

Appendix: Elliptic regularity for first order operators

First order operators.

Definition 1. Let E_1, E_2 be complex vector bundles over Y .

A first order differential operator is a \mathbb{C} -linear map

$$L : \Gamma(E_1) \rightarrow \Gamma(E_2)$$

with the property that for all \mathbb{C} -valued functions f

$$[L, f] = L \circ f - f \circ L : \Gamma(E_1) \rightarrow \Gamma(E_2)$$

is given by a tensor, i.e. is induced by a section of $\text{Hom}_{\mathbb{C}}(E_1, E_2)$. In this definition we use the fact that $\Gamma(E_i)$ are modules over the ring $\Omega^0(\mathbb{C})$.

Exercise 1. Prove that first order differential operators are **local** in the sense that if $s \in \Gamma(E_1)$ is supported in a closed set K , then $L(s)$ is also supported in K .

Use this to show that L is induced by a map of sheaves $\Gamma(E_1; -) \rightarrow \Gamma(E_2; -)$.

Example 2 (The structure of differential operators). Let s_1, \dots, s_n and r_1, \dots, r_m be local frames for $\Gamma(E_1)$ and $\Gamma(E_2)$, respectively. Let x_1, \dots, x_d be local coordinates on the base. If L is a first order differential operator then

$$L\left(\sum_{i=1}^n a_i s_i\right) = \sum_{i=1}^n [L, a_i] s_i + a_i L s_i.$$

We can compute the tensor $[L, a_i]_x \in \text{Hom}(E_{1,x}, E_{2,x})$ using the following trick. Near a given point p_0 , say $x(p_0) = c$ we can write

$$a_i(p) = a_i(p_0) + \sum_{j=1}^n b_{ij}(p)(x_j(p) - c_j),$$

for some smooth functions b_{ij} . Moreover it is clear that

$$da_i(p_0) = \sum_j b_{ij}(p_0) dx_j(p_0) \iff \frac{\partial a_i}{\partial x_j} = b_{ij}(p_0).$$

Now near p_0 write

$$\begin{aligned} [L, a_i] &= [L, a_i(p_0)] + \sum_{j=1}^n [L, b_{ij}(x_j - c_j)] = 0 + \sum_{j=1}^n b_{ij} [L, x_j - c_j] + [L, b_{ij}](x_j - c_j) \\ &= \sum_{j=1}^n b_{ij} [L, x_j] + [L, b_{ij}](x_j - c_j) \end{aligned}$$

Evaluating this tensor at p_0 yields (recall $x_j(p_0) = c_j$)

$$[L, a_i](p_0) = \sum_{j=1}^n b_{ij}(p_0) [L, x_j] = \sum_{j=1}^n \frac{\partial a_i}{\partial x_j}(p_0) [L, x_j].$$

But p_0 was arbitrary. It follows that

$$L(s) = L\left(\sum_{i=1}^n a_i s_i\right) = \sum_{i,j} \frac{\partial a_i}{\partial x_j} [L, x_j] s_i + a_i L(s_i).$$

In the presence of a connection, we can use the above to say that locally

$$L(s) = A(\nabla s) + B(s),$$

where A, B are tensors:

$$A \in \Gamma \operatorname{Hom}(\operatorname{Hom}(TY, E_1), E_2) \text{ and } B \in \Gamma \operatorname{Hom}(E_1, E_2).$$

We claim this actually holds globally. To see this, cover Y by a locally finite open cover $\{U_\alpha\}$, with associated partition of unity $\{\rho_\alpha\}$. Assume that for sections s_α supported in U_α we have

$$L(s_\alpha) = A_\alpha(\nabla s_\alpha) + B_\alpha(s_\alpha).$$

Then for any s

$$L(s) = L\left(\sum_\alpha \rho_\alpha s\right) = \sum_\alpha A_\alpha(\rho_\alpha \nabla s + d\rho_\alpha \otimes s) + B_\alpha(\rho_\alpha s) = A(\nabla s) + B(s),$$

provided we define

$$A = \sum_\alpha \rho_\alpha A_\alpha \text{ and } B = \sum_\alpha A_\alpha(d\rho_\alpha \otimes -) + \rho_\alpha B_\alpha(-).$$

By local finiteness, A, B are well-defined sections of the appropriate bundles. We have shown

Theorem 3. Let L be a first order differential operator $E_1 \rightarrow E_2$. Let ∇ be a connection on E_1 . Then there are sections A, B of $\operatorname{Hom}(\operatorname{Hom}(TY, E_1), E_2)$ and $\operatorname{Hom}(E_1, E_2)$, respectively, so that

$$L(s) = A(\nabla s) + B(s).$$

Exercise 2. Prove that A, B are uniquely determined by L . Hints: pick $s_0 \in E_{1,p_0}$ and a section s so $s(p_0) = s_0$, and so ∇s vanishes at p_0 (this can be achieved by parallel-transporting along radial lines). Conclude that B is completely determined by L .

Now for any element $\theta \in \operatorname{Hom}(TY, E_1)$, extend θ to a section of $\Omega^1(E_1)$. Then we can construct a section s (in a neighborhood of p_0) by parallel transporting along radial lines so that

$$\partial_r \lrcorner \nabla s = \partial_r \lrcorner \theta,$$

where ∂_r is the radial vector field. A-priori s is smooth when pulled back to the blow up $\mathbb{S}^{d-1} \times [0, \infty)$. It can be shown that s is smooth when considered as a section on \mathbb{R}^d . The key is that its (covariant) partial derivatives exist and are continuous functions on \mathbb{R}^d , and agree with θ at p_0 . Then, conclude that $A_{p_0}((\nabla s)_{p_0}) = A_{p_0}(\theta)$. Conclude that A is determined by L .

Sobolev embedding/compactness theorems.

Definition 4. Let g be a Riemannian metric on Y , an oriented Riemannian manifold (potentially with boundary), let $\langle -, - \rangle$ be an inner product on E (anti-linear in the first factor). Let ∇ denote the Levi-Civita connection on TY , and also a connection on E compatible with its metric (where there is hopefully no chance of confusion).

For codimension 0 submanifolds with boundary U , we define the space $L^p(U, E_1)$ by duality: $L^p(U, E_1)$ consists of distributions $C_c^\infty(\text{int}(U), E_1) \rightarrow \mathbb{C}$ satisfying the estimate

$$\varphi \in L^p(U, E_1) \implies |(\varphi, f)| \leq C_\varphi \left[\int_U |f|^q \, d\text{Vol}_g \right]^{1/q},$$

where $|f| = \langle f, f \rangle^{1/2}$, and q is the Hölder conjugate to p . We define the L^p norm on $L^p(U, E_1)$ by duality:

$$\|\varphi\|_{L^p} = \sup \left\{ |(\varphi, f)| : f \in C_c^\infty(\text{int}(U), E_1) \text{ and } \left[\int_U |f|^q \, d\text{Vol}_g \right]^{1/q} = 1 \right\}.$$

Exercise 3. Let φ be a continuous section of E_1 compactly supported in \overline{U} . It is clear that

$$\left[\int_U |\varphi|^p \, d\text{Vol}_g \right]^{1/p} < \infty.$$

Then φ induces an element of $L^p(U, E_1)$ by

$$(\varphi, f) = \int_U \langle \varphi(x), f(x) \rangle \, d\text{Vol}_g(x),$$

and

$$\|\varphi\|_{L^p(U, E_1)} = \left[\int_U |\varphi|^p \, d\text{Vol}_g \right]^{1/p}.$$

Hint: without loss, suppose that the integrals of $|\varphi|^p$ and $|f|^q$ are 1. Estimate

$$|\langle \varphi(x), f(x) \rangle| \leq |\varphi(x)| |f(x)| \leq \frac{1}{p} |\varphi(x)|^p + \frac{1}{q} |f(x)|^q,$$

to conclude that $\|\varphi\|_{L^p} \leq 1$. For the converse, consider smooth approximations (i.e. use convolution) to

$$\frac{\varphi}{\epsilon + |\varphi|^{2-p}}$$

and show that

$$\|\varphi\|_{L^p} \geq \int |\varphi|^p \frac{|\varphi|^2}{\epsilon |\varphi|^p + |\varphi|^2} \, d\text{Vol}_g.$$

By continuity of the integral $C^0 \rightarrow \mathbb{R}$ (on the support of φ) conclude that

$$\|\varphi\|_p \geq 1.$$

(where we are still assuming we normalized φ so $|\varphi|^p$ had integral 1).

Theorem 5. The continuous sections compactly supported in $\text{int}(U)$ are dense in $L^p(U, E_1)$.

Remark. This is a fairly deep theorem.

Corollary 6. Using convolution, prove that $C_c^\infty(\text{int}(U), E_1)$ is dense in $L^p(U, E_1)$. Conclude that the dual of $L^p(U, E_1)$ is naturally identified with $L^q(U, E_1)$. More precisely, there is a unique pairing

$$(\varphi, f) \in L^p \times L^q \mapsto \mathbb{C}$$

extending the pairing of L^p on C^∞ and vice-versa, so that

$$|(\varphi, f)| \leq \|\varphi\|_{L^p} \|f\|_{L^q},$$

and that this pairing establishes the bijection $(L^p)^* \simeq L^q$.

Definition 7 (The adjoint of a tensor). Let E, F be bundles with metrics denoted $\langle -, - \rangle$. We obtain a metric on $\text{Hom}(E, F)$ by identifying $\text{Hom}(E, F) \simeq \text{Hom}(E, \mathbb{C}) \otimes F$. The induced norm is equivalent to the operator norm

$$|\Phi|_{\text{oper}} = \sup \{ |\Phi(x)|_F : |x|_E = 1 \}.$$

Given a smooth section $\Phi \in \text{Hom}(E, F)$, we observe that

$$\int \langle \Phi(e), f \rangle \, d\text{Vol} = \int \langle e, \Phi^*(f) \rangle \, d\text{Vol},$$

where Φ^* is the pointwise adjoint. Therefore we can define the evaluation of Φ on an E -valued distribution via duality: for distributions valued in E , let $\Phi(e)$ be the distribution satisfying

$$(\Phi(e), f) = (e, \Phi^*(f)).$$

Exercise 4. If f is in $L^p(E)$, and Φ is a smooth section of $\text{Hom}(E, F)$ so that $|\Phi| < C$ is uniformly bounded on Y , then $\Phi(f)$ is in $L^p(E)$, and moreover

$$\|\Phi(f)\|_{L^p} \leq C \|f\|_{L^p}.$$

Hint: use the fact that L^p norm is defined by duality.

Exercise 5. If f is a continuous section of E , show that the weak definition of $\Phi(f)$ agrees with the definition of $\Phi(f)$ as a continuous section.

Definition 8 (connections and metrics on induced bundles). Let ∇ be a connection on E and also denote the Levi-Civita connection on (Y, g) . Suppose that ∇ is compatible with $\langle -, - \rangle$. We define a metric on $\text{Hom}(TM^{\otimes k}, E) \simeq T^*M^{\otimes k} \otimes E$ by

$$\langle \theta_1 \otimes \cdots \otimes \theta_k \otimes e, \theta'_1 \otimes \cdots \otimes \theta'_k \otimes e' \rangle = g(\theta_1, \theta'_1) \cdots g(\theta_k, \theta'_k) \langle e, e' \rangle.$$

We define a connection by

$$\nabla(\theta_1 \otimes \cdots \otimes \theta_k \otimes e) = \theta_1 \otimes \cdots \otimes \theta_k \otimes \nabla e + \sum_i \theta_1 \otimes \cdots \otimes \nabla \theta_i \otimes \cdots \otimes \theta_k \otimes e.$$

Exercise 6. Check that the induced connection on $(T^*M)^{\otimes k} \otimes E$ is compatible with the metric.

Definition 9 (the adjoint of the covariant derivative). Suppose that we have a local orthonormal frame y_1, \dots, y_n of TY . Let $\theta_i = g(y_i, -)$.

For smooth compactly supported sections $s \in \Gamma(E)$, $\lambda \otimes r \in \Gamma(\text{Hom}(TM, E))$ we compute $\nabla s = \sum \theta_i \otimes \nabla_{y_i} s$ and hence

$$\langle \nabla s, \lambda \otimes r \rangle = \sum_i g(\theta_i, \lambda) \langle \nabla_{y_i} s, r \rangle = \sum_i g(\theta_i, \lambda) (y_i \lrcorner d \langle s, r \rangle) - g(\theta_i, \lambda) \langle s, \nabla_{y_i} r \rangle,$$

so

$$\langle \nabla s, \lambda \otimes r \rangle = \int X_\lambda \lrcorner d \langle s, r \rangle d\text{Vol}_g - \langle s, \nabla_{X_\lambda} r \rangle d\text{Vol}_g,$$

where $X_\lambda = \sum_i g(\theta_i, \lambda) y_i$ (X_λ is the g -“dual” of λ).

A bit of differential form gymnastics produces

$$0 = X_\lambda \lrcorner (d \langle s, r \rangle \wedge d\text{Vol}) = (X_\lambda \lrcorner d \langle s, r \rangle) d\text{Vol} - d \langle s, r \rangle (X_\lambda \lrcorner d\text{Vol})$$

$$0 = (X_\lambda \lrcorner d \langle s, r \rangle) d\text{Vol} - d(\langle s, r \rangle (X_\lambda \lrcorner d\text{Vol})) + \langle s, r \rangle \text{div}(X_\lambda) d\text{Vol},$$

where $\text{div}(X_\lambda) d\text{Vol} = d(X_\lambda \lrcorner d\text{Vol})$. Since the integral of a compactly supported exact form is 0, we conclude

$$\langle \nabla s, \lambda \otimes r \rangle = \int \langle s, \text{div}(X_\lambda) r - \nabla_{X_\lambda} r \rangle d\text{Vol}_g.$$

Conclude that ∇_X has formal adjoint

$$(*) \quad \nabla^* : \lambda \otimes r \mapsto \text{div}(X_\lambda) r - \nabla_{X_\lambda} r.$$

One can easily check that this is a first order differential operator. While we proved this under the assumption that it has an orthonormal coordinate frame, it is clear that X_λ doesn't actually depend on the particular orthonormal coordinate frame, and hence ∇^* defined in $(*)$ is the adjoint of ∇ on any bundle E .

