Safety First, Robust Dynamic Asset Pricing, and the Cross-Section of Expected Stock Returns

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ABSTRACT

We investigate the implications of learning under Knightian uncertainty for the cross-section of average stock returns. We derive the fundamental dynamic asset pricing equation in beta regression form and provide empirical evidence that learning and Knightian uncertainty are priced in the cross-section. We also find that the component of the uncertainty premium associated with learning loses significance as the investor's investment horizon increases, in contrast to the pure robustness component of the premium. This suggests that the economic impact of time-varying betas is less important than that of model uncertainty. Results of further asset pricing tests indicate that an empirical asset pricing model that includes Knightian uncertainty survives model misspecification and explains returns better than the empirical ICAPM and the four-factor model of Fama-French-Carhart.

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Standard asset pricing models based on rational expectations perform poorly empirically (Lewellen and Nagel, 2006; Kan, Robotti, and Shanken, 2009). Rational-expectations models assume that investors know the probability law governing asset returns, whereas authors such as Knight (1921), Keynes (1921), Shackle (1949), and Roy (1952) have emphasized that investors form expectations based on vague information that cannot be quantified precisely. Related evidence from experimental studies (e.g., Ellsberg, 1961) has confirmed that individuals are averse to uncertainty regarding not only the outcome of events with known probabilities ("risk") but also the outcome of events with unknown probabilities ("Knightian uncertainty" or "ambiguity").[1] Consequently, a body of literature has emerged that studies the implications of Knightian uncertainty for portfolio selection and asset pricing.[2] So far, most of the literature has been theoretical, partly because of the difficulty of measuring uncertainty empirically.

Recently, the theoretical literature on Knightian uncertainty has been expanded to include learning.[3] In particular, Hansen and Sargent (2009) show that when investors seek to simultaneously learn the hidden state of the economy and the model driving the evolution of the hidden state space, asset pricing includes a premium for Knightian uncertainty that is both time-varying and state-dependent on the worst consumption state. Epstein and Schneider (2008) introduce a model of learning under an ambiguous signal driven by tangible and intangible news. The agent's beliefs are jointly represented by an evolving set of conditional distributions across future states of the world, and the agent evaluates the signal under the likelihood that puts the

[1] Following the literature, we use Knightian uncertainty and ambiguity interchangeably.
[2] A partial list of the asset pricing literature includes articles by Epstein and Wang (1994), Kogan and Wang (2002), Chen and Epstein (2003), Epstein and Schneider (2007, 2008), Leippold, Trojani, and Vanini (2008), Hansen (2008), Hansen and Sargent (2008, 2009), and Anderson, Ghysels, and Juergens (2009). A partial list of portfolio selection articles includes Uppal and Wang (2003), Maenhout (2004), and Garlappi, Uppal, and Wang (2007).
[3] See Epstein and Schneider (2007, 2008), Hansen (2008), Leippold, Trojani, and Vanini (2008), and Hansen and Sargent (2009).

lowest possible conditional probability of being in the good economic regime. In their model, Knightian uncertainty also introduces a premium that is time-varying and asymmetric given that investors always discount bad news more heavily than good news independently of the state of the economy.[4]

We investigate if Knightian uncertainty is priced in the cross-section of U.S. stock returns. Following the recent literature, we first derive a dynamic asset pricing model that incorporates the impact of learning when investors are ambiguity averse. More specifically, we extend the general dynamic asset pricing approach in continuous time of Merton (1973) to include the impact of the malevolent actions of nature distorting the news driving the time-varying opportunity set. To this end, we apply robust control methods (e.g., Hansen and Sargent, 2008) as well as results from nonlinear filtering theory (Wonham, 1965; Fujisaka, Kallianpur, and Kunita, 1972).

Furthermore, in the spirit of Epstein and Schneider (2008), we also derive the forward plan in discrete time of a risk-neutral but ambiguity-averse representative agent with multiple-priors and recursive preferences, who seeks to learn the state of the economy tracking an ambiguous signal. We discuss under what conditions the continuous time robust dynamic control problem can be seen as the limit of a "modified" discrete time Markov decision problem with backward induction solution matching the optimal forward plan under ambiguous signals conveying both tangible and intangible information. We show that independently of the conceptual approach used, investors' concern for Knightian uncertainty (i.e., robustness

[4] The literature distinguishes between cases in which the source of uncertainty is the signal the investor uses to learn the hidden state and the data generating process ("structured" uncertainty), and cases in which the source of uncertainty cannot be specified ("unstructured"). Taking learning into account is important because Strzalecki (2007), and Barillas, Hansen, and Sargent (2009) show that under unstructured uncertainty robust beliefs are equivalent to standard Bayesian beliefs with augmented risk aversion.

preference) adds a separate uncertainty premium to the dynamic asset pricing model in beta regression form.[5]

Empirically, we assume that robust investors use a regime-switching model for the stock market return to forecast the probability of good economic times, with transition probabilities driven by a parsimonious set of ambiguous macroeconomic news. Following the uncertainty literature, we assume investors concerned with model uncertainty doubt their forecasts, even if they know the true data generating process. Consequently, investors estimate an interval of plausible values for the a priori imprecise hidden probability of the good economic state, and select the minimum value within the interval.[6] Our proxy for Knightian uncertainty is based on the intuition in Epstein and Schneider (2008) that given some exogenous (conservative) confidence level in the approximating model, robust investors act as if they hold simultaneously multiple likelihoods for the state of nature as they process the ambiguous signal(s). The proxy we introduce is consistent with the literature and in particular is time-varying, state-dependent, and time consistent.

We test two alternative empirical implementations that are consistent with our conceptual framework. The first implementation intends to differentiate between investors' compensation for learning when using an ambiguous signal from compensation that arises from investors' desire for robustness to ambiguous news. This implementation seeks to test an empirical version of the "approximating" conditional capital asset pricing model (C-CAPM) that results from the

[5] The purpose of the model is to guide our empirical investigation, rather than theoretically propose a particular source and structure for Knightian uncertainty. Accordingly, we introduce the concept of Knightian uncertainty in a manner consistent with both the recursive multiple-priors preferences approach axiomatized in Epstein and Schneider (2003) and the multiplier preferences methodology developed in Hansen and Sargent (2008). Both approaches originate in the multiple-priors expected utility theory of Gilboa and Schmeidler's (1989).

[6] The Knightian uncertainty literature shows that robust investors act conservatively and maximize utility assuming the worst-case scenario.

dynamic robust control/filtering problem generalizing the Bayesian learning model of Veronesi (1999) under Knightian uncertainty. The second empirical implementation aims to test the model of Epstein and Schneider (2008) when investors follow a parsimonious set of ambiguous macroeconomic news.

We find support for both empirical implementations over the period 1927-2007 using as test assets the Fama-French 25 portfolios sorted by size and book-to-market and augmented with 30 portfolios sorted by industry. In particular, the results suggest that Knightian uncertainty is priced in the cross-section of expected stock returns. However, the significance of the learning premium is sensitive to misspecification and the length of the forecasting window used to estimate the time-varying rolling betas. This suggests that for "relaxed" investors the impact of model uncertainty is more important than that of time-varying betas, extending the prediction of Rogers (2001) in a pure risk framework concerning parameter uncertainty.

In a recent article, Anderson, Ghysels, and Juergens (2009) also investigate empirically the asset pricing implications of Knightian uncertainty. Using the dispersion of market return forecasts constructed from professional forecasts of aggregate corporate profits, they find that uncertainty is priced and can help explain the cross-section of stock returns. We report similar findings, using a novel proxy for Knightian uncertainty that is rooted in the investor's investment-consumption behavioral dynamic problem under learning and model misspecification. As a result, we can analyze separately whether Knightian uncertainty and learning, in addition to market risk, are priced in the cross-section. Further, we can also study the relative impact of investors' confidence in their forecasts versus the effects of time-varying betas.

Two other related papers are Zhang (2003) and Ozoguz (2009). They extend Veronesi's (1999) regime-switching asset pricing model and specify the empirical conditional capital asset pricing model (C-CAPM) of a standard Bayesian investor that seeks to learn the hidden state of the economy. In their models, uncertainty about the state of the economy generates another dimension of systematic risk. Empirically, they find support for their models and report that the Fama-French factors become insignificant after controlling for uncertainty. The signal is not ambiguous in their models, however. In the presence of ambiguous news, Knightian uncertainty matters even if investors know the true data generating process.[7]

Roy's (1952) principle of safety-first provides further motivation and meaning to our investigation.[8] Roy introduces his safety-first principle in the context of a portfolio problem under Knightian uncertainty. The principle of safety-first assumes that investors seek to minimize the effects of the worst-case scenario. Thus, investors solve a max-min portfolio problem maximizing utility under the lowest plausible consumption state. This same argument is the main intuition of the Knightian uncertainty literature in economics and finance. Consequently, we use the safety-first principle as a new rationale behind investors' conservative behavior when facing Knightian uncertainty. The literature often considers robust investors "pessimistic" because they emphasize the worst-case scenario corresponding to the lowest plausible consumption state. As mentioned, we propose a complementary interpretation: Investors are "conservative" because they worry about maintaining a "margin of safety."

Our contribution is three-fold. First, we derive and test a dynamic asset pricing model that incorporates Knightian uncertainty and learning. We show that, independently of the theoretical approach adopted, Knightian uncertainty (or a preference for robustness) generates a priced factor in the cross-section of average stock returns, unlike the standard Bayesian case with

[7] Moreover, Knightian uncertainty should still matter even in the absence of ambiguous signals, because the C-CAPM only holds each period "approximately" (Back, 2010, section 10.5, pp. 187).
[8] Lintner (1965, footnote 21, page 243) acknowledges the pioneering work of Roy in asset pricing under Knightian uncertainty.

unambiguous signals. Second, we introduce a new proxy for investors' robustness that has a strong theoretical foundation in the investor's behavioral portfolio-consumption intertemporal problem. Although the theoretical literature on Knightian uncertainty has grown quickly, there are only a few empirical studies, mainly because of the difficulty in measuring uncertainty empirically. Third, we provide two empirical implementations that generalize existing models and allow us to explore the stylized facts of Knightian uncertainty and learning in the cross-section of expected returns. We also show that an empirical asset pricing model that includes Knightian uncertainty outperforms the empirical ICAPM of Petkova (2006) and four-factor model of Fama and French (1992, 1993) and Carhart (1997).

The rest of the paper proceeds as follows. In section 1, we derive the fundamental dynamic asset pricing equation in beta regression form under Knightian uncertainty. In section 2, we introduce a feasible and easy to implement empirical proxy of investors' robustness and describe the empirical models used in the asset pricing tests. Asset pricing test results with robustness checks are provided in section 3. We conclude in section 4. Technical details and proofs are collected in appendices.

1. Conceptual Framework

In this section, we derive the dynamic optimization problem in continuous time of investors concerned with Knightian uncertainty who seek to solve simultaneously a nonlinear signal extraction problem in the stock market under the malevolent actions of "Nature".[9] Subsequently**,** in the spirit of Epstein and Schneider (2008), we derive the forward plan in discrete time of a representative agent (RA) with recursive multiple prior preferences that seeks

[9] This approach falls under the "structured" uncertainty framework, in which Knightian uncertainty does not reduce to increasing investors' coefficient of risk aversion in standard expected utility functions.

to solve the nonlinear filtering problem through an ambiguous signal. Finally, we discuss the conditions under which the continuous time robust dynamic control problem can be interpreted as the limit of a discrete Markov decision problem (MDP) with backward induction solution matching the optimal "safety first" forward plan. We show that independently of the conceptual approach used, investors' concern for Knightian uncertainty (i.e., robustness preference) adds a separate uncertainty premium to the dynamic asset pricing model in beta regression form, unlike the standard Bayesian case with unambiguous signal(s). Technical details are collected in Appendices A and B.

1.1 The continuous time robust dynamic portfolio problem with signaling

Let $X(0:T)$ be the hidden state vector representing the evolution up to a finite horizon T of technological growth in an economy with $i=1,\ldots,N$ risky assets or "technologies" and one riskless "technology". We assume that $X(0:T)$ follows a two-state Markov chain alternating between the "bad" state 0 and "good" state 1

$$dX(t)=[1-X(t-)]dQ^0(t)-X(t-)dQ^1(t) \tag{1}$$

where $X(t-)\equiv\lim_{s\uparrow t}X(s)$; and Q^i are independent Poisson processes with parameters η^i independent of the initial state $X(0)$. That is, X has an exponential distribution determining the transition from state i to state j with parameter η^i.[10] From nonlinear filtering theory (Fujisaka, Kallianpur, and Kunita, 1972), the stochastic dynamics of X follow

$$dX(t)=gdt+\delta dz_X(t) \tag{2}$$

[10] Mimicking the dynamics of the real business cycle. What we do here can be considered a special case of Cagetti et al. (2002) where technological growth is modeled as a general finite state Markov chain with multiple states, transition probability matrix $T(\tau)=\exp(\tau A)$, and jump intensity matrix between states equal to A.

where $g=[1-X(t-)]\eta^0-X(t-)\eta^1$; $dz_X(t)=[1-X(t-)]dz_X^0-X(t-)dz_X^1$ is a Brownian motion; and $z^i\equiv Q^i(t)-\eta^i(t)$ is a martingale.

Investors can solve the nonlinear filtering problem (i.e., learn *X*) using a parsimonious vector of observable (i.e., business cycle driven) variables $Y(t)$ with stochastic dynamics

$$dY(t)=y(X(t-))dt+dz_Y(t) \tag{3}$$

where dz_Y is an *M*-dimensional Brownian motion independent of η^i and $X(0)$. We write $\pi(t)$ for $\hat{X}(t)$, so that $\pi(t)$ denotes the conditional probability that $X(t)=1$ (i.e., good state) with transition dynamics equal to (Wonham, 1965)

$$d\pi(t)=\left[(1-\pi(t))\cdot\eta^0-\pi(t)\cdot\eta^1\right]dt+\kappa dz_\pi(t) \tag{4}$$

where $\kappa=\pi(t)(1-\pi(t))[y(1)-y(0)]'$; and $dz_\pi(t)=dY(t)-[(1-\pi(t))y(0)+\pi(t)y(1)]dt$.

