

# Homework 1

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**Problem 1.** What is the type of (elliptic, parabolic, hyperbolic) the following second-order PDEs:

(a)  $u_{xx} - 3u_{xy} + 2u_{yy} + u_y + 5u = 0$

(b)  $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

*Solution.*

(a)

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(2) \\ &= 9 - 8 \\ &= 1 \\ &> 0 \end{aligned}$$

Since  $D > 0$ , the PDE is hyperbolic.

(b)

$$\begin{aligned} D &= b^2 - 4ac \\ &= (6)^2 - 4(9)(1) \\ &= 36 - 36 \\ &= 0 \\ &= 0 \end{aligned}$$

Since  $D = 0$ , the PDE is parabolic.

□

**Problem 2.** Sketch the characteristic lines, and solve the transport equation

$$\begin{cases} u_t + 2u_x = 0 \\ u(x, 0) = x^3 \end{cases}$$

Sketch  $u(x, 0)$ ,  $u(x, 1)$ ,  $u(x, 2)$  on the same graph and convince yourself that the solutions are moving to the right with speed 2.

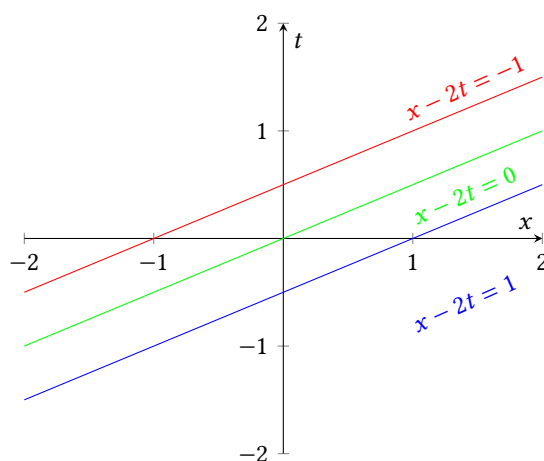
*Solution.*

$$(u_x, u_t) \cdot (2, 1) = 0$$

$u$  is constant along the lines parallel to  $(2, 1)$ . These characteristic lines are given by

$$t = \frac{x}{2} + C_1 \iff x - 2t = C_2$$

for some constants  $C_1, C_2$ . We may graph a few characteristic lines:



We can express  $u(x, y)$  as a function of  $x - 2t$ :

$$u(x, t) = f(x - 2t)$$

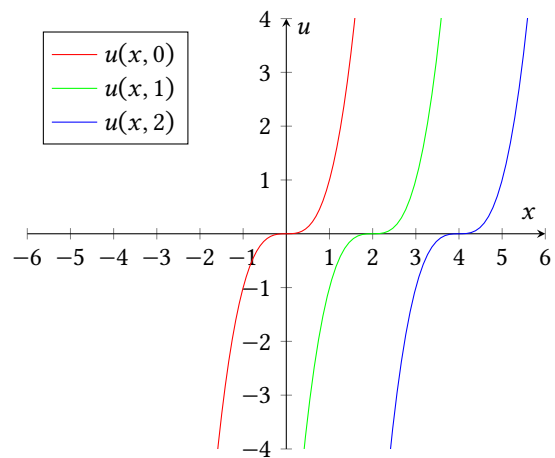
We can solve for  $f$  using the initial condition:

$$u(x, 0) = f(x) = x^3$$

Thus, the solution to the transport equation is

$$u(x, t) = (x - 2t)^3$$

We can graph  $u(x, 0)$ ,  $u(x, 1)$ ,  $u(x, 2)$  on the same graph:



We can see that the solutions are moving to the right with speed 2.

□

**Problem 3.** We define

$$u(x, t) := \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Verify that the function  $u(x, t)$  is a solution of the heat equation

$$u_t = Du_{xx}$$

for all  $(x, t)$  with  $-\infty < x < \infty$  and  $t > 0$ .

*Solution.*

$$\begin{aligned} u_t &= \frac{1}{\sqrt{4\pi D}} \left( -\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{x^2}{4Dt}} + t^{-\frac{1}{2}} e^{-\frac{x^2}{4Dt}} \frac{x^2}{4Dt} \right) \\ &= \frac{1}{2t\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left( \frac{x^2}{2Dt} - 1 \right) \\ &= \frac{1}{2t} \left( \frac{x^2}{2Dt} - 1 \right) u(x, t) \\ u_x &= \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left( -\frac{x}{2Dt} \right) \\ u_{xx} &= \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left( \frac{x^2}{4D^2 t^2} - \frac{1}{2Dt} \right) \\ &= \frac{1}{2t\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left( \frac{x^2}{2Dt} - 1 \right) \frac{1}{D} \\ &= \frac{1}{2Dt} \left( \frac{x^2}{2Dt} - 1 \right) u(x, t) \end{aligned}$$

