Some of the class notes from Thursday, October 6.

Assignment is here too.

(c) Show that an outcome in S belongs to the event $\bigcup_{n=1}^{\infty} C_n$ if and only if it belongs to all the events $A_1, A_2, \ldots$ except possibly a finite number of those events.

$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \liminf_i A_i$

1.5. THE DEFINITION OF PROBABILITY

Axioms and Basic Theorems

In this section we shall present the mathematical, or axiomatic, definition of probability. In a given experiment, it is necessary to assign to each event A in the sample space S a number $\Pr(A)$ which indicates the probability that A will occur. In order to satisfy the mathematical definition of probability, the number $\Pr(A)$ that is assigned must satisfy three specific axioms. These axioms ensure that the number $\Pr(A)$ will have certain properties which we intuitively expect a probability to have under any of the various interpretations described in Section 1.2.

The first axiom states that the probability of every event must be nonnegative.

Axiom 1. *For any event A,* $\Pr(A) \geqslant 0$.

The second axiom states that if an event is certain to occur, then the probability of that event is 1.

Axiom 2. $\Pr(S) = 1$.

Before stating Axiom 3, we shall discuss the probabilities of disjoint events. If two events are disjoint, it is natural to assume that the probability that one or the other will occur is the sum of their individual probabilities. In fact, it will be assumed that this *additive property* of probability is also true for any finite number of disjoint events and even for any infinite sequence of disjoint events. If we assume that this additive property is true only for a finite number of disjoint events, we cannot then be certain that the property will be true for an infinite sequence of disjoint events as well. However, if we assume that the additive property is true for every infinite sequence of disjoint events, then (as we shall prove) the property must also be true for any finite number of disjoint events. These considerations lead to the third axiom.

Axiom 3. *For any infinite sequence of disjoint events $A_1, A_2, \ldots,$*

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i).$$

The mathematical definition of probability can now be given as follows: A *probability distribution*, or simply a *probability*, on a sample space S is a specification of numbers $\Pr(A)$ which satisfy Axioms 1, 2, and 3.

We shall now derive two important consequences of Axiom 3. First, we shall show that if an event is impossible, its probability must be 0.

Theorem 1. $\Pr(\emptyset) = 0$.

Proof. Consider the infinite sequence of events $A_1, A_2, \ldots$ such that $A_i = \emptyset$ for $i = 1, 2, \ldots$. In other words, each of the events in the sequence is just the empty set $\emptyset$. Then this sequence is a sequence of disjoint events, since $\emptyset \cap \emptyset = \emptyset$. Furthermore, $\bigcup_{i=1}^{\infty} A_i = \emptyset$. Therefore, it follows from Axiom 3 that

$$\Pr(\emptyset) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{\infty} \Pr(\emptyset).$$

This equation states that when the number $\Pr(\emptyset)$ is added repeatedly in an infinite series, the sum of that series is simply the number $\Pr(\emptyset)$. The only real number with this property is $\Pr(\emptyset) = 0$. □

We can now show that the additive property assumed in Axiom 3 for an infinite sequence of disjoint events is also true for any finite number of disjoint events.

Theorem 2. *For any finite sequence of n disjoint events* $A_1, \ldots, A_n$,

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i).$$

Proof. Consider the infinite sequence of events $A_1, A_2, \ldots,$ in which $A_1, \ldots, A_n$ are the n given disjoint events and $A_i = \emptyset$ for $i > n$. Then the events in this infinite sequence are disjoint and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i$. Therefore, by Axiom 3,

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

$$= \sum_{i=1}^{n} \Pr(A_i) + \sum_{i=n+1}^{\infty} \Pr(A_i)$$

$$= \sum_{i=1}^{n} \Pr(A_i) + 0$$

$$= \sum_{i=1}^{n} \Pr(A_i). \quad \square$$

Further Properties of Probability

From the axioms and theorems just given, we shall now derive four other general properties of probability distributions. Because of the fundamental nature of these four properties, they will be presented in the form of four theorems, each one of which is easily proved.

Theorem 3. *For any event A,* $\Pr(A^c) = 1 - \Pr(A)$.

Proof. Since A and A^c are disjoint events and $A \cup A^c = S$, it follows from Theorem 2 that $\Pr(S) = \Pr(A) + \Pr(A^c)$. Since $\Pr(S) = 1$ by Axiom 2, then $\Pr(A^c) = 1 - \Pr(A)$. □

Theorem 4. *For any event A,* $0 \leqslant \Pr(A) \leqslant 1$.

Proof. It is known from Axiom 1 that $\Pr(A) \geqslant 0$. If $\Pr(A) > 1$, then it follows from Theorem 3 that $\Pr(A^c) < 0$. Since this result contradicts Axiom 1, which states that the probability of every event must be nonnegative, it must also be true that $\Pr(A) \leqslant 1$. □

Theorem 5. *If $A \subset B$, then* $\Pr(A) \leqslant \Pr(B)$.

