MATH 257C

CONTENTS

April 2 (Sarah)

Goals of the course:

(i) We would like to develop basic Hamiltonian Floer theory in such a way that we understand what we are blackboxing and where in the literature we would find the proofs. We would also like to keep Morse theory running in parallel for comparison's sake.

(ii) We would like to discuss some recent developments in the theory.

Fundamental analytic program for obtaining moduli spaces which lead to invariants in Floer or Morse theory (see Schwarz's book):

(1) analytic setup, definition of functional spaces of solutions/trajectories (moduli spaces are cut out by equations)

(2) analyze the index problem (ideally, we want to think of our spaces as finite-dimensional manifolds, and the index should theoretically give the dimension)

(3) transversality (we want to give our spaces "manifold-like" structures)

(4) compactness (we want to be able to count things, and counts need to be finite)

(5) gluing (if we add points to compactify, they should have neighborhoods which look appropriately manifold-like; this is a sort of converse to compactness)

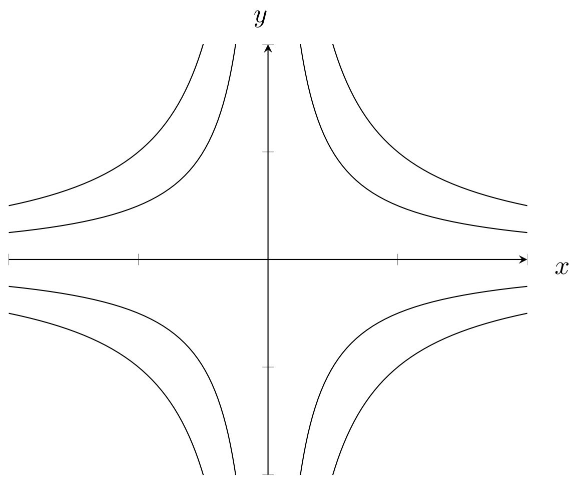
(6) coherent orientations (when we count, we want to keep track of sign)

Today we start with an exercise in Morse theory; it will illustrate steps (1), (2), and (3).

Theorem 1 (Local stable manifold theorem)**.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function whose only critical point is 0. We assume that this critical point is Morse, in the sense that the Hessian at this point is non-degenerate. Fix a Riemannian metric g on $\mathbb{R}^n$, and let grad be the gradient of ϕ with respect to g. Let $S \subset \mathbb{R}^n$ be the stable set of 0 (that is, the set of points which flow to 0 under $-\operatorname{grad}$). Then S is a submanifold of $\mathbb{R}^n$ near 0.

Remark 2. Other proofs of this theorem are also hard. For instance, look up the Hartman-Grobman theorem and note how this does not follow from it.

Example 3. Consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ given by $\phi(x, y) = x^2 - y^2$. Then ϕ is Morse, and its only critical point is $(0, 0)$. Here's a drawing of the flow lines:



Thus the stable set is the x-axis.

Now we proceed to the proof of the stable manifold theorem.

For $x \in \mathbb{R}^n$, the gradient flow line starting at x is the (unique) path $\gamma_x : [0, \delta) \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\dot{\gamma}(t) = -\operatorname{grad}(\gamma(t))$, and $\delta > 0$ is maximal. Note that for $x \in S$, $\delta = \infty$.

The plan is to do the following:

- Identify $S \subset \mathbb{R}^n$ with the set of smooth gradient flow lines starting at points in S.
- Construct a path space P (a Banach manifold) and a Banach space bundle $\mathcal{E} \to P$.
- Define a (Fredholm) section $s : P \to \mathcal{E}$, $\gamma \mapsto \dot{\gamma} + \operatorname{grad} \circ \gamma$ such that $s^{-1}(0) = S$ (this will be non-trivial because P is large).
- Prove $s \pitchfork 0$ and use the implicit function theorem. This will use a baby version of the Atiyah-Patodi-Singer index theorem.

First, we define our Banach space to be the Sobolev space $P = W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Observe that P can also be written as the equivalence classes of functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ which are square-integrable and have weak derivatives f' which are also square-integrable. In this special case, we mean that f is differentiable almost everywhere, and

$$f(t) = f(0) + \int_0^t f'(s)\, ds$$

almost everywhere.

Note that in higher dimensions, we'll have to be more mature about how we define these spaces.

Proposition 4. If $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, then $\gamma(t) \to 0$ as $t \to \infty$.

Proof. The basic idea is that the finiteness of the $(1,2)$-norm gives certain bounds on the first derivative, which forces the variance to approach zero. Then the $(1,2)$-norm also forces the values of the function itself to go to zero.

The hypothesis $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ tells us that the values

$$\left(\int_0^\infty |\gamma|^2\right)^{1/2} = \left(\sum_{N=0}^\infty \int_N^{N+1} |\gamma|^2\right)^{1/2}$$

and

$$\left(\int_0^\infty |\dot\gamma|^2\right)^{1/2} = \left(\sum_{N=0}^\infty \int_N^{N+1} |\dot\gamma|^2\right)^{1/2}$$

are finite. It follows that

$$(\dagger) \qquad \int_N^{N+1} |\gamma|^2 \quad \text{and} \quad \int_N^{N+1} |\dot\gamma|^2$$

approach zero as $N \to \infty$.

Since $\dot\gamma$ is a weak derivative, for almost every $x \in [N, N+1]$ we have

$$\begin{aligned} |\gamma(x) - \gamma(N)| &= \left|\int_N^x \dot\gamma\right| \\ &\leq \sqrt{x-N}\left(\int_N^x |\dot\gamma|^2\right)^{1/2} \\ &\leq \sqrt{x-N}\left(\int_N^{N+1} |\dot\gamma|^2\right)^{1/2} \\ &\leq \left(\int_N^{N+1} |\dot\gamma|^2\right)^{1/2} \end{aligned}$$

(the second line follows from Cauchy-Schwarz). Combining this information with the fact that both expressions in (†) go to zero yields the desired result. ☐

Remark 5. We have constructed a version of the Rellich embedding $W^{1,2}(\mathbb{R}, \cdot) \hookrightarrow C^0(\mathbb{R}, \cdot)$, which sends Sobolev spaces to Hölder spaces.

Remark 6. In the general case, we'll need to build asymptotic conditions by hand. This makes it difficult to turn more general analogues of P into Banach manifolds.

Now let $\mathcal{E} = P \times L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, so $\mathcal{E} \to P$ is a trivial bundle. Define a section $s : P \to \mathcal{E}$ by $s(\gamma) = (\gamma, \dot\gamma + \operatorname{grad} \circ \gamma)$. We do need to check that $\operatorname{grad} \circ \gamma$ is square-integrable, but this shouldn't be too bad because grad is smooth and γ has compact image by Proposition 4. We will also need to demonstrate some kind of regularity for s.

Proposition 7. If $\gamma \in S$ (i.e., γ is a smooth gradient flow line starting at a point in S), then $\gamma \in P = W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Moreover, $S = s^{-1}(0)$.

In order to prove Proposition 7, we will need the following lemma (which we will blackbox for now).

Lemma 8 (Exponential convergence of flow lines at the ends)**.** If $\gamma \in S$, then there is some $a > 0$ so that $|\gamma(t)| \leq e^{-at}$.

Remark 9. Using the gradient flow line equation we obtain the same exponential convergence result for all derivatives of flow lines. This uses a method called *bootstrapping.*

If $\gamma \in S$, then Lemma 8 tells us $\gamma \in P$, so we have $S \subset s^{-1}(0)$. On the other hand, we still need to check that if $f \in P$ satisfies $s(f) = 0$, then f is a smooth gradient flow line starting at a point in S. The tricky part is verifying that f is smooth. The Rellich embedding $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ forces f to be continuous. Then we use bootstrapping to guarantee that f is actually smooth. We know that f has a weak derivative. Since f is a continuous solution of the gradient flow line equation, we can show that f has a genuine derivative f' which must also be continuous. Repeating this process infinitely shows that f is, in fact, smooth.

April 4 (Dylan)

The goal of this lecture is to complete the proof of the "local stable manifold theorem." Recall the setting:

(i) $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Morse function with exactly one critical point at $0 \in \mathbb{R}^n$,

(ii) g is a Riemannian metric on $\mathbb{R}^n$,

(iii) $\mathrm{grad}_{\varphi,g} =: \mathrm{grad}$ is the gradient vector field associated to g and φ – recall that grad is determined by the relation $g(\mathrm{grad}, X) = \mathrm{d}\varphi(X)$, for all vector fields X.

(iv) The stable set S is defined to be the set of all $x \in \mathbb{R}^n$ so that the negative gradient flow line starting at x converges to 0.

Local Stable Manifold Theorem. There is an open neigborhood U around 0 so that $U \cap S$ is a smooth submanifold of U.

Recall the stategy for proving this theorem. We identify S with the set of gradient flow lines $\mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0, and make the following crucial observation: If $\gamma \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ satisfies the following equation

$$(*) \qquad \gamma'(t) + \mathrm{grad} \circ \gamma = 0,$$

then γ is C^∞ smooth, and conversely, if γ is a gradient flow line converging to 0, then γ lies in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and γ satisfies $(*)$. In other words, if we define

$$s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \text{ by } s(\gamma) = \gamma'(t) + \mathrm{grad} \circ \gamma,$$

then $s^{-1}(0) = S$. This was established in the previous lecture. Here are three exercises related to the content of the previous lecture.

Exercise 1 (Rellich Embedding)**.** Let $\gamma \in C_c^\infty$. Prove that for $t \geq 0$

$$\gamma(t)e^{-t} = \int_t^\infty \gamma(s)e^{-s} - \gamma'(s)e^{-s}\, ds,$$

and deduce that

$$|\gamma(t)| \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}([t,\infty),\mathbb{R}^n)}$$

In particular,

$$(\star) \qquad \|\gamma\|_{C^0} \leq \sqrt{2}\, \|\gamma\|_{W^{1,2}}\, .$$

By density of C_c^∞ functions in $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, prove that $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^0(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Moreover, conclude that $|\gamma(t)| \to 0$ as $t \to \infty$. Applying the bound $(\star)$ to the derivatives of γ, conclude that

$$W^{k,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \subset C^{k-1}(\mathbb{R}_{\geq 0}, \mathbb{R}^n),$$

and that $\|\gamma\|_{C^{k-1}} \leq C\, \|\gamma\|_{W^{k,2}}$.

Exercise 2. Show that if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{0,1}$ function (i.e. Lipshitz), and $X(0) = 0$, then the map $\gamma \mapsto X \circ \gamma$ sends $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ to $L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

More generally, if X is a $C^{k,1}$ function with $X(0)=0$, then $\gamma \mapsto X\circ\gamma$ sends $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ to $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$. Hint: for both claims, it suffices to prove it first for smooth γ, and then use the density of smooth functions in $W^{k,2}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$.

Exercise 3. Conclude that if $\gamma \in W^{k,2}$ satisfies the gradient flow line equation $(*)$, then $\gamma \in W^{k+1,2}$. Conclude that if $\gamma \in W^{1,2}$ satisfies $(*)$, then $\gamma \in W^{k,2}$ for all k. By the Rellich embedding $W^{k,2} \subset C^{k-1}$, conclude that $W^{1,2}$ solutions of $(*)$ are C^∞ smooth.

Now we turn to some new material. We will show that $s^{-1}(0)$ is a manifold via the inverse function theorem for Banach spaces.

Definition 10. Let $U \subset X$ and $V \subset Y$ be open subspaces of Banach spaces X, Y. A map $A : U \to V$ is **differentiable** at $u \in U$ provided there is a bounded linear transformation $\mathrm{d}A_u : X \to Y$ so that

$$\lim_{\xi\to 0} \frac{\|A(u+\xi) - A(u) - \mathrm{d}A_u(\xi)\|_Y}{\|\xi\|_X} = 0.$$

While this definition uses the norms on X, Y, it is clear that it only depends on the equivalence classes of the norms.

The map $A : U \to V$ is **continuously differentiable** if $u \mapsto \mathrm{d}A_u \in \mathrm{Hom}(X, Y)$ is a continous function, where the latter is given the topology induced by the “operator norm.” The set of continously differentiable functions is denoted $C^1(U, V)$.

A map A is $C^k(U, V)$, $k \geq 1$, if $\mathrm{d}A$ is $C^{k-1}(U, \mathrm{Hom}(X, Y))$.

In the appendix to this lecture, we define the terms **Banach manifold** and the **tangent bundle** of a Banach manifold.

Inverse Function Theorem. Let X, Y be Banach manifolds, and suppose $A : X \to Y$ is a C^k map so that $\mathrm{d}A_x$ is an isomorphism (i.e. is continuous in the natural topologies on TX_x and $TY_{A(x)}$). Then there are neighborhoods $U \ni x$ and $V \ni y$ so that A maps U to V diffeomorphically.

The derivatives of the vector field grad appear in the statement of the next claim. Thinking of grad is a function $\mathbb{R}^n \to \mathbb{R}^n$, it certainly has a derivative $\mathrm{dgrad}_x : \mathbb{R}^n \to \mathbb{R}^n$ at all points $x \in \mathbb{R}^n$ (warning: here we are *not* thinking of grad as a map $\mathbb{R}^n \to T\mathbb{R}^n$ when we take its derivative). Similarly, we will denote the (symmetric) second derivative matrix by

$$\mathrm{ddgrad}_x : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n.$$

This is not a coordinate invariant notion.

It is clear that if x, y are two points of $\mathbb{R}^n$, then

$$\mathrm{grad}(x+y) - \mathrm{grad}(x) = \left[\int_0^1 \mathrm{dgrad}_{x+sy}\, ds\right] \cdot y,$$

where we interpret the expression in the braces as a matrix $\mathbb{R}^n \to \mathbb{R}^n$.

Claim 11. The map $s : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ defined by $s(\gamma) = \gamma' + \mathrm{grad} \circ \gamma$ is a C^1 map, and it's derivative is given by

$$\mathrm{d}s_\gamma(\eta)(t) = \eta'(t) + \mathrm{dgrad}_{\gamma(t)} \cdot \eta(t)$$

Proof. We begin by computing

$$s(\gamma+\eta)(t)-s(\gamma)(t) = \eta'(t)+\mathrm{grad}(\gamma(t)+\eta(t))-\mathrm{grad}(\gamma(t)) = \eta'(t)+\left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)}\, ds\right]\cdot\eta(t).$$

Hence,

$$s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta) = \left[\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds\right] \cdot \eta(t).$$

It follows that

$$\left\| s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta)\right\|_{L^2} \leq \left\| \int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds \right\|_{C^0} \|\eta\|_{L^2}.$$

We compute

$$\int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds = \left[\int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds\right] \cdot \eta(t),$$

and hence

$$\left\| \int_0^1 \mathrm{dgrad}_{\gamma(t)+s\eta(t)} - \mathrm{dgrad}_{\gamma(t)}\, ds \right\|_{C^0} \leq \left\| \int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds \right\|_{C^0} \|\eta\|_{C^0}.$$

Since ddgrad is a continous function, and $\gamma(t) + rs\eta(t)$ is a bounded function of t (i.e. for $\|\eta\|_{C^0} \leq 1$, we can suppose that $\|\gamma + rs\eta\|_{C^0} \leq R$ for some large R) we conclude some C independent of t and η, $\|\eta\|_{C^0} \leq 1$, so that

$$\left\| \int_0^1 \int_0^1 s\mathrm{ddgrad}_{\gamma(t)+rs\eta(t)}\, dr\, ds \right\|_{C^0} \leq C,$$

and hence

$$\left\| s(\gamma+\eta)(t) - s(\gamma)(t) - \mathrm{d}s_\gamma(\eta)\right\|_{L^2} \leq C\|\eta\|_{L_2} \|\eta\|_{C^0} \leq C' \|\eta\|^2_{W^{1,2}},$$

where we have used the fact that $\|-\|_{C^0} \leq c\|-\|_{W^{1,2}}$ and $\|-\|_{L^2} \leq \|-\|_{W^{1,2}}$. It follows that s is differentiable and its derivative at γ is $\mathrm{d}s_\gamma$.

It is easy to show that $\mathrm{d}s_\gamma$ is a bounded function $W^{1,2} \to L^2$. Finally we show that $\gamma \to \mathrm{d}s_\gamma$ is continuous.

Given two curves γ_1, γ_2, we compute

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \mathrm{dgrad}_{\gamma_1(t)+\gamma_2(t)} \cdot \eta(t) - \mathrm{dgrad}_{\gamma_1(t)} \cdot \eta(t).$$

Arguing as we did above, we conclude

$$\mathrm{d}s_{\gamma_1+\gamma_2}(\eta) - \mathrm{d}s_{\gamma_1}(\eta) = \left[\int_0^1 \mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\, ds \cdot \gamma_2(t)\right] \cdot \eta(t),$$

and similarly to the computations above we conclude

$$\|\mathrm{d}s_{\gamma_1+\gamma_2}(\eta)-\mathrm{d}s_{\gamma_1}(\eta)\|_{L^2}\leq\left\|\int_0^1 \mathrm{ddgrad}_{\gamma_1(t)+s\gamma_2(t)}\,ds\right\|_{C^0}\|\gamma_2(t)\|_{W^{1,2}}\|\eta(t)\|_{W^{1,2}}.$$

We thereby obtain an estimate on the operator norm

$$\|\mathrm{d}s_{\gamma_1+\gamma_2}-\mathrm{d}s_{\gamma_1}\|\leq C\,\|\gamma_2(t)\|_{W^{1,2}},$$

where, for γ_2 close to γ_1, C depends only on $\|\gamma_1\|_{C^0}$ and $\|\mathrm{ddgrad}\|_{C^0(B)}$ for some large ball B. It follows that

$$\lim_{\gamma_2\to 0}\|\mathrm{d}s_{\gamma_1+\gamma_2}-\mathrm{d}s_{\gamma_1}\|=0,$$

and so we have shown that $\gamma\to\mathrm{d}s_\gamma$ is continuous. This completes the proof of the claim. $\square$

Exercise 4. Prove that $\mathrm{d}s_\gamma : W^{1,2}\to L^2$ is a bounded linear operator. Hint: if M is a bounded continuous matrix valued function, and η is in L^2, then $\|M\eta\|_{L^2}\leq\|M\|_{C^0}\|\eta\|_{L^2}$.

Our plan now is to show that the derivative of s at the zero solution $0\in W^{1,2}$ is a Fredholm operator. In fact, we will be able to show that $\mathrm{d}s_0$ is a surjective operator, and we will be able to explicitly identify the kernel of $\mathrm{d}s_0$ as the finite dimensional space spanned the positive eigenvalues of the Hessian of φ.

Definition 12. The **Hessian** of a function $\varphi:\mathbb{R}^n\to\mathbb{R}$ at x is the bilinear form made of the second partial derivatives $\mathrm{Hess}_x=\partial_i\partial_j\varphi(x)\mathrm{d}x_i\otimes\mathrm{d}x_j$. If x is a critical point, then Hess_x is coordinate independent.

In the presence of the metric g, we can define an endomorphism $\mathrm{Hess}^g_x:\mathbb{R}^n\to\mathbb{R}^n$ by

$$g(-,\mathrm{Hess}^g_x(-))=\mathrm{Hess}_x(-,-).$$

Lemma 13. Let $\mathrm{grad}=\mathrm{grad}_{\varphi,g}$ be the gradient vector field of φ, and suppose 0 is a critical point of φ. Then

$$\mathrm{dgrad}_0=\mathrm{Hess}^g_0\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n).$$

Proof. Let $g=\sum_{k,j}g_{kj}\mathrm{d}x^k\otimes\mathrm{d}x^j$, and write $\mathrm{grad}=\sum_k a_k\partial_k$. Then

$$\partial_j\varphi=g(\mathrm{grad},\partial_j)=\sum_k a_kg_{kj}\implies\partial_i\partial_j\varphi=\sum_k g_{kj}\partial_ia_k+\sum_k a_k\partial_ig_{kj}.$$

Evaluating at $x=0$, where $a\equiv 0$, we conclude

$$\partial_i\partial_j\varphi(0)=\sum_k g_{kj}\partial_ia_k=g(\partial_j,\mathrm{dgrad}_0(\partial_i))\tag{1}$$

Now we compute

$$g(\partial_j,\mathrm{Hess}^g_0(\partial_i))=\mathrm{Hess}_0(\partial_i,\partial_j)=\partial_i\partial_j\varphi(0),\tag{2}$$

comparing (1) and (2), we conclude that $\mathrm{dgrad}_0=\mathrm{Hess}^g_0$, as desired. $\square$

Now the fact that φ is a Morse function says precisely that Hess_0 is a non-degenerate bilinear form. It follows that Hess^g_0 is a g-self-adjoint operator, and hence has an eigenbasis

$v_1, \cdots, v_n$, with eigenvalues $\lambda_1, \cdots, \lambda_n$, where we suppose that

$$\lambda_1 \leq \cdots \leq \lambda_p < 0 < \lambda_{p+1} \leq \cdots \leq \lambda_n.$$

The number p is precisely the **Morse index** of the critical point (i.e. the index of the bilinear form Hess_0). Let's agree to call H_+ the subspace spanned by $v_{p+1}, \cdots, v_n$.

Define a map $F : W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \oplus H_+$ by

$$F(\gamma) = (s(\gamma), \pi_+\gamma(0)).$$

Note that evaluating a curve γ at 0 is a continuous linear map, and hence $\gamma \mapsto \pi_+\gamma(0)$ is a smooth function $W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to H_+$.

Proposition 14. $\mathrm{d}F_0$ is an isomorphism $W^{1,2} \to L^2 \oplus H_+$.

Proof. It suffices to prove that $\mathrm{d}F_0$ is a bijection, thanks to the open mapping theorem. The derivative of F at 0 is given by the formula

$$\mathrm{d}F_0(\eta) = (\eta' + \mathrm{Hess}_0 \cdot \eta\,,\, \pi_+\eta(0)).$$

This follows from Claim 11, and the fact that $\eta \mapsto \pi_+\eta(0)$ is linear.

First we prove that $\mathrm{d}F_0$ is injective. It is convenient to write η as $\eta = \sum \eta_i v_i$, where the η_i are now $W^{1,2}$ functions $\mathbb{R}_{\geq 0} \to \mathbb{R}$. Suppose that $\mathrm{d}F_0(\eta) = 0$. This is equivalent to

$$\eta_i'(t) = -\lambda_i\eta_i(t) \text{ for } i = 1, \cdots, n \text{ and } \eta_{p+1}(0) = \cdots = \eta_n(0).$$

Simple elliptic bootstrapping proves that $\eta_1, \cdots, \eta_n$ are C^∞ functions. In fact, it is clear that

$$\eta_i(t) = \eta_i(0)e^{-\lambda_i t}.$$

Since η_i is assumed to be integrable, we must have $\eta_1(0) = \cdots = \eta_p(0) = 0$, otherwise η would blow up exponentially. Since we assume $\eta_{p+1}(0) = \cdots = \eta_n(0) = 0$, we conclude that η is identically 0. It follows that $\mathrm{d}F_0$ is injective.

