Homework 5

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Problem 1.

a. *Verify* $\cos(nx)$*and* $\cos(mx)$ *are orthogonal on* $(0,\pi)$ *when* $n \neq m$ *and* $n, m \in \mathbb{N}$. *That is*

$$\int_0^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = 0.$$

b. *Verify* $\sin(nx)$ *and* $\sin(mx)$ *are orthogonal on* $(-\pi,\pi)$ *when* $n, m \in \mathbb{N}$. *That is*

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = 0.$$

Solution.

a.

$$\begin{aligned}
\int_0^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x &= \frac{1}{2}\int_0^{\pi} (\cos((n+m)x) + \cos((n-m)x))\,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{n+m}\sin((n+m)x) + \frac{1}{n-m}\sin((n-m)x)\right]_0^{\pi} \\
&= \frac{1}{2}\left[\frac{1}{n+m}\sin((n+m)\pi) + \frac{1}{n-m}\sin((n-m)\pi)\right] \\
&= 0
\end{aligned}$$

We have used the fact that $\sin(m\pi) = \sin(n\pi) = 0$ for $m, n \in \mathbb{N}$ and that $n \neq m \implies n \pm m \neq 0$.

b.

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} (\sin((n+m)x) + \sin((n-m)x))\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (\sin((n+m)x))\,\mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi} (\sin((n-m)x))\,\mathrm{d}x \\
&= 0 + 0 = 0
\end{aligned}$$

We use the fact that sin is an odd function and $(-\pi,\pi)$ is symmetric, so $\int_{-\pi}^{\pi} \sin((n+m)x)\,\mathrm{d}x = 0$ and $\int_{-\pi}^{\pi} \sin((n-m)x)\,\mathrm{d}x = 0$.

$\square$

Problem 2. *Solve*

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0, \\ u(x,0) = x^2, & 0 < x < \pi. \end{cases}$$

explicitly.

Solution.

$$u(x,t) = X(x)T(t)$$

We recall from lecture that the solution to $X''(x) = \lambda X(x)$ with the Neumann boundary conditions is

$$X(x) = c\cos(nx), \quad n \geq 0$$

where c is a constant and $\lambda = -n^2$. Then we also have

$$T'(t) = \lambda T(t) \implies T(t) = e^{\lambda t} = e^{-n^2 t}$$

So the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(nx) e^{-n^2 t}$$

We can solve for the c_n using the initial condition:

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos(nx) = x^2$$

We recognize the sum as a Fourier series and solve for a_n:

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[\frac{2x^2}{n^2}\sin(nx) + \frac{2}{n}x\cos(nx)\right]_0^\pi = \frac{4}{n^2}(-1)^n$$

We notice that for $n = 0$ our solution is undefined, so we must add the $n = 0$ term separately:

$$a_0 = \frac{1}{\pi}\int_0^\pi x^2\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{\pi^2}{3}$$

So the solution is

$$u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx) e^{-n^2 t}$$

□

Problem 3. *Write the Fourier sine series (as* $\sum a_n \sin(nx)$*) of the function*

$$f(x) = \begin{cases} -1, & 0 < x < \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < x < \pi. \end{cases}$$

For each $n \in \mathbb{N}$*, what is the value* $a_n \sin(n\pi/2)$*?*

You can separate the cases when $n = 4k, 4k+1, 4k+2, 4k+3$ *for* $k \in \mathbb{N} \cup \{0\}$*.*

Solution.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} (-1) \sin(nx)\, \mathrm{d}x + \int_{\pi/2}^{\pi} \sin(nx)\, \mathrm{d}x \right] \\ &= \frac{2}{\pi} \left[\frac{1}{n} \cos(nx) \Big|_0^{\pi/2} - \frac{1}{n} \cos(nx) \Big|_{\pi/2}^{\pi} \right] \\ &= \frac{2}{\pi} \left[\frac{\cos(n\pi/2)}{n} - \frac{\cos(0)}{n} - \frac{\cos(n\pi)}{n} + \frac{\cos(n\pi/2)}{n} \right] \\ &= \frac{2}{\pi} \left[\frac{2\cos(n\pi/2) - (-1)^n - 1}{n} \right] \end{aligned}$$

We now have

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{2\cos(n\pi/2) - (-1)^n - 1}{n} \right] \sin(nx)$$

We can separate the cases when $n = 4k, 4k+1, 4k+2, 4k+3$ for $k \in \mathbb{N} \cup \{0\}$.