The economy is subject to a $(1+N)\times 1$ vector of Brownian motions i.e., a common systematic shock from *X* plus *N* idiosyncratic shocks such that $dz(t)=\rho_i dz_\pi+\sqrt{1-\rho_i^2}\,dz_i$. Hence, the dynamics of the *N* risky asset returns follow a vector of geometric Brownian motions

$$\begin{pmatrix}\dfrac{dS_1}{S_1}\\ \vdots\\ \dfrac{dS_N}{S_N}\end{pmatrix}=\alpha_i(\pi(t))dt+\sigma_i dz(t),\quad \forall i=1,\ldots,N \tag{5}$$

and the riskless bond yield follows

$$dB(t)=r(t)B(t)dt \tag{6}$$

Under rational expectations, investors hold beliefs such that $d\hat{z}_\pi \xrightarrow{p} dz_\pi$. By contrast, introducing Knightian uncertainty, investors' beliefs are distorted as Nature perturbs the news i.e., $d\tilde{z}_\pi = d\hat{z}_\pi + h(t)dt$, where $h(t) \in H = \{h^2 \le \bar{h}\}$ given some positive constant $\bar{h}$, and consequently $d\tilde{z}(t) = \rho_i(d\hat{z}_\pi + h(t)dt) + \sqrt{1-\rho_i^2}\,dz_i$.[11, 12] The (distorted) hidden DGP now follows

$$d\tilde{X}(t) = (g + \delta h)dt + \delta dz_X \tag{7}$$

The conditional probability that $X(t) = 1$, has (distorted) transition dynamics

$$d\tilde{\pi}(t) = \left[(1-\pi(t))\cdot\eta^0 - \pi(t)\cdot\eta^1 + \kappa h\right]dt + \kappa d\tilde{z}_\pi(t) \tag{8}$$

The introduction of $h(t)$ into the Brownian motion dz_t via the learning mechanism defines a large class of models that are hard to distinguish from the approximating best fitting quasi-maximum likelihood (Q-MLE) model. The discrepancy between the approximating model and the distorted model has been represented theoretically in the related literature using the concept of conditional relative entropy between distributions. When the two models coincide, entropy is zero; when they differ, entropy is positive.[13]

Given equations (7) and (8), the vector of *N* risky asset returns follows a (bounded) distorted geometric Brownian motion, $dS(t)/S(t) = \mu(\tilde{\pi}(t))dt + \sigma d\tilde{z}(t)$ = $[\alpha(\pi(t)) + \sigma\rho h(t)]dt + \sigma\left(\rho d\hat{z}_\pi + \sqrt{1-\rho^2}\,dz_i\right)$. Robust investors' consumption-portfolio behavioral problem under Knightian uncertainty can be stated as

$$\underset{\{C,w\}}{Sup}\ \underset{\{h\}}{Inf}\left\{\hat{E}\left[\int_0^T e^{-rt}\left[u\left(C(t) + \frac{\theta}{2}h^2(t)\right)dt\right]\right]\right\} \tag{9}$$

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[11] In order to tackle Sim's critique, we model Knightian uncertainty as an objective property of the economy, which feeds-back via the shocks from the state vector outside the control of the decision maker.

[12] The only additional requirement is that investors have knowledge of sure and impossible events. Epstein and Marinacci (2006) interpret this result economically as a no-regret principle over previous choices.

[13] Bier and Engelage (2010) show that experience-based entropic risk measures are in general time-inconsistent complicating the use of backward induction to solve the dynamic programming problem. However, time-consistency can be achieved under learning if one assumes that nature acts consistently through time (Maccheroni, Marinacci, and Rustichini, 2006). In particular, we obtain time-consistency by fixing the worst-case scenario reference distribution. The striking result is that if the history of past observations is large enough, eventually Knightian uncertainty will be resolved (i.e., only risk remains).

where $C(t)$ denotes investors' consumption plan; $\theta \in \Theta = \{1 < \theta \leq \infty\}$ is some arbitrary positive penalty parameter indexing investors' robustness;[14] W denotes wealth; and w_i represents the investors' share of wealth invested in the risky asset or technology *i*. Equation (9) is subject to the (distorted) budget constraint

$$d\tilde{W}(t) = \left[W(t)\left(r + \sum_{i=1}^{N} w_i(\alpha_i - r) + \sum_{i=1}^{N} w_i\sigma_i\rho_i h(t) \right) - C(t) \right] dt + W(t)\sum_{i=1}^{N} w_i\sigma_i\left(\rho_i d\hat{z}_\pi + \sqrt{1-\rho_i^2 dz_i} \right) \quad (10)$$

and state space transition and measurement equations (7) and (8), respectively.[15]

Investors concerned with Knightian uncertainty, solve the inner minimization problem first, with Hamilton-Jacobi-Bellman (HJB) equation equal to

$$\begin{aligned} Max\Bigg[u(C(t)) + J_t + J_W\left[W(t)\left(r + \sum_{i=1}^{N} w_i(\alpha_i - r) + \sum_{i=1}^{N} w_i\sigma_i\rho_i h \right) - C(t) \right] + \frac{W^2}{2} J_{WW} \sum_{i=1}^{N}\sum_{j=1}^{N-1} w_i w_j \sigma_{ij} + \cdots + \\ + J_\pi\left((1-\pi(t))\cdot\eta^0 - \pi(t)\cdot\eta^1 + \kappa h(t) \right) + \frac{1}{2} J_{\pi\pi}\kappa^2 + J_{w_i\pi} W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i\kappa + \frac{\theta}{2}h^2 \;\Bigg] \quad (11) \end{aligned}$$

and first order necessary condition w.r.t. *h*

$$\frac{\partial HJB}{\partial h} \equiv J_W W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i + J_\pi\kappa + \theta h = 0$$

such that in equilibrium the minimizing drift distortion is equal to[16]

$$h = -\frac{J_W W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i - J_\pi\kappa}{\theta} \quad (12)$$

[14] This index measures the size of the set of alternative models that robust investors entertain. Hansen and Sargent (2008, chapter 6) show that theta is the Lagrange multiplier attached to the intertemporal constraint $h^2 \leq \bar{h}$. As long as it is assumed that $\theta > 1$, then *h* always hold with equality and one attains the inner minimization solution.

[15] The measurement equation relates the observation vector $Y(t)$ in terms of the state vector $X(t)$ through the signal and the vector of disturbed innovations.

[16] And satisfies the no-gain condition $h_t = e^{-rt}\int_{\pi^-}^{\pi^+} \pi(t)h(t+1)d\pi + \min_{\{\pi\in[\underline{\pi},\bar{\pi}]\}} h_t$ in order to attain time-consistency (Maccheroni et al., 2006).

In Appendix A, we show that investors' optimal portfolio share in risky assets in equilibrium is

$$w^* = -\frac{J_W}{W\left(J_{WW} - \frac{J_W^2}{\theta}\right)}\left[\Omega^{-1}(\alpha - r\iota)\right] + \frac{J_\pi J_W - \theta J_{W\pi}}{\theta W\left(J_{WW} - \frac{J_W^2}{\theta}\right)}\left(\Omega^{-1}\sigma\right)$$

such that

$$J_W(\alpha - r\iota) + \Omega w^* W\left(J_{WW} - \frac{J_W^2}{\theta}\right) + \sigma\left(J_{WX} - \frac{J_W J_\pi}{\theta}\right) = 0$$

where ι denotes a vector of ones; Ω is the variance-covariance matrix of risky asset returns; and $\sigma = \sigma_i \rho_i \kappa$ is a row vector of the covariances of each risky asset with innovations in the conditional belief of good economic times. The fundamental capital asset pricing equation in beta regression form for all risky assets $i = 1, \dots, N$ is (see Appendix A for details)

$$(\alpha_i - r)dt = (\lambda_M)\beta_M + (\lambda_{\tilde{\pi}})\beta_{\tilde{\pi}} + (\lambda_{M\tilde{\pi}})\beta_{M\tilde{\pi}} \tag{13}$$

where M is assumed to be the market portfolio perfectly correlated with aggregate wealth $W^A = \sum_k W^k$ in equilibrium; $\beta_M = \left(\frac{dS}{S}\right)dM$; $\beta_{\tilde{\pi}} = \left(\frac{dS}{S}\right)d\tilde{\pi}$; $\beta_{M\tilde{\pi}} = \left(\frac{dS}{S}\right)\left(dM + \frac{J_\pi}{J_W}d\tilde{\pi}\right)$; and λ is a vector of "robust" market prices for both risk and ambiguity or Knightian uncertainty.

Note that even in the absence of a concern for robustness, from the ICAPM one can derive an "approximate" version of the conditional CAPM if investors are either myopic with zero non-portfolio endowments, or one assumes that one-period-ahead portfolio returns and non-portfolio income independent, see Back (2010). The implications for our conceptual framework is that the resulting learning CAPM will be "approximate" under weak assumptions and equation 13 follows up as a "robust" version of the "approximate" learning CAPM i.e., under Knightian uncertainty.

1.2 The finite horizon forward plan under ambiguous signal(s)

In the spirit of Epstein and Schneider (2008), we now proceed to derive the optimal forward plan matching the backward induction solution of a "modified" discrete time counterpart of the robust dynamic problem in continuous time previously discussed.

Time is discrete $t = 0,1,2$ and there is a risk-neutral (but ambiguity averse) representative agent (RA) that has to decide how much to consume and invest at time 0 choosing between a single unit of the market portfolio with price P_t and a risk-free asset that we assume to be in zero net supply. Note that $P_2 = \tilde{X}$ (where $\tilde{X}$ now denotes gross dividends/earnings from the market portfolio) is paid at date 2 and not before and the riskless return is $R_f = 1$. The RA's intertemporal budget constraint at time *t+1* is

$$W_{t+1} = W_t + w_t\left(P_{t+1} - P_t\right)$$

where w_t is defined as in the previous sub-section.

Let $\tilde{X} \sim N\left(\mu, \sigma_X^2\right)$, we assume that before trading at date 1, the risk neutral (but robust) RA observes the ambiguous signal(s) $\tilde{Y}_1 = \tilde{X} + \tilde{\xi}$. That is, $\tilde{\xi}$ is normally distributed, independent of $\tilde{X}$, and has variance in the interval $\left[\underline{\sigma}_Y^2, \overline{\sigma}_Y^2\right]$. Note that ambiguity or Knightian uncertainty now arises because of the "quality" of the signal(s) rather than from the malevolent actions of "Nature". Intuitively, news from ambiguous signal(s) reflect tangible (e.g., news on earnings) and intangible (e.g., Keynes' animal spirits) information.

The RA updates each prior applying Bayes' rule. This leads to a class of posteriors for $\tilde{X}$ conditional on the signal(s) $\tilde{Y}$ with mean $\mu + \pi_1(\tilde{Y} - \mu)$ and variance $(1 - \pi_1)\sigma_X^2$.[17] As in the previous subsection, we define π as the conditional probability that $\tilde{X}_t = 1$ i.e., the good economic regime. Intuitively, average earnings/dividends are directly proportional to any news, weighted depending on the likelihood of being in the good economic regime. On the other hand, its conditional variance is inversely proportional to the likelihood of being in the good economic regime.

In particular, if $\tilde{Y}$ is defined as a vector of news from a parsimonious set of proxies tracking the real business cycle, then $\tilde{X}|\tilde{Y}$ can be assumed (as in the previous sub-section) to be driven by a two-state Markov chain mimicking the "mean-reverting" dynamics of the real business cycle. Thus

$$\pi = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_{\backslash Y}^2}$$

and $\pi \in [\pi^-, \pi^+]$. Note that the transition probability π now constitutes a "natural" measure of the "quality" of the information contained in the signal(s) $\tilde{Y} \sim N(\mu, \sigma_X^2 / \pi)$.

Hence the robust RA "fixes" ex-ante a range of priors defined by the sequences $\pi_0 \in [\pi^-, \pi^+]$ and $\pi_1 : \Re \to [\pi^-, \pi^+]$ that define the set of plausible Markov chains representing all possible dynamics or paths of the state space in the support $[\pi^-, \pi^+]$. The quasi-maximum likelihood (Q-MLE) estimate $\hat{\pi}$ must shrink this set of priors to a unique conditional distribution "pessimistically".

[17] When each prior is updated using Bayes' rule the forward planning problem satisfies the property of rectangularity defined in Epstein and Schneider (2003), which guarantees dynamic consistency. Moreover, if the optimal forward plan matches the backward induction solution, then the latter is also time consistent.

On the other hand, and simultaneously the forward-looking RA chooses an initial portfolio w_0 and "conservative" investment policy for period 1 i.e., $\underline{\kappa}(Y) \mapsto w_1(\underline{\kappa})$ that will maximize utility recursively under the worst case scenario defined by the sequence $\{\pi_0, \pi_1\}$ that corresponds to π^- each period

$$\underset{\{\pi\}}{Min}\left\{E^{\pi}\left[u(W_2)\middle|W_1, \tilde{Y}\right]\right\} = \underset{\{\pi\}}{Min}\left\{E^{\pi}\left[u\left(W_0 + w_0\left\{P_1\middle|\tilde{Y} - P_0\right\} + w_1(\underline{\kappa})\left\{\tilde{X} - P_1\middle|\tilde{Y}\right\}\right)\right]\right\} \tag{14}$$

The existence of the optimum solution $(w_0 = 1; w_1(\underline{\kappa}) = 1)$ requires that. The present value of the risky asset has to be equal to

$$P_0 = \mu - \frac{\Delta\pi}{\sqrt{2pi\pi^-}} \times \sigma_X$$

where $\Delta\pi = (\pi^+ - \pi^-)$ denotes the size of the set of priors for the good economic state; and

$$P_0 = E^{\pi^-}\left[P_1\middle|\tilde{Y}\right]$$

Concavity of the pricing function is the necessary and sufficient condition to show that the difference $P_{t+1}(\mu) - E^{\pi}\left[P_{t+1}(\tilde{Y})\right]$ is maximized when the precision of the signal is at its minimum value.

If $w_0 > 0$ then $E^{\pi}[W_1]$ is minimized when P_1 is minimized, which corresponds to the case when the signal has the lowest precision. Thus, $P_1 = \mu + \pi^-(\tilde{Y} - \mu)$ and $E^{\pi^-}\left[P_1\middle|\tilde{Y}\right] = \mu - \Delta\pi \times E^{\pi^-}\left[(\tilde{Y} - \mu) \times 1_{\{\tilde{Y} > \mu\}}\right]$. The "discounted premium" for future ambiguous news is directly proportional to the size of the set of priors and the level of volatility in the "fundamentals".

Extending the problem to that of a robust RA with infinite horizon as in Epstein and Schneider (2008) gives an average excess market return for period *t+1* that can be defined recursively as (see Epstein and Schneider, 2008 for details)

$$\overline{R}_{t+1} = P_{t+1} - R_f^{t+1} P_t = \pi^- \left(\tilde{Y} - \mu\right) + \Delta\pi \left(\frac{\sigma_X}{\sqrt{2 pi \pi^-}} \right) \tag{15}$$

where the first term is the component of the ambiguity premium that arises from the (asymmetric) response of market prices to contemporaneous intangible news i.e., bad news are always taken more seriously than good news; and the second term is the component of the ambiguity premium that investors ask in compensation for the effects of future ambiguous news.

