Final Exam

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APMA 0360 — Partial Differential Equations

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Problem 1. *Let $u(x,t)$ be the solution of the heat equation on the real line*

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where

$$f(x) = \begin{cases} 1, & -1 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Compute $\hat{u}(\kappa, t)$, the Fourier transform of $u(x,t)$, explicitly. (You do not need to find the solution $u(x,t)$.)

Solution. Recall that the Fourier transform of the heat equation in general is

$$\hat{u}(\kappa, t) = \hat{u}(\kappa, 0)e^{-D\kappa^2 t}.$$

Here, we have $D = 1$ and $\hat{u}(\kappa, 0) = \hat{f}(\kappa)$. So, it remains to compute $\hat{f}(\kappa)$. We have

$$\begin{aligned}
\hat{f}(\kappa) &= \int_{-\infty}^{\infty} e^{i\kappa x} f(x) \, \mathrm{d}x \\
&= \int_{-1}^{1} e^{i\kappa x} \, \mathrm{d}x \\
&= \frac{1}{i\kappa} \left[e^{i\kappa x}\right]_{-1}^{1} \\
&= \frac{1}{i\kappa} \left(e^{i\kappa} - e^{-i\kappa}\right) \\
&= \frac{1}{i\kappa} \left[\cos(\kappa) + i\sin(\kappa) - \cos(-\kappa) - i\sin(-\kappa)\right] \\
&= \frac{1}{i\kappa} \left[\cos(\kappa) + i\sin(\kappa) - \cos(\kappa) + i\sin(\kappa)\right] \\
&= \frac{2i\sin(\kappa)}{i\kappa} \\
&= \frac{2\sin(\kappa)}{\kappa} \\
&= 2\,\mathrm{sinc}(\kappa).
\end{aligned}$$

Thus, we have

$$\hat{u}(\kappa, t) = 2\,\mathrm{sinc}(\kappa)e^{-\kappa^2 t}.$$

□

Problem 2. *Solve the damped wave equation*

$$\begin{cases} u_{tt} + 2u_t + u = u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \dfrac{1}{1+x^2}, & x \in \mathbb{R}, \\ u_t(x,0) = 1, & x \in \mathbb{R}. \end{cases}$$

Hint. Find the equation satisfied by $v(x,t) = e^t u(x,t)$ first, and use the solution to that problem to find $u(x,t)$.

Solution.

$$\begin{aligned} v(x,t) &= e^t u(x,t), \\ v_t &= e^t u_t + e^t u, \\ v_{tt} &= e^t u_{tt} + 2e^t u_t + e^t u \\ &= e^t(u_{tt} + 2u_t + u) \\ &= e^t u_{xx} \\ &= v_{xx}. \end{aligned}$$

We see that $v(x,t)$ satisfies the undamped wave equation. The initial profiles are

$$v(x,0) = \frac{e^0}{1+x^2} = \frac{1}{1+x^2}, \quad v_t(x,0) = e^0(u_t(x,0) + u(x,0)) = 1 + \frac{1}{1+x^2}.$$

Now, we have the system

$$\begin{cases} v_{tt} = v_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v(x,0) = \dfrac{1}{1+x^2}, & x \in \mathbb{R}, \\ v_t(x,0) = 1 + \dfrac{1}{1+x^2}, & x \in \mathbb{R}. \end{cases}$$

which we can solve using d'Alembert's formula. Recall that the solution in general is

$$v(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

Here, we have $f(x) = \frac{1}{1+x^2}$, $g(x) = 1 + \frac{1}{1+x^2}$, and $c = 1$.

$$v(x,t) = \frac{1}{2}\left(\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right) + \frac{1}{2}\int_{x-t}^{x+t} 1 + \frac{1}{1+s^2}\,\mathrm{d}s.$$

We will leave the first two terms and compute the integral first.

$$\begin{aligned} \int_{x-t}^{x+t} 1 + \frac{1}{1+s^2}\,\mathrm{d}s &= \left[s + \arctan(s)\right]_{x-t}^{x+t} \\ &= (x + t + \arctan(x+t)) - (x - t + \arctan(x-t)) \\ &= 2t + \arctan(x+t) - \arctan(x-t). \end{aligned}$$