Definition 10. Let φ be a distribution $C^\infty(\text{int}(U), E) \rightarrow \mathbb{C}$. The weak derivative $\nabla \varphi$ is the distribution valued in the bundle $\text{Hom}(TM, E)$ defined by

$$(\nabla \varphi, \Phi) = (\varphi, \nabla^* \Phi),$$

where $\Phi \in \Gamma \text{Hom}(TM, E)$ is a test function.

Given a smooth vector field X , we can consider the tensor

$$X \lrcorner (-) : \Gamma \text{Hom}(TM, E) \rightarrow \Gamma E.$$

This defines a distributional evaluation $\text{Hom}(TM, E) \rightarrow E$. In this fashion, we can define the evaluations $\nabla_X \varphi$ for any distribution φ .

Exercise 7. Show that

$$(\nabla_X \varphi, \rho) = (\varphi, \nabla_X^* \rho),$$

where

$$\nabla_X^* \rho = \operatorname{div}(X)\rho - \nabla_X \rho.$$

Definition 11 (Sobolev Spaces). Let U be a codimension 0 submanifold of Y . We define the space $W^{k,p}(U, E)$ by recursion $W^{0,p}(U, E) = L^p(U, E)$, and $W^{k,p}(U, E)$ consists of all distributions φ whose weak derivative $\nabla \varphi$ lies in $W^{k-1,p}(U, \operatorname{Hom}(TY, E))$.

If the tangent bundle of U has global orthonormal frame y_1, \dots, y_n , it is equivalent to suppose that

$$\nabla_{y_1} \varphi, \dots, \nabla_{y_n} \varphi \in W^{k-1,p}(U, E),$$

Indeed the decomposition

$$\nabla \varphi = \sum_i \theta_i \otimes \nabla_{y_i} \varphi$$

identifies $\operatorname{Hom}(TY, E)$ with $E^{\oplus n}$ isometrically, and so the induced “weak” identification induces an isometry on L^p . We put a metric on $W^{k,p}(U, E)$ via it’s inclusion into

$$(*) \quad W^{k,p}(E) \rightarrow L^p(E) \oplus L^p(\operatorname{Hom}(TM, E)) \oplus \dots \oplus L^p(\operatorname{Hom}(TM^{\otimes k}, E)).$$

Exercise 8. Prove that the inclusion in $(*)$ is a closed inclusion. Hint: arguing by induction, it suffices to show that if

$$u_n \rightarrow u_\infty \text{ in } W^{k-1,p} \text{ and } \nabla u_n \rightarrow w_\infty \text{ in } W^{k-1,p},$$

then $\nabla u_\infty = w_\infty$. It will follow that $u_n \rightarrow u_\infty$ in $W^{k,p}$. The following estimate

$$(\nabla u_\infty - w_\infty, \varphi) = (u_\infty - u_n, \nabla^* \varphi) + (\nabla u_n - w_\infty, \varphi)$$

and taking the limit $n \rightarrow \infty$ proves that $\nabla u_\infty = w_\infty$.

Exercise 9. For $p \in (1, \infty)$, we have shown that L^p is reflexive. It follows that $L^p \oplus \dots \oplus L^p$ is reflexive. Conclude that $W^{k,p}$ is reflexive. Hint: this is an exercise in applying the Hahn-Banach theorem.

Definition 12. It is clear that smooth compactly supported functions induce elements of $W^{k,p}(U, E)$. We define

$$W_0^{k,p}(U, E) = \text{closure of smooth compactly supported functions in } W^{k,p}(U, E).$$

Warning: $W_0^{k,p}$ is generally not equal to $W^{k,p}$. It turns out that $W_0^{k,p}$ is much better behaved (in certain regards) compared to $W^{k,p}$.

Theorem 13. Let E be a vector bundle on Y , and let $g, \langle -, - \rangle, \nabla$ and $g', \langle -, - \rangle', \nabla'$ be two sets of data. Suppose that we can uniformly bound the C^∞ distance between the two sets of data, i.e.

$$\|g - g'\|_{C^\infty, g} + \|\langle -, - \rangle - \langle -, - \rangle'\|_{C^\infty, \langle -, - \rangle} + \|\nabla - \nabla'\|_{C^\infty, g, \langle -, - \rangle} < \infty,$$

(recall that the difference of two connections is a tensor, and hence it makes sense to talk about its C^∞ norm). Then we have an equality of sets

$$W^{k,p}(Y, g, \langle -, - \rangle, \nabla) = W^{k,p}(Y, g', \langle -, - \rangle, \nabla') \text{ and similarly for } W_0^{k,p}.$$

and moreover the two norms on $W^{k,p}$ are comparable. \square

Theorem 14. Let $U \subset Y$ be a codimension 0 manifold with boundary. Then there is an extension (isometric) embedding

$$\text{ext} : W_0^{k,p}(U, E) \rightarrow W_0^{k,p}(Y, E),$$

defined in the obvious way on smooth functions and extended by density. There is also a restriction

$$\text{res} : W^{k,p}(Y, E) \rightarrow W^{k,p}(U, E),$$

and $\text{res} \circ \text{ext} = \text{id}$.

Corollary 15. Let Y be compact with boundary. Let $f : Y \rightarrow Z$ be a smooth embedding of a codimension 0 submanifold with boundary, and let $F : E_1 \rightarrow f^*E_2$ be a bundle isomorphism. Then there is a bounded extension

$$W_0^{k,p}(Y, E_1, g_Y, \langle -, - \rangle_Y, \nabla_Y) \rightarrow W_0^{k,p}(Y, E_2, g_Z, \langle -, - \rangle_Z, \nabla_Z),$$

and a bounded restriction

$$W^{k,p}(Y, E_2, g_Z, \langle -, - \rangle_Z, \nabla_Z) \rightarrow W^{k,p}(Y, E_1, g_Y, \langle -, - \rangle_Y, \nabla_Y),$$

so that the restriction composed with the extension is the identity.

Remark. These results allow us freedom to change the data defining the Sobolev norms, without actually changing the topology of the space $W_0^k(Y, E)$. We remark that some form of compactness on Y is crucial.

Lemma 16. Let ρ be a smooth section of $\text{Hom}(E, F)$. If u is in $W^{k,p}(Y, E)$ (resp. $W_0^{k,p}$), then

$$\rho(u) \in W^{k,p}(Y, F) \text{ resp. } W_0^{k,p}$$

and

$$\|\rho(u)\|_{W^{k,p}(Y, F)} \leq C(\rho) \|u\|_{W^{k,p}(Y, E)},$$

where $C(\rho)$ depends on the sizes of the first k derivatives of ρ .

Proof. Argue by induction.

$$\nabla \rho(u) = \nabla \text{ev}(\rho \otimes u) = \text{ev}(\nabla \rho \otimes u) + \text{ev}(\rho \otimes \nabla u),$$

to conclude that

$$\|\nabla(\rho(u))\|_{W^{k-1,p}} \leq C(\nabla \rho) \|u\|_{W^{k-1,p}} + C(\rho) \|\nabla u\|_{W^{k-1,p}}.$$

\square

Sobolev Embedding. Let Y^d be compact with boundary. Let E be a vector bundle on Y . Suppose that $k > \ell$ and

$$k - \frac{p}{d} \geq \ell - \frac{r}{d}.$$

Then the inclusion of test functions into $W_0^{\ell,r}$ extends to a bounded inclusion

$$W_0^{k,p}(Y, E) \subset W_0^{\ell,r}(Y, E).$$

Proof. First we claim that it suffices to prove it locally. To see this, cover Y by finitely many open sets U_α where it is known that inclusion extends to a bounded inclusion

$$W_0^{k,p}(U_\alpha, E) \rightarrow W_0^{\ell,r}(U_\alpha, E).$$

Fix a partition of unity ρ_α . Then for any u smooth and compactly supported in Y , we have

$$\|u\|_{W^{k,p}} \leq \sum_{\alpha} C_{\alpha} \|\rho_{\alpha} u\|_{W^{\ell,r}} \leq \sum_{\alpha} C'_{\alpha} \|u\|_{W^{\ell,r}} = C(\{U_{\alpha}, \rho_{\alpha}\}) \|u\|_{W^{\ell,r}},$$

as desired.

To prove it locally, note that it does not depend on the data used to define the norms. Thus we may suppose that $U = B(r) \subset \mathbb{R}^d$, $E = \mathbb{C}^n$ is a trivial bundle, and the metrics/connections are all the standard ones. The following tricky lemma is the key result

Lemma 17 (Gagliardo-Nirenberg Inequality). Let $u \in W_0^{1,1}(\mathbb{R}^d, \mathbb{C}^n)$. Then $u \in L^{d/(d-1)}$ and

$$\|u\|_{L^{d/(d-1)}} \leq \left[\prod_1^d \|\partial_i u\|_{L^1} \right]^{1/d}.$$

Assuming this technical lemma, we can complete the proof. First we prove the following

Claim 18. Let $p < d$ and define $p^* > p$ by

$$1 - \frac{d}{p} = \frac{d}{p^*}.$$

Then there is constant C_p so that for all $u \in W_0^{1,p}$, $u \in L^{p^*}$ and

$$\|u\|_{L^{p^*}} \leq C_p \|\nabla u\|_{L^p}.$$

Proof. Suppose that u is a test function, and consider $v_\epsilon = (\epsilon + |u|^2)^{(s-1)/2} u$, where $s > 1$ will be specified later. Then

$$\partial_i v_\epsilon = \frac{(s-1)}{2} (\epsilon + |u|^2)^{(s-3)/2} \sum_j u_j \partial_i u_j u + (\epsilon + |u|^2)^{(s-1)/2} \partial_i u,$$

hence

$$|\partial_i v_\epsilon| \leq \left(\frac{(s-1)}{2} (\epsilon + |u|^2)^{(s-3)/2} |u|^2 + (\epsilon + |u|^2)^{(s-1)/2} \right) |\nabla u|.$$

Applying the lemma yields

$$\left[\int (\epsilon + |u|^2)^{(s-1)d/2(d-1)} |u|^{d/(d-1)} d\text{Vol} \right]^{(d-1)/d} \leq \left[\prod_1^d \int |\partial_i v_\epsilon| d\text{Vol} \right]^{1/d}.$$

Applying monotone convergence theorem proves

$$\left[\int |u|^{sd/(d-1)} d\text{Vol} \right]^{(d-1)/d} \leq C_s \int |u|^{s-1} |\nabla u| d\text{Vol}.$$

Now suppose that $p < d$ and p^* is defined by

$$1 - \frac{d}{p} = -\frac{d}{p^*} \implies p^*p - dp^* = -dp \implies p^* = \frac{dp}{d-p}.$$

Pick s so that

$$\frac{sd}{d-1} = p^*.$$

Then we conclude that

$$\|u\|_{L^{p^*}}^s \leq C_s \int |u|^{s-1} \|\nabla u\| d\text{Vol}.$$

Let q be Hölder conjugate to p . Then

$$\|u\|_{L^{p^*}}^s \leq C_s \left[\int |u|^{(s-1)q} d\text{Vol} \right]^{1/q} \|\nabla u\|_{L^p}.$$

Now since

$$\frac{s}{d-1} = p^* \implies (s-1)q = (d-1)p^*q - q \implies (s-1)q = \frac{d-1}{d-p}pq - q = \frac{d}{d-p}(pq - q) = p^*.$$

Therefore

$$\|u\|_{L^{p^*}}^s \leq C_s \|u\|_{L^{p^*}}^{p^*/q} \|\nabla u\|_{L^p} \implies \|u\|_{L^{p^*}}^{s-p^*/q} \leq C_s \|\nabla u\|_{L^p}.$$

However, the reader can easily check that $s - p^*/q = 1$. Therefore

$$\|u\|_{L^{p^*}} \leq C_p \|\nabla u\|_{L^p},$$

as desired. Since this holds for test functions, it extends to $W_0^{1,p}$ by density. \square

Now we can complete the proof. Suppose that

$$1 - \frac{d}{p} \geq -\frac{d}{r}.$$

Define r^* by

$$1 - \frac{d}{r^*} = -\frac{d}{r}.$$

Then $r^* \leq p$ and $r^* < d$. Therefore we can apply our claim to conclude

$$\|u\|_{L^r} \leq C_r \|\nabla u\|_{L^{r^*}}.$$

Since we are on a bounded domain,

$$\|\nabla u\|_{L^{r^*}} \leq C \|\nabla u\|_{L^p},$$

as desired. The constant C depends on r, p and the domain. We conclude a continuous embedding:

$$W^{1,p}(B) \subset L^r(B).$$

Now suppose that $k - d/p \geq \ell - d/r$. Pick $r_0, \dots, r_{k-\ell} = r$ so that

$$k - d/r_0 = (k-1) - d/r_1 = \dots = \ell - \frac{d}{r}.$$

Then by using the result for $k=1$ conclude

$$W_0^{k,r_0}(B) \subset W_0^{k-1,r_1} \dots \subset W_0^{\ell,r}(B).$$

Since $p \geq r_0$, conclude the bounded inclusion

$$W_0^{k,p}(B) \subset W_0^{\ell,r_0}(B).$$

This completes the proof of the Sobolev embedding theorem. \square

Exercise 10. Using the Gagliardo-Nirenberg inequality for $u \in W_0^{1,1}(B)$

$$C^{-1} \|u\|_{L^1} \leq \|u\|_{L^{d/(d-1)}} \leq \|\nabla u\|_{L^1},$$

prove that $1 \notin W_0^{1,1}(B)$.

Theorem 19 (Rellich-Kondrachov compactness). Suppose that $k > \ell$ and $k - d/p > \ell - d/r$, and Y^d is a compact manifold with boundary, then the Sobolev embedding

$$W_0^{k,p}(Y, E) \subset W_0^{\ell,r}(Y, E)$$

is a compact inclusion.

