

# Homework 4

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**Problem 1.** Verify the superposition principle: if  $u_1$  and  $u_2$  are two solutions to

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t > 0 \end{cases}$$

then so is  $a_1 u_1 + a_2 u_2$  for any constants  $a_1, a_2$ . Is it still true if the boundary condition is  $u(0, t) = u(\pi, t) = 1$ ?

*Solution.* We can verify the superposition principle by plugging in  $u_1$  and  $u_2$  into the heat equation and adding them together. We have

$$\begin{aligned} (a_1 u_1 + a_2 u_2)_t - (a_1 u_1 + a_2 u_2)_{xx} &= a_1 u_{1t} + a_2 u_{2t} - a_1 u_{1xx} - a_2 u_{2xx} \\ &= a_1 u_{1xx} + a_2 u_{2xx} - a_1 u_{1xx} - a_2 u_{2xx} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} (a_1 u_1 + a_2 u_2)(0, t) &= a_1 u_1(0, t) + a_2 u_2(0, t) \\ &= a_1(0) + a_2(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} (a_1 u_1 + a_2 u_2)(\pi, t) &= a_1 u_1(\pi, t) + a_2 u_2(\pi, t) \\ &= a_1(0) + a_2(0) \\ &= 0 \end{aligned}$$

so  $a_1 u_1 + a_2 u_2$  is also a solution.

If the boundary condition is  $u(0, t) = u(\pi, t) = 1$ , then we have

$$\begin{aligned} (a_1 u_1 + a_2 u_2)(0, t) &= a_1 u_1(0, t) + a_2 u_2(0, t) \\ &= a_1(1) + a_2(1) \\ &= a_1 + a_2 \end{aligned}$$

$a_1 + a_2$  is not 1 for all  $a_1, a_2$ , so the superposition principle does not hold. □

**Problem 2.** Use separation of variables to solve

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0, t) = u_x(\pi, t) = 0, & t > 0, \\ u(x, 0) = 1 + 2 \cos(3x), & 0 < x < \pi \end{cases}$$

explicitly. You don't need to go through all 3 cases.

*Solution.* We look for a solution of the form  $u(x, t) = X(x)T(t)$ . Plugging this into the heat equation, we have

$$X(x)T'(t) - X''(x)T(t) = 0$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

The boundary conditions give us

$$u_x(0, t) = X'(0)T(t) = 0 \implies X'(0) = 0$$

$$u_x(\pi, t) = X'(\pi)T(t) = 0 \implies X'(\pi) = 0$$

We now have a system of ODEs for  $X(x)$ :

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < \pi, \\ X'(0) = X'(\pi) = 0, \end{cases}$$

Let us define

$$V(x) = \begin{bmatrix} X'(x) \\ X(x) \end{bmatrix}$$

Then we have

$$V'(x) = \begin{bmatrix} X''(x) \\ X'(x) \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X'(x) \\ X(x) \end{bmatrix}$$

So we have

$$V'(x) = AV(x), \quad A = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}$$

We can find the eigenvalues and eigenvectors of  $A$ :

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & \lambda \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda = 0$$

So the eigenvalues are  $\pm \sqrt{\lambda}$ .

**Case**  $\lambda = 0$ . Then  $\lambda_1 = \lambda_2 = 0$ .

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

gives us the eigenvector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . We can find a generalized eigenvector by solving

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives us  $v_1 = 1$ . So the generalized eigenvector is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . We have the solution

$$V(x) = a_1 e^{0x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_2 e^{0x} \left( x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

which gives us

$$X(x) = a_1 + a_2x$$

We have

$$X'(x) = a_2$$

The boundary conditions give us

$$X'(0) = X'(\pi) = a_2 = 0 \implies a_2 = 0$$

So the solution is

$$X(x) = a$$

for any constant  $a$ .

