April 25 (Dylan/Lie)

Back to Morse theory to complete the analytic program. Recall the outline we intend to follow. The reader is encouraged to read the first section of Schwarz's book for more details on this outline.

(1) Functional setup.
(2) Fredholm/index theory.
(3) Transversality.
(4) Compactness.
(5) Gluing
(6) Orientation (not today).

Let's recall the setting: suppose that M is a closed manifold, f is a Morse function, and g is a Riemannian metric on M. The last time we were considering this setup, we proved the "local stable manifold theorem." In this lecture, we intend to prove a more "global" theorem – we will explore the moduli space of flow lines joining two critical points of f in M.

Let grad denote the gradient with respect to g of the Morse function f. A **flow line** $\gamma : \mathbb{R} \to M$ is a C^∞ function satisfying

$$\gamma' + \mathrm{grad}_f \circ \gamma = 0.$$

If x, y are critical points of f, and γ is a flow line satisfying the asymptotic conditions $\gamma(-\infty) = x$, $\gamma(\infty) = y$, then we call γ a **connecting trajectory**.

Define the moduli space of (parametrized) flow lines joining x to y

$$\mathcal{M}(x, y) = \{\text{connecting trajectories between x and y}\}.$$

This space $\mathcal{M}$ is what we will study using analysis techniques. Unfortunately, because translation reparametrization preserves connecting trajectories, $\mathcal{M}(x, y)$ is infinite (if it is nonempty), so ultimately we will want to consider the quotient

$$\widehat{\mathcal{M}} = \mathcal{M}/\mathbb{R}.$$

These moduli spaces feature in the definition of the "Morese complex;" one can define a chain complex (CM_*, ∂) (over $\mathbb{Z}/2\mathbb{Z}$), which is freely generated in each degree

$$CM_i := \mathbb{Z}/2\mathbb{Z}\,\langle\text{critical points of Morse index i}\rangle,$$

and whose boundary map ∂ is defined by "counting flow lines"

$$\partial : CM_{i+1} \to CM_i : \langle \partial x, y\rangle = \#\widehat{\mathcal{M}}(x, y).$$

Remark. Without continuation maps, homotopies between continuation maps etc, the theory is incomplete. We won't consider these ideas yet – for now we are preoccupied with establishing foundations for the construction of the prerequisite moduli spaces.

Strategy for construction of $\mathcal{M}(x,y)$. The idea is to construct a Banach manifold $\mathcal{P}$ so that

$$C^\infty_{x,y}(\overline{\mathbb{R}}, M) \subset \mathcal{P} \subset C^0_{x,y}(\overline{\mathbb{R}}, M)$$

(where $\overline{\mathbb{R}} \simeq [-1,1]$ via $t \mapsto \frac{2}{\pi}\arctan(t)$ is the two point compactification of $\mathbb{R}$ with smooth structure inherited from $[-1,1]$) with the following properties:

(i) Every gradient flow line joining x to y lies in $\mathcal{P}$.

(ii) The section

$$s : \gamma \mapsto \partial_t \gamma + \operatorname{grad} \circ \gamma$$

extends a smooth section of a Banach bundle $\mathcal{E} \to \mathcal{P}$, and $s^{-1}(0)$ consists precisely of the gradient flow lines joining x to y, i.e. $s^{-1}(0) = \mathcal{M}(x,y)$.

(iii) The linearization of s at $s^{-1}(0)$ is Fredholm with index equal to $\operatorname{index}(x) - \operatorname{index}(y)$.

(iv) Observe that the section s is actually a family of sections parametrized by the underlying Riemannian metric g. It can be shown that for generic metric g, the section $s : \mathcal{P} \to \mathcal{E}$ is transverse to the zero section, and hence $s^{-1}(0)$ is a smooth submanifold of $\mathcal{P}$ with dimension $\operatorname{index}(x) - \operatorname{index}(y)$. ⌟

To motivate the choice/construction of the path space $\mathcal{P}$, we recall the key analytical result which will enable us to establish (iii).

Theorem 1. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a family of matrices so that $A(-\infty)$ and $A(+\infty)$ converge, and $A(-\infty)$, $A(+\infty)$ have no imaginary eigenvalues. Then the differential operator

$$L = \partial_t + A(t) \tag{$*$}$$

induces a Fredholm operator

$$L : W^{1,p}(\mathbb{R}, \mathbb{R}^n) \to L^p(\mathbb{R}, \mathbb{R}^n),$$

whose index equals the spectral flow of the operators $A(t)$. □

The key here is that this result is stated for the spaces $W^{1,p}(\mathbb{R}, \mathbb{R}^n)$ and *so we want the tangent space $T\mathcal{P}_\gamma$ to be isomorphic to $W^{1,p}(\mathbb{R}, \mathbb{R}^n)$*, because, as we will see later, the linearization of $\partial_t + \operatorname{grad}$ at a gradient flow line is of the form $(*)$.