Thus, we have

$$v(x,t) = \frac{1}{2}\left(\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2} + 2t + \arctan(x+t) - \arctan(x-t)\right).$$

Finally, we have

$$u(x,t) = e^{-t}v(x,t) = \frac{1}{2e^t}\left(\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2} + 2t + \arctan(x+t) - \arctan(x-t)\right).$$

If we let $\xi = x + t$ and $\eta = x - t$, we can write this as

$$u(\xi,\eta) = \frac{1}{2}e^{\frac{\eta-\xi}{2}}\left(\frac{1}{1+\xi^2} + \frac{1}{1+\eta^2} + \xi - \eta + \arctan(\xi) - \arctan(\eta)\right).$$

□

Problem 3. *Find the general solution of*

$$\begin{cases} u_{xx} - 2u_x + u_{yy} = 0, & 0 < x, y < \pi, \\ u(x,0) = 0, & 0 < x < \pi, \\ u(x,\pi) = 0, & 0 < x < \pi \end{cases}$$

using separation of variables.

Solution. Let $u(x,y) = X(x)Y(y)$. Then

$$X''Y - 2X'Y + XY'' = 0 \implies \frac{2X' - X''}{X} = \frac{Y''}{Y} = \lambda.$$

We solve the ODE in y first, as we have the boundary conditions in y.

$$\begin{cases} Y'' = \lambda Y, & 0 < y < \pi, \\ Y(0) = Y(\pi) = 0. \end{cases}$$

Recall that the general solution for the homogeneous Dirichlet boundary conditions with $\lambda = -n^2$ is

$$Y_n(y) = c_n \sin(ny), \quad n = 1, 2, 3, \dots$$

where c is a constant.

Now, we solve the ODE in x.

$$\frac{2X' - X''}{X} = \lambda \implies X'' - 2X' + \lambda X = 0.$$

We have the auxiliary equation

$$r^2 - 2r + \lambda = 0 \implies r = 1 \pm \sqrt{1-\lambda} = 1 \pm \sqrt{1+n^2}.$$

Since the roots are real and distinct, the general solution is

$$\begin{aligned} X_n(x) &= \tilde{a}_n e^{(1+\sqrt{1+n^2})x} + \tilde{b}_n e^{(1-\sqrt{1+n^2})x} \\ &= e^x \left(\tilde{a}_n e^{\sqrt{1+n^2}x} + \tilde{b}_n e^{-\sqrt{1+n^2}x} \right) \\ &= e^x \left(\hat{a}_n \cosh\left(\sqrt{1+n^2}x\right) + \hat{b}_n \sinh\left(\sqrt{1+n^2}x\right) \right). \end{aligned}$$

The general solution is then

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} e^x \left(\hat{a}_n \cosh\left(\sqrt{1+n^2}x\right) + \hat{b}_n \sinh\left(\sqrt{1+n^2}x\right) \right) c_n \sin(ny) \\ &= e^x \sum_{n=1}^{\infty} \left(a_n \cosh\left(\sqrt{1+n^2}x\right) + b_n \sinh\left(\sqrt{1+n^2}x\right) \right) \sin(ny). \end{aligned}$$

□

Problem 4. *Let $u(x,t)$ be the solution of the following heat equation with mixed boundary conditions*

$$\begin{cases} u_t = u_{xx} + 3x^2, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi,t) = 1, & t > 0, \\ u(x,0) = \sin(x), & 0 < x < \pi. \end{cases}$$

Find the limiting function $u_\infty(x) = \lim_{t\to\infty} u(x,t)$.