In Appendix B, we show that in the more general case of aversion to both risk and Knightian uncertainty the "conservative" optimal forward plan $\underline{\kappa}(Y) \mapsto w_1(\underline{\kappa})$ is the backward induction solution of a "modified" discrete Markov decision problem sufficiently general to be explained within the framework of the dynamic control problem derived in section 1.1. Consequently, we get a "modified" version of equation (13)[18]

$$\forall i \quad E^{\pi^-}\left[\frac{dS_i}{S_i}\right] = \beta_M \lambda_M + \beta_{(\tilde{Y}-\mu)} \lambda_{(\tilde{Y}-\mu)} + \beta_{\Delta\pi} \lambda_{\Delta\pi} \tag{16}$$

where λ_M denotes investors' compensation for market risk; $\lambda_{(\tilde{Y}-\mu)}$ is the learning component of the Knightian uncertainty premium when investors follow ambiguous news; and $\lambda_{\Delta\pi}$ denotes investors' preference for robustness proxied by the size of the set of priors. Equation (16) represents the asset pricing model of Epstein and Schneider (2008) in beta regression form.

[18] From section 1.1 and Appendix A we have $\breve{\lambda}_M = \lambda_M + \lambda_{M\tilde{\pi}}$, and $\breve{\lambda}_{\tilde{\pi}} = \lambda_\pi + \lambda_{\Delta\pi}$. Then $\lambda_{\Delta\pi} = \breve{\lambda}_{\tilde{\pi}} - \lambda_\pi$ corresponds to the asymmetric effect in the news of ambiguous signals, and $\lambda_{M\tilde{\pi}}$ corresponds to investors' robustness to ambiguous signals proxied by the size of their set of priors instead of some penalization parameter θ.

REMARK 1: If the signal is unambiguous, then equation (16) collapses to the "approximate" Bayesian C-CAPM of Veronesi (1999), Zhang (2003), and Ozoguz (2009).

REMARK 2: If the signal(s) are unambiguous and no learning is allowed, then equation (16) collapses to the standard ICAPM of Merton (1973).

2. Empirical Methodology

Our empirical analysis proceeds in three steps. First, we present the regime-switching model investors are assumed to use to learn the hidden state of the economy. Under Knightian uncertainty, investors use the regime-switching model only as an approximation model. The approximating probability of transitioning to the good economic state is obtained via quasi maximum likelihood (Q-MLE) using a regime-switching model of the stock market return as the signal of current and future economic conditions (Perez-Quiros and Timmerman, 2000; Veronesi, 1999, 2000; Zhang, 2003; and Ozoguz, 2009). Second, we introduce our proxy for investors' Knightian uncertainty or robustness.

Third, we run asset pricing tests on two alternative empirical implementations of the R-DAPM, one defined on the transition probability of the stock market return moving into the good economic state, and the other defined on a parsimonious set of macro news driving the transition probability of the stock market return. The first implementation seeks to test a generalization of the empirical "learning" C-CAPM of Zhang (2003) and Ozoguz (2009) under Knightian uncertainty, while the second implementation seeks to test the learning model under ambiguity of Epstein and Schneider (2008) when investors follow a parsimonious set of ambiguous macroeconomic news. Our empirical analysis concludes with a model comparison between the second implementation, the empirical ICAPM of Petkova (2006), and the four-factor asset pricing model in Carhart (1997).

In this section, we describe the three steps in detail. We report the results of the asset pricing tests in the next section.

2.1 Approximating (Q-MLE) model

We assume investors model the excess market return $r^e_{m,t}$ as a first-order smooth transition autoregressive (STAR) process tracking the dynamics of the real business cycle. A STAR model is a nonlinear time series model, similar to Hamilton's regime switching model. However, a STAR model is more useful when the transition variable is assumed to be continuous (van Dijk, Teräsvirta, and Franses, 2002).[19] Specifically, the STAR model is

$$r^e_{m,t} = \gamma_0 + \phi \cdot r^e_{m,t-1} + \varepsilon_t - \varepsilon_{t-1}$$

where $r^e_{m,t-1}$ depends on the hidden state of the underlying economy with constant transition probabilities $\pi_{ij} = \Pr\{s_t = j | s_{t-1} = i\} \quad \forall i, j = 1,2$.

The evolution of the hidden state is measured using news z_{t-1} from a parsimonious set of state variables $Y = [DIV_t, RF_t, TERM_t, DEF_t]'$ that explains investors' time-varying opportunity set.

Therefore,

$$r^e_{m,t-1} | s_t = \delta | s_t \cdot z_{t-1} + \varepsilon_{t-1},$$

where $\varepsilon_t = \sqrt{\zeta_t} v_t$ follows a GARCH (1, 1) process

[19] There is a large literature that shows that stock returns are subject to low frequency jumps in their drift and/or volatility capable of generating fat-tailed and skewed marginal distributions (for a survey, see Singleton, 2006).

$$\zeta_t = \kappa_t + \psi h_{t-1} + \sigma\varepsilon_{t-1}^2$$

Thus, the *k*th test asset excess return $r_{k,t}^e$ follows a two-state regime-switching process

$$r_{k,t}^e \big| s_t = \alpha_k \big| s_t + \beta_{k,MKT} \big| s_t \cdot r_{m,t}^e + u_{k,t}$$

where β_{MKT} is market beta; and without loss of generality $u_{k,t} \sim i.i.d.$ with skewed Student's *t* distribution.[20]

2.2 A feasible and easy-to-implement proxy for investors' robustness

Robust investors concerned with Knightian uncertainty distort the probability of the good economic regime "conservatively". They fix ex ante a minimum confidence level on their approximating model(s) and, given a sample of past observations, act as econometricians and entertain several simultaneous likelihoods for observing good economic times. We note that the constant minimum "envelope" confidence level constitutes a lower bound for the time-varying confidence that investors have "individually" on their plausible models conditional on the signal. As a result, robust investors obtain an interval of possible values for the objective probability of good economic times. Further, robust investors who seek safety first will always either discount the worst-case scenario or take more seriously bad news than good news. That is, they always select the lower bound in the interval.

We assume that robust investors act as they have a set of priors $\prod_0$ about the objective transition probability π represented by the interval $\left[\pi^-, \pi^+\right]$ around the Q-MLE $\hat{\pi}$ obtained

[20] One problem with regime-switching models is parameter instability due to the strong dependence on initial conditions. We assess the stability problem using the Nyblom (1987) and Hansen (1990) test $NH = \frac{1}{T}\sum_{t=1}^{T}\eta_t'\left(\sum_{t=1}^{T} g_t g_t'\right)^{-1}\eta_t$, where $\eta_t \equiv \sum_{j=1}^{t} g_j$, $g_t \equiv \partial(-l_t)/\partial\{\alpha, \beta\}$, and l_t is the parameter vector's contribution to the log-likelihood function. Under the null hypothesis $H_0 = \{\eta_t\}_{t\geq 0}$ is a martingale, and the model is stable.

using the approximating model. We define $1-\varsigma$ as the investors' conservative "envelope" confidence level on the Q-MLE $\hat{\pi}$ with $\varsigma \in [0,1]$ being some increasing function on $h_t^{\min}$.[21]

PROPOSITION 1 (GOODMAN, 1965): *Let $\hat{\pi}_t$ be the subjective (Q-MLE) probability of transitioning to the good economic regime S=g, then given some minimum confidence level $100(1-\varsigma)\%$ on $\hat{\pi}_t(S=g)$ investors will obtain an interval of possible values for the objective probability defined by*

$$\left[\pi_t^-(S=g), \pi_t^+(S=g)\right] = \left[\frac{B-\Delta^{1/2}}{2A}, \frac{B+\Delta^{1/2}}{2A}\right] \tag{17}$$

where $A = \chi^2(1-\varsigma/K,1)+T$ *denotes the quantile of order* $1-\varsigma/K$ *of a chi-square distribution with one degree of freedom with K number of intervals and sample size* $T=\sum_{i=1}^{K} t_i$ *;* $B_i = \chi^2(1-\varsigma/K,1)+2t_i$*; and* $\Delta = B^2-4AC$ *with* $C = t_g^2 / T$.[22]

Proof: See Appendix C.

Proposition 1 gives the time-varying lower (conservative) bound representing the reference (worst-case scenario) probability, which the true (possibly but not necessarily hidden) objective probability should converge at least $100(1-\varsigma)\%$ of the time. We proxy investors' robustness calculating the normalized distance between the Q-MLE and this (worst-case) reference value: $(\hat{\pi}_t - \pi_t^-)/\hat{\pi}_t$.

To illustrate, a robust investor with a Q-MLE of 38% for the transition probability to the good state using the STAR model (likely because the economy is in the trough of the business

[21] The parameter α in Epstein and Schneider (2007, p. 1294) that can be interpreted as a confidence interval related to the interval within which a theory would be excluded from the belief set. Garlappi et al. (2007) also characterize the level of ambiguity in terms of a conservative confidence interval.
[22] As discussed in the Appendix, Goodman's result relies in the Bonferroni inequality. The approach has been criticized in the statistical literature because it leads to a very "conservative" measure of confidence. But this point constitutes our main motivation to use Goodman's approach, as is exactly the kind of behavior one should expect from a "robust" investor.

cycle), a sample of 1,000 monthly data (approximately the size of our data sample from 1927 to 2007), and having a minimum confidence level in the STAR model of 50%, would create an interval of continuum possible values for the hidden objective transition probability between 33% and 43%. A robust investor will act based on 33% instead of 38% as the conditional probability of transitioning to the good economic state. Investors' robustness using our index gives a value of 0.1316. By contrast, if the Q-MLE of the transition probability to the good state is 96% (likely because the economy is in a boom stage), the robust investor would now hold an interval of possible values for the hidden objective probability between 93% and 97%, selecting 93% instead of 96% as the probability of transitioning to the good state and robustness index equal to 0.03125. [23]

Note that the proposed empirical proxy for investors' robustness is consistent with the general framework put forth by Epstein and Schneider (2007, 2008), who allow for learning under ambiguous signal(s). It is also consistent with the general approach in Hansen and Sargent (2008, 2009) where the source of ambiguity arises from the malevolent actions of "Nature". Technically, it falls in the general class of Wald's confidence intervals for simultaneous hypothesis testing with well-known asymptotic properties in large samples. A discussion of its small sample properties is provided in Lui and Cumberland (2004).

2.3 First empirical implementation of the R-DAPM

Zhang (2003) and Ozoguz (2009) implement empirically Veronesi's (1999) learning model for stock returns when investors do not observe the stochastic process driving dividends using the following (unconditioned down) model in beta regression form

$$E\left[r_k^e\right] = \beta_{k,MKT}\lambda_{k,MKT} + \beta_{k,JUMP}\lambda_{JUMP} + \beta_{k,UNC}\lambda_{UNC} + \alpha_k \tag{18}$$

[23] Note that despite giving a "conservative" measure, the resulting interval is "tight" that is, sufficient enough to affect significantly investors' actions in the market within the neighborhood of "close models".

where *JUMP* is “learning risk” proxied by π_t, and *UNC* is “uncertainty risk,” measured in a pure risk Bayesian framework as $\pi_t(1-\pi_t)$. The probability π_t represents investors' estimate about the hidden state of the economy. When π_t is high, the economy is more likely to be in the good state; when it is low, the economy is more likely to be in the bad state. When π_t equals 0.5, investors' are most uncertain in the Bayesian sense about the true state of the economy. The product $\pi_t(1-\pi_t)$ can be interpreted as an uncertainty index that approximates zero when investors are confident about the true state of the economy, good or bad, and takes a maximum value of 0.25 when investors are most uncertain (Ozoguz, 2009). This approach assumes that the signal is unambiguous.

In order to differentiate investors’ compensation for learning using an ambiguous signal from the pure robustness component in the ambiguity premium, our first empirical implementation of the R-DAPM is specified as

$$E\left[r_k^e\right] = \beta_{k,MKT}\lambda_{k,MKT} + \beta_{k,JUMP}\lambda_{JUMP} + \beta_{k,KUNC}\lambda_{KUNC} + \alpha_k \tag{19}$$

where *JUMP* now denotes the asymmetric component in the premium when learning from ambiguous signal(s), measured by π_t^-; and *KUNC* denotes the pure ambiguity component in the premium measured by $\left(\hat{\pi}_t - \pi_t^-\right)/\hat{\pi}_t$.

2.4. Second empirical implementation of the R-DAPM

The second empirical implementation of the R-DAPM is defined on a parsimonious vector of signals driving the transition probability of the state space i.e., innovations derived from a VAR model that includes *DIV*, *TERM*, *DEF*, and *RF*. Innovations are obtained using a first-order vector autoregression (VAR).[24] Further, we also include as regressors the two Fama-French (1993) risk factors related to size (*SMB*), and book-to-market (*HML*), as well as Jegadeesh (1990), and Jegadeesh and Titman's (1993) momentum (*UMD*) risk factor. We define the *SMB*, *HML*, and *UMD* factors alternatively in levels and as innovations obtained from an augmented version of the original VAR. As discussed by Epstein and Schneider (2008), ambiguous news could arise from firm-specific variables, too.

3. Asset Pricing Tests

3.1 Test assets

The set of test assets used in the asset pricing tests include the value-weighted monthly returns on the 25 Fama-French portfolios sorted by size (ME) and book-to-market (BE/ME), obtained from Kenneth French's website.[25] The sample period runs from 1927 to 2007 but we also discuss results for the shorter COMPUSTAT period from 1962 to 2007.[26] Following Lewellen et al. (2008), we also perform a robustness check on an augmented set of test assets that includes 30 industry-sorted portfolios.

3.2 Stylized Time-Series Properties of Stock Returns

In Table 1, we report the estimates of the STAR model for the excess market return. We assume investors use the model as an approximating model. The excess market return *MKT* is measured using the value-weighted CRSP stock market index. The independent (state) variables are the lagged values of: the market dividend yield, *DIV* (Campbell and Shiller, 1988), which serves as a proxy for time-varying expected stock returns; the one-month T-bill yield, *RF*, which proxies

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[24] Campbell and Shiller (1988) show that any high order VAR can be collapsed to its first order (companion) VAR.
[25] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.
[26] We do not report them because of the space constraint, but they can be obtained by request to the authors.

the level of interest rates in the underlying economy (Fama and Schwert, 1977); the difference between the average yield of a portfolio of long-term government bonds with more than 25 years of maturity and the one-month T-bill, *TERM*, which proxies for the slope of the yield curve in the underlying economy (Campbell, 1987); and the difference between the average bond yield of a portfolio of long-term corporate bonds (Aaa/Baa) and a portfolio of long-term government bonds, *DEF*, as proxy for default risk (Fama and French, 1989).