Some ideas used in the construction of $\mathcal{P}$. When discussing "Banach manifolds of maps," Schwarz's book (and Wendl's notes on Holomorphic curves) reference the 1967 paper "Geometry of Manifolds of Maps" by Halldor Eliasson. The techincal tool Eliasson introduces is the notion of a **Manifold model** which should be thought as the necessary data needed to construct a Banach manifold of maps.

The definition of "Manifold model" we will give below attempts to incorporate asymptotic boundary conditions in some systematic way. For the purposes of constructing a *path space* we should take $\overline{X} = \overline{\mathbb{R}}$.

Definition 2. Let $\overline{X}$ be a compact manifold with boundary and let $X \subset \overline{X}$ be an open set. Let's suppose that $\overline{X} = X \cup \partial X$, for a closed submanifold ∂X. Let $C_0^\infty(\overline{X})$ be the set of continuous functions vanishing on ∂X.

A **manifold model** for $(X, \overline{X})$ is a functor $W : \mathrm{Bun}(\overline{X}) \to \mathrm{Ban}$ and the data of two natural continuous dense inclusions

$$C_0^\infty(\overline{X}, E) \to W(E) \to C_0^0(\overline{X}, E),$$

where C_0^k are functions vanishing on ∂X.

(Banach Algebra) We require the continuous multiplication maps

$$C_0^\infty(\overline{X}, E_1) \otimes C^\infty(\overline{X}, \mathrm{Hom}(E_1, E_2)) \to C_0^\infty(\overline{X}, E_2)$$

(and similarly with $\infty \mapsto 0$) to induce continuous multiplication maps

$$W(E_1) \otimes W(\mathrm{Hom}(E_1, E_2)) \to W(E_2) \text{ and } W(E_1) \otimes C^\infty(\mathrm{Hom}(E_1, E_2)) \to W(E_2)$$

(Composition is continuous) Because of the inclusion into $C_0^0(U)$, the set $W(U)$ is well-defined and open for any precompact $U \subset E$. Assuming that $C_0^0(U)$ is non-empty implies that U contains a compactly supported smooth section.

We require that there are continuous composition maps $f\circ : W(U) \to W(E_2)$ for every smooth fiber preserving map $\overline{U} \to E_2$ which vanishes on $0 \cap \partial X$ (where 0 is the zero section) making the following diagram commute

$$\begin{tikzcd}
C_0^\infty(\overline{X}, U) \arrow[r] \arrow[d, "f\circ"] & W(U) \arrow[r] \arrow[d, "f\circ"] & C_0^0(\overline{X}, U) \arrow[d, "f\circ"] \\
C_0^\infty(\overline{X}, E_2) \arrow[r] & W(E_2) \arrow[r] & C_0^0(\overline{X}, E_2).
\end{tikzcd}$$

Definition 3. Let M be a smooth manifold with Riemannian metric g and any fiberwise convex open neighborhood $\mathcal{D} \subset TM$ of the zero section on which $\exp : TM \to M \times M$ is an open embedding. For each smooth map $\gamma : \overline{X} \to M$ we can form the pullback $\gamma^*\mathcal{D} \subset \overline{X} \times TM$, which admits an open embedding

$$\exp_\gamma : \gamma^*\mathcal{D} \to \overline{X} \times M,$$

obtained by applying $\exp : TM \to M$ to the definition of $\gamma^*\mathcal{D}$. We can think of $\exp_\gamma$ as a fiber preserving smooth map which sends $0 \cap \partial X$ to $\gamma(\partial X)$. Fixing $\gamma|_{\partial X} = b$, we obtain a well-defined inclusion

$$\exp_\gamma \circ : W(\gamma^*\mathcal{D}) \to C_b^0(\overline{X}, M),$$

where C_b^k denotes functions which agree with b on ∂X.

We define

$$W(\overline{X}, M) = \bigcup \left\{ W(\gamma^*\mathcal{D}) : \gamma \in C_b^\infty(\overline{X}, M) \right\}.$$

This set of maps does not depend on the Riemannian metric or the set $\mathcal{D}$.