**Case  $\lambda < 0$ .** Let  $\lambda = -\omega^2$ , where  $\omega > 0$ . Then the eigenvalues are  $\pm i\omega$ . This gives the solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

for any constants  $a_1, a_2$ . We have

$$X'(x) = -a_1\omega \sin(\omega x) + a_2\omega \cos(\omega x)$$

and the boundary conditions give us

$$X'(0) = -a_1\omega \sin(0) + a_2\omega \cos(0) = 0 \implies a_2 = 0$$

$$X'(\pi) = -a_1\omega \sin(\omega\pi) = 0 \implies \omega = n$$

where  $n \in \mathbb{Z}^+$ . So  $X(x) = a_1 \cos(nx)$  and  $\lambda = -n^2$ . We now turn to the ODE for  $T(t)$ :

$$\frac{T'(t)}{T(t)} = \lambda = -n^2$$

$$T'(t) = -n^2 T(t)$$

$$T(t) = Ce^{-n^2 t}$$

where  $C$  is a constant. A solution is then

$$u(x, t) = X(x)T(t) = Ce^{-n^2 t} \cos(nx)$$

By the principle of superposition, the general solution is

$$u(x, t) = a + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos(nx)$$

where  $a_n$  are constants. We can find the constants by using the initial condition:

$$u(x, 0) = a + \sum_{n=1}^{\infty} a_n \cos(nx) = 1 + 2 \cos(3x)$$

We have  $a = 1$ ,  $a_3 = 2$ , and  $a_n = 0$  for all other  $n$ . So the solution is

$$u(x, t) = 1 + 2e^{-9t} \cos(3x)$$

□

**Problem 3.** Consider the following heat-like equation:

$$\begin{cases} tu_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t > 0, \end{cases}$$

Use separation of variables to write the solution as an infinite series. You don't need to go through all 3 cases.

*Solution.* Let  $u(x, t) = X(x)T(t)$ . Then

$$\begin{aligned} tX(x)T'(t) - X''(x)T(t) &= 0 \\ \frac{tT'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda \end{aligned}$$

The boundary conditions give us

$$\begin{aligned} u(0, t) = X(0)T(t) = 0 &\implies X(0) = 0 \\ u(\pi, t) = X(\pi)T(t) = 0 &\implies X(\pi) = 0 \end{aligned}$$

We now have a system of ODEs for  $X(x)$ :

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < \pi, \\ X(0) = X(\pi) = 0, \end{cases}$$

Like in the previous problem, we arrive at the solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

where  $\lambda = -\omega^2$  and  $\omega > 0$ . Using the boundary conditions, we have

$$\begin{aligned} X(0) &= a_1 \cos(0) + a_2 \sin(0) = a_1 = 0 \\ X(\pi) &= a_2 \sin(\omega\pi) = 0 \implies \omega = n \end{aligned}$$

where  $n \in \mathbb{Z}^+$ . So  $X(x) = a \sin(nx)$  and  $\lambda = -n^2$ .

We now turn to the ODE for  $T(t)$ :

$$\begin{aligned} \frac{tT'(t)}{T(t)} &= \lambda = -n^2 \\ tT'(t) &= -n^2 T(t) \\ \frac{T'(t)}{T(t)} &= -\frac{n^2}{t} \\ T'(t) &= -\frac{n^2}{t} T(t) \\ T'(t) + \frac{n^2}{t} T(t) &= 0 \end{aligned}$$

This is a first-order linear ODE with variable coefficients. We can solve it using an integrating factor:

$$\begin{aligned} \mu(t) &= e^{\int \frac{n^2}{t} dt} = e^{n^2 \ln(t)} = e^{\ln(t^{n^2})} = t^{n^2} \\ T'(t)t^{n^2} + \frac{n^2}{t}T(t)t^{n^2} &= 0 \\ (T(t)t^{n^2})' &= 0 \\ T(t)t^{n^2} &= C \\ T(t) &= Ct^{-n^2} \end{aligned}$$

where  $C$  is a constant.

A solution is then

$$u(x, t) = X(x)T(t) = C \sin(nx)t^{-n^2}$$

By the principle of superposition, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin(nx)t^{-n^2}$$

where  $a_n$  are constants.

□

**Problem 4.** Consider the heat equation with mixed boundary conditions

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi, \quad t > 0, \\ u_x(0, t) = 0, & t > 0, \\ u(\pi, t) = 0, & t > 0. \end{cases}$$

Use separation of variables and go through all 3 cases to write the solution as an infinite series.

