The goal of this lecture is to complete the proof of the "local stable manifold theorem." Recall the setting:

(i) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Morse function with exactly one critical point at $0 \in \mathbb{R}^n$,

(ii) g is a Riemannian metric on $\mathbb{R}^n$,

(iii) $\mathrm{grad}_{\varphi,g} =:$ grad is the gradient vector field associated to g and φ – recall that grad is determined by the relation $g(\mathrm{grad}, X) = \mathrm{d}\varphi(X)$, for all vector fields X.

(iv) The stable set S is defined to be the set of all $x \in \mathbb{R}^n$ so that the negative gradient flow line starting at x converges to 0.

Local Stable Manifold Theorem. There is an open neigborhood U around 0 so that $U \cap S$ is a smooth submanifold of U.

Recall the stategy for proving this theorem. We identify S with the set of gradient flow lines $\mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0, and make the following crucial observation: If $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ satisfies the following equation

$$(*) \qquad \gamma'(t) + \mathrm{grad} \circ \gamma = 0,$$

then γ is C^∞ smooth, and conversely, if γ is a gradient flow line converging to 0, then γ lies in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and γ satisfies $(*)$. In other words, if we define

$$s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \text{ by } s(\gamma) = \gamma'(t) + \mathrm{grad} \circ \gamma,$$

then $s^{-1}(0) = S$. This was established in the previous lecture. Here are three exercises related to the content of the previous lecture.

Exercise 1 (Rellich Embedding)**.** Let $\gamma \in C_c^\infty$. Prove that for $t \geq 0$

$$\gamma(t)e^{-t} = \int_t^\infty \gamma(s)e^{-s} - \gamma'(s)e^{-s}\, ds,$$

and deduce that

$$|\gamma(t)| \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}([t,\infty),\mathbb{R}^n)}$$

In particular,

$$(\star) \qquad \|\gamma\|_{C^0} \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}}\,.$$

By density of C_c^∞ functions in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, prove that $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^0(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Moreover, conclude that $|\gamma(t)| \to 0$ as $t \to \infty$. Applying the bound $(\star)$ to the derivatives of γ, conclude that

$$W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^{k-1}(\mathbb{R}_{\geq 0}, \mathbb{R}^n),$$

and that $\|\gamma\|_{C^{k-1}} \leq C\, \|\gamma\|_{W^{k,2}}$.

Exercise 2. Show that if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{0,1}$ function (i.e. Lipshitz), and $X(0) = 0$, then the map $\gamma \mapsto X \circ \gamma$ sends $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ to $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

More generally, if X is a $C^{k,1}$ function with $X(0) = 0$, then $\gamma \mapsto X \circ \gamma$ sends $W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ to $W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Hint: for both claims, it suffices to prove it first for smooth γ, and then use the density of smooth functions in $W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

Exercise 3. Conclude that if $\gamma \in W^{k,2}$ satisfies the gradient flow line equation $(*)$, then $\gamma \in W^{k+1,2}$. Conclude that if $\gamma \in W^{1,2}$ satisfies $(*)$, then $\gamma \in W^{k,2}$ for all k. By the Rellich embedding $W^{k,2} \subset C^{k-1}$, conclude that $W^{1,2}$ solutions of $(*)$ are C^∞ smooth.

Now we turn to some new material. We will show that $s^{-1}(0)$ is a manifold via the inverse function theorem for Banach spaces.

Definition 1. Let $U \subset X$ and $V \subset Y$ be open subspaces of Banach spaces X, Y. A map $A : U \to V$ is **differentiable** at $u \in U$ provided there is a bounded linear transformation $\mathrm{d}A_u : X \to Y$ so that

$$\lim_{\xi \to 0} \frac{\|A(u+\xi) - A(u) - \mathrm{d}A_u(\xi)\|_Y}{\|\xi\|_X} = 0.$$

While this definition uses the norms on X, Y, it is clear that it only depends on the equivalence classes of the norms.

The map $A : U \to V$ is **continuously differentiable** if $u \mapsto \mathrm{d}A_u \in \mathrm{Hom}(X,Y)$ is a continous function, where the latter is given the topology induced by the "operator norm." The set of continously differentiable functions is denoted $C^k(U,V)$.

A map A is $C^k(U,V)$, $k \geq 1$, if $\mathrm{d}A$ is $C^{k-1}(U, \mathrm{Hom}(X,Y))$.

