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Dylan Hu	Problem 2	3
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Problem 1. Let $u(x, t)$ be the solution of the heat equation on the real line

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where

$$f(x) = \begin{cases} 1, & -1 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Compute $\hat{u}(\kappa, t)$, the Fourier transform of $u(x, t)$, explicitly. (You do not need to find the solution $u(x, t)$.)

Solution. Recall that the Fourier transform of the heat equation in general is

$$\hat{u}(\kappa, t) = \hat{u}(\kappa, 0)e^{-D\kappa^2 t}.$$

Here, we have $D = 1$ and $\hat{u}(\kappa, 0) = \hat{f}(\kappa)$. So, it remains to compute $\hat{f}(\kappa)$. We have

$$\begin{aligned} \hat{f}(\kappa) &= \int_{-\infty}^{\infty} e^{i\kappa x} f(x) \, dx \\ &= \int_{-1}^1 e^{i\kappa x} \, dx \\ &= \frac{1}{i\kappa} [e^{i\kappa x}]_{-1}^1 \\ &= \frac{1}{i\kappa} (e^{i\kappa} - e^{-i\kappa}) \\ &= \frac{1}{i\kappa} [\cos(\kappa) + i \sin(\kappa) - \cos(-\kappa) - i \sin(-\kappa)] \\ &= \frac{1}{i\kappa} [\cos(\kappa) + i \sin(\kappa) - \cos(\kappa) + i \sin(\kappa)] \\ &= \frac{2i \sin(\kappa)}{i\kappa} \\ &= \frac{2 \sin(\kappa)}{\kappa} \\ &= 2 \operatorname{sinc}(\kappa). \end{aligned}$$

Thus, we have

$$\hat{u}(\kappa, t) = 2 \operatorname{sinc}(\kappa) e^{-\kappa^2 t}.$$

□

Problem 2. Solve the damped wave equation

$$\begin{cases} u_{tt} + 2u_t + u = u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = \frac{1}{1+x^2}, & x \in \mathbb{R}, \\ u_t(x, 0) = 1, & x \in \mathbb{R}. \end{cases}$$

Hint. Find the equation satisfied by $v(x, t) = e^t u(x, t)$ first, and use the solution to that problem to find $u(x, t)$.

Solution.

$$\begin{aligned} v(x, t) &= e^t u(x, t), \\ v_t &= e^t u_t + e^t u, \\ v_{tt} &= e^t u_{tt} + 2e^t u_t + e^t u \\ &= e^t (u_{tt} + 2u_t + u) \\ &= e^t u_{xx} \\ &= v_{xx}. \end{aligned}$$

We see that $v(x, t)$ satisfies the undamped wave equation. The initial profiles are

$$v(x, 0) = \frac{e^0}{1+x^2} = \frac{1}{1+x^2}, \quad v_t(x, 0) = e^0 (u_t(x, 0) + u(x, 0)) = 1 + \frac{1}{1+x^2}.$$

Now, we have the system

$$\begin{cases} v_{tt} = v_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v(x, 0) = \frac{1}{1+x^2}, & x \in \mathbb{R}, \\ v_t(x, 0) = 1 + \frac{1}{1+x^2}, & x \in \mathbb{R}. \end{cases}$$

which we can solve using d'Alembert's formula. Recall that the solution in general is

$$v(x, t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Here, we have $f(x) = \frac{1}{1+x^2}$, $g(x) = 1 + \frac{1}{1+x^2}$, and $c = 1$.

$$v(x, t) = \frac{1}{2} \left(\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2} \right) + \frac{1}{2} \int_{x-t}^{x+t} \left(1 + \frac{1}{1+s^2} \right) ds.$$

We will leave the first two terms and compute the integral first.

$$\begin{aligned} \int_{x-t}^{x+t} \left(1 + \frac{1}{1+s^2} \right) ds &= [s + \arctan(s)]_{x-t}^{x+t} \\ &= (x+t + \arctan(x+t)) - (x-t + \arctan(x-t)) \\ &= 2t + \arctan(x+t) - \arctan(x-t). \end{aligned}$$

Thus, we have

$$v(x, t) = \frac{1}{2} \left(\frac{1}{1 + (x + t)^2} + \frac{1}{1 + (x - t)^2} + 2t + \arctan(x + t) - \arctan(x - t) \right).$$

Finally, we have

$$u(x, t) = e^{-t} v(x, t) = \frac{1}{2e^t} \left(\frac{1}{1 + (x + t)^2} + \frac{1}{1 + (x - t)^2} + 2t + \arctan(x + t) - \arctan(x - t) \right).$$

If we let $\xi = x + t$ and $\eta = x - t$, we can write this as

$$u(\xi, \eta) = \frac{1}{2} e^{\frac{\eta - \xi}{2}} \left(\frac{1}{1 + \xi^2} + \frac{1}{1 + \eta^2} + \xi - \eta + \arctan(\xi) - \arctan(\eta) \right).$$

□

Problem 3. Find the general solution of

$$\begin{cases} u_{xx} - 2u_x + u_{yy} = 0, & 0 < x, y < \pi, \\ u(x, 0) = 0, & 0 < x < \pi, \\ u(x, \pi) = 0, & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x, y) = X(x)Y(y)$. Then

$$X''Y - 2X'Y + XY'' = 0 \implies \frac{2X' - X''}{X} = \frac{Y''}{Y} = \lambda.$$

We solve the ODE in y first, as we have the boundary conditions in y .

$$\begin{cases} Y'' = \lambda Y, & 0 < y < \pi, \\ Y(0) = Y(\pi) = 0. \end{cases}$$

Recall that the general solution for the homogeneous Dirichlet boundary conditions with $\lambda = -n^2$ is

$$Y_n(y) = c_n \sin(ny), \quad n = 1, 2, 3, \dots$$

where c is a constant.

