

Practice Midterm 2

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Equations

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Problem 1. Compute the Fourier sine series of $x^2 + 1$ on $[0, \pi]$.

Solution.

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi (x^2 + 1) \sin(nx) \, dx \\
 &= \frac{2}{\pi} \int_0^\pi (x^2 + 1) \frac{\partial}{\partial x} \left(-\frac{\cos(nx)}{n} \right) \, dx \\
 &= \frac{2}{\pi} \left[(x^2 + 1) \left(-\frac{\cos(nx)}{n} \right) \Big|_0^\pi - \int_0^\pi \left(-\frac{\cos(nx)}{n} \right) \frac{\partial}{\partial x} (x^2 + 1) \, dx \right] \\
 &= \frac{2}{\pi} \left[(\pi^2 + 1) \left(-\frac{\cos(n\pi)}{n} \right) - \left(-\frac{\cos(0)}{n} \right) \right] - \frac{2}{\pi} \int_0^\pi \left(-\frac{\cos(nx)}{n} \right) 2x \, dx \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) + \frac{4}{n\pi} \int_0^\pi x \cos(nx) \, dx \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) + \frac{4}{n\pi} \left[x \left(\frac{\sin(nx)}{n} \right) \Big|_0^\pi - \int_0^\pi \frac{\sin(nx)}{n} \, dx \right] \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) - \frac{4}{n^2\pi} \int_0^\pi \sin(nx) \, dx \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) - \frac{4}{n^2\pi} \left[-\frac{\cos(nx)}{n} \Big|_0^\pi \right] \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) - \frac{4}{n^2\pi} \left(-\frac{(-1)^n - 1}{n} \right) \\
 &= \frac{2}{n\pi} (1 - (\pi^2 + 1)(-1)^n) - \frac{4}{n^3\pi} (1 - (-1)^n)
 \end{aligned}$$

The Fourier sine series of $x^2 + 1$ on $[0, \pi]$ is then

$$\sum_{n=1}^{\infty} a_n \sin(nx).$$

We can check that the Fourier series approximates $x^2 + 1$ in $[0, \pi]$:

□

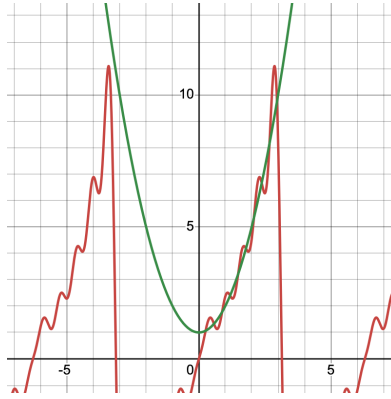


Figure 1: <https://www.desmos.com/calculator/a7wlsqmh2f>

Problem 2. Compute the Fourier sine series of $\cos(x)$ on $[0, \pi]$. Does it converge to $\cos(x)$ pointwise on $[0, \pi]$? If not, at which points does it not converge?

Solution.

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi \cos(x) \sin(nx) \, dx \\
 &= \frac{2}{\pi} \int_0^\pi \cos(x) \frac{\partial}{\partial x} \left(-\frac{\cos(nx)}{n} \right) \, dx \\
 &= \frac{2}{\pi} \left[\cos(x) \left(-\frac{\cos(nx)}{n} \right) \Big|_0^\pi - \int_0^\pi \left(-\frac{\cos(nx)}{n} \right) \frac{\partial}{\partial x} \cos(x) \, dx \right] \\
 &= \frac{2}{n\pi} (-\cos(\pi) \cos(n\pi) + \cos(0) \cos(0)) - \frac{2}{n\pi} \int_0^\pi \cos(nx) \sin(x) \, dx \\
 &= \frac{2}{n\pi} ((-1)^n + 1) - \frac{2}{n\pi} \int_0^\pi \sin(x) \frac{\partial}{\partial x} \left(\frac{\sin(nx)}{n} \right) \, dx \\
 &= \frac{2}{n\pi} ((-1)^n + 1) - \frac{2}{n\pi} \left[\sin(x) \left(\frac{\sin(nx)}{n} \right) \Big|_0^\pi - \int_0^\pi \frac{\sin(nx)}{n} \cos(x) \, dx \right] \\
 &= \frac{2}{n\pi} ((-1)^n + 1) + \frac{2}{n^2\pi} \int_0^\pi \cos(x) \sin(nx) \, dx \\
 &= \frac{2}{n\pi} ((-1)^n + 1) + \frac{1}{n^2} a_n \\
 n^2 a_n &= \frac{2n}{\pi} ((-1)^n + 1) + a_n \\
 (n^2 - 1) a_n &= \frac{2n}{\pi} ((-1)^n + 1) \\
 a_n &= \frac{2n((-1)^n + 1)}{\pi(n^2 - 1)}
 \end{aligned}$$