Theorem 4. For every manifold model W, and every smooth map $b : \partial X \to M$, there is a Banach manifold structure on $W(\overline{X}, M)$ equipped with dense inclusions

$$C_b^\infty(\overline{X}, M) \subset W(\overline{X}, M) \subset C_b^0(\overline{X}, M)$$

and so that for any $\gamma \in C_b^\infty(\overline{X}, M)$, and any choice of Riemannian metric g or open set $\mathcal{D}$ the map

$$\exp_\gamma : W(\gamma^* \mathcal{D}) \to W(\overline{X}, M)$$

is a smooth open embedding.

Remark. It follows that we can fix any $g, \mathcal{D}$ and obtain an atlas of smooth coordinate charts on $W(\overline{X}, M)$. Indeed, if one knows that $C^\infty(\overline{X}, M)$ has a countable subset dense in the C^0, one concludes that $W(\overline{X}, M)$ has a countable atlas.

The manifold model Schwarz uses to construct his path space $\mathcal{P}$. As we mentioned above, we really want the tangent spaces to our path space to be the $W^{1,p}(\mathbb{R}, \gamma^* TM)$ spaces (the $W^{1,2}$ spaces suffice), because it is for these Banach spaces that Theorem 1 applies.

Definition 5. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with the differentiable structure induced by the bijection

$$\overline{\mathbb{R}} \to [-1, 1] \qquad \varphi : t \mapsto \frac{2}{\pi} \arctan t$$

We will let t to be the smooth function defined on $\mathbb{R} \subset \overline{\mathbb{R}}$.

If $u : \overline{\mathbb{R}} \to \mathbb{R}^n$ is a C^1 function, then the derivative $\partial_t u$ is defined as usual on $\mathbb{R}$ and is defined to be 0 on $\pm\infty$. This is a C^1 function on $\overline{\mathbb{R}}$. To see why, we compute on $(-1, 1)$

$$\partial_t u(\varphi^{-1}(s)) = \frac{\partial_s (u(\varphi^{-1}(s)))}{\partial_s \varphi^{-1}(s)},$$

and use the fact that

$$(*) \qquad \varphi^{-1}(s) = \tan \frac{\pi}{2} s \implies \partial_s \varphi^{-1}(s) = \frac{\pi/2}{\cos^2(\pi s/2)} = \frac{\pi}{2}(1 + \varphi^{-1}(s)^2)$$

and thereby deduce for $u : \overline{\mathbb{R}} \to \mathbb{R}^n$ smooth we have

$$\partial_t u(\varphi^{-1}(s)) = \frac{2}{\pi} \partial_s u(\varphi^{-1}(s)) \cos^2(\pi s/2)$$

so $\partial_t u$ extends to to the boundary as a C^∞ function. We can conclude from $(*)$ that if u is C^∞ on $\overline{\mathbb{R}}$ then

$$|\partial_t u| \le \frac{\text{const}}{1 + t^2}.$$

In particular, if u is C^∞ on $\overline{\mathbb{R}}$, then $\partial_t u$ is integrable on $\mathbb{R}$ (with its standard measure $\mathrm{d}t$). If u is C^∞ on $\overline{\mathbb{R}}$ and $u(\pm\infty) = 0$, then u is in L^p for all p. To see why, observe that for $t_1 < 0$ we can compute for $t_0 < t_1$

$$u(t_1) - u(t_0) = \int_{t_0}^{t_1} \partial_t u \, \mathrm{d}t \implies |u(t_1)| \le |u(t_0)| + \frac{1}{|t_1|} - \frac{1}{|t_0|}.$$

Taking the limit as $t_0 \to 0$ concludes that $|u(t_1)| \leq |t_1|^{-1}$, and hence u is in $L^p(\mathbb{R})$. It follows that $C_0^\infty(\overline{\mathbb{R}}) \subset W^{k,p}(\mathbb{R})$ for all $p > 1$, where we agree to define

$$C_0^\infty(\overline{\mathbb{R}}) = \text{smooth functions vanishing at } \pm\infty.$$

If E is a smooth vector bundle on $\overline{\mathbb{R}}$, we define

$$W_{\mathbb{R}}^{k,p}(E) \simeq W^{k,p}(\mathbb{R}^n)$$

by choosing any smooth trivialization of E over $\overline{\mathbb{R}}$.

Remark. It is not too hard to show that $E \mapsto W_{\mathbb{R}}^{k,p}(E)$ is a Manifold model, and hence can be used to set up a path space $\mathcal{P}$ whose tangent spaces at a curve γ are precisely $W_{\mathbb{R}}^{k,p}(\gamma^*TM)$.

