April 23 (Lie/Dylan)

Consider the set up we were considering last time. On the manifold $\mathbb{S}^1 \times \mathbb{R}$, with coordinates (t, s), we considered the differential operator

$$D = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S,$$

on the trivial bundle $\mathbb{R}^{2n}$, where $S(s,t)$ is a smooth family of symmetric matrices $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which is constant as $s \to \pm\infty$, and J_0 is the "standard complex structure,"

$$J_0 = \operatorname{diag}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}).$$

Since this differential operator is of the form

$$D = \text{time } (s) \text{ derivative} + \text{self adjoint elliptic operator} + \text{lower order perturbation},$$

our previous analysis about such operators establishes the following fact

Fact. Suppose that $A(\pm\infty)$ has no imaginary eigenvalues.

Considered as a map $D : W^{1,2}(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^{2n})$, D is Fredholm, and its index is equal to the spectral flow index of the family of elliptic operators

$$A(s) = J_0 \frac{\partial}{\partial t} + S \text{ on } \mathbb{S}^1.$$

The main goal of this lecture is to compute the spectral flow index in terms of Maslov index of a certain family of symplectic matrices. To begin, let's determine the eigenvectors of $A(s)$ corresponding to the eigenvalue 0. It is clear that a section $\varphi : S^1 \to \mathbb{R}^{2n}$ is in the kernel of A

$$(*) \qquad J_0 \frac{\partial \varphi}{\partial t} + S(s,t)\varphi(t) = 0 \iff \varphi'(t) = J_0 S(s,t)\varphi(t).$$

Define $\Psi(s,t) \in \mathrm{Sp}(2n)$ by solving the ordinary differential equation

$$\frac{\partial}{\partial t}\Psi(s,t) = J_0 S(s,t)\Psi(s,t) \text{ and } \Psi(s,0) = \mathrm{id}.$$

Note that Ψ is defined on $\mathbb{R} \times \mathbb{R}$ (i.e. it is probably not periodic in the t direction). To see that Ψ is symplectic, observe that Ψ is symplectic at $t = 0$ and

$$\frac{\partial}{\partial t}\Psi^T J_0 \Psi = \Psi^T S J_0^T J_0 \Psi + \Psi^T J_0 J_0 S \Psi = 0.$$

It is clear that

$$\varphi(t) = \Psi(s,t)\varphi(0) \text{ is the unique solution to } (*) \text{ with initial condition } \varphi(0).$$

Most likely, φ will not be periodic. Since we require the domain of φ to be $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we obtain the bijection

$$\varphi \in \text{kernel of } A(s) \mapsto \varphi(0) \in \ker(1 - \Psi(s,1)).$$

As a corollary of this discussion, we conclude

$$A(\pm\infty) \text{ is hyperbolic } \iff \ker(1-\Psi(\pm\infty,1))=0.$$

To see why, observe that A is self-adjoint (because S is symmetric) and hence $A(\pm\infty)$ can only have imaginary eigenvalues if $A(\pm\infty)$ has non-zero kernel.

Claim 1. The spectral flow index of $A(s)$ is the signed intersection number of $\psi(1,s), s\in\mathbb{R}$ with the Maslov cycle.

Before we prove the claim, we will explore the Maslov cycle C in a bit more depth. Recall its definition

Definition 2. The **Maslov cycle** C is the subset of $\mathrm{Sp}(2n)$ consisting of the matrices with 1 as an eigenvalue. It has the following properties:

(i) C is a codimension 1 **stratified submanifold** of $\mathrm{Sp}(2n)$, in the sense that it can be written as a union of (non-closed) submanifolds

$$C = C_1\cup C_2\cup C_3\cup\cdots,$$

where $\dim C_{k+1} < \dim C_k$ and $\partial C_k \subset C_{k+1}\cup C_{k+2}\cdots$ (here ∂ denotes the "topological" boundary). We say that C is "codimension 1" because the "top strata" C_1 is codimension 1 inside $\mathrm{Sp}(2n)$.

(ii) The singularities of C have codimension ≥ 2 inside of C, in the sense that

$$\dim C_2 < \dim C_1 - 1.$$

This is important for obtaining a well-defined intersection number with C.

Definition 3. Let $\gamma:[0,1]\to\mathrm{Sp}(2n)$ be a smooth path which is **regular** in the sense γ has isolated intersections with C, $\gamma(0),\gamma(1)\notin C$, and whenever $\gamma(t)\in C$, the bilinear pairing

$$B_t^{\mathrm{mc}} = \langle -, V-\rangle \text{ is non-degenerate } \ker(1-\gamma(t))\otimes\ker(1-\gamma(t))\to\mathbb{R}. \tag{1}$$

where $J_0V\gamma(t)=\gamma'(t)$ for a *symmetric* matrix V. It is clear that B_t is symmetric. We define the **intersection number** of γ with C by the formula

$$\#(\gamma\cap C) = \sum_t \mathrm{signature}(B_t^{\mathrm{mc}}).$$

This sum is finite since $B_t=0$ if $\gamma(t)\notin C$ (and we have supposed isolated intersections). The bilinear form B_t^{mc} is called the **crossing form**.