In the appendix to this lecture, we define the terms **Banach manifold** and the **tangent bundle** of a Banach manifold.

Inverse Function Theorem. Let X, Y be Banach manifolds, and suppose $A : X \to Y$ is a C^k map so that $\mathrm{d}A_x$ is an isomorphism (i.e. is continuous in the natural topologies on TX_x and $TY_{A(x)}$). Then there are neighborhoods $U \ni x$ and $V \ni y$ so that A maps U to V diffeomorphically.

The derivatives of the vector field grad appear in the statement of the next claim. Thinking of grad is a function $\mathbb{R}^n \to \mathbb{R}^n$, it certainly has a derivative $\mathrm{dgrad}_x : \mathbb{R}^n \to \mathbb{R}^n$ at all points $x \in \mathbb{R}^n$ (warning: here we are *not* thinking of grad as a map $\mathbb{R}^n \to T\mathbb{R}^n$ when we take its derivative). Similarly, we will denote the (symmetric) second derivative matrix by

$$\mathrm{ddgrad}_x : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n.$$

This is not a coordinate invariant notion.

It is clear that if x, y are two points of $\mathbb{R}^n$, then

$$\mathrm{grad}(x+y) - \mathrm{grad}(x) = \left[\int_0^1 \mathrm{dgrad}_{x+sy}\, ds\right] \cdot y,$$

where we interpret the expression in the braces as a matrix $\mathbb{R}^n \to \mathbb{R}^n$.

Claim 2. The map $s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ defined by $s(\gamma) = \gamma' + \mathrm{grad} \circ \gamma$ is a C^1 map, and it's derivative is given by

$$\mathrm{d}s_\gamma(\eta)(t) = \eta'(t) + \mathrm{dgrad}_{\gamma(t)} \cdot \eta(t)$$

Proof. We begin by computing

$$s(\gamma+\eta)(t)-s(\gamma)(t) = \eta'(t)+\mathrm{grad}(\gamma(t)+\eta(t))-\mathrm{grad}(\gamma(t)) = \eta'(t)+\left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)}\, ds\right]\cdot\eta(t).$$

Hence,

$$s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta) = \left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds\right] \cdot \eta(t).$$

It follows that

$$\left\|s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta)\right\|_{L^2} \leq \left\|\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds\right\|_{C^0} \|\eta\|_{L^2}.$$

We compute

$$\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds = \left[\int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds\right] \cdot \eta(t),$$

and hence

$$\left\|\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds\right\|_{C^0} \leq \left\|\int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds\right\|_{C^0} \|\eta\|_{C^0}.$$

Since ddgrad is a continous function, and $\gamma(t) + rs\eta(t)$ is a bounded function of t (i.e. for $\|\eta\|_{C^0} \leq 1$, we can suppose that $\|\gamma + rs\eta\|_{C^0} \leq R$ for some large R) we conclude some C independent of t and η, $\|\eta\|_{C^0} \leq 1$, so that

$$\left\|\int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds\right\|_{C^0} \leq C,$$

and hence

$$\left\|s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta)\right\|_{L^2} \leq C\|\eta\|_{L_2}\|\eta\|_{C^0} \leq C'\|\eta\|^2_{W^{1,2}},$$

where we have used the fact that $\|-\|_{C^0} \leq c\|-\|_{W^{1,2}}$ and $\|-\|_{L^2} \leq \|-\|_{W^{1,2}}$. It follows that s is differentiable and its derivative at γ is $\mathrm{d}s_\gamma$.

It is easy to show that $\mathrm{d}s_\gamma$ is a bounded function $W^{1,2} \to L^2$. Finally we show that $\gamma \to \mathrm{d}s_\gamma$ is continuous.