We can handle the special case of $n = 1$ separately:

$$a_1 = \frac{2}{\pi} \int_0^\pi \cos(x) \sin(x) \, dx$$

We can use the fact that $\sin(x)$ and $\cos(x)$ are orthogonal, so $\langle \cos(x), \sin(x) \rangle = \int \cos(x) \sin(x) \, dx = 0$. Therefore, $a_1 = 0$.

The Fourier sine series of $\cos(x)$ on $[0, \pi]$ is then

$$\sum_{n=2}^{\infty} a_n \sin(nx)$$

where

$$a_n = \frac{2n((-1)^n + 1)}{\pi(n^2 - 1)}.$$

We can determine the pointwise convergence of the Fourier series by only checking the convergence of the series at the endpoints of the interval, as $\cos(x)$ is continuous everywhere:

$$\lim_{n \rightarrow \infty} \cos(0) = 1 \neq 0 = \lim_{n \rightarrow \infty} \sum_{n=2}^{\infty} a_n \sin(nx)$$

$$\lim_{n \rightarrow \infty} \cos(\pi) = -1 \neq 0 = \lim_{n \rightarrow \infty} \sum_{n=2}^{\infty} a_n \sin(nx)$$

Therefore, the Fourier series does not converge to $\cos(x)$ pointwise on $[0, \pi]$, as it does not converge at $x = 0$ and $x = \pi$. \square

Problem 3. Solve

$$\begin{cases} u_t = tu_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u(\pi, t) = 0, & t > 0, \\ u(x, 0) = x, & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x, t) = X(x)T(t)$. Then

$$u_t = tu_{xx} \implies X(x)T'(t) = tX''(x)T(t) \implies \frac{X''(x)}{X(x)} = \frac{1}{t} \frac{T'(t)}{T(t)} = \lambda$$

We recall that a solution for the homogeneous Dirichlet boundary conditions is

$$X(x) = c \sin(nx)$$

for a constant c , $n = 1, 2, 3, \dots$, and $\lambda = -n^2$. We can then solve the ODE in $T(t)$:

$$\frac{1}{t} \frac{T'(t)}{T(t)} = -n^2$$

$$\frac{T'(t)}{T(t)} = -n^2 t$$

$$\frac{d}{dt} \ln T(t) = -n^2 t$$

$$\ln T(t) = -\frac{n^2}{2} t^2 + c$$

$$T(t) = e^{-\frac{n^2}{2} t^2 + c} = Ce^{-\frac{n^2}{2} t^2}$$

The general solution is then

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2}{2} t^2} \sin(nx)$$

We can determine the coefficients a_n by using the initial condition:

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x$$

Recall from lecture that the Fourier sine series of x on $[0, \pi]$ is

$$x \stackrel{L^2}{=} \sum_{n=1}^{\infty} a_n \sin(nx)$$

where

$$a_n = (-1)^{n+1} \frac{2}{n}.$$

Therefore, the solution to the PDE is

$$u(x, t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} e^{-\frac{n^2}{2} t^2} \sin(nx).$$

□

Problem 4. Solve

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u(\pi, t) = 0, & t > 0, \\ u(x, 0) = x, & 0 < x < \pi, \\ u_t(x, 0) = \sin(4x) + 5 \sin(5x) & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x, t) = X(x)T(t)$. Then

$$u_{tt} = 4u_{xx} \implies X(x)T''(t) = 4X''(x)T(t) \implies \frac{X''(x)}{X(x)} = \frac{1}{4} \frac{T''(t)}{T(t)} = \lambda$$

