Practice Midterm 2

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Problem 1. *Compute the Fourier sine series of $x^2 + 1$ on $[0, \pi]$.*

Solution.

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} (x^2 + 1) \sin(nx) \, \mathrm{d}x \\
&= \frac{2}{\pi} \int_0^{\pi} (x^2 + 1) \frac{\partial}{\partial x} \left(-\frac{\cos(nx)}{n} \right) \mathrm{d}x \\
&= \frac{2}{\pi} \left[(x^2 + 1) \left(-\frac{\cos(nx)}{n} \right) \Big|_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos(nx)}{n} \right) \frac{\partial}{\partial x} (x^2 + 1) \, \mathrm{d}x \right] \\
&= \frac{2}{\pi} \left[(\pi^2 + 1) \left(-\frac{\cos(n\pi)}{n} \right) - \left(-\frac{\cos(0)}{n} \right) \right] - \frac{2}{\pi} \int_0^{\pi} \left(-\frac{\cos(nx)}{n} \right) 2x \, \mathrm{d}x \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) + \frac{4}{n\pi} \int_0^{\pi} x \cos(nx) \, \mathrm{d}x \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) + \frac{4}{n\pi} \left[x \left(\frac{\sin(nx)}{n} \right) \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n} \, \mathrm{d}x \right] \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) - \frac{4}{n^2\pi} \int_0^{\pi} \sin(nx) \, \mathrm{d}x \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) - \frac{4}{n^2\pi} \left[-\frac{\cos(nx)}{n} \Big|_0^{\pi} \right] \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) - \frac{4}{n^2\pi} \left(-\frac{(-1)^n - 1}{n} \right) \\
&= \frac{2}{n\pi} \left(1 - (\pi^2 + 1)(-1)^n \right) - \frac{4}{n^3\pi} (1 - (-1)^n)
\end{aligned}
$$

The Fourier sine series of $x^2 + 1$ on $[0, \pi]$ is then

$$\sum_{n=1}^{\infty} a_n \sin(nx).$$

We can check that the Fourier series approximates $x^2 + 1$ in $[0, \pi]$: □

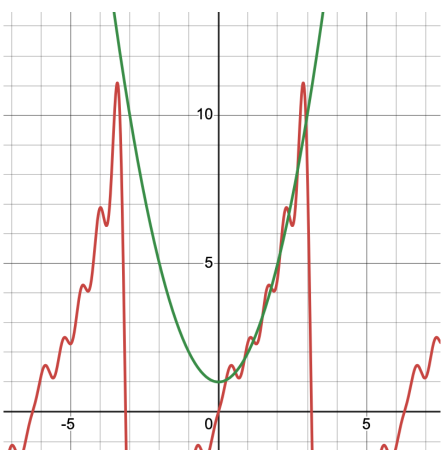

Figure 1: https://www.desmos.com/calculator/a7wlsqmh2f

Problem 2. *Compute the Fourier sine series of* $\cos(x)$ *on* $[0, \pi]$. *Does it converge to* $\cos(x)$ *pointwise on* $[0, \pi]$*? If not, at which points does it not converge?*

Solution.

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \cos(x)\frac{\partial}{\partial x}\left(-\frac{\cos(nx)}{n}\right)\mathrm{d}x \\
&= \frac{2}{\pi}\left[\cos(x)\left(-\frac{\cos(nx)}{n}\right)\Big|_0^\pi - \int_0^\pi \left(-\frac{\cos(nx)}{n}\right)\frac{\partial}{\partial x}\cos(x)\,\mathrm{d}x\right] \\
&= \frac{2}{n\pi}\big(-\cos(\pi)\cos(n\pi) + \cos(0)\cos(0)\big) - \frac{2}{n\pi}\int_0^\pi \cos(nx)\sin(x)\,\mathrm{d}x \\
&= \frac{2}{n\pi}\left((-1)^n + 1\right) - \frac{2}{n\pi}\int_0^\pi \sin(x)\frac{\partial}{\partial x}\left(\frac{\sin(nx)}{n}\right)\mathrm{d}x \\
&= \frac{2}{n\pi}\left((-1)^n + 1\right) - \frac{2}{n\pi}\left[\sin(x)\left(\frac{\sin(nx)}{n}\right)\Big|_0^\pi - \int_0^\pi \frac{\sin(nx)}{n}\cos(x)\,\mathrm{d}x\right] \\
&= \frac{2}{n\pi}\left((-1)^n + 1\right) + \frac{2}{n^2\pi}\int_0^\pi \cos(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{n\pi}\left((-1)^n + 1\right) + \frac{1}{n^2}a_n
\end{aligned}$$