Given two curves γ_1, γ_2, we compute

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \mathrm{dgrad}_{\gamma_1(t)+\gamma_2(t)} \cdot \eta(t) - \mathrm{dgrad}_{\gamma_1(t)} \cdot \eta(t).$$

Arguing as we did above, we conclude

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \left[\int_0^1 \mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\, ds \cdot \gamma_2(t)\right] \cdot \eta(t),$$

and similarly to the computations above we conclude

$$\left\| \mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) \right\|_{L^2} \leq \left\| \int_0^1 \mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\, ds \right\|_{C^0} \left\| \gamma_2(t) \right\|_{W^{1,2}} \left\| \eta(t) \right\|_{W^{1,2}} .$$

We thereby obtain an estimate on the operator norm

$$\left\| \mathrm{d}s_{\gamma_1+\gamma_2} - \mathrm{d}s_{\gamma_1} \right\| \leq C \left\| \gamma_2(t) \right\|_{W^{1,2}} ,$$

where, for γ_2 close to γ_1, C depends only on $\|\gamma_1\|_{C^0}$ and $\|\mathrm{ddgrad}\|_{C^0(B)}$ for some large ball B. It follows that

$$\lim_{\gamma_2 \to 0} \left\| \mathrm{d}s_{\gamma_1+\gamma_2} - \mathrm{d}s_{\gamma_1} \right\| = 0,$$

and so we have shown that $\gamma \to \mathrm{d}s_\gamma$ is continuous. This completes the proof of the claim. $\square$

Exercise 4. Prove that $\mathrm{d}s_\gamma : W^{1,2} \to L^2$ is a bounded linear operator. Hint: if M is a bounded continuous matrix valued function, and η is in L^2, then $\|M\eta\|_{L^2} \leq \|M\|_{C^0} \|\eta\|_{L^2}$.

Our plan now is to show that the derivative of s at the zero solution $0 \in W^{1,2}$ is a Fredholm operator. In fact, we will be able to show that $\mathrm{d}s_0$ is a surjective operator, and we will be able to explicitly identify the kernel of $\mathrm{d}s_0$ as the finite dimensional space spanned the positive eigenvalues of the Hessian of φ.

Definition 3. The **Hessian** of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at x is the bilinear form made of the second partial derivatives $\mathrm{Hess}_x = \partial_i \partial_j \varphi(x) \mathrm{d}x_i \otimes \mathrm{d}x_j$. If x is a critical point, then Hess_x is coordinate independent.

In the presence of the metric g, we can define an endomorphism $\mathrm{Hess}_x^g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(-, \mathrm{Hess}_x^g(-)) = \mathrm{Hess}_x(-,-).$$

Lemma 4. Let $\mathrm{grad} = \mathrm{grad}_{\varphi,g}$ be the gradient vector field of φ, and suppose 0 is a critical point of φ. Then

$$\mathrm{dgrad}_0 = \mathrm{Hess}_0^g \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$

Proof. Let $g = \sum_{k,j} g_{kj} \mathrm{d}x^k \otimes \mathrm{d}x^j$, and write $\mathrm{grad} = \sum_k a_k \partial_k$. Then

$$\partial_j \varphi = g(\mathrm{grad}, \partial_j) = \sum_k a_k g_{kj} \implies \partial_i \partial_j \varphi = \sum_k g_{kj} \partial_i a_k + \sum_k a_k \partial_i g_{kj}.$$

Evaluating at $x = 0$, where $a \equiv 0$, we conclude

$$\partial_i \partial_j \varphi(0) = \sum_k g_{kj} \partial_i a_k = g(\partial_j, \mathrm{dgrad}_0(\partial_i)) \tag{1}$$

Now we compute

$$g(\partial_j, \mathrm{Hess}_0^g(\partial_i)) = \mathrm{Hess}_0(\partial_i, \partial_j) = \partial_i \partial_j \varphi(0), \tag{2}$$

comparing (1) and (2), we conclude that $\mathrm{dgrad}_0 = \mathrm{Hess}_0^g$, as desired. $\square$

Now the fact that φ is a Morse function says precisely that Hess_0 is a non-degenerate bilinear form. It follows that Hess_0^g is a g-self-adjoint operator, and hence has an eigenbasis

$v_1, \cdots, v_n$, with eigenvalues $\lambda_1, \cdots, \lambda_n$, where we suppose that

$$\lambda_1 \le \cdots \le \lambda_p < 0 < \lambda_{p+1} \le \cdots \le \lambda_n.$$

The number p is precisely the **Morse index** of the critical point (i.e. the index of the bilinear form Hess_0). Let's agree to call H_+ the subspace spanned by $v_{p+1}, \cdots, v_n$.

Define a map $F : W^{1,2}(\mathbb{R}_{\ge 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\ge 0}, \mathbb{R}^n) \oplus H_+$ by

$$F(\gamma) = (s(\gamma), \pi_+\gamma(0)).$$

Note that evaluating a curve γ at 0 is a continuous linear map, and hence $\gamma \mapsto \pi_+\gamma(0)$ is a smooth function $W^{1,2}(\mathbb{R}_{\ge 0}, \mathbb{R}^n) \to H_+$.