*Solution.* As before, we let  $u(x, t) = X(x)T(t)$ . Then

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

The boundary conditions give us

$$\begin{aligned} u_x(0, t) = X'(0)T(t) = 0 &\implies X'(0) = 0 \\ u(\pi, t) = X(\pi)T(t) = 0 &\implies X(\pi) = 0 \end{aligned}$$

We now have a system of ODEs for  $X(x)$ :

$$\begin{cases} X''(x) - \lambda X(x) = 0, & 0 < x < \pi, \\ X'(0) = 0 \\ X(\pi) = 0 \end{cases}$$

As before, the eigenvalues are  $\pm\sqrt{\lambda}$ .

**Case  $\lambda > 0$ .** Let  $\lambda = \omega^2$ , where  $\omega > 0$ . Then  $\lambda_1 = \omega^2$  and  $\lambda_2 = -\omega^2$ .

$$A - \lambda_1 I = \begin{bmatrix} -\omega & \omega^2 \\ 1 & -\omega \end{bmatrix}$$

so an eigenvector corresponding to  $\lambda_1$  is  $\begin{bmatrix} \omega \\ 1 \end{bmatrix}$ .

$$A - \lambda_2 I = \begin{bmatrix} \omega & \omega^2 \\ 1 & \omega \end{bmatrix}$$

so an eigenvector corresponding to  $\lambda_2$  is  $\begin{bmatrix} -\omega \\ 1 \end{bmatrix}$ . We have the solution

$$V(x) = a_1 e^{\omega x} \begin{bmatrix} \omega \\ 1 \end{bmatrix} + a_2 e^{-\omega x} \begin{bmatrix} -\omega \\ 1 \end{bmatrix}$$

which gives us

$$X(x) = a_1 e^{\omega x} + a_2 e^{-\omega x}$$

for any constants  $a_1, a_2$ . We have

$$X'(x) = a_1 \omega e^{\omega x} - a_2 \omega e^{-\omega x}$$

The boundary conditions give us

$$X'(0) = a_1 \omega - a_2 \omega = 0 \implies a_1 = a_2$$

$$X(\pi) = a_1 e^{\omega \pi} + a_2 e^{-\omega \pi} = a_1 e^{\omega \pi} + a_1 e^{-\omega \pi} = a_1 (e^{\omega \pi} + e^{-\omega \pi}) = 0 \implies a_1 = a_2 = 0$$

So, we have no solution for  $\lambda > 0$ .

**Case  $\lambda = 0$ .** Then  $\lambda_1 = \lambda_2 = 0$ . As before, we have the general solution

$$X(x) = a_1 + a_2 x$$

We have

$$X'(x) = a_2$$

The boundary conditions give us

$$X'(0) = a_2 = 0 \implies X(x) = a_1$$

$$X(\pi) = a_1 = 0 \implies a_1 = 0$$

So, we have no solution for  $\lambda = 0$ .

**Case  $\lambda < 0$ .** As before, we have the general solution

$$X(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)$$

for any constants  $a_1, a_2$ . We have

$$X'(x) = -a_1 \omega \sin(\omega x) + a_2 \omega \cos(\omega x)$$

and the boundary conditions give us

$$X'(0) = -a_1 \omega \sin(0) + a_2 \omega \cos(0) = 0 \implies a_2 = 0$$

$$X(\pi) = a_1 \cos(\omega \pi) = 0 \implies \omega = \frac{n}{2}$$

where  $n \in \mathbb{Z}^+$ . So  $X(x) = a_1 \cos\left(\frac{n}{2}x\right)$  and  $\lambda = -\frac{n^2}{4}$ .

We now turn to the ODE for  $T(t)$ :

$$\frac{T'(t)}{T(t)} = \lambda = -\frac{n^2}{4}$$

$$T'(t) = -\frac{n^2}{4}T(t)$$

$$T(t) = Ce^{-\frac{n^2}{4}t}$$

where  $C$  is a constant.

A solution is then

$$u(x, t) = X(x)T(t) = Ce^{-\frac{n^2}{4}t} \cos\left(\frac{n}{2}x\right)$$

By the principle of superposition, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2}{4}t} \cos\left(\frac{n}{2}x\right)$$

where  $a_n$  are constants.

□