We recall that a solution for the homogeneous Dirichlet boundary conditions is

$$X(x) = c \sin(nx)$$

for a constant c , $n = 1, 2, 3, \dots$, and $\lambda = -n^2$. We can then solve the ODE in $T(t)$:

$$\frac{1}{4} \frac{T''(t)}{T(t)} = -n^2$$

$$T''(t) = -4n^2 T(t)$$

We recall that the general solution to this ODE is

$$T(t) = c_1 \cos(2nt) + c_2 \sin(2nt)$$

The general solution is then

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(2nt) + b_n \sin(2nt)) \sin(nx)$$

We can determine the coefficients a_n and b_n by using the initial conditions:

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x$$

Recall from lecture that the Fourier sine series of x on $[0, \pi]$ is

$$x \stackrel{L^2}{=} \sum_{n=1}^{\infty} a_n \sin(nx)$$

where

$$a_n = (-1)^{n+1} \frac{2}{n}.$$

Turning to the second initial condition:

$$u_t(x, t) = \sum_{n=1}^{\infty} 2n(-a_n \sin(2nt) + b_n \cos(2nt)) \sin(nx)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} 2nb_n \sin(nx) = \sin(4x) + 5 \sin(5x)$$

Comparing terms, we see that $b_4 = \frac{1}{8}$, $b_5 = \frac{1}{2}$, and $b_k = 0$ for $k \neq 4, 5$.

Therefore, the solution to the PDE is

$$u(x, t) = \frac{1}{8} \sin(8t) \sin(4x) + \frac{1}{2} \sin(10t) \sin(5x) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \cos(2nt) \sin(nx).$$

□

Problem 5. Use the energy method to show that the only solution to the heat-like equation with periodic boundary conditions

$$\begin{cases} u_t = 4u_{xx} - 3u, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = u(\pi, t), & t > 0, \\ u_x(0, t) = u_x(\pi, t), & t > 0, \\ u(x, 0) = 0, & 0 < x < \pi \end{cases}$$

is zero.

Solution. We can multiply the PDE by u and integrate over $[0, \pi]$:

$$\begin{aligned} \int_0^\pi u_t u \, dx &= \int_0^\pi (4u_{xx} - 3u)u \, dx \\ \int_0^\pi \frac{\partial}{\partial t} \left(\frac{1}{2} u^2 \right) dx &= \int_0^\pi 4u_{xx}u \, dx - 3 \int_0^\pi u^2 \, dx \\ \frac{d}{dt} \int_0^\pi \frac{1}{2} u^2 \, dx &= \int_0^\pi 4u_{xx}u \, dx - 3 \int_0^\pi u^2 \, dx \end{aligned}$$

Let $E(t) = \int_0^\pi \frac{1}{2} u^2 \, dx \geq 0$. Then

$$\begin{aligned} \frac{dE}{dt} &= 4 \int_0^\pi u_{xx}u \, dx - 3 \int_0^\pi u^2 \, dx \\ &= 4 [u_x u]_0^\pi - 4 \int_0^\pi u_x^2 \, dx - 3 \int_0^\pi u^2 \, dx \\ &= 4 [u_x(\pi, t)u(\pi, t) - u_x(0, t)u(0, t)] - 4 \int_0^\pi u_x^2 \, dx - 3 \int_0^\pi u^2 \, dx \\ &= 0 - 4 \int_0^\pi u_x^2 \, dx - 3 \int_0^\pi u^2 \, dx \\ &\leq 0 \end{aligned}$$

Then $0 \leq E(t) \leq E(0) = 0$, so $E(t) = 0$. Therefore, $u(x, t) = 0$. □

Problem 6. Consider the system of equations

$$\begin{cases} u_t = 4u_{xx} - 3u_x - 5uv^2, \\ v_t = -3uv^2. \end{cases}$$

Find a system of ODEs for U and V , where U, V are the traveling wave profiles $u(x, t) = U(z)$, $v(x, t) = V(z)$, $z = x - ct$.