Proposition 5. $\mathrm{d}F_0$ is an isomorphism $W^{1,2} \to L^2 \oplus H_+$.

Proof. It suffices to prove that $\mathrm{d}F_0$ is a bijection, thanks to the open mapping theorem. The derivative of F at 0 is given by the formula

$$\mathrm{d}F_0(\eta) = (\eta' + \mathrm{Hess}_0 \cdot \eta\,,\ \pi_+\eta(0)).$$

This follows from Claim 2, and the fact that $\eta \mapsto \pi_+\eta(0)$ is linear.

First we prove that $\mathrm{d}F_0$ is injective. It is convenient to write η as $\eta = \sum \eta_i v_i$, where the η_i are now $W^{1,2}$ functions $\mathbb{R}_{\ge 0} \to \mathbb{R}$. Suppose that $\mathrm{d}F_0(\eta) = 0$. This is equivalent to

$$\eta_i'(t) = -\lambda_i \eta_i(t) \text{ for } i = 1, \cdots, n \text{ and } \eta_{p+1}(0) = \cdots = \eta_n(0).$$

Simple elliptic bootstrapping proves that $\eta_1, \cdots, \eta_n$ are C^∞ functions. In fact, it is clear that

$$\eta_i(t) = \eta_i(0)e^{-\lambda_i t}.$$

Since η_i is assumed to be integrable, we must have $\eta_1(0) = \cdots = \eta_p(0) = 0$, otherwise η would blow up exponentially. Since we assume $\eta_{p+1}(0) = \cdots = \eta_n(0) = 0$, we conclude that η is identically 0. It follows that $\mathrm{d}F_0$ is injective.

Now we prove that $\mathrm{d}F_0$ is surjective. Given $\xi \in L^2$ and $c_{p+1}, \cdots, c_n \in H_+$, we want to define η so that

$$\eta_i(t) + \lambda_i \eta_i(t) = \xi_i(t),$$

and $\eta_i(0) = c_i$ for $i > p$. Define

$$\eta_i(t) = -e^{-\lambda_i t} \int_t^\infty e^{\lambda_i s} \xi_i(s)\, ds \text{ for } i = 1, \cdots, p,$$

and define

$$\eta_i(t) = e^{-\lambda_i t} c_i + e^{-\lambda_i t} \int_0^t e^{\lambda_i s} \xi_i(s)\, ds \text{ for } i = p+1 = \cdots = n.$$

We check that this is well-defined, i.e. the resulting η is indeed in $W^{1,2}$. First we will check that η is in L^2. Let ρ be some test function. Then for $i = 1, \cdots, p$, we compute

$$\int_0^\infty \eta_i(t)\rho(t)\, dt = -\int_0^\infty \int_t^\infty e^{\lambda_i(s-t)} \rho(t)\xi_i(s)\, ds\, dt = -\int_0^\infty \int_0^\infty e^{\lambda_i z} \rho(t) \xi_i(z+t)\, dz\, dt.$$

where have made the change of coordinates $z = s - t$. Now we switch the order of integration:

$$\int_0^\infty \int_0^\infty e^{\lambda_i z} \rho(t) \xi_i(z+t)\, dz\, dt = \int_0^\infty e^{\lambda_i z} \int_0^\infty \rho(t)\xi_i(z+t)\, dt\, dz.$$

We estimate

$$\left| \int_0^\infty \rho(t)\xi_i(z+t)\, dt \right| \le \|\rho\|_{L^2} \|\xi_i\|_{L^2},$$

and hence

$$\left| \int_0^\infty \eta_i(t)\rho(t)\, dt \right| = \left| \int_0^\infty e^{\lambda_i z} \int_0^\infty \rho(t)\xi_i(z+t)\, dt\, dz \right| \le \left\| e^{\lambda_i z} \right\|_{L^1} \|\rho\|_{L^2} \|\xi_i\|_{L^2} = C \|\rho\|_{L^2}.$$

Since $\lambda_i < 0$, the L^1 norm of $e^{\lambda_i z}$ is finite. We conclude that η_i is in L^2 since pairing it with test functions defines a bounded transformation $L^2 \to L^2$ (here we use reflexivity of L^2).