Bundles on $\mathcal{P}$. The main idea is the following:

Definition 6. A **section functor** over W will be a functor

$$L : \mathrm{Bun}(\overline{X}) \to \mathrm{Ban},$$

equipped with a continuous bilinear maps

$$L(E_1) \otimes W(\mathrm{Hom}(E_1, E_2)) \to L(E_2) \text{ and } L(E_1) \otimes C^\infty(\mathrm{Hom}(E_1, E_2)) \to L(E_2)$$

which extends the data inherent in the functor (recall the morphisms in $\mathrm{Bun}(\overline{X})$ are smooth sections of $\mathrm{Hom}(E_1, E_2)$).

Example 7. The functor $E \mapsto L_{\mathbb{R}}^p(E)$ is a section functor over $W_{\mathbb{R}}^{1,p}$.

Theorem 8. Given any section functor L and any bundle $E \to M$, there is a smooth Banach bundle

$$\mathcal{E} = L(W_b(\overline{X}, M)^*E) \to W_b(\overline{X}, M),$$

If we let $\mathcal{P} = W_b(\overline{X}, M)$, then we have

$$\mathcal{E} = L(\mathcal{P}^*E) \to \mathcal{P}.$$

Furthermore, for all $u \in W_b(\overline{X}, M)$ there are canonical identifications

$$\mathcal{E}_u \to L(u^*E).$$

Remark. The culmination of all this work is that we have a smooth bundle $L^p(\mathcal{P}^*TM)$ whose fiber at a curve u is $L^p(u^*TM)$. One can show that the "gradient flow" section $\gamma \mapsto \partial_t\gamma + \mathrm{grad}\gamma$ (defined for smooth sections) extends to a smooth section of

$$L^p(\mathcal{P}^*TM) \to \mathcal{P},$$

Proposition 9. Let $M = \mathbb{R}^n$, and let g be an arbitrary Riemannian metric on $\mathbb{R}^n$. Let f be a Morse function, and let x, y be two critical points. The linearization of the smooth section $\gamma \mapsto \gamma' + \mathrm{grad} \circ \gamma$ at a flow line is the section

$$(*) \qquad \eta \in W^{1,p}(\gamma^*TM) \mapsto \eta' + \mathrm{dgrad}_\gamma \cdot \eta \in L^p(\gamma^*TM)$$

Remark. The linearization $(*)$ of $\gamma \mapsto \gamma' + \mathrm{grad} \circ \gamma$ is precisely the kind of operator we were considering in Theorem 1. As a consequence, we conclude: the section $\gamma \mapsto \gamma' + \mathrm{grad} \circ \gamma$ has Fredholm linearizations at the gradient flow lines, and moreover, the index is precisely

$$\text{index} = \text{spectral flow of } t \mapsto \mathrm{dgrad}_{\gamma(t)} = \mathrm{index}(x) - \mathrm{index}(y).$$

Generic metrics make $\gamma \mapsto \gamma' + \mathbf{grad} \circ \gamma$ a transverse section of $L^p(\mathcal{P}^*TM) \to \mathcal{P}$. Recalling our outline (i)-(iv), we have still not said anything about how *generic metrics* will "cut" $\mathcal{M}(x, y)$ transversally. In this section we will introduce an important theorem called the **parametric transversality theorem**.

Theorem 10. Let $\mathcal{E} \to \mathcal{P}$ be a smooth Banach bundle over a Banach bundle, and let $\mathcal{G}$ be another Banach manifold. Suppose we are given a smooth map

$$\Phi : \mathcal{G} \times \mathcal{P} \to \mathcal{E}$$

so that $\Phi_g : \{g\} \times \mathcal{P} \to \mathcal{E}$ is a section for each g, i.e. Φ is a "parametrized family" of sections. Suppose the following conditions are satisfied:

(i) In some (countable) atlas of trivializations (ψ_i, U_i), $\psi_i : \mathcal{E}|_{U_i} \to U_i \times E_i$, the maps

$$\mathrm{pr} \circ \psi_i \circ \Phi : \mathcal{G} \times U_i \to E_i$$

have $0 \in E_i$ as a regular value.

(ii) The maps $\mathrm{pr} \circ \psi_i \circ \Phi_g : U_i \to E_i$ are Fredholm maps with index r (for every g).

Then the conclusion is that there exists a Baire generic set $\Sigma \subset \mathcal{G}$ so that for all $g \in \Sigma$, the section Φ_g is transverse to the zero section, i.e. the vertical differential of Φ_g is surjective on $\Phi_g^{-1}(0)$, and consequently $\Phi_g^{-1}(0)$ is a smooth submanifold, whose dimension is necessarily r.