During good economic times (Regime 1), there is a significantly positive association between future market returns and DEF at a 95% confidence level or better. During bad economic times (Regime 2), there is a significantly negative association between future market returns and *RF*, *DIV*, and *TERM* at a 95% confidence level. Note that the skewness and heteroskedasticity coefficients are statistically significant, consistent with well-known features of stock returns**.**

Table 2 shows estimates of the regime-switching time-series regressions using as test assets the 25 Fama-French stock portfolios sorted by size and book-to-market. Market betas are statistically significant in both regimes, but significantly higher during Regime 2. Based on the Nyblom-Hansen test results, we cannot reject the null that the coefficients of the time-series regressions are stable.

3.3 Asset Pricing Tests Based on the First Empirical Implementation

In Table 3, we report the results of our first empirical implementation of the R-DAPM, which generalizes the empirical learning models of Zhang (2003) and Ozoguz (2009) when the signal is suspected to be ambiguous. We show results of asset pricing tests based on two-step cross-sectional regressions (henceforth CSR; Black et al., 1972; Fama and MacBeth, 1973) that use excess returns of 25 Fama-French portfolios sorted by size and book-to-market as dependent variables over the sample period 1927-2007. We test whether asset loadings on learning (*JUMP)* and robustness (*KUNC*) are important determinants of average cross-sectional returns; if they are, there should be a significant premia for ambiguity associated with *JUMP* and *KUNC*.

Our proxies for learning and uncertainty vary with the "envelope" confidence level that investors may have on their models. Thus, we perform the tests under alternative minimum confidence levels (CL). More specifically, we decrease the typical investor's minimum confidence in her model from a 90% confidence level to a 50% level. A 50% confidence corresponds to a very robust investor who assigns a 50% probability that the family of plausible models in a close neighborhood of her forecast is correct. To assess the effects of time-varying betas, we also repeat the estimation procedure increasing the window length (T) of the first-pass rolling regressions from 36 months to 180 months. The different lengths can be interpreted as indicating how frequently investors review their investment policies based on the news. Rogers (2001) hypothesizes in a pure risk framework that parameter uncertainty is more important than the frequency of investment policy revisions, even for investors who review infrequently (i.e., "relaxed" investors) their intertemporal consumption/investment decisions.[27]

As shown in Table 3, the price of market risk is statistically significant and fairly constant around 0.7% per month across different confidence levels and rolling regression window lengths. The learning component of the ambiguity premium is statistically significant only when the typical robust investor uses predictive regressions with length less than 96 months to estimate the

[27] Because betas are parameters estimated from first pass time-series regressions, their use in second pass cross-section regressions leads to the errors-in-variables (EIV) problem. We follow Shanken (1992) in correcting for this problem. Note that VAR innovations and transition probabilities are also generated regressors that appear in first pass time-series regressions leading to a second EIV problem. Moreover, model misspecification could be severe especially when proxying for unobservable macroeconomic risk factors. Consequently, when introducing the intercept in the cross-section regression we assess the null hypothesis of zero misspricing using robust standard errors and cross-sectional goodness of fit R-squared stats ρ^2 as derived in Kan et al. (2009).

factor loadings. This finding supports Lewellen and Nagel's (2006) critique that the conditional CAPM relies on unrealistic short windows in the predictive regressions to explain asset pricing anomalies. The pure robustness component of the premium is statistically significant, particularly when the investor is "relaxed," similar to the case of parameter uncertainty in a pure risk framework (i.e., the length of the rolling regressions is above 36 months). Monthly premiums vary between 1% to and 3% and increase as investors' confidence levels decrease.

We define the composite vector Z of *f* risk factors and *R* test asset returns $Z = \left(f' \ \ R'\right)'$ with mean $E[Z]' = \left(\mu_1 \ \ \mu_2\right)'$ and variance-covariance matrix $Var[Z]' = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$. Given that in this step of the empirical analysis we do not include an intercept in the cross-section regression (Cochrane, 2001, section 12.2, page 235; Brennan, Xia, and Wang, 2004), to evaluate the null hypothesis of zero misspricing we use the composite pricing error $Q = T\overline{\boldsymbol{\alpha}}'W^{-1}\overline{\boldsymbol{\alpha}}$, where T is the size of the sample period, $\overline{\boldsymbol{\alpha}}$ denotes the average residual vector in the second pass cross-section regression, and W is a symmetric positive weighting matrix that is equal to $W = \hat{V}_{22}^{-1}$ under generalized least squares (GLS). This cross-sectional test has an asymptotic chi-squared distribution. As shown at the bottom of Table 3, we cannot reject the model at the 10% level for the sample 1927-2007.

As noted, the premium for investors' robustness becomes significant with longer predictive regressions (i.e., the more infrequent the forecast/policy review), in contrast to the component of the premium that arises from learning using an ambiguous signal. This finding extends Rogers's (2001) hypothesis beyond the pure risk setting: model uncertainty, which likely includes parameter uncertainty, is more important than infrequent policy review when the investor is relaxed. As shown in Figures 1 and 2, investors' robustness premia increases as their confidence in their models fall and they review less frequently their forecasts. In contrast, both market and learning premia remain constant. We note that the results in Zhang (2003) that correspond to the case of an unambiguous signal in a pure risk setting correspond to the special case of a very confident robust investor (i.e., 90%) in our general framework under ambiguity.

We now report test results from both second-pass Fama-MacBeth regressions and time-series regressions using observable and tradable factor mimicking portfolios tracking "orthogonalized" innovations in the state variables. We follow the general procedure in Breeden et al. (1989), Lamont (2001), and Vassalou (2003) to construct fully tradable ex-ante maximum correlated portfolios (MCP), or mimicking portfolios, of innovations in the two proposed ambiguity factors associated with learning (*JUMP*) and investors' robustness (*KUNC*). Specifically, we run multiple OLS regressions of 12-month-ahead forecasts on *JUMP* and *KUNC* on returns of observable tracking portfolios *SMB*, *HML*, and *UMD* along with control variable portfolios for *JUMP* and *KUNC* (Lamont, 2001). The control variables include *MKT*, *DIV*, *RF*, *TERM*, *DEF*, following Petkova (2006), as well as either *JUMP* or *KUNC* lagged one period.

We use different realized values of *JUMP* and *KUNC*, but because the resulting innovations are quantitatively and qualitatively similar, we report results for a typical robust investor: one who has a minimum confidence level in the worst case scenario of 50% on her family of "close neighbor" models around the approximating model and uses a forecasting window of 36 months in the predictive regressions. Figure 3 plots the factor *KUNC*. Although not shown, we also run regressions on 30 portfolios sorted by industry in the construction process with similar results. Table 4 provides summary statistics of the two full tradable MCPs tracking the news in *JUMP* and *KUNC*. Over the period 1927-2007, the MCP average monthly return for *JUMP* and *KUNC* are 0.16% and 0.38%, respectively.

Note that in this step of the analysis we include the intercept in the cross-section regressions as is common practice in the empirical asset pricing literature. Furthermore, we use full-sample instead of rolling betas following the analysis in Kan et al. (2009). This approach introduces model misspecification explicitly.

In Table 5 (Model 1), we report the asset pricing test results of Fama-MacBeth cross-sectional and time-series regressions using monthly excess returns on the 25 portfolios sorted by size and book-to-market for the whole sample period. The independent variables are the first-pass estimated factor loadings on the risk and ambiguity factors *MKT*, *JUMP* and *KUNC*, respectively. There is evidence that investors' Knightian uncertainty (*KUNC*) is "priced," which is consistent with the results of the previous section. The component of the ambiguity premium (*KUNC*) is significant even after adjustments for the EIV problem and model misspecification, following Shanken (1992), Jagannathan and Wang (1998), and Kan et al. (2009). In contrast, the learning component of the premium (*JUMP*) is now statistically insignificant. This is most likely the result of having introduced model misspecification through the inclusion of an intercept term. Note though that despite of this result, learning still may contribute to the explanatory power of the behavior of stock returns.

Additionally, we cannot reject the null that $H_0 : \rho^2 = 1$ for the sample period 1927-2007 with a 5% significance level; however, we reject it for the shorter COMPUSTAT sample period (i.e., 1962-2007). We also reject the null $H_0 : \rho^2 = 0$ at a 5% significance level for both sample periods. Therefore we conclude that although misspecified the model can explain to some extent the cross-section of expected stock returns. Results are validated using a Wald test for the joint significance of the three estimates on time-series regressions as this constitutes a cross-sectional asset pricing test when all risk factors are observable and tradable.

3.4 Asset Pricing Tests Based on the Second Empirical Implementation

As an alternative to the model that includes *MKT*, *JUMP*, and *KUNC*, we report in Table 5 (Model 2) the results of the empirical specification that includes *MKT*, *RF*, *DIV*, *TERM*, *DEF*, and *KUNC* as factors. This alternative empirical specification is used to test the asset pricing model of Epstein and Schneider (2008) when investors follow a parsimonious set of ambiguous macroeconomic news (equation 16).

Therefore, our second empirical implementation of the R-DAPM is

$$E[R_k] = \lambda_0 + \hat{\beta}_{MKT}\lambda_{MKT} + \hat{\beta}_{RF}\lambda_{RF} + \hat{\beta}_{DIV}\lambda_{DIV} + \hat{\beta}_{TERM}\lambda_{TERM} + \hat{\beta}_{DEF}\lambda_{DEF} + \hat{\beta}_{KUNC}\hat{\lambda}_{KUNC} + u_k \quad (20)$$

where λ denotes premia; and $\hat{\beta}$ factor loadings to the vector of factors.

As shown in Table 5, *TERM* and *KUNC* are "priced" in the period 1927-2007; they are also priced in the 1962-2007 (not shown). Furthermore, we cannot reject the null that $H_0 : \rho^2 = 1$ for both sample periods at a 5% significance level. These results suggest that this model seems to be correctly specified, a result that is validated using a Wald test for the joint significance of the six estimates on the time series regressions constituting a cross-sectional asset pricing test when all risk factors are observable and tradable.

Additionally, we check the robustness of the previous results using an augmented set of test assets that includes 25 Fama-French portfolios sorted by size and book-to-market and 30 portfolios sorted by industry. Table 6 provides the results for the sample period 1927-2007; results for the period 1962-2007 are similar. Overall our previous results are robust to the augmented set of test assets. Importantly, the statistical significance of investors' robustness *KUNC* in the cross-section regression is stronger using the augmented set of test assets.

3.5 Model comparisons

Kan et al. (2009) report that the empirical ICAPM of Petkova (2006) fails to pass asset pricing tests robust to model misspecification, unlike the four factor model of Fama-French (1993) and Carhart (1997), for the post-Compustat period. Consequently, we compare the asset pricing model under Knightian uncertainty with the standard ICAPM of Petkova (2006) and the Carhart (1997) four-factor model, using the robust comparative R-squared test developed by Kan et al. (2009) for nested and non-nested models.

Table 7 shows the test results proposed in Kan et al. (2009) to compare alternative asset pricing models potentially misspecified. The FF4 model includes *MKT*, *SMB*, *HML*, and *UMD* (Carhart, 1997); the ICAPM includes *MKT*, *RF*, *DIV*, *TERM*, and *DEF* (Petkova, 2006); and the R-DAPM includes *MKT*, *RF*, *DIV*, *TERM*, *DEF*, and *KUNC*. The null hypothesis is that the differences in *R*-squared statistics between the FF4 model, the ICAPM, and the R-DAPM are not statistically significant. Panel A provides the test results for GLS CSR on 25 Fama-French portfolios sorted by size and book-to-market for the sample period 1927-2007 (969 observations). Panel B provides the test results for the sample period from 1962-2007 (552 observations). The R-DAPM outperforms the two other asset pricing models. In particular, when model misspecification is taken into account, the R-DAPM outperforms the FF4 and performs as well as the ICAPM at the 1% level, but survives model misspecification tests unlike the model of Petkova (2006) for the post-Compustat period.

4. Conclusion

We derive the fundamental dynamic asset pricing equation in beta regression form of investors concerned with Knightian uncertainty who solve a portfolio choice problem seeking “safety first" while simultaneously learning the hidden state of the underlying economy from ambiguous signal(s). Asset pricing tests are implemented empirically through: 1) an extension of the empirical C-CAPM of Zhang (2003) and Ozoguz (2009) where the signal is now assumed to be ambiguous; and 2) the learning model of Epstein and Schneider (2008), when investors follow a parsimonious set of ambiguous macroeconomic news.

Importantly, we introduce a novel proxy for investors' robustness rooted in the two main "structured" approaches to Knightian uncertainty. Our proxy is time-varying, state-dependent, and time-consistent. We provide empirical evidence that learning from ambiguous signal(s) and investors' robustness are priced in the cross-section of expected stock returns. However, the component of ambiguity premia corresponding to investors' Knightian uncertainty increasingly dominates the one corresponding to learning as investors' investment horizons increase.

We also perform a model comparison test that indicates that an empirical representation of the model of Epstein and Schneider (2008) explains the cross-section of average stock returns better than the Fama-French-Carhart four-factor model. Further, the empirical representation survives model misspecification tests in shorter post-Compustat sample period, contrary to the empirical ICAPM of Petkova (2006).

