Homework 4

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Problem 1. *Verify the superposition principle: if u_1 and u_2 are two solutions to*

$$\begin{cases} u_t - u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \end{cases}$$

then so is $a_1u_1 + a_2u_2$ for any constants a_1, a_2. Is it still true if the boundary condition is $u(0,t) = u(\pi,t) = 1$?

Solution. We can verify the superposition principle by plugging in u_1 and u_2 into the heat equation and adding them together. We have

$$\begin{aligned}(a_1u_1 + a_2u_2)_t - (a_1u_1 + a_2u_2)_{xx} &= a_1u_{1t} + a_2u_{2t} - a_1u_{1xx} - a_2u_{2xx} \\ &= a_1u_{1xx} + a_2u_{2xx} - a_1u_{1xx} - a_2u_{2xx} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(a_1u_1 + a_2u_2)(0,t) &= a_1u_1(0,t) + a_2u_2(0,t) \\ &= a_1(0) + a_2(0) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(a_1u_1 + a_2u_2)(\pi,t) &= a_1u_1(\pi,t) + a_2u_2(\pi,t) \\ &= a_1(0) + a_2(0) \\ &= 0\end{aligned}$$

so $a_1u_1 + a_2u_2$ is also a solution.

If the boundary condition is $u(0,t) = u(\pi,t) = 1$, then we have

$$\begin{aligned}(a_1u_1 + a_2u_2)(0,t) &= a_1u_1(0,t) + a_2u_2(0,t) \\ &= a_1(1) + a_2(1) \\ &= a_1 + a_2\end{aligned}$$

$a_1 + a_2$ is not 1 for all a_1, a_2, so the superposition principle does not hold. □

Problem 2. *Use separation of variables to solve*

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0, \\ u(x,0) = 1 + 2\cos(3x), & 0 < x < \pi \end{cases}$$

explicitly. You don't need to go through all 3 cases.

Solution. We look for a solution of the form $u(x,t) = X(x)T(t)$. Plugging this into the heat equation, we have

$$X(x)T'(t) - X''(x)T(t) = 0$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

The boundary conditions give us

$$u_x(0,t) = X'(0)T(t) = 0 \implies X'(0) = 0$$

$$u_x(\pi,t) = X'(\pi)T(t) = 0 \implies X'(\pi) = 0$$

We now have a system of ODEs for $X(x)$:

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < \pi, \\ X'(0) = X'(\pi) = 0, \end{cases}$$

Let us define

$$V(x) = \begin{bmatrix} X'(x) \\ X(x) \end{bmatrix}$$

Then we have

$$V'(x) = \begin{bmatrix} X''(x) \\ X'(x) \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X'(x) \\ X(x) \end{bmatrix}$$

So we have

$$V'(x) = A\,V(x), \quad A = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}$$

We can find the eigenvalues and eigenvectors of A:

$$\det(A - \Lambda I) = \det \begin{bmatrix} -\Lambda & \lambda \\ 1 & -\Lambda \end{bmatrix} = \Lambda^2 - \lambda = 0$$

So the eigenvalues are $\pm\sqrt{\lambda}$.

Case $=$ 0. Then $\lambda_1 = \lambda_2 = 0$.

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

gives us the eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can find a generalized eigenvector by solving

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives us $v_1 = 1$. So the generalized eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have the solution

$$V(x) = a_1 e^{0x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_2 e^{0x} \left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

which gives us

$$X(x) = a_1 + a_2 x$$

We have

$$X'(x) = a_2$$

The boundary conditions give us

$$X'(0) = X'(\pi) = a_2 = 0 \implies a_2 = 0$$

So the solution is

$$X(x) = a$$

for any constant a.

Case < 0. Let $\lambda = -\omega^2$, where $\omega > 0$. Then the eigenvalues are $\pm i\omega$. This gives the solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

for any constants a_1, a_2. We have

$$X'(x) = -a_1\omega \sin(\omega x) + a_2\omega \cos(\omega x)$$

and the boundary conditions give us

$$X'(0) = -a_1\omega \sin(0) + a_2\omega \cos(0) = 0 \implies a_2 = 0$$

$$X'(\pi) = -a_1\omega \sin(\omega\pi) = 0 \implies \omega = n$$

where $n \in \mathbb{Z}^+$. So $X(x) = a_1 \cos(nx)$ and $\lambda = -n^2$. We now turn to the ODE for $T(t)$:

$$\frac{T'(t)}{T(t)} = \lambda = -n^2$$

$$T'(t) = -n^2 T(t)$$

$$T(t) = Ce^{-n^2 t}$$

where C is a constant. A solution is then

$$u(x,t) = X(x)T(t) = Ce^{-n^2 t} \cos(nx)$$

By the principle of superposition, the general solution is

$$u(x,t) = a + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx)$$

where a_n are constants. We can find the constants by using the initial condition:

$$u(x,0) = a + \sum_{n=1}^{\infty} a_n \cos(nx) = 1 + 2\cos(3x)$$

We have $a = 1$, $a_3 = 2$, and $a_n = 0$ for all other n. So the solution is

$$u(x,t) = 1 + 2e^{-9t} \cos(3x)$$

□

Problem 3. *Consider the following heat-like equation:*

$$\begin{cases} tu_t - u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0, \end{cases}$$

Use separation of variables to write the solution as an infinite series. You don't need to go through all 3 cases.