Proof. As in the proof of the Sobolev embedding theorem, we will reduce the theorem to a local computation in \mathbb{R}^d .

Suppose that the theorem holds locally, i.e. we can cover Y by finitely many open sets U_α where it is known that the inclusion

$$W_0^{k,p}(U_\alpha, E) \subset W_0^{\ell,r}(U_\alpha, E)$$

is compact. Let u_n be a bounded sequence in $W_0^{k,p}(Y, E)$. If $\{\rho_\alpha\}$ is a partition of unity, then $\rho_\alpha u_n$ is a bounded sequence in $W_0^{k,p}(U_\alpha, E)$, and hence, after a subsequence, converges to something $u_{\infty,\alpha} \in W_0^{\ell,r}(U_\alpha, E)$. Then

$$\left\| u_n - \sum_\alpha u_{\infty,\alpha} \right\|_{W^{\ell,r}(Y)} \leq \sum_\alpha \|\rho_\alpha u_n - u_{\infty,\alpha}\|_{W^{\ell,r}(Y)} = \sum_\alpha \|\rho_\alpha u_n - u_{\infty,\alpha}\|_{W^{\ell,r}(U_\alpha)} \rightarrow 0.$$

Therefore it suffices to prove the theorem locally. We may therefore suppose that E is trivial and $Y = B(1)$ is a ball in \mathbb{R}^n with the standard metric, connection, etc.

We can also reduce to the case $k=1$, because if we know that $W_0^{1,p} \subset W_0^{0,r}$ is compact when $1 - d/p > -d/r$, then we also know $W_0^{k,p} \subset W_0^{k-1,r}$ when $k - d/p > (k-1) - d/r$, and

by finding $r_1, \dots, r_n = r$ so that

$$k - d/p > (k-1) - d/r_1 > \dots > \ell - d/r,$$

we conclude the result for $W_0^{k,p} \subset W_0^{\ell,r}$.

The strategy of proof is the following: if \mathcal{E} is a uniformly bounded family of functions in $W_0^{1,p}(B(1))$, and ρ_δ is a mollifying function, we will show that there is a universal bound

$$\|u - \rho_\delta * u\|_{L^r} < c(\delta), \quad c(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0, \text{ for all } u \in \mathcal{E}.$$

We will then show that the family of smooth functions $\rho_\delta * \mathcal{E}$ is bounded and equicontinuous (and are all supported in $B(1+\delta)$). By Arzela-Ascoli, we conclude that $\rho_\delta * \mathcal{E}$ is pre-compact in $C^0(B(1+\delta))$ topology, and hence also in $L^q(B(1))$.

Now pick $\epsilon > 0$. Choose δ so $c(\delta) < \epsilon/3$. Since $\rho_\delta * \mathcal{E}$ is compact, we can find elements $v_1, \dots, v_n \in \mathcal{E}$ so that the $\epsilon/2$ balls around $\rho_\delta * v_i$ cover $\rho_\delta * \mathcal{E}$. Since

$$\|\mathcal{E} - \rho_\delta * \mathcal{E}\|_{L^q} < \epsilon/3,$$

we conclude that the ϵ balls around v_i cover \mathcal{E} , and since ϵ was arbitrary, we conclude \mathcal{E} is pre-compact. Any sequence in a pre-compact space inside of a Banach space converges after taking a subsequence, and so we conclude the compactness statement.

It suffices to prove the two claims:

Claim 20. If $1 - d/p > -d/q$, and \mathcal{E} is bounded in $W_0^{1,p}(B(1))$, then there is a uniform bound

$$\|v - \rho_\delta * v\|_{L^q} < c(\delta) \text{ for all } v \in \mathcal{E},$$

and $c(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Claim 21. The family $\rho_\delta * \mathcal{E}$ is equicontinuous.

□

In order to prove Claim 20, we introduce some results about convolution. We define the **translation** operator on smooth functions by

$$\tau_h(f)(x) = f(x+h).$$

It is clear that $\tau_h^* = \tau_{-h}$. It is readily checked that

$$(*) \quad \|\tau_h f\|_{W^{k,p}(B)} \leq \|f\|_{W^{k,p}},$$

and hence τ_h extends by density to $W_0^{k,p}(B)$. Moreover, for all $u \in W_0^{k,p}(B)$ we have $\tau_h u \rightarrow u$ in $W^{k,p}$ norm, as $h \rightarrow 0$.

We define the **convolution** of u with a bump function $\rho_\delta = \delta^{-d} \rho(\delta^{-1}x)$ by

$$u * \rho_\delta(x) = \int_{\mathbb{R}^d} \rho_\delta(y) \tau_{-y} u(x) \, d\text{Vol}(y).$$

This is well-defined for smooth compactly functions. By pairing with a test-function v , we conclude that

$$\begin{aligned} (u * \rho_\delta, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\delta(y) u(x-y) v(x) d\text{Vol}(y) d\text{Vol}(x) \\ &= \int_{\mathbb{R}^d} \rho_\delta(y) \int_{\mathbb{R}^d} u(x-y) v(x) d\text{Vol}(x) d\text{Vol}(y) \\ &\leq \|u\|_{L^p} \|v\|_{L^q}, \end{aligned}$$

hence $\|u * \rho_\delta\|_{L^p} \leq \|u\|_{L^p}$. Therefore convolution extends to a bounded map $L^p \rightarrow L^p$.

It is clear that $\nabla(u * \rho_\delta) = (\nabla u) * \rho_\delta$ (for smooth u) and hence

$$\|u * \rho_\delta\|_{W^{k,p}} \leq \|u\|_{W^{k,p}},$$

and this estimate shows that $\rho_\delta * (-)$ extends to a bounded map $W_0^{k,p} \rightarrow W_0^{k,p}$.

We can compute

$$\rho_\delta * u(x) = \int \rho_\delta(y) u(x-y) d\text{Vol}(y) = \int \rho_\delta(x-y) u(y) d\text{Vol}(y),$$

and so

$$\nabla(\rho_\delta * u) = (\nabla \rho_\delta) * u.$$

Consequently,

$$\|\nabla(\rho_\delta * u)\|_{C^0} \leq \|\nabla \rho_\delta\|_{C^0} \|u\|_{L^1} \leq \|\nabla \rho_\delta\|_{C^0} \|u\|_{L^p}.$$

It follows that $\rho_\delta * u$ is differentiable if $u \in L^p$ (by density, and completeness of C^1 topology) and that

$$\nabla(\rho_\delta * u) = (\nabla \rho_\delta) * u.$$

Consequently $\rho_\delta * u$ is C^∞ if $u \in L^p$.

Lemma 22. Suppose that $1 - d/p > -d/r$. Define p^* by

$$1 - d/p \geq -d/p^* > -d/r.$$

This can always be achieved. Since $1 \geq 1/r > 1/p^*$, there is $\alpha > 0$ so that

$$\frac{1}{r} = \alpha + \frac{1-\alpha}{p^*}.$$

We claim that there is a constant C so that for all $u \in W_0^{1,p}(B)$,

$$\|\tau_h u - u\|_{L^r(\mathbb{R}^d)} \leq C |h|^\alpha \|u\|_{W^{1,p}(B)}.$$

Proof. It suffices to prove this when $u \in C_0^\infty(B)$. Then we compute

$$|\tau_h u(x) - u(x)| \leq |h| \int_0^1 |\nabla u(x+th)| dt,$$

hence

$$\|\tau_h u - u\|_{L^1} \leq |h| \int_0^1 \int_{\mathbb{R}^d} |\nabla u(x+th)| d\text{Vol} dt \leq |h| \|\nabla u\|_{W^{1,p}} \text{Vol}(B)^{1/q},$$

where we use the fact that $\nabla u(x + th)$ is supported in $B(1) - th$.

Now we compute

$$|\tau_h u - u|^r = |\tau_h u - u|^{\alpha r} |\tau_h u - u|^{(1-\alpha)r},$$

and then apply Hölder's inequality, with $1/\beta$, $1/(1-\beta)$, to obtain

$$\int |\tau_h u - u|^r \leq \left[\int |\tau_h u - u|^{\frac{\alpha r}{\beta}} d\text{Vol} \right]^\beta \left[\int |\tau_h u - u|^{\frac{(1-\alpha)r}{1-\beta}} d\text{Vol} \right]^{1-\beta}.$$

Pick $\beta = \alpha r$, so that $1 - \beta = \frac{(1-\alpha)r}{p^*}$, to obtain

$$\|\tau_h u - u\|_{L^r} \leq \|\tau_h u - u\|_{L^1}^\alpha \|\tau_h u - u\|_{L^{p^*}}^{1-\alpha}.$$

Using Sobolev embedding for $1 - d/p \geq -d/p^*$, we conclude

$$\|\tau_h u - u\|_{L^{p^*}}^{1-\alpha} \leq C \|u\|_{W^{1,p}}^{1-\alpha}.$$

We also have our previous estimate

$$\|\tau_h u - u\|_{L^1}^\alpha \leq C |h|^\alpha \|u\|_{W^{1,p}}^\alpha.$$

Therefore

$$\|\tau_h u - u\|_{L^r} \leq C |h|^\alpha \|u\|_{W^{1,p}},$$

as desired. □

Lemma 23 (Claim 20). For $u \in W_0^{1,p}(B)$, we have

$$\|u - \rho_\delta * u\|_{L^q} \leq C |\delta|^\alpha \|u\|_{W^{1,p}},$$

for some C and $\alpha > 0$, dependent only p, r, B, d .

Proof. We compute

$$u - \rho_\delta * u = \int_{\mathbb{R}^d} \rho_\delta(y) (u - \tau_{-y} u) d\text{Vol}(y).$$

Pair this with a test-function φ to conclude

$$(u - \rho_\delta * u, \varphi) = \int_{\mathbb{R}^d} \rho_\delta(y) \int_{\mathbb{R}^d} \varphi(x) (u(x) - u(x - y)) d\text{Vol}(x) d\text{Vol}(y).$$

This produces the estimate

$$|(u - \rho_\delta * u, \varphi)| \leq \int_{\mathbb{R}^d} \rho_\delta(y) \|u - \tau_{-y} u\|_{L^r} d\text{Vol}(y) \|\varphi\|_{L^s},$$

where s is conjugate to r . Since $\rho_\delta(y)$ vanishes for $|y| > \delta$, we conclude

$$\int_{\mathbb{R}^d} \rho_\delta(y) \|u - \tau_{-y} u\|_{L^r} d\text{Vol}(y) \leq C |\delta|^\alpha \|u\|_{W^{1,p}},$$

hence

$$\|u - \rho_\delta * u\|_{L^q} \leq C |\delta|^\alpha \|u\|_{W^{1,p}}.$$

As a corollary, if \mathcal{E} is a bounded set in $W_0^{1,p}$, then we conclude

$$\sup_{\mathcal{E}} \|u - \rho_\delta * u\| \leq c(\delta) \quad \text{where } c(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

□

Lemma 24 (Claim 21). Let $u \in W_0^{1,p}(B(1))$, then the smooth section $u * \rho_\delta$ is supported in $B(1 + \delta)$, and

$$(1) \quad \|u * \rho_\delta\|_{C^0} \leq C(\delta) \|u\|_{W^{1,p}(B)},$$

and

$$(2) \quad \|\tau_h(u * \rho_\delta) - (u * \rho_\delta)\|_{C^0} \leq C(\delta) |h|^\alpha \|u\|_{W^{1,p}(B)},$$

for some $\alpha > 0$. As a corollary, if \mathcal{E} is a bounded set in $W_0^{1,p}(B(1))$, then $\rho_\delta * \mathcal{E}$ is a bounded equicontinuous family of functions supported on $B(1 + \delta)$.

Proof. It suffices to prove the two estimates for test u . We compute

$$|u * \rho_\delta(x)| \leq \int |u(x - y)| |\rho_\delta(y)| \, d\text{Vol}(y) \leq \|u\|_{L^p} \|\rho_\delta\|_{L^q} = C(\delta) \|u\|_{W^{1,p}}.$$

For (2), we compute

$$\tau_h(u * \rho_\delta)(x) - u * \rho_\delta(x) = \int (u(x + h - y) - u(x - y)) \rho_\delta(y) \, d\text{Vol}(y) \leq \|\tau_h u - u\|_{L^r} \|\rho_\delta\|_{L^s}.$$

Using our previous deduction,

$$\|\tau_h u - u\|_{L^r} \leq C |h|^\alpha \|u\|_{W^{1,p}},$$

for some $\alpha > 0$. Hence

$$\|\tau_h(u * \rho_\delta) - u * \rho_\delta\|_{C^0} \leq C(\delta) |h|^\alpha \|u\|_{W^{1,p}}.$$

This completes the proofs of Claims 20 and 21, and hence completes the proof of the Rellich compactness theorem. □

Morrey embedding. The goal of this section is to prove that $W_0^{k,p}$ functions automatically have a certain degree of Hölder continuity.

Definition 25. The Hölder space $C^{0,\alpha}(\mathbb{R}^d, \mathbb{C}^n)$ is defined by

$$u \in C^{0,\alpha} \iff \|\tau_h u - \tau u\|_{C^0} \leq C |h|^\alpha \text{ for } |h| < 1.$$

Defining

$$[u]_\alpha = \sup_{0 < |h| < 1} \frac{\|\tau_h u - \tau u\|_{C^0(U)}}{|h|^\alpha},$$

then $\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + [u]_\alpha$ is a complete metric on $C^{0,\alpha}$.

We define $C^{k,\alpha}$ by recursion: $C^{k,\alpha}$ consists of all C^k functions whose first derivative is $C^{k-1,\alpha}$.

Theorem 26 (Morrey Embedding). If $k > \ell$, $\alpha \in (0, 1)$, and $k - d/p \geq \ell + \alpha$, then the inclusion of test functions into $C^{\ell, \alpha}$ extends to a continuous extension $W_0^{k, p}(B) \subset C^{\ell, \alpha}$. Consequently, $W_0^{k, p}(B)$ functions are represented by ℓ times differentiable functions.

