chaos_pendulum

September 28, 2017

1 Damped Driven Pendulum

$$\frac{d^2\phi}{dt^2} + 2\beta\frac{d\phi}{dt} + \omega_o^2 \sin\phi = \gamma\omega_o^2 \cos(\omega t)$$

First some Python library stuff

```
In [1]: %matplotlib inline
        import numpy as np
        import matplotlib.pyplot as plt
        import scipy.integrate as integrate
        LARGE_FIGSIZE = (12, 8)
```

Now let's define the parameters for the problem

```
In [2]: gamma=.01
        omega=2*np.pi
        omega_o=1.5*omega
        beta=omega_o/4
```

Next we need to define the derivatives

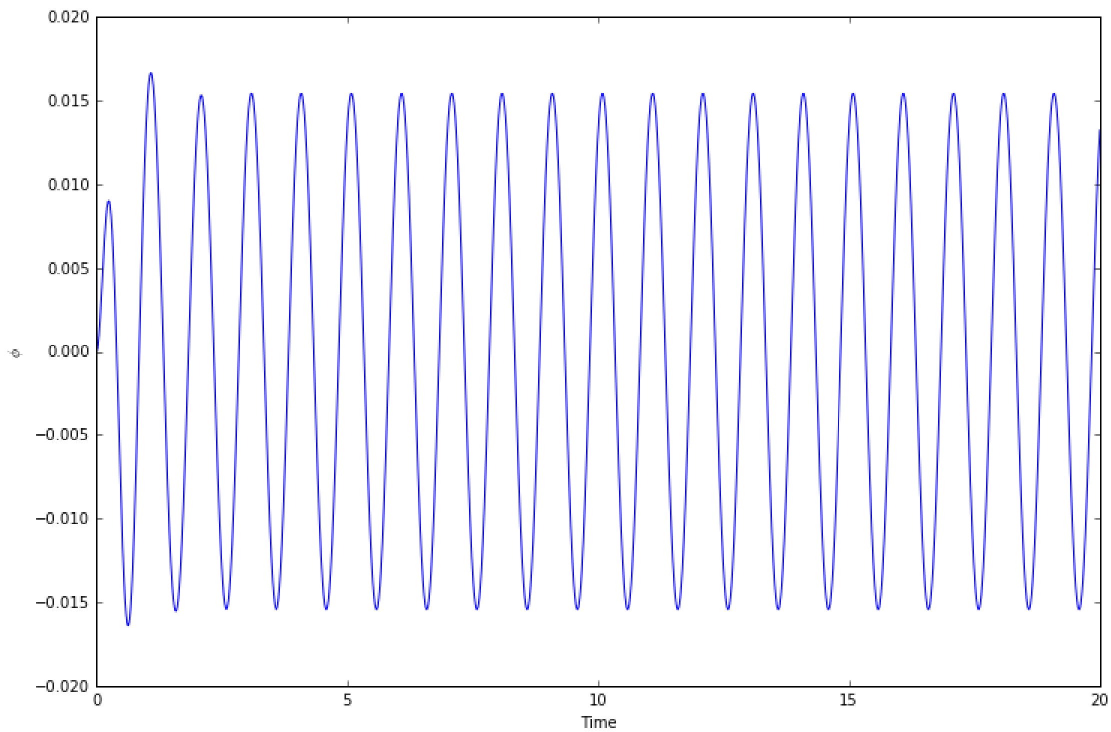
```
In [3]: def deriv(z, t):
            phi, phidot = z
            return [phidot, -2*beta*phidot-omega_o**2*np.sin(phi)+gamma*omega_o**2*np.cos(omega
```

Now we will create a vector of time values to solve the ODE and we will set an initial condition of (0,0). Then we solve the ODE

```
In [4]: t = np.linspace(0, 20, 2000)
        zinit = [0, 0]
        z = integrate.odeint(deriv, zinit, t)
```

Time to plot the solution for ϕ

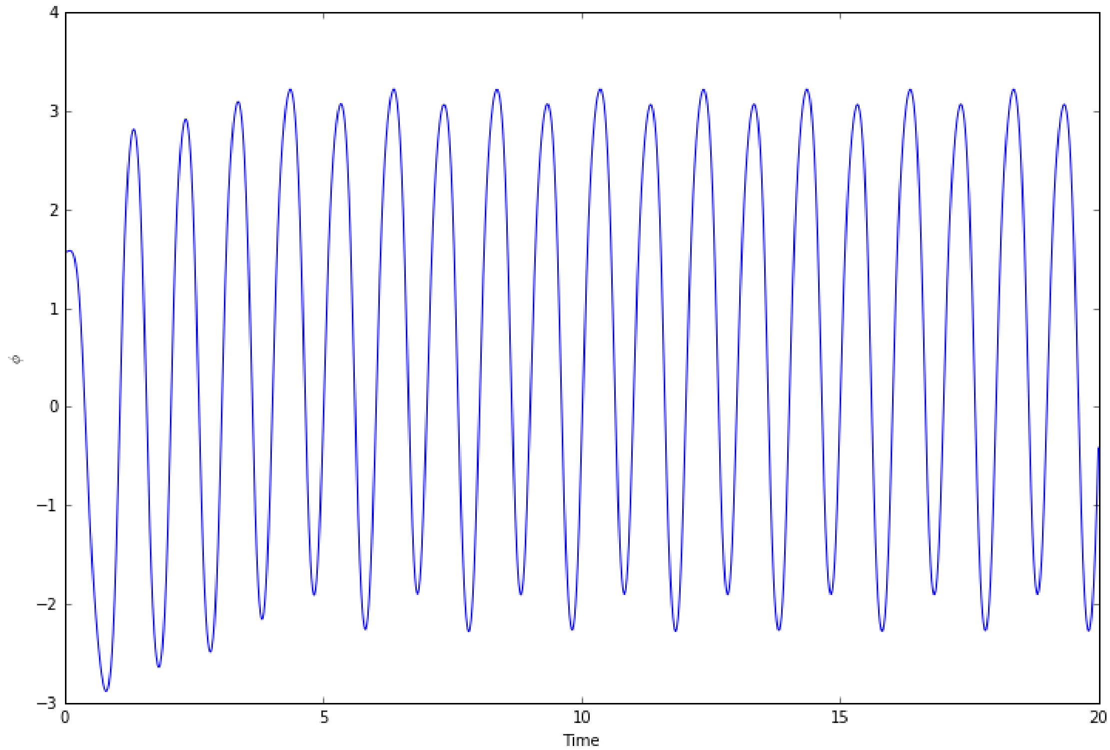
```
In [5]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
        ax.plot(t,z[0:2000,0]);
        plt.ylabel('$\phi$');
        plt.xlabel('Time');
```



Now we crank up γ

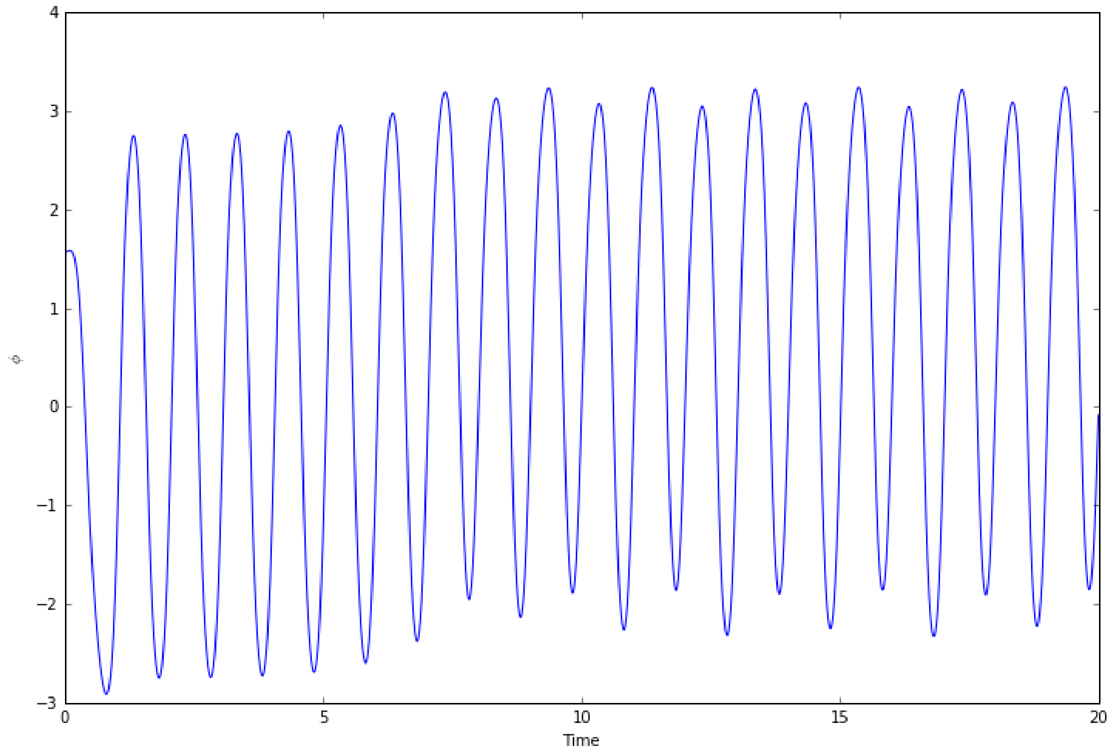
```
In [6]: gamma=1.078
```

```
In [7]: t = np.linspace(0, 20, 2000)
        zinit = [np.pi/2, 0]
        z = integrate.odeint(deriv, zinit, t)
        fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
        ax.plot(t,z[0:2000,0]);
        plt.ylabel('$\phi$');
        plt.xlabel('Time');
```



