

# Homework 5

Dylan Hu

Professor Zhuolun Yang

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**Problem 1.**

a. Verify  $\cos(nx)$  and  $\cos(mx)$  are orthogonal on  $(0, \pi)$  when  $n \neq m$  and  $n, m \in \mathbb{N}$ . That is

$$\int_0^\pi \cos(nx) \cos(mx) \, dx = 0.$$

b. Verify  $\sin(nx)$  and  $\sin(mx)$  are orthogonal on  $(-\pi, \pi)$  when  $n, m \in \mathbb{N}$ . That is

$$\int_{-\pi}^\pi \sin(nx) \cos(mx) \, dx = 0.$$

*Solution.*

a.

$$\begin{aligned} \int_0^\pi \cos(nx) \cos(mx) \, dx &= \frac{1}{2} \int_0^\pi (\cos((n+m)x) + \cos((n-m)x)) \, dx \\ &= \frac{1}{2} \left[ \frac{1}{n+m} \sin((n+m)x) + \frac{1}{n-m} \sin((n-m)x) \right]_0^\pi \\ &= \frac{1}{2} \left[ \frac{1}{n+m} \sin((n+m)\pi) + \frac{1}{n-m} \sin((n-m)\pi) \right] \\ &= 0 \end{aligned}$$

We have used the fact that  $\sin(m\pi) = \sin(n\pi) = 0$  for  $m, n \in \mathbb{N}$  and that  $n \neq m \implies n \pm m \neq 0$ .

b.

$$\begin{aligned} \int_{-\pi}^\pi \sin(nx) \cos(mx) \, dx &= \frac{1}{2} \int_{-\pi}^\pi (\sin((n+m)x) + \sin((n-m)x)) \, dx \\ &= \frac{1}{2} \int_{-\pi}^\pi \sin((n+m)x) \, dx + \frac{1}{2} \int_{-\pi}^\pi \sin((n-m)x) \, dx \\ &= 0 + 0 = 0 \end{aligned}$$

We use the fact that  $\sin$  is an odd function and  $(-\pi, \pi)$  is symmetric, so  $\int_{-\pi}^\pi \sin((n+m)x) \, dx = 0$  and  $\int_{-\pi}^\pi \sin((n-m)x) \, dx = 0$ .

□

**Problem 2.** Solve

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0, t) = u_x(\pi, t) = 0, & t > 0, \\ u(x, 0) = x^2, & 0 < x < \pi. \end{cases}$$

explicitly.

*Solution.*

$$u(x, t) = X(x)T(t)$$

We recall from lecture that the solution to  $X''(x) = \lambda X(x)$  with the von Neumann boundary conditions is

$$X(x) = c \cos(nx), \quad n \geq 0$$

where  $c$  is a constant and  $\lambda = -n^2$ . Then we also have

$$T'(t) = \lambda T(t) \implies T(t) = e^{\lambda t} = e^{-n^2 t}$$

So the general solution is

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos(nx) e^{-n^2 t}$$

We can solve for the  $c_n$  using the initial condition:

$$u(x, 0) = \sum_{n=0}^{\infty} a_n \cos(nx) = x^2$$

We recognize the sum as a Fourier series and solve for  $a_n$ :

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx = \frac{2}{\pi} \left[ \frac{2x^2}{n^2} \sin(nx) + \frac{2}{n} x \cos(nx) \right]_0^{\pi} = \frac{4}{n^2} (-1)^n$$

We notice that for  $n = 0$  our solution is undefined, so we must add the  $n = 0$  term separately:

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x^2 dx = \frac{1}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{3}$$

So the solution is

$$u(x, t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx) e^{-n^2 t}$$

□

**Problem 3.** Write the Fourier sine series (as  $\sum a_n \sin(nx)$ ) of the function

$$f(x) = \begin{cases} -1, & 0 < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$$

For each  $n \in \mathbb{N}$ , what is the value  $a_n \sin(n\pi/2)$ ?

You can separate the cases when  $n = 4k, 4k + 1, 4k + 2, 4k + 3$  for  $k \in \mathbb{N} \cup \{0\}$ .