Now we prove that $\mathrm{d}F_0$ is surjective. Given $\xi \in L^2$ and $c_{p+1}, \cdots, c_n \in H_+$, we want to define η so that

$$\eta_i(t) + \lambda_i\eta_i(t) = \xi_i(t),$$

and $\eta_i(0) = c_i$ for $i > p$. Define

$$\eta_i(t) = -e^{-\lambda_i t}\int_t^\infty e^{\lambda_i s}\xi_i(s)\,ds \text{ for } i = 1, \cdots, p,$$

and define

$$\eta_i(t) = e^{-\lambda_i t}c_i + e^{-\lambda_i t}\int_0^t e^{\lambda_i s}\xi_i(s)\,ds \text{ for } i = p+1 = \cdots = n.$$

We check that this is well-defined, i.e. the resulting η is indeed in $W^{1,2}$. First we will check that η is in L^2. Let ρ be some test function. Then for $i = 1, \cdots, p$, we compute

$$\int_0^\infty \eta_i(t)\rho(t)\,dt = -\int_0^\infty\int_t^\infty e^{\lambda_i(s-t)}\rho(t)\xi_i(s)\,ds\,dt = -\int_0^\infty\int_0^\infty e^{\lambda_i z}\rho(t)\xi_i(z+t)\,dz\,dt.$$

where have made the change of coordinates $z = s - t$. Now we switch the order of integration:

$$\int_0^\infty \int_0^\infty e^{\lambda_i z} \rho(t)\xi_i(z+t)\, dz\, dt = \int_0^\infty e^{\lambda_i z} \int_0^\infty \rho(t)\xi_i(z+t)\, dt\, dz.$$

We estimate

$$\left| \int_0^\infty \rho(t)\xi_i(z+t)\, dt \right| \le \|\rho\|_{L^2} \|\xi_i\|_{L^2},$$

and hence

$$\left| \int_0^\infty \eta_i(t)\rho(t)\, dt \right| = \left| \int_0^\infty e^{\lambda_i z} \int_0^\infty \rho(t)\xi_i(z+t)\, dt\, dz \right| \le \left\| e^{\lambda_i z} \right\|_{L^1} \|\rho\|_{L^2} \|\xi_i\|_{L^2} = C \|\rho\|_{L^2}.$$

Since $\lambda_i < 0$, the L^1 norm of $e^{\lambda_i z}$ is finite. We conclude that η_i is in L^2 since pairing it with test functions defines a bounded transformation $L^2 \to L^2$ (here we use reflexivity of L^2).

Remark. It is easy to show that η is given by a convolution of ξ with an integrable kernel. It follows that η is in L^2 by Young's inequality. Our argument essentially reproves Young's inequality in our specific setting.

Exercise 5. Prove that η_i is in L^2 for $i = p+1, \cdots, n$.

Having established that η is in L^2, we check that $\eta_i' + \lambda_i \eta_i = \xi_i$ holds weakly in L^2. Suppose $i = 1, \cdots, p$. To check that an equation holds weakly, we pair with a test function ρ. By definition of "weak" we have

$$\int_0^\infty (\eta_i'(t) + \lambda_i \eta_i(t))\rho(t)\, dt = \int_0^\infty \eta_i(t)(\lambda_i \rho(t) - \rho'(t))\, dt.$$

We write

$$\int_0^\infty \eta_i(t)(\lambda_i \rho(t) - \rho'(t))\, dt = \int_0^\infty \int_t^\infty e^{\lambda_i(s-t)} \xi_i(s)(\rho'(t) - \lambda_i \rho(t))\, ds\, dt.$$

Now we change the order of integration:

$$\int_0^\infty \int_t^\infty e^{\lambda_i(s-t)} \xi_i(s)(\rho'(t) - \lambda_i \rho(t))\, ds\, dt = \int_0^\infty \xi_i(s) e^{\lambda_i s} \left[\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i \rho(t))\, dt \right] ds.$$

We compute

$$\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i \rho(t))\, dt = \int_0^s \frac{d}{dt}\left[e^{-\lambda_i t}\rho(t) \right] dt = e^{-\lambda_i s}\rho(s),$$

where we use the fact that ρ is a test function, and hence is compactly supported in $(0, \infty)$. It follows that

$$\int_0^\infty (\eta_i'(t) + \lambda_i \eta_i(t))\rho(t)\, dt = \int_0^\infty \xi_i(s) e^{\lambda_i s} \left[\int_0^s e^{-\lambda_i t}(\rho'(t) - \lambda_i \rho(t))\, dt \right] ds = \int_0^\infty \xi_i(s)\rho(s)\, ds,$$

which demonstrates that $\eta_i' + \lambda_i \eta = \xi_i$ holds weakly (for $i = 1, \cdots, p$).

Exercise 6. Show that $\eta_i' + \lambda_i \eta_i = \xi_i$ holds weakly for $i = p+1, \cdots, n$.

Now since η_i and ξ_i are in L^2, and $\eta_i' = \xi_i - \lambda_i \eta_i$, we conclude that the weak derivative of η_i is in L^2 and hence η_i is in $W^{1,2}$.

Finally, it is clear that $\eta_i(0) = c_i$ for $i = p+1, \cdots, n$. Thus it follows that $\mathrm{d}F_0\eta = (\xi, c)$, and hence $\mathrm{d}F_0$ is surjective. This completes the proof that $\mathrm{d}F_0$ is an isomorphism. $\square$

By the inverse function theorem, it follows that F is a C^1 diffeomorphism in some neighborhood of 0. In fact, one can show without too much additional work that the map s is C^∞ (because grad is a smooth vector field). For the details involved, the reader is referred to Chris Wendl's "Lectures on Holomorphic Curves," pages 85-87. It then follows that F is a smooth diffeomorphism on some neighborhood of 0.

Consider the composite function

$$\begin{tikzcd} H_+ \arrow[r] \arrow[rrr, bend right, "\Phi"'] & 0 \oplus H_+ \arrow[r, "F^{-1}"] & W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^N) \arrow[r, "\mathrm{ev}_0"] & \mathbb{R}^n \end{tikzcd}$$

The map Φ is smooth and defined on small some disk $D(r) \subset H_+$. Since $\pi_+\Phi(x) = x$, we conclude that Φ is a section of the orthogonal projection π_+ (over $D(r)$), and hence Φ defines a smooth submanifold of $\mathbb{R}^n$ (a graph over $D(r)$).

It is clear that the unique gradient flow line starting at any point $\Phi(x) \in \Phi$ converges to 0 (by our construction). Indeed, the flow line starting at $\Phi(x)$ is $F^{-1}(0, x)$. The next lemma will establish that the graph Φ is precisely the stable set near 0.

Lemma 15. There is a neighborhood U of 0 so that any flow line starting in U and converging to 0 actually starts on $\Phi \cap U$.

Proof. First we claim that any trajectory $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ converging to 0 satisfies $\gamma(t) \in \Phi$ for t sufficiently large. We will use the result that any gradient flow line converging to 0 is automatically in $W^{1,2}$ (cf. Exercise 7).

Consider the elements $\gamma_T \in W^{1,2}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ given by $\gamma_T(t) = \gamma(t+T)$. It is clear that γ_T is still a gradient flow line, and moreover, that $\|\gamma_T\|_{W^{1,2}} \to 0$ as $T \to \infty$, since

$$\|\gamma_T\|_{W^{1,2}} = \left\|\gamma|_{[T,\infty)}\right\|_{W^{1,2}}.$$

Since our map F is a diffeomorphism on a small neighborhood of 0, we conclude that γ_T eventually enters the domain where F is a diffeomorphism, and hence $\gamma_T = F^{-1}(0, x)$ for some x (here x depends on T). Therefore $\gamma_T(0) \in \Phi$, hence $\gamma(T) \in \Phi$. This proves that γ eventually enters Φ.

Next, pick a bounded open set U' of 0 with the property that so that $\overline{\Phi \cap U'} \subset \Phi$. By compactness of $\overline{\Phi \cap U'} \subset \Phi$, it follows that there is $\delta > 0$ so that for every $x \in \Phi \cap U'$, the flow line through x can be defined on $[-\delta, \infty)$, and that this flow line remains on Φ. In other words, we can extend the flow line backwards in time by δ, while remaining on the graph Φ.

To establish the conclusion of the lemma, we will use the following we claim: there is a smaller open set $U \subset U'$ with the following property: every trajectory which starts in U either remains in U' forever, or leaves U' and never comes back inside U (a similar statement is proved on page 50 of Milnor's notes on the h-cobordism theorem). This claim is proved in Exercise 8.

Assuming this result, we can complete the proof of the lemma. If γ is a gradient flow line starting in U and γ converges to 0, then clearly γ cannot leave U'. Look at the set of times t so that $\gamma(t) \in \Phi$. Since $\gamma \to 0$, we know that $\gamma(t)$ is eventually in Φ, so this set of times is non-empty. Either (case 1) $\gamma(0) \in \Phi$, or (case 2) there is some time $t > \delta$ so $\gamma(t) \in \Phi$ and $\gamma(t-\delta) \notin \Phi$. However, since $\gamma(t) \in \Phi \cap U'$, we conclude that the flow line through $\gamma(t)$ can be extended backwards in time by amount δ *while remaining on* Φ. Therefore $\gamma(t-\delta) \in \Phi$, and so case 2 cannot happen. It follows that $\gamma(0) \in \Phi$, and since $\gamma(0) \in U$, $\gamma(0) \in \Phi \cap U$. We have shown that every flow line starting in U converging to 0 must start on $\Phi \cap U$, as desired. □

Corollary 16. Let S denote the stable set of 0, and let U be the open set furnished by the preceding lemma. Then $S \cap U = \Phi \cap U$, and so we have shown that S is a manifold near 0. The dimension of $S \cap U$ is equal to $\dim \Phi = n - p$, where p is the Morse index of the critical point. □

Here are the two exercises used in the proof of Lemma 15.

Exercise 7. Let $\mathrm{grad} : \mathbb{R}^n \to \mathbb{R}^n$ be the gradient vector field. We will use the fact that $\mathrm{grad}(0) = 0$ and dgrad_0 is an isomorphism. Suppose that $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a flow line converging to 0.

(a) Prove that γ is in L^2 if and only if $\mathrm{grad} \circ \gamma$ is in L^2. Hint: show that

$$|\gamma(t)| < c\,|\mathrm{grad} \circ \gamma(t)|$$

for t sufficiently large, for some $c > 0$.

(b) Prove that $\mathrm{grad} \circ \gamma$ is in L^2 using the relation

$$\frac{d}{dt}(\varphi \circ \gamma) = -g(\mathrm{grad} \circ \gamma, \mathrm{grad} \circ \gamma).$$

(c) Now that we know that γ is in L^2, conclude that γ' is also in L^2 using the gradient flow line equation.

(d) Conclude that any flow line γ converging to 0 is actually in $W^{1,2}$.

Exercise 8. Given a bounded open neighborhood U' of 0, one can always find a smaller open set $U \subset U'$ so that every trajectory γ starting in U either remains in U', or leaves U' and never returns to U.

(a) Pick U'' compactly supported in U' around 0 so that $g(\mathrm{grad}, \mathrm{grad}) > b$ on $U' \setminus U''$. Pick $U_\epsilon \subset U''$ so that $\max_{U_\epsilon} \varphi - \min_{U_\epsilon} \varphi < \epsilon$. Using the fact that

$$\frac{d}{dt}\varphi(t) = -g(\mathrm{grad}, \mathrm{grad}),$$

conclude that any trajectory starting and ending in U_ϵ must spend time less than $b^{-1}\epsilon$ in $U' \setminus U''$.

(b) Since $\partial U''$ is compact and contained in U', conclude a minimum amount of time needed to flow from $\partial U''$ to $\mathbb{R}^n \setminus U'$.

(c) Conclude that we can pick ϵ small enough so that any flow starting and ending in U_ϵ cannot leave U'. Taking $U = U_\epsilon$ proves the claim.

Appendix to April 4th Lecture

Definition 17. For $k \geq 1$, a C^k **Banach manifold** $\mathcal{X}$ is a topological space covered by open sets homeomorphic to open subsets of Banach spaces, where the transitions functions are C^k maps. More precisely, a Banach manifold comes equipped with a maximal atlas of coordinate charts: $c : U_c \subset \mathcal{X} \to c(U) \subset X_c$, where X_c is a Banach space, $c : U_c \to c(U)$ is homeomorphism onto an open set, and so that the transition homeomorphism

$$\rho_{21} = c_2 \circ c_1^{-1} : c_1(U_1 \cap U_2) \to c_2(U_1 \cap U_2)$$

is a C^k map.

We define a continuous map $A : \mathcal{X} \to \mathcal{Y}$ between C^k Banach spaces to be C^r $(r \leq k)$ if $c_2 \circ A \circ c_1^{-1}$ is a C^k map, for all choices of coordinates c_1, c_2 around x and $A(x)$ respectively.

Definition 18 (the tangent bundle)**.** For the purposes of this defintiion, let's agree to say that a Banach space is a topological vector space equipped with an equivalence class of complete metrics defining its topology.

For $k \geq 1$, let BMan_k be the category of C^k Banach manifolds, with C^k maps between them, let BSpace_k be the category of Banach spaces with C^k maps between them, and let Bun_k be the category of Banach space bundles over Banach manifolds. A morphism in Bun_k between bundles $E_1 \to B_1$ and $E_2 \to B_2$ is a pair (f, F) such that $f : B_1 \to B_2$ is a C^k map and F is a C^{k-1} section of the Banach space bundle $\mathrm{Hom}(B_1, f^*E_2) \to B_1$.

There is a functor $\tau : \mathrm{BSpace}_k \to \mathrm{Bun}_k$ sending a Banach space X to the trivial bundle $\tau(X) = X \times X \to X$, and which sends a morphism $f : X \to Y$ to the pair $(f, \mathrm{d}f)$, where $\mathrm{d}f$ is the C^{k-1} section of $\mathrm{Hom}(\tau(X), f^*\tau(Y)) = \mathrm{Hom}(X, Y) \times X \to X$.

The tangent bundle functor $T : \mathrm{BMan}_k \to \mathrm{Bun}_k$, is defined by three axioms:

(i) We require $T(f : X \to Y) = (f, \mathrm{d}f)$, i.e. $T(f)$ is a bundle map "over f" (where we abuse notation and use the symbol d for T as well as for τ).

(ii) The following diagram should commute up to a natural isomorphism $T \circ j \to \tau$:

$$\begin{tikzcd}
\mathrm{BSpace}_k \arrow[r, "\tau"] \arrow[d, "j"] & \mathrm{Bun}_k \\
\mathrm{BMan}_k \arrow[ur, bend right, "T"'] &
\end{tikzcd}$$

where j is the obvious inclusion functor $\mathrm{BSpace}_k \to \mathrm{BMan}_k$. This natural isomorphism should be thought of as part of the data of T.

(iii) If $i : U \to M$ is the inclusion of an open set, then the map $\mathrm{d}i : TU \to i^*TM$ is an isomorphism.

It is not very hard to show that this determines T up to unique natural isomorphism, i.e. if T' is another such functor then there is a unique natural isomorphism $T \to T'$ so that

$$\begin{tikzcd} \tau \arrow[r] \arrow[d] & T' \circ j \\ T \circ j \arrow[ur] & \end{tikzcd}$$

commutes.

April 11 (Ipsita)

Elliptic operators. We will always be looking at linear differential operators. We build on the difficulty of the operators and domains.

Constant coefficient operators on vector-valued functions on $\mathbb{R}^n$. We consider $D : (\mathbb{R}^n, \mathbb{R}^m) \to (\mathbb{R}^n, \mathbb{R}^m)$ given by

$$Du = \sum_{|\alpha \leq k} A_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha}$$

where $A_\alpha \in \mathrm{Mat}(m, n)$.

Definition 19. a. The *total symbol* of D is defined as

$$\sigma_t(\xi) = \sum_{|\alpha \leq k} A_\alpha \xi^\alpha \qquad \text{for } \xi = (\xi_1, \dots, \xi_n).$$

b. The principal symbol is given by

$$\sigma_p(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha \qquad \text{for } \xi = (\xi_1, \dots, \xi_n).$$

.

One motivation for this definition comes from looking at the Fourier transform. Note

$$\widehat{Du} = \sigma_t(i\xi)\widehat{u} \implies Du = \int e^{ix\cdot\xi} \sigma_t(i\xi) d\xi.$$

Thus, we "turned" a differential operator into an integral operator.

Ellipticity of D have different meanings in different contexts but it always has something to do with $\sigma_{t,p}(\xi)$ being invertible outside of $\xi = 0$. An example of such a condition is

$$(\star) \qquad |\sigma_t(\xi)| \geq c|\xi|^n \text{ for all } x \in \mathbb{R}^n_\xi.$$

This is possibly relevant for Schwarz spaces.

The fundamental solution of D is a $G : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $DG = \delta$. By taking Fourier transform we get $\widehat{DG} = 1$, equivalently $\sigma_t(i\xi)\widehat{G} = 1$. Ellipticity conditions let you "divide" by the symbol to obtain a tempered distribution $G = \left(\frac{1}{\sigma_t(\xi)}\right)^\vee$ which solves $DG = \delta$. Similarly, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ we can obtain a tempered distribution $G = \left(\frac{1}{\sigma_t(\xi)}\right)^\vee * f$ which solves $DG = f$.

Exercise 9. Find a condition on the total symbol $\sigma_t(\xi)$ so that

$$D : \mathcal{S} \to \mathcal{S}$$

($\mathcal{S}$ representing Schwarz functions) is bijective. What fails with $(\star)$.

Variable coefficient operators on $\mathbb{R}^n$. We consider D almost the same as above but now A_α depends on $x \in \mathbb{R}^n$. So, in the definition of the differential operator and the symbols we replace A_α by $A_\alpha(x)$. Then the symbol also depends on $x \in \mathbb{R}^n$ and we denote it by $\sigma_{t,p}(x, \xi)$.

Notice that under a change of coordinates $x \mapsto x'$, $\sigma_p(\cdot, x)$ transforms to $\sigma_p(\cdot, x')$ as a section of $\mathrm{Sym}(T^*\mathbb{R}^n) \otimes \mathrm{Mat}(m,n)$. For example,

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j},$$

$$dx_i = \sum_j \frac{\partial x'_j}{\partial x_i} dx_j.$$

In contrast σ_t does not transform so nicely because lower order terms appear. For example,

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \sum_{j,l} \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j}\left(\frac{\partial x'_l}{\partial x_k}\frac{\partial}{\partial x'_l}\right) = \left(\sum_{j,l} \frac{\partial x'_j}{\partial x_i}\frac{\partial x'_l}{\partial x_k}\frac{\partial}{\partial x'_j}\frac{\partial}{\partial x'_l}\right) + \text{(first order terms)}.$$

Ellipticity can be defined similar to the constant coefficient case, but we will not do it here because it will confuse us.

Operators on a closed manifold. Consider a closed manifold M and vector bundles E, F over M. Assume $\dim E = \dim F$. $D : C^\infty(M,E) \to C^\infty(M,F)$ is a differential operator if it looks like a differential operator on trivializations (plus some assumptions about locality). For D, $\sigma_{p,D} : \mathrm{Sym}^n(T^*M) \otimes F \to F$ is the principal symbols glued together. (Note that there exists a coordinate free definition.)

Definition 20. D is elliptic if $\sigma_{p,D}(v, \ldots, v)$ for $v \in T_mM, m \in M$ is invertible for $v \neq 0$.

Theorem 21. If D as above is elliptic, then it has finite dimensional kernel and cokernel.

We will not be proving this which is standard but non-trivial. It is crucial to this theorem that M is closed.

Definition 22. In this case, we define index of an elliptic operator D as

$$\mathrm{Ind}(D) = \dim(\ker D) - \dim(\operatorname{coker} D).$$

Index theorem for elliptic operators on closed manifolds.

Theorem 23.
a. Index of an elliptic operator (on a closed manifold) only depends on its principal symbol.
b. Index is constant on the connected components of the space of elliptic operators.

Gelfand(1960) proposed computing the index of an elliptic operator as a homotopical invariant of its symbol.

Atiyah–Singer (1963) announce they did it! The story of index calculations continues to much later.

Example 24. Consider an oriented and closed manifold M with a Riemannian metric. Then the hodge star is defined and we get

$$d + d^\star : \Omega^{even}(M) \to \Omega^{odd}(M)$$

is an elliptic differential operator. (Exercise: check!)

Let $\mathscr{H}$ denote the subspace of $\Omega^*(M)$ of harmonic forms. Note that $\omega \in \Omega^k(M)$ is harmonic if and only if

$$d\omega = d^\star\omega = 0.$$

Fact: Every de Rham cohomology class has exactly one harmonic representative. Using this fact we can conclude

$$\begin{aligned} \ker(d+d^\star) &= \mathscr{H}^{even}, \\ \operatorname{coker}(d+d^\star) &= \ker(\mathscr{H}^{odd} \to \mathscr{H}^{even}) = \mathscr{H}^{odd}, \\ \operatorname{Ind}(d+d^\star) &= \dim(\mathscr{H}^{even}) - \dim(\mathscr{H}^{odd}) = \chi(M). \end{aligned}$$

The topological side of Atiyah-Singer index theorem computes

$$\chi(M) = \int_M K$$

where K is the Euler form of TM constructed from curvature (generalised Gaussian curvature, constructed via Chern-Weil theory).

General form of the index is given by

$$\operatorname{Ind}(D) = \int_M ch(\sigma_{p,D}) Td(M)$$

where $ch(D)$ and $Td(M)$ are cohomology classes. The classes are not always easy to compute.

Later work showed that, for example for Dirac operators (which are most fundamental of elliptic operators), it is possible to construct special representatives like de Rham representatives. (One proof uses "heat kernel" techniques.)

What about manifolds with boundary?. Suppose X is a compact manifold with boundary $\partial X = Y$. Let E and F be vector bundles over X. For $D : C^\infty(X,E) \to C^\infty(X,F)$ elliptic, Fredholmness fails. (For example, $\bar{\partial} : C^\infty(\mathbb{D},\mathbb{C}) \to C^\infty(\mathbb{D},\mathbb{C})$ does not have a finite kernel.)

To solve this issue we need appropriate boundary conditions.

One **candidate** would be local boundary conditions like restricting to a subspace of $C^\infty(X,E)$ consisting of sections f such that f and/or its derivatives are prescribed point wise. (For example Dirichlet or von Neumann conditions.) It is not impossible to work with local boundary conditions. Sometimes it is possible to find nice local boundary conditions that lead to Fredholm operators. In fact, one can remove the "sometimes." But from the point of index theory we are unsure how nice the conditions can be. (Refer: "On general boundary value problems for elliptic operators" by Schulze et. al. if interested.)