Proof. As illustrated in Fig. 1.5, the event B may be treated as the union of the two disjoint events A and BA^c. Therefore, $\Pr(B) = \Pr(A) + \Pr(BA^c)$. Since $\Pr(BA^c) \geqslant 0$, then $\Pr(B) \geqslant \Pr(A)$. □

Theorem 6. *For any two events A and B,*

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB).$$

Proof. As illustrated in Fig. 1.6,

$$A \cup B = (AB^c) \cup (AB) \cup (A^cB).$$

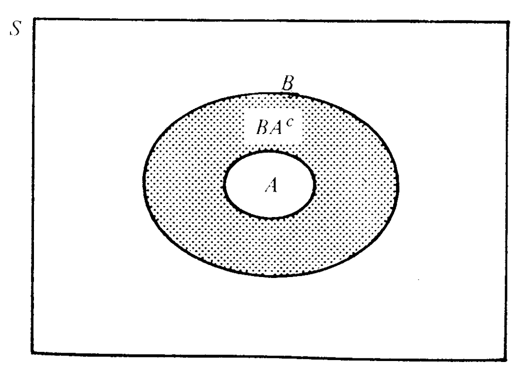

Figure 1.5 $B = A \cup (BA^c)$.

Homework Assignment #2

Due <u>Thursday</u>, October 13.

① Show $\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n \cdot \bar{X}^2$.

② Show for event A, B, and C that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

where, e.g. $AB = A \cap B$.

<u>Hint</u>: Write $A \cup B \cup C = A \cup (B \cup C)$. Then

$$P(A \cup B \cup C) = P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

....

Now do the circled exercises on the following two pages.

Recall, A^c denotes the complement of A.
Your text use $\bar{A}$.

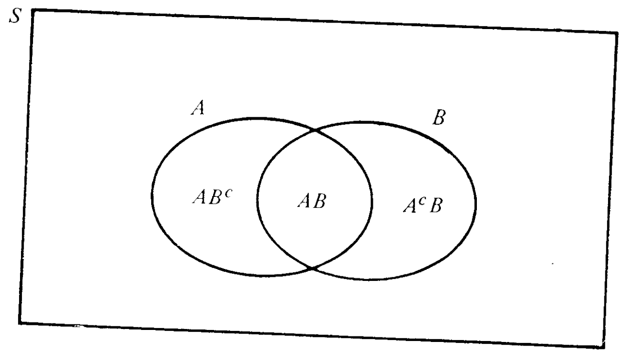

Figure 1.6 Partition of $A \cup B$.

Since the three events on the right side of this equation are disjoint, it follows from Theorem 2 that

$$\Pr(A \cup B) = \Pr(AB^c) + \Pr(AB) + \Pr(A^cB).$$

Furthermore, it is seen from Fig. 1.6 that

$$\Pr(A) = \Pr(AB^c) + \Pr(AB)$$

and

$$\Pr(B) = \Pr(A^cB) + \Pr(AB).$$

The theorem now follows from these relations. □

EXERCISES

1. A student selected from a class will be either a boy or a girl. If the probability that a boy will be selected is 0.3, what is the probability that a girl will be selected?
2. One ball is to be selected from a box containing red, white, blue, yellow, and green balls. If the probability that the selected ball will be red is 1/5 and the probability that it will be white is 2/5, what is the probability that it will be blue, yellow, or green?
3. If the probability that student A will fail a certain statistics examination is 0.5, the probability that student B will fail the examination is 0.2, and the probability that both student A and student B will fail the examination is 0.1, what is the probability that at least one of these two students will fail the examination?

4. For the conditions of Exercise 3, what is the probability that neither student A nor student B will fail the examination?

5. For the conditions of Exercise 3, what is the probability that exactly one of the two students will fail the examination?

6. Consider two events A and B such that $\Pr(A) = 1/3$ and $\Pr(B) = 1/2$. Determine the value of $\Pr(BA^c)$ for each of the following conditions: (a) A and B are disjoint; (b) $A \subset B$; (c) $\Pr(AB) = 1/8$.

7. If 50 percent of the families in a certain city subscribe to the morning newspaper, 65 percent of the families subscribe to the afternoon newspaper, and 85 percent of the families subscribe to at least one of the two newspapers, what proportion of the families subscribe to both newspapers?

8. Consider two events A and B with $\Pr(A) = 0.4$ and $\Pr(B) = 0.7$. Determine the maximum and minimum possible values of $\Pr(AB)$ and the conditions under which each of these values is attained.

9. Prove that for any two events A and B, the probability that exactly one of the two events will occur is given by the expression

$$\Pr(A) + \Pr(B) - 2\Pr(AB).$$

10. A point (x, y) is to be selected from the square S containing all points (x, y) such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Suppose that the probability that the selected point will belong to any specified subset of S is equal to the area of that subset. Find the probability of each of the following subsets: (a) the subset of points such that $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \geqslant \frac{1}{4}$; (b) the subset of points such that $\frac{1}{2} < x + y < \frac{3}{2}$; (c) the subset of points such that $y \leqslant 1 - x^2$; (d) the subset of points such that $x = y$.

11. Let $A_1, A_2, \ldots$ be any infinite sequence of events, and let $B_1, B_2, \ldots$ be another infinite sequence of events defined as follows: $B_1 = A_1$, $B_2 = A_1^c A_2$, $B_3 = A_1^c A_2^c A_3$, $B_4 = A_1^c A_2^c A_3^c A_4, \ldots$. Prove that

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(B_i) \text{ for } n = 1, 2, \ldots,$$

and that

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(B_i).$$

12. For any events $A_1, \ldots, A_n$, prove that

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} \Pr(A_i).$$