Example 11. Let's apply this theorem to the following set up. Let M be a compact manifold. Define

$$\mathcal{G} = \left\{ C^k \text{ Riemannian metrics on } M \right\}.$$

Then $\mathcal{G}$ is a convex open subset of the Banach space

$$\mathcal{G} \subset C^k(M, \mathrm{Sym}^2(T^*M)),$$

and hence $\mathcal{G}$ has a natural Banach manifold structure. Let $\mathcal{P} = W^{1,2}_{x,y}(\overline{R}, M)$, and $\mathcal{E} = L^2(\mathcal{P}^*TM)$, as above.

One can show that the map

$$\Phi : (g, \gamma) \in \mathcal{G} \times \mathcal{P} \to \gamma' + \mathrm{grad}_g \circ \gamma \in L^2(\gamma^*TM)$$

is a smooth family of sections $\Phi : \mathcal{G} \times \mathcal{P} \to \mathcal{E}$. We claim that the conditions (i) and (ii) are satisfied.

To see why, let's suppose that $(g,\gamma) \in \Phi^{-1}(0)$. Then γ is a flow line joining x to y. The linearization of Φ at (g,γ) is a linear map

$$D\Phi_{(g,\gamma)} : C^k(\mathrm{Sym}^2(T^*M)) \times W^{1,p}(\gamma^*TM) \to L^2(\gamma^*TM)$$
$$(B,\eta) \mapsto D_1\Phi_{g,\gamma}B + D_2\Phi_{g,\gamma}\eta.$$

The map $D_2\Phi_{g,\gamma}$ is simply the linearization of the section $\gamma \mapsto \gamma' + \mathrm{grad}_g \circ \gamma$. We already showed that this was Fredholm of index $\mathrm{index}(x) - \mathrm{index}(y)$. Moreover, the linearization $D_2\Phi_{g,\gamma}$ is actually an (elliptic) first order differential operator $W^{1,2}(\gamma^*TM) \to L^2(\gamma^*TM)$, and therefore satisfies unique continuation.

Since $D_2\Phi_{g,\gamma}$ has closed image and finite dimensional cokernel, the map

$$(B,\eta) \mapsto D_1\Phi_{g,\gamma}B + D_2\Phi_{g,\gamma}\eta$$

still has closed image and finite dimensional cokernel. We claim that the cokernel is zero dimensional. If not, then we can find $c \in L^2$ so that $c \perp \mathrm{im}\, D\Phi$, in particular $c \in \ker D_2\Phi^*_{g,\gamma}$, and therefore, by unique continuation for elements in the kernel of a first order elliptic operator on $\mathbb{R}$, we conclude c is everywhere non-zero.

Now we compute

$$D_1\Phi_{g,\gamma}B = D_B\mathrm{grad}_g \circ \gamma,$$

where

$$D_B\mathrm{grad}_g = \lim_{\epsilon\to 0} \epsilon^{-1}(\mathrm{grad}_{g+\epsilon B} - \mathrm{grad}_g)$$

Pick $p \in \gamma(\mathbb{R})$, and pick coordinates $z_1, \cdots, z_n$ near p. Write

$$\mathrm{d}f = \frac{\partial f}{\partial z_i}\mathrm{d}z_i \text{ and } g = g_{ij}\mathrm{d}x_i\mathrm{d}x_j \text{ and } \mathrm{grad}_g = a_i(g)\partial_i$$

Since

$$g_{ij}a_i(g) = \frac{\partial f}{\partial z_j} \implies a_i(g) = (g^{-1})_{ij}\frac{\partial f}{\partial z_j},$$

We compute

$$(g+B)^{-1} = g_{ij}^{-1}(1+g_{ij}^{-1}B_{ij})^{-1} = (g^{-1})-(g^{-2})B+O(|B|^2) \implies D_B\mathrm{grad}_g = ((g^{-2})B)_{ij}\frac{\partial f}{\partial z_j}\partial_i.$$

Since $\partial f/\partial z_j$ is a non-zero vector, and B is arbitrary, we can choose B so that $((g^{-2})B)_{ij}\frac{\partial f}{\partial z_j}\partial_i$ so that the pairing $\langle D_1\Phi_{(g,\gamma)}, c\rangle \neq 0$, unless c vanishes at z. This proves that c must vanish at z, and hence c is identically zero, contradicting our assumption that $c \neq 0$. Therefore $D\Phi_{(g,\gamma)}$ is surjective, as desired.

It follows that the family of sections $\Phi : \mathcal{G} \times \mathcal{P} \to \mathcal{E}$ satisfies the requirements (i) and (ii) needed to apply the parametric transversality theorem.