Homework 1

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APMA 0360 — Partial Differential
Equations

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Problem 1. *What is the type of (elliptic, parabolic, hyperbolic) the following second-order PDEs:*

(a) $u_{xx} - 3u_{xy} + 2u_{yy} + u_y + 5u = 0$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Solution.

(a)

$$
\begin{aligned}
D &= b^2 - 4ac \\
&= (-3)^2 - 4(1)(2) \\
&= 9 - 8 \\
&= 1 \\
&> 0
\end{aligned}
$$

Since $D > 0$, the PDE is hyperbolic.

(b)

$$
\begin{aligned}
D &= b^2 - 4ac \\
&= (6)^2 - 4(9)(1) \\
&= 36 - 36 \\
&= 0 \\
&= 0
\end{aligned}
$$

Since $D = 0$, the PDE is parabolic.

$\square$

Problem 2. *Sketch the characteristic lines, and solve the transport equation*

$$\begin{cases} u_t + 2u_x = 0 \\ u(x,0) = x^3 \end{cases}$$

Sketch $u(x,0), u(x,1), u(x,2)$ on the same graph and convince yourself that the solutions are moving to the right with speed 2.
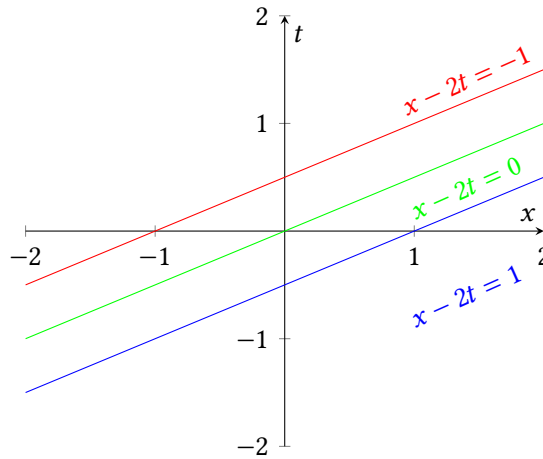
Solution.

$$(u_x, u_t) \cdot (2,1) = 0$$

u is constant along the lines parallel to $(2,1)$. These characteristic lines are given by

$$t = \frac{x}{2} + C_1 \iff x - 2t = C_2$$

for some constants C_1, C_2. We may graph a few characteristic lines:

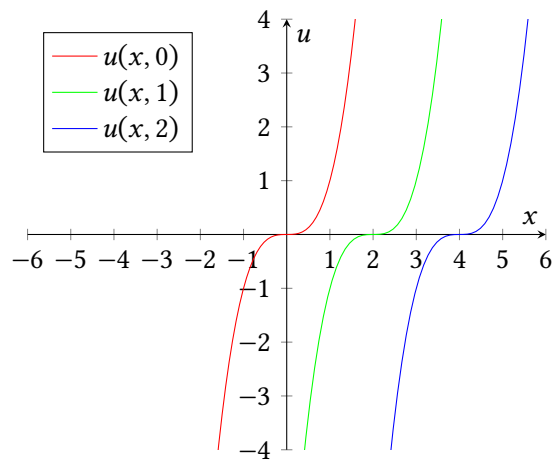We can express $u(x,y)$ as a function of $x - 2t$:

$$u(x,t) = f(x - 2t)$$

We can solve for f using the initial condition:

$$u(x,0) = f(x) = x^3$$

Thus, the solution to the transport equation is

$$u(x,t) = (x - 2t)^3$$

We can graph $u(x,0), u(x,1), u(x,2)$ on the same graph:

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We can see that the solutions are moving to the right with speed 2. □

Problem 3. *We define*

$$u(x,t) := \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Verify that the function $u(x,t)$ is a solution of the heat equation

$$u_t = Du_{xx}$$

for all (x,t) with $-\infty < x < \infty$ and $t > 0$.

Solution.

$$u_t = \frac{1}{\sqrt{4\pi D}} \left(-\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{x^2}{4Dt}} + t^{-\frac{1}{2}} e^{-\frac{x^2}{4Dt}} \frac{x^2}{4Dt} \right)$$

$$= \frac{1}{2t\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left(\frac{x^2}{2Dt} - 1 \right)$$

$$= \frac{1}{2t} \left(\frac{x^2}{2Dt} - 1 \right) u(x,t)$$

$$u_x = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left(-\frac{x}{2Dt} \right)$$

$$u_{xx} = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left(\frac{x^2}{4D^2t^2} - \frac{1}{2Dt} \right)$$

$$= \frac{1}{2t\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \left(\frac{x^2}{2Dt} - 1 \right) \frac{1}{D}$$

$$= \frac{1}{2Dt} \left(\frac{x^2}{2Dt} - 1 \right) u(x,t)$$

We can see that

$$u_t = Du_{xx}$$

for all (x,t) with $-\infty < x < \infty$ and $t > 0$. $\qquad\qquad \square$

Problem 4. *For $Q > 0$, compute*

$$\int_{-\infty}^{\infty} e^{-Qx^2} dx$$

Hint: Multiply by the identical integral but replace x variable to y, and change vari- ables to polar coordinates.

