P}{\Delta x} = \underbrace{\frac{2\mu^*(q+2)}{\pi R^4}}_C Q_o \stackrel{\text{def}}{=} C Q_o. \quad (8)$$

<sup>1</sup>Elongated particles tend to align themselves with the flow, which alters their impact on the viscosity.

<sup>2</sup>Steady flow means no acceleration.

If we fix the flow rate  $Q_o$ , the multiplier  $C$  identifies the pressure gradient needed.

### Velocity Profile Characteristics:

As the Reynolds number increases, the velocity profile will change from a quadratic ( $q = 2$ ) to a more blunted profile ( $q \gg 2$ ). The effect of a changing profile is described by representing  $q$  as a function of the centerline Reynolds' number ( $\mathcal{R}_{ec}$ ):

$$q = q(\mathcal{R}_{ec}) = c_1 \mathcal{R}_{ec} + c_2, \quad (9)$$

where  $\mathcal{R}_{ec} = \frac{\rho^* v_{\max} 2R}{\mu^*}$  and  $c_1$  and  $c_2$  are constants. Usually,  $0 \leq c_1 \ll 1$  and  $c_2 \approx 2$ . For the general case, combining Equation 4 with Equation 9 and the definition of the centerline Reynolds' number, we obtain a quadratic relationship for  $q$ ,

$$q^2 - (\gamma^* + c_2)q - 2\gamma^* = 0 \quad (10)$$

where

$$\gamma^* = \frac{2c_1 Q \rho^*}{\pi R \mu^*}$$

$\rho^*$  is the effective density, and  $\mu^*$  is the effective viscosity. This quadratic relationship can be solved for  $q$  to yield:

$$q(\mathcal{R}_{ec}) = \frac{1}{2} \left( (\gamma^* + c_2) \pm \sqrt{(\gamma^* + c_2)^2 + 8\gamma^*} \right) \quad (11)$$

### JOULE HEATING

Joule heating is the process of pumping an electrical current through a mass to directly heat it. The process can be controlled precisely and could be adapted for use in future advanced food manufacturing. From the first law of thermodynamics, we have the following description of Watts per unit volume:

$$\rho^* C^* \dot{\theta} = H - S \quad (12)$$

where  $H$  represents heating power,  $S$  represents all of the losses (conduction, convection, radiation),  $\rho^*$  is the effective slurry density,  $C^*$  is the effective slurry heat capacity, and  $\dot{\theta}$  is the time derivative of the temperature of the slurry,  $\frac{d\theta}{dt}$ . For the heating term, use Joule heating exclusively:

$$H = a \frac{J^2}{\sigma^*}, \quad (13)$$

where  $a$  is the absorption coefficient and  $J$  is the induced current within the slurry. Discretizing equation 12 and solving for the temperature yields:

$$\theta(t + \Delta t) = \theta(t) + \frac{\Delta t}{\rho^* C^*} (H - S) = \theta(t) + \frac{\Delta t}{\rho^* C^*} \left( a \frac{J^2}{\sigma^*} - S(t) \right) \quad (14)$$

If at a given time  $t$ , we want  $\theta(t + \Delta t) = \theta^*$  can solve for the necessary  $J(t)$ :

$$J(t) = \sqrt{\frac{\sigma^*}{a} \left( \rho^* C^* \frac{\theta^* - \theta(t)}{\Delta t} + S(t) \right)} \quad (15)$$

The thermal loss term  $S$  can account for any kind of thermal loss. For this assignment, assume that the dominant mode of thermal loss to the environment is convection and that the temperature is essentially uniform within a cross-section of the pipe. Convective losses are often modeled by:

$$S = hA(\theta(t) - \theta_a)$$

Where  $h$  is the convection coefficient,  $A$  is the area over which conduction occurs,  $\theta(t)$  is the temperature of the object being modeled at a given time,  $t$ , and  $\theta_a$  is the temperature of the ambient environment. Since all of our other terms are per unit volume of the slurry, we must alter this relationship to have compatible units. Assume that we are considering an arbitrary length  $\Delta x$  of the pipe. The surface area  $A$  will be  $2\pi R\Delta x$ , and the total volume will be  $^2\Delta x$ . Dividing  $A$  by volume yields  $1/R$ . As such, the convective losses per unit volume will be:

$$S = h(\theta(t) - \theta_a)/R \quad (16)$$

Note that  $S(t)$  models the transfer of heat *out of the system*, so a positive value of  $S$  represents a loss of energy.

