Homework 8

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APMA 0360 — Partial Differential Equations

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Problem 1.

(a) *Construct a twice differentiable function $v(x,t)$ such that*

$$v(0,t) = g(t), \quad v_x(\pi,t) = h(t).$$

(b) *Construct a twice differentiable function $v(x,t)$ such that*

$$v_x(0,t) = g(t), \quad v_x(\pi,t) = h(t).$$

For both parts, show some computations to justify that the functions you construct do satisfy those boundary conditions.

Solution.

(a) We can include a $\left(1 - \frac{x}{\pi}\right) g(t)$ term which has value $g(t)$ at $x = 0$ and 0 at $x = \pi$.

In order to satisfy the boundary condition $v_x(\pi,t) = h(t)$, we can solve the ODE in x:

$$v_x(\pi,t) = h(t) \implies v(\pi,t) = h(t)x + C(t).$$

We can choose $C(t) = 0$ to satisfy the boundary condition $v(0,t) = g(t)$. If we were to add the two equations, we would get

$$v(x,t) = \left(1 - \frac{x}{\pi}\right) g(t) + h(t)x$$

$$v_x(x,t) = -\frac{1}{\pi} g(t) + h(t).$$

We see that there is an extra $-\frac{1}{\pi}g(t)$ term in $v_x(x,t)$. We can eliminate this by adding a $\frac{1}{\pi}g(t)x$ term:

$$v(x,t) = \left(1 - \frac{x}{\pi}\right) g(t) + h(t)x + \frac{1}{\pi} g(t)x.$$

We can verify that this function satisfies the boundary conditions:

$$v(0,t) = g(t) + 0 + 0 = g(t),$$

$$v_x(\pi,t) = -\frac{1}{\pi} g(t) + h(t) + \frac{1}{\pi} g(t) = h(t).$$

(b) We construct a function $v_x(x,t)$ such that it is a linear interpolation between $g(t)$ and $h(t)$ at $x = 0$ and $x = \pi$, respectively.

$$v_x(x,t) = \left(1 - \frac{x}{\pi}\right) g(t) + \frac{x}{\pi} h(t) = g(t) + \frac{x}{\pi}\big(h(t) - g(t)\big).$$

We can integrate with respect to x to find $v(x,t)$:

$$v(x,t) = \int g(t) + \frac{x}{\pi}\big(h(t) - g(t)\big)\,\mathrm{d}x = g(t)x + \frac{x^2}{2\pi}\big(h(t) - g(t)\big) + C(t).$$

We can choose $C(t) = 0$. We can verify that this function satisfies the boundary conditions:

$$v_x(0,t) = g(t) + 0 = g(t),$$

$$v_x(\pi,t) = g(t) + \frac{\pi}{\pi}\big(h(t) - g(t)\big) = h(t).$$

$\square$

Problem 2. *Solve the heat equation with source:*

$$\begin{cases} u_t - u_{xx} = e^{-t}\sin(3x), & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = 1, & 0 < x < \pi. \end{cases}$$

Solution. Recall that for the heat equation with source $Q(x,t)$, homogeneous Dirichlet boundary conditions, and initial profile $f(x)$

$$\begin{cases} u_t - u_{xx} = Q(x,t), & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < \pi, \end{cases}$$

the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(nx),$$

where

$$\begin{aligned} A_n(t) &= e^{-n^2 t}\int_0^t e^{n^2 s} q_n(s)\,\mathrm{d}s + e^{-n^2 t} a_n, \\ q_n(t) &= \frac{2}{\pi}\int_0^{\pi} Q(x,t)\sin(nx)\,\mathrm{d}x \\ a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x. \end{aligned}$$

We first compute $q_n(t)$.

$$\begin{aligned} q_n(t) &= \frac{2}{\pi}\int_0^{\pi} e^{-t}\sin(3x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}e^{-t}\int_0^{\pi}\sin(3x)\sin(nx)\,\mathrm{d}x. \end{aligned}$$

By orthogonality of the sine functions, we know that the integral is $\frac{\pi}{2}$ when $n = 3$ and 0 otherwise. This gives

$$q_n(t) = \begin{cases} e^{-t}, & n = 3, \\ 0, & n \neq 3. \end{cases}$$

We now compute a_n.