Theorem 4. Suppose that γ_0,γ_1 are homotopic regular paths, where we require the homotopy γ_t to always satisfy $\gamma_t(0),\gamma_t(1)\notin C$. Then

$$\#(\gamma_0\cap C) = \#(\gamma_1\cap C).$$

Exercise 1 (stratifications)**.** In this exercise we will stratify the set of singular matrices $\mathbb{R}^{n\times m}$, $m\leq n$, to give some idea of how one might stratify C. Let $\Sigma_k=\{A:\dim\ker A = k\}$, so that $\Sigma_1 \cup \cdots \cup \Sigma_m$ is the set of singular matrices. We will prove that each Σ_i is a manifold, and compute their codimensions inside of $\mathbb{R}^{n\times m}$.

Pick an ℓ-dimensional subspace $\Lambda \subset \mathbb{R}^m$, and consider the open set U_Λ of matrices which are injective on Λ. We can form an $n-\ell$-dimensional bundle Φ on U whose fiber at A is $\mathbb{R}^n/A(\Lambda)$. The restriction of A to $\Lambda^\perp$ gives a section s of $\mathrm{Hom}(\Lambda^\perp, \Phi)$, and we make the crucial observation that $s = 0$ if and only if A has rank equal to exactly ℓ (this only holds on U_Λ).

Moreover, the section s is transverse to 0. This is fairly easy to see, since, we can arbitrarily perturb A on $\Lambda^\perp$. Therefore we have locally expressed the set of matrices of rank ℓ as the zero locus of the section of an $(m-\ell)(n-\ell)$ dimensional bundle.

Since $\dim\ker A + \mathrm{rank}(A) = m$, we conclude that $k = m-\ell$, and hence we have locally expressed the matrices of corank k as the zero locus of a transverse section of a $k(n-m+k)$ dimensional bundle. It follows that

$$\Sigma_k \text{ has codimension } k(n-m+k) \text{ inside of } \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n).$$

If $m = n$, then Σ_k has codimension k^2.

Remark. We leave it to the reader to ponder how similar ideas may be used to obtain a stratification of C with the advertised properties.

Exercise 2 (intersection numbers)**.** Let $C = C_1 \cup C_2 \cup \cdots$ be a closed stratified submanifold of M, and suppose $\mathrm{codim}(C_1) = k$ and $\dim C_2 < \dim C_1 + 1$. Suppose the top face C_1 has a co-orientation

Given a compact oriented k-dimensional manifold $(N, \partial N)$ and a map

$$f : (N, \partial N) \to (M, M \setminus C) \tag{$*$}$$

show that the intersection number of f with C is well-defined if we follow the recipe:

Homotope f through maps of the form $(*)$ so that f becomes disjoint from $C_2 \cup C_3 \cup \cdots$, and so that f is transverse to the top stata C_1, and then count the number of intersection points, with signs according to the orientation on N and the coorientation on C_1.

Why is the assumption that $\dim C_2 < \dim C_1 + 1$ necessary? Give an example of a stratified submanifold where this assumption fails, and where a homotopy invariant intersection number cannot be defined.

We can also define a crossing form for the spectral flow

Definition 5. Let $A(s)$ be a one parameter family of self-adjoint operators. Define the crossing form at time s by

$$B_s^{\mathrm{sf}} = \int_{\mathbb{S}^1} \langle -, \frac{\partial A}{\partial s}(-)\rangle \mathrm{d}t \text{ on } \ker A(s).$$

Claim 6. The bijection

$$\varphi \in \ker A(s) \mapsto \varphi(0) \ker(1 - \Psi(1, s))$$

respects the crossing forms B^{sf} and B^{mc}.

Proof. Fix some $s_0 \in \mathbb{R}$ and compute

$$
\begin{aligned}
B^{\mathrm{sf}}_{s_0}(\varphi) = \int_{S^1} \langle \varphi, \frac{\partial A}{\partial s}(s_0)\varphi\rangle dt &= \int_{S^1} \langle \varphi, \frac{\partial S}{\partial s}(s_0)\varphi\rangle dt \\
&= \int_{S^1} \langle \Psi(s_0,t)\varphi(0), \frac{\partial S}{\partial s}(s_0)\Psi(s_0,t)\varphi(0)\rangle \\
&= \int_{S^1} \langle \varphi(0), \Psi^*(s_0,t)\frac{\partial S}{\partial s}(s_0)\Psi(s_0,t)\varphi(0)\rangle.
\end{aligned}
\tag{1}
$$

