MATH 257C

Contents

April 2 (Sarah)

Goals of the course:

(i) We would like to develop basic Hamiltonian Floer theory in such a way that we understand what we are blackboxing and where in the literature we would find the proofs. We would also like to keep Morse theory running in parallel for comparison's sake.

(ii) We would like to discuss some recent developments in the theory.

Fundamental analytic program for obtaining moduli spaces which lead to invariants in Floer or Morse theory (see Schwarz's book):

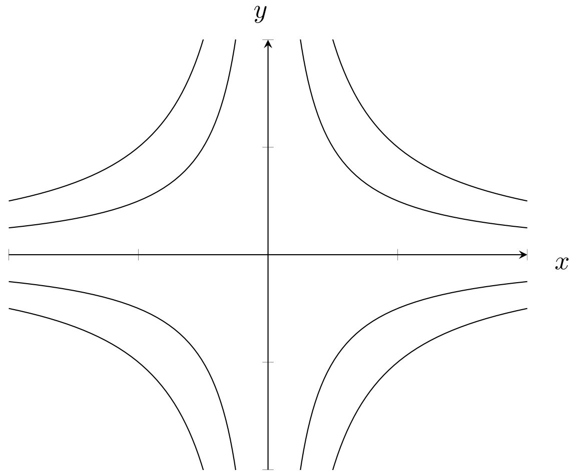
(1) analytic setup, definition of functional spaces of solutions/trajectories (moduli spaces are cut out by equations)
(2) analyze the index problem (ideally, we want to think of our spaces as finite-dimensional manifolds, and the index should theoretically give the dimension)
(3) transversality (we want to give our spaces "manifold-like" structures)
(4) compactness (we want to be able to count things, and counts need to be finite)
(5) gluing (if we add points to compactify, they should have neighborhoods which look appropriately manifold-like; this is a sort of converse to compactness)
(6) coherent orientations (when we count, we want to keep track of sign)

Today we start with an exercise in Morse theory; it will illustrate steps (1), (2), and (3).

Theorem 1 (Local stable manifold theorem)**.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function whose only critical point is 0. We assume that this critical point is Morse, in the sense that the Hessian at this point is non-degenerate. Fix a Riemannian metric g on $\mathbb{R}^n$, and let grad be the gradient of ϕ with respect to g. Let $S \subset \mathbb{R}^n$ be the stable set of 0 (that is, the set of points which flow to 0 under $-\mathrm{grad}$). Then S is a submanifold of $\mathbb{R}^n$ near 0.

Remark 2. Other proofs of this theorem are also hard. For instance, look up the Hartman-Grobman theorem and note how this does not follow from it.

Example 3. Consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ given by $\phi(x, y) = x^2 - y^2$. Then ϕ is Morse, and its only critical point is $(0, 0)$. Here's a drawing of the flow lines:



Thus the stable set is the x-axis.

Now we proceed to the proof of the stable manifold theorem.

For $x \in \mathbb{R}^n$, the gradient flow line starting at x is the (unique) path $\gamma_x : [0, \delta) \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\dot{\gamma}(t) = -\operatorname{grad}(\gamma(t))$, and $\delta > 0$ is maximal. Note that for $x \in S$, $\delta = \infty$.

The plan is to do the following:

- Identify $S \subset \mathbb{R}^n$ with the set of smooth gradient flow lines starting at points in S.
- Construct a path space P (a Banach manifold) and a Banach space bundle $\mathcal{E} \to P$.
- Define a (Fredholm) section $s : P \to \mathcal{E}$, $\gamma \mapsto \dot{\gamma} + \operatorname{grad} \circ \gamma$ such that $s^{-1}(0) = S$ (this will be non-trivial because P is large).
- Prove $s \pitchfork 0$ and use the implicit function theorem. This will use a baby version of the Atiyah-Patodi-Singer index theorem.

First, we define our Banach space to be the Sobolev space $P = W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Observe that P can also be written as the equivalence classes of functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ which are square-integrable and have weak derivatives f' which are also square-integrable. In this special case, we mean that f is differentiable almost everywhere, and

$$f(t) = f(0) + \int_0^t f'(s)\, ds$$

almost everywhere.

Note that in higher dimensions, we'll have to be more mature about how we define these spaces.

Proposition 4. If $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, then $\gamma(t) \to 0$ as $t \to \infty$.

Proof. The basic idea is that the finiteness of the $(1,2)$-norm gives certain bounds on the first derivative, which forces the variance to approach zero. Then the $(1,2)$-norm also forces the values of the function itself to go to zero.

The hypothesis $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ tells us that the values

$$\left(\int_0^\infty |\gamma|^2\right)^{1/2} = \left(\sum_{N=0}^\infty \int_N^{N+1} |\gamma|^2\right)^{1/2}$$

and

$$\left(\int_0^\infty |\dot\gamma|^2\right)^{1/2} = \left(\sum_{N=0}^\infty \int_N^{N+1} |\dot\gamma|^2\right)^{1/2}$$

are finite. It follows that

$$(\dagger) \qquad \int_N^{N+1} |\gamma|^2 \qquad \text{and} \qquad \int_N^{N+1} |\dot\gamma|^2$$

approach zero as $N \to \infty$.

Since $\dot\gamma$ is a weak derivative, for almost every $x \in [N, N+1]$ we have

$$\begin{aligned} |\gamma(x) - \gamma(N)| &= \left|\int_N^x \dot\gamma\right| \\ &\leq \sqrt{x-N}\left(\int_N^x |\dot\gamma|^2\right)^{1/2} \\ &\leq \sqrt{x-N}\left(\int_N^{N+1} |\dot\gamma|^2\right)^{1/2} \\ &\leq \left(\int_N^{N+1} |\dot\gamma|^2\right)^{1/2} \end{aligned}$$

(the second line follows from Cauchy-Schwarz). Combining this information with the fact that both expressions in (†) go to zero yields the desired result. □

Remark 5. We have constructed a version of the Rellich embedding $W^{1,2}(\mathbb{R}, \cdot) \hookrightarrow C^0(\mathbb{R}, \cdot)$, which sends Sobolev spaces to Hölder spaces.