## EFFECTIVE PROPERTIES OF PARTICLE-LADEN FLUIDS

### Thermal effects:

The primary properties of each phase which change due to temperature changes of the slurry are the viscosity and electrical conductivity as follows:

$$\mu_f(\theta) = \mu_{fo}e^{-k_1 \frac{\theta(t) - \theta_o}{\theta_o}}, \quad (17)$$

$$\sigma_f(\theta) = \sigma_{fo}e^{-k_2 \frac{\theta(t) - \theta_o}{\theta_o}}, \quad (18)$$

$$\sigma_p(\theta) = \sigma_{po}e^{-k_3 \frac{\theta(t) - \theta_o}{\theta_o}}. \quad (19)$$

where  $\mu_{fo}$  is the nominal viscosity of the fluid at  $\theta = \theta_o$ ,  $\sigma_{fo}$  and  $\sigma_{po}$  represent the nominal electrical conductivity of the fluid and particles respectively at  $\theta = \theta_o$ ,  $\theta_o$  is the initial temperature of the slurry, and  $[k_1, , k_2, k_3]$  are experimentally determined thermal "softening" parameters. All other properties are assumed to change negligibly with temperature over the range of temperatures that will be encountered in this problem.

### Effective Density:

It is important to characterize the effective properties of a particle-laden fluid as a function of the volume fraction of particles and the baseline (interstitial) fluid properties. The density of the particle-laden fluid is actually an "effective density", since it actually is a mixture of materials (particles and interstitial fluid). Effective properties are defined through volume averages. For example, the effective density of the mixture is:

$$\rho^* \stackrel{\text{def}}{=} \langle \rho(\mathbf{x}) \rangle_V \stackrel{\text{def}}{=} \frac{1}{V} \int_V \rho(\mathbf{x}) dV = \frac{1}{V} \left( \int_{V_f} \rho_f dV + \int_{V_p} \rho_p dV \right) = v_f \rho_f + v_p \rho_p \quad (20)$$

where  $v_f$  and  $v_p$  are the volume fractions of the fluid and particles, respectively. The volume fractions have to sum to unity, (i.e.  $v_f + v_p = 1 \Rightarrow v_f = 1 - v_p$ ).

### Effective Heat Capacity:

Similarly:

$$C^* \stackrel{\text{def}}{=} \langle C(\mathbf{x}) \rangle_V \stackrel{\text{def}}{=} \frac{1}{V} \int_V C(\mathbf{x}) dV = \frac{1}{V} \left( \int_{V_f} C_f dV + \int_{V_p} C_p dV \right) = v_f C_f + v_p C_p \quad (21)$$

where  $v_f$  and  $v_p$  are the volume fractions of the fluid and particles, respectively. The volume fractions have to sum to unity, (i.e.  $v_f + v_p = 1 \Rightarrow v_f = 1 - v_p$ ).

### Effective Viscosity:

Similar approaches can be used to calculate other physical properties of mixtures, such as the effective viscosity. However, to calculate them is a bit more complicated, since they require estimating mechanical interactions between constituents. There are a number of models which provide expressions for the effective viscosity of the fluid containing particles. For the purposes of this flow analysis, the particles are considered to be rigid relative to the surrounding fluid (e.g. almond bits in molten chocolate). There are several models for the effective properties of mixtures with different assumptions about the interaction of the constituent phases. As such, we would normally need to use the Hashin and Shtrikman bounds to obtain an lower and upper bound for the slurry viscosities ( $\mu^{*, -}$  and  $\mu^{*, +}$ ). However, we can take advantage of the large deviation of mechanical properties between the fluid and particles. One can take the limit of the particle phase becoming rigid, i.e. the bulk and shear moduli tending towards infinity,  $\kappa_p \rightarrow \infty$  and  $\mu_p \rightarrow \infty$ , signifying that the particles are much stiffer than the interstitial fluid, while simultaneously specifying that the interstitial fluid is incompressible, i.e.  $\kappa_f/\mu_f \rightarrow \infty$  with  $\mu_f$  being finite. This yields:

$$\mu^* \geq \mu^{*, -} = \mu_f (1 + 2.5 \frac{v_p}{1 - v_p}). \quad (22)$$

The expression in Equation 22 is the tightest known lower bound on the effective viscosity of a two-phase material comprised of rigid particles in a surrounding incompressible fluid. This expression remains quite accurate up to about  $v_p = 0.25$ , which is sufficient for most applications.

### Effective Conductivity:

Unlike the effective viscosities shown above, the electrical conductivity is not as directly correlated to the properties of the fluid and particles in the slurry. As such, we can no longer simplify the Hashin-Shtrikman bounds as we did above for the viscosities, and must use the full bounds. The effective thermal conductivity estimates are, for two isotropic materials with an overall isotropic response:

$$\underbrace{\sigma_f + \frac{v_p}{\frac{1}{\sigma_p - \sigma_f} + \frac{1 - v_p}{3\sigma_f}}}_{\sigma^{*, -}} \leq \sigma^* \leq \underbrace{\sigma_p + \frac{1 - v_p}{\frac{1}{\sigma_f - \sigma_p} + \frac{v_p}{3\sigma_p}}}_{\sigma^{*, +}}, \quad (23)$$

Such bounds are the tightest known on isotropic effective responses, with isotropic two phase microstructures, where only the volume fractions and physical properties of the constituents are known. The typical use of these bounds is to form a convex combination of them as follows:

$$\sigma^* \approx \phi \sigma^{*, -} + (1 - \phi) \sigma^{*, +} \stackrel{\text{def}}{=} \sigma^{*, \phi} \quad (24)$$

- If  $\phi = 0$  we have the lower bound,
- If  $\phi = 1$  we have the upper bound,
- If  $\phi = 1/2$  we have the average of the upper and lower bounds.

## NOZZLE WEAR

The next property we want to model is the impact of wear on the nozzle over time. There will be a trade-off between the volume fraction of particles in the fluid, the print resolution and the expected lifetime of the nozzle. We consider a nozzle with radius  $R_o$  with a circular cross-sectional area of  $A_0 = \pi R_o^2$ . The objective is to describe the mechanism by which  $R$  changes, due to nozzle wall wear caused by fluid-induced shear stress. As a simplification, we assume that the wear is uniform around the circumference of the channel.

### Interface wear:

We assume that wear of the nozzle wall is controlled by the intensity of the fluid-induced shear stress near the wall, where higher shear stresses increase the wear at the wall. We model the wear rate as being proportional to the normalized difference between the shear stress at the wall and the critical “wear” stress ( $\tau^{crit} > 0$ ):

$$\frac{dR}{dt} = \eta * \max\left(\frac{\tau_w - \tau^{crit}}{\tau^{crit}}, 0\right) = \eta * \max\left(\frac{\mu^* Q_o (q + 2)}{\pi \tau^{crit} R^3} - 1, 0\right) \quad (25)$$

where  $\eta$  is a rate constant representing the wear per unit time. If we assume fully developed laminar flow, then we can assume  $q$ . The steady state value of the channel radius, denoted  $R^{ss}$ , can then be determined analytically.

Note that, if  $q$  is not assumed constant, it will be a function of the radius by equation 11. This makes it very challenging to write a closed-form solution for  $R^{ss}$ . You can use a nonlinear solver such as Matlab’s `fsolve()`, and include  $q$  as a function of  $R$  to solve the problem for non-constant  $q$ . This will **not** be required for this project, but this is a good extension if you would like to enhance the functionality of your simulation in the future.