$$\begin{aligned}
n^2 a_n &= \frac{2n}{\pi}\left((-1)^n + 1\right) + a_n \\
(n^2 - 1)a_n &= \frac{2n}{\pi}\left((-1)^n + 1\right) \\
a_n &= \frac{2n\left((-1)^n + 1\right)}{\pi(n^2 - 1)}
\end{aligned}$$

We can handle the special case of $n = 1$ separately:

$$a_1 = \frac{2}{\pi}\int_0^\pi \cos(x)\sin(x)\,\mathrm{d}x$$

We can use the fact that $\sin(x)$ and $\cos(x)$ are orthogonal, so $\langle \cos(x), \sin(x)\rangle = \int \cos(x)\sin(x)\,\mathrm{d}x = 0$. Therefore, $a_1 = 0$.

The Fourier sine series of $\cos(x)$ on $[0, \pi]$ is then

$$\sum_{n=2}^{\infty} a_n \sin(nx)$$

where

$$a_n = \frac{2n\left((-1)^n + 1\right)}{\pi(n^2 - 1)}.$$

We can determine the pointwise convergence of the Fourier series by only checking the convergence of the series at the endpoints of the interval, as $\cos(x)$ is continuous everywhere:

$$\lim_{n\to\infty} \cos(0) = 1 \neq 0 = \lim_{n\to\infty} \sum_{n=2}^{\infty} a_n \sin(nx)$$

$$\lim_{n\to\infty} \cos(\pi) = -1 \neq 0 = \lim_{n\to\infty} \sum_{n=2}^{\infty} a_n \sin(nx)$$

Therefore, the Fourier series does not converge to $\cos(x)$ pointwise on $[0, \pi]$, as it does not converge at $x = 0$ and $x = \pi$. □

Problem 3. *Solve*

$$\begin{cases} u_t = t u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = x, & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x,t) = X(x)T(t)$. Then

$$u_t = t u_{xx} \implies X(x)T'(t) = tX''(x)T(t) \implies \frac{X''(x)}{X(x)} = \frac{1}{t}\frac{T'(t)}{T(t)} = \lambda$$

We recall that a solution for the homogeneous Dirichlet boundary conditions is

$$X(x) = c\sin(nx)$$

for a constant c, $n = 1, 2, 3, \ldots$, and $\lambda = -n^2$. We can then solve the ODE in $T(t)$:

$$\begin{aligned} \frac{1}{t}\frac{T'(t)}{T(t)} &= -n^2 \\ \frac{T'(t)}{T(t)} &= -n^2 t \\ \frac{\mathrm{d}}{\mathrm{d}t}\ln T(t) &= -n^2 t \\ \ln T(t) &= -\frac{n^2}{2}t^2 + c \\ T(t) &= e^{-\frac{n^2}{2}t^2 + c} = Ce^{-\frac{n^2}{2}t^2} \end{aligned}$$

The general solution is then

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2}{2}t^2}\sin(nx)$$

We can determine the coefficients a_n by using the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x$$

Recall from lecture that the Fourier sine series of x on $[0,\pi]$ is

$$x \overset{L^2}{=} \sum_{n=1}^{\infty} a_n \sin(nx)$$

where

$$a_n = (-1)^{n+1}\frac{2}{n}.$$

Therefore, the solution to the PDE is

$$u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}e^{-\frac{n^2}{2}t^2}\sin(nx).$$

□

Problem 4. *Solve*

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = x, & 0 < x < \pi, \\ u_t(x,0) = \sin(4x) + 5\sin(5x) & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x,t) = X(x)T(t)$. Then

$$u_{tt} = 4u_{xx} \implies X(x)T''(t) = 4X''(x)T(t) \implies \frac{X''(x)}{X(x)} = \frac{1}{4}\frac{T''(t)}{T(t)} = \lambda$$