Remark 6. In the general case, we'll need to build asymptotic conditions by hand. This makes it difficult to turn more general analogues of P into Banach manifolds.

Now let $\mathcal{E} = P \times L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, so $\mathcal{E} \to P$ is a trivial bundle. Define a section $s : P \to \mathcal{E}$ by $s(\gamma) = (\gamma, \dot\gamma + \operatorname{grad} \circ \gamma)$. We do need to check that $\operatorname{grad} \circ \gamma$ is square-integrable, but this shouldn't be too bad because grad is smooth and γ has compact image by Proposition 4. We will also need to demonstrate some kind of regularity for s.

Proposition 7. If $\gamma \in S$ (i.e., γ is a smooth gradient flow line starting at a point in S), then $\gamma \in P = W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Moreover, $S = s^{-1}(0)$.

In order to prove Proposition 7, we will need the following lemma (which we will blackbox for now).

Lemma 8 (Exponential convergence of flow lines at the ends)**.** If $\gamma \in S$, then there is some $a > 0$ so that $|\gamma(t)| \leq e^{-at}$.

Remark 9. Using the gradient flow line equation we obtain the same exponential convergence result for all derivatives of flow lines. This uses a method called *bootstrapping.*

If $\gamma \in S$, then Lemma 8 tells us $\gamma \in P$, so we have $S \subset s^{-1}(0)$. On the other hand, we still need to check that if $f \in P$ satisfies $s(f) = 0$, then f is a smooth gradient flow line starting at a point in S. The tricky part is verifying that f is smooth. The Rellich embedding $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ forces f to be continuous. Then we use bootstrapping to guarantee that f is actually smooth. We know that f has a weak derivative. Since f is a continuous solution of the gradient flow line equation, we can show that f has a genuine derivative f' which must also be continuous. Repeating this process infinitely shows that f is, in fact, smooth.

April 4 (Dylan)

The goal of this lecture is to complete the proof of the "local stable manifold theorem." Recall the setting:

(i) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Morse function with exactly one critical point at $0 \in \mathbb{R}^n$,

(ii) g is a Riemannian metric on $\mathbb{R}^n$,

(iii) $\mathrm{grad}_{\varphi,g} =: \mathrm{grad}$ is the gradient vector field associated to g and φ – recall that grad is determined by the relation $g(\mathrm{grad}, X) = \mathrm{d}\varphi(X)$, for all vector fields X.

(iv) The stable set S is defined to be the set of all $x \in \mathbb{R}^n$ so that the negative gradient flow line starting at x converges to 0.

Local Stable Manifold Theorem. There is an open neigborhood U around 0 so that $U \cap S$ is a smooth submanifold of U.

Recall the stategy for proving this theorem. We identify S with the set of gradient flow lines $\mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0, and make the following crucial observation: If $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ satisfies the following equation

$$(*) \qquad \gamma'(t) + \mathrm{grad} \circ \gamma = 0,$$

then γ is C^∞ smooth, and conversely, if γ is a gradient flow line converging to 0, then γ lies in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and γ satisfies $(*)$. In other words, if we define

$$s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \text{ by } s(\gamma) = \gamma'(t) + \mathrm{grad} \circ \gamma,$$

then $s^{-1}(0) = S$. This was established in the previous lecture. Here are three exercises related to the content of the previous lecture.

Exercise 1 (Rellich Embedding)**.** Let $\gamma \in C_c^\infty$. Prove that for $t \geq 0$

$$\gamma(t)e^{-t} = \int_t^\infty \gamma(s)e^{-s} - \gamma'(s)e^{-s}\, ds,$$

and deduce that

$$|\gamma(t)| \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}([t,\infty),\mathbb{R}^n)}$$

In particular,

$$(\star) \qquad \|\gamma\|_{C^0} \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}}\, .$$

By density of C_c^∞ functions in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, prove that $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^0(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Moreover, conclude that $|\gamma(t)| \to 0$ as $t \to \infty$. Applying the bound $(\star)$ to the derivatives of γ, conclude that

$$W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^{k-1}(\mathbb{R}_{\geq 0}, \mathbb{R}^n),$$

and that $\|\gamma\|_{C^{k-1}} \leq C\, \|\gamma\|_{W^{k,2}}$.

Exercise 2. Show that if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{0,1}$ function (i.e. Lipshitz), and $X(0) = 0$, then the map $\gamma \mapsto X \circ \gamma$ sends $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ to $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

More generally, if X is a $C^{k,1}$ function with $X(0)=0$, then $\gamma \mapsto X\circ\gamma$ sends $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ to $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$. Hint: for both claims, it suffices to prove it first for smooth γ, and then use the density of smooth functions in $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$.

Exercise 3. Conclude that if $\gamma \in W^{k,2}$ satisfies the gradient flow line equation $(*)$, then $\gamma \in W^{k+1,2}$. Conclude that if $\gamma \in W^{1,2}$ satisfies $(*)$, then $\gamma \in W^{k,2}$ for all k. By the Rellich embedding $W^{k,2} \subset C^{k-1}$, conclude that $W^{1,2}$ solutions of $(*)$ are C^∞ smooth.

Now we turn to some new material. We will show that $s^{-1}(0)$ is a manifold via the inverse function theorem for Banach spaces.

Definition 10. Let $U \subset X$ and $V \subset Y$ be open subspaces of Banach spaces X, Y. A map $A : U \to V$ is **differentiable** at $u \in U$ provided there is a bounded linear transformation $\mathrm{d}A_u : X \to Y$ so that

$$\lim_{\xi\to 0} \frac{\|A(u+\xi) - A(u) - \mathrm{d}A_u(\xi)\|_Y}{\|\xi\|_X} = 0.$$

While this definition uses the norms on X, Y, it is clear that it only depends on the equivalence classes of the norms.

The map $A : U \to V$ is **continuously differentiable** if $u \mapsto \mathrm{d}A_u \in \mathrm{Hom}(X,Y)$ is a continous function, where the latter is given the topology induced by the "operator norm." The set of continosly differentiable functions is denoted $C^1(U,V)$.

A map A is $C^k(U,V)$, $k \geq 1$, if $\mathrm{d}A$ is $C^{k-1}(U,\mathrm{Hom}(X,Y))$.