Remark. It is easy to show that η is given by a convolution of ξ with an integrable kernel. It follows that η is in L^2 by Young's inequality. Our argument essentially reproves Young's inequality in our specific setting.

Exercise 5. Prove that η_i is in L^2 for $i = p+1, \cdots, n$.

Having established that η is in L^2, we check that $\eta_i' + \lambda_i \eta_i = \xi_i$ holds weakly in L^2. Suppose $i = 1, \cdots, p$. To check that an equation holds weakly, we pair with a test function ρ. By definition of "weak" we have

$$\int_0^\infty (\eta_i'(t) + \lambda_i \eta_i(t))\rho(t)\, dt = \int_0^\infty \eta_i(t)(\lambda_i \rho(t) - \rho'(t))\, dt.$$

We write

$$\int_0^\infty \eta_i(t)(\lambda_i\rho(t) - \rho'(t))\, dt = \int_0^\infty \int_t^\infty e^{\lambda_i(s-t)} \xi_i(s)(\rho'(t) - \lambda_i \rho(t))\, ds\, dt.$$

Now we change the order of integration:

$$\int_0^\infty \int_t^\infty e^{\lambda_i(s-t)}\xi_i(s)(\rho'(t) - \lambda_i\rho(t))\, ds\, dt = \int_0^\infty \xi_i(s) e^{\lambda_i s} \left[\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i \rho(t))\, dt \right] ds.$$

We compute

$$\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i\rho(t))\, dt = \int_0^s \frac{d}{dt}\left[e^{-\lambda_i t}\rho(t) \right] dt = e^{-\lambda_i s}\rho(s),$$

where we use the fact that ρ is a test function, and hence is compactly supported in $(0, \infty)$. It follows that

$$\int_0^\infty (\eta_i'(t) + \lambda_i\eta_i(t))\rho(t)\, dt = \int_0^\infty \xi_i(s) e^{\lambda_i s}\left[\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i\rho(t))\, dt \right] ds = \int_0^\infty \xi_i(s)\rho(s)\, ds,$$

which demonstrates that $\eta_i' + \lambda_i \eta = \xi_i$ holds weakly (for $i = 1, \cdots, p$).

Exercise 6. Show that $\eta_i' + \lambda_i\eta_i = \xi_i$ holds weakly for $i = p+1, \cdots, n$.

Now since η_i and ξ_i are in L^2, and $\eta_i' = \xi_i - \lambda_i \eta_i$, we conclude that the weak derivative of η_i is in L^2 and hence η_i is in $W^{1,2}$.

Finally, it is clear that $\eta_i(0) = c_i$ for $i = p+1, \cdots, n$. Thus it follows that $\mathrm{d}F_0\eta = (\xi, c)$, and hence $\mathrm{d}F_0$ is surjective. This completes the proof that $\mathrm{d}F_0$ is an isomorphism. $\square$

By the inverse function theorem, it follows that F is a C^1 diffeomorphism in some neighborhood of 0. In fact, one can show without too much additional work that the map s is C^∞ (because grad is a smooth vector field). For the details involved, the reader is referred to Chris Wendl's "Lectures on Holomorphic Curves," pages 85-87. It then follows that F is a smooth diffeomorphism on some neighborhood of 0.

Consider the composite function

$$\begin{tikzcd} H_+ \arrow[r] \arrow[rrr, bend right, "\Phi"'] & 0 \oplus H_+ \arrow[r, "F^{-1}"] & W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^N) \arrow[r, "\mathrm{ev}_0"] & \mathbb{R}^n \end{tikzcd}$$

The map Φ is smooth and defined on small some disk $D(r) \subset H_+$. Since $\pi_+\Phi(x) = x$, we conclude that Φ is a section of the orthogonal projection π_+ (over $D(r)$), and hence Φ defines a smooth submanifold of $\mathbb{R}^n$ (a graph over $D(r)$).

It is clear that the unique gradient flow line starting at any point $\Phi(x) \in \Phi$ converges to 0 (by our construction). Indeed, the flow line starting at $\Phi(x)$ is $F^{-1}(0, x)$. The next lemma will establish that the graph Φ is precisely the stable set near 0.

Lemma 6. There is a neighborhood U of 0 so that any flow line starting in U and converging to 0 actually starts on $\Phi \cap U$.