Proof. First we claim that it suffices to prove the case $k = 1$, $\ell = 0$. To see why, observe that if we know it for $1 - d/p \geq \alpha$, then if $u \in W^{k, p}$, all the first $k - 1$ derivatives of u are in $W^{1, p}$ and hence in $C^{0, \alpha}$. It follows that u is $C^{k-1, \alpha}$. Therefore

$$W^{k, p} \subset C^{k-1, \alpha}.$$

Now if $p < d$, then we can find $p < p^* < \infty$ so that

$$k - d/p = (k - 1) - d/p^*.$$

By continuing, we conclude that we can find $\ell < j \leq k$ so that

$$k - d/p = j - d/p^* \geq \ell + \alpha,$$

where $p^* \geq d$. To see why $\ell < j$, observe that if not, then we would conclude $\alpha < 0$. Since $p^* \geq d$, we can find $\beta \in [0, 1)$ so that

$$j - d/p^* = (j - 1) + \beta \geq \ell + \alpha.$$

To see why $\beta \neq 1$, use $p^* < \infty$. If $\beta \neq 0$, then by applying the composite $W^{k, p} \subset W^{j, p^*} \subset C^{j-1, \beta} \subset C^{\ell, \alpha}$, we conclude the desired result.

If $\beta = 0$, then we need to be a bit more careful. We essentially are in the case when $p = d$ and

$$k - d/p \geq \ell + \alpha,$$

so $k - 2 \geq \ell$, since $\alpha > 0$. Since $\alpha < 1$, the inequality is strict:

$$k - d/p > \ell + \alpha.$$

Then we can find $d < p^* < \infty$ so

$$k - d/p > k - 1 - d/p^* = k - 2 + \beta > \ell + \alpha,$$

where $\beta > 0$, and then use the inclusion $W^{k, p} \subset W^{k-1, p^*} \subset W^{k-2, \beta} \subset W^{\ell, \alpha}$.

Now we prove the case when $k = 1$, and $1 - d/p \geq \alpha$.

Proposition 27. Let u be a smooth function (not necessarily a test function). There is a constant C depending only on d (and not the radius r , etc) so that

$$\|u(z) - \bar{u}(x)\|_{C^0(B(x; r))} \leq C(d) \int_{B(x; r)} \frac{|\nabla u(y)|}{|z - y|^{d-1}} d\text{Vol}(y)$$

where $\bar{u}(x, r)$ denotes the average of u over $B(x; r)$.

Proof. This is a fairly straightforward computation using the fundamental theorem of calculus. We compute

$$u(z_2) - u(z_1) = \int_0^{|z_2 - z_1|} \nabla u(z_1 + t \frac{z_2 - z_1}{|z_2 - z_1|}) \frac{z_2 - z_1}{|z_2 - z_1|} dt.$$

Integrating over z_2 , and dividing by the volume of the ball, yields

$$\bar{u}(x) - u(z_1) = \frac{1}{\text{Vol}} \int_{B(x;r)} \int_0^{|z_2 - z_1|} \nabla u(z_1 + t \frac{z_2 - z_1}{|z_2 - z_1|}) \frac{z_2 - z_1}{|z_2 - z_1|} d\text{Vol}(z_2) dt.$$

Introduce polar coordinates $z_2 = z_1 + \rho\theta$, where ρ ranges from 0 to $\ell(\theta) < 2r$.

$$\bar{u}(x) - u(z_1) = \frac{1}{\text{Vol}} \int_{\mathbb{S}^{d-1}} \int_0^{\ell(\theta)} \int_0^\rho \nabla u(z_1 + t\theta) \theta \rho^{d-1} dt d\rho d\theta.$$

The region $\{(\rho, t) : t < \rho, \rho < \ell(\theta)\}$ can be reparametrized to give

$$\bar{u}(x) - u(z_1) = \frac{1}{\text{Vol}} \int_{\mathbb{S}^{d-1}} \int_0^{\ell(\theta)} \frac{u(z_1 + t\theta)}{t^{d-1}} t^{d-1} \int_t^{\ell(\theta)} \rho^{d-1} d\rho dt d\theta.$$

Changing back to regular coordinates $y = z_1 + t\theta$, we conclude,

$$|\bar{u}(x) - u(z_1)| \leq \frac{D^d}{\text{Vol} d} \int \frac{|\nabla u(y)|}{|y - z_1|^{d-1}} d\text{Vol}(y),$$

Where D is the diameter of the ball. Note that D^d/Vol is a constant independent of r . This completes the proof of the proposition. \square

Now suppose that u is a test function on $B(1)$. For each $z \in B$, we use the previous proposition to conclude

$$|u(z)| \leq |\bar{u}| + C \int_B \frac{|\nabla u(y)|}{|y - z|^{d-1}} d\text{Vol}(y).$$

We can estimate $\bar{u} \leq C \|u\|_{L^p}$. Now apply Hölder's inequality with p, q to conclude

$$|u(z)| \leq C \|u\|_{L^p} + \|\nabla u\|_{L^p} \left[\int \frac{1}{|y - z|^{(d-1)q}} d\text{Vol} \right]^{1/q}.$$

We conclude

$$\int \frac{1}{|y - z|^{(d-1)q}} d\text{Vol} = C \int_0^1 \rho^{(d-1)(1-q)} d\rho.$$

We compute

$$1 - \frac{d}{p} > \alpha \implies 1 - d + \frac{d}{q} > \alpha \implies (q-1)(1-d) + 1 > q\alpha,$$

which implies $\rho^{(d-1)(1-q)}$ is integrable over $[0, 1]$, and hence:

$$(1) \quad |u(z)| \leq C(d, p) \|u\|_{W^{1,p}}.$$

This proves that $W^{1,p}(B)$ functions satisfy a uniform C^0 bound. We now to establish oscillation estimates to conclude that $W^{1,p}(B)$ functions lie in $C^{0,\alpha}$. As before, let u be a test

function supported in B . Pick $x \in B$. For $z \in B_r(x)$, we apply the proposition to conclude

$$|u(z) - \bar{u}| \leq C \int_{B_r(x)} \frac{|\nabla u(y)|}{|z - y|^{d-1}} d\text{Vol}(y).$$

We conclude, for $r < 1$,

$$|u(z) - \bar{u}| \leq C \|\nabla u\|_{L^p} \int_0^r \rho^{(d-1)(1-q)} d\rho \leq C \|\nabla u\|_{L^p} r^\alpha,$$

This was independent of x , and hence:

$$|u(z_1) - u(z_2)| \leq 2C \|\nabla u\|_{L^p} |z_1 - z_2|^\alpha.$$

Thus, combined with (1), we conclude

$$(2) \quad \|u\|_{C^{0,\alpha}} \leq C \|u\|_{W^{1,p}}.$$

This proves that $W^{1,p}$ includes into $C^{0,\alpha}$, as desired. \square

Exercise 11. If $k + \beta > \ell + \alpha$, use Arzela Ascoli to conclude $C^{k,\beta}(B) \subset C^{\ell,\alpha}(B)$ is a compact inclusion, where $C^{k,\beta}(B)$ are the $C^{k,\beta}$ functions which vanish outside of B .

Exercise 12 (Morrey compactness). If $k - d/p > \ell + \alpha$, $\alpha \in [0, 1)$, then the inclusion

$$W_0^{k,p}(B) \subset C^{\ell,\alpha}$$

is compact.

Corollary 28. Let Y be a compact manifold with boundary, and suppose $k - d/p > \ell$. Then the inclusion of test sections of E into $C^\ell(E)$ extends to a compact inclusion of

$$W_0^{k,p}(Y, E) \subset C^\ell(E).$$

Proof. As usual, it suffices to prove this locally. However, locally, it is a consequence of the Morrey embedding theorem, and the previous exercise. \square

Example 29. Let Y be a Riemann surface. Then Sobolev class functions $W^{1,p}(Y, \mathbb{C}^n)$ are automatically continuous provided $p > 2$. This is a crucial fact when trying to define $W^{1,p}$ Sobolev spaces of maps $Y \rightarrow M$, where M is another manifold.

Remark. The two results: (i) Sobolev embedding/Rellich compactness and (ii) Morrey embedding/compactness are fundamental results in the study of partial differential operators on manifolds.

Exercise 13. Let $f \in W^{k,p}(Y, E)$, where we do not assume that $f \in W_0^{k,p}(Y, E)$. Suppose that $k - p/d > \ell$. Prove that f is of class C^ℓ .

Hint: First prove that $\rho f \in W_0^{k,p}(Y, E)$, if ρ is compactly supported in the interior of Y . Conclude that f is a sum of smooth sections, and hence is smooth.

Elliptic operators.

Definition 30. Let $E_i \rightarrow Y$, $i = 1, 2$, be vector bundles over Y . Given a differential operator $L : E_1 \rightarrow E_2$, we can consider the assignment

$$f \in \Gamma(\mathbb{C}) \mapsto [L, f] \in \Gamma \operatorname{Hom}(E_1, E_2).$$

This assignment is actually a first-order linear differential operator $\mathbb{C} \rightarrow \operatorname{Hom}(E_1, E_2)$. In fact, there exists a unique tensor $\sigma : T^*M \rightarrow \operatorname{Hom}(E_1, E_2)$ so that

$$[L, f] = \sigma(df).$$

This tensor σ is called the **symbol** of L . To see why there exists such a tensor σ , we review some results proved in the first section.

Exercise 14. Prove that $[L, -]$ is a first order linear differential operator $f \mapsto [L, f]$. Hint: $[L, \varphi f] - \varphi[L, f] = [L, \varphi]f$.

Exercise 15. Since d is a connection on the trivial bundle \mathbb{C} , we already proved that we can express

$$[L, f] = A(df) + B(f)$$

for some tensors A and B . Use this to prove that $B = 0$.

We proved that $A : T^*Y \rightarrow \operatorname{Hom}(E_1, E_2)$ and $B : \mathbb{C} \rightarrow \operatorname{Hom}(E_1, E_2)$ are uniquely determined by L . Conclude that there is unique tensor σ so that $[L, f] = \sigma(df)$.

Definition 31. A first order operator $L : E_1 \rightarrow E_2$ is called elliptic if its symbol σ maps $(T^*Y)^\times$ into $\operatorname{GL}(E_1, E_2)$. Note that this forces $\dim E_1 = \dim E_2$.

Example 32. Let Y be a manifold with $\dim Y \geq 2$. There are no elliptic first order operators on the trivial bundle \mathbb{R} . To see why, note that $\operatorname{Hom}(\mathbb{R}, \mathbb{R})$ is one-dimensional, while T^*Y is greater than 2 dimensional, so the tensor $\sigma : T^*Y \rightarrow \operatorname{Hom}(\mathbb{R}, \mathbb{R})$ must have some kernel. This is not true for higher order operators. This partially explains the prevalence of *second* order operators when working with real-valued functions.

Example 33. Let $\dim M > n$. Then there are no first order differential operators from $E_1 \rightarrow E_2$, if $\dim E_1 = \dim E_2 = n$.

To see why, consider the space $\operatorname{Hom}(E_1, E_2) \simeq \mathbb{R}^{n \times n}$, and consider a $> n$ -dimensional subspace Π of $\mathbb{R}^{n \times n}$. Suppose that Π doesn't touch the singular set. Then we can find matrices A_0, \dots, A_n so that

$$\sum_i x_i A_i \text{ is non-singular}$$

for every x_0, \dots, x_n , not all 0. In particular, for any vector v ,

$$\sum_i x_i A_i v \neq 0 \text{ for all } x_0, \dots, x_n.$$

But this implies that we have found a linearly independent set with $> n$ elements in E_2 , contrary to the requirement that $\dim E_1 = \dim E_2 = n$.

Example 34. As a corollary to the previous example, consider a > 2 dimensional manifold X with almost complex structure J . The equation for holomorphic functions $X \rightarrow \mathbb{C}$ is not elliptic, because $\dim X > \dim \mathbb{C} = 2$.

Constant coefficient first-order elliptic operators on \mathbb{T}^d . Let $\mathbb{C}^n \rightarrow \mathbb{T}^d$ be a trivial line bundle endowed the the standard metric $\langle -, - \rangle$, and suppose that \mathbb{T}^d has the Euclidean metric g and connection ∇ inherited from \mathbb{R}^d . The trivial bundle has connection $\nabla = d$ (let's agree to use the symbols ∇ and d interchangeably for sections of a trivial bundle).

We have shown that any first order operator is of the form

$$Lu = A(\nabla u) + B(u),$$

Observe that

$$L(fu) - fLu = A(df \otimes u).$$

Therefore

$$\sigma(df) = A(df \otimes -).$$

Now recall that

$$\nabla u = dx_i \otimes \frac{\partial u}{\partial x_i},$$

and hence

$$L(u) = \sum_i \sigma(dx_i) \frac{\partial u}{\partial x_i} + B(u) = \sum_i A_i \frac{\partial u}{\partial x_i} + Bu.$$

Example 35. Let $\sigma : T^*\mathbb{T}^2 \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ by the symbol

$$\sigma(adx + bdy) = a + bJ,$$

where J is a complex structure. This is clearly elliptic, because

$$(a + bJ)^{-1} = \frac{a - bJ}{a^2 + b^2}.$$

This induces a partial differential operator on $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by

$$\bar{\partial}_U(u) = \frac{du}{dx} + J \frac{du}{dy}.$$

Definition 36. We say that a differential operator $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ on the torus \mathbb{T}^d has **constant coefficients**, provided

$$Lu = A(du) + B(u)$$

where A, B are constant tensors. The goal of the next section is to prove the interior elliptic estimates for constant coefficient linear elliptic operators on \mathbb{T}^d .

Interior elliptic estimates. Here is the general statement we will ultimately prove.

Theorem 37 (Interior elliptic estimates). Let L be a first-order elliptic operator $E_1 \rightarrow E_2$, and let Y be a compact manifold with boundary. For all $p \in (1, \infty)$, and all $k = 1, 2, \dots$, there is constant $C = C(p, k)$ so that

$$\|u\|_{W^{k,p}(Y, E_1)} \leq C \left[\|Lu\|_{W^{k-1,p}(Y, E_2)} + \|u\|_{L^p(Y, E_1)} \right].$$

for all $u \in W_0^{k,p}(Y, E_1)$.