Possibly one of the main insights of Atiyah–Patodi–Singer is that one could consider global boundary conditions for the index theory to work well.

Theorem 25. (Atiyah–Patodi–Singer I)
Consider X, Y, E, F and D as above.

Assume

a. D first order.

b. In a neighbourhood $I_u \times Y$ of Y (where parameter u is decreasing towards $\partial X = Y$), D should look like
$$D = \sigma_0(\frac{\partial}{\partial u} + A)$$
where σ is a bundle homeomorphism $E|_Y \to F|_Y$ ($E|_{I\times Y}$ is pullback from $E|_Y$, similar for F) and A is self-adjoint (with respect to inner product $\langle s, s' \rangle = \int\limits_Y h(s(y), s'(y))dy, s, s' \in C^\infty(X, E)$). Note here h is a fixed Hermitian metric on E and dy is with respect to a fixed measure on Y.

Let $C^\infty(X, E, P)$ denote the sections f such that the projection of $f|_Y$ to the non-negative eigenspace of A is zero.

Then $D : C^\infty(X, E, P) \to C^\infty(X, F)$ is Fredholm. Moreover
$$\mathrm{Ind}(D) = \int\limits_X \alpha_0(x)dx - \frac{h + \eta(0)}{2},$$
where

(i) $\alpha_0(x)$ is the constant term in the asymptotic expansion (as $t \to 0$) of
$$\sum e^{-t\nu'} |\phi'_\mu(x)|^2 - \sum e^{-t\mu''} |\phi''_\mu(x)|^2,$$
where $\mu', \phi'_\mu, (resp.\ \mu'', \phi''_\mu)$ denote the eigenvalues of $D^*D(resp.\ DD^*)$ on the double of X. Note that D and D^* naturally extend to operators on the double of X using special form of D near ∂.

(ii) $h = \dim\ker A$.

(iii) $\eta(s) = \sum\limits_{\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}$, where λ runs over eigenvalues of A. Here $\eta(s)$ converges absolutely for $Re(s) \gg 1$ and extends to a meromorphic function on the entire plane with finite value at $s = 0$.

Remark Turns out this case is not the most important for us, so don't worry too much about it. Next time we cover spectral flows, which is more important for us.

Example 26. Let us consider $\bar\partial : C^\infty(\mathbb{D}, \mathbb{C}) \to C^\infty(\mathbb{D}, \mathbb{C})$ given by
$$\bar\partial = \frac{d}{dt} - A$$
where $-A : C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C}); e^{2\pi in\theta} \mapsto 2\pi n e^{2\pi in\theta}$ for e^t radial coordinates and $e^{i\theta}$ angular coordinates. Then $C^\infty(\mathbb{D}, \mathbb{C}, P)$ consists of those sections which have only negative Fourier coefficients. So, no holomorphic functions are in $C^\infty(\mathbb{D}, \mathbb{C}, P)$. Also, we can compute $\eta(s) = 0$ as $Re(s) \gg 1$ implies $\eta(0) = 0$.

April 16 (Daren)

Today we will talk about index of operators of certain simple form on cylinders, following Kronheimer-Mrowka.

Two warnings:

(1) Index theory is for linear operators, but we will deal with non-linear operators later. What we want to compute indices of are the linearisations of the non-linear operators at the solutions of the non-linear equations.

(2) Index theory, and more generally Fredholm theory, is generally done in the setting of Hilbert space $W^{k,2}$.

For J-holomorphic curve related problems, we will need to use $W^{k,p}$ spaces where $p > 2$, for the following reason:

- we need $kp > 2$, so that
  - the definition of the relevant completion in this non-linear case requiresthis.
  - $W^{k,p}$ is closed under multiplication.
  - $W^{k,p} \hookrightarrow C^0$ compactly.
- k=1, because of elliptic regularity.

Hence, the following results, especially the ones regarding Fredholmness of operators, are not good enough for what we need later.

Quick reminders on Fredholm operators: H_1, H_2 are Hilbert spaces.

- **Definition**: A bounded operator $F : H_1 \to H_2$ is called *Fredholm* if its kernel and cokernel are both finite dimensional and it has closed range. The *index* of F is dim ker(F) − dim coker(F).
- **Definition**: A bounded operator $F : H_1 \to H_2$ is called *compact* if it is the limit of a sequence of finite rank operators in the operator norm. Equivalently, if it maps bounded set B in H_1 to precompact set $F(B)$ in H_2.
- **Lemma**: $F : H_1 \to H_2$ is Fredholm if and only if $\exists P : H_1 \to H_2$, such that $PF - id_{H_1}$ and $FP - id_{H_2}$ are compact operators. Such P is called a parametrix of F.
- **Lemma**: If F_t is a continuous family of Fredholm operators for $t \in [0, 1]$, then $ind(F_0) = ind(F_1)$.

Main proposition today:

Let Y be a closed manifold, $Z = \mathbb{R} \times Y$, where t is the coordinate in the $\mathbb{R}$-direction. $E \to Y$

is a vector bundle, with an Euclidean metric on the fiber. The pullback of E to Z along projection to Y is called E again.

Proposition 14.2.1(in Kronheimer-Mrowka): Let L_0 be a first-order, self-adjoint elliptic operator acting on sections of a vector bundle $E \to Y$, and let h_t be a time-dependent bounded operator on $L^2(Y, E)$, varying continuously in the operator norm topology and equal to constants $h_\pm$ on each ends. Suppose $L_0 + h_\pm$ are *hyperbolic*. Then the operator

$$Q = \frac{d}{dt} + L_0 + h_t : L^2_1(Z, E) \to L^2(Z, E)$$

is Fredholm and has index equal to the *spectral flow* of the path of operators $L_0 + h_t$, where $L^2_1(Z, E) = W^{1,2}(Z, E)$.

Definition: An operator L is called *hyperbolic* if it has no eigenvalue on the imaginary axis.

Definition: For a family of operators $L_0 + h_t$ satisfying the condition in the proposition,

the *spectral flow* is defined as the **net** number of eigenvalues whose real part changing from negative to positive.

To be more precise, first deform the path L_0+h_t so that it is smooth over $(0, 1)$. Then consider

the set $S = \{(t, \lambda) \mid \lambda \in Spec(L_0 + h_t)\}$. By spectrum theory, the spectrum of $L_0 + h_t$ is discrete, and the generalized eigenspaces are finite dimensional. (See Lemma 12.2.4 in K-W). We say (t, λ) is a *simple point* if the generalized λ-eigenspace of $L_0 + h_t$ is 1-dimensional. At simple points, S is a smooth 1-manifold, on which t is a local coordinate; we use the coordinate t to orient S at such points. In the space of bounded operators on L_2, the set of those h for which the spectrum of $L_0 + h$ has a non-simple eigenvalue lying on the imaginary axis is a locally finite union of submanifolds of codimension at least 2.(see Lemma 12.2.4 again) The path h_t can therefore be moved so that the intersection of S with $(0, 1) \times i\mathbb{R}$ consists entirely of simple points; and any two such paths can be joined by a homotopy of paths with the same property. For such a path, we define the spectral flow as the intersection number of S with $(0, 1) \times i\mathbb{R}$. From this perspective, we can see the the spectral flow depends only on the starting and ending points $h_\pm$.

Lemma 12.2.4 in K-W says the following:

Lemma 12.2.4 If $L = L_0 + h : L^2_1(Y, E) \to L^2(Y, E)$ satisfies the condition in the previous

proposition, then

- There are only finitely many eigenvalues of the complexification $L \otimes \mathbb{C}$ in any compact subset of the complex plane $\mathbb{C}$, and the generalized eigenspaces of the complexification are finite-dimensional. All the generalized eigenvectors belong to $L^2_1(Y, E)$.
- If h, like L_0, is self-adjoint, then the eigenvalues are real, and there is a complete orthonormal system of eigenvectors $e_n$in $L^2(Y, E)$. The span of the eigenvectors is dense in $L^2_1(Y, E)$.
- If h is not self-adjoint, the imaginary parts of the eigenvalues λ of $L \otimes 1_{\mathbb{C}}$ are bounded by the L_2-operator norm of $(h - h^*)/2$.

Proof of the proposition:

Step 1: We consider the translational invariant case, *e.g.* when $h_t = h, \forall t$. In this case, the

spectral flow is obviously 0, so we need to show the index of D is 0. Assume E is a complex vector bundle, (for the general case, note that $C^\infty(E)$ are the real sections of $C^\infty(E \otimes \mathbb{C})$, and the Fourier transform that we will do below sends them to sections that are fixed under some other involution).

Let F_1 be the completion of $C_c^\infty(E \otimes \mathbb{C})$, the space of compactly supported smooth sections of $E \otimes \mathbb{C}$, with respect to the norm

$$\|\widehat{u}\|^2 = \sum_{i=0}^{1} \int_{-\infty}^{\infty} \|\xi\|^{2(1-i)} \|\widehat{u}\|^2_{L^2_1(\{\xi\}\times Y)}\, d\xi$$

Consider the Fourier transform of sections of $E \to \mathbb{R}_t \times Y$ with respect to the t-variable.

$$\widehat{u}(\xi, y) = \int_{-\infty}^{\infty} e^{-it\xi} u(t, y) dt$$

The Fourier transform sends $L^2(Z, E)$ to $L^2(Z, E)$, and $L^2_1(Z, E)$ to F_1. The Fourier transform of $D = \dfrac{d}{dt} + L_0 + h$ is $\widehat{D} = L_0 + h + i\xi : F_1 \to L^2(Z, E)$. We will show $\widehat{D}$ is invertible.

Let's consider what the operator does on each slice first.

Let $\widehat{D}_\xi = L_0 + h + i\xi : L^2_1(Y, E) \to L^2(Y, E)$ for $\xi \in \mathbb{R}$. By hyperbolicity of the operator $L_0 + h$, $\widehat{D}_\xi$ is injective. (This doesn't hold in the general case, where we only know the hyperbolicity at two ends) Moreover, by standard elliptic theory on closed manifolds, $\widehat{D}_\xi$ is Fredholm. By invariance of index under deformation, we have $\mathrm{ind}(\widehat{D}_\xi) = \mathrm{ind}(L_0)$, and

$$\begin{aligned}
\mathrm{ind}(L_0) &= \dim \mathrm{Ker}(L_0) - \dim \mathrm{Coker}(L_0) \\
&= \dim \mathrm{Ker}(L_0) - \dim \mathrm{Ker}\ (L_0^*) \\
&= \dim \mathrm{Ker}(L_0) - \dim \mathrm{Ker}(L_0) = 0
\end{aligned}$$

Therefore, $\widehat{D}_\xi$ is invertible.

Then, we define the inverse $\widehat{D}^{-1}$: $C^\infty(Z,E) \to L^2_{1,loc}(Z,E)$ by applying $(L_0 + h + i\xi)^{-1}$ slicewise. We need to give a bound of F_1-norm of $(L_0 + h + i\xi)^{-1}(f)$ in terms of L^2-norm of f, so that we can extend it to a bounded operator from $L^2(Z,E)$ to F_1. (Check Lemma 14.1.3 in K-W for details)

Step 2: For the general case $D = \dfrac{d}{dt} + h_t + L_0$, we show D is Fredholm first, using the *parametrix gluing technique.* (which is important in index theory)

Choose $a' < a < b < b'$ in $\mathbb{R}$, such that $h(t) = h_-$ for $t < a$, $h(t) = h_+$ for $t > b$.

Choose partition of unity $\eta_\pm : \mathbb{R} \to [0,1]$, such that $\eta_- + \eta_+ = 1$, $\eta_-(t) = 1$ for $t \le a$, and $\eta_+(t) = 1$ for $t \ge b$.

Choose bump functions $\gamma_\pm$, such that $\gamma_-(t) = 1$ for $t \le b$, and $\gamma_+(t) = 0$ for $t \ge b'$. Similarly, $\gamma_+(t) = 0$ for $t \le a'$, and $\gamma_+(t) = 1$ for $t \ge a$.

Let $G_\pm : L^2(Z,E) \to L^2_1(Z,E)$ be the inverse of $D_\pm = \frac{d}{dt} + L_0 + h_\pm$ as in Step 1. Define

$$P : L^2(Z,E) \to L^2_1(Z,E)$$

$$e \ \to \ \gamma_- G_- \eta_- e + \gamma_+ G_+ \eta_+ e$$

Claim: P is a parametrix for D. Proof: We compute DP explicitly.

$$\begin{aligned}
DPe =& D(\sum \gamma_\pm G_\pm \eta_\pm e) \qquad \text{(summing over + and -)} \\
=& \sum \gamma'_\pm G_\pm \eta_\pm e + \sum \gamma_\pm D(G_\pm \eta_\pm e) \\
=& \sum \gamma'_\pm G_\pm \eta_\pm e + \sum \gamma_\pm (D - D_\pm) G_\pm \eta_\pm e + e \quad \text{(As } D_+G_+ = D_-G_- = id\text{, and} \gamma_-\eta_- + \gamma_+\eta_+ = 1) \\
=& e + \sum (\gamma'_\pm + \gamma_\pm (h - h_\pm)) G_\pm \eta_\pm e
\end{aligned}$$

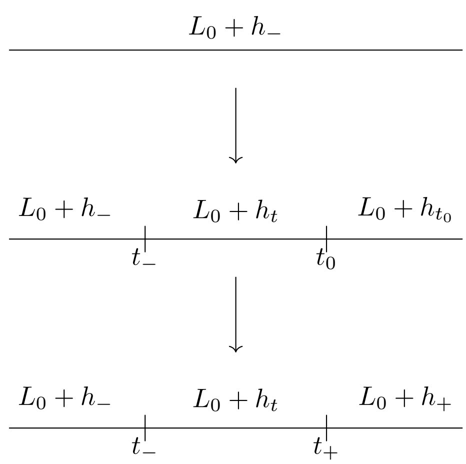
Therefore $id - DP$ is compact, as the maps $G_\pm \eta_\pm :\ L^2(Z,E) \to L^2_1(Z,E)$ are bounded, and $\gamma'_\pm + \gamma_\pm(h - h_\pm)$ factor through $L^2_1(Z,E) \hookrightarrow L^2_c(Z,E)$, which is compact by Rellich embedding theorem.

We will start with Step 3 in the next lecture, which does the actual computation of index.

April 18 (Sarah)

We've shown that D is invertible in the special case $h_t = h$ is constant, and we've shown that D is Fredholm in general. We still need to check that the index of D is equal to the spectral flow of $L_0 + h_t$. We'll only deal with two special cases.

Case 1: h_t is hyperbolic for all t. By definition, the eigenvalues of h_t never cross $i\mathbb{R}$, so the spectral flow is zero. To check $\mathrm{Ind}(D) = 0$, we deform D to something invertible. We define a family D_s so that $D_0 = \frac{d}{dt} + L_0 + h_-$ and $D_1 = D$.



In particular, this is a deformation through Fredholm operators, and since index is locally constant it follows that $\mathrm{Ind}(D) = \mathrm{Ind}(D_0)$. But D_0 is invertible, so the index is 0.

Case 2: We assume that the spectrum of $L_0 + h_t$ is simple (each eigenspace has dimension one) and h_t are symmetric. Let $\lambda_1(t), \ldots, \lambda_n(t)$ be the eigenvalues which ever cross 0, with eigenvectors $u_1(t), \ldots, u_n(t)$ (these eigenvectors are functions of y, but we'll ignore that to keep notation relatively simple). If we pick an appropriate basis, we can write

$$D = \frac{d}{dt} + \left(\begin{array}{c|c} A(t) & \\ \hline & \lambda(t) \end{array}\right),$$

where

$$\lambda(t) = \begin{pmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{pmatrix}$$

and A is some infinite-dimensional diagonal matrix.

We now compute $\ker(D)$. Let $f \in \ker(D)$, and $f = c_1 u_1 + \ldots + c_n u_n$ for $c_1, \ldots, c_n$ functions of t. The argument below can be used to show that any $f \in \ker(D)$ has to be of this form as well.

We have

$$\begin{aligned} 0 &= Df(t) \\ &= \sum_{k=1}^{n} \frac{d}{dt}(c_k(t)u_k(t)) + (L_0 + h_t)(c_k(t)u_k(t)) \\ &= \sum_{k=1}^{n} \frac{dc_k}{dt}(t)u_k(t) + c(t)\frac{du}{dt}(t) + c_k(t)\lambda_k(t)u_k(t). \end{aligned}$$

But λ_i and u_i are constant at infinity (by hypothesis, h_t is constant at infinity), so we can write

$$\sum_{k=1}^{n} \left(\frac{dc_k}{dt}(t) + c_k(t)\lambda_k(\pm\infty) \right) u_k(\pm\infty) = 0$$

for large enough t. Since $u_k(\pm\infty)$ are eigenvectors for $h_\pm$, they are linearly independent, which implies

$$\frac{dc}{dt}(t) + c_k(t)\lambda_k(\pm\infty) = 0$$

for sufficiently large t. At $-\infty$, we get $c_k(t) = d_k^- e^{-\lambda_k(-\infty)t}$, and at $+\infty$ we have $c_k(t) = d_k^+ e^{-\lambda_k(+\infty)t}$. In order for f to be L^2, we need $\mathrm{Re}(\lambda_k(-\infty)) < 0$ and $\mathrm{Re}(\lambda_k(+\infty)) > 0$. Thus the dimension of $\ker(D)$ is the number of eigenvalues whose real part goes from negative to positive, which is precisely the number of intersections which count positively toward spectral flow.

A similar analysis will show that the dimension of $\mathrm{coker}(D)$ is the number of intersections which count negatively toward spectral flow. The basic idea is that we can define some kind of adjoint:

$$D^* = -\frac{d}{dt} + L_0 + h_t^*.$$

Then the dimension of $\mathrm{coker}(D)$ is the dimension of $\ker(D^*)$, so we only need to relate $\ker(D^*)$ to the spectral flow. The relationship of elements of $\ker(D^*)$ to changes in signs of eigenvectors is nearly identical to the argument above, except that some signs are switched. Thus we add to the dimension of $\ker(D^*)$ when eigenvalues cross negatively. This completes the proof of case 2.

Remark 27. To see that the spectral index is finite, we need to note that the imaginary parts of the eigenvalues are bounded by the L^2 operator norm of $h - h^*$. We know that the norms of $h_t - h_t^*$ are uniformly bounded by continuity and compactness. Hence the eigenvalues in the spectral flow are contained in a band, finite in the imaginary direction. It then follows from the spectral theorem and basic principles of generic deformation that we can deform D to something which fits into case 2, up to the assumption that h_t is symmetric, which is probably not necessary for the argument.

We now proceed to a case which is of special interest in symplectic geometry. In what follows, we use (s,t) coordinates in $\mathbb{R} \times S^1$. We consider operators $D : L_1^2(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to$

$L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ of the form

$$D = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S,$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}$ and $S = S(s,t)$ is a smooth family of symmetric matrices satisfying

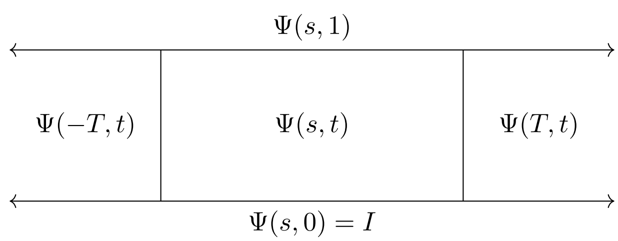
$$\frac{\partial S}{\partial s}(s,t) = 0$$

for $|s| \gg 1$. We will see later that these operators arise as linearizations of the Floer equation.

We endow $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_{st} = \langle \cdot, J_0 \cdot \rangle$. Let $Sp(2n)$ denote the Lie group of symplectic matrices, allow with a map $sp(2n) \xrightarrow{\sim} \mathrm{Symm}$, $A \mapsto J_0 A$. By integrating $S(s,t)$ in the t-direction we obtain a map $\Psi : \mathbb{R} \times \mathbb{R} \to Sp(2n)$ such that

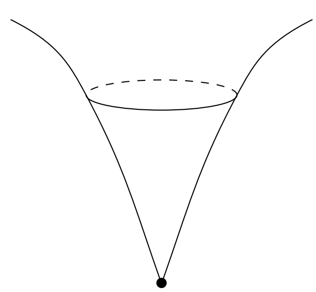
$$\frac{d\Psi}{dt} = J_0 S \Psi \qquad \text{and} \qquad \Psi(s,0) = I.$$

Then there is some T such that $\Psi(-s,t) = \Psi(-T,t)$ and $\Psi(s,t) = \Psi(T,t)$ for all $s > T$.



Let $\gamma : [0,1] \to Sp(2n)$ be a path such that $\gamma(0) = I$ and 1 is not an eigenvalue of $\gamma(1)$. Then we can define a Maslov (or Conley-Zehnder) index $\mu_{CZ}(\gamma) \in \mathbb{Z}$. There are two ways to define it:

(1) Let $Sp^*(2n)$ be the set of symplectic matrices for which 1 is not an eigenvalue. The Maslov cycle $Sp(2n) \setminus Sp^*(2n)$ is a singular codimension one subset, with singularities in codimension two.



The Maslov cycle is co-orientable, which means that any transverse crossing of it can be given a sign. Then we can define μ_{CZ} to be the signed count of intersections with the Maslov cycle, for generic γ.

(2) $Sp^*(2n)$ has two connected components; the component in which M lies is determined by the sign of $\det(I-M)$. We identify a special point in each of these, $B_+ = -I$ and

$$B_- = \begin{pmatrix} 2 & & & & \\ & \frac{1}{2} & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

Any path γ with $\gamma(1) \in Sp^*(2n)$ can be completed to a path $\tilde{\gamma}$ such that $\tilde{\gamma} \setminus \gamma$ doesn't cross the Maslov cycle and $\tilde{\gamma}(1) = B_\pm$.

We know $U(n) = Sp(2n) \cap O(2n)$, and $U(n)$ is a maximal compact subgroup of $Sp(2n)$. Inclusion is a homotopy equivalence. If $\rho : Sp(2n) \to U(n)$ is a homotopy inverse to inclusion (in particular we use the one given by the polar decomposition of matrices in $Sp(2n)$), then $\det^2 \circ \rho \circ \tilde{\gamma}$ is a loop in S^1. Note that the *det* here is the complex determinant of matrices in $U(n)$, its absolute value squared gives the real determinant. If we fix an isomorphism $\pi_1(S^1, 1) \cong \mathbb{Z}$, we can define $\mu_{CZ}(\gamma) = [\det^2 \circ \rho \circ \tilde{\gamma}]$.

The determinant map yields an isomorphism $\pi_1(Sp(2n), I) \to \pi_1(S^1, 1)$. If $\gamma(1) = I$, we can define $\mu_{CZ}(\gamma) = 2[\gamma]_{\pi_1}$. In particular, if γ is contractible then $\mu_{CZ}(\gamma) = 0$.

