1. Elliptic differential operators

Definition 1. Let E_1, E_2 be complex vector bundles over Y.

A first order differential operator is a $\mathbb{C}$-linear map

$$L : \Gamma(E_1) \to \Gamma(E_2)$$

with the property that for all $\mathbb{C}$-valued functions f

$$[L, f] = L \circ f - f \circ L : \Gamma(E_1) \to \Gamma(E_2)$$

is given by a tensor, i.e. is induced by a section of $\mathrm{Hom}_{\mathbb{C}}(E_1, E_2)$.

Exercise 1. Prove that first order differential operators are **local** in the sense that if s is supported in a closed set K, then $L(s)$ is also supported in K.

Example 2. A (linear) **connection** on $E \to Y$ is a first order differential operator

$$\nabla : E \to \mathrm{Hom}(TY, E).$$

satisfying

$$[\nabla, f]u = \mathrm{d}f \otimes u. \text{ Notice that } u \mapsto \mathrm{d}f \otimes u \text{ is a tensor.}$$

Interestingly, we define a connection by specifying the tensor we get by commutating with functions, $[\nabla, f]$.

Exercise 2. Let $E_1, E_2 \to Y$ be two vector bundles over Y. Let L be a first order linear differential operator. Let ∇ be a connection on E_1. Then there are two tensors

$$A : \mathrm{Hom}(TY, E_1) \to E_2 \text{ and } B : E_1 \to E_2$$

so that

$$L(u) = A(\nabla u) + B(u).$$

Moreover, A and B are uniquely determined by L.

Definition 3. Let $E_i \to Y$, $i = 1, 2$, be vector bundles over Y. Given a differential operator $L : E_1 \to E_2$, we can consider the assignment

$$f \in \Gamma(\mathbb{C}) \mapsto [L, f] \in \Gamma \, \mathrm{Hom}(E_1, E_2).$$

This assignment is actually a first-order linear differential operator $\mathbb{C} \to \mathrm{Hom}(E_1, E_2)$. Since d is a connection on the trivial bundle $\mathbb{C}$, we already proved that we can express

$$[L, f] = A(\mathrm{d}f) + B(f)$$

for some tensors A and B. Use this to prove that $B = 0$.

We said that $A : T^*Y \to \mathrm{Hom}(E_1, E_2)$ and $B : \mathbb{C} \to \mathrm{Hom}(E_1, E_2)$ are uniquely determined by L. Conclude that there is unique tensor σ so that $[L, f] = \sigma(\mathrm{d}f)$.