Proof. First we claim that any trajectory $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0 satisfies $\gamma(t) \in \Phi$ for t sufficiently large. We will use the result that any gradient flow line converging to 0 is automatically in $W^{1,2}$ (cf. Exercise 7).

Consider the elements $\gamma_T \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ given by $\gamma_T(t) = \gamma(t+T)$. It is clear that γ_T is still a gradient flow line, and moreover, that $\|\gamma_T\|_{W^{1,2}} \to 0$ as $T \to \infty$, since

$$\|\gamma_T\|_{W^{1,2}} = \left\|\gamma|_{[T,\infty)}\right\|_{W^{1,2}}.$$

Since our map F is a diffeomorphism on a small neighborhood of 0, we conclude that γ_T eventually enters the domain where F is a diffeomorphism, and hence $\gamma_T = F^{-1}(0, x)$ for some x (here x depends on T). Therefore $\gamma_T(0) \in \Phi$, hence $\gamma(T) \in \Phi$. This proves that γ eventually enters Φ.

Next, pick a bounded open set U' of 0 with the property that so that $\overline{\Phi \cap U'} \subset \Phi$. By compactness of $\overline{\Phi \cap U'} \subset \Phi$, it follows that there is $\delta > 0$ so that for every $x \in \Phi \cap U'$, the flow line through x can be defined on $[-\delta, \infty)$, and that this flow line remains on Φ. In other words, we can extend the flow line backwards in time by δ, while remaining on the graph Φ.

To establish the conclusion of the lemma, we will use the following we claim: there is a smaller open set $U \subset U'$ with the following property: every trajectory which starts in U either remains in U' forever, or leaves U' and never comes back inside U (a similar statement is proved on page 50 of Milnor's notes on the h-cobordism theorem). This claim is proved in Exercise 8.

Assuming this result, we can complete the proof of the lemma. If γ is a gradient flow line starting in U and γ converges to 0, then clearly γ cannot leave U'. Look at the set of times t so that $\gamma(t) \in \Phi$. Since $\gamma \to 0$, we know that $\gamma(t)$ is eventually in Φ, so this set of times is non-empty. Either (case 1) $\gamma(0) \in \Phi$, or (case 2) there is some time $t > \delta$ so $\gamma(t) \in \Phi$ and $\gamma(t-\delta) \notin \Phi$. However, since $\gamma(t) \in \Phi \cap U'$, we conclude that the flow line through $\gamma(t)$ can be extended backwards in time by amount δ *while remaining on* Φ. Therefore $\gamma(t-\delta) \in \Phi$, and so case 2 cannot happen. It follows that $\gamma(0) \in \Phi$, and since $\gamma(0) \in U$, $\gamma(0) \in \Phi \cap U$. We have shown that every flow line starting in U converging to 0 must start on $\Phi \cap U$, as desired. $\square$

Corollary 7. Let S denote the stable set of 0, and let U be the open set furnished by the preceding lemma. Then $S \cap U = \Phi \cap U$, and so we have shown that S is a manifold near 0. The dimension of $S \cap U$ is equal to $\dim \Phi = p$, the Morse index of the critical point. $\square$

Here are the two exercises used in the proof of Lemma 6.

Exercise 7. Let $\mathrm{grad} : \mathbb{R}^n \to \mathbb{R}^n$ be the gradient vector field. We will use the fact that $\mathrm{grad}(0) = 0$ and dgrad_0 is an isomorphism. Suppose that $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a flow line converging to 0.

(a) Prove that γ is in L^2 if and only if $\mathrm{grad} \circ \gamma$ is in L^2. Hint: show that

$$|\gamma(t)| < c\,|\mathrm{grad} \circ \gamma(t)|$$

for t sufficiently large, for some $c > 0$.

(b) Prove that $\mathrm{grad} \circ \gamma$ is in L^2 using the relation

$$\frac{d}{dt}(\varphi \circ \gamma) = -g(\mathrm{grad} \circ \gamma, \mathrm{grad} \circ \gamma).$$

(c) Now that we know that γ is in L^2, conclude that γ' is also in L^2 using the gradient flow line equation.

(d) Conclude that any flow line γ converging to 0 is actually in $W^{1,2}$.

Exercise 8. Given a bounded open neighborhood U' of 0, one can always find a smaller open set $U \subset U'$ so that every trajectory γ starting in U either remains in U', or leaves U' and never returns to U.