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} \\ &= \frac{2}{n\pi}\left(1 - (-1)^n\right). \end{aligned}$$

We can now compute $A_n(t)$. For $n \neq 3$, the integral is 0, so we can write

$$
\begin{aligned}
u(x,t) &= e^{-3^2 t} \int_0^t e^{3^2 s} q_3(s)\, \mathrm{d}s \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} a_n \sin(nx) \\
&= e^{-9t} \int_0^t e^{9s} e^{-s}\, \mathrm{d}s \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \frac{2}{n\pi} (1 - (-1)^n) \sin(nx) \\
&= e^{-9t} \int_0^t e^{8s}\, \mathrm{d}s \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \frac{2}{n\pi} (1 - (-1)^n) \sin(nx) \\
&= e^{-9t} \left[\frac{1}{8} e^{8s} \right]_0^t \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \frac{2}{n\pi} (1 - (-1)^n) \sin(nx) \\
&= \frac{1}{8}(e^{-t} - e^{-9t}) \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \frac{2}{n\pi} (1 - (-1)^n) \sin(nx).
\end{aligned}
$$

□

Problem 3. *Solve the heat equation with inhomogeneous Dirichlet boundary condition:*

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = t, & t > 0, \\ u(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Solution. We devise a function $v(x,t)$ that satisfies the inhomogeneous boundary conditions. We choose a function that is a linear interpolation between 0 and t at $x = 0$ and $x = \pi$, respectively:

$$v(x,t) = \frac{x}{\pi}t.$$

Let $w(x,t) = u(x,t) - v(x,t)$. Then

$$\begin{aligned} w_t - w_{xx} &= (u_t - v_t) - (u_{xx} - v_{xx}) \\ &= (u_t - u_{xx}) - v_t + v_{xx} \\ &= 0 - \frac{x}{\pi} + 0 \\ &= -\frac{x}{\pi}. \end{aligned}$$

We can see that $w(x,t)$ satisfies

$$\begin{cases} w_t - w_{xx} = -\frac{x}{\pi}, & 0 < x < \pi, \quad t > 0, \\ w(0,t) = w(\pi,t) = 0, & t > 0, \\ w(x,0) = 0, & 0 < x < \pi. \end{cases}$$

We now have a heat equation with source $Q(x) = -\frac{x}{\pi}$ which is independent of t, homogeneous Dirichlet boundary conditions, and initial profile $f(x) = 0$. We recall the general solution is given by

$$w(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(nx),$$

where

$$\begin{aligned} A_n(t) &= \frac{q_n}{n^2}\left(1 - e^{-n^2 t}\right) + e^{-n^2 t}a_n, \\ q_n &= \frac{2}{\pi}\int_0^\pi Q(x)\sin(nx)\,\mathrm{d}x, \\ a_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x. \end{aligned}$$

Since $a_n = \frac{2}{\pi}\int_0^\pi 0\sin(nx)\,\mathrm{d}x = 0$, we have

$$A_n(t) = \frac{q_n}{n^2}\left(1 - e^{-n^2 t}\right).$$

We can compute q_n.

$$
\begin{aligned}
q_n &= \frac{2}{\pi}\int_0^\pi -\frac{x}{\pi}\sin(nx)\,\mathrm{d}x \\
&= -\frac{2}{\pi^2}\int_0^\pi x\sin(nx)\,\mathrm{d}x \\
&= -\frac{2}{\pi^2}\left[\left[-\frac{1}{n}x\cos(nx)\right]_0^\pi - \int_0^\pi -\frac{1}{n}\cos(nx)\,\mathrm{d}x\right] \\
&= -\frac{2}{\pi^2}\left(-\frac{1}{n}\pi(-1)^n\right) - \frac{2}{\pi^2}\left[\frac{1}{n^2}\sin(nx)\right]_0^\pi \\
&= \frac{2}{n\pi}(-1)^n.
\end{aligned}
$$