### Numerical Simulation:

In order to study the time-evolution of the wear, equation 25 is solved using forward Euler time integration:

$$R(t + \Delta t) = R(t) + \Delta t \mathcal{F}(R(t)). \quad (26)$$

## PROJECT INSTRUCTIONS AND DELIVERABLES

Please address the following specific questions and incorporate them into a brief technical report that would be sufficient for an undergraduate student not enrolled in this class to understand the theory and interpret your results. Use the resources on bCourses or refer to previous assignments if you need a refresher on the expectations for technical reports and plots in this class.

### PART 1: PUMPING PRESSURE & REYNOLDS NUMBER

For the first part of this assignment, you will determine the pressure gradient required to maintain a sinusoidally-varying flow rate while the temperature of the material also varies. You may assume the following:

- The flow is quasi-static (acceleration may be neglected).
- The fluid is heated uniformly (imagine that the flow is moving through an infinitely long pipe and that the heating current is uniform everywhere and started simultaneously).
- Heat sources other than the Joule heating (i.e. friction) are negligible.

The flow rate varies with time according to:

$$Q = Q_o(1 + b\sin(\omega t)) \quad (27)$$

Assume that we want the mixture to heat up at a constant rate of 50 K/s.

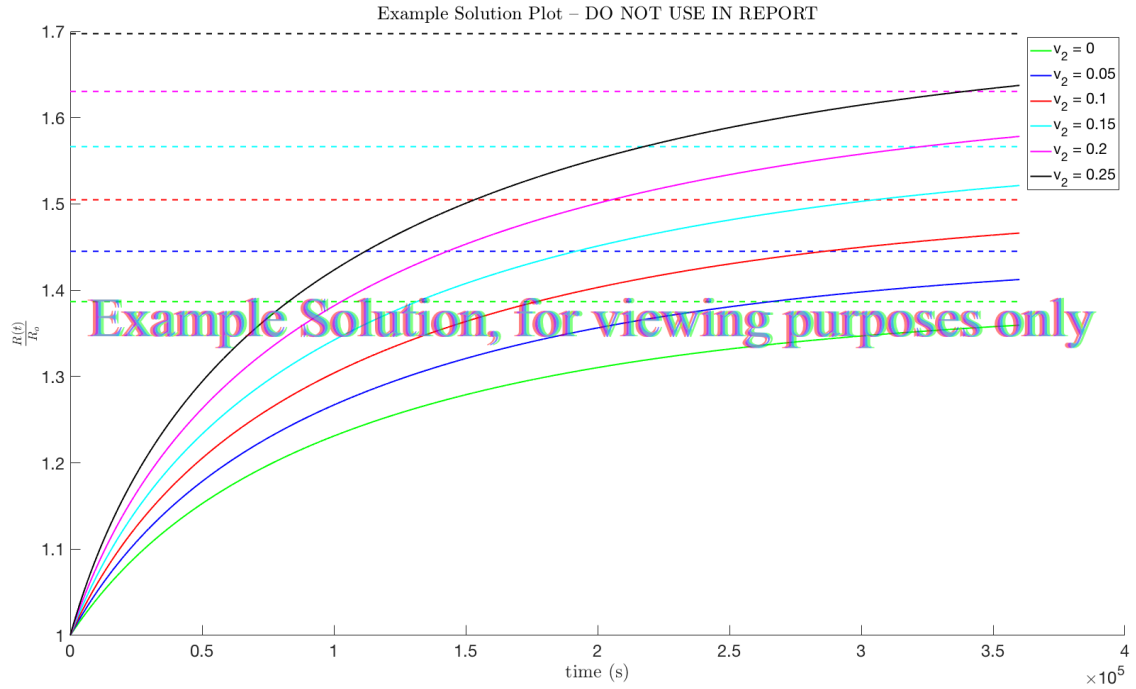
1. Explain the impact of the simplifying assumptions made in modeling this problem. What is left out?
2. When presenting the formula for  $\tau_w$ , explain which terms influence it and how.
3. Rederive and explain the pressure gradient expression. A figure or illustration may help.
4. Explain equation 11. Are both roots physically valid? If not, which one is correct and why? (*Hint: you should be able to answer this question by inspecting the equation and looking at how  $q$  is used. Something will go very wrong if you pick the incorrect root.*)
5. Use the fixed temperature increase rate and assume that the mixture loses heat to the environment by convection only. Show how to use equations 15 and 16 to solve for  $J(t)$ .
6. Plot the current of the mixture as a function of time. Use a time step of  $\Delta t_1 = .01s$ , and a final time of  $T = 2.5s$ .
7. Plot the pressure gradient to maintain the time-dependent flow for  $v_2 \in \{0, 0.05, 0.10, 0.15, 0.20, 0.25\}$ . This should be *one figure* with 6 curves. Use a time step of  $\Delta t_1 = .01s$ , and a final time of  $T = 2.5s$ . Remember that the material properties are a function of temperature!
8. Plot the Reynolds number of the time-dependent flow for  $v_2 \in \{0, 0.05, 0.10, 0.15, 0.20, 0.25\}$ . This should be *one figure* with 6 curves. Use a time step of  $\Delta t_1 = .01s$ , and a final time of  $T = 2.5s$ . Remember that the material properties are a function of temperature!