In the appendix to this lecture, we define the terms **Banach manifold** and the **tangent bundle** of a Banach manifold.

Inverse Function Theorem. Let X, Y be Banach manifolds, and suppose $A : X \to Y$ is a C^k map so that $\mathrm{d}A_x$ is an isomorphism (i.e. is continuous in the natural topologies on TX_x and $TY_{A(x)}$). Then there are neighborhoods $U \ni x$ and $V \ni y$ so that A maps U to V diffeomorphically.

The derivatives of the vector field grad appear in the statement of the next claim. Thinking of grad is a function $\mathbb{R}^n \to \mathbb{R}^n$, it certainly has a derivative $\mathrm{dgrad}_x : \mathbb{R}^n \to \mathbb{R}^n$ at all points $x \in \mathbb{R}^n$ (warning: here we are *not* thinking of grad as a map $\mathbb{R}^n \to T\mathbb{R}^n$ when we take its derivative). Similarly, we will denote the (symmetric) second derivative matrix by

$$\mathrm{ddgrad}_x : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n.$$

This is not a coordinate invariant notion.

It is clear that if x, y are two points of $\mathbb{R}^n$, then

$$\mathrm{grad}(x+y) - \mathrm{grad}(x) = \left[\int_0^1 \mathrm{dgrad}_{x+sy}\, ds\right] \cdot y,$$

where we interpret the expression in the braces as a matrix $\mathbb{R}^n \to \mathbb{R}^n$.

Claim 11. The map $s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ defined by $s(\gamma) = \gamma' + \mathrm{grad} \circ \gamma$ is a C^1 map, and it's derivative is given by

$$\mathrm{d}s_\gamma(\eta)(t) = \eta'(t) + \mathrm{dgrad}_{\gamma(t)} \cdot \eta(t)$$

Proof. We begin by computing

$$s(\gamma+\eta)(t) - s(\gamma)(t) = \eta'(t) + \mathrm{grad}(\gamma(t)+\eta(t)) - \mathrm{grad}(\gamma(t)) = \eta'(t) + \left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)}\, ds\right] \cdot \eta(t).$$

Hence,

$$s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta) = \left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds\right] \cdot \eta(t).$$

It follows that

$$\left\| s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta) \right\|_{L^2} \leq \left\| \int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds \right\|_{C^0} \|\eta\|_{L^2}.$$

We compute

$$\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds = \left[\int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds\right] \cdot \eta(t),$$

and hence

$$\left\| \int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds \right\|_{C^0} \leq \left\| \int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds \right\|_{C^0} \|\eta\|_{C^0}.$$

Since ddgrad is a continous function, and $\gamma(t) + rs\eta(t)$ is a bounded function of t (i.e. for $\|\eta\|_{C^0} \leq 1$, we can suppose that $\|\gamma + rs\eta\|_{C^0} \leq R$ for some large R) we conclude some C independent of t and η, $\|\eta\|_{C^0} \leq 1$, so that

$$\left\| \int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds \right\|_{C^0} \leq C,$$

and hence

$$\left\| s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta) \right\|_{L^2} \leq C \|\eta\|_{L_2} \|\eta\|_{C^0} \leq C' \|\eta\|^2_{W^{1,2}},$$

where we have used the fact that $\|-\|_{C^0} \leq c \|-\|_{W^{1,2}}$ and $\|-\|_{L^2} \leq \|-\|_{W^{1,2}}$. It follows that s is differentiable and its derivative at γ is $\mathrm{d}s_\gamma$.

It is easy to show that $\mathrm{d}s_\gamma$ is a bounded function $W^{1,2} \to L^2$. Finally we show that $\gamma \to \mathrm{d}s_\gamma$ is continuous.

Given two curves γ_1, γ_2, we compute

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \mathrm{dgrad}_{\gamma_1(t)+\gamma_2(t)} \cdot \eta(t) - \mathrm{dgrad}_{\gamma_1(t)} \cdot \eta(t).$$

Arguing as we did above, we conclude

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \left[\int_0^1 \mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\, ds \cdot \gamma_2(t)\right] \cdot \eta(t),$$

and similarly to the computations above we conclude

$$\left\|\mathrm{d}s_{\gamma_1+\gamma_2}(\eta)-\mathrm{d}s_{\gamma_1}(\eta)\right\|_{L^2}\leq\left\|\int_0^1\mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\,ds\right\|_{C^0}\left\|\gamma_2(t)\right\|_{W^{1,2}}\left\|\eta(t)\right\|_{W^{1,2}}.$$

We thereby obtain an estimate on the operator norm

$$\left\|\mathrm{d}s_{\gamma_1+\gamma_2}-\mathrm{d}s_{\gamma_1}\right\|\leq C\left\|\gamma_2(t)\right\|_{W^{1,2}},$$

where, for γ_2 close to γ_1, C depends only on $\|\gamma_1\|_{C^0}$ and $\|\mathrm{ddgrad}\|_{C^0(B)}$ for some large ball B. It follows that

$$\lim_{\gamma_2\to 0}\left\|\mathrm{d}s_{\gamma_1+\gamma_2}-\mathrm{d}s_{\gamma_1}\right\|=0,$$

and so we have shown that $\gamma\to\mathrm{d}s_\gamma$ is continuous. This completes the proof of the claim. □

Exercise 4. Prove that $\mathrm{d}s_\gamma : W^{1,2}\to L^2$ is a bounded linear operator. Hint: if M is a bounded continuous matrix valued function, and η is in L^2, then $\|M\eta\|_{L^2}\leq\|M\|_{C^0}\|\eta\|_{L^2}$.

Our plan now is to show that the derivative of s at the zero solution $0\in W^{1,2}$ is a Fredholm operator. In fact, we will be able to show that $\mathrm{d}s_0$ is a surjective operator, and we will be able to explicitly identify the kernel of $\mathrm{d}s_0$ as the finite dimensional space spanned the positive eigenvalues of the Hessian of φ.

