Homework 7

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Problem 1. *Find the solution to the boundary value problem on a wedge:*

$$\begin{cases} \Delta u = 0, & 0 < r < 1, \quad 0 < \theta < \pi/2, \\ u(r,0) = 0, & 0 < r < 1, \\ u(r,\pi/2) = 0, & 0 < r < 1, \\ u(1,\theta) = f(\theta), & 0 < \theta < \pi/2. \end{cases}$$

Hint. Be careful that $\theta \in (0,\pi/2)$, we do not have periodic boundary condition in θ. And the formulas for Fourier coefficients are slightly different than what we have when $\theta \in (0,\pi)$ or $\theta \in (0,2\pi)$.

Solution. First, we recall that the Laplacian in polar coordinates is given by

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

Using separation of variables, we assume that $u(r,\theta) = R(r)\Theta(\theta)$. Then Laplace's equation becomes

$$R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2} = 0 \iff \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

The boundary conditions in θ give $\Theta(0) = \Theta\left(\frac{\pi}{2}\right) = 0$. We solve the ODE in θ first.

$$\begin{cases} \Theta'' + \lambda\Theta = 0, & 0 < \theta < \frac{\pi}{2}, \\ \Theta(0) = 0, & 0 < \theta < \frac{\pi}{2}, \\ \Theta\left(\frac{\pi}{2}\right) = 0, & 0 < \theta < \frac{\pi}{2}. \end{cases}$$

Case 1: $\lambda < 0$. Let $\lambda = -\omega^2$ where $\omega > 0$.

$$\Theta''(\theta) = \omega^2\Theta(\theta) \implies \Theta(\theta) = Ae^{\omega\theta} + Be^{-\omega\theta}.$$

The boundary conditions give

$$\begin{cases} \Theta(0) = A + B = 0, \\ \Theta\left(\frac{\pi}{2}\right) = Ae^{\omega\pi/2} + Be^{-\omega\pi/2} = 0. \end{cases}$$

By substituting the first equation into the second, we find that $A\left(e^{\omega\pi/2} - e^{-\omega\pi/2}\right) = 0$, and we see that the only solution is $A = B = 0 \implies \Theta(\theta) = 0$.

Case 2: $\lambda = 0$. Then $\Theta'' = 0 \implies \Theta(\theta) = A + B\theta$. The boundary conditions give $A = A + B\frac{\pi}{2} = 0$, and we find that $A = B = 0 \implies \Theta(\theta) = 0$.

Case 3: $\lambda > 0$. Let $\lambda = \omega^2$ where $\omega > 0$.

$$\Theta''(\theta) = -\omega^2\Theta(\theta) \implies \Theta(\theta) = A\cos(\omega\theta) + B\sin(\omega\theta).$$

Using the boundary conditions:

$$\Theta(0) = A = 0, \quad \Theta\left(\frac{\pi}{2}\right) = B\sin\left(\frac{\omega\pi}{2}\right) = 0.$$

Then $\omega\frac{\pi}{2} = n\pi$ for some $n \in \mathbb{Z}$, and we find that $\omega = 2n$. Thus, $\Theta(\theta) = b_n\sin(2n\theta)$ for $n = 1,2,3,\ldots$ and $\lambda = 4n^2$.

Now we solve the ODE in r.

$$\frac{r^2R'' + rR'}{R} = 4n^2 \implies r^2R'' + rR' - 4n^2R = 0, \quad n = 1,2,3,\ldots$$

We recognize this as a Cauchy-Euler equation, and we make the substitution $R(r) = r^\alpha$.

$$\begin{aligned} r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - 4n^2r^\alpha = 0 \\ r^\alpha(\alpha^2 - \alpha + \alpha - 4n^2) = 0 \\ \alpha^2 - 4n^2 = 0 \implies \alpha = \pm 2n. \end{aligned}$$

This gives us two linearly independent solutions r^{2n} and r^{-2n}, so the general solution is

$$R(r) = Cr^{-2n} + Dr^{2n}, \quad n = 1, 2, 3, \ldots$$

However, we must have that the solution is bounded at $r = 0$, so we must have $C = 0$.

