STAT451 HW5

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November 2, 2011

3.27 The time to failure in hours of an important piece of electronic equipment used in a manufactured DVD player has the density function

$$f(x) = \begin{cases} \left(\frac{1}{2000}\right) e^{\frac{-x}{2000}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

a. Find $F(x)$.

$$F(x) = \int f(x)dx = \begin{cases} 0, & x < 0 \\ -e^{\frac{-x}{2000}}, & x \geq 0 \end{cases}$$

b. Determine the probability that the component (and thus the DVD player) lasts more than 1000 hours before the component needs to be replaced.

$$P(X > 1000) = \int_{1000}^{\infty} \frac{e^{\frac{-x}{2000}}}{2000} dx = .6065$$

c. Determine the probability that the component fails before 2000 hours.

$$P(X < 2000) = \int_{0}^{2000} \frac{e^{\frac{-x}{2000}}}{2000} dx = .6321$$

3.29 An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From production data in the past, it has been determined that the particle size (in micrometers) distribution is characterized by

$$f(x) = \begin{cases} 3x^{-4}, & x > 1, \\ 0, & \text{elsewhere} \end{cases}$$

a. Verify that this is a valid density function.

From my notes, a Probability Density Function is any function $f(x)$ that satisfies:

- $f(x) \geq 0$ for all real x,
- $\int_{-\infty}^{\infty} f(x)dx = 1$,
- For event A
  $P(x \in A) = \int_{x \in A} f(x)dx$

The first bullet is satisfied, there are no negative values of $f(x)$. The second bullet can be verified by

$$\int_{1}^{\infty} f(x)dx = \frac{-1}{\infty} - \frac{-1}{1^4} = 1$$

Since we're not given any events or probabilities, this is sufficient.

b. Evaluate $F(x)$.

$$F(x) = \begin{cases} 0, & x < 1 \\ -x^{-3}, & x \geq 1 \end{cases}$$

c. What is the probability that a random particle from the manufactured fuel exceeds 4 micrometers?

$$P(X > 4) = \int_4^\infty 3x^{-4} dx = \frac{1}{64}$$

3.30 Measurements of scientific systems are always subject to variation, some more than others. There are many structures for measurement error and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function

$$f(x) = \begin{cases} k(3 - x^2), & -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Determine k that renders $f(x)$ a valid density function.
In order for $f(x)$ to be a valid density function, the following must hold.

$$\begin{aligned} \int_{-1}^{1} k(3 - x^2) dx &= 1 \\ \int_{-1}^{1} (3k - kx^2) dx &= 1 \\ 3kx - \frac{x^3}{3k}\Big|_{-1}^{1} &= 1 \\ k(3x - \frac{x^3}{3})\Big|_{-1}^{1} &= 1 \\ \frac{8k}{3} - \frac{-8k}{3} &= 1 \\ \frac{16k}{3} &= 1 \\ k &= \frac{3}{16} \end{aligned}$$

And we'll double check by plugging it in

$$\int_{-1}^{1} \left(\frac{3}{16}\right) (3 - x^2) dx = 1$$

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3.36 On a labratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

a. Calculate $P(X \leq \frac{1}{3})$.

$$\begin{aligned} P(X \leq \frac{1}{3}) &= F(\frac{1}{3}) \\ &= \int_0^{\frac{1}{3}} 2(1-x)dx \\ &= 2x - x^2\Big|_0^{\frac{1}{3}} \\ &= \left(\frac{2}{3} - \frac{1}{9}\right) - 0 \\ &= \frac{5}{9} \end{aligned}$$

b. What is the probability that X will exceed 0.5?

$$\begin{aligned} P(X > 0.5) &= F(1) - F(0.5) \\ &= \int_{0.5}^{1} 2(1-x)dx \\ &= 2x - x^2\Big|_{0.5}^{1} \\ &= 1 - \frac{3}{4} \\ &= \frac{1}{4} \end{aligned}$$

c. Given that $X \geq 0.5$, what is the probability that X will be less than 0.75?

$$P(X < 0.75 \mid X \geq 0.5) = \frac{P(0.5 \leq X < 0.75)}{P(X \geq 0.5)}$$

Recall that part **b** asked for $P(X > 0.5)$ which is the same as $P(X \geq 0.5)$.

$$\begin{aligned} &= \frac{\int_{0.5}^{0.75} 2(1-x)dx}{\frac{1}{4}} \\ &= 4(2x - x^2\Big|_{0.5}^{0.75}) \\ &= 4\left(\left(\frac{6}{4} - \frac{9}{16}\right) - \left(2 - \frac{1}{4}\right)\right) \\ &= 4\left(\frac{15}{16} - \frac{3}{4}\right) \\ &= 4\left(\frac{3}{16}\right) \\ &= \frac{3}{4} \end{aligned}$$

3.37 Determine the values of c so that the following functions represent joint probability distributions of the random variables X and Y.

a. $f(x,y) = cxy$, for $x = 1,2,3$; $y = 1,2,3$
Need the following to be true.

$$\sum_x \sum_y f(x,y) = 1$$

Plugging in $f(x,y)$.

