Homework 2

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Problem 1. *Let a and c be real numbers.*

(a) Show that if $g(x) = f(x-a)$, then $\hat{g}(k) = e^{ika}\hat{f}(k)$.

(b) Use the Fourier transform and part (a) to solve the transport PDE

$$\begin{cases} u_t + cu_x = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution.

(a)

$$\begin{aligned}
\hat{g}(k) &= \int_{-\infty}^{\infty} g(x)e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} f(x-a)e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} f(y)e^{-ik(y+a)}dy \quad (\text{let } y = x-a) \\
&= e^{ika}\int_{-\infty}^{\infty} f(y)e^{-iky}dy \\
&= e^{ika}\hat{f}(k).
\end{aligned}$$

(b)

$$\begin{aligned}
u_t(x,t) &= -cu_x(x,t) \\
\hat{u}_t(k,t) &= \int_{-\infty}^{\infty} -cu_x(x,t)e^{-ikx}dx \\
&= -c\int_{-\infty}^{\infty} u_x(x,t)e^{-ikx}dx \\
&= -c\Big[u(x,t)e^{-ikx}\Big]_{-\infty}^{\infty} + c\int_{-\infty}^{\infty} u(x,t)ike^{-ikx}dx \\
&= cik\hat{u}(k,t).
\end{aligned}$$

We have the ODE $\hat{u}_t(k,t) = cik\hat{u}(k,t)$, which has the solution $\hat{u}(k,t) = \hat{u}(k,0)e^{cikt} = \hat{f}(k)e^{cikt}$. Using part (a), we have

$$u(x,t) = f(x-ct)$$

which is the solution to the transport PDE.

□

Problem 2. *Consider the heat equation $u_t = Du_{xx}$ with $x \in \mathbb{R}$ and $D > 0$.*

(a) For each $k \in \mathbb{R}$, find $\lambda = \lambda(k) \in \mathbb{R}$ so that $u(x,t) = e^{\lambda t}e^{ikx}$ satisfies the heat equation. Here $i = \sqrt{-1}$.

(b) Assume that $a(k)$ is Schwartz class in k. Show that $u(x,t) = \int_{-\infty}^{\infty} a(k)e^{\lambda(k)t}e^{ikx}dk$ satisfies the heat equation for $t \geq 0$.

Solution.

(a)

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}e^{\lambda t}e^{ikx} \\ &= \lambda e^{\lambda t}e^{ikx} \\ u_{xx}(x,t) &= \frac{\partial^2}{\partial x^2}e^{\lambda t}e^{ikx} \\ &= -k^2 e^{\lambda t}e^{ikx}. \end{aligned}$$

$$u_t(x,t) = Du_{xx}(x,t) \implies \lambda e^{\lambda t}e^{ikx} = -Dk^2 e^{\lambda t}e^{ikx}.$$

$$\lambda(k) = -Dk^2.$$

(b)

$$u_t(x,t) = \int_{-\infty}^{\infty} a(k)\lambda(k)e^{\lambda(k)t}e^{ikx}dk = \int_{-\infty}^{\infty} a(k)(-Dk^2)e^{-Dk^2t}e^{ikx}dk.$$

$$u_{xx}(x,t) = \int_{-\infty}^{\infty} a(k)(-k^2)e^{-Dk^2t}e^{ikx}dk.$$

$$u_t(x,t) = Du_{xx}(x,t).$$

Note that $t \geq 0$ so that e^{-Dk^2t} is either 1 or Schwartz class.
Thus, $u(x,t) = \int_{-\infty}^{\infty} a(k)e^{\lambda(k)t}e^{ikx}dk$ satisfies the heat equation for $t \geq 0$.

□

Problem 3. *Consider the equation $u_t = u_{xxxx}$ for $x \in \mathbb{R}$ with $u(x,0) = f(x)$ where $f(x)$ is Schwartz class. Derive an expression for the Fourier transform $\hat{u}(k,t)$ of the solution $u(x,t)$ in terms of k, t, and $\hat{f}(k)$. (You do not need to find $u(x,t)$ from $\hat{u}(k,t)$).*

Solution.

$$\begin{aligned}
u_t(x,t) &= u_{xxxx}(x,t) \\
\hat{u}_t(k,t) &= \int_{-\infty}^{\infty} u_{xxxx}(x,t)e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} u(x,t)(-ik)^4 e^{-ikx}dx \\
&= (-ik)^4 \hat{u}(k,t) \\
&= k^4 \hat{u}(k,t).
\end{aligned}$$

We have the ODE $\hat{u}_t(k,t) = k^4\hat{u}(k,t)$, which has the solution $\hat{u}(k,t) = \hat{u}(k,0)e^{k^4 t} = \hat{f}(k)e^{k^4 t}$.

$\square$

Problem 4.

(a) For given $D > 0$ and $a, c \in \mathbb{R}$, consider the equation $u_t = Du_{xx} + cu_x - au$ for $x \in \mathbb{R}$ and $t > 0$. Assume that $u(x,t)$ is a solution of this equation. Verify that $v(x,t) := u(y - ct, t)e^{at}$ is a solution of $v_t = Dv_{yy}$ for $y \in \mathbb{R}$ and $t > 0$.

(b) Recall that (from HW1)

$$v(y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}$$

satisfies $v_t = v_{yy}$ for $y \in \mathbb{R}$ and $t > 0$. Use the preceding part to find the solution $u(x,t)$ to $u_t = u_{xx} + 2u_x - 0.5u$.

Solution.

(a)

$$\begin{aligned}
v_t(x,t) &= \frac{\partial}{\partial t} u(y - ct, t)e^{at} \\
&= u_t(y - ct, t)e^{at} - cu_x(y - ct, t)e^{at} + au(y - ct, t)e^{at} \\
&= (Du_{xx}(y - ct, t) + cu_x(y - ct, t) - au(y - ct, t))\, e^{at} - cu_x(y - ct, t)e^{at} + au(y - ct, t)e^{at} \\
&= Du_{xx}(y - ct, t)e^{at} \\
&= Dv_{yy}(x,t)
\end{aligned}$$

(b) We have that $D = 1$, $a = 2$, and $c = 0$. Let $v(x,t) = u(y - ct, t)e^{at}$. Then

$$u_t = Du_{xx} + cu_x - au \implies v_t = Dv_{yy}.$$

Let $y = x$, $c = 0$, and $a = 2$. Then

$$v(x,t) = u(x,t)e^{2t}.$$

$$u(x,t) = v(x,t)e^{-2t} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{-2t} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t} - 2t}$$

$\square$