Homework 3

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Problem 1. *Solve*

$$\begin{cases} u_t - Du_{xx} = 0 \\ u(x,0) = e^x \end{cases}$$

explicitly.

Remark. You are allowed to use the solution formula even though $u(x,0)$ is not in Schwartz class.

Solution. Recall that the fundamental solution to the heat equation is

$$G(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

and the solution to the heat equation is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)u(y,0)\,dy$$

so

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} e^y\,dy.$$

Focusing on the exponent, we have

$$-\frac{(x-y)^2}{4Dt} + y = \frac{4Dty - (x-y)^2}{4Dt}$$

Focusing on the numerator, we complete the square with respect to y:

$$\begin{aligned} 4Dty - (x-y)^2 &= 4Dty - x^2 + 2xy - y^2 \\ &= -y^2 + 2xy + 4Dty - x^2 \\ &= -y^2 + (2x + 4Dt)y - x^2 \\ &= -\left(y^2 - (2x+4Dt)y\right) - x^2 \\ &= -(y - x - 2Dt)^2 - x^2 + (x+2Dt)^2 \\ &= -(y - x - 2Dt)^2 + 4Dtx + 4D^2t^2 \\ &= -(y - x - 2Dt)^2 + 4Dt(x + Dt) \end{aligned}$$

So we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-2Dt)^2}{4Dt}} e^{x+Dt}\,dy \\ &= \frac{e^{x+Dt}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x-2Dt)^2}{4Dt}}\,dy \\ &= \frac{e^{x+Dt}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{y-x-2Dt}{\sqrt{4Dt}}\right)^2}\,dy \end{aligned}$$

Let $u = \frac{y-x-2Dt}{\sqrt{4Dt}}$, so $du = \frac{1}{\sqrt{4Dt}}\,dy$, and

$$u(x,t) = \frac{e^{x+Dt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

which is the Gaussian integral, so

$$u(x,t) = \frac{e^{x+Dt}}{\sqrt{\pi}} \sqrt{\pi} = e^{x+Dt}$$

□

Problem 2. *Use the Fourier transform to find the general solution of the wave equation*

$$u_{tt} = c^2 u_{xx}$$

Hint. You may use the fact from ODE that:

$$y'' + a^2 y = 0 \implies y(t) = Ae^{iat} + Be^{-iat}$$

and Question 1 from HW2.

Solution. We may first take the Fourier transform of both sides:

$$\mathcal{F}\{u_{tt}\} = \mathcal{F}\{c^2 u_{xx}\}$$

$$\hat{u}_{tt} = c^2(ik)^2\hat{u}$$

$$\hat{u}_{tt} = -c^2k^2\hat{u}$$

This is a second order ODE in $\hat{u}$, and we can use the provided fact to obtain

$$\hat{u}'' + c^2k^2\hat{u} = 0 \implies \hat{u}(t) = A(k)e^{ickt} + B(k)e^{-ickt}$$

Now we take the inverse Fourier transform of $\hat{u}$:

$$u(x,t) = \mathcal{F}^{-1}\{\hat{u}(t)\} = \mathcal{F}^{-1}\{Ae^{ickt} + Be^{-ickt}\}$$

Recall from Question 1 of HW2 (using v to distinguish from u) that

$$\hat{v}_1(k,t) = \hat{v}_1(k,0)e^{ickt} = \hat{f}_1(k)e^{ickt} \implies v_1(x,t) = f_1(x-ct)$$

And similarly we have

$$\hat{v}_2(k,t) = \hat{v}_2(k,0)e^{-ickt} = \hat{f}_2(k)e^{-ickt} \implies v_2(x,t) = f_2(x+ct)$$

So, we may write

$$u(x,t) = \mathcal{F}^{-1}\{A(k)e^{ickt} + B(k)e^{-ickt}\} = \mathcal{F}^{-1}\{\hat{u}(t)\} = \mathcal{F}^{-1}\{A(k)\hat{u}(k,0)e^{ickt} + B(k)\hat{u}(k,0)e^{-ickt}\}$$

Letting $f(k) := A(k)\hat{u}(k,0)$ and $g(k) := B(k)\hat{u}(k,0)$, we have

$$u(x,t) = \mathcal{F}^{-1}\{f(k)e^{ickt} + g(k)e^{-ickt}\} = f(x-ct) + g(x+ct)$$

$\square$

Problem 3. *Solve*

$$\begin{cases} u_{tt} - 3u_{xt} - 4u_{xx} = 0 \\ u(x,0) = 2x^2 \\ u_t(x,0) = e^{4x} \end{cases}$$

Solution.

$$u_{tt} - 3u_{xt} - 4u_{xx} \iff \left(\frac{\partial}{\partial t} - 4\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u = 0$$

So we have

$$\begin{cases} \xi = x - t \\ \eta = x + 4t \end{cases}$$

and

$$\begin{cases} x = \dfrac{4\xi + \eta}{5} \\ t = \dfrac{\eta - \xi}{5} \end{cases}$$

Then

$$\frac{\partial}{\partial \xi} = \frac{4}{5}\frac{\partial}{\partial x} - \frac{1}{5}\frac{\partial}{\partial t} = -\frac{1}{5}\left(\frac{\partial}{\partial t} - 4\frac{\partial}{\partial x}\right)$$

and

$$\frac{\partial}{\partial \eta} = \frac{1}{5}\frac{\partial}{\partial x} + \frac{1}{5}\frac{\partial}{\partial t} = \frac{1}{5}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

Let us define

$$V(\xi,\eta) = u\left(\frac{4\xi + \eta}{5}, \frac{\eta - \xi}{5}\right)$$

As in lecture, we can compute to find

$$V_{\xi\eta} = 0 \implies V(\xi,\eta) = F(\xi) + G(\eta) = F(x-t) + G(x+4t)$$

so

$$u(x,t) = F(x-t) + G(x+4t)$$

Now we need to find F and G using the initial conditions.

□

Problem 4. *Choose $\alpha > 0$. Let $u(x,t)$ be a solution of Schwartz to the damped wave equation $u_{tt} + \alpha u_t = u_{xx}$. Show that the energy*

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \left[(u_t(x,t))^2 + (u_x(x,t))^2\right] dx$$

is non-increasing ($\frac{d}{dt}E(t) \leq 0$).

Solution.

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[(u_t(x,t))^2 + (u_x(x,t))^2\right] dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} 2u_t(x,t)u_{tt}(x,t) + 2u_x(x,t)u_{xt}(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)(u_{xx}(x,t) - \alpha u_t(x,t)) + u_x(x,t)u_{xt}(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)u_{xx}(x,t)dx - \alpha\int_{-\infty}^{\infty} (u_t(x,t))^2dx + \int_{-\infty}^{\infty} u_x(x,t)u_{xt}(x,t)dx \\
&= -\int_{-\infty}^{\infty} u_x(x,t)u_t(x,t)dx - \alpha\int_{-\infty}^{\infty} (u_t(x,t))^2dx + \int_{-\infty}^{\infty} u_x(x,t)u_{xt}(x,t)dx \\
&= -\int_{-\infty}^{\infty} u_x(x,t)u_{xt}(x,t)dx - \alpha\int_{-\infty}^{\infty} (u_t(x,t))^2dx + \int_{-\infty}^{\infty} u_x(x,t)u_{xt}(x,t)dx \\
&= -\alpha\int_{-\infty}^{\infty} (u_t(x,t))^2dx \leq 0
\end{aligned}$$

where the last inequality follows from the fact that $\alpha > 0$ and the integrand is non-negative. $\square$