April 22 (Lie/Dylan)

Consider the set up we were considering last time. On the manifold $\mathbb{S}^1 \times \mathbb{R}$, with coordinates (t,s), we considered the differential operator

$$D = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S,$$

on the trivial bundle $\mathbb{R}^{2n}$, where $S(s,t)$ is a smooth family of symmetric matrices $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which is constant as $s \to \pm\infty$, and J_0 is the "standard complex structure,"

$$J_0 = \operatorname{diag}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}).$$

Since this differential operator is of the form

$$D = \text{time } (s) \text{ derivative} + \text{self adjoint elliptic operator} + \text{lower order perturbation},$$

our previous analysis about such operators establishes the following fact

Fact. Considered as a map $D : W^{1,2}(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^{2n})$, D is Fredholm, and its index is equal to the spectral flow index of the family of elliptic operators

$$A(s) = J_0 \frac{\partial}{\partial t} + S \text{ on } \mathbb{S}^1.$$

We also have $\psi(s,t)$ $\mathbb{R}^2$ family of symplectic matrices.

Want to compute the spectral flow index in terms of Maslov matrices. What are the eigenvectors of $A(s)$ corresponding to the eigenvalue 0?

i.e. find $\phi : S^1 \to \mathbb{R}^{2n}$ such that

$$J_0 \frac{\partial \phi}{\partial t} + s(s,t)\phi(t) = 0$$

$$\phi' = J_0 S(s,t)\phi(t)$$

Claim: $\phi(t) = \psi(s,t)\phi(0)$

This is easy to verify:

$$\phi'(t) = \frac{\partial \psi}{\partial t}\phi(0) = J_0 S(s,t)\psi\phi(0) = J_0 S(s,t)\phi(t)$$

Also need $\phi(0) = \phi(1) = \psi(s,1)\phi(0)$

So should be eigenvector of $\psi(s,1)$ of eigenvector 1, $\ker(A(s)) = \ker(1 - \psi(s,1))$. Thus hyperbolicity can be defined by saying that $\psi_\pm(1)$ has no eigenvalue 1. (Because ψ_t is symmetric, and thus has real eigenvalues and hyperbolicity means 0 is not an eigenvalue).

Claim: Spectral flow index of $A(s)$ = signed intersection nnumber of $\psi(1,s), s \in \mathbb{R}$ with the Maslov cycle.

More about the Maslov cycle:

$C = Sp(2n) - Sp^*(2n)$,

- Codim 1 inside $p(2n)$.
- Singularity in codim 2.
- We can assign contribution for intersection of a path γ under a mild transversality assumption (regular). View tangent vectors $\gamma'(t)$ as symmetric matrices S. We say that γ at a intersextion at time x is regular if $< \cdot, S \cdot >$ is non-degenerate at $\ker(1 - \gamma(x))$. If this is the case, then the contribution is defined to be the signature of this nondegenerate bilinear form.

$< \cdot, S \cdot >$ on $\ker(1 - \psi(1, s))$ is called the crossing form. Crossing form for spectral flow is $\int_{S^1} < \cdot, \frac{\partial A}{\partial s}() \cdot >$ on $\ker(A(s))$.

Claim:

$$\ker(Id - \psi(1, s)) \xrightarrow{\cong} \ker(A(s))$$
$$\phi(0) \mapsto \psi(s, t)\phi(0)$$

repsects these crossing form. Take s_0, then

$$\begin{aligned} \int_{S^1} < \phi, \frac{\partial A}{\partial s}(s_0)\phi > dt &= \int_{S^1} < \phi, \frac{\partial S}{\partial s}(s_0)\phi > dt \\ &= \int_{S^1} < \psi(s_0, t)\phi(0), \frac{\partial S}{\partial s}(s_0)\psi(s_0, t)\phi(0) > \\ &= \int_{S^1} < \phi(0), \phi^*(s_0, t)\frac{\partial S}{\partial s}(s_0)\psi(s_0, t)\phi(0) > . \end{aligned}$$

