

Homework 2

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Equations

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Problem 1. Let a and c be real numbers.

(a) Show that if $g(x) = f(x - a)$, then $\hat{g}(k) = e^{ika} \hat{f}(k)$.

(b) Use the Fourier transform and part (a) to solve the transport PDE

$$\begin{cases} u_t + cu_x = 0 \\ u(x, 0) = f(x) \end{cases}$$

Solution.

(a)

$$\begin{aligned} \hat{g}(k) &= \int_{-\infty}^{\infty} g(x) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} f(x - a) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} f(y) e^{-ik(y+a)} dy \quad (\text{let } y = x - a) \\ &= e^{ika} \int_{-\infty}^{\infty} f(y) e^{-iky} dy \\ &= e^{ika} \hat{f}(k). \end{aligned}$$

(b)

$$\begin{aligned} u_t(x, t) &= -cu_x(x, t) \\ \hat{u}_t(k, t) &= \int_{-\infty}^{\infty} -cu_x(x, t) e^{-ikx} dx \\ &= -c \int_{-\infty}^{\infty} u_x(x, t) e^{-ikx} dx \\ &= -c [u(x, t) e^{-ikx}]_{-\infty}^{\infty} + c \int_{-\infty}^{\infty} u(x, t) ike^{-ikx} dx \\ &= cik \hat{u}(k, t). \end{aligned}$$

We have the ODE $\hat{u}_t(k, t) = cik \hat{u}(k, t)$, which has the solution $\hat{u}(k, t) = \hat{u}(k, 0) e^{cikt} = \hat{f}(k) e^{cikt}$.

Using part (a), we have

$$u(x, t) = f(x - ct)$$

which is the solution to the transport PDE.

□

Problem 2. Consider the heat equation $u_t = Du_{xx}$ with $x \in \mathbb{R}$ and $D > 0$.

- (a) For each $k \in \mathbb{R}$, find $\lambda = \lambda(k) \in \mathbb{R}$ so that $u(x, t) = e^{\lambda t} e^{ikx}$ satisfies the heat equation. Here $i = \sqrt{-1}$.
- (b) Assume that $a(k)$ is Schwartz class in k . Show that $u(x, t) = \int_{-\infty}^{\infty} a(k) e^{\lambda(k)t} e^{ikx} dk$ satisfies the heat equation for $t \geq 0$.

Solution.

(a)

$$\begin{aligned} u_t(x, t) &= \frac{\partial}{\partial t} e^{\lambda t} e^{ikx} \\ &= \lambda e^{\lambda t} e^{ikx} \\ u_{xx}(x, t) &= \frac{\partial^2}{\partial x^2} e^{\lambda t} e^{ikx} \\ &= -k^2 e^{\lambda t} e^{ikx}. \end{aligned}$$

$$\begin{aligned} u_t(x, t) = Du_{xx}(x, t) &\implies \lambda e^{\lambda t} e^{ikx} = -Dk^2 e^{\lambda t} e^{ikx}. \\ \lambda(k) &= -Dk^2. \end{aligned}$$

(b)

$$\begin{aligned} u_t(x, t) &= \int_{-\infty}^{\infty} a(k) \lambda(k) e^{\lambda(k)t} e^{ikx} dk = \int_{-\infty}^{\infty} a(k) (-Dk^2) e^{-Dk^2 t} e^{ikx} dk. \\ u_{xx}(x, t) &= \int_{-\infty}^{\infty} a(k) (-k^2) e^{-Dk^2 t} e^{ikx} dk. \\ u_t(x, t) &= Du_{xx}(x, t). \end{aligned}$$

Note that $t \geq 0$ so that $e^{-Dk^2 t}$ is either 1 or Schwartz class.

Thus, $u(x, t) = \int_{-\infty}^{\infty} a(k) e^{\lambda(k)t} e^{ikx} dk$ satisfies the heat equation for $t \geq 0$.

□

Problem 3. Consider the equation $u_t = u_{xxxx}$ for $x \in \mathbb{R}$ with $u(x, 0) = f(x)$ where $f(x)$ is Schwartz class. Derive an expression for the Fourier transform $\hat{u}(k, t)$ of the solution $u(x, t)$ in terms of k , t , and $\hat{f}(k)$. (You do not need to find $u(x, t)$ from $\hat{u}(k, t)$).

Solution.

$$\begin{aligned}
 u_t(x, t) &= u_{xxxx}(x, t) \\
 \hat{u}_t(k, t) &= \int_{-\infty}^{\infty} u_{xxxx}(x, t) e^{-ikx} dx \\
 &= \int_{-\infty}^{\infty} u(x, t) (-ik)^4 e^{-ikx} dx \\
 &= (-ik)^4 \hat{u}(k, t) \\
 &= k^4 \hat{u}(k, t).
 \end{aligned}$$

We have the ODE $\hat{u}_t(k, t) = k^4 \hat{u}(k, t)$, which has the solution $\hat{u}(k, t) = \hat{u}(k, 0) e^{k^4 t} = \hat{f}(k) e^{k^4 t}$.

□

Problem 4.

(a) For given $D > 0$ and $a, c \in \mathbb{R}$, consider the equation $u_t = Du_{xx} + cu_x - au$ for $x \in \mathbb{R}$ and $t > 0$. Assume that $u(x, t)$ is a solution of this equation. Verify that $v(x, t) := u(y - ct, t)e^{at}$ is a solution of $v_t = Dv_{yy}$ for $y \in \mathbb{R}$ and $t > 0$.

(b) Recall that (from HW1)

$$v(y, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}$$

satisfies $v_t = v_{yy}$ for $y \in \mathbb{R}$ and $t > 0$. Use the preceding part to find the solution $u(x, t)$ to $u_t = u_{xx} + 2u_x - 0.5u$.

Solution.

(a)

$$\begin{aligned} v_t(x, t) &= \frac{\partial}{\partial t} u(y - ct, t) e^{at} \\ &= u_t(y - ct, t) e^{at} - cu_x(y - ct, t) e^{at} + au(y - ct, t) e^{at} \\ &= (Du_{xx}(y - ct, t) + cu_x(y - ct, t) - au(y - ct, t)) e^{at} - cu_x(y - ct, t) e^{at} + au(y - ct, t) e^{at} \\ &= Du_{xx}(y - ct, t) e^{at} \\ &= Dv_{yy}(x, t) \end{aligned}$$

(b) We have that $D = 1$, $a = 2$, and $c = 0$. Let $v(x, t) = u(y - ct, t)e^{at}$. Then

$$u_t = Du_{xx} + cu_x - au \implies v_t = Dv_{yy}.$$

Let $y = x$, $c = 0$, and $a = 2$. Then

$$v(x, t) = u(x, t)e^{2t}.$$

$$u(x, t) = v(x, t)e^{-2t} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} e^{-2t} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t} - 2t}$$

□