Remark. These estimates are called **interior** because they deal with the space $W_0^{k,p}$, which is the closure of $C_c^\infty(\text{int}(Y), E)$ in $W^{k,p}$. We still require Y to be compact.

Exercise 16. Let L be a first order differential operator on Y . The very first thing to observe (which we have not said yet) is that linear differential operators extend to all distributions via duality – this is possible since every operator has an adjoint (which is immediate since $L = A \circ \nabla + B$, where A, ∇, B all have adjoints). Moreover, A, ∇, B all extend to distributions, and the extensions satisfy

$$L = A \circ \nabla + B.$$

By the $W^{\bullet,p}$ boundedness properties of ∇ and tensors, we conclude a constant $C = C(L, k)$ so

$$\|Lu\|_{W^{k-1,p}} \leq C \|u\|_{W^{k,p}}.$$

Suppose that L is constant coefficients on \mathbb{T}^d . We will prove that the elliptic estimates hold for L using Fourier analysis on \mathbb{T}^d . We recall the necessary definitions.

Define $e_\ell = \exp(i \langle \ell, - \rangle) : \mathbb{T}^d \simeq \mathbb{R}^d / \mathbb{Z}^d \rightarrow \mathbb{C}$, for $\ell \in \mathbb{Z}^d$. Then e_ℓ are test functions which are orthonormal with respect to the L^2 inner product.

Theorem 38. If $f \in L^2(\mathbb{T}^d, \mathbb{C}^n)$, then

$$(*) \quad f = \sum_{\ell \in \mathbb{Z}} (e_\ell, f) e_\ell.$$

Here the pairing (e_ℓ, f) is vector valued.

Proof. This is standard, by now. By continuity of both sides, it suffices to prove it for smooth f . It is clear that the set of linear combinations of e_k is an unital algebra closed under conjugation, and moreover the algebra separates points in the sense that for any two points x, y we can find a function φ in the algebra generated by e_k so that $\varphi(x) \neq \varphi(y)$. Then by the Stone-Weierstrass theorem, the span of $\{e_k\}$ is dense in C^0 . Since the C^0 norm controls the L^2 norm, and C^0 is dense in L^2 , we conclude e_ℓ are dense in L^2 , and hence they form an orthonormal basis for the Hilbert space L^2 .

To prove that $(*)$ holds, we apply the fact that e_ℓ are a basis to each of the component functions of f . □

Proposition 39. Suppose that a section f has a weak derivative $\nabla f \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. Then

$$(\mathrm{d}x_j \otimes s_i e_\ell, \nabla f) = (\nabla^*(\mathrm{d}x_j \otimes s_i e_\ell), f) = -i\ell_j(e_\ell, f_i),$$

where s_1, \dots, s_n be an orthonormal frame \mathbb{C}^n , and f_i denotes $\langle s_i, f \rangle$.

We remark that the sections $\mathrm{d}x_j \otimes s_i e_\ell$ form an orthonormal basis for $L^2(T^*\mathbb{T}^d)$, since they are certainly orthonormal and contain a generating set of smooth sections

$$\mathrm{d}x_j \otimes u,$$

where u is a smooth \mathbb{C}^n valued function in the closure of their span.

As a consequence, if $f \in W^{1,2}$, then

$$\|\nabla f\|_{L^2}^2 = \sum_{\ell} |\ell|^2 |(e_\ell, f)|^2 < \infty.$$

Conversely, if f is a distribution and the Fourier coefficients of ∇f are in ℓ^2 , then f is $W^{1,2}$. Note that (e_ℓ, f) is a vector, and so $|(e_\ell, f)|$ is measured using the inner product on \mathbb{C}^n .

Proof. Since f is a distribution, $(e_0, f) < \infty$, and hence the sum

$$\sum_{\ell} |\ell|^2 |(e_\ell, f)|^2 < \infty$$

implies that $\|f\|_{L^2} < \infty$, and so f is represented by a L^2 function. By assumption that the fourier series for the distribution ∇f converges, we conclude that ∇f is an L^2 section of $\text{Hom}(T\mathbb{T}, \mathbb{C}^n)$, and hence f is in $W^{1,2}$. We also observe that

$$\|f\|_{W^{1,2}}^2 = \sum_{\ell} (1 + |\ell|^2) |(e_\ell, f)|^2.$$

□

More generally, the sections

$$\mathrm{d}x_{j_1} \otimes \dots \otimes \mathrm{d}x_{j_k} \otimes e_\ell s_i$$

form an orthonormal basis for $L^2(\text{Hom}(TM^{\otimes k}, \mathbb{C}^n))$. It is easy to check that

$$\nabla^*(\mathrm{d}x_{j_1} \otimes \mathrm{d}x_{j_2} \otimes \dots \otimes \mathrm{d}x_{j_k} \otimes e_\ell s_i) = \mathrm{d}x_{j_2} \otimes \dots \otimes \mathrm{d}x_{j_k} \ell_{j_1} e_\ell s_i,$$

(i.e. this can be figured out fairly easily remembering our general formula for the adjoint of a connection). Therefore, if f is a section of \mathbb{C}^n , then the coefficients of $\nabla^k f$ are

$$(\mathrm{d}x_{j_1} \otimes \mathrm{d}x_{j_2} \otimes \dots \otimes \mathrm{d}x_{j_k} \otimes e_\ell s_i, \nabla^k f) = \ell_{j_1} \dots \ell_{j_k} (e_\ell, \langle s_i, f \rangle).$$

Therefore

$$\|\nabla^k f\|_{L^2}^2 = \sum |\ell|^{2k} |(e_\ell, f)|^2.$$

Hence for $f \in W^{k,2}$, we have

$$\|f\|_{W^{k,2}}^2 = \sum_{\ell} (1 + |\ell|^2 + \dots + |\ell|^{2k}) |(e_\ell, f)|^2.$$

Now suppose that L is an first-order elliptic operator with constant coefficients,

$$L(u) = A(\nabla u) + B(u).$$

We compute

$$(e_\ell s_i, L(u)) = (e_\ell s_i, A(\nabla u) + B(u)) = (\nabla^* A^*(e_\ell s_i) + B^*(e_\ell s_i), u),$$

where, recall, s_1, \dots, s_n are the constant orthonormal frame for \mathbb{C}^n . Since A is constant, so is A^* , and hence

$$\nabla^* e_\ell A^*(s_i) = -e_\ell i\ell \lrcorner A^*(s_i) + e_\ell \nabla^* A^*(s_i)$$

Exercise 17. Prove this relation by exploring the commutator $[\nabla^*, e_\ell]$.

Then, we conclude

$$(e_\ell s_i, L(u)) = (-e_\ell i\ell \lrcorner A^*(s_i) + e_\ell \nabla^* A^*(s_i) + e_\ell B(s_i), u).$$

Since $(e_\ell, L(u)) = (e_\ell s_i, L(u))s_i$, conclude

$$(e_\ell, L(u)) = \sum_i (-e_\ell, \langle i\ell \lrcorner A^*(s_i), u \rangle) s_i + (e_\ell, \langle \nabla^* A^*(s_i) + B(s_i), u \rangle) s_i.$$

Exercise 18. If s is a constant section, then $(e_\ell, \langle s, u \rangle) = \langle s, (e_\ell, u) \rangle$.

We therefore obtain the nice formula

$$(e_\ell, L(u)) = \sum_i \langle i\ell \lrcorner A^*(s_i), \widehat{u}(\ell) \rangle + \langle \nabla^* A^*(s_i) + B(s_i), \widehat{u}(\ell) \rangle s_i.$$

We conclude

$$\sum_{i,\ell} |\langle \ell \lrcorner A^*(s_i), \widehat{u}(\ell) \rangle|^2 \leq \|L(u)\|_{L^2}^2 + c \|u\|_{L^2}^2.$$

To conclude the elliptic estimate for $W^{1,2}$, we will try to bound

$$\sum_\ell |\ell|^2 |\widehat{u}(\ell)|^2 \leq C \sum_{i,\ell} |\langle \ell \lrcorner A^*(s_i), \widehat{u}(\ell) \rangle|^2.$$

Claim 40. There is a constant c so that for all vectors v

$$c |\ell|^2 |v|^2 \leq \sum_i |\langle \ell \lrcorner A^*(s_i), v \rangle|^2.$$

This is where ellipticity of L comes in.

Proof. Since L is elliptic, the map $v \mapsto A(\ell \otimes v)$ is an isomorphism for each non-zero ℓ .

The adjoint of A is defined by formally taking the adjoint of the map

$$A : \text{Hom}(TM, \mathbb{C}^n) \rightarrow \mathbb{C}^n,$$

so

$$\langle A(\ell \otimes v), s_i \rangle = \langle \ell \otimes v, A^*(s_i) \rangle = \langle v, \ell \lrcorner A^*(s_i) \rangle.$$

Therefore we conclude that

$$v \mapsto \langle v, \ell \lrcorner A^*(s_i) \rangle s_i$$

is an injection. Define c by

$$c = \inf_{|v|=|\ell|=1} \sum_i |\langle v, \ell \lrcorner A^*(s_i) \rangle|^2,$$

and note that c is the infimum of a continuous function on a compact set. Since c is never vanishing we conclude $c(v) > 0$. By bilinearity, we conclude

$$c |\ell|^2 |v|^2 \leq \sum_i |\langle v, \ell \lrcorner A^*(s_i) \rangle|^2.$$

Here it was useful to temporarily allow ℓ to be a non-integer. This completes the proof. \square

Corollary 41 (interior estimates for $p = 2$). Let L be a constant coefficient first order elliptic operator on the torus. Then for all $u \in W^{1,2}$, we have

$$\|u\|_{W^{1,2}} \leq C [\|Lu\|_{L^2} + \|u\|_{L^2}].$$

Exercise 19. Prove that the interior elliptic estimates for the trivial complex bundle \mathbb{C}^n imply the interior estimates for the trivial real bundle \mathbb{R}^n .

We will see later that the interior elliptic estimates + a general elliptic regularity theorem are enough to construct a parametrix (in general). We can be more explicit in the constant coefficient case and explicitly describe a parametrix. We have shown above that

$$(e_\ell, L(u)) = \sum_i \langle i\ell \lrcorner A^*(s_i), \widehat{u}(\ell) \rangle s_i + \langle \nabla^* A^*(s_i) + B(s_i), \widehat{u}(\ell) \rangle s_i.$$

Consider

$$v \mapsto \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle s_i.$$

as a map $\mathbb{C}^n \rightarrow \mathbb{C}^n$. We have shown that the norm (squared) of the first term is bounded below by $c |\ell|^2 |v|^2$. It is clear that the norm of the second term is bounded above by $C |v|$, and hence for all but finitely many ℓ , the map

$$v \mapsto \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle s_i.$$

is invertible. Indeed, by pick ℓ sufficiently large, we may suppose that there is a neighborhood of infinity $\Lambda \subset \mathbb{Z}^d$ so that

$$|\ell| |v| < C \left| \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle s_i \right|,$$

for all $\ell \in \Lambda$, and all v . Given any collection of vectors b_ℓ , we can find unique a_ℓ , for $\ell \in \Lambda$ so that

$$(e_\ell, L(a_\ell)) = b_\ell,$$

and moreover,

$$(*) \quad |a_\ell| \leq C \frac{|b_\ell|}{\ell} \text{ for } \ell \in \Lambda.$$

Suppose now that $w = \sum_\ell b_\ell e_\ell$, and w lies in L^2 . Let Π be finite dimensional subspace spanned by e_ℓ , $\ell \notin \Lambda$.

Define

$$u := P(w) = \sum_\Lambda a_\ell e_\ell \text{ so that } L(u) - w \text{ lies in } \Pi.$$

This prescription defines a bounded linear map $L^2 \rightarrow W^{1,2}$, since we have shown $(*)$.

Note that $LP(w) = w - F(w)$, where F is a finite rank operator (projection onto Π). Similarly, observe that $PL(u - F(u)) = u - F(u)$, so

$$PL(u) = u - G(u),$$

where G is some other finite rank operator. Therefore we have inverted L up to finite rank operators. Observe that

$$\|u\|_{W^{1,2}} \leq \|u - G(u)\|_{W^{1,2}} + \|G(u)\|_{W^{1,2}} \leq C \|Lu\|_{L^2} + \|G(u)\|_{W^{1,2}}.$$

Note that for u restricted to the orthogonal complement of $\ker G$, we know that G is continuous with respect to any topology on it's finite rank target. We therefore have

$$\|G(u)\|_{W^{1,2}} \leq C \|u\|_{L^2},$$

for $u \perp \ker G$, and hence this holds for all u . Thus we conclude

$$\|u\|_{W^{1,2}} \leq C [\|Lu\|_{L^2} + \|u\|_{L^2}].$$

Thus we see that the existence of a parametrix implies the interior estimates.

Theorem 42. Let L be a first order constant coefficient elliptic operator on \mathbb{T}^d . Then for all $u \in W^{1,p}$, and all $p > 0$, we have

$$\|u\|_{W^{1,p}} \leq C [\|Lu\|_{L^p} + \|u\|_{L^p}].$$

Remark 43. We won't prove this theorem, because I think it is quite hard. Even for the standard Cauchy-Riemann operator $\partial_x + i\partial_y$, I think one needs to use "Calderon-Zygmund" theory.

One method to prove it would be to try to show that our parametrix P is bounded from $L^p \rightarrow W^{1,p}$, but this seems hard.

However, assuming this theorem, we can prove the interior elliptic estimates in general without too much additional work.

It is desirable to have the elliptic estimates for $p > 2$, because the Sobolev spaces $W^{1,p}(\Sigma, -)$ are made of continuous functions when $p > 2$ – this is important for developing the non-linear theory. We could try to develop the non-linear theory using the spaces $W^{2,2}(\Sigma, -)$. One meta-reason this is difficult is that “the first estimate is the hardest to establish,” and it is generally easier to establish estimates which begin with lower regularity.