Define $\widehat{S}(s, t)$ to be the tangent vector of $\psi(s, t)$ in the s-direction:

$$\frac{\partial \psi}{\partial s} = J_0 \widehat{S} \psi$$

Crossing form on $\ker(1 - \psi(1, s))$ is

$$\begin{aligned} < \phi(0), \widehat{S}(s_0, 1)\phi(0) > &=< \psi(s_0, 1)\phi(0), \widehat{S}(s_0, 1)\psi(s_0, 1)\phi(0) > \\ &=< \phi(0), \psi^*(s_0, 1)\widehat{S}(s_0, 1)\psi(s_0, 1)\phi(0) > \\ &= \int_{S^1} < \phi(0), \frac{\partial}{\partial t}(\psi^*(s_0, t)\widehat{S}(s_0, t)\psi(s_0, t))\phi(0) > \end{aligned}$$

since $\widehat{S}(s, 0) = 0$ as $\psi(s, 0) = id$.

Claim: $\frac{\partial}{\partial t}(\psi^* \widehat{S} \psi) = \psi^* \frac{\partial S}{\partial s} \psi$

Proof.

$$\begin{aligned}
\text{LHS} &= \frac{\partial \psi^*}{\partial t}\widehat{S}\psi + \psi^* \frac{\partial}{\partial t}(\widehat{S}\psi) \\
&= -\psi^* S J_0 \widehat{S}\psi - \psi^* J_0 \frac{\partial}{\partial t}((J_0\widehat{S}\psi) \\
&= -\psi^* S J_0 \widehat{S}\psi - \psi^* J_0 \frac{\partial}{\partial t}\frac{\partial}{\partial s}\psi \\
&= -\psi^* S J_0 \widehat{S}\psi - \psi^* J_0 \frac{\partial}{\partial s} J_0 S\psi \\
&= -\psi^* S J_0 \widehat{S}\psi + \psi^* \frac{\partial}{\partial s}(S\psi) \\
&= -\psi^* S J_0 \widehat{S}\psi + \psi^* \frac{\partial S}{\partial s}\psi + \psi^* S J_0 \widehat{S}\psi
\end{aligned}$$

□

Exercise 1. Check that in the generic situation, the sign of the intersection of the spectral flow with the zero line is given by the sign of the crossing form.

To finish the proof of the computation of spectral flow index,

$$\mu_{CZ}(\psi_-) + \mu(\psi(s,1)) - \mu_{CZ}(\psi_+) = 0$$

gives spectral flow index $= \mu_{CZ}(\psi_+) - \mu_{CZ}(\psi_-)$.

Maslow cycle in $Sp(2)$.

Aside in polar decomposition: for any matrix $A \in GL_n(\mathbb{R})$ there is unique $U \in O(n)$ and P positive definite symmetric matrix such that $A = UP$.

Claim: Let $A \in Sp(2n)$ and $A = UP$ be its polar decompostion. Then $J_0U = UJ_0$, i.e. $U \in U(n)$.

Proof. $-J_0AJ_0 = (A^*)^{-1} = UP^{-1}$ $-J_0AJ_0 = J_0UJ_0J_0PJ_0$ is also a polar decompostio except that J_0PJ_0 is negative definite now $< J_0PJ_0v, v >= -< PJ_0v, J_0v >< 0$. Thus $-J_0UJ_0 = U$.

□

The generalization to general Lie group is Cartan decomposition.

Maslow ncycle in $Sp(2)$: $SL(2) \cong SO(2) \times$symmetric positive definite matrices of determinant1 $\cong S^1 \times \mathbb{R}^2$. Thus $SL_2(\mathbb{R})$ is an open solid torus. Maslove cycle $= \{A \in SL_2(\mathbb{R}) | trA = 2\}$.

Back to Morse theory to ocomplete the analytic program.

(1) functional step
(2) Fredholm/indexx theory
(3) transversality
(4) compactness
(5) gluing

(6) orientation(not today)

M clsoed manifold, f Morse function, g (generic) Riemannian metric.

$$\gamma : \mathbb{R}_+ \to M gamma' + \mathrm{grad}_f \circ \gamma = 0$$

Let x, y be critical point of f, a $\gamma \to x, t \to -\infty$ and $\gamma \to y, t \to +\infty$ and satisified the gradient flow equation is called a connecting trajectory.