We recall that a solution for the homogeneous Dirichlet boundary conditions is

$$X(x) = c\sin(nx)$$

for a constant c, $n = 1, 2, 3, \ldots$, and $\lambda = -n^2$. We can then solve the ODE in $T(t)$:

$$\frac{1}{4}\frac{T''(t)}{T(t)} = -n^2$$

$$T''(t) = -4n^2 T(t)$$

We recall that the general solution to this ODE is

$$T(t) = c_1\cos(2nt) + c_2\sin(2nt)$$

The general solution is then

$$u(x,t) = \sum_{n=1}^{\infty}\big(a_n\cos(2nt) + b_n\sin(2nt)\big)\sin(nx)$$

We can determine the coefficients a_n and b_n by using the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} a_n\sin(nx) = x$$

Recall from lecture that the Fourier sine series of x on $[0,\pi]$ is

$$x \overset{L^2}{=} \sum_{n=1}^{\infty} a_n\sin(nx)$$

where

$$a_n = (-1)^{n+1}\frac{2}{n}.$$

Turning to the second initial condition:

$$u_t(x,t) = \sum_{n=1}^{\infty} 2n\big(-a_n\sin(2nt) + b_n\cos(2nt)\big)\sin(nx)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} 2nb_n\sin(nx) = \sin(4x) + 5\sin(5x)$$

Comparing terms, we see that $b_4 = \frac{1}{8}$, $b_5 = \frac{1}{2}$, and $b_k = 0$ for $k \neq 4, 5$.

Therefore, the solution to the PDE is

$$u(x,t) = \frac{1}{8}\sin(8t)\sin(4x) + \frac{1}{2}\sin(10t)\sin(5x) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\cos(2nt)\sin(nx).$$

□

Problem 5. *Use the energy method to show that the only solution to the heat-like equation with periodic boundary conditions*

$$\begin{cases} u_t = 4u_{xx} - 3u, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t), & t > 0, \\ u_x(0,t) = u_x(\pi,t), & t > 0, \\ u(x,0) = 0, & 0 < x < \pi \end{cases}$$

is zero.

Solution. We can multiply the PDE by u and integrate over $[0,\pi]$:

$$\int_0^\pi u_t u \, \mathrm{d}x = \int_0^\pi (4u_{xx} - 3u) u \, \mathrm{d}x$$

$$\int_0^\pi \frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) \mathrm{d}x = \int_0^\pi 4u_{xx} u \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\pi \frac{1}{2}u^2 \, \mathrm{d}x = \int_0^\pi 4u_{xx} u \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x$$

Let $E(t) = \int_0^\pi \frac{1}{2}u^2 \, \mathrm{d}x \geq 0$. Then

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= 4\int_0^\pi u_{xx} u \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x \\ &= 4\, u_x u\big|_0^\pi - 4\int_0^\pi u_x^2 \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x \\ &= 4\left[u_x(\pi,t)u(\pi,t) - u_x(0,t)u(0,t)\right] - 4\int_0^\pi u_x^2 \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x \\ &= 0 - 4\int_0^\pi u_x^2 \, \mathrm{d}x - 3\int_0^\pi u^2 \, \mathrm{d}x \\ &\leq 0 \end{aligned}$$

Then $0 \leq E(t) \leq E(0) = 0$, so $E(t) = 0$. Therefore, $u(x,t) = 0$. □

Problem 6. *Consider the system of equations*

$$\begin{cases} u_t = 4u_{xx} - 3u_x - 5uv^2, \\ v_t = -3uv^2. \end{cases}$$

Find a system of ODEs for U and V, where U, V are the traveling wave profiles $u(x,t) = U(z),\ v(x,t) = V(z),\ z = x - ct$.

Solution. We have the following system:

$$\begin{cases} u(x,t) = U(x-ct) & = U(z), \\ v(x,t) = V(x-ct) & = V(z). \end{cases} \implies \begin{cases} u_t = \dfrac{\mathrm{d}U}{\mathrm{d}z}\dfrac{\mathrm{d}z}{\mathrm{d}t} & = -cU'(z), \\ v_t = \dfrac{\mathrm{d}V}{\mathrm{d}z}\dfrac{\mathrm{d}z}{\mathrm{d}t} & = -cV'(z), \\ u_x = \dfrac{\mathrm{d}U}{\mathrm{d}z}\dfrac{\mathrm{d}z}{\mathrm{d}x} & = U'(z), \\ u_{xx} = \dfrac{\mathrm{d}U'}{\mathrm{d}z}\dfrac{\mathrm{d}z}{\mathrm{d}x} & = U''(z). \end{cases}$$