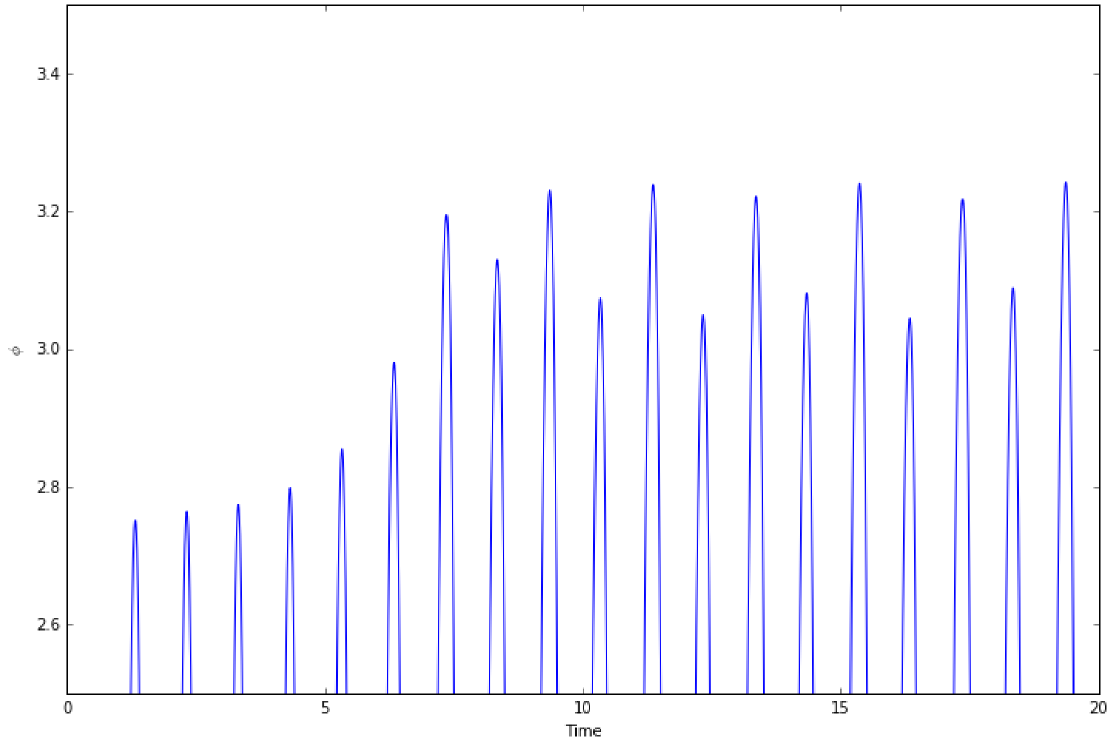
Now we crank up γ some more...

```
In [8]: gamma=1.081
        t = np.linspace(0, 20, 2000)
        zinit = [np.pi/2, 0]
        z = integrate.odeint(deriv, zinit, t)
        fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
        ax.plot(t,z[0:2000,0]);
        plt.ylabel('$\phi$');
        plt.xlabel('Time');
```



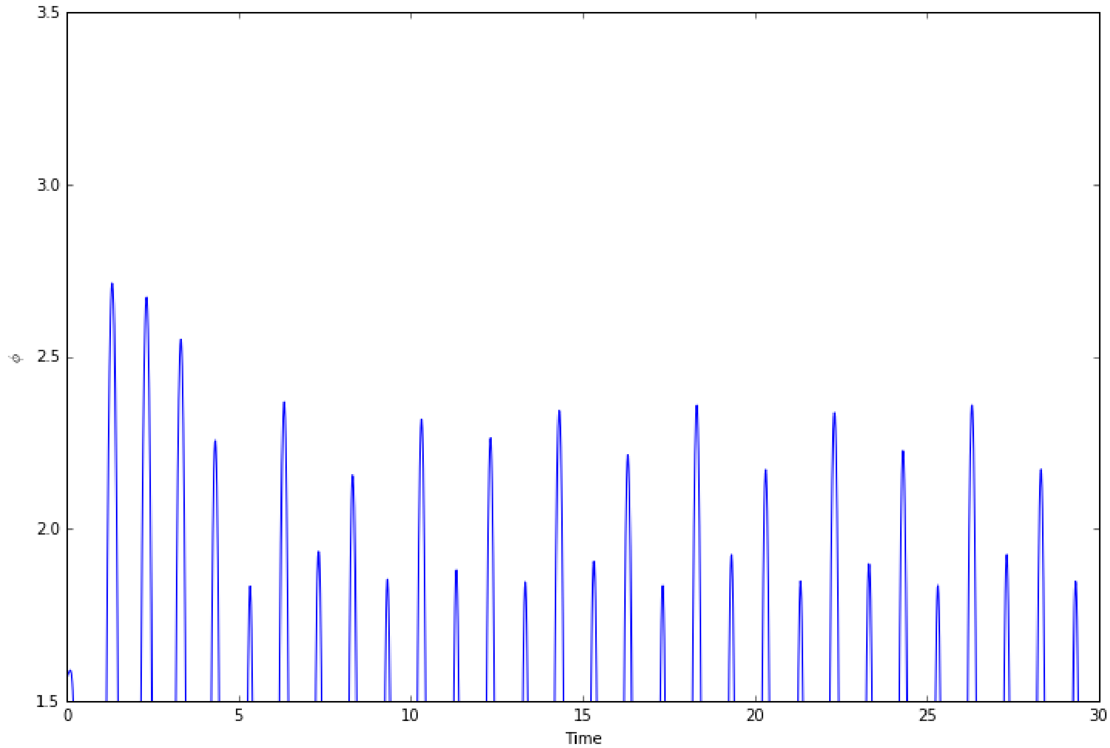
Zooming in

```
In [9]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
        ax.plot(t,z[0:2000,0]);
        plt.ylim([2.5,3.5]);
        plt.xlim([0,20]);
        plt.ylabel('$\phi$');
        plt.xlabel('Time');
```



Now we crank up γ even more...

In [10]:
```
gamma=1.0826
t = np.linspace(0, 30, 2000)
zinit = [np.pi/2, 0]
z = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(t,z[0:2000,0]);
plt.ylim([1.5,3.5]);
plt.xlim([0,30]);
plt.ylabel('$\phi$');
plt.xlabel('Time');
```

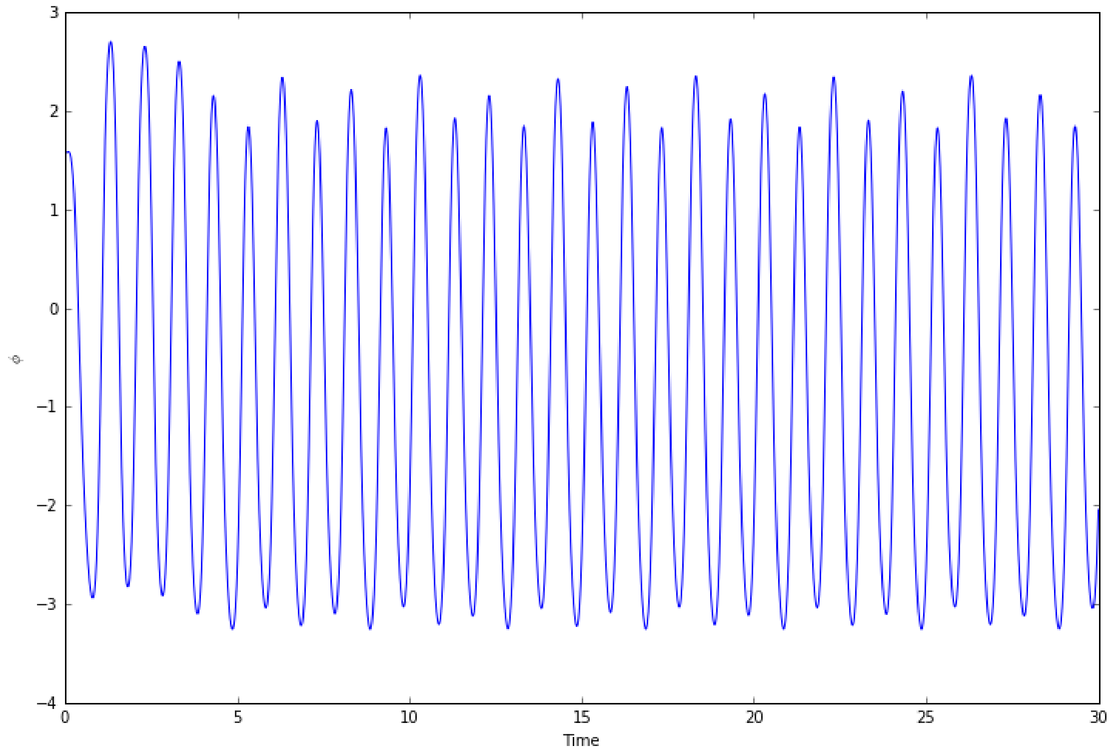


Notice that each increase in γ for the next period doubling was smaller. This is called a period doubling cascade.