Note that the spaces $W^{1,2}(\mathbb{S}^1, -)$ and $W^{1,2}([0, 1], -)$ consist of continuous function (when $p = 2$). This implies that non-linear theory for elliptic equations on one-dimensional manifolds can be approached using the $W^{k,2}$ spaces. Non-linear elliptic equations on one-dimensional manifolds can be thought of as the theory of ordinary differential equations.

Exercise 20. Construct a parametrix $P : L^2(\mathbb{T}^2) \rightarrow W^{1,2}(\mathbb{T}^2)$ for the constant coefficient elliptic differential operator

$$u \mapsto \bar{\partial}u = u_x + iu_y.$$

Hint: the fourier coefficients of Lu are

$$(e_\ell, Lu) = i(\ell_1 + i\ell_2)\widehat{u}(\ell).$$

Therefore we can invert L whenever $\ell \neq 0$. Conclude a unique parametrix $P : L^2 \rightarrow W^{1,2}$ so that

$$u - LPu = \int u \, d\text{Vol},$$

and so that

$$u - PLu = \int u \, d\text{Vol}.$$

Conclude that the index of $\bar{\partial}$ is 0.

Interior elliptic estimates in the non-constant coefficient case. First we will generalize to the case when we have a differential operator on the torus which does not have constant coefficients. Write

$$L(u) = A(\nabla u) + B(u).$$

Recall that we are trying to prove

$$\|u\|_{W^{1,p}} \leq C [\|Lu\|_{L^p} + \|u\|_{L^p}].$$

Since

$$\|A(\nabla u)\|_{L^p} \leq \|L(u)\|_{L^p} + C \|u\|_{L^p},$$

it suffices to bound $\|\nabla u\|_{L^p}$ by $\|A(\nabla u)\|_{L^p}$.

Now cover \mathbb{T}^d by N^d overlapping squares U_α of size $(2/N)^d$, and let $x_\alpha \in U_\alpha$ be the center of the square.

Let $L_\alpha(u) = A_\alpha(\nabla u) + B_\alpha(u)$. This defines a constant coefficient differential operator on the entire torus \mathbb{T}^d .

Choose a partition of unity ρ_α for U_α , with the property that

$$|\nabla \rho_\alpha| < CN,$$

where C is independent of N .

Suppose we can prove that

$$(*) \quad \|\rho_\alpha u\|_{W^{1,p}} \leq C_\alpha [\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p}]$$

Then we estimate

$$\|u\|_{W^{1,p}} \leq \sum_{\alpha} \|\rho_\alpha u\|_{W^{1,p}} \leq \sum_{\alpha} C_\alpha [\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p}].$$

Now write

$$L\rho_\alpha u = \rho_\alpha Lu + [L, \rho_\alpha]u$$

and thereby conclude

$$\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p} \leq \|\rho_\alpha Lu\|_{L^p} + \|[L, \rho_\alpha]u\|_{L^p} + \|u\|_{L^p} \leq [\|Lu\|_{L^p} + c(N) \|u\|_{L^p}].$$

The constant $c(N)$ may be quite large (it involves the derivatives of ρ_α , which are bounded by N). We ultimately conclude

$$\|u\|_{W^{1,p}} \leq C(N) [\|Lu\|_{L^p} + \|u\|_{L^p}],$$

for some large constant $C(N)$. Therefore it suffices to prove that $(*)$ holds for some finite N .

Now is where we use the constant coefficient operators L_α . Consider the operators $L_x = A_x(\nabla u) + B_x(u)$ as x ranges over \mathbb{T}^d . Each one satisfies an estimate of the form

$$\|u\|_{W^{1,p}} \leq C_x (\|L_x u\|_{L^p} + \|u\|_{L^p}).$$

We claim that there is a constant C independent of x so that

$$\|u\|_{W^{1,p}} \leq C (\|L_x u\|_{L^p} + \|u\|_{L^p})$$

holds for all x . This is a compactness argument. To see why, observe that for each $x \in \mathbb{T}^d$, there is a ball B around x and a constant C so that $y \in B$ implies

$$\|u\|_{W^{1,p}} \leq C (\|L_y u\|_{L^p} + \|u\|_{L^p}).$$

Then by covering \mathbb{T}^d by finitely many such balls and taking the maximum of the constants C proves the claim.

We know that

$$\|\rho_\alpha u\|_{W^{1,p}} \leq C (\|L_\alpha \rho_\alpha u\|_{L^p} + \|u\|_{L^p}) \leq C (\|L\rho_\alpha u\|_{L^p} + \|(L_\alpha - L)\rho_\alpha u\|_{L^p} + \|u\|_{L^p}).$$

We estimate

$$\|(L_\alpha - L)\rho_\alpha u\|_{L^p} \leq \frac{1}{N} \|L\|_{C^1} \|\rho_\alpha u\|_{W^{1,p}}.$$

Combining everything, we obtain

$$\|\rho_\alpha u\|_{W^{1,p}} \leq C \left[\|L\rho_\alpha u\|_{L^p} + \frac{\|L\|_{C^1}}{N} \|\rho_\alpha u\|_{W^{1,p}} + C(N) \|u\|_{L^p} \right].$$

Picking N large enough so $C \|L\|_{C^1} / N < 1/2$, we conclude

$$\|\rho_\alpha u\|_{W^{1,p}} \leq 2C [\|L\rho_\alpha u\| + C(N) \|u\|_{L^p}],$$

as desired. This completes the proof of the elliptic estimates for first order operators on \mathbb{T}^d .

We can now complete the proof of the interior elliptic estimates:

Theorem 44 (interior elliptic estimates for $W^{1,p}$). Let Y be a compact manifold with boundary, and let L be a first order elliptic operator $E_1 \rightarrow E_2$. There is a constant $C(p)$ so that for all $u \in W_0^{1,p}(Y, E_1)$, we have

$$\|u\|_{W^{1,p}} \leq C(\|Lu\|_{L^p} + \|u\|_{L^p}).$$

Proof. First we show that it holds locally on Y . Cover Y by finitely many compact balls (or half balls) with boundary U_α , each equipped with an embedding $\varphi_\alpha : U_\alpha \rightarrow \mathbb{T}^d$ trivializing E_1, E_2 .

Using the connection and metric U_α inherits from \mathbb{T}^d and the trivial bundle on \mathbb{T}^d , we can write

$$L = A_\alpha(\nabla u_\alpha) + B_\alpha(u_\alpha) \text{ for } u_\alpha \in W_0^{1,p}(U_\alpha).$$

The tensors A_α, B_α are defined on the codimension 0 image of U_α in \mathbb{T}^d . Taking a tubular neighborhood of $U_\alpha \subset \mathbb{T}^d$, we can extend A_α, B_α smoothly to global sections on \mathbb{T}^d of the appropriate bundles. Then we conclude for $u \in W_0^{1,p}(U_\alpha) \subset W_0^{1,p}(\mathbb{T}^d)$

$$(*) \quad \|u\|_{W^{1,p}} \leq C(\|Lu\|_{L^p} + \|u\|_{L^p}),$$

this implies the same result when we use the norms that $W_0^{1,p}(B_\alpha)$ inherits from $W_0^{1,p}(Y)$, since the Sobolev norms defined on a compact manifold do not depend on the metric/connection data used (up to norm equivalence).

Now let ρ_α be a partition of unity for the balls U_α . We estimate

$$\|u\|_{W^{1,p}} \leq \sum_\alpha \|\rho_\alpha u\|_{W^{1,p}} \leq C \sum_\alpha \|\rho_\alpha Lu\|_p + \|[L, \rho_\alpha]u\|_{L^p} + \|\rho_\alpha u\|_{L^p},$$

for some large C (the maximum of the constants we were guaranteed in $(*)$). We can estimate the three terms appearing in the sum

$$\begin{aligned} \sum_\alpha \|\rho_\alpha Lu\|_{L^p} &\leq (\text{number of balls}) \|Lu\|_{L^p} \\ \sum_\alpha \|[L, \rho_\alpha]u\|_{L^p} &\leq \left[\sum_\alpha \|[L, \rho_\alpha]\|_{C^0} \right] \|u\|_{L^p} \\ \sum_\alpha \|\rho_\alpha u\|_{L^p} &\leq (\text{number of balls}) \|u\|_{L^p}, \end{aligned}$$

we conclude for $u \in W_0^{1,p}$ that

$$\|u\|_{W^{1,p}} \leq C(\|Lu\|_{L^p} + \|u\|_{L^p}).$$

This completes the proof. \square

Exercise 21. Prove that if $u \in W_0^{1,p}(Y)$ then $\rho_\alpha u \in W_0^{1,p}(B_\alpha)$. Hint: the boundary case requires some proof since, since ρ_α is potentially non-vanishing on the boundary face $B_\alpha \cap \partial Y$.

Elliptic bootstrapping.

Theorem 45. Let $u \in L^p(Y)$ be supported away from ∂Y , and suppose that $Lu \in L^p(Y)$ (where Lu is defined by duality using the adjoint L^*). Then $u \in W_0^{1,p}(Y)$.

Remark. Here u being “supported away from ∂Y ” is equivalent to the existence of a bump function ρ supported in $\text{int}(Y)$ so that $\rho u = u$.

Remark. The idea for the proof (in the case when $Y = B(1)$) will be to consider the convolutions $\rho_\delta * u$ (which are well defined elements of $W_0^{1,p}(B(1))$ for δ sufficiently small because of our assumption on the support of u). If we can show that $\rho_\delta * u$ is uniformly bounded in $W_0^{1,p}$, then reflexivity of $W_0^{1,p}$, and the fact that $\rho_\delta * u \rightarrow u$ in L^p , we can deduce that u is in fact in $W_0^{1,p}$.

Definition 46. Let $T_\delta u = \rho_\delta * u$, i.e.

$$T_\delta(u) = \int_{\mathbb{R}^d} u(y)(\tau^{-x}\rho_\delta)(y) dy,$$

where $\rho : B(1) \rightarrow [0, \infty)$ is a radially symmetric bump function with integral 1. This formula uses the antipodal symmetry of ρ_δ .

This defines a continuous linear map $L^p(B(1)) \rightarrow C^\infty(B(1+\delta))$, and moreover

$$\rho_\delta * u \rightarrow u \text{ in } L^p.$$

The key part of the proof is the following “commutator estimate”

Lemma 47. Let L be a first order differential operator on \mathbb{R}^d . There is a constant C independent of δ so that for all $u \in L^p(B(1))$ we have

$$\|[L, T_\delta]u\|_{L^p} \leq C \|u\|_{L^p}.$$

Proof (of lemma). We will prove the case when $L = A(\nabla u)$. The reader can prove the case $L = B(u)$, and thereby obtain it for all first order differential operators. Let g be a test section in L^q .

We compute

$$\langle A(\nabla T_\delta(u)), g \rangle = \langle T_\delta(\nabla u), A^*g \rangle = \int \langle \nabla u(y), \tau^{-x}\rho_\delta(y)A_x^*g(x) \rangle dy.$$

Using the adjoint of ∇ (with respect to the y -variable, we consider x to be fixed for the moment), we conclude

$$\langle LT_\delta u, g \rangle = \langle A(\nabla T_\delta(u)), g \rangle = \int \langle u(y), \tau^{-x}[\nabla^*, \rho_\delta]A_x^*g(x) \rangle dy,$$

where we use the fact that ∇ commutes with τ^{-x} and that $\nabla^* A_x^* g(x) = 0$, since we consider x fixed.

Exercise 22. Prove that the symbol of $\nabla^* : \text{Hom}(TM, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is

$$[\nabla^*, f] = \text{grad}(f) \lrcorner (-).$$

This is a tensor, and hence ∇^* is a first order operator.

Exercise 23. Prove that if w is a constant section of $\text{Hom}(TM, \mathbb{R}^n)$, then $\nabla^* w = 0$. Hint:

$$\int \langle \nabla^* w, u \rangle \, d\text{Vol} = \int \langle w, \nabla u \rangle \, d\text{Vol}$$

Write

$$w = \sum dx_i \otimes v_i \text{ and } \nabla u = \sum dx_i \otimes \frac{\partial u}{\partial x_i}.$$

Conclude that

$$\int \langle w, \nabla u \rangle \, d\text{Vol} = \sum_i \int \left\langle v_i, \frac{\partial u}{\partial x_i} \right\rangle \, d\text{Vol} = 0,$$

since v_i are constant (use compatibility of the connection with $\langle -, - \rangle$).

Continuing with the proof of the claim, we now explore the term $T_\delta Lu$.

$$\begin{aligned} \langle T_\delta Lu, g \rangle &= \int_{\mathbb{R}^d} \langle A_y \nabla_y u, \tau^{-x} \rho_\delta(y) g(x) \rangle \, dy \\ &= \int_{\mathbb{R}^d} \langle u, \tau^{-x} [\nabla^*, \rho_\delta] A_y^* g(x) \rangle \, dy + \int_{\mathbb{R}^d} \langle u, \tau^{-x} \rho_\delta \cdot \nabla^* (A_y^* \cdot g(x)) \rangle \, dy. \end{aligned}$$

Therefore

$$\langle T_\delta Lu - LT_\delta u, g \rangle = \int_{\mathbb{R}^d} \langle u, \tau^{-x} [\nabla^*, \rho_\delta] (A_y^* - A_x^*) g(x) \rangle \, dy + \int_{\mathbb{R}^d} \langle u, \tau^{-x} \rho_\delta \cdot \nabla^* (A_y^* \cdot g(x)) \rangle \, dy.$$

Taking absolute values produces

$$(*) \quad |\langle T_\delta Lu - LT_\delta u, g \rangle| \leq [\|\rho_\delta\|_{C^1} \cdot \text{osc}(A^*; \delta) |g(x)| + \|A^*\|_{C^1} |g(x)|] \int_{x+B(\delta)} |u(y)| \, dy.$$

where

$$\text{osc}(A^*; \delta) = \sup_{|x-y| < \delta} |A^*(y) - A^*(x)|.$$

Integrating $(*)$ over $B(1)$ conclude

$$|(T_\delta Lu - LT_\delta u, g)| \leq [\|\rho_\delta\|_{C^1} \text{osc}(A^*; \delta) + \|A^*\|_{C^1}] \left\| \int_{x+B(\delta)} u(y) \, dy \right\|_{L^p} \|g\|_{L^q},$$

and hence, conclude

$$\| [T_\delta, L] u \|_{L^p} \leq C [\|\rho_\delta\|_{C^1} \text{osc}(A^*; \delta) + \|A^*\|_{C^1}] \|u\|_{L^p},$$

where we have used the (easy) fact that

$$\left\| \int_{x+B(\delta)} u(y) \, dy \right\|_{L^p} \leq \|u\|_{L^p}.$$

This can be proved by pairing with a test function. Now observe that $\text{osc}(A^*; \delta) < \|A^*\|_{C^1} \delta$, and $\|\rho_\delta\|_{C^1} = \delta^{-1} \|\rho_1\|_{C^1}$, so

$$\|[T_\delta, L]u\|_{L^p} \leq C(1 + \|\rho_1\|_{C^1}) \|A^*\|_{C^1} \|u\|_{L^p}.$$

This proves the claim, as we can take C to be $C(1 + \|\rho_1\|_{C^1}) \|A^*\|_{C^1}$ which does not depend on δ . \square

Now we can prove the theorem.