Case 1: $n = 4k$ (note that we can ignore $k = 0$ in this case since $n = 0$ is not in the sum)

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[\frac{2\cos(2k\pi) - (-1)^{4k} - 1}{4k} \right] \sin(2k\pi) = \frac{2}{\pi} \left[\frac{2 - 1 - 1}{4k} \right] \sin(2k\pi) = 0$$

Case 2: $n = 4k + 1$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[\frac{2\cos(2k\pi + \pi/2) - (-1)^{4k+1} - 1}{4k+1} \right] \sin(2k\pi + \pi/2) = \frac{2}{\pi} \left[\frac{2(0) + 1 - 1}{4k+1} \right] \sin(2k\pi + \pi/2) = 0$$

Case 3: $n = 4k + 2$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[\frac{2\cos(2k\pi + \pi) - (-1)^{4k+2} - 1}{4k+2} \right] \sin(2k\pi + \pi) = \frac{2}{\pi} \left[\frac{2(-1) - 1 - 1}{4k+2} \right] (0) = 0$$

Case 4: $n = 4k + 3$

$$a_n \sin(n\pi/2) = \frac{2}{\pi} \left[\frac{2\cos(2k\pi + 3\pi/2) - (-1)^{4k+3} - 1}{4k+3} \right] \sin(2k\pi + 3\pi/2) = \frac{2}{\pi} \left[\frac{2(0) - (-1) - 1}{4k+3} \right] \sin(2k\pi + 3\pi/2) = 0$$

Therefore, we have $a_n \sin(n\pi/2) = 0$ for all $n \in \mathbb{N}$.

$\square$

Problem 4. *Solve*

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = x^2, & 0 < x < \pi, \\ u_t(x,0) = sin(2x), & 0 < x < \pi. \end{cases}$$

explicitly.

Solution.

$$u(x,t) = X(x)T(t)$$
$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

Recall from lecture that the general solution to $X''(x) = \lambda X(x)$ with the Dirichlet boundary conditions is

$$X(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

where $\lambda = -n^2, n \in \mathbb{N}$.

$$T''(t) = -n^2 T(t) \implies T(t) = c_1 \cos(nt) + c_2 \sin(nt)$$

So the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) \sin(nx) + b_n \sin(nt) \sin(nx))$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = x^2$$

We recognize the sum as a Fourier sine series:

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx)\,\mathrm{d}x$$

and solve for a_n using integration by parts:

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{-x^2 \cos(nx)}{n} + \frac{2x \sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{-\pi^2 \cos(n\pi)}{n} + \frac{2\cos(n\pi)}{n^3} - \frac{2}{n^3} \right] \\ &= \frac{2}{\pi} \left[\frac{-\pi^2 (-1)^n}{n} + \frac{2(-1)^n - 2}{n^3} \right] \\ &= \frac{-2\pi(-1)^n}{n} + \frac{4(-1)^n - 4}{\pi n^3} \\ &= (-1)^n \left(\frac{4}{\pi n^3} - \frac{2\pi}{n} \right) - \frac{4}{\pi n^3} \end{aligned}$$

Now we solve for b_n using the initial condition:

$$u_t(x,t) = \sum_{n=1}^{\infty} (-na_n \sin(nt) \sin(nx) + nb_n \cos(nt) \sin(nx)) = \sin(2x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} nb_n \sin(nx) = \sin(2x)$$

We can simply match the coefficients to find that $b_2 = 1/2$, and $b_n = 0$ for all other n.

So the solution is

$$u(x,t) = \frac{1}{2}\sin(nt)\sin(nx) + \sum_{n=1}^{\infty} \left((-1)^n \left(\frac{4}{\pi n^3} - \frac{2\pi}{n}\right) - \frac{4}{\pi n^3}\right) \cos(nt)\sin(nx)$$

$\square$