Definition 12. The **Hessian** of a function $\varphi:\mathbb{R}^n\to\mathbb{R}$ at x is the bilinear form made of the second partial derivatives $\mathrm{Hess}_x=\partial_i\partial_j\varphi(x)\mathrm{d}x_i\otimes\mathrm{d}x_j$. If x is a critical point, then Hess_x is coordinate independent.

In the presence of the metric g, we can define an endomorphism $\mathrm{Hess}^g_x:\mathbb{R}^n\to\mathbb{R}^n$ by

$$g(-,\mathrm{Hess}^g_x(-))=\mathrm{Hess}_x(-,-).$$

Lemma 13. Let $\mathrm{grad}=\mathrm{grad}_{\varphi,g}$ be the gradient vector field of φ, and suppose 0 is a critical point of φ. Then

$$\mathrm{dgrad}_0=\mathrm{Hess}^g_0\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n).$$

Proof. Let $g=\sum_{k,j}g_{kj}\mathrm{d}x^k\otimes\mathrm{d}x^j$, and write $\mathrm{grad}=\sum_k a_k\partial_k$. Then

$$\partial_j\varphi=g(\mathrm{grad},\partial_j)=\sum_k a_kg_{kj}\implies\partial_i\partial_j\varphi=\sum_k g_{kj}\partial_ia_k+\sum_k a_k\partial_ig_{kj}.$$

Evaluating at $x=0$, where $a\equiv 0$, we conclude

$$\partial_i\partial_j\varphi(0)=\sum_k g_{kj}\partial_ia_k=g(\partial_j,\mathrm{dgrad}_0(\partial_i))\tag{1}$$

Now we compute

$$g(\partial_j,\mathrm{Hess}^g_0(\partial_i))=\mathrm{Hess}_0(\partial_i,\partial_j)=\partial_i\partial_j\varphi(0),\tag{2}$$

comparing (1) and (2), we conclude that $\mathrm{dgrad}_0=\mathrm{Hess}^g_0$, as desired. □

Now the fact that φ is a Morse function says precisely that Hess_0 is a non-degenerate bilinear form. It follows that Hess^g_0 is a g-self-adjoint operator, and hence has an eigenbasis

$v_1, \cdots, v_n$, with eigenvalues $\lambda_1, \cdots, \lambda_n$, where we suppose that

$$\lambda_1 \leq \cdots \leq \lambda_p < 0 < \lambda_{p+1} \leq \cdots \leq \lambda_n.$$

The number p is precisely the **Morse index** of the critical point (i.e. the index of the bilinear form Hess_0). Let's agree to call H_+ the subspace spanned by $v_{p+1}, \cdots, v_n$.

Define a map $F : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \oplus H_+$ by

$$F(\gamma) = (s(\gamma), \pi_+\gamma(0)).$$

Note that evaluating a curve γ at 0 is a continuous linear map, and hence $\gamma \mapsto \pi_+\gamma(0)$ is a smooth function $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to H_+$.

Proposition 14. $\mathrm{d}F_0$ is an isomorphism $W^{1,2} \to L^2 \oplus H_+$.

Proof. It suffices to prove that $\mathrm{d}F_0$ is a bijection, thanks to the open mapping theorem. The derivative of F at 0 is given by the formula

$$\mathrm{d}F_0(\eta) = (\eta' + \mathrm{Hess}_0 \cdot \eta\,,\, \pi_+\eta(0)).$$

This follows from Claim 11, and the fact that $\eta \mapsto \pi_+\eta(0)$ is linear.

First we prove that $\mathrm{d}F_0$ is injective. It is convenient to write η as $\eta = \sum \eta_i v_i$, where the η_i are now $W^{1,2}$ functions $\mathbb{R}_{\geq 0} \to \mathbb{R}$. Suppose that $\mathrm{d}F_0(\eta) = 0$. This is equivalent to

$$\eta_i'(t) = -\lambda_i \eta_i(t) \text{ for } i = 1, \cdots, n \text{ and } \eta_{p+1}(0) = \cdots = \eta_n(0).$$

Simple elliptic bootstrapping proves that $\eta_1, \cdots, \eta_n$ are C^∞ functions. In fact, it is clear that

$$\eta_i(t) = \eta_i(0)e^{-\lambda_i t}.$$

Since η_i is assumed to be integrable, we must have $\eta_1(0) = \cdots = \eta_p(0) = 0$, otherwise η would blow up exponentially. Since we assume $\eta_{p+1}(0) = \cdots = \eta_n(0) = 0$, we conclude that η is identically 0. It follows that $\mathrm{d}F_0$ is injective.

Now we prove that $\mathrm{d}F_0$ is surjective. Given $\xi \in L^2$ and $c_{p+1}, \cdots, c_n \in H_+$, we want to define η so that

$$\eta_i(t) + \lambda_i \eta_i(t) = \xi_i(t),$$

and $\eta_i(0) = c_i$ for $i > p$. Define

$$\eta_i(t) = -e^{-\lambda_i t} \int_t^\infty e^{\lambda_i s}\xi_i(s)\, ds \text{ for } i = 1, \cdots, p,$$

and define

$$\eta_i(t) = e^{-\lambda_i t}c_i + e^{-\lambda_i t} \int_0^t e^{\lambda_i s}\xi_i(s)\, ds \text{ for } i = p+1 = \cdots = n.$$

We check that this is well-defined, i.e. the resulting η is indeed in $W^{1,2}$. First we will check that η is in L^2. Let ρ be some test function. Then for $i = 1, \cdots, p$, we compute

$$\int_0^\infty \eta_i(t)\rho(t)\, dt = -\int_0^\infty \int_t^\infty e^{\lambda_i(s-t)}\rho(t)\xi_i(s)\, ds\, dt = -\int_0^\infty \int_0^\infty e^{\lambda_i z}\rho(t)\xi_i(z+t)\, dz\, dt.$$

where have made the change of coordinates $z = s-t$. Now we switch the order of integration:

$$\int_0^\infty \int_0^\infty e^{\lambda_i z}\rho(t)\xi_i(z+t)\,dz\,dt = \int_0^\infty e^{\lambda_i z}\int_0^\infty \rho(t)\xi_i(z+t)\,dt\,dz.$$

We estimate

$$\left|\int_0^\infty \rho(t)\xi_i(z+t)\,dt\right| \leq \|\rho\|_{L^2}\|\xi_i\|_{L^2},$$

and hence

$$\left|\int_0^\infty \eta_i(t)\rho(t)\,dt\right| = \left|\int_0^\infty e^{\lambda_i z}\int_0^\infty \rho(t)\xi_i(z+t)\,dt\,dz\right| \leq \left\|e^{\lambda_i z}\right\|_{L^1}\|\rho\|_{L^2}\|\xi_i\|_{L^2} = C\|\rho\|_{L^2}.$$

Since $\lambda_i < 0$, the L^1 norm of $e^{\lambda_i z}$ is finite. We conclude that η_i is in L^2 since pairing it with test functions defines a bounded transformation $L^2 \to L^2$ (here we use reflexivity of L^2).