Proof. We are given $u, Lu \in L^p$ and we know that u is supported away from the boundary, and we want to show that $u \in W_0^{1,p}$.

First suppose that u is supported in $B(r) \subset B(1)$ for some $r < 1$. For $\delta < r$, $\rho_\delta * u$ is supported in $B(1)$. We compute

$$L(\rho_\delta * u) = \rho_\delta * (Lu) + [L, T_\delta]u,$$

and hence

$$(*) \quad \|L(\rho_\delta * u)\|_{L^p} \leq \|Lu\|_{L^p} + C \|u\|_{L^p},$$

where the constant C is given by our “commutator estimate.”

By the interior elliptic estimates applied to $\rho_\delta * u \in W_0^{1,p}$, together with $(*)$ we conclude

$$\|\rho_\delta * u\|_{W^{1,p}} \leq C(\|Lu\|_{L^p} + \|u\|_{L^p}).$$

Therefore $\rho_\delta * u$ is uniformly bounded in $W_0^{1,p}$. It follows (by reflexivity of $W_0^{1,p}$) that $\rho_\delta * u$ converges weakly in $W_0^{1,p}$ (after taking a subsequence). Since we know that $\rho_\delta * u$ converges to u in L^p , we conclude that u must be equal to the weak limit of $\rho_\delta * u$ in $W_0^{1,p}$, and hence u is in $W_0^{1,p}$, as desired.

More generally, suppose that u is defined on an arbitrary compact manifold with boundary Y . Since the support of u is separated from the boundary, we can find balls

$$B_1(1), \dots, B_n(1) \subset \text{int}(Y)$$

so that the balls $B_1(r), \dots, B_n(r)$ cover the support of u . Picking a partition of unity ρ_1, \dots, ρ_n so ρ_i is supported in $B_i(r)$, conclude that

$$L\rho_i u = \rho_i Lu + [L, \rho_i]u \in L^p \implies \rho_i u \in W_0^{1,p} \implies u \in W_0^{1,p}.$$

This proves the theorem. \square

Differentiating the equation. The goal of this section will be to prove the following result

Theorem 48 (interior bootstrapping). Let Y be a compact manifold with boundary, and $L : E_1 \rightarrow E_2$ a first order elliptic operator. Then for sections u supported away from ∂Y we

have

$$u \in L^p(Y) \text{ and } Lu \in W_0^{k,p}(Y) \implies u \in W_0^{k+1,p}(Y).$$

The key idea needed to prove this is described in the following exercise:

Exercise 24. If L is elliptic, then there exists an elliptic first order operator

$$L' : \text{Hom}(TM, E_1) \rightarrow \text{Hom}(TM, E_2)$$

so that

$$\nabla L(u) = L'(\nabla u) + \text{first order operator}.$$

Hint: writing $L(u) = A(\nabla u) + B(u)$, observe that

$$\nabla L(u) = A(\nabla \nabla u) + B(\nabla u) + \nabla A(\nabla u) + \nabla B(u),$$

therefore we can define

$$L'(\Phi) = A(\nabla \Phi) + B(\Phi).$$

We claim that L' is still elliptic. To see why, compute

$$[L', f](\Phi) = A(df \otimes \Phi),$$

and since $A(\xi \otimes -) : E_1 \rightarrow E_2$ is an isomorphism if $\xi \neq 0$, we conclude the induced map on $\text{Hom}(TM, E_1) \rightarrow \text{Hom}(TM, E_2)$ is still an isomorphism.

We adopt the convention that we have chosen extensions of L to elliptic operators

$$\text{Hom}(TM^{\otimes n}, E_1) \rightarrow \text{Hom}(TM^{\otimes n} E_2)$$

so that $[L, \nabla]$ is a first order operator.

Proof (interior bootstrapping). We will prove that

$$(*) \quad u \text{ supported away from } \partial Y \text{ and } u, Lu \in W_0^{k,p} \implies u \in W_0^{k+1,p}.$$

by induction. We already know it for $k = 0$. Now suppose that $u, Lu \in W_0^{k,p}$. Then $\nabla u \in W_0^{k-1,p}$, and

$$L'\nabla u = \nabla Lu + [L', \nabla]u \in W_0^{k-1,p}.$$

Since L' is elliptic, and we suppose that we know the $k-1$ case of $(*)$ for all elliptic operators (in particular, for L'), we conclude that

$$\nabla u \in W_0^{k,p},$$

and hence $u \in W_0^{k+1,p}$, as desired. This proves $(*)$ for all k . It is clear that knowing $(*)$ for all k implies

$$u \in L^p(Y) \text{ and } Lu \in W_0^{k,p}(Y) \implies u \in W_0^{k+1,p}(Y).$$

for sections u supported away from ∂Y . This completes the proof. \square

Using the same “differentiating the equation” trick, we can upgrade the interior elliptic estimates:

Theorem 49 (interior elliptic estimates, for all k). Let L be an elliptic differential operator. There is a constant C_k (depending on L and k) so that for all $u \in W_0^{k+1,p}(Y)$ we have

$$(*) \quad \|u\|_{W^{k+1,p}} \leq C_k (\|Lu\|_{W^{k,p}} + \|u\|_{L^p}).$$

Proof. The case when $k = 0$ is already known. Suppose we have found constants C_i so that $(*)$ holds for $i < k$.

Let $u \in W_0^{k+1,p}(Y)$. Then $\nabla u \in W_0^{k,p}(Y)$, and hence we can apply the $k-1$ elliptic estimate to ∇u

$$\|\nabla u\|_{W^{k,p}} \leq C_{k-1} (\|L'\nabla u\|_{W^{k-1,p}} + \|\nabla u\|_{L^p}).$$

It follows that

$$\begin{aligned} \|\nabla u\|_{W^{k,p}} &\leq C_{k-1} (\|\nabla(Lu)\|_{W^{k-1,p}} + \|[L, \nabla]u\|_{W^{k-1,p}} + \|\nabla u\|_{L^p}) \\ &\leq C' (\|Lu\|_{W^{k,p}} + \|u\|_{W^{k,p}} + \|u\|_{W^{1,p}}) \\ &\leq C_k (\|Lu\|_{W^{k,p}} + \|u\|_{L^p}), \end{aligned}$$

where we have used the fact that $[L, \nabla]$ is first order operator, and then used the $0, k-1$ cases of the elliptic estimates. It is clear that constant C_k does not depend on the u we started with. This completes the proof. \square

Local elliptic regularity.

Definition 50. Let Y be a potentially non-compact manifold. We define

$$(*) \quad W_{\text{loc}}^{k,p}(Y) = \lim W^{k,p}(Z) \text{ where } Z \text{ compactly supported in } \text{int}(Y).$$

Note that the set $W_{\text{loc}}^{k,p}(Y)$ can be thought of as a subset of distributions $C_c^\infty(Y, E_1) \rightarrow \mathbb{R}$ (or \mathbb{C}). The definition in $(*)$ also tells us the topology of $W_{\text{loc}}^{k,p}(Y)$.

Theorem 51. Let L be an elliptic first order operator on Y (potentially non-compact). Then

$$(1) \quad u \in L_{\text{loc}}^p(Y) \text{ and } Lu \in W_{\text{loc}}^{k,p}(Y) \iff u \in W_{\text{loc}}^{k+1,p}(Y)$$

and the map

$$(2) \quad u \in W_{\text{loc}}^{k+1,p}(Y) \rightarrow u \times Lu \in L_{\text{loc}}^p(Y) \times W_{\text{loc}}^{k,p}(Y),$$

is a closed embedding.

Remark. The fact that (2) defines a closed embedding is closely related to the interior elliptic estimates.

Proof. The proof of (1) is quite easy, since we already know a similar statement in the interior case. Here is the argument: if $u \in L_{\text{loc}}^p$ and $Lu \in W_{\text{loc}}^{k,p}$, then for any test function ρ we have

$$(*) \quad \rho u \in L^p \text{ and } L\rho u = \rho Lu + [L, \rho]u \in L^p,$$

(since $[L, \rho]$ is a tensor with compact support, $[L, \rho]u \in L^p$). By interior bootstrapping we conclude $\rho u \in W_0^{1,p}$. By taking a locally finite partition of unity, we conclude $u \in W_{\text{loc}}^{1,p}$.

Now we repeat the argument, but now that we know that $u \in W_{\text{loc}}^{1,p}$, we can conclude that $L\rho u \in W_0^{1,p}$, and hence conclude $\rho u \in W_0^{2,p}$, etc. We conclude that (1) holds.

To prove (2), we will use the interior elliptic estimates. It is easy to show that $W_{\text{loc}}^{k,p}$ is a metrizable space, and hence to prove that the map in (2) is a closed embedding, it suffices to consider sequences of elements. Let u_n be a sequence in $W_{\text{loc}}^{k+1,p}$, and suppose that $u_n \rightarrow u_\infty$ in L_{loc}^p and Lu_n converges in $W_{\text{loc}}^{k,p}$. Let ρ be some bump function, say $\rho = 1$ on the compact set Z . By the interior elliptic estimates we conclude that ρu_n is Cauchy in $W^{1,p}$, and hence u_n converges in $W_{\text{loc}}^{1,p}$. By repeating the argument, conclude that ρu_n is Cauchy in $W^{2,p}$, etc. Ultimately deduce that u_n converges in $W_{\text{loc}}^{k+1,p}$. This completes the proof. \square

Definition 52. We define the topology on the space C^∞ by a limit

$$C_{\text{loc}}^\infty(Y) = \lim_{k \rightarrow \infty} C_{\text{loc}}^k(Y),$$

where loc indicates that we are not thinking of C^k as a normed space, but rather as the limit of the normed spaces $C^k(Z)$ as Z ranges over compact subdomains of Y .

An easy consequence of the Morrey embedding theorem, we conclude that

$$(*) \quad C_{\text{loc}}^\infty(Y) \simeq \lim_k W_{\text{loc}}^{k,p}(Y).$$

Theorem 53. Let Y be a manifold, and let L be a first order elliptic operator on Y . Then the map

$$u \in C_{\text{loc}}^\infty(Y) \mapsto u \times Lu \in L_{\text{loc}}^p \times C_{\text{loc}}^\infty(Y)$$

is a closed embedding.

Proof. Let $u_n \in C_{\text{loc}}^\infty(Y)$ be a sequence so that u_n converges in L_{loc}^p and Lu_n converges in C_{loc}^∞ .

Clearly Lu_n converges in $W_{\text{loc}}^{k,p}$, for every k , and hence u_n converges in $W_{\text{loc}}^{k,p}$ for every k by Theorem 51. By (*), we conclude that u_n converges in C_{loc}^∞ , as desired. \square

Example 54. Let u_n be a sequence of holomorphic functions in $D(1)$ so that u_n converges to u_∞ in L^p . Then u_∞ is C^∞ smooth, holomorphic, and $u_n \rightarrow u_\infty$ converges in C_{loc}^∞ .

To see that u_∞ is C^∞ smooth, use the fact that $u \in C_{\text{loc}}^\infty \mapsto u \times \bar{\partial}u \in L^p \times C_{\text{loc}}^\infty$ is a closed embedding. Therefore the sequence u_n of holomorphic functions converges in C_{loc}^∞ . The limit must obviously equal u , and hence $u \in C_{\text{loc}}^\infty$ and $u_n \rightarrow u$ in C_{loc}^∞ . It is now clear that u is holomorphic.

Fredholm theory of elliptic operators on closed manifolds. In this section we will work with $p = 2$. It will simplify things without any loss of generality concerning the final statement we will prove.

Consider an elliptic operator $L : E_1 \rightarrow E_2$ on a closed manifold and its adjoint

$$L^* : E_2 \rightarrow E_1,$$

and consider the spaces $W^{k,2}(E_i)$, $k = 0, 1, 2, \dots$.

Let's agree to write L_k for the induced map $W^{k+1,2} \rightarrow W^{k,2}$.

Lemma 55. The map L_k has closed range and the kernel of L_k is finite dimensional.

Proof. Since

$$\|u\|_{W^{k+1,2}} \leq C_k(\|Lu\|_{W^k} + \|u\|_{L^p}),$$

we can easily deduce that L_k has finite kernel and closed range. We leave the proof to the reader, with the hint that to prove first that L has finite dimensional kernel, and then to consider L restricted to a complement of the kernel. \square

Lemma 56. The cokernel of L_k is canonically identified with $\text{Hom}(\ker L_0^*, \mathbb{R})$ via the L^2 inner product, and so the cokernel of L_k is also finite dimensional. In other words:

$$\text{im } L_k = (\ker L_0^*)^\perp \cap W^{k,2}.$$

Proof. It is clear that if $w = Lu$ for $u \in W^{k+1,2}$, then $w \in (\ker L_0^*)^\perp$. The reverse inclusion is a bit deeper. Suppose that $w \in (\ker L_0^*)^\perp \cap W^{k,2}$.

