

April 25 (Dylan/Lie)

Back to Morse theory to complete the analytic program. Recall the outline we intend to follow. The reader is encouraged to read the first section of Schwarz’s book for more details on this outline.

- (1) Functional setup.
- (2) Fredholm/index theory.
- (3) Transversality.
- (4) Compactness.
- (5) Gluing
- (6) Orientation (not today).

Let’s recall the setting: suppose that M is a closed manifold, f is a Morse function, and g is a Riemannian metric on M . The last time we were considering this setup, we proved the “local stable manifold theorem.” In this lecture, we intend to prove a more “global” theorem – we will explore the moduli space of flow lines joining two critical points of f in M .

Let grad denote the gradient with respect to g of the Morse function f . A **flow line** $\gamma : \mathbb{R} \rightarrow M$ is a C^∞ function satisfying

$$\gamma' + \text{grad}_f \circ \gamma = 0.$$

If x, y are critical points of f , and γ is a flow line satisfying the asymptotic conditions $\gamma(-\infty) = x$, $\gamma(\infty) = y$, then we call γ a **connecting trajectory**.

Define the moduli space of (parametrized) flow lines joining x to y

$$\mathcal{M}(x, y) = \{\text{connecting trajectories between } x \text{ and } y\}.$$

This space \mathcal{M} is what we will study using analysis techniques. Unfortunately, because translation reparametrization preserves connecting trajectories, $\mathcal{M}(x, y)$ is infinite (if it is non-empty), so ultimately we will want to consider the quotient

$$\widehat{\mathcal{M}} = \mathcal{M}/\mathbb{R}.$$

These moduli spaces feature in the definition of the “Morese complex;” one can define a chain complex (CM_*, ∂) (over $\mathbb{Z}/2\mathbb{Z}$), which is freely generated in each degree

$$CM_i := \mathbb{Z}/2\mathbb{Z} \langle \text{critical points of Morse index } i \rangle,$$

and whose boundary map ∂ is defined by “counting flow lines”

$$\partial : CM_{i+1} \rightarrow CM_i : \langle \partial x, y \rangle = \# \widehat{\mathcal{M}}(x, y).$$

Remark. Without continuation maps, homotopies between continuation maps etc, the theory is incomplete. We won’t consider these ideas yet – for now we are preoccupied with establishing foundations for the construction of the prerequisite moduli spaces.

Strategy for construction of $\mathcal{M}(x, y)$. The idea is to construct a Banach manifold \mathcal{P} so that

$$C_{x,y}^\infty(\overline{\mathbb{R}}, M) \subset \mathcal{P} \subset C_{x,y}^0(\overline{\mathbb{R}}, M)$$

(where $\overline{\mathbb{R}} \simeq [-1, 1]$ via $t \mapsto \frac{2}{\pi} \arctan(t)$ is the two point compactification of \mathbb{R} with smooth structure inherited from $[-1, 1]$) with the following properties:

- (i) Every gradient flow line joining x to y lies in \mathcal{P} .
- (ii) The section

$$s : \gamma \mapsto \partial_t \gamma + \text{grad} \circ \gamma$$

extends a smooth section of a Banach bundle $\mathcal{E} \rightarrow \mathcal{P}$, and $s^{-1}(0)$ consists precisely of the gradient flow lines joining x to y , i.e. $s^{-1}(0) = \mathcal{M}(x, y)$.

- (iii) The linearization of s at $s^{-1}(0)$ is Fredholm with index equal to $\text{index}(x) - \text{index}(y)$.

(iv) Observe that the section s is actually a family of sections parametrized by the underlying Riemannian metric g . It can be shown that for generic metric g , the section $s : \mathcal{P} \rightarrow \mathcal{E}$ is transverse to the zero section, and hence $s^{-1}(0)$ is a smooth submanifold of \mathcal{P} with dimension $\text{index}(x) - \text{index}(y)$.

Our can construct a Banach bundle $\mathcal{E} \rightarrow \mathcal{P}$ where \mathcal{P} is a Banach manifold and a section s and such that $s^{-1}(0) = \mathcal{M}(x, y)$. Here \mathcal{P} is a Banach manifold of paths $x \rightarrow y$. The charts of \mathcal{P} are

Assume $\gamma : \mathbb{R} \rightarrow M$ smooth for simplicity with some kind of asymptotics condition at $\pm\infty$ which makes them converge to x and y .

$U \subset X(\gamma^* TM)$ open where X denotes a Banach space containing $C_c(\mathbb{R}; \gamma^* TM)$ (I assume they are dense).

$\implies U \xrightarrow{\text{arrow}} \mathcal{P}$. Moreover the asymptotic condition that γ satisfied should also be satisfied for smooth paths inside the image of U .

Two solutions:

- (1) Schwarz's book: roughly, replace the domain \mathbb{R} with $[-1, 1]$, noting that $(-1, 1)$ is diffeomorphic to \mathbb{R} and use the smooth structure on $[-1, 1]$, asymptotic condition is now $-1 \mapsto x, +1 \mapsto y$.

$$X = H_{\mathbb{R}}^{1,2}(\gamma^* \text{smallenoughdiskin} TM)$$

- (2) Floer's approach: Use slightly modified Sobolev norm. $W^{k,p,\sigma,\sigma_+}(\mathbb{R}_t, \mathbb{R}) := \{f \in W_{loc}^{k,p} \mid \|f\|_{k,p,\sigma,\sigma_+} < \infty\}$

$$\|f\|_{k,p,\sigma,\sigma_+} = \|\exp((\beta(t)\sigma_+ + \beta(-t)\sigma_-)t) \cdot f\|_{k,p}$$

This is a Banach space.

Asymptotic condition near $\pm\infty$ is γ should be given by exponentiating a family of vector of at TM_x on $(-\infty, \rho)$. and satisfy true exponential decay condition for some $\sigma_- < 0$.

The above characterization of chart of Banach space gives tangent spaces of pathes that satisfy the asymptotic condition. (Need Proof).

The vector bundle is locally just $W^{k-1,p,\sigma,\sigma+}(\mathbb{R}_t, \mathbb{R})$ and the section goes as $\partial/\partial t + \text{grado}$

2) With a lot of pain, most previous work generalize here. But the linearization is a little more involved because because we are exponentially sections not just adding them: the right operator is $\partial/\partial t + \Gamma$ the Christoffel symbols are involved.

3) Consequences of parametric transversality theorem for Banach manifolds.

Proposition 1. G, M Banach manifolds, $E \rightarrow M$ is a Banach space bundle. $\varphi : G \times M \rightarrow E$ smooth G -parametric sections (Family of sections, try to find a generic one such that intersection with 0 is a true manifold), there is a countable trivialization, i.e.

$$\{(U, \psi | U \subset M \text{ open}, \psi : E|_U \rightarrow E_u \times U)\}$$

such that for every member:

- 0 is a regular value of $pr \circ \psi \circ \varphi : G \times U \rightarrow E_u$
- $pr \circ \psi \circ \varphi_g : U \rightarrow E_u$ is a Fredholm map of Banach manifold (All tangent map are) of index r for every $g \in G$.

Then there is a G -set (Baire set) $\Sigma \subset G$ such that $Z_g = \{\varphi_g(m) = 0\}$ is a closed submanifold of M for $g \in \Sigma$

For us, $G = \text{Banach manifold of Riemannian metric}$, If we choose G large enough, 1 is indeed satisfied but it is not a general theorem, need to make computations.