$$\gamma_{n+1} - \gamma_n = \frac{1}{\delta_f}(\gamma_n - \gamma_{n-1})$$

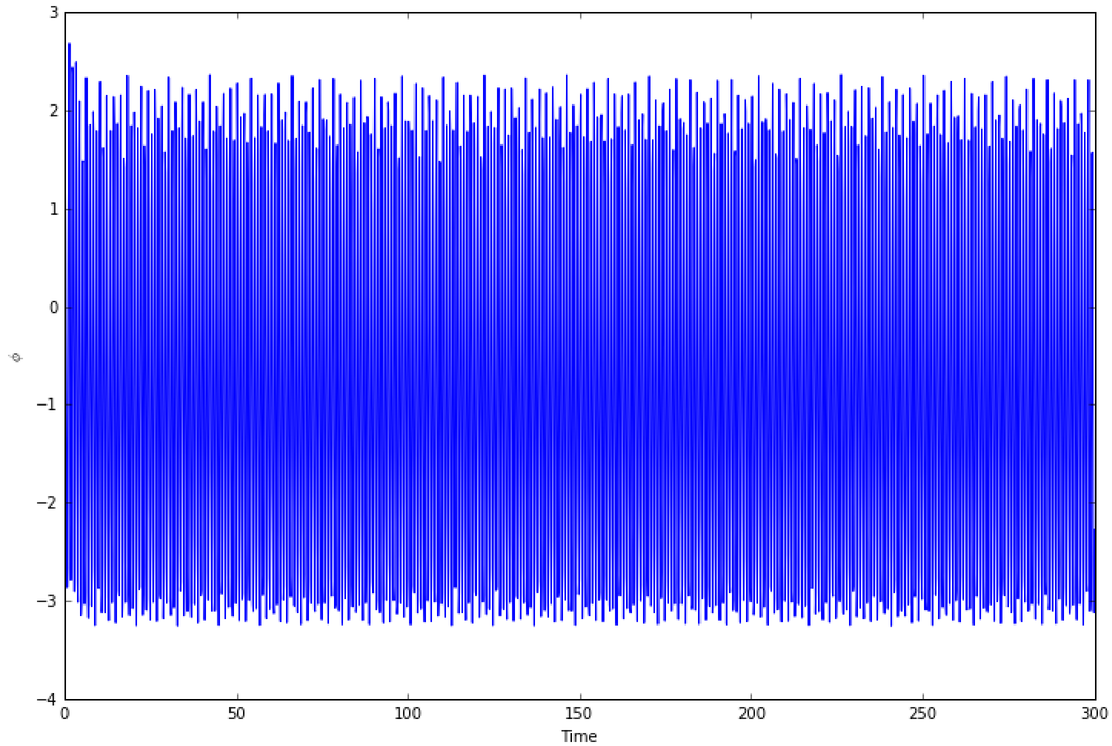
And notice for this geometric series, that γ goes to a fixed value as n goes to ∞.
That fixed value is 1.0829 and the name given to the δ_f is the Fiegenbaum Number.
So what happens to the pendulum when you go over 1.0829?

```
In [11]: gamma=1.0829
         t = np.linspace(0, 30, 2000)
         zinit = [np.pi/2, 0]
         z = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,z[0:2000,0]);
         plt.ylabel('$\phi$');
         plt.xlabel('Time');
```



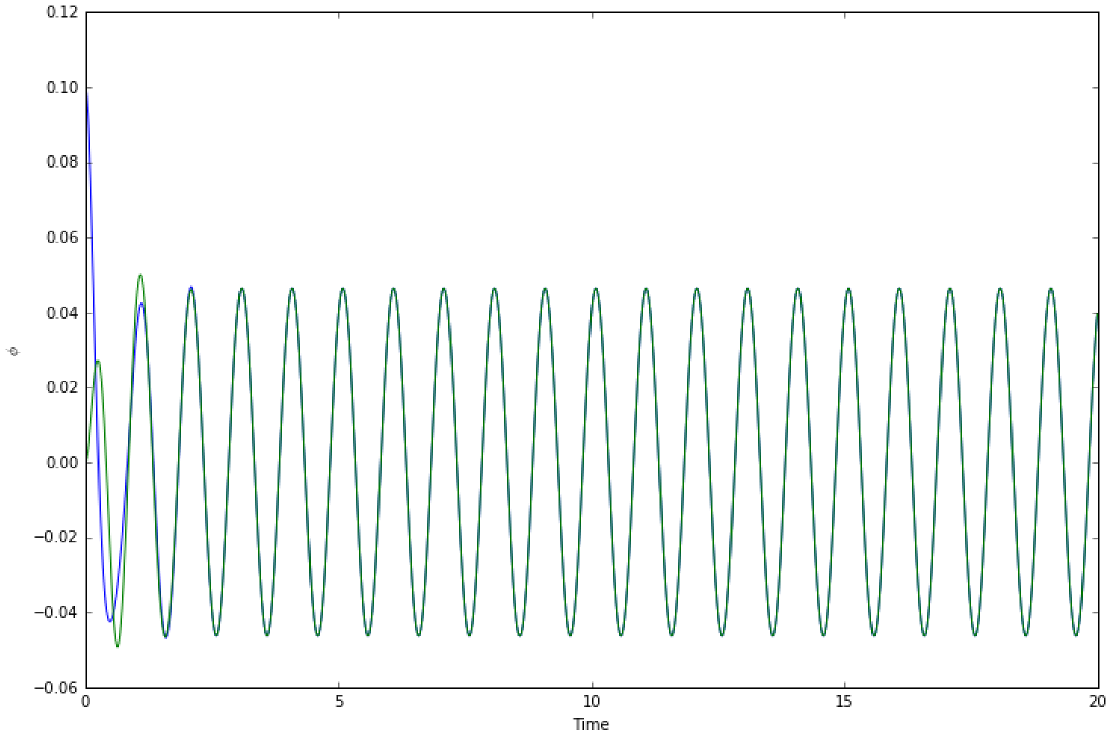
And if we look longer in time, there is no sign of a repeating pattern.

```
In [12]: gamma=1.0829
         t = np.linspace(0, 300, 2000)
         zinit = [np.pi/2, 0]
         z = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,z[0:2000,0]);
         plt.ylabel('$\phi$');
         plt.xlabel('Time');
```



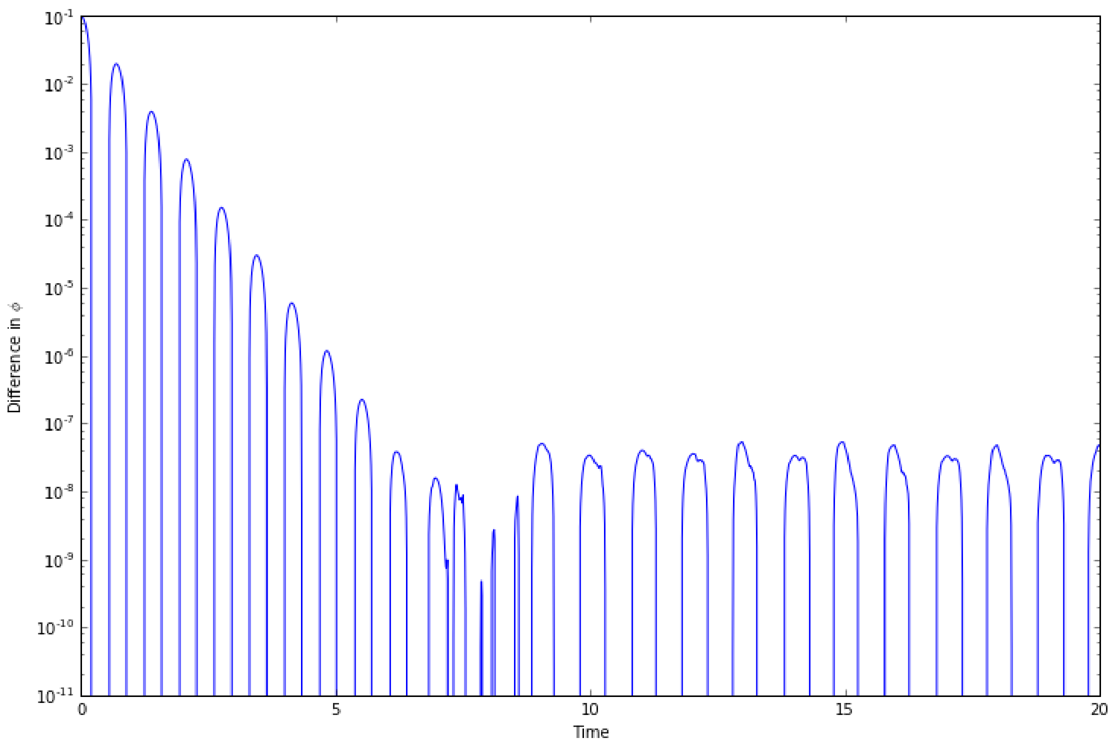
Next let's look at the impact of changing the initial conditions. First let's do it for the system prior to going chaotic.

In [13]:
```
gamma=.03
t = np.linspace(0, 20, 2000)
zinit = [0, 0]
z = integrate.odeint(deriv, zinit, t)
zinit = [0+.1, 0]
zz = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(t,zz[0:2000,0]);
ax.plot(t,z[0:2000,0]);
plt.ylabel('$\phi$');
plt.xlabel('Time');
```



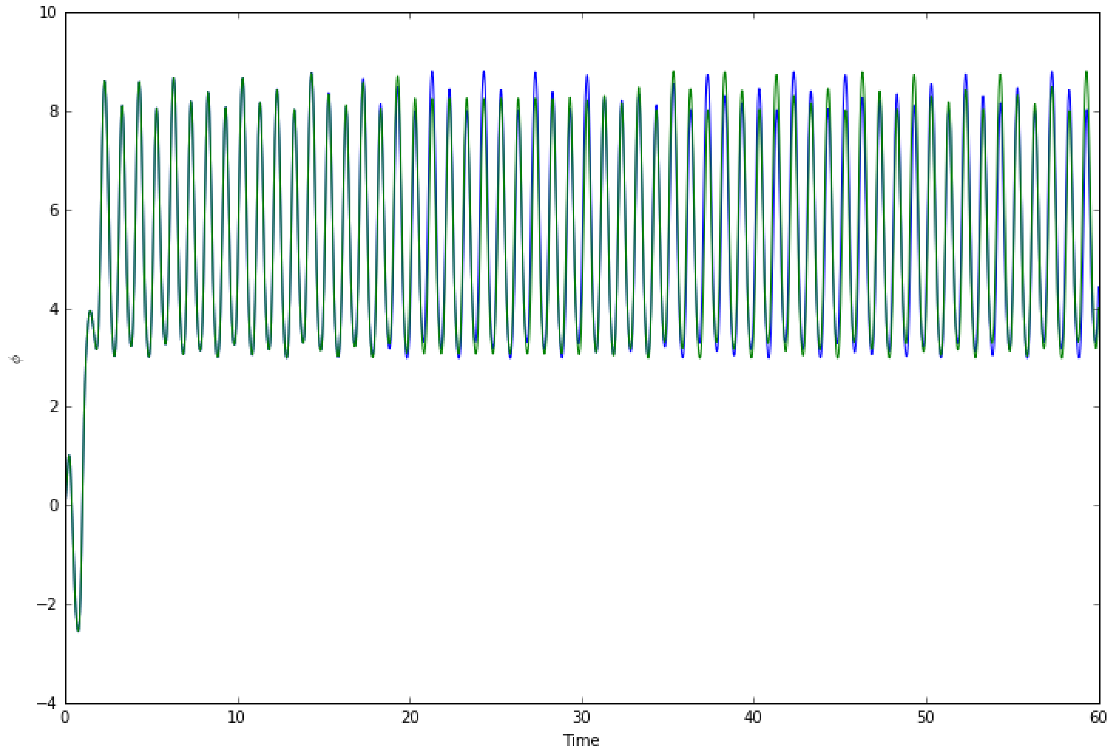
We can plot the difference between the two solutions through time

```
In [14]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,zz[0:2000,0]-z[0:2000,0]);
         plt.yscale('log')
         plt.ylabel('Difference in $\phi$');
         plt.xlabel('Time');
```

