COS 210 Worksheet 1 Dylan Kapnias (u18108467)

Question 1

Assume: p is prime & $\sqrt{p} = \frac{a}{b}$ $\exists\, a,b \in \mathbb{Z}$, $b \neq 0$ $\therefore \gcd(a,b) = 1$ (co-prime)

$$\Rightarrow p = \frac{a^2}{b^2}$$
$$\Rightarrow a^2 = pb^2 \quad \therefore p \text{ divides } a^2$$
Let $a = pk$, $k \in \mathbb{Z}$
$$\Rightarrow a^2 = p^2k^2 = pb^2$$
$$\Rightarrow b^2 = pk^2 \quad \therefore p \text{ divides } b^2$$

Thus a and b have atleast p as a common multiple.
This is a contradiction to our assumption that a & b are co-prime.
Thus our assumtion is incorrect and $\sqrt{p}$ is irrational.

Question 2

Basis Step: $n = 1$, LHS $= 1$
RHS $= \left(\frac{1 \cdot 2}{2}\right)^2$
$= 1$
$\therefore$ LHS = RHS
$\therefore$ Basis Step holds.

Induction Hypothesis: Let $k \in \mathbb{Z}$ be such that the series is true for $n = k$, $k \geq 1$:
$$1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{(k).(k+1)}{2}\right)^2 \quad (*)$$

Induction Step: $1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$
$$= \left(\frac{(k).(k+1)}{2}\right)^2 + (k+1)^3 \quad (*)$$
$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$= \frac{(k+1)^2(k^2 + 4k + k)}{4}$$
$$= \left(\frac{(k+1).((k+1)+1)}{2}\right)^2$$

$\therefore$ By the principle of weak mathematical induction, the series is true for $n \geq 1$.

Question 3

Let $P(x)$ be the property that: $f(n) = 11^{n+2} + 12^{2n+1}$ & 133 divides $f(n)$, $n \geq 1$

Basis Step: $f(1) = 11^3 + 12^3$
$= 3059$
$= 23 \cdot 133$
$\therefore f(1) \in P(x)$
Thus basis step is true.

Induction Hypothesis: Let $k \in \mathbb{Z}$ be such that $f(k) \in P(x)$

For $k = n$, $k \geq 1$:

$$f(k) = 133r, \quad r \in \mathbb{Z}$$
$$= 11^{k+2} + 12^{2k+1} \qquad (*)$$

Induction Step:
$$f(k+1) = 11^{k+3} + 12^{2k+3}$$
$$= 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 12^2$$
$$= 133r + 133 \cdot 12^{k+1} \qquad (*)$$
$$= 133(11 \cdot r + 12^{2k+1})$$

$\therefore$ $f(k+1) \in P(x)$

$\therefore$ By the principle of weak mathematical induction $f(n) \in P(x)$ for $n \geq 1$.