Homework 6

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Problem 1. *Assume that $H(x,t)$, $I(x,t)$ satisfy*

$$\begin{aligned} H_t &= -bHI, \\ I_t &= bHI - \gamma I + DI_{xx}, \end{aligned}$$

Define

$$\tau := \gamma t, \quad h := \frac{H}{N}, \quad i := \frac{I}{N}, \quad R_0 := \frac{bN}{\gamma}, \quad d := \frac{D}{\gamma},$$

where b, γ, N, D are some positive constants. Show that $h(x,\tau)$, $i(x,\tau)$ satisfy the system

$$\begin{aligned} h_\tau &= -R_0 hi, \\ i_\tau &= R_0 hi - i + di_{xx}. \end{aligned}$$

Hint: chain rule.

Solution.

$$\begin{aligned} h_\tau &= \frac{\partial h}{\partial \tau} \\ &= \frac{\partial h}{\partial t}\frac{\partial t}{\partial \tau} \\ &= \left(\frac{\partial}{\partial t}\left(\frac{H}{N}\right)\right)\frac{1}{\gamma} \\ &= \frac{1}{N}\frac{\partial H}{\partial t}\frac{1}{\gamma} \\ &= \frac{1}{N}(-bHI)\frac{1}{\gamma} \\ &= -\frac{bHI}{N}\frac{R_0}{bN} \\ &= -R_0\frac{H}{N}\frac{I}{N} \\ &= -R_0 hi \end{aligned}$$

$$\begin{aligned} i_\tau &= \frac{\partial i}{\partial \tau} \\ &= \frac{\partial i}{\partial t}\frac{\partial t}{\partial \tau} \\ &= \left(\frac{\partial}{\partial t}\left(\frac{I}{N}\right)\right)\frac{1}{\gamma} \\ &= \frac{1}{N}\frac{\partial I}{\partial t}\frac{1}{\gamma} \\ &= \frac{1}{N}(bHI - \gamma I + DI_{xx})\frac{1}{\gamma} \\ &= \frac{bHI}{N}\frac{R_0}{bN} - \frac{\gamma I}{N}\frac{1}{\gamma} + \frac{DI_{xx}}{N}\frac{1}{\gamma} \\ &= R_0 hi - \frac{I}{N} + \frac{DI_{xx}}{N\gamma} \\ &= R_0 hi - i + d\frac{I_{xx}}{N} \\ &= R_0 hi - i + di_{xx} \qquad \left(i_{xx} = \frac{I_{xx}}{N}\right) \end{aligned}$$

$\square$

Problem 2. *For $(x, y) \neq (0, 0)$, compute the Laplacian of*

$$u(x, y) = \ln\left(x^2 + y^2\right)$$

and conclude whether it satisfies the Laplace equation for $(x, y) \neq (0, 0)$.

Solution.

$$\begin{aligned}
u_x &= \frac{2x}{x^2 + y^2}, \\
u_y &= \frac{2y}{x^2 + y^2}, \\
u_{xx} &= \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, \\
u_{yy} &= \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}, \\
\Delta u = u_{xx} + u_{yy} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = 0.
\end{aligned}$$

Thus, $u(x, y) = \ln\left(x^2 + y^2\right)$ satisfies the Laplace equation for $(x, y) \neq (0, 0)$. □

Problem 3. *Use separation of variables to solve*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi, \\ u_y(x, 0) = 0, & 0 < x < \pi, \\ u_y(x, \pi) = 0, & 0 < x < \pi, \\ u(0, y) = 0, & 0 < y < \pi, \\ u(\pi, y) = 1 + 3\cos(2y), & 0 < y < \pi. \end{cases}$$

Solution. Let $u(x, y) = X(x)Y(y)$. Then the Laplace equation gives

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda.$$

We consider the system

$$\begin{cases} Y''(y) = \lambda Y(y), & 0 < y < \pi, \\ Y'(0) = Y'(\pi) = 0. \end{cases}$$

Recall that the general solution for the homogeneous Neumann boundary conditions with $\lambda = -n^2$ is

$$Y(y) = c\cos(ny), \quad n = 0, 1, 2, \ldots$$

for some constant c.