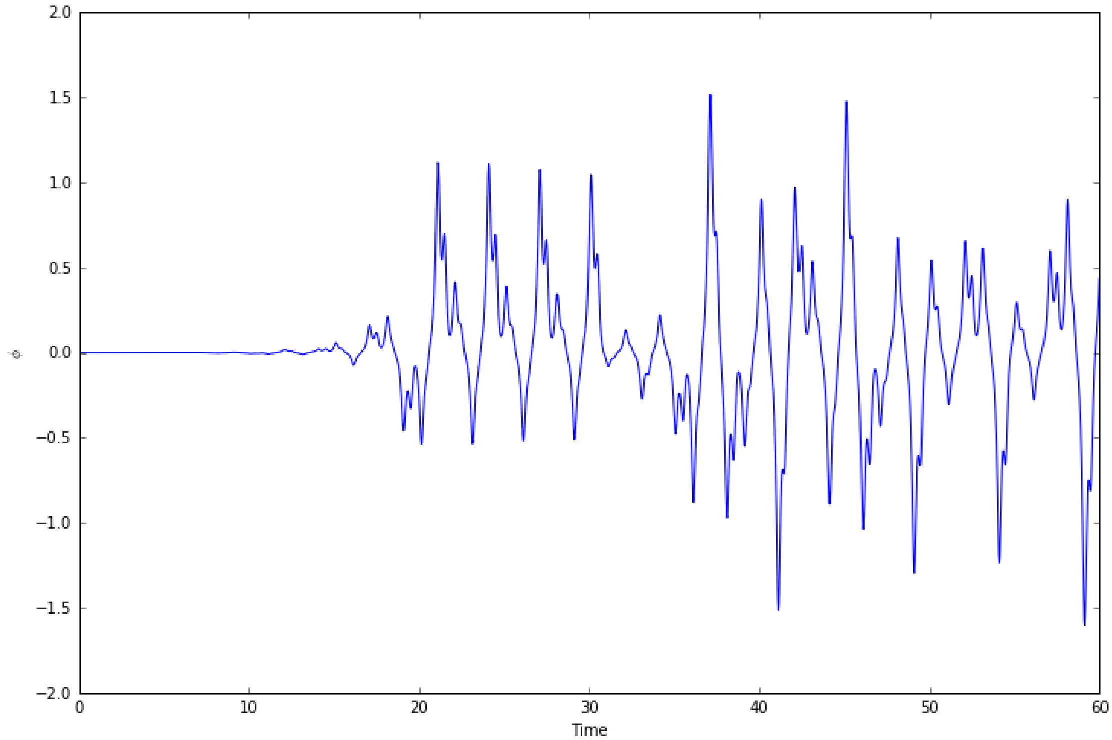


Now let's do the same thing, but when the system is chaotic (the larger value for the drive strength)

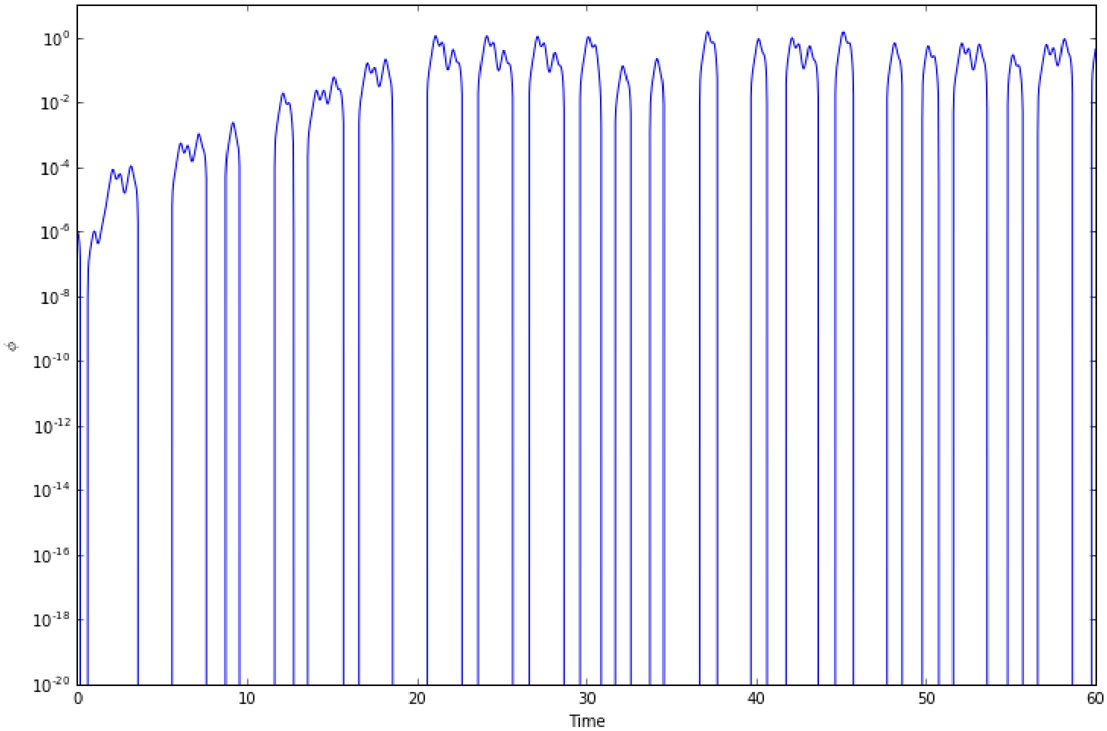
```
In [15]: gamma=1.09
         t = np.linspace(0, 60, 2000)
         zinit = [0, 0]
         z = integrate.odeint(deriv, zinit, t)
         zinit = [0+.000001, 0]
         zz = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,zz[0:2000,0]);
         ax.plot(t,z[0:2000,0]);
         plt.ylabel('$\phi$');
         plt.xlabel('Time');
```




```
In [16]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,zz[0:2000,0]-z[0:2000,0]);
         #plt.yscale('log')
         #plt.ylim([10**-20,10**1]);
         plt.ylabel('$\phi$');
         plt.xlabel('Time');
```




```
In [17]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(t,zz[0:2000,0]-z[0:2000,0]);
         plt.yscale('log')
         plt.ylim([10**-20,10**1]);
         plt.ylabel('$\phi$');
         plt.xlabel('Time');
```



We can show that in the linear case we expect different initial conditions to converge to the same solution:

$$\Delta\phi = \phi_2 - \phi_1$$

$$\phi_1(t) = A\cos(\omega t) + C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\phi_2(t) = A\cos(\omega t) + D_1 e^{r_1 t} + D_2 e^{r_2 t}$$

$$\Delta\phi = B_1 e^{(-\beta + \sqrt{\beta^2 - \omega_o^2})t} + B_2 e^{(-\beta + \sqrt{\beta^2 - \omega_o^2})t}$$

$$\Delta\phi = D e^{-\beta t}\cos(\omega_1 t - \delta)$$

$$ln \mid \Delta\phi(t) \mid = lnD - \beta t - ln \mid \cos\omega_1 t - \delta \mid$$

But this is all just fancy physics stuff right? Who cares about pendulums? We live in NC! Tell me about something I care about like Hurricanes.

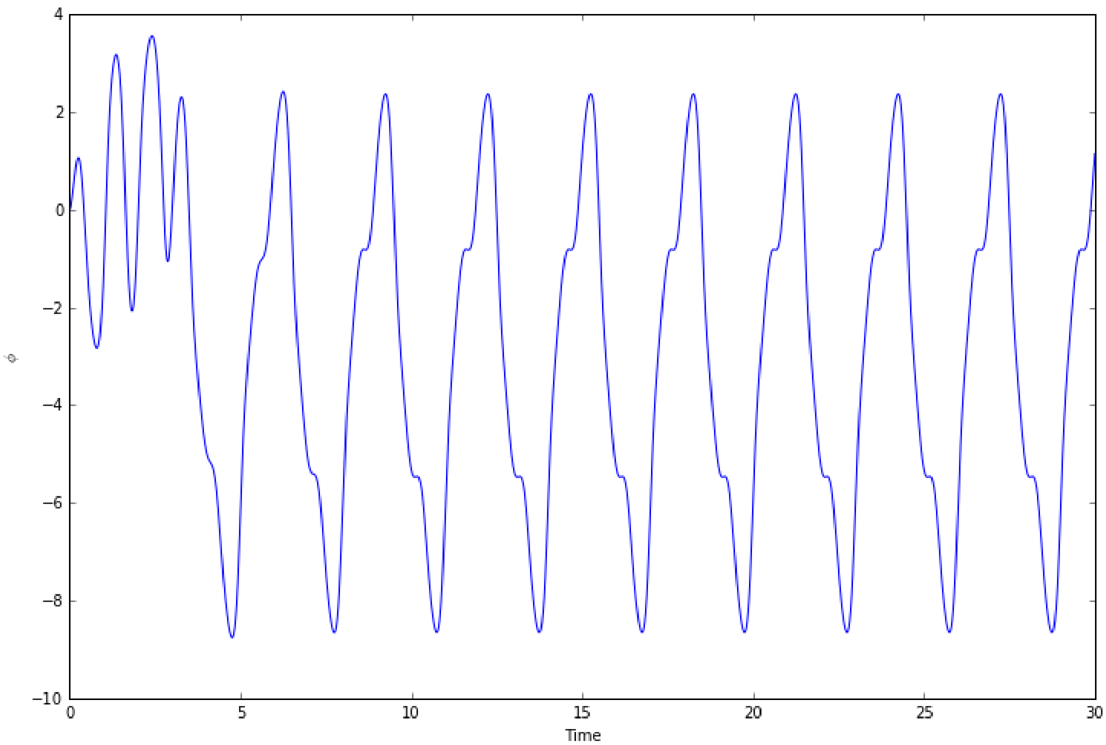
For the sensitivity to initial conditions, the difference between two solutions explodes expoentially:

$$\mid \Delta\phi(t) \mid \approx k e^{\lambda} t$$

λ is called the Lyapunov exponent

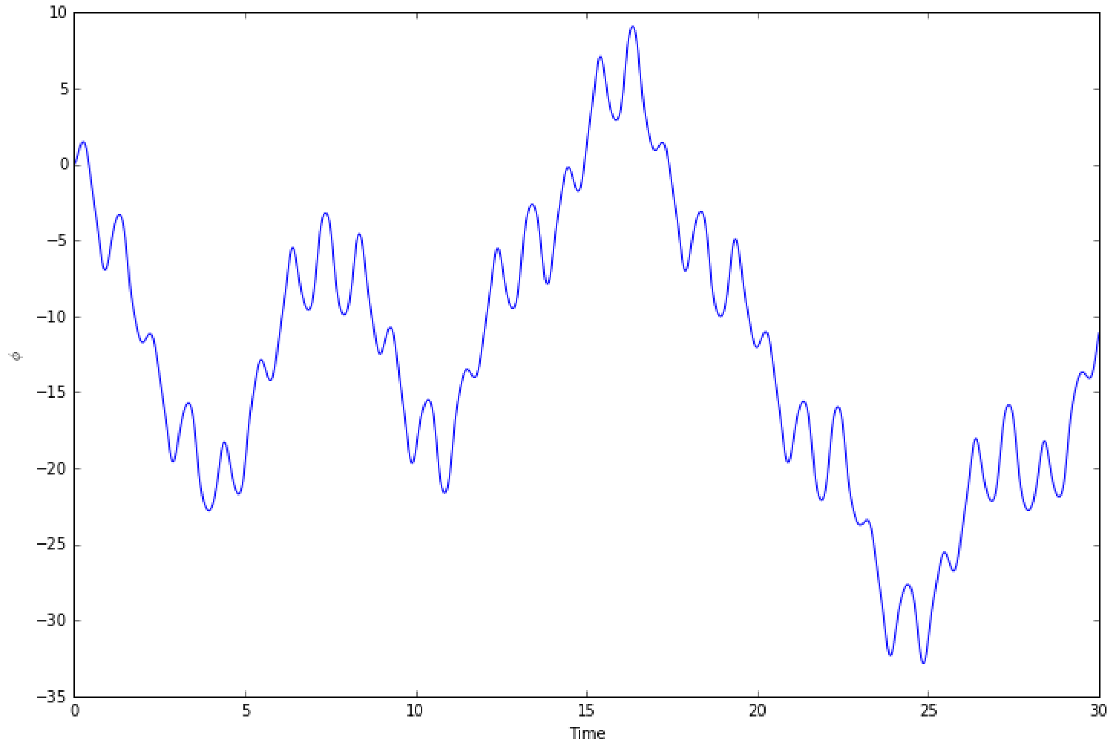
So is it all just chaos above the critical γ ?

In [18]:
```
gamma=1.13
t = np.linspace(0, 30, 2000)
zinit = [0, 0]
z = integrate.odeint(deriv, zinit, t)
zinit = [0+.000001, 0]
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(t,z[0:2000,0]);
plt.ylabel('$\phi$');
plt.xlabel('Time');
```

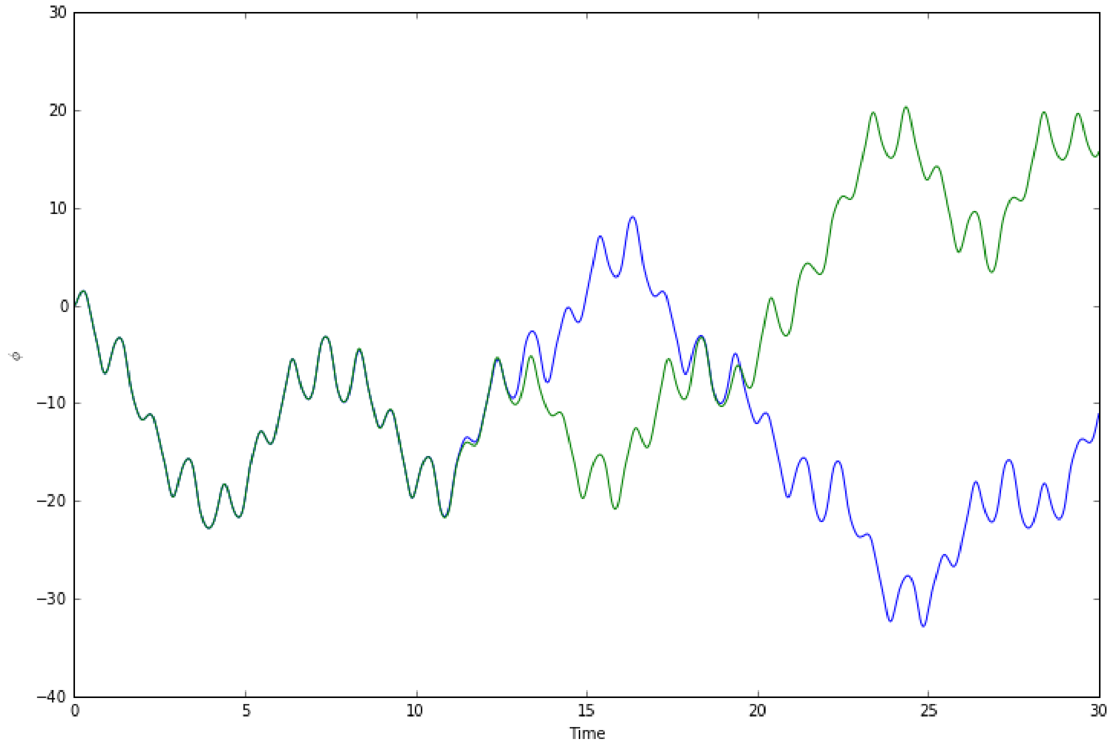


Now crank γ up some more

In [21]:
```
gamma=1.503
t = np.linspace(0, 30, 2000)
zinit = [0, 0]
z = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(t,z[0:2000,0]);
plt.ylabel('$\phi$');
plt.xlabel('Time');
```



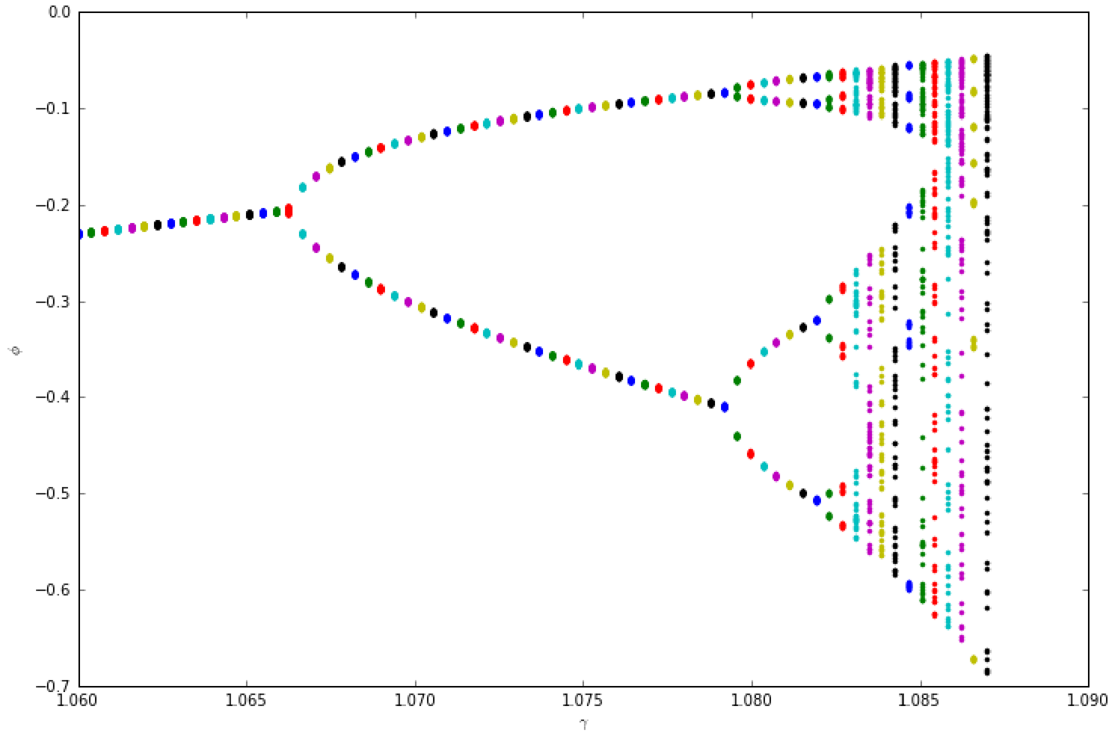
In [22]:
```
gamma=1.503
t = np.linspace(0, 30, 2000)
zinit = [0+.0001, 0]
zz = integrate.odeint(deriv, zinit, t)
zinit = [0+.000001, 0]
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(t,z[0:2000,0]);
ax.plot(t,zz[0:2000,0]);
plt.ylabel('$\phi$');
plt.xlabel('Time');
```