$$\begin{aligned} \sum_{x=1}^{3}\sum_{y=1}^{3} cxy &= 1 \\ c\sum_{x=1}^{3}\sum_{y=1}^{3} xy &= 1 \\ c((1)(1)+(1)(2)+(1)(3)+(2)(1)+(2)(2)+(2)(3)+(3)(1)+(3)(2)+(3)(3)) &= 1 \\ c &= \frac{1}{36} \end{aligned}$$

b. $f(x,y) = c \mid x - y \mid$, for $x = -2,0,2$; $y = -2,3$

$$\begin{aligned} c(\mid -2-(-2) \mid + \mid -2-3 \mid + \mid 0-(-2) \mid + \mid 0-3 \mid + \mid 2-(-2) \mid + \mid 2-3 \mid) &= 1 \\ c(0+5+2+3+4+1) &= 1 \\ c &= \frac{1}{15} \end{aligned}$$

3.39 From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, find

a. the joint probability distribution of X and Y
There are $\binom{8}{4} = 70$ ways to take a random sample of 4 pieces of fruit from the sack containing a total of 8. This will be the denominator for all probabilities in the table. Now we use the product rule to choose the given number of each fruit.
For example, the ways to get 2 oranges and 2 apples is $\binom{3}{2}\binom{2}{2} = 3$ and the probability of this happening is therefore $\frac{3}{70}$.
For the case when $X = 0$ and $Y = 0$, this assumes the only remaining fruit are bananas, but it's impossible to select 4 fruit because there are only 3 bananas. So the probability of this happening is 0.
Another type of case is where $0 < X + Y < 4$. We will need to borrow from the bananas to complete these, by multiplying $\binom{3}{x}\binom{2}{y}\binom{3}{4-(x+y)}$ and dividing that by 70.

		x			
		0	1	2	3
	0	0	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$
y	1	$\frac{2}{70}$	$\frac{18}{70}$	$\frac{18}{70}$	$\frac{2}{70}$
	2	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$	0

b. $P[(X,Y)\in A]$**, where** A **is the region that is given by** $\{(x,y) \mid x+y \leq 2\}$
The values (x,y) that satisfy this equation are (0,0),(0,1),(0,2),(1,0),(1,1),(2,0). Their values from the table are 0, $\frac{3}{70}$, $\frac{9}{70}$, $\frac{2}{70}$, $\frac{18}{70}$, $\frac{3}{70}$. The total probability for A is the sum of these.

$$P[(X,Y)\in A] = \tfrac{35}{70} = \tfrac{1}{2}$$

3.41 A candy company distributes boxes of chocolates with a mixture of creams, toffees, and cordials. Suppose that the weight of each box is 1 kilogram, but the individual weights of the creams, toffees, and cordials vary from box to box. For a randomly selected box, let X **and** Y **represent the weights of the creams and the toffees, respectively, and suppose that the joint density function of these variables is**

$$f(x,y) = \begin{cases} 24xy, & 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\ & x+y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the probability that in a given box the cordials account for more than 1/2 of the weight.

$$\begin{aligned} P(X+Y<\frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-y} 24xydxdy \\ &= \int_0^{\frac{1}{2}} 24y \left(\int_0^{\frac{1}{2}-y} xdx \right) dy \\ &= \int_0^{\frac{1}{2}} 24y \frac{(2y-1)^2}{8} dy \\ &= 3\int_0^{\frac{1}{2}} y(2y-1)^2 dy \\ &= \frac{1}{16} \end{aligned}$$

b. Find the marginal density for the weight of the creams.
Let $f_1(x)$ represent the marginal density for X.

$$\begin{aligned} f_1(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^{1-x} 24xydy \\ &= 24x \int_0^{1-x} ydy \\ &= 24x\frac{(x-1)^2}{2} \\ &= 12x(x-1)^2 \\ f_1(x) &= \begin{cases} 12x(x-1)^2, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

c. Find the probability that the weight of the toffees in a box is less than 1/8 of a kilogram if it is known that creams constitute 3/4 of the weight.

First we need to find $f(y \mid x)$.

$$\begin{aligned} f(y \mid x) &= \frac{f(x,y)}{f_1(x)} \\ &= \frac{24xy}{12x(x-1)^2} \\ &= \frac{12y}{(x-1)^2} \\ P(Y < \frac{1}{8} \mid X = \frac{3}{4}) &= \int_0^{\frac{1}{8}} \frac{12y}{((\frac{3}{4})-1)^2} dy \\ &= \int_0^{\frac{1}{8}} 192y dy \\ &= \frac{3}{2} \end{aligned}$$

3.51 Consider an experiment that consists of 2 rolls of a balanced die. If X is the number of 4s and Y is the number of 5s obtained in the 2 rolls of the die, find

a. the joint probability distribution of X and Y;

The probability of rolling 2 dice and not getting a 4 or a 5 on either roll is $\left(\frac{4}{6}\right)^2$ because of the other 4 possible rolls each time. To get a 4 but not a 5, or a 5 but not a 4 there are two ways: getting the 4 first (or 5), or second. The probability of rolling a 4 is just $\frac{1}{6}$, then you cannot roll a 5 or another 4, so multiply by $\frac{4}{6}$. Considering the reverse order, you can just multiply this by 2. Getting the same roll twice is just $\left(\frac{1}{6}\right)^2$. Getting one of each is also $\left(\frac{1}{6}\right)^2$, but you multiply by 2 for the reverse order.

		x		
		0	1	2
	0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$
y	1	$\frac{8}{36}$	$\frac{2}{36}$	0
	2	$\frac{1}{36}$	0	0

b. $P[(X,Y) \in A]$, where A is the region $\{(x,y) \mid 2x+y<3\}$.