Appendix: Elliptic regularity for first order operators

First order operators.

Definition 28. Let E_1, E_2 be complex vector bundles over Y.

A first order differential operator is a $\mathbb{C}$-linear map

$$L : \Gamma(E_1) \to \Gamma(E_2)$$

with the property that for all $\mathbb{C}$-valued functions f

$$[L, f] = L \circ f - f \circ L : \Gamma(E_1) \to \Gamma(E_2)$$

is given by a tensor, i.e. is induced by a section of $\mathrm{Hom}_{\mathbb{C}}(E_1, E_2)$. In this definition we use the fact that $\Gamma(E_i)$ are modules over the ring $\Omega^0(\mathbb{C})$.

Exercise 10. Prove that first order differential operators are **local** in the sense that if $s \in \Gamma(E_1)$ is supported in a closed set K, then $L(s)$ is also supported in K.

Use this to show that L is induced by a map of sheaves $\Gamma(E_1; -) \to \Gamma(E_2; -)$.

Example 29 (The structure of differential operators)**.** Let $s_1, \cdots, s_n$ and $r_1, \cdots, r_m$ be local frames for $\Gamma(E_1)$ and $\Gamma(E_2)$, respectively. Let $x_1, \cdots, x_d$ be local coordinates on the base. If L is a first order differential operator then

$$L(\sum_{i=1}^{n} a_i s_i) = \sum_{i=1}^{n} [L, a_i] s_i + a_i L s_i.$$

We can compute the tensor $[L, a_i]_x \in \mathrm{Hom}(E_{1,x}, E_{2,x})$ using the following trick. Near a given point p_0, say $x(p_0) = c$ we can write

$$a_i(p) = a_i(p_0) + \sum_{j=1}^{n} b_{ij}(p)(x_j(p) - c_j),$$

for some smooth functions b_{ij}. Moreover it is clear that

$$\mathrm{d}a_i(p_0) = \sum_j b_{ij}(p_0) \mathrm{d}x_j(p_0) \iff \frac{\partial a_i}{\partial x_j} = b_{ij}(p_0).$$

Now near p_0 write

$$\begin{aligned}
[L, a_i] = [L, a_i(p_0)] + \sum_{j=1}^{n} [L, b_{ij}(x_j - c_j)] &= 0 + \sum_{j=1}^{n} b_{ij}[L, x_j - c_j] + [L, b_{ij}](x_j - c_j) \\
&= \sum_{j=1}^{n} b_{ij}[L, x_j] + [L, b_{ij}](x_j - c_j)
\end{aligned}$$

Evaluating this tensor at p_0 yields (recall $x_j(p_0) = c_j$)

$$[L, a_i](p_0) = \sum_{j=1}^{n} b_{ij}(p_0)[L, x_j] = \sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j}(p_0)[L, x_j].$$

But p_0 was arbitrary. It follows that

$$L(s) = L(\sum_{i=1}^{n} a_i s_i) = \sum_{i,j} \frac{\partial a_i}{\partial x_j}[L, x_j]s_i + a_i L(s_i).$$

In the presence of a connection, we can use the above to say that locally

$$L(s) = A(\nabla s) + B(s),$$

where A, B are tensors:

$$A \in \Gamma \operatorname{Hom}(\operatorname{Hom}(TY, E_1), E_2) \text{ and } B \in \Gamma \operatorname{Hom}(E_1, E_2).$$

We claim this actually holds globally. To see this, cover Y by a locally finite open cover $\{U_\alpha\}$, with associated partition of unity $\{\rho_\alpha\}$. Assume that for sections s_α supported in U_α we have

$$L(s_\alpha) = A_\alpha(\nabla s_\alpha) + B_\alpha(s_\alpha).$$

Then for any s

$$L(s) = L(\sum_\alpha \rho_\alpha s) = \sum_\alpha A_\alpha(\rho_\alpha \nabla s + \mathrm{d}\rho_\alpha \otimes s) + B_\alpha(\rho_\alpha s) = A(\nabla s) + B(s),$$

provided we define

$$A = \sum_\alpha \rho_\alpha A_\alpha \text{ and } B = \sum_\alpha A_\alpha(\mathrm{d}\rho_\alpha \otimes -) + \rho_\alpha B_\alpha(-).$$

By local finiteness, A, B are well-defined sections of the appropriate bundles. We have shown

Theorem 30. Let L be a first order differential operator $E_1 \to E_2$. Let ∇ be a connection on E_1. Then there are sections A, B of $\operatorname{Hom}(\operatorname{Hom}(TY, E_1), E_2)$ and $\operatorname{Hom}(E_1, E_2)$, respectively, so that

$$L(s) = A(\nabla s) + B(s).$$

Exercise 11. Prove that A, B are uniquely determined by L. Hints: pick $s_0 \in E_{1,p_0}$ and a section s so $s(p_0) = s_0$, and so ∇s vanishes at p_0 (this can be achieved by parallel-transporting along radial lines). Conclude that B is completely determined by L.

Now for any element $\theta \in \operatorname{Hom}(TY, E_1)$, extend θ to a section of $\Omega^1(E_1)$. Then we can construct a section s (in a neighborhood of p_0) by parallel transporting along radial lines so that

$$\partial_r \lrcorner \nabla s = \partial_r \lrcorner \theta,$$

where ∂_r is the radial vector field. A-priori s is smooth when pulled back to the blow up $\mathbb{S}^{d-1} \times [0, \infty)$. It can be shown that s is smooth when considered as a section on $\mathbb{R}^d$. The key is that its (covariant) partial derivatives exist and are continuous functions on $\mathbb{R}^d$, and agree with θ at p_0. Then, conclude that $A_{p_0}((\nabla s)_{p_0}) = A_{p_0}(\theta)$. Conclude that A is determined by L.

Sobolev embedding/compactness theorems.

Definition 31. Let g be a Riemannian metric on Y, an oriented Riemannian manifold (potentially with boundary), let $\langle -,-\rangle$ be an inner product on E (anti-linear in the first factor). Let ∇ denote the Levi-Civita connection on TY, and also a connection on E compatible with its metric (where there is hopefully no chance of confusion).

For codimension 0 submanifolds with boundary U, we define the space $L^p(U, E_1)$ by duality: $L^p(U, E_1)$ consists of distributions $C_c^\infty(\mathrm{int}(U), E_1) \to \mathbb{C}$ satisfying the estimate

$$\varphi \in L^p(U, E_1) \implies |(\varphi, f)| \leq C_\varphi \left[\int_U |f|^q \ \mathrm{dVol}_g\right]^{1/q},$$

where $|f| = \langle f, f\rangle^{1/2}$, and q is the Hölder conjugate to p. We define the L^p norm on $L^p(U, E_1)$ by duality:

$$\|\varphi\|_{L^p} = \sup\left\{|(\varphi, f)| : f \in C_c^\infty(\mathrm{int}(U), E_1) \text{ and } \left[\int_U |f|^q \ \mathrm{dVol}_g\right]^{1/q} = 1\right\}.$$

Exercise 12. Let φ be a continuous section of E_1 compactly supported in $\overline{U}$. It is clear that

$$\left[\int_U |\varphi|^p \ \mathrm{dVol}_g\right]^{1/p} < \infty.$$

Then φ induces an element of $L^p(U, E_1)$ by

$$(\varphi, f) = \int_U \langle \varphi(x), f(x)\rangle \,\mathrm{dVol}_g(x),$$

and

$$\|\varphi\|_{L^p(U,E_1)} = \left[\int_U |\varphi|^p \ \mathrm{dVol}_g\right]^{1/p}.$$

Hint: without loss, suppose that the integrals of $|\varphi|^p$ and $|f|^q$ are 1. Estimate

$$|\langle \varphi(x), f(x)\rangle| \leq |\varphi(x)|\,|f(x)| \leq \frac{1}{p}|\varphi(x)|^p + \frac{1}{q}|f(x)|^q,$$

to conclude that $\|\varphi\|_{L^p} \leq 1$. For the converse, consider smooth approximations (i.e. use convolution) to

$$\frac{\varphi}{\epsilon + |\varphi|^{2-p}}$$

and show that

$$\|\varphi\|_{L^p} \geq \int |\varphi|^p \frac{|\varphi|^2}{\epsilon|\varphi|^p + |\varphi|^2} \mathrm{dVol}_g.$$

By continuity of the integral $C^0 \to \mathbb{R}$ (on the support of φ) conclude that

$$\|\varphi\|_p \geq 1.$$

(where we are still assuming we normalized φ so $|\varphi|^p$ had integral 1).

Theorem 32. The continuous sections compactly supported in $\mathrm{int}(U)$ are dense in $L^p(U, E_1)$.

Remark. This is a fairly deep theorem.

Corollary 33. Using convolution, prove that $C_c^\infty(\mathrm{int}(U), E_1)$ is dense in $L^p(U, E_1)$. Conclude that the dual of $L^p(U, E_1)$ is naturally identified with $L^q(U, E_1)$. More precisely, there is a unique pairing

$$(\varphi, f) \in L^p \times L^q \mapsto \mathbb{C}$$

extending the pairing of L^p on C^∞ and vice-versa, so that

$$|(\varphi, f)| \leq \|\varphi\|_{L^p} \|f\|_{L^q},$$

and that this pairing establishes the bijection $(L^p)^* \simeq L^q$.

Definition 34 (The adjoint of a tensor)**.** Let E, F be bundles with metrics denoted $\langle -, - \rangle$. We obtain a metric on $\mathrm{Hom}(E, F)$ by identifying $\mathrm{Hom}(E, F) \simeq \mathrm{Hom}(E, \mathbb{C}) \otimes F$. The induced norm is equivalent to the operator norm

$$|\Phi|_{\mathrm{oper}} = \sup\left\{|\Phi(x)|_F : |x|_E = 1\right\}.$$

Given a smooth section $\Phi \in \mathrm{Hom}(E, F)$, we observe that

$$\int \langle \Phi(e), f \rangle \, \mathrm{dVol} = \int \langle e, \Phi^*(f) \rangle \, \mathrm{dVol},$$

where Φ^* is the pointwise adjoint. Therefore we can define the evaluation of Φ on an E-valued distribution via duality: for distributions valued in E, let $\Phi(e)$ be the distribution satisfying

$$(\Phi(e), f) = (e, \Phi^*(f)).$$

Exercise 13. If f is in $L^p(E)$, and Φ is a smooth section of $\mathrm{Hom}(E, F)$ so that $|\Phi| < C$ is uniformly bounded on Y, then $\Phi(f)$ is in $L^p(E)$, and moreover

$$\|\Phi(f)\|_{L^p} \leq C \|f\|_{L^p}.$$

Hint: use the fact that L^p norm is defined by duality.

Exercise 14. If f is a continuous section of E, show that the weak definition of $\Phi(f)$ agrees with the definition of $\Phi(f)$ as a continuous section.

Definition 35 (connections and metrics on induced bundles)**.** Let ∇ be a connection on E and also denote the Levi-Civita connection on (Y, g). Suppose that ∇ is compatible with $\langle -, - \rangle$. We define a metric on $\mathrm{Hom}(TM^{\otimes k}, E) \simeq T^*M^{\otimes k} \otimes E$ by

$$\langle \theta_1 \otimes \cdots \otimes \theta_k \otimes e, \theta_1' \otimes \cdots \otimes \theta_k' \otimes e' \rangle = g(\theta_1, \theta_1') \cdots g(\theta_k, \theta_k') \langle e, e' \rangle.$$

We define a connection by

$$\nabla(\theta_1 \otimes \cdots \otimes \theta_k \otimes e) = \theta_1 \otimes \cdots \otimes \theta_k \otimes \nabla e + \sum_i \theta_1 \otimes \cdots \otimes \nabla\theta_i \otimes \cdots \theta_k \otimes e.$$

Exercise 15. Check that the induced connection on $(T^*M)^{\otimes k} \otimes E$ is compatible with the metric.

Definition 36 (the adjoint of the covariant derivative)**.** Suppose that we have a local orthonormal frame $y_1, \cdots, y_n$ of TY. Let $\theta_i = g(y_i, -)$.

For smooth compactly supported sections $s \in \Gamma(E)$, $\lambda \otimes r \in \Gamma(\mathrm{Hom}(TM, E))$ we compute $\nabla s = \sum \theta_i \otimes \nabla_{y_i} s$ and hence

$$\langle \nabla s, \lambda \otimes r \rangle = \sum_i g(\theta_i, \lambda) \langle \nabla_{y_i} s, r \rangle = \sum_i g(\theta_i, \lambda)(y_i \lrcorner \, \mathrm{d} \langle s, r \rangle) - g(\theta_i, \lambda) \langle s, \nabla_{y_i} r \rangle ,$$

so

$$(\nabla s, \lambda \otimes r) = \int X_\lambda \lrcorner \, \mathrm{d} \langle s, r \rangle \, \mathrm{dVol}_g - \langle s, \nabla_{X_\lambda} r \rangle \, \mathrm{dVol}_g,$$

where $X_\lambda = \sum_i g(\theta_i, \lambda) y_i$ (X_λ is the g-"dual" of λ).

A bit of differential form gymnastics produces

$$0 = X_\lambda \lrcorner \, (\mathrm{d} \langle s, r \rangle \wedge \mathrm{dVol}) = (X_\lambda \lrcorner \, \mathrm{d} \langle s, r \rangle)\mathrm{dVol} - \mathrm{d} \langle s, r \rangle \, (X_\lambda \lrcorner \, \mathrm{dVol})$$

$$0 = (X_\lambda \lrcorner \, \mathrm{d} \langle s, r \rangle)\mathrm{dVol} - \mathrm{d}(\langle s, r \rangle \, (X_\lambda \lrcorner \, \mathrm{dVol})) + \langle s, r \rangle \, \mathrm{div}(X_\lambda)\mathrm{dVol},$$

whre $\mathrm{div}(X_\lambda)\mathrm{dVol} = \mathrm{d}(X_\lambda \lrcorner \, \mathrm{dVol})$. Since the integral of a compactly supported exact form is 0, we conclude

$$(\nabla s, \lambda \otimes r) = \int \langle s, \mathrm{div}(X_\lambda) r - \nabla_{X_\lambda} r \rangle \, \mathrm{dVol}_g.$$

Conclude that ∇_X has formal adjoint

$$(*) \qquad \nabla^* : \lambda \otimes r \mapsto \mathrm{div}(X_\lambda) r - \nabla_{X_\lambda} r.$$

One can easily check that this is a first order differential operator. While we proved this under the assumption that it has an orthonormal coordinate frame, it is clear that X_λ doesn't actually depend on the particular orthonormal coordinate frame, and hence ∇^* defined in $(*)$ is the adjoint of ∇ on any bundle E.

Definition 37. Let φ be a distribution $C^\infty(\mathrm{int}(U), E) \to \mathbb{C}$. The weak derivative $\nabla \varphi$ is the distribution valued in the bundle $\mathrm{Hom}(TM, E)$ defined by

$$(\nabla \varphi, \Phi) = (\varphi, \nabla^* \Phi),$$

where $\Phi \in \Gamma \, \mathrm{Hom}(TM, E)$ is a test function.

Given a smooth vector field X, we can consider the tensor

$$X \lrcorner \, (-) : \Gamma \, \mathrm{Hom}(TM, E) \to \Gamma E.$$

This defines a distributional evaluation $\mathrm{Hom}(TM, E) \to E$. In this fashion, we can define the evaluations $\nabla_X \varphi$ for any distribution φ.

Exercise 16. Show that

$$(\nabla_X \varphi, \rho) = (\varphi, \nabla_X^* \rho),$$

where

$$\nabla_X^* \rho = \operatorname{div}(X)\rho - \nabla_X \rho.$$

Definition 38 (Sobolev Spaces)**.** Let U be a codimension 0 submanifold of Y. We define the space $W^{k,p}(U,E)$ by recursion $W^{0,p}(U,E) = L^p(U,E)$, and $W^{k,p}(U,E)$ consists of all distributions φ whose weak derivative $\nabla\varphi$ lies in $W^{k-1,p}(U, \operatorname{Hom}(TY,E))$.

If the tangent bundle of U has global orthonormal frame $y_1, \cdots, y_n$, it is equivalent to suppose that

$$\nabla_{y_1}\varphi, \cdots, \nabla_{y_n}\varphi \in W^{k-1,p}(U,E),$$

Indeed the decomposition

$$\nabla\varphi = \sum_i \theta_i \otimes \nabla_{y_i}\varphi$$

identifies $\operatorname{Hom}(TY,E)$ with $E^{\oplus n}$ isometrically, and so the induced "weak" identification induces an isometry on L^p. We put a metric on $W^{k,p}(U,E)$ via it's inclusion into

$$(*) \qquad W^{k,p}(E) \to L^p(E) \oplus L^p(\operatorname{Hom}(TM,E)) \oplus \cdots \oplus L^p(\operatorname{Hom}(TM^{\otimes k}, E)).$$

Exercise 17. Prove that the inclusion in $(*)$ is a closed inclusion. Hint: arguing by induction, it suffices to show that if

$$u_n \to u_\infty \text{ in } W^{k-1,p} \text{ and } \nabla u_n \to w_\infty \text{ in } W^{k-1,p},$$

then $\nabla u_\infty = w_\infty$. It will follow that $u_n \to u_\infty$ in $W^{k,p}$. The following estimate

$$(\nabla u_\infty - w_\infty, \varphi) = (u_\infty - u_n, \nabla^*\varphi) + (\nabla u_n - w_\infty, \varphi)$$

and taking the limit $n \to \infty$ proves that $\nabla u_\infty = w_\infty$.

Exercise 18. For $p \in (1,\infty)$, we have shown that L^p is reflexive. It follows that $L^p \oplus \cdots \oplus L^p$ is reflexive. Conclude that $W^{k,p}$ is reflexive. Hint: this is an exercise in applying the Hahn-Banach theorem.

Definition 39. It is clear that smooth compactly supported functions induce elements of $W^{k,p}(U,E)$. We define

$$W_0^{k,p}(U,E) = \text{closure of smooth compactly supported functions in } W^{k,p}(U,E).$$

Warning: $W_0^{k,p}$ is generally not equal to $W^{k,p}$. It turns out that $W_0^{k,p}$ is much better behaved (in certain regards) compared to $W^{k,p}$.

Theorem 40. Let E be a vector bundle on Y, and let $g, \langle -,-\rangle, \nabla$ and $g', \langle -,-\rangle', \nabla'$ be two sets of data. Suppose that we can uniformly bound the C^∞ distance between the two sets of data, i.e.

$$\|g - g'\|_{C^\infty, g} + \left\|\langle -,-\rangle - \langle -,-\rangle'\right\|_{C^\infty, \langle -,-\rangle} + \|\nabla - \nabla'\|_{C^\infty, g, \langle -,-\rangle} < \infty,$$

(recall that the difference of two connections is a tensor, and hence it makes sense to talk about its C^∞ norm). Then we have an equality of sets

$$W^{k,p}(Y, g, \langle -, - \rangle, \nabla) = W^{k,p}(Y, g', \langle -, - \rangle, \nabla') \text{ and similarly for } W_0^{k,p}.$$

and moreover the two norms on $W^{k,p}$ are comparable. □

Theorem 41. Let $U \subset Y$ be a codimension 0 manifold with boundary. Then there is an extension (isometric) embedding

$$\mathrm{ext} : W_0^{k,p}(U, E) \to W_0^{k,p}(Y, E),$$

defined in the obvious way on smooth functions and extended by density. There is also a restriction

$$\mathrm{res} : W^{k,p}(Y, E) \to W^{k,p}(U, E),$$

and $\mathrm{res} \circ \mathrm{ext} = \mathrm{id}$.

Corollary 42. Let Y be compact with boundary. Let $f : Y \to Z$ be a smooth embedding of a codimension 0 submanifold with boundary, and let $F : E_1 \to f^*E_2$ be a bundle isomorphism. Then there is a bounded extension

$$W_0^{k,p}(Y, E_1, g_Y, \langle -, - \rangle_Y, \nabla_Y) \to W_0^{k,p}(Y, E_2, g_Z, \langle -, - \rangle_Z, \nabla_Z),$$

and a bounded restriction

$$W^{k,p}(Y, E_2, g_Z, \langle -, - \rangle_Z, \nabla_Z) \to W^{k,p}(Y, E_1, g_Y, \langle -, - \rangle_Y, \nabla_Y),$$

so that the restriction composed with the extension is the identity.

Remark. These results allow us freedom to change the data defining the Sobolev norms, without actually changing the topology of the space $W_0^k(Y, E)$. We remark that some form of compactness on Y is crucial.

Lemma 43. Let ρ be a smooth section of $\mathrm{Hom}(E, F)$. If u is in $W^{k,p}(Y, E)$ (resp. $W_0^{k,p}$), then

$$\rho(u) \in W^{k,p}(Y, F) \text{ resp. } W_0^{k,p}$$

and

$$\|\rho(u)\|_{W^{k,p}(Y,F)} \leq C(\rho) \|u\|_{W^{k,p}(Y,E)},$$

where $C(\rho)$ depends on the sizes of the first k derivatives of ρ.

Proof. Argue by induction.

$$\nabla \rho(u) = \nabla \mathrm{ev}(\rho \otimes u) = \mathrm{ev}(\nabla \rho \otimes u) + \mathrm{ev}(\rho \otimes \nabla u),$$

to conclude that

$$\|\nabla(\rho(u))\|_{W^{k-1,p}} \leq C(\nabla \rho) \|u\|_{W^{k-1,p}} + C(\rho) \|\nabla u\|_{W^{k-1,p}}.$$

□

Sobolev Embedding. Let Y^d be compact with boundary. Let E be a vector bundle on Y. Suppose that $k > \ell$ and

$$k - \frac{p}{d} \geq \ell - \frac{r}{d}.$$

Then the inclusion of test functions into $W_0^{\ell,r}$ extends to a bounded inclusion

$$W_0^{k,p}(Y,E) \subset W_0^{\ell,r}(Y,E).$$

Proof. First we claim that is suffices to prove it locally. To see this, cover Y by finitely many open sets U_α where it is known that inclusion extends to a bounded inclusion

$$W_0^{k,p}(U_\alpha, E) \to W_0^{\ell,r}(U_\alpha, E).$$

Fix a partition of unity ρ_α. Then for any u smooth and compactly supported in Y, we have

$$\|u\|_{W^{k,p}} \leq \sum_\alpha C_\alpha \|\rho_\alpha u\|_{W^{\ell,r}} \leq \sum_\alpha C'_\alpha \|u\|_{W^{\ell},r} = C(\{U_\alpha, \rho_\alpha\}) \|u\|_{W^{\ell,r}},$$

as desired.