Next we need to think about how to see all of this varying behavior as we change the drive strength.

```
In [23]: store_vals = np.zeros([100,70])
         dum_gam = np.linspace(1.06, 1.087, 70)
         for i in range(70):
             gamma = dum_gam[i]
             t = np.linspace(0, 500, 500000)
             zinit = [-np.pi/2, 0]
             z = integrate.odeint(deriv, zinit, t)
             store_vals[:,i] = z[400000:500000:1000,0]
```

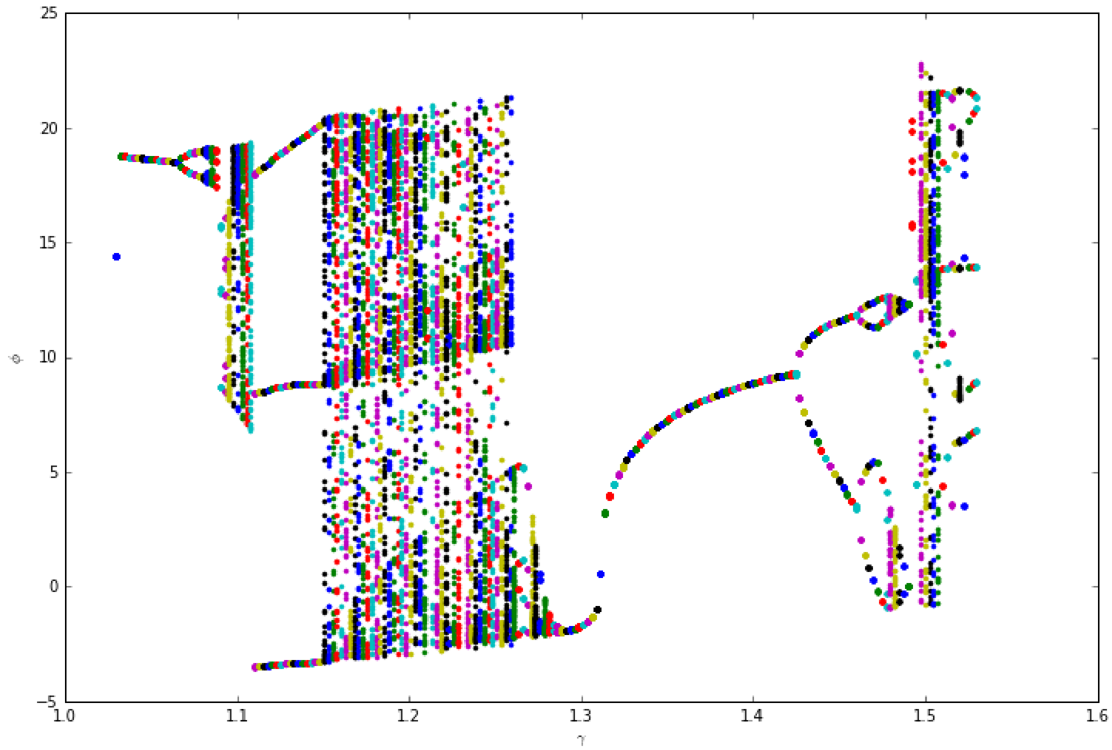
```
In [24]: import numpy.matlib
         dum = np.matlib.repmat(dum_gam,100,1)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(dum,store_vals,'.');
         ax.set_ylabel('$\phi$');
         ax.set_xlabel('$\gamma$');
```



How about we plot the velocity and go to much larger values of gamma - what do you think it will look like?

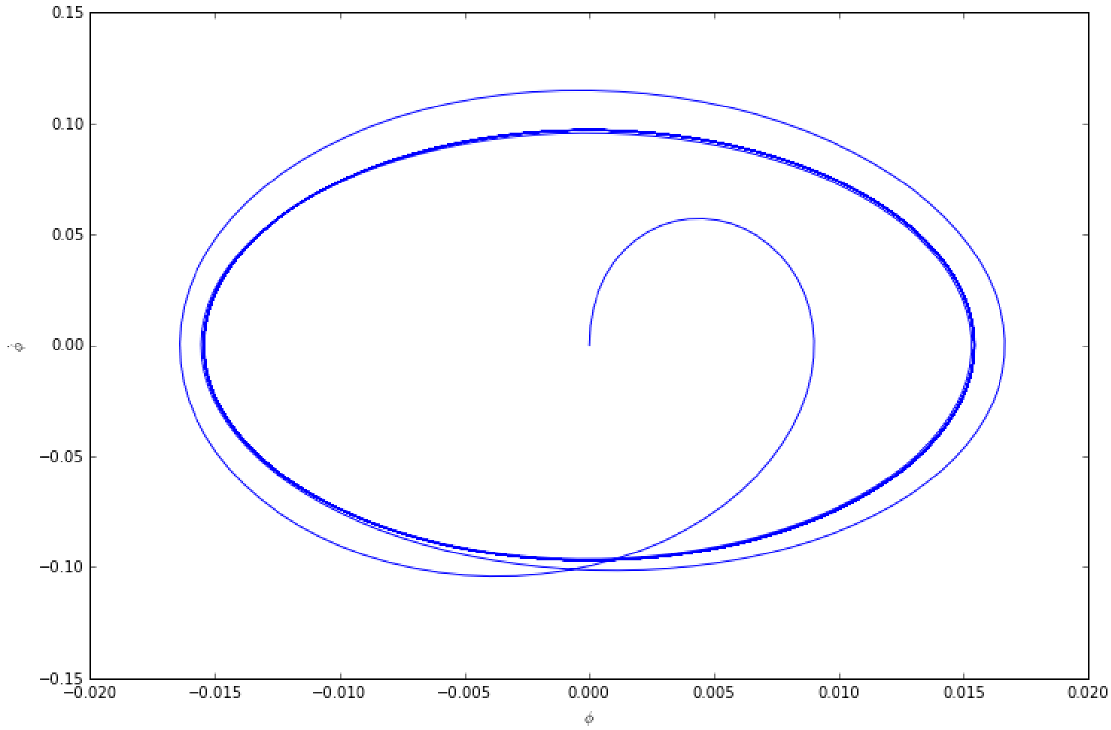
```
In [25]: store_vals = np.zeros([100,200])
         dum_gam = np.linspace(1.03, 1.53, 200)
         for i in range(200):
             gamma = dum_gam[i]
             t = np.linspace(0, 500, 500000)
             zinit = [-np.pi/2, 0]
             z = integrate.odeint(deriv, zinit, t)
             store_vals[:,i] = z[400000:500000:1000,1]
```

```
In [26]: dum = np.matlib.repmat(dum_gam,100,1)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(dum,store_vals,'.');
         ax.set_ylabel('$\phi$');
         ax.set_xlabel('$\gamma$');
```



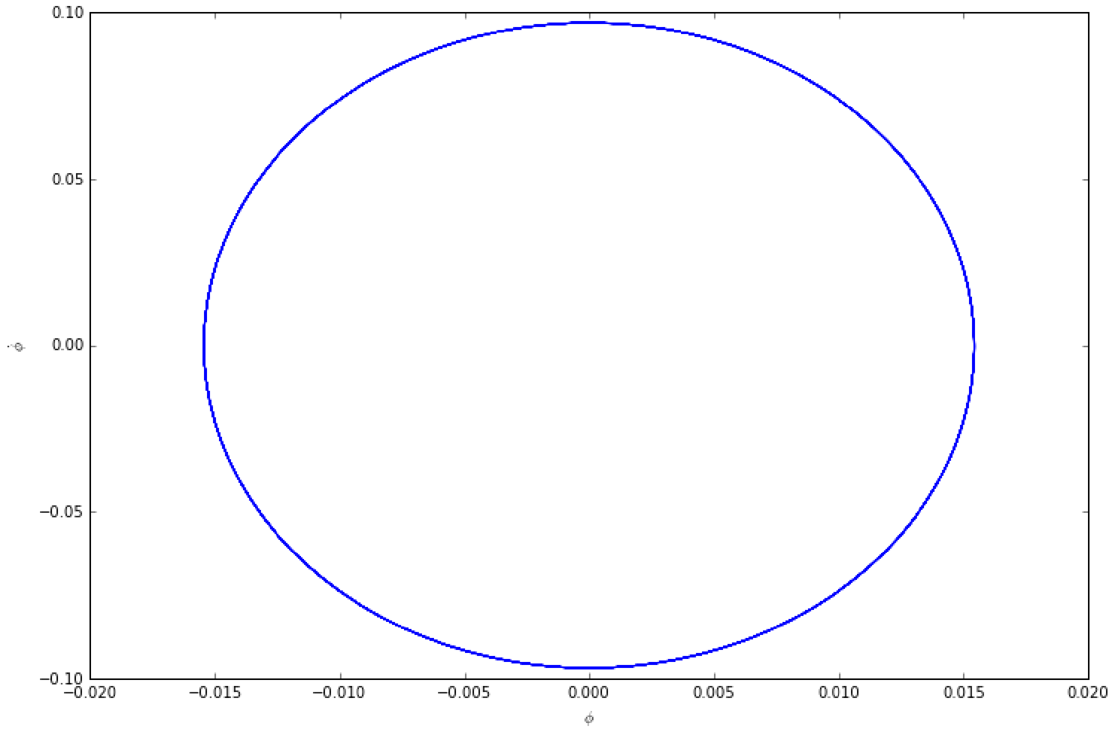
Another way to look at the systems behavior as we have seen, is the phase space plot: What are the phase space axes of the DDP?