Solution. Let $I = \int_{-\infty}^{\infty} e^{-Qx^2} dx = \int_{-\infty}^{\infty} e^{-Qy^2} dy$. Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-Qx^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-Qy^2} dy\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-Q(x^2+y^2)} dx dy$$

We can change variables to polar coordinates:

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-Qr^2} r \, dr \, d\theta$$

$$= 2\pi \int_0^{\infty} e^{-Qr^2} r \, dr$$

Let $u = -Qr^2$, then $du = -2Qr dr$. Thus,

$$I^2 = 2\pi \left[\frac{1}{-2Q} e^{-Qr^2}\right]_0^{\infty}$$

$$= \frac{\pi}{Q}$$

Thus,

$$I = \int_{-\infty}^{\infty} e^{-Qx^2} dx = \sqrt{\frac{\pi}{Q}}$$

$\square$

Problem 5. *We will derive the transport equation $u_t + cu_x = 0$ for $x, t \in \mathbb{R}$ using an approach via compartment models similar to what we used for the derivation of the heat equation in class: assume that $u(x,t)$ is the concentration of particles at position x and time t. Let h and τ be positive and small. First, divide up the real line into intervals of length h centered at positions $..., x - 2h, x - h, x, x + h, x + 2h, ...$ Next, we assume that during the time interval from t to $t + \tau$ all particles in an interval move to the nearest interval to their right.*

(a) *Write down a balance equation that reflects this situation. Please provide a clear description of the various terms in the balance equation.*

(b) *Rearrange the terms in your balance equation and substitute them from Taylor expansions appropriately.*

(c) *Choose an appropriate way to let h, t go to zero, and show that the limit is the transport equation we want.*

Solution.

(a) Number of particles at time $t + \tau$ = Number of particles at time t + Change:

$$hu(x, t + \tau) = hu(x,t) + \text{Change}$$

$$\begin{aligned} \text{Change} &= (\text{Number of particles that moved from } x - h \text{ to } x) \\ &\quad - (\text{Number of particles that moved from } x \text{ to } x + h). \\ &= h\left(u(x - h, t) - u(x,t)\right) \end{aligned}$$

We have

$$hu(x, t + \tau) = hu(x,t) + h\left(u(x - h, t) - u(x,t)\right)$$

(b) Rearranging the terms and dividing by $h\tau$, we have

$$\frac{u(x, t + \tau) - u(x,t)}{\tau} = \frac{h}{\tau} \frac{u(x - h, t) - u(x,t)}{h} \tag{1}$$

We can use a Taylor expansion to approximate $u(x - h, t)$

$$u(x - h, t) = u(x,t) - hu_x(x,t) + O(h^2)$$

so that we have

$$u(x - h, t) - u(x,t) = -hu_x(x,t) + O(h^2)$$

Substituting into (1), we have

$$\frac{u(x, t + \tau) - u(x,t)}{\tau} = \frac{h}{\tau} \frac{-hu_x(x,t) + O(h^2)}{h}$$

$$\frac{u(x, t + \tau) - u(x,t)}{\tau} = \frac{h}{\tau}\left(-u_x(x,t) + O(h)\right)$$

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(c) Let $h \to 0$ and $\tau \to 0$. Let us also choose $\tau = \frac{h}{c}$ so that $\tau \to 0$ as $h \to 0$. Then

$$\lim_{h \to 0} \frac{u(x, t + \tau) - u(x, t)}{\tau} = \lim_{h \to 0} \frac{h}{\tau} \left(-u_x(x, t) + O(h) \right)$$

$$\lim_{\tau \to 0} \frac{u(x, t + \tau) - u(x, t)}{\tau} = \lim_{h \to 0} \frac{h}{\frac{h}{c}} \left(-u_x(x, t) + O(h) \right)$$

$$u_t(x, t) = -cu_x(x, t)$$

$$u_t + cu_x = 0$$

Thus, the limit is the transport equation.

□