

Explanation of algorithm for sampling from $s_k(x) = u_k(x) / \int_D u_k(x') dx' = c u_k(x)$. First we need to find the normalizing constant $c := 1 / \int_D u_k(x') dx'$:

$$\begin{aligned} \int_D u_k(x') dx' &= \sum_{j=1}^k \int_{z_{j-1}}^{z_j} \exp(h(x_j) + (x' - x_j)h'(x_j)) dx' \\ &= \sum_{j=1}^k \frac{1}{h'(x_j)} \exp(h(x_j) + (x' - x_j)h'(x_j)) \Big|_{x'=z_{j-1}}^{z_j} \\ &= \sum_{j=1}^k \frac{\exp(u_k(z_j)) - \exp(u_k(z_{j-1}))}{h'(x_j)} \end{aligned}$$

Next, in order to use the inverse CDF method for sampling, we must find the CDF for $s_k(x)$, $S_k(x) = c \int_{z_0}^x u_k(x') dx'$:

$$\begin{aligned} S_k(x) &= c \left(\sum_{j=1}^{t-1} \int_{z_{j-1}}^{z_j} \exp(h(x_j) + (x' - x_j)h'(x_j)) dx' + \int_{z_{t-1}}^x \exp(h(x_t) + (x' - x_t)h'(x_t)) dx' \right) \\ &= c \left(\sum_{j=1}^{t-1} \frac{u_k(z_j) - u_k(z_{j-1})}{h'(x_j)} + \frac{u_k(x) - u_k(z_{t-1})}{h'(x_t)} \right) \end{aligned}$$

where t is the index of which interval of z 's that x lies in. Formally, it is $t(x) = i : x \in (z_{i-1}, z_i)$. For convenience, we call the sum term above `partialSums[t-1]`. Note that our normalizing constant $c = 1/\text{partialSums}[k]$. Thus,

$$S_k(x) = c \left(\text{partialSums}[t-1] + \frac{u_k(x) - u_k(z_{t-1})}{h'(x_t)} \right)$$

Now we can determine the inverse transform $S_k^{-1}(U)$, where $U \sim \text{Uniform}[0,1]$. Because t is actually a function of x , it too must be inverted. Intuitively, we want to pick the biggest t such that $S_k(z_{t-1}) < U$. Formally, $t(U) = i : U \in (c \text{ partialSums}[i-1], c \text{ partialSums}[i])$. Solving for the inverse, we have:

$$S_k^{-1}(U) = \frac{\log(h'(x_t)(\frac{U}{c} - \text{partialSums}[t-1]) + u_k(z_{t-1})) - h(x_t)}{h'(x_t)} + x_t$$