Solution. We have the following system:

$$\begin{cases} u(x, t) = U(x - ct) = U(z), \\ v(x, t) = V(x - ct) = V(z). \end{cases} \implies \begin{cases} u_t = \frac{dU}{dz} \frac{dz}{dt} = -cU'(z), \\ v_t = \frac{dV}{dz} \frac{dz}{dt} = -cV'(z), \\ u_x = \frac{dU}{dz} \frac{dz}{dx} = U'(z), \\ u_{xx} = \frac{dU'}{dz} \frac{dz}{dx} = U''(z). \end{cases}$$

Now we can substitute these into the original system:

$$\begin{cases} -cU'(z) = 4U''(z) - 3U'(z) - 5UV^2, \\ -cV'(z) = -3UV^2. \end{cases}$$

□

Problem 7. Use separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi, \\ u(x, 0) = 0, & 0 < x < \pi, \\ u(x, \pi) = 0, & 0 < x < \pi, \\ u_x(0, y) = \sin(2y), & 0 < y < \pi, \\ u_x(\pi, y) = 0, & 0 < y < \pi, \end{cases}$$

Note: $(\cosh(x))' = \sinh(x)$, $(\sinh(x))' = \cosh(x)$. No negative signs for the derivatives of hyperbolic sine/cosine functions.

Solution. Let $u(x, y) = X(x)Y(y)$. Then

$$u_{xx} + u_{yy} = 0 \implies X''Y + XY'' = 0 \implies -\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

We first solve the ODE in $Y(y)$:

$$\begin{cases} Y''(y) = \lambda Y(y), \\ Y(0) = 0, \\ Y(\pi) = 0. \end{cases}$$

Recall that a solution to the homogeneous Dirichlet boundary conditions is

$$Y(y) = c \sin(ny)$$

for a constant c , $n = 1, 2, 3, \dots$, and $\lambda = -n^2$. We can then solve the ODE in $X(x)$:

$$X''(x) = -\lambda X(x) = n^2 X(x)$$

Recall the general solution

$$X(x) = Ae^{nx} + Be^{-nx}$$

which can be rewritten using the hyperbolic sine and cosine functions as

$$X(x) = \tilde{A} \sinh(nx) + \tilde{B} \cosh(nx).$$

The solution with the homogeneous Dirichlet boundary conditions is then

$$u(x, y) = \sum_{n=1}^{\infty} (a_n \sinh(nx) + b_n \cosh(nx)) \sin(ny).$$

We now determine the coefficients a_n and b_n with the other boundary conditions.

First, we compute the partial derivative of u with respect to x :

$$u_x(x, y) = \sum_{n=1}^{\infty} n(a_n \cosh(nx) + b_n \sinh(nx)) \sin(ny)$$

With the $u_x(0, y) = \sin(2y)$ boundary condition:

$$u_x(0, y) = \sum_{n=1}^{\infty} n a_n \sin(ny) = \sin(2y)$$

Comparing terms, we see that $a_2 = \frac{1}{2}$ and $a_k = 0$ for $k \neq 2$.

With the $u_x(\pi, y) = 0$ boundary condition:

$$u_x(\pi, y) = \sum_{n=1}^{\infty} n(a_n \cosh(n\pi) + b_n \sinh(n\pi)) \sin(ny) = 0$$

We can let $c_n = n(a_n \cosh(n\pi) + b_n \sinh(n\pi))$ and rewrite it as

$$\sum_{n=1}^{\infty} c_n \sin(ny) = 0$$

which is a Fourier sine series that implies $c_n = 0$ for all n .

Now, we have

$$c_n = n(a_n \cosh(n\pi) + b_n \sinh(n\pi)) = 0$$

However, we know that n is strictly positive, so we must have

$$a_n \cosh(n\pi) + b_n \sinh(n\pi) = 0$$

From earlier, we recall that $a_2 = \frac{1}{2}$ and $a_k = 0$ for $k \neq 2$. Therefore, $b_k = 0$ for $k \neq 2$, and $b_2 < 0$. We can solve for b_2 :

$$\begin{aligned} \frac{1}{2} \cosh(2\pi) + b_2 \sinh(2\pi) &= 0 \\ b_2 &= -\frac{1}{2} \frac{\cosh(2\pi)}{\sinh(2\pi)} = -\frac{1}{2} \coth(2\pi) \end{aligned}$$

Therefore, the solution to the PDE is

$$\begin{aligned} u(x, y) &= \frac{1}{2} \sinh(2x) - \frac{1}{2} \frac{\cosh(2\pi)}{\sinh(2\pi)} \cosh(2x) \sin(2y) \\ &= \frac{1}{2} \sinh(2x) - \frac{1}{2} \coth(2\pi) \cosh(2x) \sin(2y). \end{aligned}$$

□