Solution. We can devise functions $w(x,t)$ and $v(x)$ such that

$$w(x,t) = u(x,t) - v(x,t)$$

and

$$\begin{cases} w_t = w_{xx} + Q(x), & 0 < x < \pi, \quad t > 0, \\ w(0,t) = 0, & t > 0, \\ w_x(\pi,t) = 0, & t > 0 \end{cases}$$

so that we can find the limiting function $w_\infty(x) = \lim_{t\to\infty} w(x,t)$ by solving the ODE

$$\begin{cases} -w_\infty''(x) = Q(x), & 0 < x < \pi, \\ w_\infty(0) = 0, \\ w_\infty'(\pi) = 0. \end{cases}$$

which leads to the limiting function $u_\infty(x) = w_\infty(x) + v(x)$.

Let $v(x) = x$. Then $v(0) = 0,\ v'(x) = 1,\ v''(x) = 0$, and so

$$\begin{cases} w_t = w_{xx} + 3x^2, & 0 < x < \pi, \quad t > 0, \\ w(0,t) = 0, & t > 0, \\ w_x(\pi,t) = 0, & t > 0, \end{cases}$$

and

$$\begin{cases} -w_\infty''(x) = 3x^2, & 0 < x < \pi, \\ w_\infty(0) = 0, \\ w_\infty'(\pi) = 0. \end{cases}$$

$$w_\infty''(x) = -3x^2 \implies w_\infty'(x) = -x^3 + c_1 \implies w_\infty(x) = -\frac{x^4}{4} + c_1 x + c_2.$$

$$w_\infty(0) = 0 \implies c_2 = 0, \quad w_\infty'(\pi) = 0 \implies -\pi^3 + c_1 = 0 \implies c_1 = \pi^3.$$

So we have

$$w_\infty(x) = \pi^3 x - \frac{x^4}{4}.$$

and

$$
\begin{aligned}
u_\infty(x) &= w_\infty(x) + v(x) \\
&= \pi^3 x - \frac{x^4}{4} + x \\
&= \left(\pi^3 + 1\right) x - \frac{x^4}{4}.
\end{aligned}
$$

□

Problem 5. *Solve the following wave equation with inhomogeneous mixed boundary conditions*

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(\pi, t) = \sin\left(\frac{3t}{2}\right), & t > 0, \\ u(x,0) = 0, & 0 < x < \pi, \\ u_t(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Solution. First, we devise the functions $w(x,t)$ and $v(x)$ such that

$$w(x,t) = u(x,t) - v(x,t)$$

and

$$\begin{cases} w_{tt} - w_{xx} = Q(x,t), & 0 < x < \pi, \quad t > 0, \\ w(0,t) = 0, & t > 0, \\ w_x(\pi, t) = 0, & t > 0. \end{cases}$$

We choose $v(x,t) = \sin\left(\frac{3t}{2}\right) x$. Then

$$\begin{cases} w_{tt} - w_{xx} = \frac{9}{4}\sin\left(\frac{3t}{2}\right) x, & 0 < x < \pi, \quad t > 0, \\ w(0,t) = 0, & t > 0, \\ w_x(\pi, t) = 0, & t > 0, \\ w(x,0) = 0, & 0 < x < \pi, \\ w_t(x,0) = -\frac{3x}{2}, & 0 < x < \pi. \end{cases}$$

We can now solve this using the eigenfunction method.

First, we show that with these homogeneous mixed boundary conditions in x, the solutions are of the form

$$w(x,t) \stackrel{L^2}{=} \sum_{n=1}^{\infty} A_n(t) \sin(nx).$$

Using separation of variables and letting $w(x,t) = X(x)T(t)$, we arrive at the ODE in x

$$\begin{cases} X''(x) = \lambda X(x), & 0 < x < \pi, \\ X(0) = 0, & x = 0, \\ X'(\pi) = 0, & x = \pi. \end{cases}$$

We can let $\omega > 0$ and consider the three cases $\lambda = 0$, $\lambda = \omega^2 > 0$, and $\lambda = -\omega^2 < 0$.

Case 1: $\lambda = 0$.

$$X''(x) = 0 \implies X(x) = c_1 + c_2 x$$

$$X(0) = 0 \implies c_1 = 0, \quad X'(\pi) = 0 \implies c_2 = 0.$$

So, there are only trivial solutions in this case.