Now we can substitute these into the original system:

$$\begin{cases} -cU'(z) = 4U''(z) - 3U'(z) - 5UV^2, \\ -cV'(z) = -3UV^2. \end{cases}$$

□

Problem 7. *Use separation of variables to solve*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi, \\ u(x,0) = 0, & 0 < x < \pi, \\ u(x,\pi) = 0, & 0 < x < \pi, \\ u_x(0,y) = \sin(2y), & 0 < y < \pi, \\ u_x(\pi,y) = 0, & 0 < y < \pi, \end{cases}$$

Note: $(\cosh(x))' = \sinh(x)$, $(\sinh(x))' = \cosh(x)$. *No negative signs for the derivatives of hyperbolic sine/cosine functions.*

Solution. Let $u(x,y) = X(x)Y(y)$. Then

$$u_{xx} + u_{yy} = 0 \implies X''Y + XY'' = 0 \implies -\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

We first solve the ODE in $Y(y)$:

$$\begin{cases} Y''(y) = \lambda Y(y), \\ Y(0) = 0, \\ Y(\pi) = 0. \end{cases}$$

Recall that a solution to the homogeneous Dirichlet boundary conditions is

$$Y(y) = c\sin(ny)$$

for a constant c, $n = 1, 2, 3, \ldots$, and $\lambda = -n^2$. We can then solve the ODE in $X(x)$:

$$X''(x) = -\lambda X(x) = n^2 X(x)$$

Recall the general solution

$$X(x) = Ae^{nx} + Be^{-nx}$$

which can be rewritten using the hyperbolic sine and cosine functions as

$$X(x) = \tilde{A}\sinh(nx) + \tilde{B}\cosh(nx).$$

The solution with the homogeneous Dirichlet boundary conditions is then

$$u(x,y) = \sum_{n=1}^{\infty} \big(a_n \sinh(nx) + b_n \cosh(nx)\big)\sin(ny).$$

We now determine the coefficients a_n and b_n with the other boundary conditions.

First, we compute the partial derivative of u with respect to x:

$$u_x(x,y) = \sum_{n=1}^{\infty} n\big(a_n \cosh(nx) + b_n \sinh(nx)\big)\sin(ny)$$

With the $u_x(0,y) = \sin(2y)$ boundary condition:

$$u_x(0,y) = \sum_{n=1}^{\infty} na_n \sin(ny) = \sin(2y)$$

Comparing terms, we see that $a_2 = \frac{1}{2}$ and $a_k = 0$ for $k \neq 2$.

With the $u_x(\pi, y) = 0$ boundary condition:

$$u_x(\pi, y) = \sum_{n=1}^{\infty} n\big(a_n \cosh(n\pi) + b_n \sinh(n\pi)\big) \sin(ny) = 0$$

We can let $c_n = n\big(a_n \cosh(n\pi) + b_n \sinh(n\pi)\big)$ and rewrite it as

$$\sum_{n=1}^{\infty} c_n \sin(ny) = 0$$

which is a Fourier sine series that implies $c_n = 0$ for all n.

Now, we have

$$c_n = n\big(a_n \cosh(n\pi) + b_n \sinh(n\pi)\big) = 0$$

However, we know that n is strictly positive, so we must have

$$a_n \cosh(n\pi) + b_n \sinh(n\pi) = 0$$

From earlier, we recall that $a_2 = \frac{1}{2}$ and $a_k = 0$ for $k \neq 2$. Therefore, $b_k = 0$ for $k \neq 2$, and $b_2 < 0$. We can solve for b_2:

$$\frac{1}{2} \cosh(2\pi) + b_2 \sinh(2\pi) = 0$$

$$b_2 = -\frac{1}{2} \frac{\cosh(2\pi)}{\sinh(2\pi)} = -\frac{1}{2} \coth(2\pi)$$

Therefore, the solution to the PDE is

$$\begin{aligned} u(x, y) &= \frac{1}{2} \sinh(2x) - \frac{1}{2} \frac{\cosh(2\pi)}{\sinh(2\pi)} \cosh(2x) \sin(2y) \\ &= \frac{1}{2} \sinh(2x) - \frac{1}{2} \coth(2\pi) \cosh(2x) \sin(2y). \end{aligned}$$

□