Now, we solve the ODE in x .

$$\frac{2X' - X''}{X} = \lambda \implies X'' - 2X' + \lambda X = 0.$$

We have the auxiliary equation

$$r^2 - 2r + \lambda = 0 \implies r = 1 \pm \sqrt{1 - \lambda} = 1 \pm \sqrt{1 + n^2}.$$

Since the roots are real and distinct, the general solution is

$$\begin{aligned} X_n(x) &= \tilde{a}_n e^{(1+\sqrt{1+n^2})x} + \tilde{b}_n e^{(1-\sqrt{1+n^2})x} \\ &= e^x \left(\tilde{a}_n e^{\sqrt{1+n^2}x} + \tilde{b}_n e^{-\sqrt{1+n^2}x} \right) \\ &= e^x \left(\hat{a}_n \cosh(\sqrt{1+n^2}x) + \hat{b}_n \sinh(\sqrt{1+n^2}x) \right). \end{aligned}$$

The general solution is then

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} e^x \left(\hat{a}_n \cosh(\sqrt{1+n^2}x) + \hat{b}_n \sinh(\sqrt{1+n^2}x) \right) c_n \sin(ny) \\ &= e^x \sum_{n=1}^{\infty} \left(a_n \cosh(\sqrt{1+n^2}x) + b_n \sinh(\sqrt{1+n^2}x) \right) \sin(ny). \end{aligned}$$

□

Problem 4. Let $u(x, t)$ be the solution of the following heat equation with mixed boundary conditions

$$\begin{cases} u_t = u_{xx} + 3x^2, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(\pi, t) = 1, & t > 0, \\ u(x, 0) = \sin(x), & 0 < x < \pi. \end{cases}$$

Find the limiting function $u_\infty(x) = \lim_{t \rightarrow \infty} u(x, t)$.

Solution. We can devise functions $w(x, t)$ and $v(x)$ such that

$$w(x, t) = u(x, t) - v(x)$$

and

$$\begin{cases} w_t = w_{xx} + Q(x), & 0 < x < \pi, \quad t > 0, \\ w(0, t) = 0, & t > 0, \\ w_x(\pi, t) = 0, & t > 0. \end{cases}$$

so that we can find the limiting function $w_\infty(x) = \lim_{t \rightarrow \infty} w(x, t)$ by solving the ODE

$$\begin{cases} -w_\infty''(x) = Q(x), & 0 < x < \pi, \\ w_\infty(0) = 0, \\ w_\infty'(\pi) = 0. \end{cases}$$

which leads to the limiting function $u_\infty(x) = w_\infty(x) + v(x)$.

Let $v(x) = \frac{x^2}{2\pi}$. Then $v'(x) = \frac{x}{\pi}$, $v''(x) = \frac{1}{\pi}$, and $v(0) = 0$, $v'(\pi) = 1$. We have

$$\begin{cases} w_t = w_{xx} + 3x^2 + \frac{1}{\pi}, & 0 < x < \pi, \quad t > 0, \\ w(0, t) = 0, & t > 0, \\ w_x(\pi, t) = 0, & t > 0, \end{cases}$$

and

$$\begin{cases} -w_\infty''(x) = 3x^2 + \frac{1}{\pi}, & 0 < x < \pi, \\ w_\infty(0) = 0, \\ w_\infty'(\pi) = 0. \end{cases}$$

$$w_\infty''(x) = -3x^2 - \frac{1}{\pi} \implies w_\infty'(x) = -x^3 - \frac{x}{\pi} + c_1 \implies w_\infty(x) = -\frac{x^4}{4} - \frac{x^2}{2\pi} + c_1x + c_2.$$

$$w_\infty(0) = 0 \implies c_2 = 0, \quad w_\infty(\pi) = 0 \implies -\frac{\pi^4}{4} - \frac{\pi^2}{2\pi} + c_1\pi = 0 \implies c_1 = \frac{\pi^3 + 2}{4}.$$

So we have

$$w_{\infty}(x) = -\frac{x^4}{4} - \frac{x^2}{2\pi} + \frac{\pi^3 + 2}{4}x.$$

and

$$\begin{aligned} u_{\infty}(x) &= w_{\infty}(x) + v(x) \\ &= -\frac{x^4}{4} - \frac{x^2}{2\pi} + \frac{\pi^3 + 2}{4}x + \frac{x^2}{2\pi} \\ &= \frac{(\pi^3 + 2)x - x^4}{4}. \end{aligned}$$

□

Problem 5. Solve the following wave equation with inhomogeneous mixed boundary conditions

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(\pi, t) = \sin\left(\frac{3t}{2}\right), & t > 0, \\ u(x, 0) = 0, & 0 < x < \pi, \\ u_t(x, 0) = 0, & 0 < x < \pi. \end{cases}$$

Solution.

□