Note that in the case of $n = 0$, we have $Y(y) = c$, which admits a solution

$$X(x) = a_0 + b_0 x.$$

We also have

$$X''(x) = -\lambda X(x) = n^2 X(x),$$

and we recall the general solution

$$X(x) = Ae^{nx} + Be^{-nx}$$

where A, B are arbitrary constants. Using the hyperbolic sine and cosine functions, we can rewrite this as

$$X(x) = \tilde{A}\cosh(nx) + \tilde{B}\sinh(nx).$$

We can now write the general solution with $u_y(x, 0) = u_y(x, \pi) = 0$ as

$$u(x, y) = a_0 + b_0 x + \sum_{n=1}^{\infty} \big(a_n \cosh(nx) + b_n \sinh(nx)\big)\cos(ny)$$

The boundary condition $u(0, y) = 0$ gives $x = 0,\ \cosh(0) = 1,\ \sinh(0) = 0$, so

$$\begin{aligned} u(0, y) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) \\ &= \sum_{n=0}^{\infty} a_n \cos(ny) \\ &= 0. \end{aligned}$$

We recognize this as a Fourier series, so

$$a_n = \frac{2}{\pi}\int_0^{\pi} 0 \cdot \cos(ny)\,\mathrm{d}y = 0.$$

With the boundary condition $u(\pi, y) = 1 + 3\cos(2y)$, we have

$$u(\pi, y) = \pi b_0 + \sum_{n=1}^{\infty} b_n \sinh(n\pi)\cos(ny) = 1 + 3\cos(2y).$$

As it is a constant, we can let $b_n \sinh(n\pi) = \tilde{b}_n$. Then

$$\begin{aligned}
\tilde{b}_n &= \frac{2}{\pi}\int_0^{\pi} \big(1 + 3\cos(2y) - \pi b_0\big)\cos(ny)\,\mathrm{d}y \\
&= \frac{2}{\pi}\int_0^{\pi} \cos(ny)\,\mathrm{d}y + \frac{6}{\pi}\int_0^{\pi} \cos(2y)\cos(ny)\,\mathrm{d}y - \pi b_0 \frac{2}{\pi}\int_0^{\pi} \cos(ny)\,\mathrm{d}y \\
&= \frac{2}{\pi}\left[\frac{1}{n}\sin(ny)\right]_0^{\pi} + \frac{6}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos((n+2)y) + \cos((n-2)y)\right)\mathrm{d}y - 2b_0\left[\frac{1}{n}\sin(ny)\right]_0^{\pi} \\
&= 0 + \frac{3}{\pi}\left[\frac{1}{n+2}\sin((n+2)y) + \frac{1}{n-2}\sin((n-2)y)\right]_0^{\pi} - 0 \\
&= 0.
\end{aligned}$$

We see that $b_n = \frac{\tilde{b}_n}{\sinh(n\pi)} = 0$ for all $n \neq 2$. Thus, we have

$$u(x, y) = b_0 x + b_2 \sinh(2x)\cos(2y).$$

We can again apply the $u(\pi, y) = 1 + 3\cos(2y)$ boundary condition to solve for b_0:

$$u(\pi, y) = \pi b_0 + b_2 \sinh(2\pi)\cos(2y) = 1 + 3\cos(2y).$$

Comparing terms, we see that $b_0 = \frac{1}{\pi}$ and $b_2 = \frac{3}{\sinh(2\pi)}$. Thus, the solution is

$$u(x, y) = \frac{x}{\pi} + \frac{3\sinh(2x)}{\sinh(2\pi)}\cos(2y).$$

□

Problem 4. *Use separation of variables to solve*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x, y < \pi, \\ u(x,0) = 0, & 0 < x < \pi, \\ u(x,\pi) = 100, & 0 < x < \pi, \\ u(0,y) = 0, & 0 < y < \pi, \\ u(\pi,y) = 100, & 0 < y < \pi. \end{cases}$$

Solution. As we do not have either the homogeneous Dirichlet or Neumann boundary conditions, we first use the superposition principle to write

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

where

$$\begin{cases} u_{1xx} + u_{1yy} = 0, & 0 < x, y < \pi, \\ u_1(x,0) = 0, & 0 < x < \pi, \\ u_1(x,\pi) = 100, & 0 < x < \pi, \\ u_1(0,y) = 0, & 0 < y < \pi, \\ u_1(\pi,y) = 0, & 0 < y < \pi. \end{cases}$$

and

$$\begin{cases} u_{2xx} + u_{2yy} = 0, & 0 < x, y < \pi, \\ u_2(x,0) = 0, & 0 < x < \pi, \\ u_2(x,\pi) = 0, & 0 < x < \pi, \\ u_2(0,y) = 0, & 0 < y < \pi, \\ u_2(\pi,y) = 100, & 0 < y < \pi. \end{cases}$$

Then we can proceed using separation of variables to solve each of these equations separately.