The general solution for $u(r,\theta)$ is

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin(2n\theta).$$

The boundary condition $u(1,\theta) = f(\theta)$ gives

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin(2n\theta).$$

We wish to find the Fourier coefficients b_n. If we multiply both sides by $\sin(2k\theta)$ and integrate from 0 to $\pi/2$, we find that

$$\int_0^{\pi/2} f(\theta)\sin(2k\theta)\,\mathrm{d}\theta = \sum_{n=1}^{\infty} b_n \int_0^{\pi/2} \sin(2n\theta)\sin(2k\theta)\,\mathrm{d}\theta.$$

We can rewrite the integral on the right-hand side to be from 0 to π.

$$\int_0^{\pi/2} f(\theta)\sin(2k\theta)\,\mathrm{d}\theta = \sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin(n\theta)\sin(k\theta)\,\mathrm{d}\theta.$$

Now we can recall that for the integral on the right hand side

$$\int_0^{\pi} \sin(n\theta)\sin(k\theta)\,\mathrm{d}\theta = \langle \sin(n\theta), \sin(k\theta)\rangle_{(0,\pi)} = \begin{cases} 0 & n \neq k \\ \pi/2 & n = k \end{cases}.$$

So we find that

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\sin(2n\theta)\,\mathrm{d}\theta.$$

and the solution is

$$u(r,\theta) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\int_0^{\pi/2} f(\theta)\sin(2n\theta)\,\mathrm{d}\theta\right] r^{2n}\sin(2n\theta).$$

□

Problem 2. *Find the solution to the following boundary value problem on an annulus:*

$$\begin{cases} \Delta u = 0, & 1 < r < 2, \quad 0 < \theta < 2\pi, \\ u(1,\theta) = \sin(\theta), & 0 < \theta < 2\pi, \\ u(2,\theta) = \cos(\theta), & 0 < \theta < 2\pi. \end{cases}$$

Hint. Since we are working on an annulus, we do not need boundedness condition for u at r $= 0$.

Solution. As in the previous problem, we arrive at

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

We solve the ODE in θ first with the implicit periodic boundary conditions. We recall from lecture that the general solution is

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n = 0, 1, 2, \dots, \quad \lambda = n^2.$$

We also recall that the general solution for $R(r)$ given $\lambda = n^2$ is

$$R = \begin{cases} C_n r^{-n} + D_n r^n, & n = 1, 2, 3, \dots, \\ C_0 \ln(r) + D, & n = 0. \end{cases}$$

Here, we do not set $C_n = 0$ or $C_0 = 0$ as we do not need to enforce boundedness at $r = 0$ since the annulus is bounded by $1 < r < 2$.

The general solution for $u(r,\theta)$ is

$$u(r,\theta) = A + B\ln(r) + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] \left[C_n r^{-n} + D_n r^n\right]$$

Using the first boundary condition, we have:

$$u(1,\theta) = A + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] \left[C_n + D_n\right] = \sin(\theta).$$

We recognize this as a Fourier series, and we can compute the Fourier coefficients.

$$A = \int_0^{2\pi} \sin(\theta)\, \mathrm{d}\theta = 0$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} \sin(\theta)\cos(n\theta)\, \mathrm{d}\theta = 0$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} \sin(\theta)\sin(n\theta)\, \mathrm{d}\theta = \begin{cases} \frac{1}{C_1 + D_1} & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Now we have

$$u(r,\theta) = B\ln(r) + \frac{1}{C_1 + D_1} \sin(\theta)\left[C_1 + D_1\right] = B\ln(r) + \sin(\theta)$$

Using the second boundary condition, we have:

$$u(2,\theta) = B\ln(2) + \sin(\theta) = \cos(\theta) \implies B = \frac{\cos(\theta) - \sin(\theta)}{\ln(2)}$$