Appendix A: The dynamic robust control problem in continuous time

Substituting back the solution in equation (12) into the HJB equation (11) we get

$$J_W W\sum_{i=1}^{N} w_i(\alpha_i - r) + J_W W\sum_{i=1}^{N} w_i\sigma_i\rho_i\left(-\frac{J_W W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i - J_\pi\kappa}{\theta}\right) + \frac{W^2}{2}J_{WW}\sum_{i=1}^{N}\sum_{j=1}^{N-1} w_i w_j\sigma_{ij} + \cdots +$$

$$J_\pi\left[\alpha + \kappa\left(-\frac{J_W W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i - J_\pi\kappa}{\theta}\right)\right] + J_{W\pi}W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i\kappa + \frac{\theta}{2}\left(-\frac{J_W W(t)\sum_{i=1}^{N} w_i\sigma_i\rho_i - J_\pi\kappa}{\theta}\right)^2 \quad \text{(A.1)}$$

Isolating terms in w_i and substituting back into the HJB equation (11) we get

$$J_W W(t)(\alpha_i - r)\left(-2\frac{J_W^2 W^2(t)\sum_{j=1}^{N-1} w_j^*\sigma_{ij}}{\theta}\right) - \frac{WJ_W J_\pi\kappa\sigma_i\rho_i}{\theta} + J_{WW}W^2\sum_{j=1}^{N-1} w_j^*\sigma_{ij} + J_{\pi W}W(t)\sigma_i\rho_i\kappa +$$

$$+\frac{J_W^2 W^2\sum_{j=1}^{N-1} w_j^*\sigma_{ij}}{\theta} = 0 \quad \text{(A.2)}$$

and

$$J_W W(t)(\alpha_i - r) + J_{WW}W^2\sum_{j=1}^{N-1} w_j^*\sigma_{ij} - J_W^2 W^2\sum_{j=1}^{N-1} w_j^*\sigma_{ij} + J_{\pi W}W(t)\sigma_i\rho_i\kappa - \frac{W(t)J_W J_\pi\delta\rho_i\sigma_i}{\theta} = 0 \quad \text{(A.3)}$$

Dividing by *W* we get

$$J_W(\alpha - r\iota) + \Omega w^*\left(J_{WW}W - \frac{J_W^2 W}{\theta}\right) + J_{W\pi}\sigma - \frac{J_W J_\pi}{\theta}\sigma = 0 \quad \text{(A.4)}$$

where ι denotes a vector of ones; Ω is the variance-covariance matrix of risky asset returns; and $\sigma = \sigma_i\rho_i\kappa$ is a row vector of the covariances of each risky asset with innovations in the conditional belief of good economic times. In equilibrium, the investors' optimal portfolio share in risky assets is equal to

$$w^* = -\frac{J_W}{W\left(J_{WW} - \frac{J_W^2}{\theta}\right)}\left[\Omega^{-1}(\alpha - r\iota)\right] + \frac{J_\pi J_W - \theta J_{W\pi}}{\theta W\left(J_{WW} - \frac{J_W^2}{\theta}\right)}\left(\Omega^{-1}\sigma\right) \tag{A.5}$$

such that

$$J_W(\alpha - r\iota) + \Omega w^* W\left(J_{WW} - \frac{J_W^2}{\theta}\right) + \sigma\left(J_{WX} - \frac{J_W J_\pi}{\theta}\right) = 0 \tag{A.6}$$

But this is the fundamental capital asset pricing equation written in general form as[28]

$$(\alpha - r\iota)dt = -\left(\frac{dS}{S}\right)\left(\frac{dJ_W}{J_W}\right) \tag{A.7}$$

which holds approximately for every investor k and risky asset i. Thus

$$(\alpha - r\iota)dt \approx \frac{-\left(J_{WW} - \frac{J_W^2}{\theta}\right)}{J_W}\left(\frac{dS}{S}\right)dM - \frac{\left(J_{W\pi} - \frac{J_W J_\pi}{\theta}\right)}{J_W}\left(\frac{dS}{S}\right)d\tilde{\pi} \tag{A.8}$$

where M is the market portfolio perfectly correlated with aggregate wealth $W^A = \sum_k W^k$ in equilibrium. Then, for all $i = 1, \ldots, N$ it must be that

$$(\alpha_i - r)dt \approx \breve{\lambda}_M(\theta)\left(\frac{dS}{S}\right)dM + \breve{\lambda}_{\tilde{\pi}}(\theta)\left(\frac{dS}{S}\right)d\tilde{\pi} \tag{A.9}$$

where $\breve{\lambda}_M \equiv -\sum_k J_{WW}/J_W + \sum_k J_W/\theta$; and $\breve{\lambda}_{\tilde{\pi}} \equiv -\sum_k J_{W\pi}/J_W + \sum_k J_\pi/\theta$. We define $\lambda_M \equiv -\sum_k J_{WW}/J_W$, $\lambda_{M\tilde{\pi}} \equiv \sum_k J_W/\theta$, $\lambda_\pi \equiv -\sum_k J_{W\pi}/J_W$, and $\lambda_{\Delta\pi} \equiv J_\pi/J_W\ \lambda_{M\tilde{\pi}}$. So that

$$(\alpha_i - r)dt = (\lambda_M)\beta_M + (\lambda_{\tilde{\pi}})\beta_{\tilde{\pi}} + (\lambda_{M\tilde{\pi}})\beta_{M\tilde{\pi}} \tag{A.10}$$

where $\beta_M = \left(\frac{dS}{S}\right)dM$; $\beta_{\tilde{\pi}} = \left(\frac{dS}{S}\right)d\tilde{\pi}$; and $\beta_{M\tilde{\pi}} = \left(\frac{dS}{S}\right)\left(dM + \frac{J_\pi}{J_W}d\tilde{\pi}\right)$. *Q.E.D.*

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[28] By Ito's lemma

$dJ_W(W(t), \tilde{\pi}(t), t) = \left(J_{WW} - J_W^2/\theta\right)dW + \left(J_{W\pi} - J_W J_\pi/\theta\right)d\tilde{\pi} + J_{Wt}dt + \tfrac{1}{2}J_{WWW}(dW)^2$ *+ something dt,*

and $(dS/S)(dJ_W/J_W) = \left(J_{WW} - J_W^2/\theta\right)(dS/S)dW + \left(J_{W\pi} - J_W J_\pi/\theta\right)(dS/S)d\tilde{\pi} + O(t)$.

Appendix B: The "modified" robust Markov decision problem (MDP) in discrete time

Let time be discrete with finite time horizon T i.e., $t \in [0,T]$. Investors information structure is represented by the event tree $\{\mathbf{F}_t\}_{t=0,\ldots,T}$, where $\mathbf{F}_0 \equiv \Omega$ with $\Omega \equiv \otimes_{t=0}^{T} S_t$ the set of all plausible states S; $\mathbf{F}_T \equiv \{\{\omega\} : \omega \in \Omega\}$; $\omega = (\omega_1, \ldots, \omega_T)$ are all possible observation paths generated by the sequence $(X_t)_{t\geq 0}$; $\mathbf{F}_{t+1} \subset \mathbf{F}_t$; and the tree has a 2-valued splitting function mimicking the dynamics of the real business cycle. Investors solve a Markov decision problem (MDP) at each point in time i.e., the stochastic dynamic problem of consuming today or postponing consumption to the future and investing, while seeking to learn simultaneously the state and the data generating process $(X_t)_{t\geq 0}$ driving the time-variant investment opportunity set.

Given the assumptions of: 1) a constant and finite splitting function; and 2) finite time horizon we define $(S, A, \Gamma) \in \mathrm{B}(\Re^n)$, where $A \in \mathrm{B}(\Re^m)$ is the space of investors' X-adapted (state dependent) consumption plans; S is the finite discrete state space endowed with the σ-algebra $\sum$ i.e., the set of all Borel-measurable investment strategies $\kappa : S \to A$ mapping each possible $\omega \in S$ into a feasible consumption plan $a \in \Gamma(\omega)$; and $\Gamma(\omega)$ is a continuous and compact valued mapping from $\omega \in S$ into $\mathrm{B}(A)$, that is the collection of all feasible consumption plans available to investors given $\omega \in S$.

We define $gr\Gamma \equiv \{(\omega, u) \in S \times A : a \in \Gamma(\omega)\}$ as the graph of Γ or the set of all feasible pairs of states and consumption plans. Let $Z \in \mathrm{B}(\Re^k)$ be some arbitrary space defined by sequences $(\xi_t)_{t\geq 1} \sim \phi \in P(Z = z)$ of *i.i.d.* shocks, each one driving the investment opportunity set with transition function $\Pi : gr\Gamma \times Z \ni (\omega, a, z) \mapsto \pi(\omega, a, z) \in S$. We can follow the robust control literature, where $(h_t)_t$ is generally interpreted as a dynamic penalty function satisfying

$h_t \in H = \{h^2 \le \overline{h}\}$ for some positive constant $\overline{h}$. Nature distorts the news $(\xi_t)_{t\ge 1} \equiv (\hat{\xi}_t + h_t)_{t\ge 1}$, where $\hat{\xi}$ are the news given investors' best (Q-MLE) look ahead (common) subjective belief $\hat{\phi} \in P(Z = z)$. Alternatively, in the spirit of Epstein and Schneider (2008) we can modify the model and assume that news encompass tangible and intangible information given the range of precisions entailed in the ambiguous signal(s).

We fix $\psi^0 \in P(\Omega)$ as the reference worst-case marginal distribution corresponding to investors' confidence in their models while solving their stochastic consumption plans. And define the set of all possible distributions equivalent to ψ^0 as $\Phi(\psi^0) = \{\phi | Z = z\}$ with conditional distribution equal to $\hat{\phi} \in \mathrm{P}(Z = z)$. That is, the reference distribution only defines the null sets (i.e., sure and impossible events).

Under the first approach, at each time *t*, Nature picks $S \in \Omega$ with probability $\pi(S)$ and cost

$$h_\Omega(\pi) = \inf_{\{P:P(S)=\pi(S)\ \forall S\in\Omega\}} \left\{ \delta \sum_{S\in\Omega} \pi(S) \times h_S(\pi_S) + h_\Omega(P) \right\} \tag{B.1}$$

where $\pi(S) \equiv \sum_{\omega\in S} \pi(\omega)$; $\delta \in (0,1)$ is the time discount factor; and

$\pi_S(\omega) \equiv \begin{cases} \pi(\omega)/\pi(S), if \quad \omega \in S \\ 0, o.w. \end{cases}$ (Maccheroni et al., 2006). Investors observe $X = \omega$, update their beliefs conservatively consistent with some minimum $h_t^{\min}$ and respond with action $a_t \in \Gamma(\omega_t) \subset A$. After choosing $a_t \in \Gamma(\omega_t) \subset A$ investors receive the reward $u_t(\omega_t, a_t)$ and the state is updated recursively i.e., $X_{t+1} = \pi_t(\omega_t, a_t, z_t)$.

Note that if condition (B.1) holds at each point in time *t*, then Nature will act consistently. That is, if it tries to foul investors there will be no gain in doing so. We show that this condition can follow from the properties of Markov chains using the coupling method. The proof is by contradiction.

We start assuming that Nature draws from two independent alternative Markov chains $(X_t)_{t\geq 0}$ and $(X_t^*)_{t\geq 0}$ with transition functions and marginal distributions equal to $\pi(\omega_t, \xi_{t+1})$ $X_0 \sim \psi$, and $\pi(\omega_t, \xi_{t+1}^*)$ $X_0^* \sim \psi^*$, respectively. Given that every Markov chain in a finite space has at least one stationary distribution, we also assume that $\psi^* \in P(S)$ is stationary. We assume that Nature actually seeks to foul investors switching the source of shocks at some coupling time $\tau = \min\{t \geq 0 : X_t = X_t^*\}$ with positive gain equal to $(1-\varepsilon)$ for some arbitrarily small ε equal to $P(X_t' = X_t^*)$. That is, the observed process $(X_t')_{t\geq 0}$ follows $(X_t)_{t\geq 0}$ for $t \leq \tau$ and then switches to $(X_t^*)_{t\geq 0}$ for $t > \tau$ while investors keep updating $(X_t')_{t\geq 0}$ using the stochastic kernel

$$\sum 1_{\{B\}}\left[\pi(\omega_{t,}\xi_{t+1})\right]\cdot \phi(\Delta\xi), \text{ for all } \omega \in S \text{ and } B \in \mathrm{B}(S) \tag{B.2}$$

based on $\pi(\omega_t, \xi_{t+1})$ $X_0 \sim \psi$, with corresponding Markov operator M^t such that $X_t' \sim \psi M^t$. Given $\psi^* \in P(S)$, at each time *t* we fix $\{\pi^-, \pi^+\}$ two probability measures corresponding to the worst and best-case scenarios. Note that one can obtain a similar result if the signal is suspected to be ambiguous as in the second approach (see sub-section 1.2). Independently of the interpretation adopted, we can define the metric

$$\left\|\hat{\pi} - \pi^-\right\| \equiv \sup_{B\in B(S)}\left|\hat{\pi}(B) - \pi^-(B)\right| \text{ on } \Phi(\psi^0) \tag{B.3}$$

which is proportional to the total variation distance between the quasi-MLE and the reference (worst-case scenario) distributions in the set $\Phi(\psi^0)$. Note that (B.3) implies that $\left\|\psi M^t - \psi^*\right\|_{t\to\infty} \leq P(X_t' \neq X_t^*)$. Intuitively, if at each time *t* the probability that $(X_t')_{t\geq 0}$ and

$\left(X_t^*\right)_{t\geq 0}$ differ is small, then the distance between their distributions should be also small, which is generally assumed in the Knightian uncertainty literature.

If the stochastic kernel for the joint process $\left(X_t', X_t^*\right)_{t\geq 0}$ on $S \times S$ satisfies the condition $\varepsilon \equiv \min\left\{P\left(X_{t+1}' = X_{t+1}^*\right)\right\} > 0$, independently of the current state, then for all $j \leq t$ it must be that $P\left(X_t' \neq X_t^*\right) \leq P \cap_{j \leq T} \left(X_j \neq X_j^*\right) \leq \left(1 - \in\right)^t$ for all $t \in [0, T]$. The first part of the inequality arises because the event $X_t' \neq X_t^*$ implies occurrence of $X_j \neq X_j^*$. Define $x = (\omega, S)$ and $D \equiv \{x \in S \times S : \omega = S\}$. If we expand the probability $P \cap_{j \leq T} \left(X_j \neq X_j^*\right)$ in terms of sums of stochastic kernels over D^C with last term (from time *t-1* to *t*) equal to $\sum_{x^t} k\left(x^{t-1}, x^t\right) \leq 1 - \varepsilon$. Then working backwards through the expression gives the second part of the inequality.

Then, under a sufficiently large sample of observations it must be that in the limit as $t \to \infty$, $\left\|\psi M^t - \psi^*\right\|_{t \to \infty} \to 0$ and Nature's gain is $(1-\varepsilon)^t = 0$, which contradicts our initial assumption.