To prove it locally, note that it does not depend on the data used to define the norms. Thus we may suppose that $U = B(r) \subset \mathbb{R}^d$, $E = \mathbb{C}^n$ is a trivial bundle, and the metrics/connections are all the standard ones. The following tricky lemma is the key result

Lemma 44 (Gagliardo-Nirenberg Inequality)**.** Let $u \in W_0^{1,1}(\mathbb{R}^d, \mathbb{C}^n)$. Then $u \in L^{d/(d-1)}$ and

$$\|u\|_{L^{d/(d-1)}} \leq \left[\prod_1^d \|\partial_i u\|_{L^1}\right]^{1/d}.$$

Assuming this technical lemma, we can complete the proof. First we prove the following

Claim 45. Let $p < d$ and define $p^* > p$ by

$$1 - \frac{d}{p} = \frac{d}{p^*}.$$

Then there is constant C_p so that for all $u \in W_0^{1,p}$, $u \in L^{p^*}$ and

$$\|u\|_{L^{p^*}} \leq C_p \|\nabla u\|_{L^p}.$$

Proof. Suppose that u is a test function, and consider $v_\epsilon = (\epsilon + |u|^2)^{(s-1)/2} u$, where $s > 1$ will be specified later. Then

$$\partial_i v_\epsilon = \frac{(s-1)}{2}(\epsilon + |u|^2)^{(s-3)/2} \sum_j u_j \partial_i u_j u + (\epsilon + |u|^2)^{(s-1)/2} \partial_i u,$$

hence

$$|\partial_i v_\epsilon| \leq (\frac{(s-1)}{2}(\epsilon + |u|^2)^{(s-3)/2} |u|^2 + (\epsilon + |u|^2)^{(s-1)/2}) |\nabla u|.$$

Applying the lemma yields

$$\left[\int (\epsilon + |u|^2)^{(s-1)d/2(d-1)} |u|^{d/(d-1)} \,\mathrm{dVol}\right]^{(d-1)/d} \leq \left[\prod_1^d \int |\partial_i v_\epsilon| \,\mathrm{dVol}\right]^{1/d}.$$

Applying monotone convergence theorem proves

$$\left[\int |u|^{sd/(d-1)} \,\mathrm{dVol}\right]^{(d-1)/d} \leq C_s \int |u|^{s-1} |\nabla u| \,\mathrm{dVol}.$$

Now suppose that $p < d$ and p^* is defined by

$$1 - \frac{d}{p} = -\frac{d}{p^*} \implies p^* p - dp^* = -dp \implies p^* = \frac{dp}{d-p}.$$

Pick s so that

$$\frac{sd}{d-1} = p^*.$$

Then we conclude that

$$\|u\|_{L^{p^*}}^s \leq C_s \int |u|^{s-1} \|\nabla u\| \,\mathrm{dVol}.$$

Let q be Holdër conjugate to p. Then

$$\|u\|_{L^{p^*}}^s \leq C_s \left[\int |u|^{(s-1)q} \,\mathrm{dVol}\right]^{1/q} \|\nabla u\|_{L^p}.$$

Now since

$$\frac{s}{d-1} = p^* \implies (s-1)q = (d-1)p^* q - q \implies (s-1)q = \frac{d-1}{d-p} pq - q = \frac{d}{d-p}(pq-q) = p^*.$$

Therefore

$$\|u\|_{L^{p^*}}^s \leq C_s \|u\|_{L^{p^*}}^{p^*/q} \|\nabla u\|_{L^p} \implies \|u\|_{L^{p^*}}^{s-p^*/q} \leq C_s \|\nabla u\|_{L^p}.$$

However, the reader can easily check that $s - p^*/q = 1$. Therefore

$$\|u\|_{L^{p^*}} \leq C_p \|\nabla u\|_{L^p},$$

as desired. Since this holds for test functions, it extends to $W_0^{1,p}$ by density. □

Now we can complete the proof. Suppose that

$$1 - \frac{d}{p} \geq -\frac{d}{r}.$$

Define r^* by

$$1 - \frac{d}{r^*} = -\frac{d}{r}.$$

Then $r^* \leq p$ and $r^* < d$. Therefore we can apply our claim to conclude

$$\|u\|_{L^r} \leq C_r \|\nabla u\|_{L^{r^*}}.$$

Since we are on a bounded domain,

$$\|\nabla u\|_{L^{r^*}} \leq C \|\nabla u\|_{L^p},$$

as desired. The constant C depends on r, p and the domain. We conclude a continuous embedding:

$$W^{1,p}(B) \subset L^r(B).$$

Now suppose that $k - d/p \geq \ell - d/r$. Pick $r_0, \cdots, r_{k-\ell} = r$ so that

$$k - d/r_0 = (k-1) - d/r_1 = \cdots = \ell - \frac{d}{r}.$$

Then by using the result for $k = 1$ conclude

$$W_0^{k,r_0}(B) \subset W_0^{k-1,r_1} \cdots \subset W_0^{\ell,r}(B).$$

Since $p \geq r_0$, conclude the bounded inclusion

$$W_0^{k,p}(B) \subset W_0^{\ell,r_0}(B).$$

This completes the proof of the Sobolev embedding theorem. $\square$

Exercise 19. Using the Gagliardo-Nirenberg inequality for $u \in W_0^{1,1}(B)$

$$C^{-1} \|u\|_{L^1} \leq \|u\|_{L^{d/(d-1)}} \leq \|\nabla u\|_{L^1},$$

prove that $1 \notin W_0^{1,1}(B)$.

Theorem 46 (Rellich-Kondrachov compactness)**.** Suppose that $k > \ell$ and $k - d/p > \ell - d/r$, and Y^d is a compact manifold with boundary, then the Sobolev embedding

$$W_0^{k,p}(Y, E) \subset W_0^{\ell,r}(Y, E)$$

is a compact inclusion.

Proof. As in the proof of the Sobolev embedding theorem, we will reduce the theorem to a local computation in $\mathbb{R}^d$.

Suppose that the theorem holds locally, i.e. we can cover Y by finitely many open sets U_α where it is know that the inclusion

$$W_0^{k,p}(U_\alpha, E) \subset W_0^{\ell,r}(U_\alpha, E)$$

is compact. Let u_n be a bounded sequence in $W_0^{k,p}(Y, E)$. If $\{\rho_\alpha\}$ is a partition of unity, then $\rho_\alpha u_n$ is a bounded sequence in $W_0^{k,p}(U_\alpha, E)$, and hence, after a subsequence, converges to something $u_{\infty,\alpha} \in W_0^{\ell,r}(U_\alpha, E)$. Then

$$\left\| u_n - \sum_\alpha u_{\infty,\alpha} \right\|_{W^{\ell,r}(Y)} \leq \sum_\alpha \|\rho_\alpha u_n - u_{\infty,\alpha}\|_{W^{\ell,r}(Y)} = \sum_\alpha \|\rho_\alpha u_n - u_{\infty,\alpha}\|_{W^{\ell,r}(U_\alpha)} \to 0.$$

Therefore it suffices to prove the theorem locally. We may therefore suppose that E is trivial and $Y = B(1)$ is a ball in $\mathbb{R}^n$ with the standard metric, connection, etc.

We can also reduce to the case $k = 1$, because if we know that $W_0^{1,p} \subset W_0^{0,r}$ is compact when $1 - d/p > -d/r$, then we also know $W_0^{k,p} \subset W_0^{k-1,r}$ when $k - d/p > (k-1) - d/r$, and by finding $r_1, \cdots, r_n = r$ so that

$$k - d/p > (k-1) - d/r_1 > \cdots > \ell - d/r,$$

we conclude the result for $W_0^{k,p} \subset W_0^{\ell,r}$.

The strategy of proof is the following: if $\mathcal{E}$ is a uniformly bounded family of functions in $W_0^{1,p}(B(1))$, and ρ_δ is a mollifying function, we will show that there is a universal bound

$$\|u - \rho_\delta * u\|_{L^r} < c(\delta),\ c(\delta) \to 0 \text{ as } \delta \to 0, \text{ for all } u \in \mathcal{E}.$$

We will then show that the family of smooth functions $\rho_\delta * \mathcal{E}$ is bounded and equicontinuous (and are all supported in $B(1+\delta)$). By Arzéla-Ascoli, we conclude that $\rho_\delta * \mathcal{E}$ is pre-compact in $C^0(B(1+\delta))$ topology, and hence also in $L^q(B(1))$.

Now pick $\epsilon > 0$. Choose δ so $c(\delta) < \epsilon/3$. Since $\rho_\delta * \mathcal{E}$ is compact, we can find elements $v_1, \cdots, v_n \in \mathcal{E}$ so that the $\epsilon/2$ balls around $\rho_\delta * v_i$ cover $\rho_\delta * \mathcal{E}$. Since

$$\|\mathcal{E} - \rho_\delta * \mathcal{E}\|_{L^q} < \epsilon/3,$$

we conclude that the ϵ balls around v_i cover $\mathcal{E}$, and since ϵ was arbitrary, we conclude $\mathcal{E}$ is pre-compact. Any sequence in a pre-compact space inside of a Banach space converges after taking a subsequence, and so we conclude the compactness statement.

It suffices to prove the two claims:

Claim 47. If $1 - d/p > -d/q$, and $\mathcal{E}$ is bounded in $W_0^{1,p}(B(1))$, then there is a uniform bound

$$\|v - \rho_\delta * v\|_{L^q} < c(\delta) \text{ for all } v \in \mathcal{E},$$

and $c(\delta) \to 0$ as $\delta \to 0$.

Claim 48. The family $\rho_\delta * \mathcal{E}$ is equicontinuous.

□

In order to prove Claim 47, we introduce some results about convolution. We define the **translation** operator on smooth functions by

$$\tau_h(f)(x) = f(x+h).$$

It is clear that $\tau_h^* = \tau_{-h}$. It is readily checked that

$$(*) \qquad \|\tau_h f\|_{W^{k,p}(B)} \leq \|f\|_{W^{k,p}},$$

and hence τ_h extends by density to $W_0^{k,p}(B)$. Moreover, for all $u \in W_0^{k,p}(B)$ we have $\tau_h u \to u$ in $W^{k,p}$ norm, as $h \to 0$.

We define the **convolution** of u with a bump function $\rho_\delta = \delta^{-d}\rho(\delta^{-1}x)$ by

$$u * \rho_\delta(x) = \int_{\mathbb{R}^d} \rho_\delta(y) \tau_{-y} u(x) \, \mathrm{dVol}(y).$$

This is well-defined for smooth compactly functions. By pairing with a test-function v, we conclude that

$$\begin{aligned}(u*\rho_\delta, v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_\delta(y)u(x-y)v(x)\mathrm{dVol}(y)\mathrm{dVol}(x)\\ &= \int_{\mathbb{R}^d}\rho_\delta(y)\int_{\mathbb{R}^d}u(x-y)v(x)\mathrm{dVol}(x)\mathrm{dVol}(y)\\ &\le \|u\|_{L^p}\|v\|_{L^q},\end{aligned}$$

hence $\|u*\rho_\delta\|_{L^p} \le \|u\|_{L^p}$. Therefore convolution extends to a bounded map $L^p \to L^p$.

It is clear that $\nabla(u*\rho_\delta) = (\nabla u)*\rho_\delta$ (for smooth u) and hence

$$\|u*\rho_\delta\|_{W^{k,p}} \le \|u\|_{W^{k,p}},$$

and this estimate shows that $\rho_\delta * (-)$ extends to a bounded map $W_0^{k,p} \to W_0^{k,p}$.

We can compute

$$\rho_\delta * u(x) = \int \rho_\delta(y)u(x-y)\mathrm{dVol}(y) = \int \rho_\delta(x-y)u(y)\mathrm{dVol}(y),$$

and so

$$\nabla(\rho_\delta * u) = (\nabla\rho_\delta)*u.$$

Consequently,

$$\|\nabla(\rho_\delta * u)\|_{C^0} \le \|\nabla\rho_\delta\|_{C^0}\|u\|_{L^1} \le \|\nabla\rho_\delta\|_{C^0}\|u\|_{L^p}.$$

It follows that $\rho_\delta * u$ is differentiable if $u \in L^p$ (by density, and completeness of C^1 topology) and that

$$\nabla(\rho_\delta * u) = (\nabla\rho_\delta)*u.$$

Consequently $\rho_\delta * u$ is C^∞ if $u \in L^p$.

Lemma 49. Suppose that $1 - d/p > -d/r$. Define p^* by

$$1 - d/p \ge -d/p^* > -d/r.$$

This can always be achieved. Since $1 \ge 1/r > 1/p^*$, there is $\alpha > 0$ so that

$$\frac{1}{r} = \alpha + \frac{1-\alpha}{p^*}.$$

We claim that there is a constant C so that for all $u \in W_0^{1,p}(B)$,

$$\|\tau_h u - u\|_{L^r(\mathbb{R}^d)} \le C|h|^\alpha \|u\|_{W^{1,p}(B)}.$$

Proof. It suffices to prove this when $u \in C_0^\infty(B)$. Then we compute

$$|\tau_h u(x) - u(x)| \le |h|\int_0^1 |\nabla u(x+th)|\,\mathrm{d}t,$$

hence

$$\|\tau_h u - u\|_{L^1} \le |h|\int_0^1\int_{\mathbb{R}^d}|\nabla u(x+th)|\,\mathrm{dVol}\,\mathrm{d}t \le |h|\,\|\nabla u\|_{W^{1,p}}\,\mathrm{Vol}(B)^{1/q},$$

where we use the fact that $\nabla u(x+th)$ is supported in $B(1)-th$.

Now we compute

$$|\tau_h u - u|^r = |\tau_h u - u|^{\alpha r} |\tau_h u - u|^{(1-\alpha)r},$$

and then apply Hölder's inequality, with $1/\beta$, $1/(1-\beta)$, to obtain

$$\int |\tau_h u - u|^r \le \left[\int |\tau_h u - u|^{\frac{\alpha r}{\beta}} \mathrm{dVol}\right]^{\beta} \left[\int |\tau_h u - u|^{\frac{(1-\alpha)r}{1-\beta}} \mathrm{dVol}\right]^{1-\beta}.$$

Pick $\beta = \alpha r$, so that $1-\beta = \frac{(1-\alpha)r}{p^*}$, to obtain

$$\|\tau_h u - u\|_{L^r} \le \|\tau_h u - u\|_{L^1}^{\alpha} \|\tau_h u - u\|_{L^{p^*}}^{1-\alpha}.$$

Using Sobolev embedding for $1-d/p \ge -d/p^*$, we conclude

$$\|\tau_h u - u\|_{L^{p^*}}^{1-\alpha} \le C \|u\|_{W^{1,p}}^{1-\alpha}.$$

We also have our previous estimate

$$\|\tau_h u - u\|_{L^1}^{\alpha} \le C |h|^{\alpha} \|u\|_{W^{1,p}}^{\alpha}.$$

Therefore

$$\|\tau_h u - u\|_{L^r} \le C |h|^{\alpha} \|u\|_{W^{1,p}},$$

as desired. □

Lemma 50 (Claim 47)**.** For $u \in W_0^{1,p}(B)$, we have

$$\|u - \rho_\delta * u\|_{L^q} \le C |\delta|^{\alpha} \|u\|_{W^{1,p}},$$

for some C and $\alpha > 0$, dependent only p, r, B, d.

Proof. We compute

$$u - \rho_\delta * u = \int_{\mathbb{R}^d} \rho_\delta(y)(u - \tau_{-y} u) \mathrm{dVol}(y).$$

Pair this with a test-function φ to conclude

$$(u - \rho_\delta * u, \varphi) = \int_{\mathbb{R}^d} \rho_\delta(y) \int_{\mathbb{R}^d} \varphi(x)(u(x) - u(x-y)) \mathrm{dVol}(x) \mathrm{dVol}(y).$$

This produces the estimate

$$|(u - \rho_\delta * u, \varphi)| \le \int_{\mathbb{R}^d} \rho_\delta(y) \|u - \tau_{-y} u\|_{L^r} \mathrm{dVol}(y) \|\varphi\|_{L^s},$$

where s is conjugate to r. Since $\rho_\delta(y)$ vanishes for $|y| > \delta$, we conclude

$$\int_{\mathbb{R}^d} \rho_\delta(y) \|u - \tau_{-y} u\|_{L^r} \mathrm{dVol}(y) \le C |\delta|^{\alpha} \|u\|_{W^{1,p}},$$

hence

$$\|u - \rho_\delta * u\|_{L^q} \le C |\delta|^{\alpha} \|u\|_{W^{1,p}}.$$

As a corollary, if $\mathcal{E}$ is a bounded set in $W_0^{1,p}$, then we conclude

$$\sup_{\mathcal{E}} \|u - \rho_\delta * u\| \leq c(\delta) \text{ where } c(\delta) \to 0 \text{ as } \delta \to 0.$$

□

Lemma 51 (Claim 48)**.** Let $u \in W_0^{1,p}(B(1))$, then the smooth section $u * \rho_\delta$ is supported in $B(1+\delta)$, and

$$\|u * \rho_\delta\|_{C^0} \leq C(\delta) \|u\|_{W^{1,p}(B)}, \tag{1}$$

and

$$\|\tau_h(u * \rho_\delta) - (u * \rho_\delta)\|_{C^0} \leq C(\delta) |h|^\alpha \|u\|_{W^{1,p}(B)}, \tag{2}$$

for some $\alpha > 0$. As a corollary, if $\mathcal{E}$ is a bounded set in $W_0^{1,p}(B(1))$, then $\rho_\delta * \mathcal{E}$ is a bounded equicontinuous family of functions supported on $B(1+\delta)$.

Proof. It suffices to prove the two estimates for test u. We compute

$$|u * \rho_\delta(x)| \leq \int |u(x-y)| |\rho_\delta(y)| \,\mathrm{dVol}(y) \leq \|u\|_{L^p} \|\rho_\delta\|_{L^q} = C(\delta) \|u\|_{W^{1,p}}.$$

For (2), we compute

$$\tau_h(u * \rho_\delta)(x) - u * \rho_\delta(x) = \int (u(x+h-y) - u(x-y))\rho_\delta(y) \mathrm{dVol}(y) \leq \|\tau_h u - u\|_{L^r} \|\rho_\delta\|_{L^s}.$$

Using our previous deduction,

$$\|\tau_h u - u\|_{L^r} \leq C |h|^\alpha \|u\|_{W^{1,p}},$$

for some $\alpha > 0$. Hence

$$\|\tau_h(u * \rho_\delta) - u * \rho_\delta\|_{C^0} \leq C(\delta) |h|^\alpha \|u\|_{W^{1,p}}.$$

This completes the proofs of Claims 47 and 48, and hence completes the proof of the Rellich compactness theorem. □

Morrey embedding. The goal of this section is to prove that $W_0^{k,p}$ functions automatically have a certain degree of Hölder continuity.

Definition 52. The Hölder space $C^{0,\alpha}(\mathbb{R}^d, \mathbb{C}^n)$ is defined by

$$u \in C^{0,\alpha} \iff \|\tau_h u - \tau u\|_{C^0} \leq C |h|^\alpha \text{ for } |h| < 1.$$

Defining

$$[u]_\alpha = \sup_{0<|h|<1} \frac{\|\tau_h u - \tau u\|_{C^0(U)}}{|h|^\alpha},$$

then $\|u\|_{C^{0,\alpha}} = \|u\|_{C^0} + [u]_\alpha$ is a complete metric on $C^{0,\alpha}$.

We define $C^{k,\alpha}$ by recursion: $C^{k,\alpha}$ consists of all C^k functions whose first derivative is $C^{k-1,\alpha}$.

Theorem 53 (Morrey Embedding)**.** If $k > \ell$, $\alpha \in (0,1)$, and $k - d/p \geq \ell + \alpha$, then the inclusion of test functions into $C^{\ell,\alpha}$ extends to a continuous extension $W_0^{k,p}(B) \subset C^{\ell,\alpha}$. Consequently, $W_0^{k,p}(B)$ functions are represented by ℓ times differentiable functions.

Proof. First we claim that it suffices to prove the case $k = 1$, $\ell = 0$. To see why, observe that if we know it for $1 - d/p \geq \alpha$, then if $u \in W^{k,p}$, all the first $k-1$ derivatives of u are in $W^{1,p}$ and hence in $C^{0,\alpha}$. It follows that u is $C^{k-1,\alpha}$. Therefore

$$W^{k,p} \subset C^{k-1,\alpha}.$$

Now if $p < d$, then we can find $p < p^* < \infty$ so that

$$k - d/p = (k-1) - d/p^*.$$

By continuing, we conclude that we can find $\ell < j \leq k$ so that

$$k - d/p = j - d/p^* \geq \ell + \alpha,$$

where $p^* \geq d$. To see why $\ell < j$, observe that if not, then we would conclude $\alpha < 0$. Since $p^* \geq d$, we can find $\beta \in [0,1)$ so that

$$j - d/p^* = (j-1) + \beta \geq \ell + \alpha.$$

To see why $\beta \neq 1$, use $p^* < \infty$. If $\beta \neq 0$, then by applying the composite $W^{k,p} \subset W^{j,p^*} \subset C^{j-1,\beta} \subset C^{\ell,\alpha}$, we conclude the desired result.

If $\beta = 0$, then we need to a bit more careful. We essentially are in the case when $p = d$ and

$$k - d/p \geq \ell + \alpha,$$

so $k - 2 \geq \ell$, since $\alpha > 0$. Since $\alpha < 1$, the inequality is strict:

$$k - d/p > \ell + \alpha.$$

Then we can find $d < p^* < \infty$ so

$$k - d/p > k - 1 - d/p^* = k - 2 + \beta > \ell + \alpha,$$

where $\beta > 0$, and then use the inclusion $W^{k,p} \subset W^{k-1,p^*} \subset W^{k-2,\beta} \subset W^{\ell,\alpha}$.

Now we prove the case when $k = 1$, and $1 - d/p \geq \alpha$.

Proposition 54. Let u be a smooth function (not necessarily a test function). There is a constant C depending only on d (and not the radius r, etc) so that

$$\|u(z) - \bar{u}(x)\|_{C^0(B(x;r))} \leq C(d) \int_{B(x;r)} \frac{|\nabla u(y)|}{|z-y|^{d-1}} \mathrm{dVol}(y)$$

where $\bar{u}(x,r)$ denotes the average of u over $B(x;r)$.