Solution. Let $u(x,t) = X(x)T(t)$. Then

$$tX(x)T'(t) - X''(x)T(t) = 0$$
$$\frac{tT'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

The boundary conditions give us

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$
$$u(\pi,t) = X(\pi)T(t) = 0 \implies X(\pi) = 0$$

We now have a system of ODEs for $X(x)$:

$$\begin{cases} X''(x) - \lambda X(x) = 0, \quad 0 < x < \pi, \\ X(0) = X(\pi) = 0, \end{cases}$$

Like in the previous problem, we arrive at the solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

where $\lambda = -\omega^2$ and $\omega > 0$. Using the boundary conditions, we have

$$X(0) = a_1 \cos(0) + a_2 \sin(0) = a_1 = 0$$
$$X(\pi) = a_2 \sin(\omega\pi) = 0 \implies \omega = n$$

where $n \in \mathbb{Z}^+$. So $X(x) = a\sin(nx)$ and $\lambda = -n^2$.

We now turn to the ODE for $T(t)$:

$$\frac{tT'(t)}{T(t)} = \lambda = -n^2$$
$$tT'(t) = -n^2 T(t)$$
$$\frac{T'(t)}{T(t)} = -\frac{n^2}{t}$$
$$T'(t) = -\frac{n^2}{t}T(t)$$
$$T'(t) + \frac{n^2}{t}T(t) = 0$$

This is a first-order linear ODE with variable coefficients. We can solve it using an integrating factor:

$$\mu(t) = e^{\int \frac{n^2}{t}\,dt} = e^{n^2 \ln(t)} = e^{\ln(t^{n^2})} = t^{n^2}$$

$$T'(t)t^{n^2} + \frac{n^2}{t}T(t)t^{n^2} = 0$$
$$\left(T(t)t^{n^2}\right)' = 0$$
$$T(t)t^{n^2} = C$$
$$T(t) = Ct^{-n^2}$$

where C is a constant.

A solution is then

$$u(x,t) = X(x)T(t) = C\sin(nx)t^{-n^2}$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)t^{-n^2}$$

where a_n are constants.

□

Problem 4. *Consider the heat equation with mixed boundary conditions*

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0. \end{cases}$$

Use separation of variables and go through all 3 cases to write the solution as an infinite series.

Solution. As before, we let $u(x,t) = X(x)T(t)$. Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

The boundary conditions give us

$$u_x(0,t) = X'(0)T(t) = 0 \implies X'(0) = 0$$

$$u(\pi,t) = X(\pi)T(t) = 0 \implies X(\pi) = 0$$

We now have a system of ODEs for $X(x)$:

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < \pi, \\ X'(0) = 0 \\ X(\pi) = 0 \end{cases}$$

As before, the eigenvalues are $\pm\sqrt{\lambda}$.

Case >0. Let $\lambda = \omega^2$, where $\omega > 0$. Then $\lambda_1 = \omega^2$ and $\lambda_2 = -\omega^2$.

$$A - \lambda_1 I = \begin{bmatrix} -\omega & \omega^2 \\ 1 & -\omega \end{bmatrix}$$

so an eigenvector corresponding to λ_1 is $\begin{bmatrix} \omega \\ 1 \end{bmatrix}$.

$$A - \lambda_2 I = \begin{bmatrix} \omega & \omega^2 \\ 1 & \omega \end{bmatrix}$$

so an eigenvector corresponding to λ_2 is $\begin{bmatrix} -\omega \\ 1 \end{bmatrix}$. We have the solution

$$V(x) = a_1 e^{\omega x} \begin{bmatrix} \omega \\ 1 \end{bmatrix} + a_2 e^{-\omega x} \begin{bmatrix} -\omega \\ 1 \end{bmatrix}$$

which gives us

$$X(x) = a_1 e^{\omega x} + a_2 e^{-\omega x}$$

for any constants a_1, a_2. We have

$$X'(x) = a_1 \omega e^{\omega x} - a_2 \omega e^{-\omega x}$$

The boundary conditions give us

$$X'(0) = a_1\omega - a_2\omega = 0 \implies a_1 = a_2$$

$$X(\pi) = a_1 e^{\omega\pi} + a_2 e^{-\omega\pi} = a_1 e^{\omega\pi} + a_1 e^{-\omega\pi} = a_1(e^{\omega\pi} + e^{-\omega\pi}) = 0 \implies a_1 = a_2 = 0$$

So, we have no solution for $\lambda > 0$.

Case $=0$. Then $\lambda_1 = \lambda_2 = 0$. As before, we have the general solution

$$X(x) = a_1 + a_2 x$$

We have

$$X'(x) = a_2$$

The boundary conditions give us

$$X'(0) = a_2 = 0 \implies X(x) = a_1$$

$$X(\pi) = a_1 = 0 \implies a_1 = 0$$

So, we have no solution for $\lambda = 0$.

Case < 0. As before, we have the general solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

for any constants a_1, a_2. We have

$$X'(x) = -a_1 \omega \sin(\omega x) + a_2 \omega \cos(\omega x)$$

and the boundary conditions give us

$$X'(0) = -a_1 \omega \sin(0) + a_2 \omega \cos(0) = 0 \implies a_2 = 0$$

$$X(\pi) = a_1 \cos(\omega \pi) = 0 \implies \omega = n + \frac{1}{2}$$

where $n = 0, 1, 2, \dots$.

So $X(x) = a_1 \cos\left(\left(n + \frac{1}{2}\right) x\right)$ and $\lambda = -\frac{n^2}{4}$.

We now turn to the ODE for $T(t)$:

$$\frac{T'(t)}{T(t)} = \lambda = -\frac{n^2}{4}$$

$$T'(t) = -\frac{n^2}{4} T(t)$$

$$T(t) = Ce^{-\frac{n^2}{4}t}$$

where C is a constant.

A solution is then

$$u(x,t) = X(x)T(t) = Ce^{-\frac{n^2}{4}t} \cos\left(\frac{n}{2}x\right)$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2}{4}t} \cos\left(\frac{n}{2}x\right)$$

where a_n are constants.

$\square$