Consistent with Nature, or alternatively because robust investors update each prior using Bayes' rule, they will always act time consistently when choosing the conservative investment policy $\underline{\kappa} \in \Sigma$ that corresponds to the stochastic recursive sequence

$$\tilde{X}_{t+1} = \pi\left(\omega_t, \underline{\kappa}(\omega_t), \xi_{t+1}\right) \tag{B.4}$$

with stochastic kernel on *S* equal to

$$\sum 1_{\{B\}}\left[\pi\left(\omega, \underline{\kappa}(\omega), \xi\right)\right] \cdot \phi(\Delta\xi), \text{ for all } \omega \in S \text{ and } B \in \mathrm{B}(S) \tag{B.5}$$

We define $M_{\underline{\kappa}}$ as the corresponding Markov operator and define the investors' conservative reward function as

$$u_{\underline{\kappa}} : S \ni \omega \mapsto r\left(\omega, \underline{\kappa}(\omega)\right) \in \Re \tag{B.6}$$

where $u : gr\Gamma \to \Re$ is assumed to be continuous and bounded. The next period expected reward under the conservative investment policy $\underline{\kappa}$ is

$$M_{\underline{\kappa}} u_{\underline{\kappa}}(\omega) = \sum r_{\underline{\kappa}}\left[\pi(\omega, \underline{\kappa}(\omega), \xi)\right] \cdot \phi(d\xi), \text{ for all } \omega \in S \tag{B.7}$$

yielding the transition path

$$\tilde{X}_{t+1}(S) = \pi(\omega_t(S), \underline{\kappa}(X_t(S)), \xi_{t+1}(S)) \text{ for all } X_0(S) = \omega \tag{B.8}$$

with total reward

$$U_{\underline{\sigma}}(\omega) \equiv \sum_{t=0}^{T} \delta^t r_{\underline{\kappa}}\left(\tilde{X}_t(\omega)\right), \text{ for all } \omega \in S \tag{B.9}$$

The intertemporal behavioral problem of a robust investor that seeks safety first is

$$\underset{\{\underline{\kappa}\}}{Max}\left\{E\left[U_{\underline{\kappa}}\right]\right\} \tag{B.10}$$

We define $\upsilon_{\underline{\kappa}}(x) \equiv EU_{\underline{\kappa}}$ and let the value function $\upsilon^* : S \to \Re$ be equal to $\upsilon^*(x) = \sup_{\underline{\kappa}}\{\upsilon_{\underline{\kappa}}(\omega)\}$ for all $\omega \in S$. The investors' optimal conservative policy $\underline{\kappa}^*$ attains the supremum for every $\omega \in S$ as υ^* is unique and satisfies the Bellman equation.

Note that the optimal investment policy should be $\underline{\kappa}(\xi) \mapsto w_t(\underline{\kappa}) = 1$ when the risky asset is the market portfolio as illustrated in section 1.2.

Q.E.D.

Appendix C: The boundaries of Knightian uncertainty

Define $\hat{\pi}_t$ as the Q-MLE of the probability of transitioning to the good state assuming that the state space follows a two-state Markov chain. Given a sample of size *T*, let $f_t = t_g / T$ be the likelihood $\hat{\pi}_t = \pi_g$ of good economic times with limiting confidence interval $1-\varsigma$. Robust investors acknowledge that observed frequencies may be drawn from a hidden (alternatively imprecise) distribution of multinomial proportions. Hence, they will hold simultaneous likelihood ratio tests for their forecast. We follow Goodman (1965) and find the asymptotic interval for the forecast using the Bonferroni inequality. The Bonferroni inequality is a useful rule in probability theory that can be used to find the limiting interval of plausible values of the most likely value drawn from a distribution without assuming independence. Then, as $T \to +\infty$, f_t is approximately normally distributed with mean $\hat{\pi}_t$ and variance $\hat{\pi}(1-\hat{\pi})/T$ so that

$$Z_t = \frac{\sqrt{T}\left(f_t - \hat{\pi}\right)}{\sqrt{\hat{\pi}(1-\hat{\pi})}} \tag{C.1}$$

is distributed as a standard normal variate. The interval for $\hat{\pi}_t$ has bounds equal to the solutions of the quadratic equation

$$T\left(f - \hat{\pi}\right)^2 = X^2(1-\varsigma,1)\hat{\pi}(1-\hat{\pi}) \tag{C.2}$$

or what is the same

$$\pi^2\left(T + X^2(1-\varsigma,1)\right) - \left(2t_g + X^2(1-\varsigma,1)\right)\pi + t_g^2/T = 0 \tag{C.3}$$

where $X^2(1-\varsigma,1)$ denotes the quantile of order $1-\varsigma$ of a chi-square distribution with one degree of freedom. Equations (C.2) and (C.3) have two solutions that define the lower and upper bounds defining the interval that $\hat{\pi}_t = \pi_g$ with coverage probability equal to $1-T\varsigma$. *Q.E.D.*

References

Anderson, E. W., E., Ghysels, and J. L., Juergens, 2009, "The Impact of Risk and Uncertainty on Expected Returns," *Journal of Financial Economics*, 94, 233-263.

Back, K., 2010, *Asset Pricing and Portfolio Choice Theory*, Oxford University Press, NY.

Barillas, F., L. P., Hansen, and T. J., Sargent, 2009, "Doubts or Variability?" *Journal of Economic Theory*, 144, 2388-2418.

Bier, M., and D., Engelage, 2010, "Merging of Opinions under Uncertainty," working paper No. 433, Institute of Mathematical Economics.

Black, F., M. C., Jensen, and M., Scholes, 1972, "The Capital Asset Pricing Model: Some Empirical Finding," in M.C. Jensen, ed., *Studies in the Theory of Capital Markets*, Praeger, New York.

Breeden, D. T., M. R., Gibbons, and R. H., Litzenberger, 1989, "Empirical Tests of the Consumption-oriented CAPM," *Journal of Finance*, 44, 1595-1616.

Brennan, M. J., A. W., Wang, and Y., Xia, 2004, "Estimation and Test of a Simple Model of Intertemporal Capital Asset Pricing," *Journal of Finance*, 59, 1743-1776.

Cagetti, M., L. P., Hansen, T. J., Sargent, and N., Williams, 2002, "Robustness and Pricing with Uncertain Growth," *Review of Financial Studies*, 15, 363-404.

Campbell, J. Y., 1987, "Stock Returns and the Term Structure," *Journal of Financial Economics*, 18, 373-399.

Campbell, J. Y., and R. J., Shiller, 1988, "Stock Prices, Earnings, and Expected Dividends," *Journal of Finance*, 43, 661-676.

Carhart, M. M., 1997, "On Persistence of Mutual Fund Performance," *Journal of Finance*, 52, 57-82.

Chen, Z., and L. G., Epstein, 2002, “Ambiguity, Risk and Asset Returns in Continuous Time,” *Econometrica*, 70, 1403-1443.

Cochrane, J. H., 2001, *Asset Pricing*, Princeton University Press, Princeton, NJ.

Ellsberg, D., 1961, “Risk, Ambiguity, and the Savage Axioms,” *Quarterly Journal of Economics*, 75, 643-669.

Epstein, L. G., 1999, “A Definition of Uncertainty Aversion,” *Review of Economic Studies*, 66, 579-608.

Epstein, L. G., and M., Marinacci, 2006, “Mutual Absolute Continuity of Multiple Priors,” *Journal of Economic Theory*, 137, 716-720.

Epstein, L. G., and M., Schneider, 2008, “Ambiguity, Information Quality and Asset Pricing, *Journal of Finance*, 63, 197-228.

Epstein, L. G., and M., Schneider, 2007, “Learning under Ambiguity,” *Review of Economic Studies*, 74, 1275-1303.

Epstein, L. G., M., Schneider, 2003, “Recursive Multiple Priors,” *Journal of Economic Theory*, 113, 1-13.

Epstein, L. G., and T., Wang, 1994, “Intertemporal Asset Pricing under Knightian Uncertainty,” *Econometrica*, 62, 283-322.

Fama, E. F., and K. R., French, 1992, “The Cross-section of Expected Returns,” *Journal of Finance*, 47, 427-465.

Fama, E. F., and K. R., French, 1993, “Common Risk Factors in the Returns on Stocks and Bonds,” *Journal of Financial Economics*, 33, 3-56.

Fama, E. F., and K. R., French, 1989, “Business Conditions and Expected Returns on Stocks and Bonds,” *Journal of Financial Economics*, 25, 23-49.

Fama, E. F., and J. D., MacBeth, 1973, “Risk, Return, and Equilibrium: Empirical Tests,” *Journal of Political Economy*, 81, 607–636.

Fama, E. F., and W., Schwert, 1977, “Asset Returns and Inflation,” *Journal of Financial Economics*, 5, 115-146.

Fujisaki, M., G., Kallianpur, and H., Kunita, 1972, “Stochastic Differential Equations for the Non-linear Filtering Problem,” *Osaka Journal of Mathematics*, 9, 19-40.

Garlappi, L., R., Uppal, and T., Wang, 2007, “Portfolio Selection with Parameter and Model Uncertainty: A Multi-prior Approach,” *Review of Financial Studies*, 20, 41-81.

Gilboa, I., and D., Schmeidler, 1989, “Maxmin Expected Utility with Nonunique Prior,” *Journal of Mathematical Economics*, 18, 141-153.

Goodman, L. A., 1965, On Simultaneous Confidence Intervals for Multinomial Proportions,” *Technometrics*, 7, 249-275.

Hansen, B. E., 1990, “Lagrange Multiplier Tests for Parameter Instability in Non-linear Models,” working paper, University of Rochester.

Hansen, L. P., 2008, Beliefs, Doubts and Learning: Valuing Macroeconomic Risk,” *American Economic Review*, 97, 1-30.

Hansen, L. P., and T. J., Sargent, 2008, *Robustness*, Princeton University Press, Princeton, NJ.

Hansen, L. P., and T. J., Sargent, 2009, Fragile Beliefs and the Price of Uncertainty, working paper, University of Chicago, New York University, and Hoover Institution.

Jagannathan, R., and Z., Wang, 1998, An Asymptotic Theory for Estimating Beta-pricing Models using Cross-sectional Regression, *Journal of Finance*, 53, 1285-1309.

Jegadeesh, N., 1990, Evidence of Predictable Behavior of Security Returns, *Journal of Finance*, 45, 881-898.

Jegadeesh, N., and S., Titman, 1993, “Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency,” *Journal of Finance*, 48, 65-91.

Kan, R., C., Robotti, and J., Shanken, 2009, “Pricing Model Performance and the Two-pass Cross-sectional Regression Methodology,” working paper, University of Toronto, Federal Reserve Bank of Atlanta, Emory University, and NBER.

Keynes, J. M., 1921, *A Treatise on Probability*, Macmillan Co., Cambridge, UK.

Knight, F. H., 1921, Risk, *Uncertainty and Profit*, University Chicago Press, Chicago, IL.

Kogan, L., and T., Wang, 2002, “A Simple Theory of Asset Pricing under Model Uncertainty,” working paper, MIT, and University of British Columbia.

Lamont, O. A., 2001, “Economic Tracking Portfolios,” *Journal of Econometrics*, 105, 161-184.

Leippold, M., F., Trojani, and P., Vanini, 2008, “Learning and Asset Prices under Ambiguous Information,” *Review of Financial Studies*, 21, 2565-2597.

Lewellen, J., and S., Nagel, 2006, “The Conditional CAPM does not Explain Asset-pricing Anomalies,” *Journal of Financial Economics*, 82, 289-314.

Lewellen, J., S., Nagel, and J., Shanken, 2008, “A Skeptical Appraisal of Asset Pricing Tests,” working paper No. 12360, NBER.

Lintner, J., 1965, “The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets,” *The Review of Economics and Statistics*, 47, 233-275.

Lui, K.-J., and W. G., Cumberland, 2004, “Interval Estimation of Gamma for an "RxS" Table,” *Psychometrika*, 69, 275-290.

Maccheroni, F., M., Marinacci, and A., Rustichini, 2006, “Dynamic Variational Preferences,” *Journal of Economic Theory*, 128, 4-44.

Maenhout, P. J., 2004, “Robust Portfolio Rules and Asset Pricing,” *Review of Financial Studies*, 17, 951-983.

Merton, R. C., 1973, “An Intertemporal Capital Asset Pricing Model,” *Econometrica*, 41, 867-887.

Nyblom, J., 1989, “Testing for the Constancy of Parameters Over Time,” *Journal of the American Statistical Association*, 84, 223-230.

Ozoguz, A., 2009, “Good Times or Bad Times? Investors’ Uncertainty and Stock Returns,” *Review of Financial Studies*, 22, 4377-4422.

Perez-Quiros, and A., G., Timmermann, 2000, “Firm Size and Cyclical Variations in Stock Returns,” *Journal of Finance*, 55, 1229-1262.

Petkova, R., 2006, “Do the Fama-French Factors Proxy for Innovations in Predictive Variables?” *Journal of Finance*, 61, 581-612.

Roy, A. D., 1952, “Safety First and the Holding of Assets,” *Econometrica*, 20, 431-449.

Rogers, L. C. G., 2001, “The Relaxed Investor and Parameter Uncertainty,” *Finance Stochastics*, 5, 131-154.

Shackle, G., L., S., 1949, *Expectation in Economics*, Cambridge University Press, Cambridge.

Shanken J., 1992, “On the Estimation of Beta Pricing Models,” *Review of Financial Studies*, 5, 1-34.

Singleton, K. J., 2006, *Empirical Dynamic Asset Pricing*, Princeton Univ. Press, Princeton, NJ.

Strzalecki, T., 2007, “Axiomatic Foundations of Multiplier Preferences,” working paper, Stanford University.

Uppal, R., and T., Wang, 2003, Model Misspecification and Underdiversification,” *Journal of Finance*, 58, 6, 2465-2486.

van Dijk, D., T., Teräsvirta, and P. H., Franses, 2002, "Smooth Transition Autoregressive Models-A Survey of Recent Developments," *Econometric Reviews*, 21, 1-47.

Vassalou, M., 2003, "News Related to Future GDP Growth as Risk Factors in Equity Returns," *Journal of Financial Economics*, 68, 47-73**.**

Veronesi, P., 1999, "Stock Market Overreaction to Bad News in Good Times: A Rational Expectation Equilibrium Model," *Review of Financial Studies*, 12, 975-1007.

Veronesi, P., 2000, "How does Information Quality Affect Stock Returns?" *Journal of Finance*, 55, 2, 807-837

Wonham, M. W., 1965, "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering," *SIAM Journal of Control*, *Series A*, 2, 347-369.

Zhang, Z., 2003, "State Uncertainty and the Cross-sectional Returns: Theory and Evidence," working paper, Singapore Management University.

Table 1
Approximate Time-Series Regressions: Time-Varying Excess Market Return

This table reports the conditional maximum likelihood (ML) estimates of the smooth transition (STAR) model with two regimes for the excess market return, which is assumed to follow an ARMA (1,1)-GARCH(1,1) process. The explanatory variables are *RF(-1)*, *DIV(-1)*, *TERM(-1)*, and *DEF(-1)* lagged one period. The MA process is: 1 - a_1 L -...- a_q L^q. The GARCH parameters are in ARMA-in-squares form. Standard errors are calculated following Newey-West HAC formula and using a Parzen kernel and plug-in bandwidth equal to 5. The sample period is 1927-2007.