Proof. This is a fairly straightforward computation using the fundamental theorem of calculus. We compute

$$u(z_2) - u(z_1) = \int_0^{|z_2-z_1|} \nabla u(z_1 + t\frac{z_2 - z_1}{|z_2 - z_1|})\frac{z_2 - z_1}{|z_2 - z_1|}\mathrm{d}t.$$

Integrating over z_2, and dividing by the volume of the ball, yields

$$\bar{u}(x) - u(z_1) = \frac{1}{\mathrm{Vol}}\int_{B(x;r)}\int_0^{|z_2-z_1|} \nabla u(z_1 + t\frac{z_2 - z_1}{|z_2 - z_1|})\frac{z_2 - z_1}{|z_2 - z_1|}\mathrm{dVol}(z_2)\mathrm{d}t.$$

Introduce polar coordinates $z_2 = z_1 + \rho\theta$, where ρ ranges from 0 to $\ell(\theta) < 2r$.

$$\bar{u}(x) - u(z_1) = \frac{1}{\mathrm{Vol}}\int_{\mathbb{S}^{d-1}}\int_0^{\ell(\theta)}\int_0^{\rho} \nabla u(z_1 + t\theta)\theta\rho^{d-1}\,\mathrm{d}t\,d\rho\,\mathrm{d}\theta.$$

The region $\{(\rho, t) : t < \rho, \rho < \ell(\theta)\}$ can be reparametrized to give

$$\bar{u}(x) - u(z_1) = \frac{1}{\mathrm{Vol}}\int_{\mathbb{S}^{d-1}}\int_0^{\ell(\theta)} \frac{u(z_1 + t\theta)}{t^{d-1}}t^{d-1}\int_t^{\ell(\theta)} \rho^{d-1}\mathrm{d}\rho\,\mathrm{d}t\,d\theta.$$

Changing back to regular coordinates $y = z_1 + t\theta$, we conclude,

$$|\bar{u}(x) - u(z_1)| \leq \frac{D^d}{\mathrm{Vol}\,d}\int \frac{|\nabla u(y)|}{|y - z_1|^{d-1}}\mathrm{dVol}(y),$$

Where D is the diameter of the ball. Note that D^d/Vol is a constant independent of r. This completes the proof of the proposition. □

Now suppose that u is a test function on $B(1)$. For each $z \in B$, we use the previous proposition to conclude

$$|u(z)| \leq |\bar{u}| + C\int_B \frac{|\nabla u(y)|}{|y - z|^{d-1}}\mathrm{dVol}(y).$$

We can estimate $\bar{u} \leq C\|u\|_{L^p}$. Now apply Hölder's inequality with p, q to conclude

$$|u(z)| \leq C\|u\|_{L^p} + \|\nabla u\|_{L^p}\left[\int \frac{1}{|y - z|^{(d-1)q}}\mathrm{dVol}\right]^{1/q}.$$

We conclude

$$\int \frac{1}{|y - z|^{(d-1)q}}\mathrm{dVol} = C\int_0^1 \rho^{(d-1)(1-q)}d\rho.$$

We compute

$$1 - \frac{d}{p} > \alpha \implies 1 - d + \frac{d}{q} > \alpha \implies (q-1)(1-d) + 1 > q\alpha,$$

which implies $\rho^{(d-1)(1-q)}$ is integrable over $[0, 1]$, and hence:

$$|u(z)| \leq C(d, p)\|u\|_{W^{1,p}}. \tag{1}$$

This proves that $W^{1,p}(B)$ functions satisfy a uniform C^0 bound. We now to establish oscillation estimates to conclude that $W^{1,p}(B)$ functions lie in $C^{0,\alpha}$. As before, let u be a test function supported in B. Pick $x \in B$. For $z \in B_r(x)$, we apply the proposition to conclude

$$|u(z) - \bar{u}| \le C \int_{B_r(x)} \frac{|\nabla u(y)|}{|z-y|^{d-1}} \mathrm{dVol}(y).$$

We conclude, for $r < 1$,

$$|u(z) - \bar{u}| \le C \|\nabla u\|_{L^p} \int_0^r \rho^{(d-1)(1-q)}\, d\rho \le C \|\nabla u\|_{L^p} r^{\alpha},$$

This was independent of x, and hence:

$$|u(z_1) - u(z_2)| \le 2C \|\nabla u\|_{L^p} |z_1 - z_2|^{\alpha}.$$

Thus, combined with (1), we conclude

$$\|u\|_{C^{0,\alpha}} \le C \|u\|_{W^{1,p}}. \tag{2}$$

This proves that $W^{1,p}$ includes into $C^{0,\alpha}$, as desired. □

Exercise 20. If $k+\beta > \ell+\alpha$, use Arzéla Ascoli to conclude $C^{k,\beta}(B) \subset C^{\ell,\alpha}(B)$ is a compact inclusion, where $C^{k,\beta}(B)$ are the $C^{k,\beta}$ functions which vanish outside of B.

Exercise 21 (Morrey compactness)**.** If $k - d/p > \ell + \alpha$, $\alpha \in [0,1)$, then the inclusion

$$W_0^{k,p}(B) \subset C^{\ell,\alpha}$$

is compact.

Corollary 55. Let Y be a compact manifold with boundary, and suppose $k - d/p > \ell$. Then the inclusion of test sections of E into $C^{\ell}(E)$ extends to a compact inclusion of

$$W_0^{k,p}(Y,E) \subset C^{\ell}(E).$$

Proof. As usual, it suffices to prove this locally. However, locally, it is a consequence of the Morrey embedding theorem, and the previous exercise. □

Example 56. Let Y be a Riemann surface. Then Sobolev class functions $W^{1,p}(Y, \mathbb{C}^n)$ are automatically continuous provided $p > 2$. This is a crucial fact when trying to define $W^{1,p}$ Sobolev spaces of maps $Y \to M$, where M is another manifold.

Remark. The two results: (i) Sobolev embedding/Rellich compactness and (ii) Morrey embedding/compactness are fundamental results in the study of partial differential operators on manifolds.

Exercise 22. Let $f \in W^{k,p}(Y,E)$, where we do not assume that $f \in W_0^{k,p}(Y,E)$. Suppose that $k - p/d > \ell$. Prove that f is of class C^{ℓ}.

Hint: First prove that $\rho f \in W_0^{k,p}(Y,E)$, if ρ is compactly supported in the interior of Y. Conclude that f is a sum of smooth sections, and hence is smooth.

Elliptic operators.

Definition 57. Let $E_i \to Y$, $i = 1, 2$, be vector bundles over Y. Given a differential operator $L : E_1 \to E_2$, we can consider the assignment

$$f \in \Gamma(\mathbb{C}) \mapsto [L, f] \in \Gamma \operatorname{Hom}(E_1, E_2).$$

This assignment is actually a first-order linear differential operator $\mathbb{C} \to \operatorname{Hom}(E_1, E_2)$. In fact, there exists a unique tensor $\sigma : T^*M \to \operatorname{Hom}(E_1, E_2)$ so that

$$[L, f] = \sigma(\mathrm{d}f).$$

This tensor σ is called the **symbol** of L. To see why there exists such a tensor σ, we review some results proved in the first section.

Exercise 23. Prove that $[L, -]$ is a first order linear differential operator $f \mapsto [L, f]$. Hint: $[L, \varphi f] - \varphi[L, f] = [L, \varphi]f$.

Exercise 24. Since d is a connection on the trivial bundle $\mathbb{C}$, we already proved that we can express

$$[L, f] = A(\mathrm{d}f) + B(f)$$

for some tensors A and B. Use this to prove that $B = 0$.

We proved that $A : T^*Y \to \operatorname{Hom}(E_1, E_2)$ and $B : \mathbb{C} \to \operatorname{Hom}(E_1, E_2)$ are uniquely determined by L. Conclude that there is unique tensor σ so that $[L, f] = \sigma(\mathrm{d}f)$.

Definition 58. A first order operator $L : E_1 \to E_2$ is called elliptic if its symbol σ maps $(T^*Y)^\times$ into $\mathrm{GL}(E_1, E_2)$. Note that this forces $\dim E_1 = \dim E_2$.

Example 59. Let Y be a manifold with $\dim Y \geq 2$. There are no elliptic first order operators on the trivial bundle $\mathbb{R}$. To see why, note that $\operatorname{Hom}(\mathbb{R}, \mathbb{R})$ is one-dimensional, while T^*Y is greater than 2 dimensional, so the tensor $\sigma : T^*Y \to \operatorname{Hom}(\mathbb{R}, \mathbb{R})$ must have some kernel. This is not true for higher order operators. This partially explains the prevalence of *second* order operators when working with real-valued functions.

Example 60. Let $\dim M > n$. Then there are no first order differential operators from $E_1 \to E_2$, if $\dim E_1 = \dim E_2 = n$.

To see why, consider the space $\operatorname{Hom}(E_1, E_2) \simeq \mathbb{R}^{n \times n}$, and consider a $> n$-dimensional subspace Π of $\mathbb{R}^{n\times n}$. Suppose that Π doesn't touch the singular set. Then we can find matrices $A_0, \cdots, A_n$ so that

$$\sum_i x_i A_i \text{ is non-singular}$$

for every $x_0, \cdots, x_n$, not all 0. In particular, for any vector v,

$$\sum_i x_i A_i v \neq 0 \text{ for all } x_0, \cdots, x_n.$$

But this implies that we have found a linearly independent set with $> n$ elements in E_2, contrary to the requirement that $\dim E_1 = \dim E_2 = n$.

Example 61. As a corollary to the previous example, consider a > 2 dimensional manifold X with almost complex structure J. The equation for holomorphic functions $X \to \mathbb{C}$ is not elliptic, because $\dim X > \dim \mathbb{C} = 2$.

Constant coefficient first-order elliptic operators on $\mathbb{T}^d$. Let $\mathbb{C}^n \to \mathbb{T}^d$ be a trivial line bundle endowed the the standard metric $\langle -, - \rangle$, and suppose that $\mathbb{T}^d$ has the Euclidean metric g and connection ∇ inherited from $\mathbb{R}^d$. The trivial bundle has connection $\nabla = \mathrm{d}$ (let's agree to use the symbols ∇ and d interchangeably for sections of a trivial bundle).

We have shown that any first order operator is of the form

$$Lu = A(\nabla u) + B(u),$$

Observe that

$$L(fu) - fLu = A(\mathrm{d}f \otimes u).$$

Therefore

$$\sigma(\mathrm{d}f) = A(\mathrm{d}f \otimes -).$$

Now recall that

$$\nabla u = \mathrm{d}x_i \otimes \frac{\partial u}{\partial x_i},$$

and hence

$$L(u) = \sum_i \sigma(\mathrm{d}x_i) \frac{\partial u}{\partial x_i} + B(u) = \sum_i A_i \frac{\partial u}{\partial x_i} + Bu.$$

Example 62. Let $\sigma : T^*\mathbb{T}^2 \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ by the symbol

$$\sigma(a\mathrm{d}x + b\mathrm{d}y) = a + bJ,$$

where J is a complex structure. This is clearly elliptic, because

$$(a + bJ)^{-1} = \frac{a - bJ}{a^2 + b^2}.$$

This induces a partial differential operator on $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\overline{\partial}_U(u) = \frac{du}{dx} + J\frac{du}{dy}.$$

Definition 63. We say that a differential operator $L : \mathbb{C}^n \to \mathbb{C}^n$ on the torus $\mathbb{T}^d$ has **constant coefficients**, provided

$$Lu = A(\mathrm{d}u) + B(u)$$

where A, B are constant tensors. The goal of the next section is to prove the interior elliptic estimates for constant coefficient linear elliptic operators on $\mathbb{T}^d$.

Interior elliptic estimates. Here is the general statement we will ultimately prove.

Theorem 64 (Interior elliptic estimates)**.** Let L be a first-order elliptic operator $E_1 \to E_2$, and let Y be a compact manifold with boundary. For all $p \in (1, \infty)$, and all $k = 1, 2, \cdots$, there is constant $C = C(p, k)$ so that

$$\|u\|_{W^{k,p}(Y,E_1)} \le C \left[\|Lu\|_{W^{k-1,p}(Y,E_2)} + \|u\|_{L^p(Y,E_1)} \right].$$

for all $u \in W_0^{k,p}(Y, E_1)$.

Remark. These estimates are called **interior** because they deal with the space $W_0^{k,p}$, which is the closure of $C_c^\infty(\mathrm{int}(Y), E)$ in $W^{k,p}$. We still require Y to be compact.

Exercise 25. Let L be a first order differential operator on Y. The very first thing to observe (which we have not said yet) is that linear differential operators extend to all distributions via duality – this is possible since every operator has an adjoint (which is immediate since $L = A \circ \nabla + B$, where A, ∇, B all have adjoints). Moreover, A, ∇, B all extend to distributions, and the extensions satisfy

$$L = A \circ \nabla + B.$$

By the $W^{\bullet,p}$ boundedness properties of ∇ and tensors, we conclude a constant $C = C(L, k)$ so

$$\|Lu\|_{W^{k-1,p}} \le C \|u\|_{W^{k,p}}.$$

Suppose that L is constant coefficients on $\mathbb{T}^d$. We will prove that the elliptic estimates hold for L using Fourier analysis on $\mathbb{T}^d$. We recall the necessary definitions.

Define $e_\ell = \exp(i \langle \ell, - \rangle) : \mathbb{T}^d \simeq \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}$, for $\ell \in \mathbb{Z}^d$. Then e_ℓ are test functions which are orthonormal with respect to the L^2 inner product.

Theorem 65. If $f \in L^2(\mathbb{T}^d, \mathbb{C}^n)$, then

$$f = \sum_{\ell \in \mathbb{Z}} (e_\ell, f) e_\ell. \tag{$*$}$$

Here the pairing (e_ℓ, f) is vector valued.

Proof. This is standard, by now. By continuity of both sides, it suffices to prove it for smooth f. It is clear that the set of linear combinations of e_k is an unital algebra closed under conjugation, and moreover the algebra seperates points in the sense that for any two points x, y we can find a function φ in the algebra generated by e_k so that $\varphi(x) \neq \varphi(y)$. Then by the Stone-Weierstrass theorem, the span of $\{e_k\}$ is dense in C^0. Since the C^0 norm controls the L^2 norm, and C^0 is dense in L^2, we conclude e_ℓ are dense in L^2, and hence they form an orthonormal basis for the Hilbert space L^2.

To prove that $(*)$ holds, we apply the fact that e_ℓ are a basis to each of the component functions of f. $\square$

Proposition 66. Suppose that a section f has a weak derivative $\nabla f \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. Then

$$(\mathrm{d}x_j \otimes s_i e_\ell, \nabla f) = (\nabla^*(\mathrm{d}x_j \otimes s_i e_\ell), f) = -i\ell_j (e_\ell, f_i),$$

where $s_1, \cdots, s_n$ be an orthonormal frame $\mathbb{C}^n$, and f_i denotes $\langle s_i, f\rangle$.

We remark that the sections $\mathrm{d}x_j \otimes s_i e_\ell$ form an orthonormal basis for $L^2(T^*\mathbb{T}^d)$, since they are certainly orthonormal and contain a generating set of smooth sections

$$\mathrm{d}x_j \otimes u,$$

where u is a smooth $\mathbb{C}^n$ valued function in the closure of their span.

As a consequence, if $f \in W^{1,2}$, then

$$\|\nabla f\|_{L^2}^2 = \sum_\ell |\ell|^2 \left|(e_\ell, f)\right|^2 < \infty.$$

Conversely, if f is a distribution and the Fourier coefficients of ∇f are in ℓ^2, then f is $W^{1,2}$. Note that (e_ℓ, f) is a vector, and so $|(e_\ell, f)|$ is measured using the inner product on $\mathbb{C}^n$.

Proof. Since f is a distribution, $(e_0, f) < \infty$, and hence the sum

$$\sum_\ell |\ell|^2 \left|(e_\ell, f)\right|^2 < \infty$$

implies that $\|f\|_{L^2} < \infty$, and so f is represented by a L^2 function. By assumption that the fourier series for the distribution ∇f converges, we conclude that ∇f is an L^2 section of $\mathrm{Hom}(T\mathbb{T}, \mathbb{C}^n)$, and hence f is in $W^{1,2}$. We also observe that

$$\|f\|_{W^{1,2}}^2 = \sum_\ell (1 + |\ell|^2) \left|(e_\ell, f)\right|^2.$$

$\square$

More generally, the sections

$$\mathrm{d}x_{j_1} \otimes \cdots \otimes \mathrm{d}x_{j_k} \otimes e_\ell \, s_i$$

form an orthonormal basis for $L^2(\mathrm{Hom}(TM^{\otimes k}, \mathbb{C}^n))$. It is easy to check that

$$\nabla^*(\mathrm{d}x_{j_1} \otimes \mathrm{d}x_{j_2} \otimes \cdots \otimes \mathrm{d}x_{j_k} \otimes e_\ell \, s_i) = \mathrm{d}x_{j_2} \otimes \cdots \otimes \mathrm{d}x_{j_k} \ell_{j_1} \, e_\ell \, s_i,$$

(i.e. this can be figured out fairly easily remembering our general formula for the adjoint of a connection). Therefore, if f is a section of $\mathbb{C}^n$, then the coefficients of $\nabla^k f$ are

$$(\mathrm{d}x_{j_1} \otimes \mathrm{d}x_{j_2} \otimes \cdots \otimes \mathrm{d}x_{j_k} \otimes e_\ell \, s_i, \nabla^k f) = \ell_{j_1} \cdots \ell_{j_k} (e_\ell, \langle s_i, f\rangle).$$

Therefore

$$\left\|\nabla^k f\right\|_{L^2}^2 = \sum |\ell|^{2k} \left|(e_\ell, f)\right|^2.$$

Hence for $f \in W^{k,2}$, we have

$$\|f\|_{W^{k,2}}^2 = \sum_\ell (1 + |\ell|^2 + \cdots + |\ell|^{2k}) \left|(e_\ell, f)\right|^2.$$

Now suppose that L is an first-order elliptic operator with constant coefficients,

$$L(u) = A(\nabla u) + B(u).$$

We compute

$$(e_\ell s_i, L(u)) = (e_\ell s_i, A(\nabla u) + B(u)) = (\nabla^* A^*(e_\ell s_i) + B^*(e_\ell s_i), u),$$

where, recall, $s_1, \cdots, s_n$ are the constant orthonormal frame for $\mathbb{C}^n$. Since A is constant, so is A^*, and hence

$$\nabla^* e_\ell A^*(s_i) = -e_\ell\, i\ell \lrcorner A^*(s_i) + e_\ell \nabla^* A^*(s_i)$$

Exercise 26. Prove this relation by exploring the commutator $[\nabla^*, e_\ell]$.

Then, we conclude

$$(e_\ell s_i, L(u)) = (-e_\ell\, i\ell \lrcorner A^*(s_i) + e_\ell \nabla^* A^*(s_i) + e_\ell B(s_i), u).$$

Since $(e_\ell, L(u)) = (e_\ell s_i, L(u)) s_i$, conclude

$$(e_\ell, L(u)) = \sum_i (-e_\ell, \langle i\ell \lrcorner A^*(s_i), u\rangle) s_i + (e_\ell, \langle \nabla^* A^*(s_i) + B(s_i), u\rangle) s_i.$$

Exercise 27. If s is a constant section, then $(e_\ell, \langle s, u\rangle) = \langle s, (e_\ell, u)\rangle$.

We therefore obtain the nice formula

$$(e_\ell, L(u)) = \sum_i \langle i\ell \lrcorner A^*(s_i), \widehat{u}(\ell)\rangle + \langle \nabla^* A^*(s_i) + B(s_i), \widehat{u}(\ell)\rangle\, s_i.$$

We conclude

$$\sum_{i,\ell} |\langle \ell \lrcorner A^*(s_i), \widehat{u}(\ell)\rangle|^2 \le \|L(u)\|_{L^2}^2 + c\,\|u\|_{L^2}^2\,.$$

To conclude the elliptic estimate for $W^{1,2}$, we will try to bound

$$\sum_\ell |\ell|^2\, |\widehat{u}(\ell)|^2 \le C \sum_{i,\ell} |\langle \ell \lrcorner A^*(s_i), \widehat{u}(\ell)\rangle|^2\,.$$

Claim 67. There is a constant c so that for all vectors v

$$c\,|\ell|^2\,|v|^2 \le \sum_i |\langle \ell \lrcorner A^*(s_i), v\rangle|^2\,.$$

This is where ellipticity of L comes in.

Proof. Since L is elliptic, the map $v \mapsto A(\ell \otimes v)$ is an isomorphism for each non-zero ℓ.

The adjoint of A is defined by formally taking the adjoint of the map

$$A : \mathrm{Hom}(TM, \mathbb{C}^n) \to \mathbb{C}^n,$$

so
$$\langle A(\ell \otimes v), s_i \rangle = \langle \ell \otimes v, A^*(s_i) \rangle = \langle v, \ell \lrcorner A^*(s_i) \rangle .$$
Therefore we conclude that
$$v \mapsto \langle v, \ell \lrcorner A^*(s_i) \rangle \, s_i$$
is an injection. Define c by
$$c = \inf_{|v|=|\ell|=1} \sum_i |\langle v, \ell \lrcorner A^*(s_i) \rangle|^2 ,$$
and note that c is the infimum of a continuous function on a compact set. Since c is never vanishing we conclude $c(v) > 0$. By bilinearity, we conclude
$$c \, |\ell|^2 \, |v|^2 \leq \sum_i |\langle v, \ell \lrcorner A^*(s_i) \rangle|^2 .$$
Here it was useful to temporarily allow ℓ to be a non-integer. This completes the proof. $\square$

Corollary 68 (interior estimates for $p = 2$)**.** Let L be a constant coefficient first order elliptic operator on the torus. Then for all $u \in W^{1,2}$, we have
$$\|u\|_{W^{1,2}} \leq C \left[\, \|Lu\|_{L^2} + \|u\|_{L^2} \, \right] .$$

Exercise 28. Prove that the interior elliptic estimates for the trivial complex bundle $\mathbb{C}^n$ imply the interior estimates for the trivial real bundle $\mathbb{R}^n$.

We will see later that the interior elliptic estimates + a general elliptic regularity theorem are enough to construct a parametrix (in general). We can be more explicit in the constant coefficient case and explicitly describe a parametrix. We have shown above that
$$(e_\ell, L(u)) = \sum_i \langle i\ell \lrcorner A^*(s_i), \widehat{u}(\ell) \rangle \, s_i + \langle \nabla^* A^*(s_i) + B(s_i), \widehat{u}(\ell) \rangle \, s_i.$$
Consider
$$v \mapsto \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle \, s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle \, s_i.$$
as a map $\mathbb{C}^n \to \mathbb{C}^n$. We have shown that the norm (squared) of the first term is bounded below by $c \, |\ell|^2 \, |v|^2$. It is clear that the norm of the second term is bounded above by $C \, |v|$, and hence for all but finitely many ℓ, the map
$$v \mapsto \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle \, s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle \, s_i.$$
is invertible. Indeed, by pick ℓ sufficiently large, we may suppose that there is a neighborhood of infinity $\Lambda \subset \mathbb{Z}^d$ so that
$$|\ell| \, |v| < C \left| \sum_i \langle i\ell \lrcorner A^*(s_i), v \rangle \, s_i + \sum_i \langle \nabla^* A^*(s_i) + B(s_i), v \rangle \, s_i \right| ,$$

for all $\ell \in \Lambda$, and all v. Given any collection of vectors b_ℓ, we can find unique a_ℓ, for $\ell \in \Lambda$ so that

$$(e_\ell, L(a_\ell)) = b_\ell,$$

and moreover,

$$(*) \qquad |a_\ell| \leq C \frac{|b_\ell|}{\ell} \text{ for } \ell \in \Lambda.$$

Suppose now that $w = \sum_\ell b_\ell e_\ell$, and w lies in L^2. Let Π be finite dimensional subspace spanned by e_ℓ, $\ell \notin \Lambda$.