(a) Pick U'' compactly supported in U' around 0 so that $g(\mathrm{grad}, \mathrm{grad}) > b$ on $U' \setminus U''$. Pick $U_\epsilon \subset U''$ so that $\max_{U_\epsilon} \varphi - \min_{U_\epsilon} \varphi < \epsilon$. Using the fact that

$$\frac{d}{dt}\varphi(t) = -g(\mathrm{grad}, \mathrm{grad}),$$

conclude that any trajectory starting and ending in U_ϵ must spend time less than $b^{-1}\epsilon$ in $U' \setminus U''$.

(b) Since $\partial U''$ is compact and contained in U', conclude a minimum amount of time needed to flow from $\partial U''$ to $\mathbb{R}^n \setminus U'$.

(c) Conclude that we can pick ϵ small enough so that any flow starting and ending in U_ϵ cannot leave U'. Taking $U = U_\epsilon$ proves the claim.

Appendix

Definition 8. For $k \geq 1$, a C^k **Banach manifold** $\mathcal{X}$ is a topological space covered by open sets homeomorphic to open subsets of Banach spaces, where the transitions functions are C^k maps. More precisely, a Banach manifold comes equipped with a maximal atlas of coordinate charts: $c : U_c \subset \mathcal{X} \to c(U) \subset X_c$, where X_c is a Banach space, $c : U_c \to c(U)$ is homeomorphism onto an open set, and so that the transition homeomorphism

$$\rho_{21} = c_2 \circ c_1^{-1} : c_1(U_1 \cap U_2) \to c_2(U_1 \cap U_2)$$

is a C^k map.

We define a continuous map $A : \mathcal{X} \to \mathcal{Y}$ between C^k Banach spaces to be C^r $(r \leq k)$ if $c_2 \circ A \circ c_1^{-1}$ is a C^k map, for all choices of coordinates c_1, c_2 around x and $A(x)$ respectively.

Definition 9 (the tangent bundle)**.** For the purposes of this defintiion, let's agree to say that a Banach space is a topological vector space equipped with an equivalence class of complete metrics defining its topology.

For $k \geq 1$, let BMan_k be the category of C^k Banach manifolds, with C^k maps between them, let BSpace_k be the category of Banach spaces with C^k maps between them, and let Bun_k be the category of Banach space bundles over Banach manifolds. A morphism in Bun_k between bundles $E_1 \to B_1$ and $E_2 \to B_2$ is a pair (f, F) such that $f : B_1 \to B_2$ is a C^k map and F is a C^{k-1} section of the Banach space bundle $\mathrm{Hom}(B_1, f^*E_2) \to B_1$.

There is a functor $\tau : \mathrm{BSpace}_k \to \mathrm{Bun}_k$ sending a Banach space X to the trivial bundle $\tau(X) = X \times X \to X$, and which sends a morphism $f : X \to Y$ to the pair $(f, \mathrm{d}f)$, where $\mathrm{d}f$ is the C^{k-1} section of $\mathrm{Hom}(\tau(X), f^*\tau(Y)) = \mathrm{Hom}(X, Y) \times X \to X$.

The tangent bundle functor $T : \mathrm{BMan}_k \to \mathrm{Bun}_k$, is defined by three axioms:

(i) We require $T(f : X \to Y) = (f, \mathrm{d}f)$, i.e. $T(f)$ is a bundle map "over f" (where we abuse notation and use the symbol d for T as well as for τ).

(ii) The following diagram should commute up to a natural isomorphism $T \circ j \to \tau$:

$$\begin{tikzcd}
\mathrm{BSpace}_k \arrow[r, "\tau"] \arrow[d, "j"] & \mathrm{Bun}_k \\
\mathrm{BMan}_k \arrow[ur, bend right, "T"'] &
\end{tikzcd}$$

where j is the obvious inclusion functor $\mathrm{BSpace}_k \to \mathrm{BMan}_k$. This natural isomorphism should be thought of as part of the data of T.

(iii) If $i : U \to M$ is the inclusion of an open set, then the map $\mathrm{d}i : TU \to i^*TM$ is an isomorphism.

It is not very hard to show that this determines T up to unique natural isomorphism, i.e. if T' is another such functor then there is a unique natural isomorphism $T \to T'$ so that

$$\begin{tikzcd}
\tau \arrow[r] \arrow[d] & T' \circ j \\
T \circ j \arrow[ur] &
\end{tikzcd}$$

commutes.