Remark. It is easy to show that η is given by a convolution of ξ with an integrable kernel. It follows that η is in L^2 by Young's inequality. Our argument essentially reproves Young's inequality in our specific setting.

Exercise 5. Prove that η_i is in L^2 for $i = p+1,\cdots,n$.

Having established that η is in L^2, we check that $\eta_i'+\lambda_i\eta_i = \xi_i$ holds weakly in L^2. Suppose $i = 1,\cdots,p$. To check that an equation holds weakly, we pair with a test function ρ. By definition of "weak" we have

$$\int_0^\infty (\eta_i'(t)+\lambda_i\eta_i(t))\rho(t)\,dt = \int_0^\infty \eta_i(t)(\lambda_i\rho(t)-\rho'(t))\,dt.$$

We write

$$\int_0^\infty \eta_i(t)(\lambda_i\rho(t)-\rho'(t))\,dt = \int_0^\infty\int_t^\infty e^{\lambda_i(s-t)}\xi_i(s)(\rho'(t)-\lambda_i\rho(t))\,ds\,dt.$$

Now we change the order of integration:

$$\int_0^\infty\int_t^\infty e^{\lambda_i(s-t)}\xi_i(s)(\rho'(t)-\lambda_i\rho(t))\,ds\,dt = \int_0^\infty \xi_i(s)e^{\lambda_i s}\left[\int_0^s e^{-\lambda_i t}(\rho'(t)-\lambda_i\rho(t))\,dt\right]ds.$$

We compute

$$\int_0^s e^{-\lambda_i t}(\rho'(t)-\lambda_i\rho(t))\,dt = \int_0^s \frac{d}{dt}\left[e^{-\lambda_i t}\rho(t)\right]dt = e^{-\lambda_i s}\rho(s),$$

where we use the fact that ρ is a test function, and hence is compactly supported in $(0,\infty)$. It follows that

$$\int_0^\infty (\eta_i'(t)+\lambda_i\eta_i(t))\rho(t)\,dt = \int_0^\infty \xi_i(s)e^{\lambda_i s}\left[\int_0^s e^{-\lambda_i t}(\rho'(t)-\lambda_i\rho(t))\,dt\right]ds = \int_0^\infty \xi_i(s)\rho(s)\,ds,$$

which demonstrates that $\eta_i'+\lambda_i\eta = \xi_i$ holds weakly (for $i = 1,\cdots,p$).

Exercise 6. Show that $\eta_i'+\lambda_i\eta_i = \xi_i$ holds weakly for $i = p+1,\cdots,n$.

Now since η_i and ξ_i are in L^2, and $\eta_i' = \xi_i - \lambda_i\eta_i$, we conclude that the weak derivative of η_i is in L^2 and hence η_i is in $W^{1,2}$.

Finally, it is clear that $\eta_i(0) = c_i$ for $i = p+1, \cdots, n$. Thus it follows that $\mathrm{d}F_0\eta = (\xi, c)$, and hence $\mathrm{d}F_0$ is surjective. This completes the proof that $\mathrm{d}F_0$ is an isomorphism. $\square$

By the inverse function theorem, it follows that F is a C^1 diffeomorphism in some neighborhood of 0. In fact, one can show without too much additional work that the map s is C^∞ (because grad is a smooth vector field). For the details involved, the reader is referred to Chris Wendl's "Lectures on Holomorphic Curves," pages 85-87. It then follows that F is a smooth diffeomorphism on some neighborhood of 0.

Consider the composite function

$$\begin{tikzcd} H_+ \arrow[r] \arrow[rrr, bend right, "\Phi"'] & 0 \oplus H_+ \arrow[r, "F^{-1}"] & W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^N) \arrow[r, "\mathrm{ev}_0"] & \mathbb{R}^n \end{tikzcd}$$

The map Φ is smooth and defined on small some disk $D(r) \subset H_+$. Since $\pi_+\Phi(x) = x$, we conclude that Φ is a section of the orthogonal projection π_+ (over $D(r)$), and hence Φ defines a smooth submanifold of $\mathbb{R}^n$ (a graph over $D(r)$).

It is clear that the unique gradient flow line starting at any point $\Phi(x) \in \Phi$ converges to 0 (by our construction). Indeed, the flow line starting at $\Phi(x)$ is $F^{-1}(0, x)$. The next lemma will establish that the graph Φ is precisely the stable set near 0.

Lemma 15. There is a neighborhood U of 0 so that any flow line starting in U and converging to 0 actually starts on $\Phi \cap U$.

Proof. First we claim that any trajectory $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0 satisfies $\gamma(t) \in \Phi$ for t sufficiently large. We will use the result that any gradient flow line converging to 0 is automatically in $W^{1,2}$ (cf. Exercise 7).

Consider the elements $\gamma_T \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ given by $\gamma_T(t) = \gamma(t+T)$. It is clear that γ_T is still a gradient flow line, and moreover, that $\|\gamma_T\|_{W^{1,2}} \to 0$ as $T \to \infty$, since

$$\|\gamma_T\|_{W^{1,2}} = \left\|\gamma|_{[T,\infty)}\right\|_{W^{1,2}}.$$

Since our map F is a diffeomorphism on a small neighborhood of 0, we conclude that γ_T eventually enters the domain where F is a diffeomorphism, and hence $\gamma_T = F^{-1}(0, x)$ for some x (here x depends on T). Therefore $\gamma_T(0) \in \Phi$, hence $\gamma(T) \in \Phi$. This proves that γ eventually enters Φ.