Panel A: Smooth Transition Model: $r^e_{m,t} = \gamma_0 + \phi \cdot r^e_{m,t-1} + \varepsilon_t - \varepsilon_{t-1}$ with $\zeta_t = \kappa_t + \psi h_{t-1} + \sigma\varepsilon^2_{t-1}$

	Estimate	Std. Err.	t-Ratio	p-value
Regime 1				
Student *t* d.f.^(1/2)	3.6946	0.7294	-	-
Log (Skewness)	-0.1939	0.0591	-3.2810	0.0010
Intercept	0.0023	0.0083	0.3540	0.7230
AR1	-0.9524	0.0302	-31.5260	0.0000
MA1	-0.9638	0.0219	-44.0680	0.0000
GARCH Intercept	0.0438	0.0088	-	-
GARCH AR1	0.9499	0.0198	48.0250	0.0000
GARCH MA1	0.8534	0.0248	34.3830	0.0000
Regime 2				
Intercept	0.0176	0.0096	1.8240	0.0680
GARCH Intercept	0.0572	0.0142	4.0180	0.0000

Log Likelihood:1650.28

Panel B: Indicator Function *MKT (-1)*: $r^e_{m,t-1} = \delta\ z_{t-1} + \varepsilon_{t-1}$

	Estimate	Std. Err.	t-Ratio	p-value
Regime 1				
Intercept	-0.6871	0.2454	-2.7990	0.0050
RF(-1)	0.0022	0.0146	0.1500	0.8810
DIV(-1)	-0.0164	0.0251	-0.6560	0.5120
TERM(-1)	-0.0477	0.0245	-1.9450	0.0520
DEF(-1)	0.2252	0.0816	2.7610	0.0060
Regime 2				
Intercept	-0.7142	0.1512	-4.7240	0.0000
RF(-1)	-0.0401	0.0177	-2.2640	0.0240
DIV(-1)	-0.3615	0.0923	-3.9180	0.0000
TERM(-1)	-0.1538	0.0382	-4.0290	0.0000
DEF(-1)	0.0231	0.0156	1.4820	0.1390
Smooth (gamma)	36.5380	17.6785	2.0670	0.0390

Table 2
Approximate Time-Series Regressions: State-Dependent Factor Loadings

This table reports the estimates of the two-state regime switching time-series regressions for the 25 Fama-French test assets. Standard errors are calculated following Newey-West HAC with Parzen kernel and plug-in bandwidth equal to five. Highlighted t-stats in black denote 5% significance, in italics 10% significance level. The sample period is 1927-2007.

Time-Series Regression: $r_{i,t}^{e}\left|s_t = \alpha_i\right|s_t + \beta_{i,MKT}\left|s_t \cdot r_{m,t}^{e} + u_t\right.$

	Low	2	3	4	High	Low	2	3	4	High
			α/Regime1					*t*		
Small	-0.0093	-0.0685	0.0432	0.0057	-0.0022	-1.1150	**-3.0180**	0.8270	0.6350	-0.2520
2	-0.0057	0.0006	0.0025	0.0070	0.0102	**-3.3500**	0.4620	*1.6730*	1.3380	0.8330
3	-0.0145	0.0085	0.0029	0.0122	0.0111	-0.5170	**4.3450**	**3.1810**	*1.8730*	**4.4260**
4	0.0048	0.0015	-0.0047	0.0014	0.0043	0.9690	0.1050	-0.5440	0.9700	0.0910
Large	0.0017	0.0008	0.0326	-0.0015	0.0002	**2.2190**	0.1890	**2.2630**	-0.6570	*1.7470*
			α/Regime2					*t*		
Small	-0.0047	0.0729	-0.04563	0.0034	0.0160	0.2490	**3.315**	-0.805	0.144	0.922
2	0.0003	0.0015	0.0022	0.0007	-0.0031	**2.615**	0.456	0.184	0.741	0.599
3	0.0052	0.0001	-0.0018	-0.0053	0.0016	0.466	**-4.038**	**-2.651**	1.444	**-3.682**
4	-0.0048	-0.0003	0.0108	0.0019	0.0016	-0.847	-0.063	*0.721*	0.303	1.05
Large	-0.0016	-0.0006	-0.0280	0.0010	0.0034	**-3.769**	-0.143	**-2.146**	1.062	1.175
			β/Regime1					*t*		
Small	0.8054	1.3740	1.9026	0.5499	1.7011	**3.0010**	**16.6040**	**9.4050**	**4.5520**	**35.7680**
2	1.2944	1.1419	1.0354	0.6648	0.3406	**22.9550**	**39.4160**	**22.1330**	**6.2170**	**3.3640**
3	0.8238	1.0381	1.0873	0.5216	0.9852	**2.6390**	**23.0800**	**44.9110**	**6.7850**	**12.7250**
4	0.8519	0.7453	0.9103	1.1386	1.0743	**13.8700**	**7.8940**	**11.7980**	**21.3490**	**31.4480**
Large	0.9806	0.6942	0.3684	0.8362	0.7700	**59.5010**	**4.9160**	**3.6980**	**9.1520**	**14.4710**
			β/Regime2					*t*		
Small	2.3710	1.1767	0.2998	1.4085	0.3984	**2.8390**	0.9290	**4.2760**	**5.1950**	**8.8320**
2	1.2844	1.3135	1.1586	1.3708	1.9145	0.1430	**3.2120**	**2.3930**	**5.1670**	**8.1680**
3	1.4270	1.1533	1.2570	1.4925	1.2243	1.1820	**2.3690**	**5.5430**	**9.2490**	1.0590
4	1.4842	1.3408	1.1340	1.0602	1.2958	**5.1460**	**3.3690**	**2.3870**	**3.4280**	**8.4380**
Large	0.9635	1.2680	1.3729	1.0095	1.0730	0.6840	**2.0980**	**5.4280**	*1.7930*	**3.6760**
			St. Dev.					*t*		
Small	0.1444	0.0626	0.0584	0.0511	0.0608	-	-	-	-	-
2	0.0434	0.0349	0.0327	0.0379	0.0485	-	-	-	-	-
3	0.0315	0.0240	0.0266	0.0295	0.0479	-	-	-	-	-
4	0.0217	0.0198	0.0230	0.0293	0.0457	-	-	-	-	-
Large	0.0157	0.0153	0.0222	0.0398	0.0516	-	-	-	-	-
			(Skewness)					*t*		
Small	0.0947	0.1037	0.1237	0.1568	0.1948	**2.4370**	*1.7180*	*1.7120*	**2.9660**	**4.3730**
2	0.0623	0.0576	0.0659	0.0703	0.0850	1.4470	1.3530	1.3570	1.5390	*1.6070*
3	-0.0237	0.0017	0.0987	0.0509	0.1362	-0.4660	0.0370	*2.0490*	1.0960	**3.1620**
4	0.0675	0.0451	0.0267	0.0846	0.1175	*1.7130*	1.0280	0.5010	1.5410	**2.5500**
Large	-0.0103	0.0134	0.0568	0.0563	0.0911	-0.2450	0.2950	1.1670	1.2230	**2.0320**
			R^2					*Log Likelihood*		
Small	0.4946	0.5456	0.6742	0.6693	0.5969	1326.64	1580.13	1660.84	1797.54	1700.15
2	0.7074	0.7632	0.7612	0.7693	0.7209	1787.09	2003.39	2071.43	2026.48	1803.01
3	0.8191	0.8644	0.8502	0.8057	0.7587	2077.51	2292.83	2292.51	2175.72	1839.68
4	0.8754	0.9117	0.8710	0.8241	0.7558	2413.38	2567.71	2428.48	2196.27	1818.03
Large	0.9188	0.9138	0.8630	0.7870	0.6584	2686.38	2735.95	2477.28	2195.47	1811.71

Table 3
Fama-MacBeth Second-Pass GLS-CSR Test Results (1927-2007)

This table reports the second-pass Fama-MacBeth GLS-CSR test results on excess returns on 25 portfolio returns sorted by size and book-to-market. The results are presented across different confidence levels (CL) ranging from 50% to 90%, and different length in the rolling regression windows (T) ranging from 36 months to 180 months. The GLS-CSR test has null hypothesis $H_0 : \hat{Q} = T\overline{\boldsymbol{\alpha}}'\left(\hat{V}_{22}^{-1}\right)\overline{\boldsymbol{\alpha}} = 0$, with $\hat{\boldsymbol{\alpha}} = \overline{\mathbf{R} - R_f} - \hat{\beta}\,\boldsymbol{\lambda}$. Individual *t*-statistics are presented with Shanken's correction, $\hat{Q}$ are presented with and without Shanken's correction. Highlighted t-stats in black denote statistical significance at the 5% level, in italics at the 10% level.

Cross-Sectional Regression: $\overline{\mathbf{R} - R_f} = \hat{\beta}\,\boldsymbol{\lambda} + \boldsymbol{\alpha}$

		$\hat{\lambda}$ *MKT*						*t-stat*		
T/CL	90	80	70	60	50	T/CL	90	80	70	60	50
180	0.0072	0.0072	0.0072	0.0072	0.0072	180	**4.39**	**4.38**	**4.38**	**4.38**	**4.38**
120	0.0077	0.0077	0.0077	0.0077	0.0077	120	**3.77**	**3.76**	**3.76**	**3.76**	**3.76**
96	0.0070	0.0070	0.0070	0.0070	0.0070	96	**3.41**	**3.41**	**3.41**	**3.41**	**3.41**
60	0.0070	0.0070	0.0070	0.0070	0.0070	60	**3.53**	**3.53**	**3.53**	**3.53**	**3.52**
36	0.0071	0.0071	0.0071	0.0071	0.0071	36	**3.60**	**3.60**	**3.60**	**3.60**	**3.60**

		$\hat{\lambda}$ *JUMP*						*t-stat*			
T/CL	90	80	70	60	50	T/CL	90	80	70	60	50
180	0.0346	0.0339	0.0336	0.0335	0.0335	180	0.66	0.64	0.64	0.64	0.64
120	0.0608	0.0604	0.0601	0.0599	0.0597	120	1.19	1.18	1.18	1.18	1.17
96	0.0968	0.0960	0.0951	0.0942	0.0933	96	**2.11**	**2.09**	**2.08**	**2.06**	**2.04**
60	0.0864	0.0865	0.0867	0.0868	0.0869	60	**2.30**	**2.31**	**2.31**	**2.32**	**2.32**
36	0.0590	0.0604	0.0613	0.0618	0.0622	36	**2.19**	**2.24**	**2.27**	**2.29**	**2.30**

		$\hat{\lambda}$ *KUNC*						*t-stat*			
T/CL	90	80	70	60	50	T/CL	90	80	70	60	50
180	0.0069	0.0135	0.0195	0.0248	0.0297	180	**5.20**	**5.18**	**5.13**	**5.03**	**4.89**
120	0.0057	0.0111	0.0160	0.0206	0.0247	120	**4.16**	**4.15**	**4.10**	**4.00**	**3.83**
96	0.0047	0.0095	0.0143	0.0190	0.0236	96	**3.57**	**3.72**	**3.80**	**3.81**	**3.75**
60	0.0033	0.0072	0.0115	0.0162	0.0211	60	**2.52**	**2.82**	**3.04**	**3.14**	**3.16**
36	0.0015	0.0041	0.0075	0.0116	0.0159	36	1.19	1.69	**2.10**	**2.35**	**2.46**

CL	90	80	70	60	50
$\hat{Q}$	34.41	34.36	34.32	34.29	34.25
p-value	0.10	0.10	0.10	0.11	0.11
Adj. $\hat{Q}$	48.50	48.44	48.38	48.33	48.95
p-value	0.07	0.07	0.07	0.08	0.08

Table 4
Summary Statistics: Maximum Correlated Portfolios (1927-2007)

This table reports the summary statistics of the full tradable ex-ante maximum correlated portfolios (MCPs) mimicking innovations in the two risk factors *JUMP* and *KUNC*. The portfolio weights are obtained running multiple ordinary least squares (OLS) regressions between the 12-month forecast of *JUMP* and *KUNC* and the excess returns of *SMB*, *HML*, and *UMD* as tracking portfolios. For each OLS regression we control for *MKT*, *DIV*, *RF*, *TERM*, *DEF* and *JUMP* or *KUNC* (depending on the dependent variable) lagged one period. Summary Statistics are for monthly MCP Returns.

	JUMP	*KUNC*
Mean	0.0016	0.0038
Std. dev.	0.0385	0.0210
Min	-0.1642	-0.0918
Max	0.4361	0.2504
Median	-0.0016	0.0026
Kurtosis	26.3043	22.3208
Skewness	2.8600	2.0409

Table 5
CSR and GRS Test Results on 25 Fama-French Portfolios (1927-2007)

This table reports the sample cross-section and GRS asset pricing tests for two versions of the R-DAPM: 1) a three factor model with *MKT, JUMP*, and *KUNC*; 2) An augmented version of the multi factor model with *MKT, RF, DIV, TERM, DEF*, and *KUNC*. Both models are estimated using monthly returns on the 25 Fama-French portfolios ranked by size and book-to-market. The sample data is from 1927-2007 (969 observations). ρ^2 are cross-sectional R^2, $p(\rho^2 = 1)$ is the p-value for the specification test $H_0 : \rho^2 = 1$, *rse* is the standard error of ρ^2 assuming $0 < \rho^2 < 1$, $p(\rho^2 = 0)$ is the p-value for the test $H_0 : \rho^2 = 0$, *Wald* is the joint test of $H_0 : \hat{\lambda} = 0$ with p-value $p(Wald)$. We report also the estimates $\hat{\lambda}_n$ for the nth risk factor with Fama-Macbeth (1973) t-stats assuming the model is correctly specified (t_{FM}), Shanken's (1992) adjusted t-stats given the EIV problem (t_S), Jagannathan and Wang (1998) adjusted t-stats (t_{JW}) both assuming the model is well specified, and Kan et al. (2009) robust t-stats under model misspecification (t_{KRS}) .