## PART 2: NOZZLE WEAR:

In order to simulate wear, you will calculate the radius at a function of time for 100 hours of continuous operation. For this part, we assume fully developed, laminar flow such that  $q$  is constant. We also steady state thermal conditions for the differential element in question. As such, the temperature will be a fixed constant  $\theta(t) = \bar{\theta}$  for all time.

1. Show how to analytically solve for the steady state radius  $R_{ss}$  using equation 26, and assuming that  $q = 2$ . (*Hint: What is the rate of change of the nozzle radius at steady state?*)
2. Plot the normalized radius  $\frac{R(t)}{R_o}$  as a function of time for  $v_2 \in \{0, 0.05, 0.10, 0.15, 0.20, 0.25\}$ . and  $\bar{\theta} = 500 \text{ K}$ . On the same figure, plot  $R_{ss}/R_o$  as a horizontal dashed line of the same color as its associated normalized radius curve. This should be *one figure* with 6 curves and 6 horizontal dashed lines. Please see example plot below for formatting. Use a time step of  $\Delta t_2 = 3.6s$ , and a final time of  $T_2 = (100 \times 3600)s$ . Remember that the material properties are now constant, since the temperature is now constant!
3. Plot the normalized radius  $\frac{R(t)}{R_o}$  as a function of time for  $\bar{\theta} \in \{300, 400, 500, 600, 700, 800\} \text{ K}$ . and  $v_2 = 0.15$ . On the same figure, plot  $R_{ss}/R_o$  as a horizontal dashed line of the same color as its associated normalized radius curve. This should be *one figure* with 6 curves and 6 horizontal dashed lines. Please see example plot below for formatting. Use a time step of  $\Delta t_2 = 3.6s$ , and a final time of  $T_2 = (100 \times 3600)s$ . Remember that the material properties are now constant since the temperature is constant!

- Plot the analytically determined  $R_{ss}$  as a function of the temperature for the range  $\bar{\theta} \in \{300, 400, 500, 600, 700, 800\} K$  and for  $v_2 \in \{0, 0.05, 0.10, 0.15, 0.20, 0.25\}$ . This should be *one figure* with 6 curves. Based on this figure, does the volume fraction or the slurry temperature appear to have a larger effect on the nozzle wear? How are these results consistent with the theory developed in lecture?
- Create a table comparing the percent difference<sup>3</sup> between  $R(T)$  and  $R_{ss}$  for each combination of  $v_2$  values and  $\bar{\theta}$  values. In which cases is the difference smaller? What does a smaller difference physically represent?