The values (x,y) that satisfy this equation are (0,0),(0,1),(0,2),(1,0). Their values from the table are $\frac{16}{36}, \frac{8}{36}, \frac{1}{36}, \frac{8}{36}$. Just need to sum them.

$$P[(X,Y) \in A] = \frac{33}{36} = \frac{11}{12}$$

3.53 Three cards are drawn without replacement from the 12 face cards (jacks, queens, and kings) of an ordinary deck of 52 playing cards. Let X be the number of kings selected and Y the number of jacks. Find

a. the joint probability distribution of X and Y;

The probability of drawing a queen on the first card is $\frac{4}{12}$. The probability of drawing a second queen from the remaining cards is $\frac{3}{11}$, and a third $\frac{2}{10}$. This product is the probability of getting no kings or jacks. There are 3 ways to get one jack and no kings, because you can get it on the first, second, or third draw. The probability of drawing it first is $\frac{4}{12}$, and the probability of not getting a king or jack next, since there are still 4 kings and 3 jacks, is $\frac{4}{11}$. This means it had to be a queen, of which there were 4. Now there are 3 for the final draw, resulting in $\frac{3}{10}$. This sort of calculation will continue in the following table.

Composition	Resulting Hand	Probability
0J/0K	QQQ	$\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{1}{55}$
1J/0K	JQQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QJQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QQJ	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
2J/0K	JJQ	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
	JQJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QJJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
3J/0K	JJJ	$\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{1}{55}$
0J/1K	KQQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QKQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QQK	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
1J/1K	JKQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
	JQK	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
	QJK	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
	KJQ	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
	KQJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
	QKJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} = \frac{8}{165}$
2J/1K	JJK	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
	JKJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	KJJ	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
3J/1K	—	0
0J/2K	KKQ	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
	KQK	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	QKK	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
1J/2K	KKJ	$\frac{4}{12} \times \frac{3}{11} \times \frac{4}{10} = \frac{2}{55}$
	KJK	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
	JKK	$\frac{4}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{2}{55}$
2J/2K	—	0
3J/2K	—	0
0J/3K	KKK	$\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{1}{55}$
1J/3K	—	0
2J/3K	—	0
3J/3K	—	0

f(x,y)		x			
		0	1	2	3
y	0	$\frac{1}{55}$	$\frac{6}{55}$	$\frac{6}{55}$	$\frac{1}{55}$
	1	$\frac{6}{55}$	$\frac{16}{55}$	$\frac{6}{55}$	0
	2	$\frac{6}{55}$	$\frac{6}{55}$	0	0
	3	$\frac{1}{55}$	0	0	0

b. $P[(X,Y) \in A]$**, where** A **is the region** $\{(x,y) \mid x+y \geq 2\}$**.**
The values (x,y) that satisfy this equation are (1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),(3,0),(3,1),(3,2),(3,3). Their values from the table are $\frac{16}{55}, \frac{6}{55}, \frac{6}{55}, \frac{6}{55}, \frac{6}{55}, \frac{1}{55}, \frac{1}{55}, 0, 0, 0, 0, 0, 0$. Just need the sum of them.

$$P[(X,Y) \in A] = \tfrac{42}{55}$$

3.55 Given the joint density function

$$f(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0 < x < 2, \quad 2 < y < 4, \\ 0, & \text{elsewhere,} \end{cases}$$

find $P(1 < Y < 3 \mid X = 1)$**.**

$$P(1 < Y < 3 \mid X = 1) = \int_2^3 g(y \mid X = 1)dy$$

Let $f_1(x)$ be the marginal of X.

$$\begin{aligned}
f_1(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_2^4 \frac{6-x-y}{8}dy \\
&= \frac{6-x}{8}y - \frac{y^2}{16}\bigg|_2^4 \\
&= \left(\frac{4(6-x)}{8} - 1\right) - \left(\frac{2(6-x)}{8} - \frac{1}{4}\right) \\
&= \frac{3-x}{4} \\
g(y \mid X = 1) &= \frac{f(x,y)}{f_1(x)} \\
&= \frac{\frac{6-1-y}{8}}{\frac{1}{2}} \\
&= \frac{5-y}{4} \\
P(1 < Y < 3 \mid X = 1) &= \int_2^3 \frac{5-y}{4}dy \\
&= \frac{5y}{4} - \frac{y^2}{8}\bigg|_2^3 \\
&= \frac{15}{4} - \frac{9}{8} - \frac{10}{4} + \frac{4}{8} \\
&= \frac{5}{8}
\end{aligned}$$