Define

$$u := P(w) = \sum_\Lambda a_\ell e_\ell \text{ so that } L(u) - w \text{ lies in } \Pi.$$

This prescription defines a bounded linear map $L^2 \to W^{1,2}$, since we have shown $(*)$.

Note that $LP(w) = w - F(w)$, where F is a finite rank operator (projection onto Π). Similarly, observe that $PL(u - F(u)) = u - F(u)$, so

$$PL(u) = u - G(u),$$

where G is some other finite rank operator. Therefore we have inverted L up to finite rank operators. Observe that

$$\|u\|_{W^{1,2}} \leq \|u - G(u)\|_{W^{1,2}} + \|G(u)\|_{W^{1,2}} \leq C \|Lu\|_{L^2} + \|G(u)\|_{W^{1,2}}.$$

Note that for u restricted to the orthogonal complement of $\ker G$, we know that G is continuous with respect to any topology on it's finite rank target. We therefore have

$$\|G(u)\|_{W^{1,2}} \leq C \|u\|_{L^2},$$

for $u \perp \ker G$, and hence this holds for all u. Thus we conclude

$$\|u\|_{W^{1,2}} \leq C \left[\|Lu\|_{L^2} + \|u\|_{L^2} \right].$$

Thus we see that the existence of a parametrix implies the interior estimates.

Theorem 69. Let L be a first order constant coefficient elliptic operator on $\mathbb{T}^d$. Then for all $u \in W^{1,p}$, and all $p > 0$, we have

$$\|u\|_{W^{1,p}} \leq C \left[\|Lu\|_{L^p} + \|u\|_{L^p} \right].$$

Remark 70. We won't prove this theorem, because I think it is quite hard. Even for the standard Cauchy-Riemann operator $\partial_x + i\partial_y$, I think one needs to use "Calderon-Zygmund" theory.

One method to prove it would be to try to show that our parametrix P is bounded from $L^p \to W^{1,p}$, but this seems hard.

However, assuming this theorem, we can prove the interior elliptic estimates in general without too much additional work.

It is desirable to have the elliptic estimates for $p > 2$, because the Sobolev spaces $W^{1,p}(\Sigma, -)$ are made of continuous functions when $p > 2$ – this is important for developing the non-linear theory. We could try to develop the non-linear theory using the spaces $W^{2,2}(\Sigma, -)$. One meta-reason this is difficult is that "the first estimate is the hardest to establish," and it is generally easier to establish estimates which begin with lower regularity.

Note that the spaces $W^{1,2}(\mathbb{S}^1, -)$ and $W^{1,2}([0,1], -)$ consist of continuous function (when $p = 2$). This implies that non-linear theory for elliptic equations on one-dimensional manifolds can be approached using the $W^{k,2}$ spaces. Non-linear elliptic equations on one-dimensional manifolds can be thought of as the theory of ordinary differential equations.

Exercise 29. Construct a parametrix $P : L^2(\mathbb{T}^2) \to W^{1,2}(\mathbb{T}^2)$ for the constant coefficient elliptic differential operator

$$u \mapsto \overline{\partial} u = u_x + i u_y.$$

Hint: the fourier coefficients of Lu are

$$(e_\ell, Lu) = i(\ell_1 + i\ell_2)\widehat{u}(\ell).$$

Therefore we can invert L whenever $\ell \neq 0$. Conclude a unique parametrix $P : L^2 \to W^{1,2}$ so that

$$u - LPu = \int u \, \mathrm{dVol},$$

and so that

$$u - PLu = \int u \, \mathrm{dVol}.$$

Conclude that the index of $\overline{\partial}$ is 0.

Interior elliptic estimates in the non-constant coefficient case. First we will generalize to the case when we have a differential operator on the torus which does not have constant coefficients. Write

$$L(u) = A(\nabla u) + B(u).$$

Recall that we are trying to prove

$$\|u\|_{W^{1,p}} \leq C \left[\|Lu\|_{L^p} + \|u\|_{L^p} \right].$$

Since

$$\|A(\nabla u)\|_{L^p} \leq \|L(u)\|_{L^p} + C \|u\|_{L^p},$$

it suffices to bound $\|\nabla u\|_{L^p}$ by $\|A(\nabla u)\|_{L^p}$.

Now cover $\mathbb{T}^d$ by N^d overlapping squares U_α of size $(2/N)^d$, and let $x_\alpha \in U_\alpha$ be the center of the square.

Let $L_\alpha(u) = A_\alpha(\nabla u) + B_\alpha(u)$. This defines a constant coefficient differential operator on the entire torus $\mathbb{T}^d$.

Choose a partition of unity ρ_α for U_α, with the property that

$$|\nabla \rho_\alpha| < CN,$$

where C is independent of N.

Suppose we can prove that

$$\|\rho_\alpha u\|_{W^{1,p}} \leq C_\alpha \left[\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p}\right] \tag{$*$}$$

Then we estimate

$$\|u\|_{W^{1,p}} \leq \sum_\alpha \|\rho_\alpha u\|_{W^{1,p}} \leq \sum_\alpha C_\alpha \left[\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p}\right].$$

Now write

$$L\rho_\alpha u = \rho_\alpha Lu + [L, \rho_\alpha]u$$

and thereby conclude

$$\|L\rho_\alpha u\|_{L^p} + \|u\|_{L^p} \leq \|\rho_\alpha Lu\|_{L^p} + \|[L, \rho_\alpha]u\|_{L^p} + \|u\|_{L^p} \leq \left[\|Lu\|_{L^p} + c(N)\|u\|_{L^p}\right].$$

The constant $c(N)$ may be quite large (it involves the derivatives of ρ_α, which are bounded by N). We ultimately conclude

$$\|u\|_{W^{1,p}} \leq C(N)\left[\|Lu\|_{L^p} + \|u\|_{L^p}\right],$$

for some large constant $C(N)$. Therefore it suffices to prove that $(*)$ holds for some finite N.

Now is where we use the constant coefficient operators L_α. Consider the operators $L_x = A_x(\nabla u) + B_x(u)$ as x ranges over $\mathbb{T}^d$. Each one satisfies an estimate of the form

$$\|u\|_{W^{1,p}} \leq C_x(\|L_x u\|_{L^p} + \|u\|_{L^p}).$$

We claim that there is a constant C independent of x so that

$$\|u\|_{W^{1,p}} \leq C(\|L_x u\|_{L^p} + \|u\|_{L^p})$$

holds for all x. This is a compactness argument. To see why, observe that for each $x \in \mathbb{T}^d$, there is a ball B around x and a constant C so that $y \in B$ implies

$$\|u\|_{W^{1,p}} \leq C(\|L_y u\|_{L^p} + \|u\|_{L^p}).$$

Then by covering $\mathbb{T}^d$ by finitely many such balls and taking the maximum of the constants C proves the claim.

We know that

$$\|\rho_\alpha u\|_{W^{1,p}} \leq C(\|L_\alpha \rho_\alpha u\|_{L^p} + \|u\|_{L^p}) \leq C(\|L\rho_\alpha u\|_{L^p} + \|(L_\alpha - L)\rho_\alpha u\|_{L^p} + \|u\|_{L^p}).$$

We estimate

$$\|(L_\alpha - L)\rho_\alpha u\|_{L^p} \leq \frac{1}{N}\|L\|_{C^1}\|\rho_\alpha u\|_{W^{1,p}}.$$

Combining everything, we obtain

$$\|\rho_\alpha u\|_{W^{1,p}} \leq C\left[\|L\rho_\alpha u\| + \frac{\|L\|_{C^1}}{N}\|\rho_\alpha u\|_{W^{1,p}} + C(N)\|u\|_{L^p}\right].$$

Picking N large enough so $C\left\|L\right\|_{C^1}/N < 1/2$, we conclude

$$\left\|\rho_\alpha u\right\|_{W^{1,p}} \leq 2C\left[\left\|L\rho_\alpha u\right\| + C(N)\left\|u\right\|_{L^p}\right],$$

as desired. This completes the proof of the elliptic estimates for first order operators on $\mathbb{T}^d$.

We can now complete the proof of the interior elliptic estimates:

Theorem 71 (interior elliptic estimates for $W^{1,p}$)**.** Let Y be a compact manifold with boundary, and let L be a first order elliptic operator $E_1 \to E_2$. There is a constant $C(p)$ so that for all $u \in W_0^{1,p}(Y, E_1)$, we have

$$\left\|u\right\|_{W^{1,p}} \leq C(\left\|Lu\right\|_{L^p} + \left\|u\right\|_{L^p}).$$

Proof. First we show that it holds locally on Y. Cover Y by finitely many compact balls (or half balls) with boundary U_α, each equipped with an embedding $\varphi_\alpha : U_\alpha \to \mathbb{T}^d$ trivializing E_1, E_2.

Using the connection and metric U_α inherits from $\mathbb{T}^d$ and the trivial bundle on $\mathbb{T}^d$, we can write

$$L = A_\alpha(\nabla u_\alpha) + B_\alpha(u_\alpha) \text{ for } u_\alpha \in W_0^{1,p}(U_\alpha).$$

The tensors A_α, B_α are defined on the codimension 0 image of U_α in $\mathbb{T}^d$. Taking a tubular neighborhood of $U_\alpha \subset \mathbb{T}^d$, we can extend A_α, B_α smoothly to global sections on $\mathbb{T}^d$ of the appropriate bundles. Then we conclude for $u \in W_0^{1,p}(U_\alpha) \subset W_0^{1,p}(\mathbb{T}^d)$

$$\left\|u\right\|_{W^{1,p}} \leq C(\left\|Lu\right\|_{L^p} + \left\|u\right\|_{L^p}), \tag{$*$}$$

this implies the same result when we use the norms that $W_0^{1,p}(B_\alpha)$ inherits from $W_0^{1,p}(Y)$, since the Sobolev norms defined on a compact manifold do not depend on the metric/connection data used (up to norm equivalence).

Now let ρ_α be a partition of unity for the balls U_α. We estimate

$$\left\|u\right\|_{W^{1,p}} \leq \sum_\alpha \left\|\rho_\alpha u\right\|_{W^{1,p}} \leq C\sum_\alpha \left\|\rho_\alpha Lu\right\|_p + \left\|[L,\rho_\alpha]u\right\|_{L^p} + \left\|\rho_\alpha u\right\|_{L^p},$$

for some large C (the maximum of the constants we were guaranteed in $(*)$). We can estimate the three terms appearing in the sum

$$\sum_\alpha \left\|\rho_\alpha Lu\right\|_{L^p} \leq (\text{number of balls})\left\|Lu\right\|_{L^p}$$

$$\sum_\alpha \left\|[L,\rho_\alpha]u\right\|_{L^p} \leq \left[\sum_\alpha \left\|[L,\rho_\alpha]\right\|_{C^0}\right]\left\|u\right\|_{L^p}$$

$$\sum_\alpha \left\|\rho_\alpha u\right\|_{L^p} \leq (\text{number of balls})\left\|u\right\|_{L^p},$$

we conclude for $u \in W_0^{1,p}$ that

$$\left\|u\right\|_{W^{1,p}} \leq C(\left\|Lu\right\|_{L^p} + \left\|u\right\|_{L^p}).$$

This completes the proof. □

Exercise 30. Prove that if $u \in W_0^{1,p}(Y)$ then $\rho_\alpha u \in W_0^{1,p}(B_\alpha)$. Hint: the boundary case requires some proof since, since ρ_α is potentially non-vanishing on the boundary face $B_\alpha \cap \partial Y$.

Elliptic bootstrapping.

Theorem 72. Let $u \in L^p(Y)$ be supported away from ∂Y, and suppose that $Lu \in L^p(Y)$ (where Lu is defined by duality using the adjoint L^*). Then $u \in W_0^{1,p}(Y)$.

Remark. Here u being "supported away from ∂Y" is equivalent to the existence of a bump function ρ supported in $\operatorname{int}(Y)$ so that $\rho u = u$.

Remark. The idea for the proof (in the case when $Y = B(1)$) will be to consider the convolutions $\rho_\delta * u$ (which are well defined elements of $W_0^{1,p}(B(1))$ for δ sufficiently small because of our assumption on the support of u). If we can show that $\rho_\delta * u$ is uniformly bounded in $W_0^{1,p}$, then reflexivity of $W_0^{1,p}$, and the fact that $\rho_\delta * u \to u$ in L^p, we can deduce that u is in fact in $W_0^{1,p}$.

Definition 73. Let $T_\delta u = \rho_\delta * u$, i.e.

$$T_\delta(u) = \int_{\mathbb{R}^d} u(y)(\tau^{-x}\rho_\delta)(y)\,\mathrm{d}y,$$

where $\rho : B(1) \to [0, \infty)$ is a radially symmetric bump function with integral 1. This formula uses the antipodal symmetry of ρ_δ.

This defines a continuous linear map $L^p(B(1)) \to C^\infty(B(1+\delta))$, and moreover

$$\rho_\delta * u \to u \text{ in } L^p.$$

The key part of the proof is the following "commutator estimate"

Lemma 74. Let L be a first order differential operator on $\mathbb{R}^d$. There is a constant C independent of δ so that for all $u \in L^p(B(1))$ we have

$$\|[L, T_\delta]u\|_{L^p} \le C\,\|u\|_{L^p}.$$

Proof (of lemma)**.** We will prove the case when $L = A(\nabla u)$. The reader can prove the case $L = B(u)$, and thereby obtain it for all first order differential operators. Let g be a test section in L^q.

We compute

$$\langle A(\nabla T_\delta(u)), g\rangle = \langle T_\delta(\nabla u), A^* g\rangle = \int \left\langle \nabla u(y), \tau^{-x}\rho_\delta(y) A_x^* g(x)\right\rangle \mathrm{d}y.$$

Using the adjoint of ∇ (with respect to the y-variable, we consider x to be fixed for the moment), we conclude

$$\langle LT_\delta u, g\rangle = \langle A(\nabla T_\delta(u)), g\rangle = \int \left\langle u(y), \tau^{-x}[\nabla^*, \rho_\delta] A_x^* g(x)\right\rangle \mathrm{d}y,$$

where we use the fact that ∇ commutes with τ^{-x} and that $\nabla^* A_x^* g(x) = 0$, since we consider x fixed.

Exercise 31. Prove that the symbol of $\nabla^* : \mathrm{Hom}(TM, \mathbb{R}^n) \to \mathbb{R}^n$ is

$$[\nabla^*, f] = \mathrm{grad}(f) \lrcorner (-).$$

This is a tensor, and hence ∇^* is a first order operator.

Exercise 32. Prove that if w is a constant section of $\mathrm{Hom}(TM, \mathbb{R}^n)$, then $\nabla^* w = 0$. Hint:

$$\int \langle \nabla^* w, u \rangle \,\mathrm{dVol} = \int \langle w, \nabla u \rangle \,\mathrm{dVol}$$

Write

$$w = \sum \mathrm{d}x_i \otimes v_i \text{ and } \nabla u = \sum \mathrm{d}x_i \otimes \frac{\partial u}{\partial x_i}.$$

Conclude that

$$\int \langle w, \nabla u \rangle \,\mathrm{dVol} = \sum_i \int \left\langle v_i, \frac{\partial u}{\partial x_i} \right\rangle \mathrm{dVol} = 0,$$

since v_i are constant (use compatibility of the connection with $\langle -, - \rangle$).

Continuing with the proof of the claim, we now explore the term $T_\delta L u$.

$$\begin{aligned}\langle T_\delta L u, g\rangle &= \int_{\mathbb{R}^d} \left\langle A_y \nabla_y u, \tau^{-x} \rho_\delta(y) g(x) \right\rangle \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left\langle u, \tau^{-x} [\nabla^*, \rho_\delta] A_y^* g(x) \right\rangle \mathrm{d}y + \int_{\mathbb{R}^d} \left\langle u,\, \tau^{-x} \rho_\delta \cdot \nabla^* (A_y^* \cdot g(x)) \right\rangle \mathrm{d}y.\end{aligned}$$

Therefore

$$\langle T_\delta L u - L T_\delta u, g\rangle = \int_{\mathbb{R}^d} \left\langle u, \tau^{-x} [\nabla^*, \rho_\delta](A_y^* - A_x^*) g(x) \right\rangle \mathrm{d}y + \int_{\mathbb{R}^d} \left\langle u,\, \tau^{-x} \rho_\delta \cdot \nabla^*(A_y^* \cdot g(x)) \right\rangle \mathrm{d}y.$$

Taking absolute values produces

$$(*) \qquad |\langle T_\delta L u - L T_\delta u, g\rangle| \leq \left[\|\rho_\delta\|_{C^1} \cdot \mathrm{osc}(A^*; \delta)\, |g(x)| + \|A^*\|_{C^1} |g(x)| \right] \int_{x + B(\delta)} |u(y)| \,\mathrm{d}y.$$

where

$$\mathrm{osc}(A^*; \delta) = \sup_{|x-y|<\delta} |A^*(y) - A^*(x)|\,.$$

Integrating $(*)$ over $B(1)$ conclude

$$|(T_\delta L u - L T_\delta u, g)| \leq \left[\|\rho_\delta\|_{C^1} \mathrm{osc}(A^*; \delta) + \|A^*\|_{C^1}\right] \left\| \int_{x+B(\delta)} u(y)\,\mathrm{d}y \right\|_{L^p} \|g\|_{L^q}\,,$$

and hence, conclude

$$\|[T_\delta, L] u\|_{L^p} \leq C\left[\|\rho_\delta\|_{C^1} \mathrm{osc}(A^*; \delta) + \|A^*\|_{C^1}\right] \|u\|_{L^p}\,,$$

where we have used the (easy) fact that

$$\left\| \int_{x+B(\delta)} u(y)\,\mathrm{d}y \right\|_{L^p} \leq \|u\|_{L^p}\,.$$

This can be proved by pairing with a test function. Now observe that $\operatorname{osc}(A^*;\delta) < \|A^*\|_{C^1}\delta$, and $\|\rho_\delta\|_{C^1} = \delta^{-1}\|\rho_1\|_{C^1}$, so

$$\|[T_\delta, L]u\|_{L^p} \leq C(1 + \|\rho_1\|_{C^1})\|A^*\|_{C^1}\|u\|_{L^p}.$$

This proves the claim, as we can take C to be $C(1 + \|\rho_1\|_{C^1})\|A^*\|_{C^1}$ which does not depend on δ. □

Now we can prove the theorem.

Proof. We are given $u, Lu \in L^p$ and we know that u is supported away from the boundary, and we want to show that $u \in W_0^{1,p}$.

First suppose that u is supported in $B(r) \subset B(1)$ for some $r < 1$. For $\delta < r$, $\rho_\delta * u$ is supported in $B(1)$. We compute

$$L(\rho_\delta * u) = \rho_\delta * (Lu) + [L, T_\delta]u,$$

and hence

$$(*) \qquad \|L(\rho_\delta * u)\|_{L^p} \leq \|Lu\|_{L^p} + C\|u\|_{L^p},$$

where the constant C is given by our "commutator estimate."

By the interior elliptic estimates applied to $\rho_\delta * u \in W_0^{1,p}$, together with $(*)$ we conclude

$$\|\rho_\delta * u\|_{W^{1,p}} \leq C(\|Lu\|_{L^p} + \|u\|_{L^p}).$$

Therefore $\rho_\delta * u$ is uniformly bounded in $W_0^{1,p}$. It follows (by reflexivity of $W_0^{1,p}$) that $\rho_\delta * u$ converges weakly in $W_0^{1,p}$ (after taking a subsequence). Since we know that $\rho_\delta * u$ converges to u in L^p, we conclude that u must be equal to the weak limit of $\rho_\delta * u$ in $W_0^{1,p}$, and hence u is in $W_0^{1,p}$, as desired.

More generally, suppose that u is defined on an arbitrary compact manifold with boundary Y. Since the support of u is separated from the boundary, we can find balls

$$B_1(1), \cdots, B_n(1) \subset \operatorname{int}(Y)$$

so that the balls $B_1(r), \cdots, B_n(r)$ cover the support of u. Picking a partition of unity $\rho_1, \cdots, \rho_n$ so ρ_i is supported in $B_i(r)$, conclude that

$$L\rho_i u = \rho_i Lu + [L, \rho_i]u \in L^p \implies \rho_i u \in W_0^{1,p} \implies u \in W_0^{1,p}.$$

This proves the theorem. □

Differentiating the equation. The goal of this section will be to prove the following result

Theorem 75 (interior bootstrapping)**.** Let Y be a compact manifold with boundary, and $L : E_1 \to E_2$ a first order elliptic operator. Then for sections u supported away from ∂Y we have

$$u \in L^p(Y) \text{ and } Lu \in W_0^{k,p}(Y) \implies u \in W_0^{k+1,p}(Y).$$

The key idea needed to prove this is described in the following exercise:

Exercise 33. If L is elliptic, then there exists an elliptic first order operator

$$L' : \mathrm{Hom}(TM, E_1) \to \mathrm{Hom}(TM, E_2)$$

so that

$$\nabla L(u) = L'(\nabla u) + \text{first order operator}.$$

Hint: writing $L(u) = A(\nabla u) + B(u)$, observe that

$$\nabla L(u) = A(\nabla\nabla u) + B(\nabla u) + \nabla A(\nabla u) + \nabla B(u),$$

therefore we can define

$$L'(\Phi) = A(\nabla\Phi) + B(\Phi).$$

We claim that L' is still elliptic. To see why, compute

$$[L', f](\Phi) = A(\mathrm{d}f \otimes \Phi),$$

and since $A(\xi \otimes -) : E_1 \to E_2$ is an isomorphism if $\xi \neq 0$, we conclude the induced map on $\mathrm{Hom}(TM, E_1) \to \mathrm{Hom}(TM, E_2)$ is still an isomorphism.

We adopt the convention that we have chosen extensions of L to elliptic operators

$$\mathrm{Hom}(TM^{\otimes n}, E_1) \to \mathrm{Hom}(TM^{\otimes n} E_2)$$

so that $[L, \nabla]$ is a first order operator.