In [28]:
```
gamma=.01
t = np.linspace(0, 20, 2000)
zinit = [0, 0]
z = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(z[0:2000,0],z[0:2000,1]);
plt.xlabel("$\phi$");
plt.ylabel("$\dot{\phi}$");
```

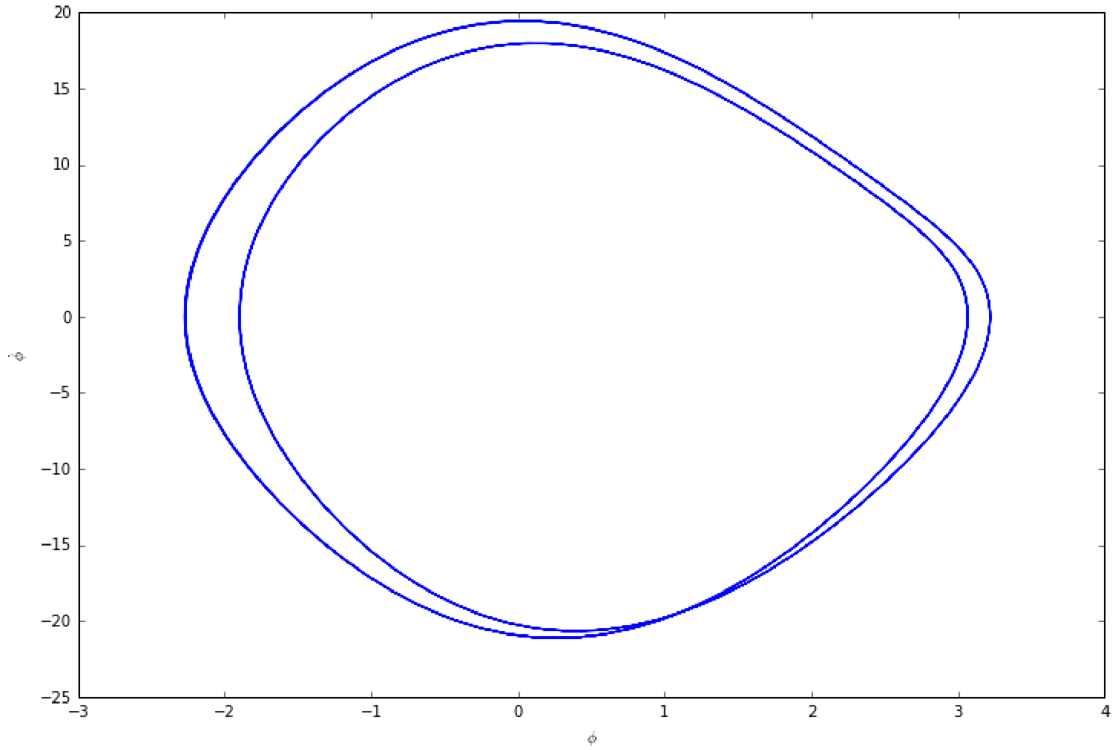


Let's just look at the attractor

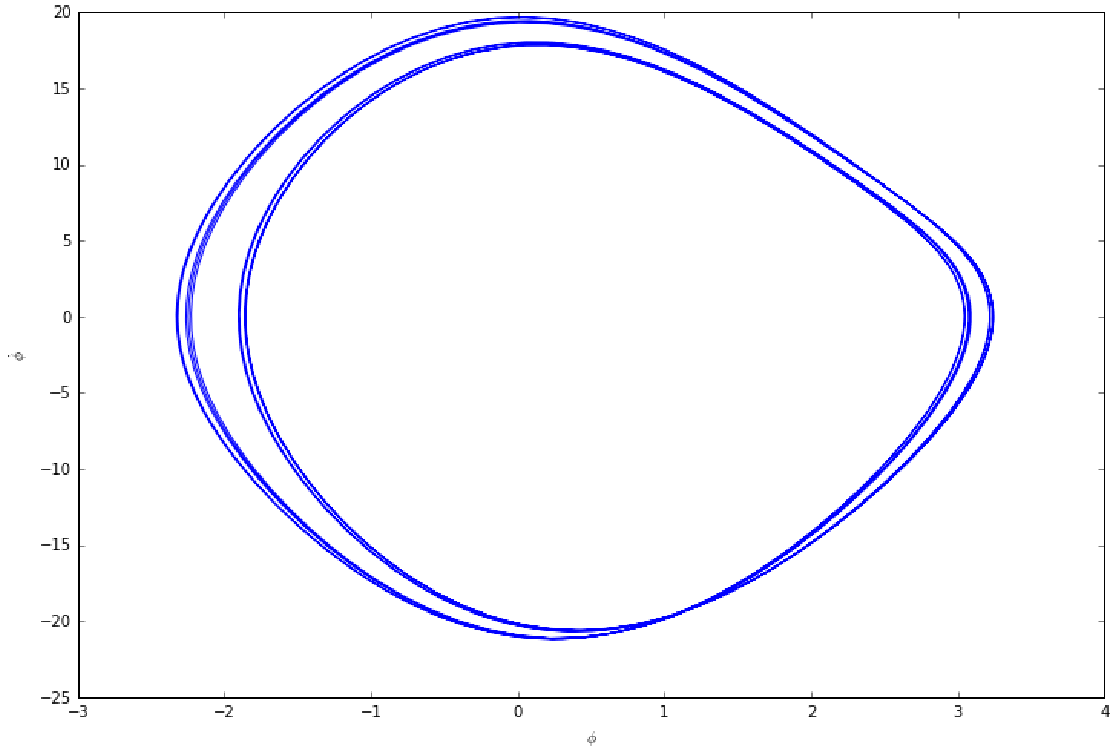
```
In [29]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(z[1000:2000,0],z[1000:2000,1]);
         plt.xlabel("$\phi$");
         plt.ylabel("$\dot{\phi}$");
```



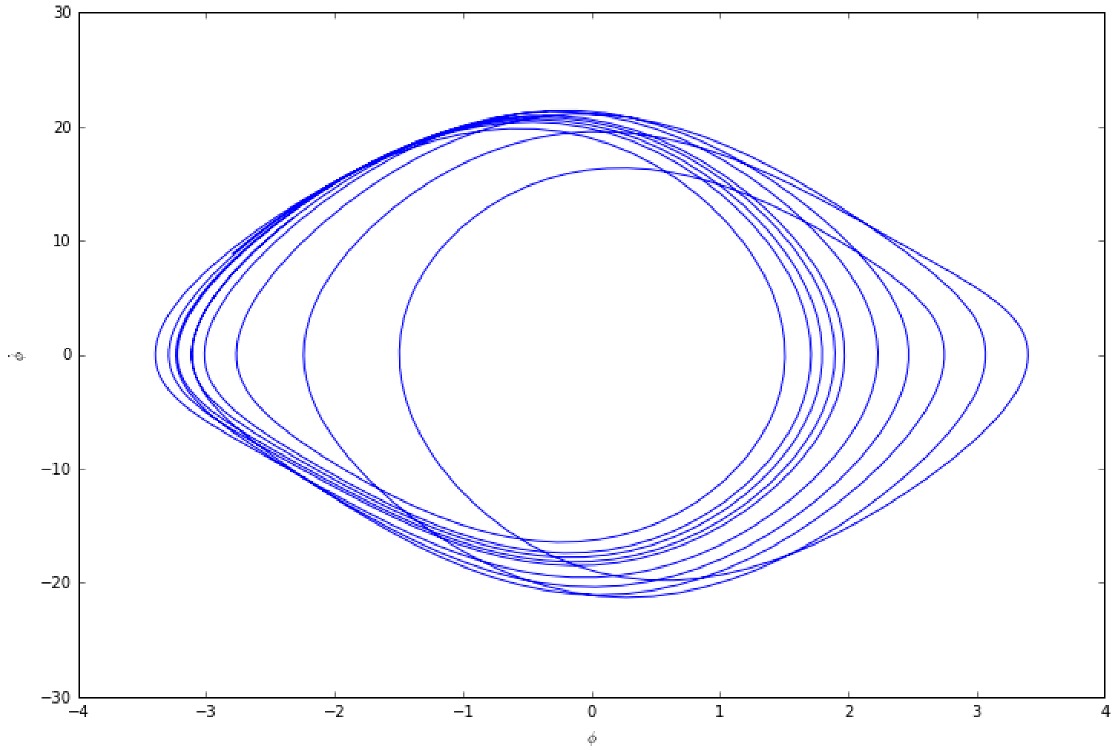
In [30]:
```
gamma=1.078
t = np.linspace(0, 20, 2000)
zinit = [np.pi/2, 0]
z = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(z[1000:2000,0],z[1000:2000,1]);
plt.xlabel("$\phi$");
plt.ylabel("$\dot{\phi}$");
```



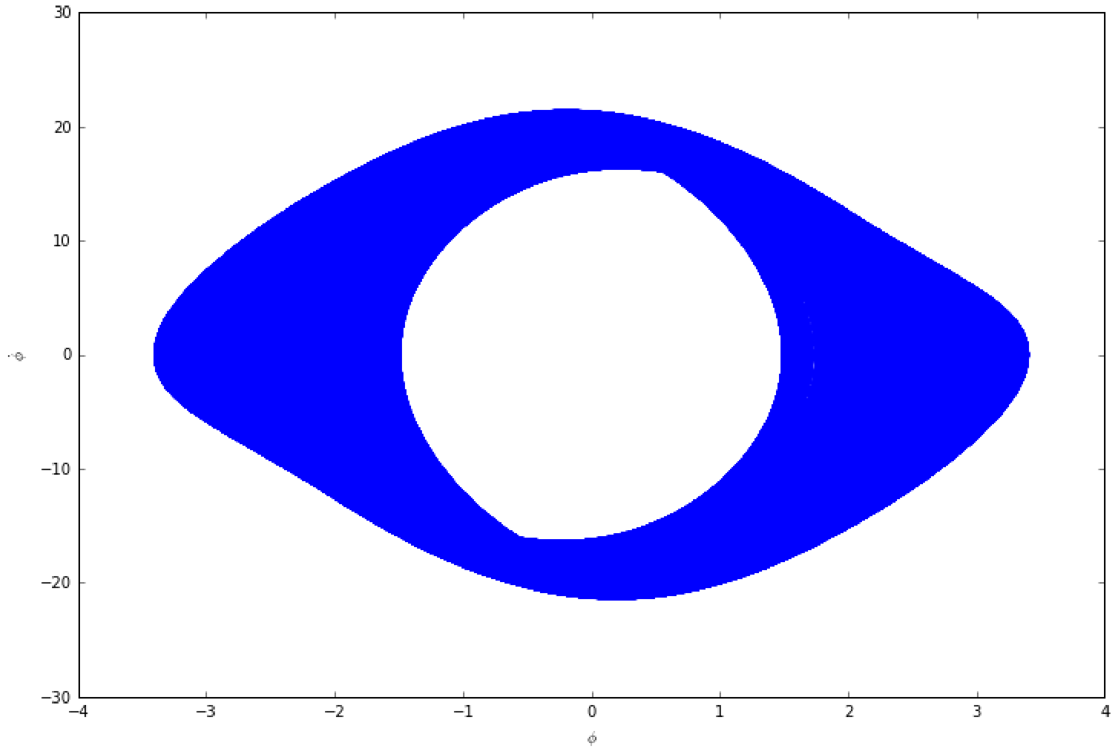

```
In [31]: gamma=1.081
         t = np.linspace(0, 20, 2000)
         zinit = [np.pi/2, 0]
         z = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(z[1000:2000,0],z[1000:2000,1]);
         plt.xlabel("$\phi$");
         plt.ylabel("$\dot{\phi}$");
```




```
In [32]: gamma=1.105
         t = np.linspace(0, 20, 2000)
         zinit = [np.pi/2, 0]
         z = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(z[1000:2000,0],z[1000:2000,1]);
         plt.xlabel("$\phi$");
         plt.ylabel("$\dot{\phi}$");
```