Case 2: $\lambda = \omega^2 > 0$.

$$X''(x) = \omega^2 X(x) \implies X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

$$X'(x) = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$$

$$X(0) = 0 \implies c_1 + c_2 = 0 \implies c_2 = -c_1$$

$$X'(\pi) = 0 \implies c_1 \omega e^{\omega \pi} + c_1 \omega e^{-\omega \pi} = 0 \implies c_1 = 0$$

So, there are only trivial solutions in this case.

Case 3: $\lambda = -\omega^2 < 0$.

$$X''(x) = -\omega^2 X(x) \implies X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

$$X'(x) = -c_1 \omega \sin(\omega x) + c_2 \omega \cos(\omega x)$$

$$X(0) = 0 \implies c_1 = 0$$

$$X'(\pi) = 0 \implies c_2 \omega \cos(\omega \pi) = 0 \implies \omega = n + \frac{1}{2},\ n = 0, 1, 2, \ldots$$

This admits the solutions

$$X_n(x) = c_n \sin\left(\left(n + \frac{1}{2}\right)x\right).$$

With the ODE in t, we now have

$$T''(t) = -\left(n + \frac{1}{2}\right)^2 T(t) \implies T(t) = a_n \cos\left(\left(n + \frac{1}{2}\right)t\right) + b_n \sin\left(\left(n + \frac{1}{2}\right)t\right).$$

and the general solution is

$$w(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\left(n + \frac{1}{2}\right)t\right) + b_n \sin\left(\left(n + \frac{1}{2}\right)t\right)\right) \sin\left(\left(n + \frac{1}{2}\right)x\right).$$

Now that we have shown that $w(x,t)$ can be represented as a Fourier sine series in x, we can find the coefficients $A_n(t)$ using the formula derived in lecture for the wave equation with a source and inhomogeneous mixed initial conditions. Recall that for $\tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(nx)$, we have

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = Q(x,t), & 0 < x < \pi, \quad t > 0, \\ \tilde{u}(x,0) = f(x), & 0 < x < \pi, \\ \tilde{u}_t(x,0) = g(x), & 0 < x < \pi. \end{cases}$$

where we decompose $Q(x,t)$, $f(x)$, and $g(x)$ into their Fourier sine series as

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin(nx), \quad f(x) = \sum_{n=1}^{\infty} a_n \sin(nx), \quad g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

which gives us the ODE

$$\begin{cases} A_n''(t) + n^2 A_n(t) = q_n(t), \quad t > 0, \\ A_n(0) = a_n, \\ A_n'(0) = b_n. \end{cases}$$

We recall the formula for $A_n(t)$ is

$$A_n(t) = \left(a_n - \frac{1}{n}\int_0^t q_n(s)\sin(ns)\,\mathrm{d}s\right)\cos(nt) + \left(\frac{b_n}{n} + \frac{1}{n}\int_0^t q_n(s)\cos(ns)\,\mathrm{d}s\right)\sin(nt).$$

Here, we have $Q(x,t) = \frac{9}{4}\sin\left(\frac{3t}{2}\right)x$, $f(x) = 0$, and $g(x) = -\frac{3x}{2}$. Trivially, $a_n = 0$. We can find the other Fourier coefficients:

$$\begin{aligned} q_n(t) &= \frac{2}{\pi}\int_0^{\pi} \frac{9}{4}\sin\left(\frac{3t}{2}\right)x\sin(nx)\,\mathrm{d}x \\ &= \frac{9}{2\pi}\sin\left(\frac{3t}{2}\right)\int_0^{\pi} x\sin(nx)\,\mathrm{d}x \\ &= \frac{9}{2\pi}\sin\left(\frac{3t}{2}\right)\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi} \\ &= \frac{9}{2\pi}\sin\left(\frac{3t}{2}\right)\left(-\frac{\pi}{n}(-1)^n\right) \\ &= -(-1)^n\frac{9}{2n}\sin\left(\frac{3t}{2}\right). \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} -\frac{3x}{2}\sin(nx)\,\mathrm{d}x \\ &= -\frac{3}{\pi}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x \\ &= -\frac{3}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi} \\ &= -\frac{3}{\pi}\left(-\frac{\pi}{n}(-1)^n\right) \\ &= \frac{3(-1)^n}{n}. \end{aligned}$$