**To submit your assignment:**

- Upload a PDF of your report to bcourses. **DO NOT PUT THE PDF IN A ZIP FILE!** To save paper, electronic-only submission of your report is encouraged. If you prefer feedback on a printed copy, you may submit one to the box outside 6102 Etcheverry Hall or during lecture in addition to your electronic submission.
- Upload a zip file to bCourses containing your code (as a .m file, .py file, etc) and any supporting files required to understand and run your code.

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<sup>3</sup>% difference =  $100 \times (R_{ss} - R(T))/R_{ss}$



## PART 1 VARIABLE GLOSSARY

Symbol	Type	Units	Value	Description
$\mu_1 = \mu_{fo}$	Scalar	Pa s	.001	Phase 1 viscosity
$\sigma_{fo}$	Scalar	$\Omega^{-1}m^{-1}$	0.617	Reference temp. conductivity of fluid.
$\sigma_{po}$	Scalar	$\Omega^{-1}m^{-1}$	0.13	Reference temp. conductivity of particles.
$\rho_1 = \rho_{fo}$	Scalar	kg/m <sup>3</sup>	2000	Phase 1 density
$\rho_2 = \rho_{po}$	Scalar	kg/m <sup>3</sup>	4000	Phase 1 density
$C_f$	scalar	J/kg K	1600	Fluid heat capacity
$C_p$	scalar	J/kg K	3800	Particle heat capacity
$c_1$	Scalar	none	0.01	Flow profile parameter
$c_2$	Scalar	none	2	Flow profile parameter
$R$	Scalar	m	1e-3	Channel radius
$Q_o$	Scalar	m <sup>3</sup> /s	1e-6	Base flow rate for part A
$b$	Scalar	none	.25	Flow rate parameter for part A
$\omega$	Scalar	s <sup>-1</sup>	12	Flow angular frequency.
$h$	scalar	W/(m <sup>2</sup> K)	10	Convective Heat transfer coefficient
$\theta_o$	scalar	K	300	Initial Slurry Temperature
$\theta_a$	scalar	K	300	Ambient Temperature
$\dot{\theta}$	scalar	K/s	50	Time Rate of change of temperature
$\Delta t_1$	scalar	s	0.01	Time Step Size
$T$	scalar	s	2.5	Total simulation time
$a$	scalar	none	0.8	absorption coefficient
$k_1, k_2, k_3$	scalar	unitless	0.5, 1, 2	Thermal softening parameters
$v_2$	scalar	none	{0, 0.05, 0.10, 0.15, 0.20, 0.25}	Particle Volume Fractions
$\phi$	scalar	none	0.5	Hashin-Shtrikman Weight

## PART 2 VARIABLE GLOSSARY

Symbol	Type	Units	Value	Description
$R(t = 0) = R_0$	Scalar	m	0.001	Starting channel radius.
$q$	Scalar	unitless	2	Velocity Profile Coefficient
$v_m(t = 0)$	Scalar	m / s	0.01	Initial mean velocity
$Q_o$	Scalar	m <sup>3</sup> /s	$\int_{A_o} v dA = \pi R_o^2 v_m(t = 0)$	Volumetric flow rate
$\mu_{fo}$	Scalar	Pa s	0.001	Fluid viscosity
$\rho_1 = \rho_{fo}$	Scalar	kg/m <sup>3</sup>	2000	Phase 1 density
$\rho_2 = \rho_{po}$	Scalar	kg/m <sup>3</sup>	4000	Phase 1 density
$\theta$	Scalar	K	See Instructions	Slurry Temperature
$v_2$	Scalar	unitless	See Instructions	Particle Volume Fraction
$\theta_o$	Scalar	K	300	Initial Slurry Temperature
$\tau^{crit}$	scalar	Pa	0.001	Wall detachment threshold stress
$\eta$	scalar	m/s	$10^{-5}/3600$	Wear rate constant
$c_1$	Scalar	none	0.01	Flow profile parameter
$c_2$	Scalar	none	2	Flow profile parameter
$\Delta t_2$	scalar	s	3.6	Time Step Size
$T$	scalar	s	$3600 \times 100$	Time Step Size
$k_1, k_2, k_3$	scalar	unitless	0.5, 1, 2	Thermal softening parameters