We can see that

$$u_t = Du_{xx}$$

for all  $(x, t)$  with  $-\infty < x < \infty$  and  $t > 0$ . □

**Problem 4.** For  $Q > 0$ , compute

$$\int_{-\infty}^{\infty} e^{-Qx^2} dx$$

*Hint: Multiply by the identical integral but replace  $x$  variable to  $y$ , and change variables to polar coordinates.*

*Solution.* Let  $I = \int_{-\infty}^{\infty} e^{-Qx^2} dx = \int_{-\infty}^{\infty} e^{-Qy^2} dy$ . Then

$$\begin{aligned} I^2 &= \left( \int_{-\infty}^{\infty} e^{-Qx^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-Qy^2} dy \right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(x^2+y^2)} dx dy \end{aligned}$$

We can change variables to polar coordinates:

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-Qr^2} r dr d\theta \\ &= 2\pi \int_0^{\infty} e^{-Qr^2} r dr \end{aligned}$$

Let  $u = -Qr^2$ , then  $du = -2Qr dr$ . Thus,

$$\begin{aligned} I^2 &= 2\pi \left[ \frac{1}{-2Q} e^{-Qr^2} \right]_0^{\infty} \\ &= \frac{\pi}{Q} \end{aligned}$$

Thus,

$$I = \int_{-\infty}^{\infty} e^{-Qx^2} dx = \sqrt{\frac{\pi}{Q}}$$

□

**Problem 5.** We will derive the transport equation  $u_t + cu_x = 0$  for  $x, t \in \mathbb{R}$  using an approach via compartment models similar to what we used for the derivation of the heat equation in class: assume that  $u(x, t)$  is the concentration of particles at position  $x$  and time  $t$ . Let  $h$  and  $\tau$  be positive and small. First, divide up the real line into intervals of length  $h$  centered at positions  $\dots, x - 2h, x - h, x, x + h, x + 2h, \dots$ . Next, we assume that during the time interval from  $t$  to  $t + \tau$  all particles in an interval move to the nearest interval to their right.

- (a) Write down a balance equation that reflects this situation. Please provide a clear description of the various terms in the balance equation.
- (b) Rearrange the terms in your balance equation and substitute them from Taylor expansions appropriately.
- (c) Choose an appropriate way to let  $h, t$  go to zero, and show that the limit is the transport equation we want.

*Solution.*

- (a) Number of particles at time  $t + \tau$  = Number of particles at time  $t$  + Change:

$$hu(x, t + \tau) = hu(x, t) + \text{Change}$$

$$\begin{aligned} \text{Change} &= (\text{Number of particles that moved from } x - h \text{ to } x) \\ &\quad - (\text{Number of particles that moved from } x \text{ to } x + h). \\ &= h(u(x - h, t) - u(x, t)) \end{aligned}$$

We have

$$hu(x, t + \tau) = hu(x, t) + h(u(x - h, t) - u(x, t))$$

- (b) Rearranging the terms and dividing by  $h\tau$ , we have

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{h}{\tau} \frac{u(x - h, t) - u(x, t)}{h} \quad (1)$$

We can use a Taylor expansion to approximate  $u(x - h, t)$

$$u(x - h, t) = u(x, t) - hu_x(x, t) + O(h^2)$$

so that we have

$$u(x - h, t) - u(x, t) = -hu_x(x, t) + O(h^2)$$

Substituting into (1), we have

$$\begin{aligned} \frac{u(x, t + \tau) - u(x, t)}{\tau} &= \frac{h}{\tau} \frac{-hu_x(x, t) + O(h^2)}{h} \\ \frac{u(x, t + \tau) - u(x, t)}{\tau} &= \frac{h}{\tau} (-u_x(x, t) + O(h)) \end{aligned}$$

- (c) Let  $h \rightarrow 0$  and  $\tau \rightarrow 0$ . Let us also choose  $\tau = \frac{h}{c}$  so that  $\tau \rightarrow 0$  as  $h \rightarrow 0$ . Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(x, t + \tau) - u(x, t)}{\tau} &= \lim_{h \rightarrow 0} \frac{h}{\tau} (-u_x(x, t) + O(h)) \\ \lim_{\tau \rightarrow 0} \frac{u(x, t + \tau) - u(x, t)}{\tau} &= \lim_{h \rightarrow 0} \frac{h}{\frac{h}{c}} (-u_x(x, t) + O(h)) \\ u_t(x, t) &= -cu_x(x, t) \\ u_t + cu_x &= 0 \end{aligned}$$

Thus, the limit is the transport equation.

□