1) $E[R_k] = \lambda_0 + \hat{\beta}_{MKT}\lambda_{MKT} + \hat{\beta}_{JUMP}\lambda_{JUMP} + \hat{\beta}_{KNUC}\hat{\lambda}_{KUNC} + u_k$

2) $E[R_k] = \lambda_0 + \hat{\beta}_{MKT}\lambda_{MKT} + \hat{\beta}_{RF}\lambda_{RF} + \hat{\beta}_{DIV}\lambda_{DIV} + \hat{\beta}_{TERM}\lambda_{TERM} + \hat{\beta}_{DEF}\lambda_{DEF} + \hat{\beta}_{KUNC}\hat{\lambda}_{KUNC} + u_k$

	$\hat{\lambda}_0$	$\hat{\lambda}_{MKT}$	$\hat{\lambda}_{JUMP}$	$\hat{\lambda}_{RF}$	$\hat{\lambda}_{DIV}$	$\hat{\lambda}_{TERM}$	$\hat{\lambda}_{DEF}$	$\hat{\lambda}_{KUNC}$
Model 1	0.0146	-0.0048	-0.0001					0.0054
t_{FM}	5.1421	-1.3992	-0.0522					3.5954
t_S	4.6518	-1.3010	-0.0499					3.3433
t_{JW}	4.5634	-1.3075	-0.0479					3.4024
t_{KRS}	3.3389	-0.9744	-0.0351					2.0986
ρ^2	0.3619							
$p(\rho^2 = 1)$	0.0528							
rse	0.1569							
$p(\rho^2 = 0)$	0.0109							
Wald	23.4127							
$p(Wald)$	0.0002							
Model 2	0.0132	-0.0017		-1.2565	0.3309	1.1544	-0.2409	0.0019
t_{FM}	5.2265	-0.5401		-4.3728	1.5185	4.8875	-1.8323	2.5214
t_S	2.8059	-0.3305		-2.3581	0.8216	2.6413	-1.0053	2.0559
t_{JW}	2.5900	-0.3044		-2.4449	0.7512	2.7896	-0.9369	2.1559
t_{KRS}	2.4044	-0.2852		-1.7292	0.5578	1.9862	-0.7227	1.9783
ρ^2	0.5954							
$p(\rho^2 = 1)$	0.9612							
rse	0.2934							
$p(\rho^2 = 0)$	0.0108							
Wald	20.0815							
$p(Wald)$	0.0006							

Table 6
CSR and GRS Test Results on 55 Portfolios (1927-2007)

This table reports the sample cross-section and GRS asset pricing tests for two versions of the R-DAPM: 1) a three factor model with *MKT*, *JUMP*, and *KUNC*; 2) An augmented version of the multi factor model with *MKT*, *RF*, *DIV*, *TERM*, *DEF*, and *KUNC*. Both models are estimated using monthly returns on 55 portfolios ranked by size, book-to-market, and industry. The sample data is from 1927-2007 (969 observations). ρ^2 are cross-sectional R^2, $p(\rho^2 = 1)$ is the *p*-value for the specification test $H_0 : \rho^2 = 1$, *rse* is the standard error of ρ^2 assuming $0 < \rho^2 < 1$, $p(\rho^2 = 0)$ is the *p*-value for the test $H_0 : \rho^2 = 0$, *Wald* is the joint test of $H_0 : \hat{\lambda} = 0$ with *p*-value $p(Wald)$. We report also the estimates $\hat{\lambda}_n$ for the *n*th risk factor with Fama-Macbeth (1973) *t*-stats assuming the model is correctly specified (t_{FM}), Shanken's (1992) adjusted *t*-stats given the EIV problem (t_S), Jagannathan and Wang (1998) adjusted *t*-stats (t_{JW}) both assuming the model is well specified, and Kan et al. (2009) robust *t*-stats under model misspecification (t_{KRS}) .

1) $E[R_k] = \lambda_0 + \hat{\beta}_{MKT}\lambda_{MKT} + \hat{\beta}_{JUMP}\lambda_{JUMP} + \hat{\beta}_{KNUC}\hat{\lambda}_{KUNC} + u_k$

2) $E[R_k] = \lambda_0 + \hat{\beta}_{MKT}\lambda_{MKT} + \hat{\beta}_{RF}\lambda_{RF} + \hat{\beta}_{DIV}\lambda_{DIV} + \hat{\beta}_{TERM}\lambda_{TERM} + \hat{\beta}_{DEF}\lambda_{DEF} + \hat{\beta}_{KUNC}\hat{\lambda}_{KUNC} + u_k$

	$\hat{\lambda}_0$	$\hat{\lambda}_{MKT}$	$\hat{\lambda}_{JUMP}$	$\hat{\lambda}_{RF}$	$\hat{\lambda}_{DIV}$	$\hat{\lambda}_{TERM}$	$\hat{\lambda}_{DEF}$	$\hat{\lambda}_{KUNC}$
Model 1	0.0090	0.0012	0.0020					0.0036
t_{FM}	6.7836	0.5357	1.4717					4.3529
t_S	6.6064	0.5304	1.4659					4.3150
t_{JW}	5.8953	0.4949	1.4375					4.4339
t_{KRS}	5.1524	0.4701	1.3039					3.7591
ρ^2	0.1395							
$p(\rho^2 = 1)$	0.0000							
Rse	0.0710							
$p(\rho^2 = 0)$	0.0045							
Wald	19.7767							
$p(Wald)$	0.0017							
Model 2	0.0074	0.0029		-0.0236	0.0260	0.0096	0.0008	0.0021
t_{FM}	4.9020	1.2650		-2.8865	1.1560	3.1885	0.8365	3.3024
t_S	4.2680	1.1848		-2.5289	1.0181	2.7927	0.7460	3.2310
t_{JW}	4.0923	1.1431		-2.5986	1.0558	2.8395	0.7400	3.4017
t_{KRS}	3.1944	0.9939		-1.6274	0.7569	1.6478	0.5984	3.2703
ρ^2	0.2073							
$p(\rho^2 = 1)$	0.0028							
Rse	0.1087							
$p(\rho^2 = 0)$	0.0342							
Wald	25.7683							
$p(Wald)$	0.0046							

Table 7
Tests of Equality of Cross-Sectional R^2s

This table reports pair-wise tests of equality of the GLS cross-sectional ρ^2 between the model in row *i* and model in column *j* with null $H_0 : \hat{\rho}_i^2 - \hat{\rho}_j^2 = 0$ with p-values in parenthesis computed without (p-value (1)) and with (p-value (2)) the assumption of model misspecification. The FF-4 model includes *MKT*, *SMB*, *HML*, and *UMD* as risk factors,. The ICAPM includes *MKT*, *RF*, *DIV*, *TERM*, and *DEF* as risk factors. The R-DAPM includes *MKT*, *DIV*, *TERM*, *DEF*, and *KUNC* as risk factors. The models are estimated using monthly returns on 25 Fama-French portfolios ranked by size and book-to-market. The sample periods are from 1927-2007 (969 observations) and 1962-2007 (552 observations).

Panel A: (1927-2007)

	FF-4	R-ICAPM
ICAPM	0.2242	-0.0320
p-value(1)	(0.0764)	(0.9943)
p-value(2)	(0.0000)	(0.1244)
FF-4		-0.2562
p-value(1)		(0.0683)
p-value(2)		(0.0000)

Panel B: (1962-2007)

	FF-4	R-ICAPM
ICAPM	0.0590	-0.0320
p-value(1)	(0.1406)	(0.2411)
p-value(2)	(0.0827)	(0.1846)
FF-4		-0.2562
p-value(1)		(0.4199)
p-value(2)		(0.0001)

Figure 1. Equity Risk Premia Across Investors' Confidence Level

Solid black lines denote the period 1927-2007 and solid grey lines the period 1962-2007. The top chart plots market risk premia, the middle chart plots learning premia, and the bottom chart plots Knightian uncertainty risk premia.

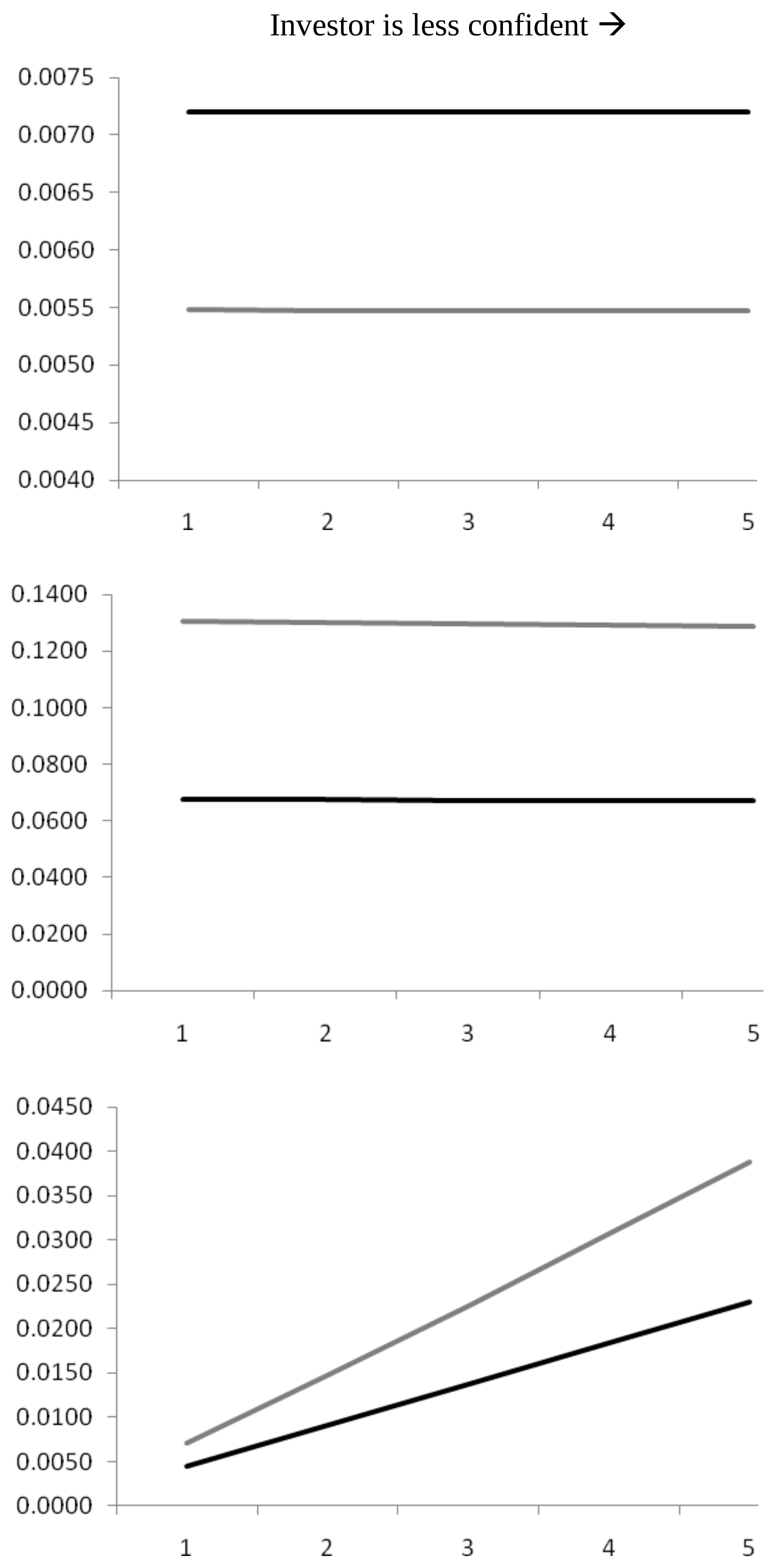

Figure 2. Equity Risk Premia Across Rolling Regression Windows

Solid black lines denote the period 1927-2007 and solid grey lines the period 1962-2007. The top chart plots market risk premia, the middle chart plots learning premia, and the bottom chart plots Knightian uncertainty risk premia.

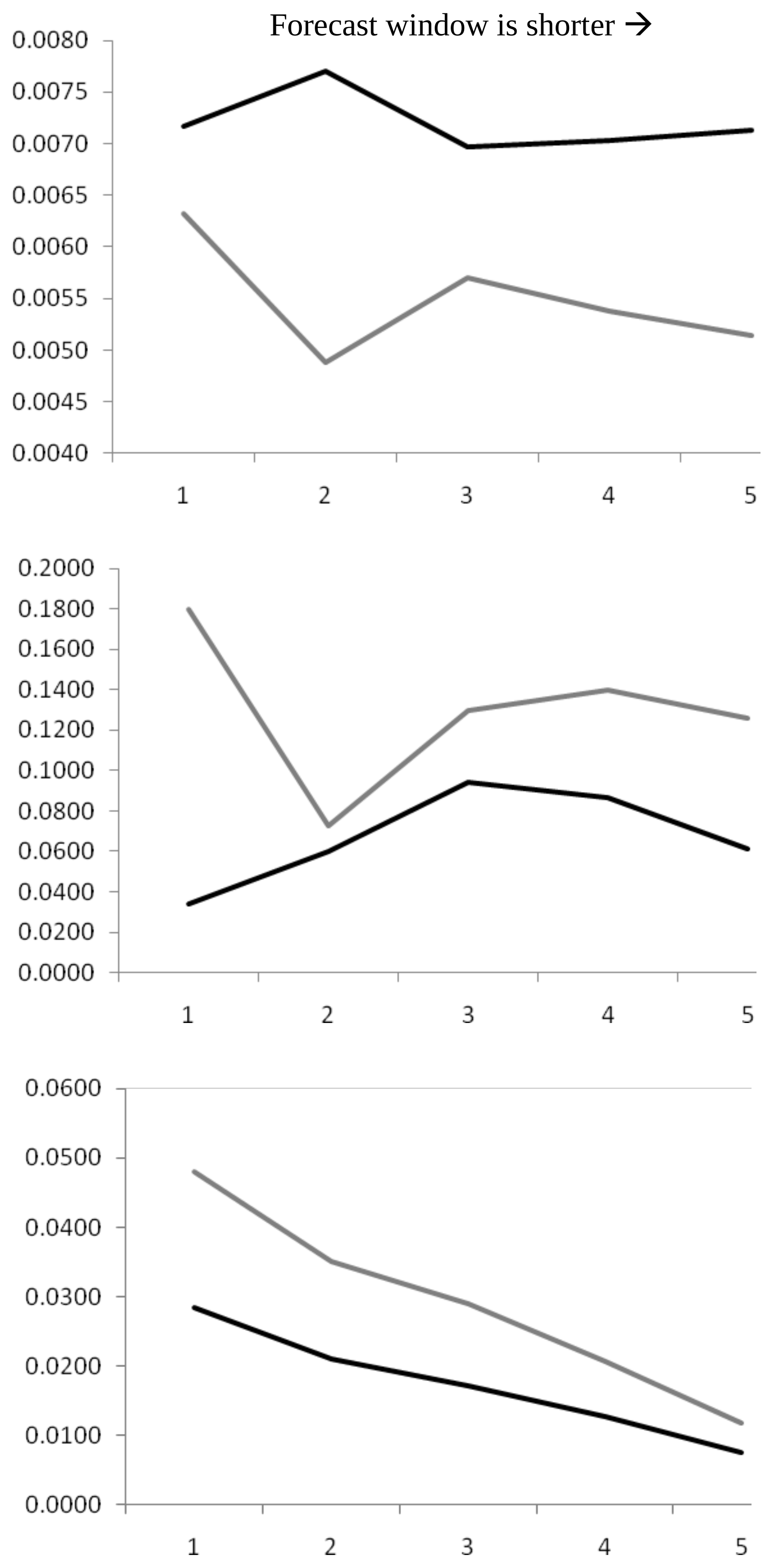

Figure 3. ***KUNC*** **(CF = 50%, N = 36 months)**

The top chart plots the 12-month moving average of the Knightian uncertainty risk factor in levels for the period 1927-1961. The bottom chart plots the 12-month moving average of the Knightian uncertainty risk factor in levels for the period 1962-2007.