*Solution.*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \left[ \int_0^{\pi/2} (-1) \sin(nx) dx + \int_{\pi/2}^{\pi} \sin(nx) dx \right] \\ &= \frac{2}{\pi} \left[ \frac{1}{n} \cos(nx) \Big|_0^{\pi/2} - \frac{1}{n} \cos(nx) \Big|_{\pi/2}^{\pi} \right] \\ &= \frac{2}{\pi} \left[ \frac{\cos(n\pi/2)}{n} - \frac{\cos(0)}{n} - \frac{\cos(n\pi)}{n} + \frac{\cos(n\pi/2)}{n} \right] \\ &= \frac{2}{\pi} \left[ \frac{2 \cos(n\pi/2) - (-1)^n - 1}{n} \right] \end{aligned}$$

We now have

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left[ \frac{2 \cos(n\pi/2) - (-1)^n - 1}{n} \right] \sin(nx)$$

We can separate the cases when  $n = 4k, 4k + 1, 4k + 2, 4k + 3$  for  $k \in \mathbb{N} \cup \{0\}$ .

**Case 1:**  $n = 4k$  (note that we can ignore  $k = 0$  in this case since  $n = 0$  is not in the sum)

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[ \frac{2 \cos(2k\pi) - (-1)^{4k} - 1}{4k} \right] \sin(2k\pi) = \frac{2}{\pi} \left[ \frac{2 - 1 - 1}{4k} \right] \sin(2k\pi) = 0$$

**Case 2:**  $n = 4k + 1$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[ \frac{2 \cos(2k\pi + \pi/2) - (-1)^{4k+1} - 1}{4k + 1} \right] \sin(2k\pi + \pi/2) = \frac{2}{\pi} \left[ \frac{2(0) + 1 - 1}{4k + 1} \right] \sin(2k\pi + \pi/2) = 0$$

**Case 3:**  $n = 4k + 2$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[ \frac{2 \cos(2k\pi + \pi) - (-1)^{4k+2} - 1}{4k + 2} \right] \sin(2k\pi + \pi) = \frac{2}{\pi} \left[ \frac{2(-1) - 1 - 1}{4k + 2} \right] (0) = 0$$

**Case 4:**  $n = 4k + 3$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[ \frac{2 \cos(2k\pi + 3\pi/2) - (-1)^{4k+3} - 1}{4k + 3} \right] \sin(2k\pi + 3\pi/2) = \frac{2}{\pi} \left[ \frac{2(0) - (-1) - 1}{4k + 3} \right] \sin(2k\pi + 3\pi/2) = 0$$

Therefore, we have  $a_n \sin(n\pi/2) = 0$  for all  $n \in \mathbb{N}$ .

□

**Problem 4.** Solve

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t > 0, \\ u(x, 0) = x^2, & 0 < x < \pi, \\ u_t(x, 0) = \sin(2x), & 0 < x < \pi. \end{cases}$$

explicitly.

*Solution.*

$$u(x, t) = X(x)T(t)$$

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Recall from lecture that the general solution to  $X''(x) = \lambda X(x)$  with the Dirichlet boundary conditions is

$$X(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

where  $\lambda = -n^2, n \in \mathbb{N}$ .

$$T''(t) = -n^2 T(t) \implies T(t) = c_1 \cos(nt) + c_2 \sin(nt)$$

So the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nt) \sin(nx) + b_n \sin(nt) \sin(nx))$$

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x^2$$

We recognize the sum as a Fourier sine series:

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx) dx$$

and solve for  $a_n$  using integration by parts:

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[ \frac{-x^2 \cos(nx)}{n} + \frac{2x \sin(nx)}{n^2} + \frac{2 \cos(nx)}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[ \frac{-\pi^2 \cos(n\pi)}{n} + \frac{2 \cos(n\pi)}{n^3} - \frac{2}{n^3} \right] \\ &= \frac{2}{\pi} \left[ \frac{-\pi^2 (-1)^n}{n} + \frac{2(-1)^n - 2}{n^3} \right] \\ &= \frac{-2\pi(-1)^n}{n} + \frac{4(-1)^n - 4}{\pi n^3} \end{aligned}$$

Now we solve for  $b_n$  using the initial condition:

$$u_t(x, t) = \sum_{n=1}^{\infty} (-na_n \sin(nt) \sin(nx) + nb_n \cos(nt) \sin(nx)) = \sin(2x)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} n b_n \sin(nx) = \sin(2x)$$

We can simply match the coefficients to find that  $b_2 = 1/2$ , and  $b_n = 0$  for all other  $n$ .

So the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left( \left( \frac{-2\pi(-1)^n}{n} + \frac{4(-1)^n - 4}{\pi n^3} \right) \cos(nt) \sin(nx) + \frac{1}{2} \sin(nt) \sin(nx) \right)$$

□