Next, pick a bounded open set U' of 0 with the property that so that $\overline{\Phi \cap U'} \subset \Phi$. By compactness of $\overline{\Phi \cap U'} \subset \Phi$, it follows that there is $\delta > 0$ so that for every $x \in \Phi \cap U'$, the flow line through x can be defined on $[-\delta, \infty)$, and that this flow line remains on Φ. In other words, we can extend the flow line backwards in time by δ, while remaining on the graph Φ.

To establish the conclusion of the lemma, we will use the following we claim: there is a smaller open set $U \subset U'$ with the following property: every trajectory which starts in U either remains in U' forever, or leaves U' and never comes back inside U (a similar statement is proved on page 50 of Milnor's notes on the h-cobordism theorem). This claim is proved in Exercise 8.

Assuming this result, we can complete the proof of the lemma. If γ is a gradient flow line starting in U and γ converges to 0, then clearly γ cannot leave U'. Look at the set of times t so that $\gamma(t) \in \Phi$. Since $\gamma \to 0$, we know that $\gamma(t)$ is eventually in Φ, so this set of times is non-empty. Either (case 1) $\gamma(0) \in \Phi$, or (case 2) there is some time $t > \delta$ so $\gamma(t) \in \Phi$ and $\gamma(t-\delta) \notin \Phi$. However, since $\gamma(t) \in \Phi \cap U'$, we conclude that the flow line through $\gamma(t)$ can be extended backwards in time by amount δ *while remaining on* Φ. Therefore $\gamma(t-\delta) \in \Phi$, and so case 2 cannot happen. It follows that $\gamma(0) \in \Phi$, and since $\gamma(0) \in U$, $\gamma(0) \in \Phi \cap U$. We have shown that every flow line starting in U converging to 0 must start on $\Phi \cap U$, as desired. □

Corollary 16. Let S denote the stable set of 0, and let U be the open set furnished by the preceding lemma. Then $S \cap U = \Phi \cap U$, and so we have shown that S is a manifold near 0. The dimension of $S \cap U$ is equal to $\dim \Phi = n - p$, where p is the Morse index of the critical point. □

Here are the two exercises used in the proof of Lemma 15.

Exercise 7. Let $\mathrm{grad} : \mathbb{R}^n \to \mathbb{R}^n$ be the gradient vector field. We will use the fact that $\mathrm{grad}(0) = 0$ and dgrad_0 is an isomorphism. Suppose that $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a flow line converging to 0.

(a) Prove that γ is in L^2 if and only if $\mathrm{grad} \circ \gamma$ is in L^2. Hint: show that

$$|\gamma(t)| < c\,|\mathrm{grad} \circ \gamma(t)|$$

for t sufficiently large, for some $c > 0$.

(b) Prove that $\mathrm{grad} \circ \gamma$ is in L^2 using the relation

$$\frac{d}{dt}(\varphi \circ \gamma) = -g(\mathrm{grad} \circ \gamma, \mathrm{grad} \circ \gamma).$$

(c) Now that we know that γ is in L^2, conclude that γ' is also in L^2 using the gradient flow line equation.

(d) Conclude that any flow line γ converging to 0 is actually in $W^{1,2}$.

Exercise 8. Given a bounded open neighborhood U' of 0, one can always find a smaller open set $U \subset U'$ so that every trajectory γ starting in U either remains in U', or leaves U' and never returns to U.

(a) Pick U'' compactly supported in U' around 0 so that $g(\mathrm{grad}, \mathrm{grad}) > b$ on $U' \setminus U''$. Pick $U_\epsilon \subset U''$ so that $\max_{U_\epsilon} \varphi - \min_{U_\epsilon} \varphi < \epsilon$. Using the fact that

$$\frac{d}{dt}\varphi(t) = -g(\mathrm{grad}, \mathrm{grad}),$$

conclude that any trajectory starting and ending in U_ϵ must spend time less than $b^{-1}\epsilon$ in $U' \setminus U''$.

(b) Since $\partial U''$ is compact and contained in U', conclude a minimum amount of time needed to flow from $\partial U''$ to $\mathbb{R}^n \setminus U'$.

(c) Conclude that we can pick ϵ small enough so that any flow starting and ending in U_ϵ cannot leave U'. Taking $U = U_\epsilon$ proves the claim.

Appendix to April 4th Lecture

Definition 17. For $k \geq 1$, a C^k **Banach manifold** $\mathfrak{X}$ is a topological space covered by open sets homeomorphic to open subsets of Banach spaces, where the transitions functions are C^k maps. More precisely, a Banach manifold comes equipped with a maximal atlas of coordinate charts: $c : U_c \subset \mathfrak{X} \to c(U) \subset X_c$, where X_c is a Banach space, $c : U_c \to c(U)$ is homeomorphism onto an open set, and so that the transition homeomorphism

$$\rho_{21} = c_2 \circ c_1^{-1} : c_1(U_1 \cap U_2) \to c_2(U_1 \cap U_2)$$

is a C^k map.

We define a continuous map $A : \mathfrak{X} \to \mathfrak{Y}$ between C^k Banach spaces to be C^r $(r \leq k)$ if $c_2 \circ A \circ c_1^{-1}$ is a C^k map, for all choices of coordinates c_1, c_2 around x and $A(x)$ respectively.

Definition 18 (the tangent bundle)**.** For the purposes of this defintiion, let's agree to say that a Banach space is a topological vector space equipped with an equivalence class of complete metrics defining its topology.

For $k \geq 1$, let BMan_k be the category of C^k Banach manifolds, with C^k maps between them, let BSpace_k be the category of Banach spaces with C^k maps between them, and let Bun_k be the category of Banach space bundles over Banach manifolds. A morphism in Bun_k between bundles $E_1 \to B_1$ and $E_2 \to B_2$ is a pair (f, F) such that $f : B_1 \to B_2$ is a C^k map and F is a C^{k-1} section of the Banach space bundle $\mathrm{Hom}(B_1, f^*E_2) \to B_1$.