Part 1: $u_1(x,y)$

Let $u_1(x,y) = X(x)Y(y)$. Then the Laplace equation gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

We consider the system

$$\begin{cases} X''(x) = \lambda X(x), & 0 < x < \pi, \\ X(0) = X(\pi) = 0, \end{cases}$$

The general solution for the homogeneous Dirichlet boundary conditions with $\lambda = -n^2$ is

$$X(x) = c\sin(nx), \quad n = 1, 2, 3, \ldots$$

for some constant c.

We also have

$$Y''(y) = -\lambda Y(y) = n^2 Y(y),$$

and we recall the general solution

$$Y(y) = Ae^{ny} + Be^{-ny}$$

where A, B are arbitrary constants. Using the hyperbolic sine and cosine functions, we can rewrite this as

$$Y(y) = \tilde{A}\cosh(ny) + \tilde{B}\sinh(ny).$$

We can now write the general solution with $u_1(0,y) = u_1(\pi,y) = 0$ as

$$u_1(x,y) = \sum_{n=1}^{\infty} \big(a_n \cosh(ny) + b_n \sinh(ny)\big) \sin(nx)$$

The boundary condition $u_1(x,0) = 0$ gives $y = 0,\ \cosh(0) = 1,\ \sinh(0) = 0$, so

$$u_1(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = 0$$

We recognize this as a Fourier sine series, so

$$a_n = \frac{2}{\pi} \int_0^{\pi} 0 \cdot \sin(nx)\,\mathrm{d}x = 0.$$

With the boundary condition $u_1(x,\pi) = 100$, we have

$$u_1(x,\pi) = \sum_{n=1}^{\infty} b_n \sinh(n\pi) \sin(nx) = 100.$$

As it is a constant, we can let $b_n \sinh(n\pi) = \tilde{b}_n$. Then

$$\begin{aligned}
\tilde{b}_n &= \frac{2}{\pi} \int_0^{\pi} 100 \cdot \sin(nx)\,\mathrm{d}x \\
&= \frac{200}{\pi} \int_0^{\pi} \sin(nx)\,\mathrm{d}x \\
&= \frac{200}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi} \\
&= \frac{200}{\pi} \left[-\frac{1}{n} \cos(n\pi) + \frac{1}{n} \cos(0) \right] \\
&= \frac{200}{\pi} \left[\frac{1 - (-1)^n}{n} \right].
\end{aligned}$$

Then

$$b_n = \frac{\tilde{b}_n}{\sinh(n\pi)} = \frac{200}{\pi} \left[\frac{1 - (-1)^n}{n} \right] \frac{1}{\sinh(n\pi)}$$

for all n.

So, our solution is

$$u_1(x,y) = \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n} \right] \frac{\sinh(ny)}{\sinh(n\pi)} \sin(nx).$$

Part 2: $u_2(x,y)$

By symmetry, we can see that the solution to $u_2(x,y)$ will be the same as $u_1(x,y)$ but with x and y interchanged:

$$u_2(x,y) = \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n} \right] \frac{\sinh(nx)}{\sinh(n\pi)} \sin(ny).$$

Combining the solutions

By the superposition principle, the solution to the original problem is

$$\begin{aligned}
u(x, y) &= u_1(x, y) + u_2(x, y) \\
&= \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n}\right] \frac{\sinh(ny)}{\sinh(n\pi)} \sin(nx) + \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n}\right] \frac{\sinh(nx)}{\sinh(n\pi)} \sin(ny) \\
&= \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n}\right] \frac{\sinh(ny)\sin(nx) + \sinh(nx)\sin(ny)}{\sinh(n\pi)}.
\end{aligned}$$

□