The solution is

$$u(r,\theta) = \frac{\ln(r)}{\ln(2)}\big(\cos(\theta) - \sin(\theta)\big) + \sin(\theta).$$

$\square$

Problem 3. *Consider the following harmonic function in a unit disk with boundary value:*

$$\begin{cases} \Delta u = 0, & 0 < r < 1, \quad 0 < \theta < 2\pi, \\ u(1, \theta) = 1 + 3\sin(2\theta), & 0 < \theta < 2\pi. \end{cases}$$

Without *solving the solution explicitly, answer the following questions:*

(a) What is the maximum value of u in the unit disk $\{x^2 + y^2 \leq 1\}$?

(b) What is the value of u at the origin?

Solution.

(a) By the maximum principle, the maximum value of u in the unit disk is the maximum value of u on the boundary. The maximum value of u on the boundary is $1 + 3$, so the maximum value of u in the unit disk is 4.

(b) The origin is the center of the disk, so the value of u at the origin is the average at the boundary by the mean value property.

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} (1 + 3\sin(2\theta))\, d\theta \\ &= \frac{1}{2pi} \int_0^{2\pi} 1\, d\theta + \frac{3}{2\pi} \int_0^{2\pi} \sin(2\theta)\, d\theta \\ &= 1 + 0 \\ &= 1. \end{aligned}$$

□

Problem 4. *Let Ω be a bounded connected open set. The goal of this problem is to show the uniqueness of harmonic function with Dirichlet boundary condition by an energy method.*

(a) *Recall that for any vector field $\overrightarrow{A} = (A_1, A_2)$, $\operatorname{div} \overrightarrow{A} = A_{1x} + A_{2y}$, and for any scalar function f, $\nabla f = (f_x, f_y)$. Verify that*
$$\operatorname{div}(u\nabla u) = u_x^2 + u_y^2 + u\Delta u.$$

(b) *Recall the divergence theorem from multivariable calculus:*
$$\int_\Omega \operatorname{div} \overrightarrow{A} \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega} \overrightarrow{A} \cdot n \, \mathrm{d}S,$$
where n is the unit outer normal vector on $\partial\Omega$. Use the divergence theorem with $\overrightarrow{A} = u\nabla u$ to show that the only function satisfying
$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
is 0.

Solution.

(a)
$$\begin{aligned}
\operatorname{div}(u\nabla u) &= \operatorname{div}\left(u\left(u_x, u_y\right)\right) \\
&= \operatorname{div}\left(uu_x, uu_y\right) \\
&= (uu_x)_x + (uu_y)_y = u_x^2 + uu_{xx} + u_y^2 + uu_{yy} \\
&= u_x^2 + u_y^2 + u\left(u_{xx} + u_{yy}\right) \\
&= u_x^2 + u_y^2 + u\Delta u.
\end{aligned}$$

(b) By the divergence theorem,
$$\int_\Omega \operatorname{div}(u\nabla u) \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega} u\nabla u \cdot n \, \mathrm{d}S.$$
Since $u = 0$ on $\partial\Omega$, we have
$$\int_\Omega \operatorname{div}(u\nabla u) \, \mathrm{d}x \, \mathrm{d}y = 0.$$
By part (a), we have
$$\int_\Omega \left(u_x^2 + u_y^2 + u\Delta u\right) \mathrm{d}x \, \mathrm{d}y = 0.$$
Since u is harmonic, $\Delta u = 0$, so we have
$$\int_\Omega \left(u_x^2 + u_y^2\right) \mathrm{d}x \, \mathrm{d}y = 0.$$
Since $u_x^2 + u_y^2 \geq 0$, we must have that $u_x^2 + u_y^2 = 0$, so $u_x = u_y = 0$. Thus, u is constant, and since $u = 0$ on $\partial\Omega$, we must have that $u = 0$ in Ω.

$\square$