There is a functor $\tau : \mathrm{BSpace}_k \to \mathrm{Bun}_k$ sending a Banach space X to the trivial bundle $\tau(X) = X \times X \to X$, and which sends a morphism $f : X \to Y$ to the pair $(f, \mathrm{d}f)$, where $\mathrm{d}f$ is the C^{k-1} section of $\mathrm{Hom}(\tau(X), f^*\tau(Y)) = \mathrm{Hom}(X, Y) \times X \to X$.

The tangent bundle functor $T : \mathrm{BMan}_k \to \mathrm{Bun}_k$, is defined by three axioms:

(i) We require $T(f : X \to Y) = (f, \mathrm{d}f)$, i.e. $T(f)$ is a bundle map "over f" (where we abuse notation and use the symbol d for T as well as for τ).

(ii) The following diagram should commute up to a natural isomorphism $T \circ j \to \tau$:

$$\begin{tikzcd} \mathrm{BSpace}_k \arrow[r, "\tau"] \arrow[d, "j"] & \mathrm{Bun}_k \\ \mathrm{BMan}_k \arrow[ur, bend right, "T"'] & \end{tikzcd}$$

where j is the obvious inclusion functor $\mathrm{BSpace}_k \to \mathrm{BMan}_k$. This natural isomorphism should be thought of as part of the data of T.

(iii) If $i : U \to M$ is the inclusion of an open set, then the map $\mathrm{d}i : TU \to i^*TM$ is an isomorphism.

It is not very hard to show that this determines T up to unique natural isomorphism, i.e. if T' is another such functor then there is a unique natural isomorphism $T \to T'$ so that

$$\begin{tikzcd} \tau \arrow[r] \arrow[d] & T' \circ j \\ T \circ j \arrow[ur] & \end{tikzcd}$$

commutes.

April 11 (Ipsita)

Elliptic operators. We will always be looking at linear differential operators. We build on the difficulty of the operators and domains.

Constant coefficient operators on vector-valued functions on $\mathbb{R}^n$. We consider $D : (\mathbb{R}^n, \mathbb{R}^m) \to (\mathbb{R}^n, \mathbb{R}^m)$ given by

$$Du = \sum_{|\alpha \leq k} A_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}$$

where $A_\alpha \in \mathrm{Mat}(m, n)$.

Definition 19. a. The *total symbol* of D is defined as

$$\sigma_t(\xi) = \sum_{|\alpha \leq k} A_\alpha \xi^\alpha \qquad \text{for } \xi = (\xi_1, \ldots, \xi_n).$$

b. The principal symbol is given by

$$\sigma_p(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha \qquad \text{for } \xi = (\xi_1, \ldots, \xi_n).$$

.

One motivation for this definition comes from looking at the Fourier transform. Note

$$\widehat{Du} = \sigma_t(i\xi)\widehat{u} \implies Du = \int e^{ix\cdot\xi}\sigma_t(i\xi)d\xi.$$

Thus, we "turned" a differential operator into an integral operator.

Ellipticity of D have different meanings in different contexts but it always has something to do with $\sigma_{t,p}(\xi)$ being invertible outside of $\xi = 0$. An example of such a condition is

$$(\star) \qquad |\sigma_t(\xi)| \geq c|\xi|^n \text{ for all } x \in \mathbb{R}^n_\xi.$$

This is possibly relevant for Schwarz spaces.

The fundamental solution of D is a $G : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $DG = \delta$. By taking Fourier transform we get $\widehat{DG} = 1$, equivalently $\sigma_t(i\xi)\widehat{G} = 1$. Ellipticity conditions let you "divide" by the symbol to obtain a tempered distribution $G = \left(\frac{1}{\sigma_t(\xi)}\right)^\vee$ which solves $DG = \delta$. Similarly, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ we can obtain a tempered distribution $G = \left(\frac{1}{\sigma_t(\xi)}\right)^\vee * f$ which solves $DG = f$.

Exercise 9. Find a condition on the total symbol $\sigma_t(\xi)$ so that

$$D : \mathcal{S} \to \mathcal{S}$$

($\mathcal{S}$ representing Schwarz functions) is bijective. What fails with $(\star)$.

Variable coefficient operators on $\mathbb{R}^n$. We consider D almost the same as above but now A_α depends on $x \in \mathbb{R}^n$. So, in the definition of the differential operator and the symbols we replace A_α by $A_\alpha(x)$. Then the symbol also depends on $x \in \mathbb{R}^n$ and we denote it by $\sigma_{t,p}(x, \xi)$.

Notice that under a change of coordinates $x \mapsto x'$, $\sigma_p(\cdot, x)$ transforms to $\sigma_p(\cdot, x')$ as a section of $\mathrm{Sym}(T^*\mathbb{R}^n) \otimes \mathrm{Mat}(m, n)$. For example,

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j},$$

$$dx_i = \sum_j \frac{\partial x'_j}{\partial x_i} dx_j.$$

In contrast σ_t does not transform so nicely because lower order terms appear. For example,

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} = \sum_{j,l} \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j} \left(\frac{\partial x'_l}{\partial x_k} \frac{\partial}{\partial x'_l} \right) = \left(\sum_{j,l} \frac{\partial x'_j}{\partial x_i} \frac{\partial x'_l}{\partial x_k} \frac{\partial}{\partial x'_j} \frac{\partial}{\partial x'_l} \right) + (\text{first order terms}).$$

Ellipticity can be defined similar to the constant coefficient case, but we will not do it here because it will confuse us.

Operators on a closed manifold. Consider a closed manifold M and vector bundles E, F over M. Assume $\dim E = \dim F$. $D : C^\infty(M, E) \to C^\infty(M, F)$ is a differential operator if it looks like a differential operator on trivializations (plus some assumptions about locality). For D, $\sigma_{p,D} : \mathrm{Sym}^n(T^*M) \otimes F \to F$ is the principal symbols glued together. (Note that there exists a coordinate free definition.)

Definition 20. D is elliptic if $\sigma_{p,D}(v, \ldots, v)$ for $v \in T_m M, m \in M$ is invertible for $v \neq 0$.

Theorem 21. If D as above is elliptic, then it has finite dimensional kernel and cokernel.