First we will prove that $w \in \text{im } L_0$ (i.e. there is a $W^{1,2}$ section u so $L_0 u = w$).

Let's suppose that $w \notin \text{im } L_0$ in search of a contradiction. Since $\text{im } L_0$ has closed range in L^2 , we can find a section v so that $(w, v) \neq 0$, but $(\text{im } L_0, v) = 0$. It follows that $L^* v = 0$ weakly and hence by elliptic regularity, v is of class $W^{1,2}$ and $v \in \ker L_0^*$, contradicting the fact that $w \in (\ker L_0^*)^\perp$. Therefore $w = Lu$ for $u \in W^{1,2}$. By elliptic regularity, since $u \in L^p$ and $Lu \in W^{k,2}$, we conclude $u \in W^{k+1,2}$, and hence $w \in \text{im } L_k$, as desired. \square

Proposition 57. For all k ,

$$\ker L_k = \{u \in C^\infty : Lu = 0\} = \{u \in L^2 : Lu = 0 \text{ weakly}\}.$$

Proof. It is clear that

$$\{u \in C^\infty : Lu = 0\} \subset \ker L_k \subset \{u \in L^2 : Lu = 0 \text{ weakly}\}.$$

Therefore it suffices to show that

$$\{u \in L^2 : Lu = 0 \text{ weakly}\} \subset \{u \in C^\infty : Lu = 0\}.$$

This is a straightforward consequence of elliptic regularity:

$$u \in L_{\text{loc}}^2 \text{ and } Lu \in C_{\text{loc}}^\infty \implies u \in C_{\text{loc}}^\infty.$$

This completes the proof. \square

Corollary 58. Let L be a first order elliptic operator on a closed manifold Y . There are canonical L^2 -orthogonal decompositions

$$W^{k,2}(E_2) = \text{im } L_k \oplus \ker L^* \text{ and } W^{k,2}(E_1) = \text{im } L_k^* \oplus \ker L,$$

including the case when $k = \infty$ (where $W^{\infty,2} = C^\infty$).

Proof. We already know that $\ker L^* \subset W^{k,2}(E_2)$ and $\ker L \subset W^{k,2}(E_1)$ for all k . To prove that

$$W^{k,2}(E_2) = \operatorname{im} L_k \oplus \ker L^*,$$

it suffices to observe that if $u \in W^{k,2}(E_2)$ is orthogonal to $\ker L^*$, then $u \in \operatorname{im} L_k$.

Now when $k = \infty$, we once again notice that it suffices to prove that if $u \in C^\infty(E_2)$, and $u \perp \ker L^*$, then $u \in \operatorname{im} L$. This is easy, since we know that $u = Lw$ for some $w \in W^{1,2}$, and hence by regularity w is actually in $C^\infty(E_2)$, as desired. This completes the proof. \square

We may care about similar decompositions of $W^{k,p}$ in the case when $p \neq 2$. A very similar argument proves the following:

Theorem 59. Consider the split injections

$$\ker L \rightarrow W^{k,p}(E_1) \text{ and } \ker L^* \rightarrow W^{k,p}(E_2),$$

where the splittings are given by the L^2 orthogonal projections, i.e.

$$u \mapsto (u, v_1)v_1 + \cdots + (u, v_n)v_n.$$

These splittings are obviously continuous in the $W^{k,p}$ topology. With respect to these splittings, we have

$$W^{k,p}(E_2) = \operatorname{im} L_k \oplus \ker L^* \text{ and } W^{k,p}(E_1) = \operatorname{im} L_k^* \oplus \ker L.$$

Example 60. Consider $L = \bar{\partial}$ on \mathbb{T}^2 . As we have seen, this is an elliptic operator, and $L^* = -\partial$. It is easy to show using Fourier coefficients that

$$\ker L = \ker L^* = \mathbb{C} \text{ (the constants)}.$$

As corollary of the preceding decomposition theorem, we conclude that for $u \in W^{k,p}$

$$\text{if } \int u \, d\operatorname{Vol} = 0 \text{ then } u = \bar{\partial}w \text{ for unique } W^{k+1,p} \text{ satisfying } \int w \, d\operatorname{Vol} = 0.$$

As a consequence, we can prove the following local surjectivity result: the map of sheaves

$$\bar{\partial} : W_{\operatorname{loc}}^{k+1,p} \rightarrow W_{\operatorname{loc}}^{k,p}.$$

is locally surjective (on any Riemann surface).

Proof. It suffices to prove that for $w \in W_{\operatorname{loc}}^{k,p}(D(1))$, we can find $u \in W_{\operatorname{loc}}^{k+1,p}(D(r))$ so $\bar{\partial}u = w$ on $D(r)$. Let ρ be a bump function equal to 1 on a neighborhood of $D(r)$ and which is supported in $D(1)$. Then $\rho u \in W_0^{k,p}(D(1))$. By embedding $D(1)$ in the torus, it follows that there is a $W^{k+1,p}(\mathbb{T}^2)$ section v so that

$$\bar{\partial}v = \rho u + c$$

for some constant c . Let $\varphi(z) = cz$ on a neighborhood of $D(r)$. Then

$$\bar{\partial}(v - \rho\varphi) = u \text{ on } D(r),$$

as desired. □

Example 61 (the parametrix). Consider the splitting

$$W^{k,p}(E_2) = \text{im } L_k \oplus \ker L^*.$$

Let's define p to be the projection onto $\ker L^*$, and q to be the projection onto $\ker L$, then

$$P_k = (1 - q) \circ L_k^{-1} \circ (1 - p)$$

is well-defined: i.e. there is a unique element in $u \in W^{k+1,p}(E_2)$ so that $u \perp \ker L$ and $L_k u = (1 - p)w$ for each $w \in W^{k,p}(E_2)$. The open mapping theorem guarantees that P_k is continuous, i.e. L_k^{-1} is continuous on $(\ker L^*)^\perp \rightarrow (\ker L)^\perp$.

It is clear, by uniqueness, that $P_k = P_{k+1}$ on $W^{k+1,p} \subset W^{k,p}$. Therefore we obtain a well-defined continuous map $P : C^\infty \rightarrow C^\infty$ which equals P_k on $W^{k,p}$. Note that

$$1 - PL = \text{projection onto } \ker L$$

$$1 - LP = \text{projection onto } \ker L^*.$$

Homotopy invariance of the Fredholm index. Let $L : E_1 \rightarrow E_2$ be an elliptic operator on a closed manifold Y . Define the index of L to be $\dim \ker L - \dim \ker L^*$.

The goal of this section will be to show that the index is constant on a continuously varying family of elliptic operators.

More generally, if X, Y are any Banach spaces, consider the space $\mathcal{F}(X, Y)$ of operators $L : X \rightarrow Y$, equipped with the distance induced by the operator norm

$$d(L, L') = \|L - L'\|$$

and which satisfy the Fredholm condition:

L has finite dimensional kernel, closed image, and finite dimensional cokernel.

Theorem 62. The index function $\mathcal{F}(X, Y) \rightarrow \mathbb{Z}$ is continuous.

There is a slight trick to the proof. It is useful to introduce one concept related to Fredholm maps.

Definition 63. Let $L : X \rightarrow Y$ be a bounded linear operator. A **regularization** of L is the data of two finite dimensional vector spaces and maps a, b, c so that

$$(*) \quad \begin{bmatrix} L & a \\ b & c \end{bmatrix} : X \oplus V \rightarrow Y \oplus W$$

is an isomorphism.

Proposition 64. A map $L : X \rightarrow Y$ is Fredholm if and only if it admits a regularization.

Proof. Suppose L is Fredholm. Take $V = \text{coker } L$, $W = \ker L$. Choosing a splitting j of $\pi : Y \rightarrow \text{coker } L$ and a splitting p of $i : \ker L \rightarrow X$ gives a map

$$\begin{bmatrix} L & j \\ p & 0 \end{bmatrix} : X \oplus \text{coker } L \rightarrow Y \oplus \ker L$$

which the reader can check is bijective, and hence is an isomorphism.

Conversely, suppose that L admits a regularization of the form $(*)$. Since $(*)$ is an isomorphism, the map $x \in X \mapsto Lx \times bx \in Y \times W$ is a closed embedding, and hence we conclude a bound

$$(1) \quad \|x\|_X \leq C(\|Lx\|_Y + \|bx\|_W).$$

Since $x \mapsto bx$ is a finite rank operator (in particular, is compact), (1) implies that L has finite dimensional kernel and closed image.

We will prove that $\text{coker } L$ is finite dimensional arguing by contradiction. Suppose that $\text{coker } L$ is infinite dimensional. Then we can find unit vectors y_1, y_2, y_3, \dots in Y so that

$$\|y_i - \text{im } L - \text{span}(y_1, \dots, y_{i-1})\| > 1/2.$$

Since $(*)$ is an isomorphism, we can find vectors x_i, v_i so $\|x_i\| + \|v_i\|$ is bounded so that

$$Lx_i + a(v_i) = y_i \implies \|a(v_i) - a(v_j)\| > 1/2 \text{ for all } i \neq j,$$

in particular

$$(2) \quad \|a\| \|v_i - v_j\| > 1/2 \text{ for all } i \neq j.$$

Since any bounded ball in V is precompact, a subsequence of v_1, v_2, \dots must converge, contradicting (2). Therefore $\text{coker } L$ is finite dimensional. \square

Proposition 65. Suppose that $L : X \rightarrow Y$ admits a regularization a, b, c, V, W . There is $\epsilon > 0$ so that any map L' within ϵ of L is also regularized by a, b, c, V, W .

Proof. This is easy, since we know that being an isomorphism is open condition in the space of maps, so that there is ϵ so that

$$\left\| \Lambda - \begin{bmatrix} L & a \\ b & c \end{bmatrix} \right\| < \epsilon \implies \Lambda \text{ is an isomorphism.}$$

In particular, if $\|L' - L\| < \epsilon$, then

$$\left\| \begin{bmatrix} L' & a \\ b & c \end{bmatrix} - \begin{bmatrix} L & a \\ b & c \end{bmatrix} \right\| < \epsilon \implies \begin{bmatrix} L' & a \\ b & c \end{bmatrix} \text{ is an isomorphism.}$$

As a corollary, the space of Fredholm maps $\mathcal{F}(X, Y)$ is open in the operator norm. \square

Proposition 66. Let $L : X \rightarrow Y$ be a Fredholm operator and suppose a, b, c, V, W is a regularization for L . Then

$$\dim \ker L - \dim \operatorname{coker} L = \dim W - \dim V.$$

Proof. Since X is Fredholm, we can split X, Y as follows

$$X = X' \oplus \ker L \text{ and } Y = \operatorname{im} L \oplus \operatorname{coker} L,$$

where X' is complimentary to $\ker L$, and so that $L' = L|_{X'} : X' \rightarrow \operatorname{im} L$ is an isomorphism. Then the matrix of the regularization takes the form

$$R = \begin{bmatrix} L' & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1 & b_2 & c \end{bmatrix} : X' \oplus \ker L \oplus V \rightarrow \operatorname{im} L \oplus \operatorname{coker} L \oplus W$$

We will construct an isomorphism $\varphi : \ker L \oplus V \rightarrow \operatorname{coker} L \oplus W$. We begin by computing

$$\begin{bmatrix} L' & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1 & b_2 & c \end{bmatrix} \begin{bmatrix} x \\ k \\ v \end{bmatrix} = \begin{bmatrix} L'x + a_1(v) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha(x, k, v) \\ \beta(x, k, v) \end{bmatrix}.$$

Motivated by this computation, define $x(v) \in X'$ *uniquely* by the equation $L'x(v) + a_1(v) = 0$. Then define φ by

$$\varphi \begin{bmatrix} k \\ v \end{bmatrix} = R \begin{bmatrix} x(v) \\ k \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha(x(v), k, v) \\ \beta(x(v), k, v) \end{bmatrix}$$

We claim that φ is a bijection. Injectivity is easy: if $\varphi(k, v) = 0$, then

$$R \begin{bmatrix} x(v) \\ k \\ v \end{bmatrix} = 0 \implies \begin{bmatrix} k \\ v \end{bmatrix} = 0.$$

Conversely, since R is surjective, there is some x, k, v so that

$$R \begin{bmatrix} x \\ k \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \implies x = x(v) \implies \varphi \begin{bmatrix} k \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Therefore φ is an isomorphism $\ker L \oplus V \rightarrow \operatorname{coker} L \oplus W$. It follows that

$$\dim \ker L + \dim V = \dim \operatorname{coker} L + \dim W \implies \dim \ker L - \dim \operatorname{coker} L = \dim W - \dim V,$$

as desired. \square

Theorem 67 (homotopy invariance of the Fredholm index). Let X, Y be Banach spaces and consider the space $\mathcal{F}(X, Y)$ of Fredholm maps $X \rightarrow Y$. Then the index function $\mathcal{F}(X, Y) \rightarrow \mathbb{Z}$

defined by

$$\text{index}(L) = \dim \ker L - \dim \text{coker } L$$

is continuous.

Proof. If L has index n , then we can regularize L to an isomorphism

$$\begin{bmatrix} L & a \\ b & c \end{bmatrix} : X \oplus V \rightarrow Y \oplus W.$$

In particular

$$(1) \quad \text{index}(L) = \dim W - \dim V$$

For L' sufficiently close to L , we know that

$$\begin{bmatrix} L' & a \\ b & c \end{bmatrix} : X \oplus V \rightarrow Y \oplus W.$$

is still an isomorphism. It follows that

$$(2) \quad \text{index}(L') = \dim W - \dim V$$

Comparing (1) and (2), we see that $\text{index}(L) = \text{index}(L')$, as desired. \square

Corollary 68. Let L_t be a continuous family of Elliptic operators on a closed manifold Y . It is easy to show that L_t is continuous in the operator norm topology on $W^{1,p}(Y) \rightarrow L^p(Y)$. It follows that

$$\text{index}(L_1) = \text{index}(L_0).$$