Now define $\widehat{S}(s,t)$ to be the tangent vector of $\Psi(s,t)$ in the s-direction; more precisely, we use the identification of the tangent space at $\Psi(s,t)$ with the tangent space at $1 \in \mathrm{Sp}(2n)$:

$$
\frac{\partial \Psi}{\partial s} = J_0 \widehat{S}\Psi.
$$

By definition, the Maslov cycle crossing form on $\ker(1 - \Psi(1, s_0))$ is

$$
\begin{aligned}
B^{\mathrm{mc}}_{s_0}(\varphi(0)) = \langle \varphi(0), \widehat{S}(s_0,1)\varphi(0)\rangle &= \langle \Psi(s_0,1)\varphi(0), \widehat{S}(s_0,1)\Psi(s_0,1)\varphi(0)\rangle \\
&= \langle \varphi(0), \Psi^*(s_0,1)\widehat{S}(s_0,1)\Psi(s_0,1)\varphi(0)\rangle \\
&= \int_{S^1} \langle \varphi(0), \frac{\partial}{\partial t}(\Psi^*(s_0,t)\widehat{S}(s_0,t)\Psi(s_0,t))\varphi(0)\rangle
\end{aligned}
\tag{2}
$$

since $\widehat{S}(s,0) = 0$ as $\Psi(s,0)$ is identically 1.

We claim that

$$
\frac{\partial}{\partial t}(\Psi^*\widehat{S}\Psi) = \Psi^*\frac{\partial S}{\partial s}\Psi \tag{$*$}
$$

To see this, we simply compute

$$
\begin{aligned}
\mathrm{LHS} &= \frac{\partial \Psi^*}{\partial t}\widehat{S}\Psi + \Psi^*\frac{\partial}{\partial t}(\widehat{S}\Psi) \\
&= -\Psi^* S J_0 \widehat{S}\Psi - \Psi^* J_0 \frac{\partial}{\partial t}((J_0\widehat{S}\Psi) \\
&= -\Psi^* S J_0 \widehat{S}\Psi - \Psi^* J_0 \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Psi \\
&= -\Psi^* S J_0 \widehat{S}\Psi - \Psi^* J_0 \frac{\partial}{\partial s}J_0 S\Psi \\
&= -\Psi^* S J_0 \widehat{S}\Psi + \Psi^* \frac{\partial}{\partial s}(S\Psi) \\
&= -\Psi^* S J_0 \widehat{S}\Psi + \Psi^* \frac{\partial S}{\partial s}\Psi + \Psi^* S J_0 \widehat{S}\Psi \\
&= \mathrm{RHS},
\end{aligned}
$$

as desired. Comparing (1) and (2), and using ($*$), completes the proof of the claim. □

Exercise 3. Check that in the generic situation, the sign of the intersection of the spectral flow with the zero line is given by the sign of the B^{sf} crossing form.

To finish the proof of the computation of spectral flow index, consider the contractible loop of sympletic matrices $\Psi_- = \Psi(-\infty, t)$, $\Psi(s, 1)$, $\Psi_+(+\infty, t)$. The additivity of the Conley-Zehnder index applied to this path produces

$$\mu_{CZ}(\Psi_-) + \mu_{CZ}(\Psi(s,1)) - \mu_{CZ}(\Psi_+) = 0$$

which gives spectral flow index $\mu_{CZ}(\Psi_+) - \mu_{CZ}(\Psi_-)$.

Maslov cycle in $\mathrm{Sp}(2)$**.** Recall the polar decomposition: for any matrix $A \in GL_n(\mathbb{R})$ there is unique $U \in O(n)$ and P positive definite symmetric matrix such that $A = UP$.

Claim 7. Let $A \in Sp(2n)$ and $A = UP$ be its polar decompostion. Then $J_0 U = U J_0$, i.e. $U \in U(n) = O(2n) \cap \mathrm{GL}_n(\mathbb{C})$.

Proof. $-J_0 A J_0 = (A^*)^{-1} = UP^{-1}$ $-J_0 A J_0 = J_0 U J_0 J_0 P J_0$ is also a polar decompostion except that $J_0 P J_0$ is negative definite now $\langle J_0 P J_0 v, v\rangle = -\langle P J_0 v, J_0 v\rangle < 0$. By changing signs, we see that we have another polar decomposition. Since there is a unique polar decomposition, we conclude $-J_0 U J_0 = U$. □

Using this polar decomposition (and recalling that $\mathrm{Sp}(2) = \mathrm{SL}_2(\mathbb{R})$), we obtain the diffeomorphism

$$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}(2) \times \text{symmetric positive definite matrices of determinant } 1 \cong S^1 \times \mathbb{R}^2.$$

This can also be seen by recalling the transitive group action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane by Möbius transformations. Thus $\mathrm{SL}_2(\mathbb{R})$ is an open solid torus.

We can be quite explicit about the Maslov cycle in this low dimensional case. If A is a symplectic matrix, we compute

$$\det(1 - A) = 2 - \mathrm{tr}(A),$$

and hence $C = \{A : \mathrm{tr}(A) = 2\}$.

Exercise 4. Let C be the Maslov cycle in $\mathrm{Sp}(2)$. Prove that $C_1 = \{A : \dim\ker(1 - A) = 1\}$ is cut transversally by the function $A \mapsto \det(1 - A)$. Prove that $C_2 = \{A : \dim\ker(1 - A) = 2\}$ is $\{1\}$. Thus we have a stratification

$$C = C_1 \cup \{1\}\,.$$

What does the Maslov cycle look like in a neighborhood of 1?