$\mathcal{M}(x, y) = \{connectingtrajectorybetweenxandy\}$. And $\widehat{\mathcal{M}}$ is the unparametrized connecting trajectory.

We want to define a chain complex (M_i, ∂). $CM := \mathbb{Z}_2 < criticalpointsofMorseindexi >$. $\partial : CM_{i+1} \to CM_i :< \partial x, y >= \#\widehat{M}(x, y)$

Need to prove ∂^2 is 0.

Remark. Without continuation maps, homotopies between continuation maps etc, the theory is noncomplete.

1) Need to construct a Banach bundle $\mathcal{E} \to \mathcal{P}$ where $\mathcal{P}$ is a Banach manifold and a section s and such that $s^{-1}(0) = M(x, y)$.

$\mathcal{P}$ is a Banach manifold of pathes $x \to y$. The charts of $\mathcal{P}$ looks llike the following:

Assume $\gamma : \mathbb{R} \to M$ smooth for simplicity with some kind of asymptotics condition at $\pm\infty$ which makes them converge to x and y.

$U \subset X(\gamma^* TM)$ open where X denotes a Banach space containing $C_c(\mathbb{R}; \gamma^* TM)$ (I assume they are dense).

$\implies U \xrightarrow{arrow} \mathcal{P}$. Moreover the aymptotic condition that γ satisfied should also be satified for smooth paths inside the image of U.

Two solutions:

(1) Schwarz's book: roughly, replace the domain $\mathbb{R}$ with $[-1, 1]$, noting that $(-1, 1)$ is diffeo to $\mathbb{R}$ and use the smooth structure on $[-1, 1]$, asyptotic condition is now $-1 \mapsto x, +1 \mapsto y$.

$$X = H^{1,2}_{\mathbb{R}}(\gamma^* smallenoughdiskinTM)$$

(2) Floer's approach: Usse slightly modified Sobolev norm. $W^{k,p,\sigma,\sigma_+}(\mathbb{R}_t, \mathbb{R}) := \{f \in W^{k,p}_{loc} | \, \|f\|_{k,p,\sigma,\sigma_+} < \infty\}$
  $\|f\|_{k,p,\sigma,\sigma_+} = \|exp((\beta(t)\sigma_+ + \beta(-t)\sigma_-)t) \cdot f\|_{k,p}$
  This is a Banach space.
  Aymptotic condition near $\pm\infty$ is γ should be given by exponentiating a family of vector of at TM_x on $(-\infty, \rho)$. and satisfy true exponential decay condition for some $\sigma_- < 0$.
  The above characterization of chart of Banach space gives tangent spaces of pathes that satisfy the aymptotic condition. (Need Proof).

The vector bundle is locally just $W^{k-1,p,\sigma,\sigma_+}(\mathbb{R}_t, \mathbb{R})$ and the section goes as $\partial / \partial t + \mathrm{grad} \circ$

2) With a lot of pain, most previous work generalize here. But the linearization is a little more involved because because we are exponentiallly sections not just adding them: the right operator is $\partial/\partial t + \Gamma$ the Christoffel symbols are involved.

3) Consequences of parametric transversality theorem for Banach manifolds.

Proposition 1. G, M Banach manifolds, $E \to M$ is a Banach space bundle. $\Phi : G \times M \to E$ smooth G-parametric sections (Family of sections, try to find a generic one such that intersection with 0 is a true manifold), there is a countable trivialization, i.e.

$$\{(U, \psi | U \subset M open, \psi : E|_U \to E_u \times U\}$$

such that for every member:

- 0 is a regualr value of $pr \circ \psi \circ \Phi : G \times U \to E_u$
- $pr \circ \psi \circ \Phi_g : U \to E_u$ is a Fredholm map of Banach manifold (All tangent map are) of index r for every $g \in G$.

Then there is a G-set (Baire set) $\Sigma \subset G$ such that $Z_g = \{\Phi_g(m) = 0\}$ is a closed submanifold of M for $g \in \Sigma$

For us, G = Banach manifold of Riemannian metric, If we choose G large enough, 1 is indeed satified but it is not a general theorem, need to make computations.