Proof (interior bootstrapping)**.** We will prove that

$$(*) \qquad u \text{ supported away from } \partial Y \text{ and } u, Lu \in W_0^{k,p} \implies u \in W_0^{k+1,p}.$$

by induction. We already know it for $k = 0$. Now suppose that $u, Lu \in W_0^{k,p}$. Then $\nabla u \in W_0^{k-1,p}$, and

$$L'\nabla u = \nabla L u + [L', \nabla]u \in W_0^{k-1,p}.$$

Since L' is elliptic, and we suppose that we know the $k-1$ case of $(*)$ for all elliptic operators (in particular, for L'), we conclude that

$$\nabla u \in W_0^{k,p},$$

and hence $u \in W_0^{k+1,p}$, as desired. This proves $(*)$ for all k. It is clear that knowing $(*)$ for all k implies

$$u \in L^p(Y) \text{ and } Lu \in W_0^{k,p}(Y) \implies u \in W_0^{k+1,p}(Y).$$

for sections u supported away from ∂Y. This completes the proof. □

Using the same "differentiating the equation" trick, we can upgrade the interior elliptic estimates:

Theorem 76 (interior elliptic estimates, for all k)**.** Let L be an elliptic differential operator. There is a constant C_k (depending on L and k) so that for all $u \in W_0^{k+1,p}(Y)$ we have

$$\|u\|_{W^{k+1,p}} \leq C_k(\|Lu\|_{W^{k,p}} + \|u\|_{L^p}). \tag{$*$}$$

Proof. The case when $k = 0$ is already known. Suppose we have found constants C_i so that $(*)$ holds for $i < k$.

Let $u \in W_0^{k+1,p}(Y)$. Then $\nabla u \in W_0^{k,p}(Y)$, and hence we can apply the $k-1$ elliptic estimate to ∇u

$$\|\nabla u\|_{W^{k,p}} \leq C_{k-1}(\|L'\nabla u\|_{W^{k-1,p}} + \|\nabla u\|_{L^p}).$$

It follows that

$$\begin{aligned}\|\nabla u\|_{W^{k,p}} &\leq C_{k-1}(\|\nabla(Lu)\|_{W^{k-1,p}} + \|[L,\nabla]u\|_{W^{k-1,p}} + \|\nabla u\|_{L^p})\\ &\leq C'(\|Lu\|_{W^{k,p}} + \|u\|_{W^{k,p}} + \|u\|_{W^{1,p}})\\ &\leq C_k(\|Lu\|_{W^{k,p}} + \|u\|_{L^p}),\end{aligned}$$

where we have used the fact that $[L,\nabla]$ is first order operator, and then used the $0, k-1$ cases of the elliptic estimates. It is clear that constant C_k does not depend on the u we started with. This completes the proof. $\square$

Local elliptic regularity.

Definition 77. Let Y be a potentially non-compact manifold. We define

$$W^{k,p}_{\text{loc}}(Y) = \lim W^{k,p}(Z) \text{ where } Z \text{ compactly supported in } \operatorname{int}(Y). \tag{$*$}$$

Note that the set $W^{k,p}_{\text{loc}}(Y)$ can be thought of as a subset of distributions $C^\infty_c(Y, E_1) \to \mathbb{R}$ (or $\mathbb{C}$). The definition in $(*)$ also tells us the topology of $W^{k,p}_{\text{loc}}(Y)$.

Theorem 78. Let L be an elliptic first order operator on Y (potentially non-compact). Then

$$u \in L^p_{\text{loc}}(Y) \text{ and } Lu \in W^{k,p}_{\text{loc}}(Y) \iff u \in W^{k+1,p}_{\text{loc}}(Y) \tag{1}$$

and the map

$$u \in W^{k+1,p}_{\text{loc}}(Y) \to u \times Lu \in L^p_{\text{loc}}(Y) \times W^{k,p}_{\text{loc}}(Y), \tag{2}$$

is a closed embedding.

Remark. The fact that (2) defines a closed embedding is closely related to the interior elliptic estimates.

Proof. The proof of (1) is quite easy, since we already know a similar statement in the interior case. Here is the argument: if $u \in L^p_{\text{loc}}$ and $Lu \in W^{k,p}_{\text{loc}}$, then for any test function ρ we have

$$\rho u \in L^p \text{ and } L\rho u = \rho L u + [L,\rho]u \in L^p, \tag{$*$}$$

(since $[L,\rho]$ is a tensor with compact support, $[L,\rho]u \in L^p$). By interior bootstrapping we conclude $\rho u \in W^{1,p}_0$. By taking a locally finite partition of unity, we conclude $u \in W^{1,p}_{\text{loc}}$.

Now we repeat the argument, but now that we know that $u \in W^{1,p}_{\mathrm{loc}}$, we can conclude that $L\rho u \in W^{1,p}_0$, and hence conclude $\rho u \in W^{2,p}_0$, etc. We conclude that (1) holds.

To prove (2), we will use the interior elliptic estimates. It is easy to show that $W^{k,p}_{\mathrm{loc}}$ is a metrizable space, and hence to prove that the map in (2) is a closed embedding, it suffices to consider sequences of elements. Let u_n be a sequence in $W^{k+1,p}_{\mathrm{loc}}$, and suppose that $u_n \to u_\infty$ in L^p_{loc} and Lu_n converges in $W^{k,p}_{\mathrm{loc}}$. Let ρ be some bump function, say $\rho = 1$ on the compact set Z. By the interior elliptic estimates we conclude that ρu_n is Cauchy in $W^{1,p}$, and hence u_n converges in $W^{1,p}_{\mathrm{loc}}$. By repeating the argument, conclude that ρu_n is Cauchy in $W^{2,p}$, etc. Ultimately deduce that u_n converges in $W^{k+1,p}_{\mathrm{loc}}$. This completes the proof. □

Definition 79. We define the topology on the space C^∞ by a limit

$$C^\infty_{\mathrm{loc}}(Y) = \lim_{k\to\infty} C^k_{\mathrm{loc}}(Y),$$

where loc indicates that we are not thinking of C^k as a normed space, but rather as the limit of the normed spaces $C^k(Z)$ as Z ranges over compact subdomains of Y.

An easy consequence of the Morrey embedding theorem, we conclude that

$$(*) \qquad C^\infty_{\mathrm{loc}}(Y) \simeq \lim_k W^{k,p}_{\mathrm{loc}}(Y).$$

Theorem 80. Let Y be a manifold, and let L be a first order elliptic operator on Y. Then the map

$$u \in C^\infty_{\mathrm{loc}}(Y) \mapsto u \times Lu \in L^p_{\mathrm{loc}} \times C^\infty_{\mathrm{loc}}(Y)$$

is a closed embedding.

Proof. Let $u_n \in C^\infty_{\mathrm{loc}}(Y)$ be a sequence so that u_n converges in L^p_{loc} and Lu_n converges in C^∞_{loc}.

Clearly Lu_n converges in $W^{k,p}_{\mathrm{loc}}$, for every k, and hence u_n converges in $W^{k,p}_{\mathrm{loc}}$ for every k by Theorem 78. By $(*)$, we conclude that u_n converges in C^∞_{loc}, as desired. □

Example 81. Let u_n be a sequence of holomorphic functions in $D(1)$ so that u_n converges to u_∞ in L^p. Then u_∞ is C^∞ smooth, holomorphic, and $u_n \to u_\infty$ converges in C^∞_{loc}.

To see that u_∞ is C^∞ smooth, use the fact that $u \in C^\infty_{\mathrm{loc}} \mapsto u \times \overline{\partial} u \in L^p \times C^\infty_{\mathrm{loc}}$ is a closed embedding. Therefore the sequence u_n of holomorphic functions converges in C^∞_{loc}. The limit must obviously equal u, and hence $u \in C^\infty_{\mathrm{loc}}$ and $u_n \to u$ in C^∞_{loc}. It is now clear that u is holomorphic.

Fredholm theory of elliptic operators on closed manifolds. In this section we will work with $p = 2$. It will simplify things without any loss of generality concerning the final statement we will prove.

Consider an elliptic operator $L : E_1 \to E_2$ on a closed manifold and its adjoint

$$L^* : E_2 \to E_1,$$

and consider the spaces $W^{k,2}(E_i)$, $k = 0, 1, 2, \cdots$.

Let's agree to write L_k for the induced map $W^{k+1,2} \to W^{k,2}$.

Lemma 82. The map L_k has closed range and the kernel of L_k is finite dimensional.

Proof. Since

$$\|u\|_{W^{k+1,2}} \leq C_k(\|Lu\|_{W^k} + \|u\|_{L^p}),$$

we can easily deduce that L_k has finite kernel and closed range. We leave the proof to the reader, with the hint that to prove first that L has finite dimensional kernel, and then to consider L restricted to a complement of the kernel. □

Lemma 83. The cokernel of L_k is canonically identified with $\mathrm{Hom}(\ker L_0^*, \mathbb{R})$ via the L^2 inner product, and so the cokernel of L_k is also finite dimensional. In other words:

$$\operatorname{im} L_k = (\ker L_0^*)^\perp \cap W^{k,2}.$$

Proof. It is clear that if $w = Lu$ for $u \in W^{k+1,2}$, then $w \in (\ker L_0^*)^\perp$. The reverse inclusion is a bit deeper. Suppose that $w \in (\ker L_0^*)^\perp \cap W^{k,2}$.

First we will prove that $w \in \operatorname{im} L_0$ (i.e. there is a $W^{1,2}$ section u so $L_0 u = w$).

Let's suppose that $w \notin \operatorname{im} L_0$ in search of a contradiction. Since $\operatorname{im} L_0$ has closed range in L^2, we can find a section v so that $(w, v) \neq 0$, but $(\operatorname{im} L_0, v) = 0$. It follows that $L^* v = 0$ weakly and hence by elliptic regularity, v is of class $W^{1,2}$ and $v \in \ker L_0^*$, contradicting the fact that $w \in (\ker L_0^*)^\perp$. Therefore $w = Lu$ for $u \in W^{1,2}$. By elliptic regularity, since $u \in L^p$ and $Lu \in W^{k,2}$, we conclude $u \in W^{k+1,2}$, and hence $w \in \operatorname{im} L_k$, as desired. □

Proposition 84. For all k,

$$\ker L_k = \{u \in C^\infty : Lu = 0\} = \left\{u \in L^2 : Lu = 0 \text{ weakly}\right\}.$$

Proof. It is clear that

$$\{u \in C^\infty : Lu = 0\} \subset \ker L_k \subset \left\{u \in L^2 : Lu = 0 \text{ weakly}\right\}.$$

Therefore it suffices to show that

$$\left\{u \in L^2 : Lu = 0 \text{ weakly}\right\} \subset \{u \in C^\infty : Lu = 0\}.$$

This is a straightforward consequence of elliptic regularity:

$$u \in L^2_{\mathrm{loc}} \text{ and } Lu \in C^\infty_{\mathrm{loc}} \implies u \in C^\infty_{\mathrm{loc}}.$$

This completes the proof. □

Corollary 85. Let L be a first order elliptic operator on a closed manifold Y. There are canonical L^2-orthogonal decompositions

$$W^{k,2}(E_2) = \operatorname{im} L_k \oplus \ker L^* \text{ and } W^{k,2}(E_1) = \operatorname{im} L_k^* \oplus \ker L,$$

including the case when $k = \infty$ (where $W^{\infty,2} = C^\infty$).

Proof. We already know that $\ker L^* \subset W^{k,2}(E_2)$ and $\ker L \subset W^{k,2}(E_1)$ for all k. To prove that

$$W^{k,2}(E_2) = \operatorname{im} L_k \oplus \ker L^*,$$

it suffices to observe that if $u \in W^{k,2}(E_2)$ is orthogonal to $\ker L^*$, then $u \in \operatorname{im} L_k$.

Now when $k = \infty$, we once again notice that it suffices to prove that if $u \in C^\infty(E_2)$, and $u \perp \ker L^*$, then $u \in \operatorname{im} L$. This is easy, since we know that $u = Lw$ for some $w \in W^{1,2}$, and hence by regularity w is actually in $C^\infty(E_2)$, as desired. This completes the proof. □

We may care about similar decompositions of $W^{k,p}$ in the case when $p \neq 2$. A very similar argument proves the following:

Theorem 86. Consider the split injections

$$\ker L \to W^{k,p}(E_1) \text{ and } \ker L^* \to W^{k,p}(E_2),$$

where the splittings are given by the L^2 orthogonal projections, i.e.

$$u \mapsto (u, v_1)v_1 + \cdots + (u, v_n)v_n.$$

These splittings are obviously continuous in the $W^{k,p}$ topology. With respect to these splittings, we have

$$W^{k,p}(E_2) = \operatorname{im} L_k \oplus \ker L^* \text{ and } W^{k,p}(E_1) = \operatorname{im} L_k^* \oplus \ker L.$$

Example 87. Consider $L = \overline{\partial}$ on $\mathbb{T}^2$. As we have seen, this is an elliptic operator, and $L^* = -\partial$. It is easy to show using Fourier coefficients that

$$\ker L = \ker L^* = \mathbb{C} \text{ (the constants)}.$$

As corollary of the preceding decomposition theorem, we conclude that for $u \in W^{k,p}$

$$\text{if } \int u \,\mathrm{dVol} = 0 \text{ then } u = \overline{\partial} w \text{ for unique } W^{k+1,p} \text{ satisfying } \int w \,\mathrm{dVol} = 0.$$

As a consequence, we can prove the following local surjectivity result: the map of sheaves

$$\overline{\partial} : W^{k+1,p}_{\mathrm{loc}} \to W^{k,p}_{\mathrm{loc}}.$$

is locally surjective (on any Riemann surface).

Proof. It suffices to prove that for $w \in W^{k,p}_{\mathrm{loc}}(D(1))$, we can find $u \in W^{k+1,p}_{\mathrm{loc}}(D(r))$ so $\overline{\partial} u = w$ on $D(r)$. Let ρ be a bump function equal to 1 on a neighborhood of $D(r)$ and which is supported in $D(1)$. Then $\rho u \in W^{k,p}_0(D(1))$. By embedding $D(1)$ in the torus, it follows that there is a $W^{k+1,p}(\mathbb{T}^2)$ section v so that

$$\overline{\partial} v = \rho u + c$$

for some constant c. Let $\varphi(z) = cz$ on a neighborhood of $D(r)$. Then

$$\overline{\partial}(v - \rho\varphi) = u \text{ on } D(r),$$

as desired. □

Example 88 (the parametrix)**.** Consider the splitting

$$W^{k,p}(E_2) = \operatorname{im} L_k \oplus \ker L^*.$$

Let's define p to be the projection onto $\ker L^*$, and q to be the projection onto $\ker L$, then

$$P_k = (1-q) \circ L_k^{-1} \circ (1-p)$$

is well-defined: i.e. there is a unique element in $u \in W^{k+1,p}(E_2)$ so that $u \perp \ker L$ and $L_k u = (1-p)w$ for each $w \in W^{k,p}(E_2)$. The open mapping theorem guarantees that P_k is continuous, i.e. L_k^{-1} is continuous on $(\ker L^*)^\perp \to (\ker L)^\perp$.

It is clear, by uniqueness, that $P_k = P_{k+1}$ on $W^{k+1,p} \subset W^{k,p}$. Therefore we obtain a well-defined continuous map $P : C^\infty \to C^\infty$ which equals P_k on $W^{k,p}$. Note that

$$\begin{aligned} 1 - PL &= \text{ projection onto } \ker L \\ 1 - LP &= \text{ projection onto } \ker L^*. \end{aligned}$$

Homotopy invariance of the Fredholm index. Let $L : E_1 \to E_2$ be an elliptic operator on a closed manifold Y. Define the index of L to be $\dim \ker L - \dim \ker L^*$.

The goal of this section will be to show that the index is constant on a continuously varying family of elliptic operators.

More generally, if X, Y are any Banach spaces, consider the space $\mathcal{F}(X,Y)$ of operators $L : X \to Y$, equipped with the distance induced by the operator norm

$$d(L, L') = \|L - L'\|$$

and which satisfy the Fredholm condition:

L has finite dimensional kernel, closed image, and finite dimensional cokernel.

Theorem 89. The index function $\mathcal{F}(X,Y) \to \mathbb{Z}$ is continuous.

There is a slight trick to the proof. It is useful to introduce one concept related to Fredholm maps.

Definition 90. Let $L : X \to Y$ be a bounded linear operator. A **regularization** of L is the data of two finite dimensional vector spaces and maps a, b, c so that

$$(*) \qquad \begin{bmatrix} L & a \\ b & c \end{bmatrix} : X \oplus V \to Y \oplus W$$

is an isomorphism.

Proposition 91. A map $L : X \to Y$ is Fredholm if and only if it admits a regularization.

Proof. Suppose L is Fredholm. Take $V = \operatorname{coker} L$, $W = \ker L$. Choosing a splitting j of $\pi : Y \to \operatorname{coker} L$ and a splitting p of $i : \ker L \to X$ gives a map

$$\begin{bmatrix} L & j \\ p & 0 \end{bmatrix} : X \oplus \operatorname{coker} L \to Y \oplus \ker L$$

which the reader can check is bijective, and hence is an isomorphism.

Conversely, suppose that L admits a regularization of the form $(*)$. Since $(*)$ is an isomorphism, the map $x \in X \mapsto Lx \times bx \in Y \times W$ is a closed embedding, and hence we conclude a bound

$$\|x\|_X \le C(\|Lx\|_Y + \|bx\|_W). \tag{1}$$

Since $x \mapsto bx$ is a finite rank operator (in particular, is compact), (1) implies that L has finite dimensional kernel and closed image.

We will prove that $\operatorname{coker} L$ is finite dimensional arguing by contradiction. Suppose that $\operatorname{coker} L$ is infinite dimensional. Then we can find unit vectors $y_1, y_2, y_3, \cdots$ in Y so that

$$\|y_i - \operatorname{im} L - \operatorname{span}(y_1, \cdots, y_{i-1})\| > 1/2.$$

Since $(*)$ is an isomorphism, we can find vectors x_i, v_i so $\|x_i\| + \|v_i\|$ is bounded so that

$$Lx_i + a(v_i) = y_i \implies \|a(v_i) - a(v_j)\| > 1/2 \text{ for all } i \neq j,$$

in particular

$$\|a\|\,\|v_i - v_j\| > 1/2 \text{ for all } i \neq j. \tag{2}$$

Since any bounded ball in V is precompact, a subsequence of $v_1, v_2, \cdots$ must converge, contradicting (2). Therefore $\operatorname{coker} L$ is finite dimensional. □

Proposition 92. Suppose that $L : X \to Y$ admits a regularization a, b, c, V, W. There is $\epsilon > 0$ so that any map L' within ϵ of L is also regularized by a, b, c, V, W.

Proof. This is easy, since we know that being an isomorphism is open condition in the space of maps, so that there is ϵ so that

$$\left\| \Lambda - \begin{bmatrix} L & a \\ b & c \end{bmatrix} \right\| < \epsilon \implies \Lambda \text{ is an isomorphism.}$$

In particular, if $\|L' - L\| < \epsilon$, then

$$\left\| \begin{bmatrix} L' & a \\ b & c \end{bmatrix} - \begin{bmatrix} L & a \\ b & c \end{bmatrix} \right\| < \epsilon \implies \begin{bmatrix} L' & a \\ b & c \end{bmatrix} \text{ is an isomorphism.}$$

As a corollary, the space of Fredholm maps $\mathcal{F}(X, Y)$ is open in the operator norm. □

Proposition 93. Let $L : X \to Y$ be a Fredholm operator and suppose a, b, c, V, W is a regularization for L. Then

$$\dim \ker L - \dim \operatorname{coker} L = \dim W - \dim V.$$

Proof. Since X is Fredholm, we can split X, Y as follows

$$X = X' \oplus \ker L \text{ and } Y = \operatorname{im} L \oplus \operatorname{coker} L,$$

where X' is complimentary to $\ker L$, and so that $L' = L|_{X'} : X' \to \operatorname{im} L$ is an isomorphism. Then the matrix of the regularization takes the form

$$R = \begin{bmatrix} L' & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1 & b_2 & c \end{bmatrix} : X' \oplus \ker L \oplus V \to \operatorname{im} L \oplus \operatorname{coker} L \oplus W$$

We will construct an isomorphism $\varphi : \ker L \oplus V \to \operatorname{coker} L \oplus W$. We begin by computing

$$\begin{bmatrix} L' & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1 & b_2 & c \end{bmatrix} \begin{bmatrix} x \\ k \\ v \end{bmatrix} = \begin{bmatrix} L'x + a_1(v) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha(x,k,v) \\ \beta(x,k,v) \end{bmatrix}.$$

Motivated by this computation, define $x(v) \in X'$ *uniquely* by the equation $L'x(v) + a_1(v) = 0$. Then define φ by

$$\varphi \begin{bmatrix} k \\ v \end{bmatrix} = R \begin{bmatrix} x(v) \\ k \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha(x(v),k,v) \\ \beta(x(v),k,v) \end{bmatrix}$$

We claim that φ is a bijection. Injectivity is easy: if $\varphi(k, v) = 0$, then

$$R \begin{bmatrix} x(v) \\ k \\ v \end{bmatrix} = 0 \implies \begin{bmatrix} k \\ v \end{bmatrix} = 0.$$

Conversely, since R is surjective, there is some x, k, v so that

$$R \begin{bmatrix} x \\ k \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \implies x = x(v) \implies \varphi \begin{bmatrix} k \\ v \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Therefore φ is an isomorphism $\ker L \oplus V \to \operatorname{coker} L \oplus W$. It follows that

$$\dim \ker L + \dim V = \dim \operatorname{coker} L + \dim W \implies \dim \ker L - \dim \operatorname{coker} L = \dim W - \dim V,$$

as desired. $\square$

Theorem 94 (homotopy invariance of the Fredholm index)**.** Let X, Y be Banach spaces and consider the space $\mathcal{F}(X, Y)$ of Fredholm maps $X \to Y$. Then the index function $\mathcal{F}(X, Y) \to \mathbb{Z}$

defined by

$$\mathrm{index}(L) = \dim\ker L - \dim\operatorname{coker} L$$

is continuous.

Proof. If L has index n, then we can regularize L to an isomorphism

$$\begin{bmatrix} L & a \\ b & c \end{bmatrix} : X \oplus V \to Y \oplus W.$$

In particular

$$\mathrm{index}(L) = \dim W - \dim V \tag{1}$$

For L' sufficiently close to L, we know that

$$\begin{bmatrix} L' & a \\ b & c \end{bmatrix} : X \oplus V \to Y \oplus W.$$

is still an isomorphism. It follows that

$$\mathrm{index}(L') = \dim W - \dim V \tag{2}$$

Comparing (1) and (2), we see that $\mathrm{index}(L) = \mathrm{index}(L')$, as desired. □

Corollary 95. Let L_t be a continuous family of Elliptic operators on a closed manifold Y. It is easy to show that L_t is continuous in the operator norm topology on $W^{1,p}(Y) \to L^p(Y)$. It follows that

$$\mathrm{index}(L_1) = \mathrm{index}(L_0).$$