We will not be proving this which is standard but non-trivial. It is crucial to this theorem that M is closed.

Definition 22. In this case, we define index of an elliptic operator D as

$$\mathrm{Ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D).$$

Index theorem for elliptic operators on closed manifolds.

Theorem 23.
a. Index of an elliptic operator (on a closed manifold) only depends on its principal symbol.
b. Index is constant on the connected components of the space of elliptic operators.

Gelfand(1960) proposed computing the index of an elliptic operator as a homotopical invariant of its symbol.

Atiyah–Singer (1963) announce they did it! The story of index calculations continues to much later.

Example 24. Consider an oriented and closed manifold M with a Riemannian metric. Then the hodge star is defined and we get

$$d + d^\star : \Omega^{even}(M) \to \Omega^{odd}(M)$$

is an elliptic differential operator. (Exercise: check!)

Let $\mathscr{H}$ denote the subspace of $\Omega^*(M)$ of harmonic forms. Note that $\omega \in \Omega^k(M)$ is harmonic if and only if

$$d\omega = d^\star \omega = 0.$$

Fact: Every de Rham cohomology class has exactly one harmonic representative. Using this fact we can conclude

$$\begin{aligned}
\ker(d + d^\star) &= \mathscr{H}^{even}, \\
\operatorname{coker}(d + d^\star) &= \ker(\mathscr{H}^{odd} \to \mathscr{H}^{even}) = \mathscr{H}^{odd}, \\
\operatorname{Ind}(d + d^\star) &= \dim(\mathscr{H}^{even}) - \dim(\mathscr{H}^{odd}) = \chi(M).
\end{aligned}$$

The topological side of Atiyah-Singer index theorem computes

$$\chi(M) = \int_M K$$

where K is the Euler form of TM constructed from curvature (generalised Gaussian curvature, constructed via Chern-Weil theory).

General form of the index is given by

$$\operatorname{Ind}(D) = \int_M ch(\sigma_{p,D}) Td(M)$$

where $ch(D)$ and $Td(M)$ are cohomology classes. The classes are not always easy to compute.

Later work showed that, for example for Dirac operators (which are most fundamental of elliptic operators), it is possible to construct special representatives like de Rham representatives. (One proof uses "heat kernel" techniques.)

What about manifolds with boundary?. Suppose X is a compact manifold with boundary $\partial X = Y$. Let E and F be vector bundles over X. For $D : C^\infty(X, E) \to C^\infty(X, F)$ elliptic, Fredholmness fails. (For example, $\bar{\partial} : C^\infty(\mathbb{D}, \mathbb{C}) \to C^\infty(\mathbb{D}, \mathbb{C})$ does not have a finite kernel.)

To solve this issue we need appropriate boundary conditions.

One **candidate** would be local boundary conditions like restricting to a subspace of $C^\infty(X, E)$ consisting of sections f such that f and/or its derivatives are prescribed point wise. (For example Dirichlet or von Neumann conditions.) It is not impossible to work with local boundary conditions. Sometimes it is possible to find nice local boundary conditions that lead to Fredholm operators. In fact, one can remove the "sometimes." But from the point of index theory we are unsure how nice the conditions can be. (Refer: "On general boundary value problems for elliptic operators" by Schulze et. al. if interested.)

Possibly one of the main insights of Atiyah–Patodi–Singer is that one could consider global boundary conditions for the index theory to work well.

Theorem 25. (Atiyah–Patodi–Singer I)
Consider X, Y, E, F and D as above.

Assume

a. D first order.
b. In a neighbourhood $I_u \times Y$ of Y (where parameter u is decreasing towards $\partial X = Y$), D should look like
$$D = \sigma_0(\frac{\partial}{\partial u} + A)$$
where σ is a bundle homeomorphism $E|_Y \to F|_Y$ ($E|_{I\times Y}$ is pullback from $E|_Y$, similar for F) and A is self-adjoint (with respect to inner product $\langle s, s'\rangle = \int\limits_Y h(s(y), s'(y))dy, s, s' \in C^\infty(X, E)$). Note here h is a fixed Hermitian metric on E and dy is with respect to a fixed measure on Y.

Let $C^\infty(X, E, P)$ denote the sections f such that the projection of $f|_Y$ to the non-negative eigenspace of A is zero.

Then $D : C^\infty(X, E, P) \to C^\infty(X, F)$ is Fredholm. Moreover
$$\mathrm{Ind}(D) = \int\limits_X \alpha_0(x)dx - \frac{h + \eta(0)}{2},$$
where

(i) $\alpha_0(x)$ is the constant term in the asymptotic expansion (as $t \to 0$) of
$$\sum e^{-t\nu'}|\phi'_\mu(x)|^2 - \sum e^{-t\mu''}|\phi''_\mu(x)|^2,$$
where $\mu', \phi'_\mu, (resp.\ \mu'', \phi''_\mu)$ denote the eigenvalues of $D^*D(resp.\ DD^*)$ on the double of X. Note that D and D^* naturally extend to operators on the double of X using special form of D near ∂.

(ii) $h = \dim\ker A$.

(iii) $\eta(s) = \sum\limits_{\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}$, where λ runs over eigenvalues of A. Here $\eta(s)$ converges absolutely for $Re(s) \gg 1$ and extends to a meromorphic function on the entire plane with finite value at $s = 0$.

Remark Turns out this case is not the most important for us, so don't worry too much about it. Next time we cover spectral flows, which is more important for us.

Example 26. Let us consider $\bar\partial : C^\infty(\mathbb{D}, \mathbb{C}) \to C^\infty(\mathbb{D}, \mathbb{C})$ given by
$$\bar\partial = \frac{d}{dt} - A$$
where $-A : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C}); e^{2\pi in\theta} \mapsto 2\pi ne^{2\pi in\theta}$ for e^t radial coordinates and $e^{i\theta}$ angular coordinates. Then $C^\infty(\mathbb{D}, \mathbb{C}, P)$ consists of those sections which have only negative Fourier coefficients. So, no holomorphic functions are in $C^\infty(\mathbb{D}, \mathbb{C}, P)$. Also, we can compute $\eta(s) = 0$ as $Re(s) \gg 1$ implies $\eta(0) = 0$.