Now, let us calculate the coefficient of the $\cos(nt)$ term in $A_n(t)$.

$$\begin{aligned}
-\frac{1}{n}\int_0^t q_n(s)\sin(ns)\,\mathrm{d}s &= -\frac{1}{n}\int_0^t -(-1)^n\frac{9}{2n}\sin\left(\frac{3s}{2}\right)\sin(ns)\,\mathrm{d}s \\
&= (-1)^n\frac{9}{2n^2}\int_0^t \sin\left(\frac{3s}{2}\right)\sin(ns)\,\mathrm{d}s \\
&= (-1)^n\frac{9}{2n^2}\int_0^t \frac{1}{2}\left[\cos\left(\frac{3s}{2}-ns\right)-\cos\left(\frac{3s}{2}+ns\right)\right]\mathrm{d}s \\
&= (-1)^n\frac{9}{4n^2}\int_0^t \cos\left(\frac{3s}{2}-ns\right)-\cos\left(\frac{3s}{2}+ns\right)\mathrm{d}s \\
&= (-1)^n\frac{9}{4n^2}\left[\frac{2}{3-2n}\sin\left(\frac{3s}{2}-ns\right)-\frac{2}{3+2n}\sin\left(\frac{3s}{2}+ns\right)\right]_0^t \\
&= (-1)^n\frac{9}{2n^2}\left(\frac{1}{3-2n}\sin\left(\frac{3t}{2}-nt\right)-\frac{1}{3+2n}\sin\left(\frac{3t}{2}+nt\right)\right).
\end{aligned}$$

And, the coefficient of the $\sin(nt)$ term in $A_n(t)$.

$$\begin{aligned}
\frac{b_n}{n}+\frac{1}{n}\int_0^t q_n(s)\cos(ns)\,\mathrm{d}s &= \frac{3(-1)^n}{n^2}+\frac{1}{n}\int_0^t -(-1)^n\frac{9}{2n}\sin\left(\frac{3s}{2}\right)\cos(ns)\,\mathrm{d}s \\
&= \frac{3(-1)^n}{n^2}-(-1)^n\frac{9}{2n^2}\int_0^t \sin\left(\frac{3s}{2}\right)\cos(ns)\,\mathrm{d}s \\
&= \frac{3(-1)^2}{n^2}\left(1-\frac{3}{2}\int_0^t \sin\left(\frac{3s}{2}\right)\cos(ns)\,\mathrm{d}s\right) \\
&= \frac{3(-1)^2}{n^2}\left(1-\frac{3}{4}\int_0^t \sin\left(\frac{3s}{2}-ns\right)+\sin\left(\frac{3s}{2}+ns\right)\mathrm{d}s\right) \\
&= \frac{3(-1)^2}{n^2}\left(1+\frac{3}{2}\left[\frac{1}{3-2n}\cos\left(\frac{3s}{2}-ns\right)+\frac{1}{3+2n}\cos\left(\frac{3s}{2}+ns\right)\right]_0^t\right) \\
&= \frac{3(-1)^2}{n^2}\left(1+\frac{3}{2}\left[\frac{1}{3-2n}\left(\cos\left(\frac{3t}{2}-nt\right)-1\right)+\frac{1}{3+2n}\left(\cos\left(\frac{3t}{2}+nt\right)-1\right)\right]\right).
\end{aligned}$$

So, we have

$$w(x,t)=\sum_{n=1}^{\infty}A_n(t)\sin(nx)$$

with A_n given with the coefficients calculated above.

Finally, we have

$$u(x,t)=w(x,t)+v(x,t)=\sin\left(\frac{3t}{2}\right)x+\sum_{n=1}^{\infty}A_n(t)\sin(nx).$$

□