We now compute $A_n(t)$.

$$
\begin{aligned}
A_n(t) &= \frac{q_n}{n^2}\left(1-e^{-n^2t}\right) \\
&= \frac{2}{n^3\pi}(-1)^n\left(1-e^{-n^2t}\right).
\end{aligned}
$$

We have

$$
w(x,t) = \sum_{n=1}^{\infty}\frac{2}{n^3\pi}(-1)^n\left(1-e^{-n^2t}\right)\sin(nx).
$$

and

$$
u(x,t) = w(x,t) + v(x,t) = \sum_{n=1}^{\infty}\frac{2}{n^3\pi}(-1)^n\left(1-e^{-n^2t}\right)\sin(nx) + \frac{x}{\pi}t.
$$

□

Problem 4. *Solve the wave equation with a constant gravitational force:*

$$\begin{cases} u_{tt} - u_{xx} = -1, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = u_t(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Solution. Recall that for the wave equation with source $Q(x,t)$, homogeneous Dirichlet boundary conditions, and initial profiles $f(x)$ and $g(x)$

$$\begin{cases} u_{tt} - u_{xx} = Q(x,t), & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < \pi, \\ u_t(x,0) = g(x), & 0 < x < \pi, \end{cases}$$

the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(nx),$$

where

$$\begin{aligned} A_n(t) &= \left(a_n - \frac{1}{n} \int_0^t q_n(s) \sin(ns) \, \mathrm{d}s\right) \cos(nt) + \left(\frac{b_n}{n} + \frac{1}{n} \int_0^t q_n(s) \cos(ns) \, \mathrm{d}s\right) \sin(nt), \\ q_n(t) &= \frac{2}{\pi} \int_0^{\pi} Q(x,t) \sin(nx) \, \mathrm{d}x, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x, \\ b_n &= \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) \, \mathrm{d}x. \end{aligned}$$

Since $f(x) = 0$ and $g(x) = 0$, we have $a_n = b_n = 0$. We can compute q_n.

$$\begin{aligned} q_n &= \frac{2}{\pi} \int_0^{\pi} -1 \sin(nx) \, \mathrm{d}x \\ &= -\frac{2}{\pi} \left[-\frac{1}{n} \cos(nx)\right]_0^{\pi} \\ &= \frac{2}{n\pi} \left((-1)^n - 1\right). \end{aligned}$$

We can now compute $A_n(t)$.

$$
\begin{aligned}
A_n(t) &= \left(0 - \frac{1}{n}\int_0^t q_n \sin(ns)\,\mathrm{d}s\right)\cos(nt) + \left(\frac{0}{n} + \frac{1}{n}\int_0^t q_n \cos(ns)\,\mathrm{d}s\right)\sin(nt) \\
&= \left(-\frac{1}{n}q_n\int_0^t \sin(ns)\,\mathrm{d}s\right)\cos(nt) + \left(\frac{1}{n}q_n\int_0^t \cos(ns)\,\mathrm{d}s\right)\sin(nt) \\
&= \frac{q_n}{n}\left(\int_0^t \cos(ns)\,\mathrm{d}s\sin(nt) - \int_0^t \sin(ns)\,\mathrm{d}s\cos(nt)\right) \\
&= \frac{q_n}{n}\left(\left[\frac{1}{n}\sin(ns)\right]_0^t \sin(nt) - \left[-\frac{1}{n}\cos(ns)\right]_0^t \cos(nt)\right) \\
&= \frac{q_n}{n^2}\left(\sin^2(nt) + \cos^2(nt) - \cos(nt)\right) \\
&= \frac{q_n}{n^2}\left(1 - \cos(nt)\right) \\
&= \frac{2}{n^3\pi}\left((-1)^n - 1\right)\left(1 - \cos(nt)\right).
\end{aligned}
$$

We have the solution

$$
u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi}\left((-1)^n - 1\right)\left(1 - \cos(nt)\right)\sin(nx).
$$

□