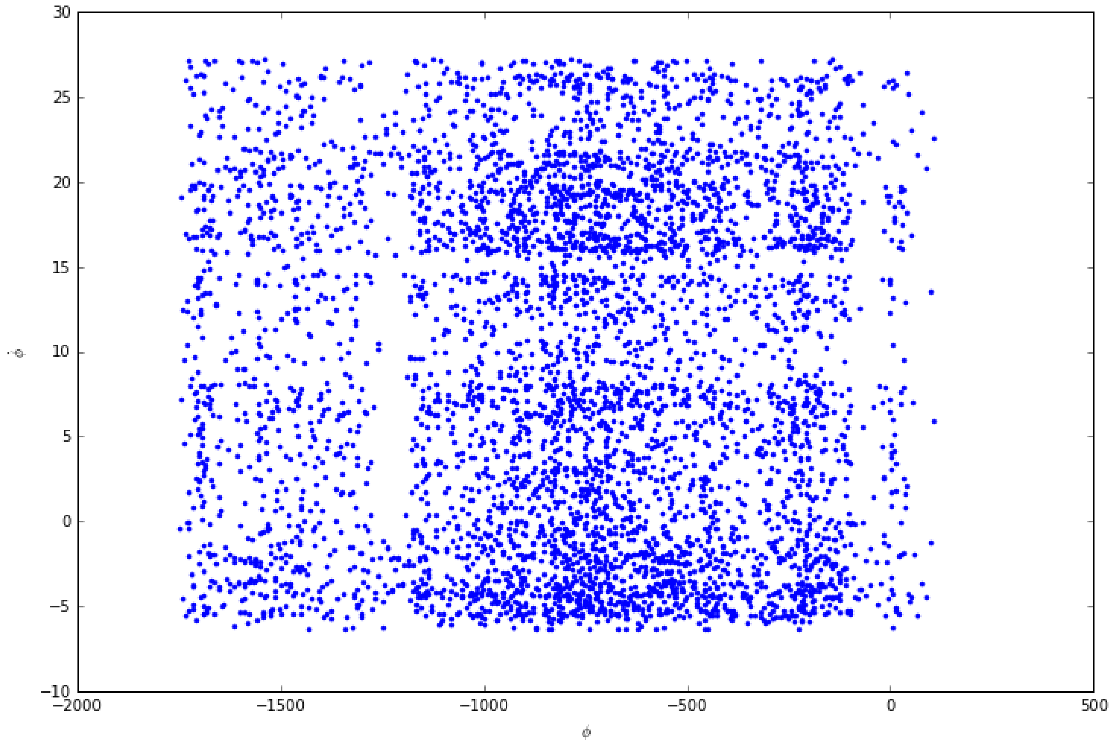
In [33]:
```
gamma=1.105
t = np.linspace(0, 2000, 200000)
zinit = [np.pi/2, 0]
z = integrate.odeint(deriv, zinit, t)
fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
ax.plot(z[1000:200000,0],z[1000:200000,1]);
plt.xlabel("$\phi$");
plt.ylabel("$\dot{\phi}$");
```



There is a nice way to clean this up. Take sections/slices. Said another way, iterate at fixed intervals and see where the system is

What do you think this looks like for the chaotic pendulum?

```
In [34]: gamma=1.5
         beta=omega_o/8
         t = np.linspace(0, 5000, 5000000)
         zinit = [-np.pi/2, 0]
         z = integrate.odeint(deriv, zinit, t)
         fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(z[10000:5000000:1000,0],z[10000:5000000:1000,1],'.');
         plt.xlabel("$\phi$");
         plt.ylabel("$\dot{\phi}$");
```



Let's zoom in...
And let's zoom in somre more...

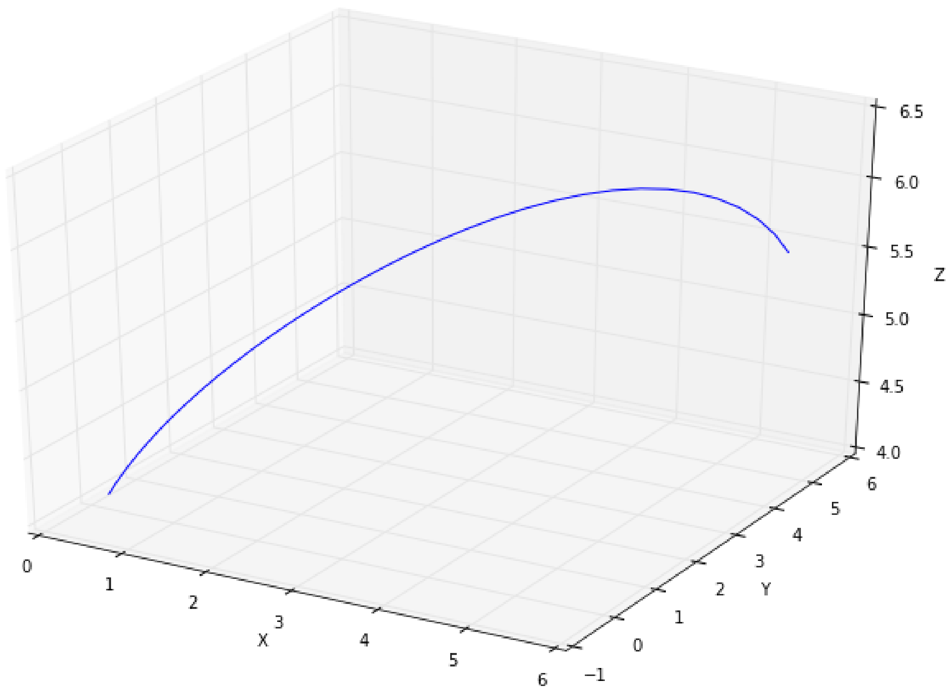
2 Other Systems

Ball Bouncing on a plate
Lorenz System

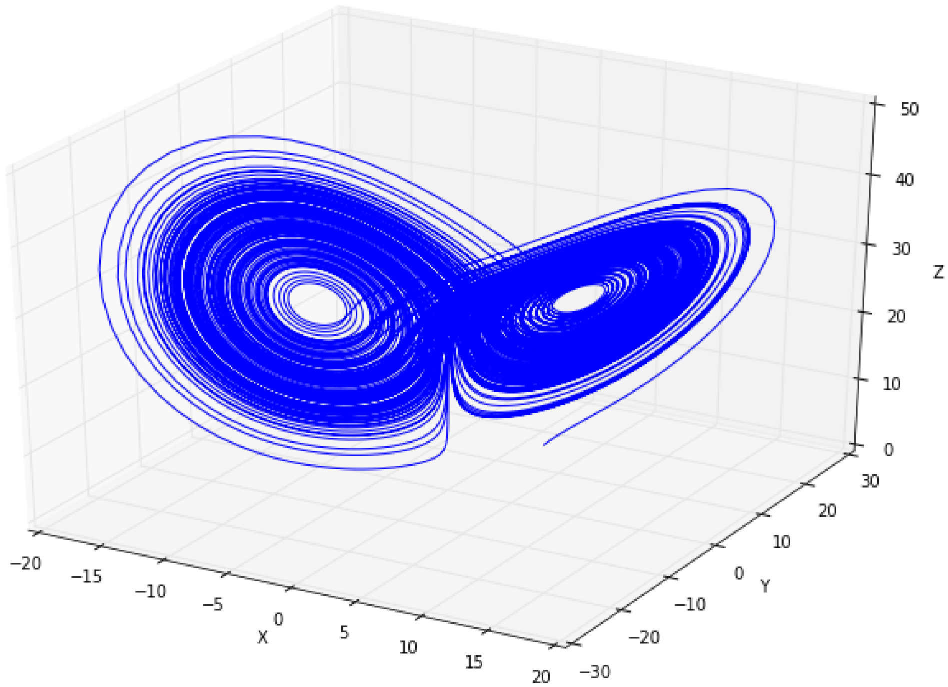
$$\frac{dX}{dt} = pr(-X+Y)$$
$$\frac{dY}{dt} = rX - Y - XZ$$
$$\frac{dZ}{dt} = XY - bZ$$

In [35]:
```
p=0.5
sigma=10
beta=8/3
def solvr_long(Y, t):
    return [sigma*(Y[1]-Y[0]), Y[0]*(p-Y[2])-Y[1], Y[0]*Y[1]-beta*Y[2]]
a_t_long = np.arange(0, 1000, 0.01)
asol_long = integrate.odeint(solvr_long, [5.5, 5.5, 5.5], a_t_long)
```

In [38]:
```
from mpl_toolkits.mplot3d import Axes3D
fig = plt.figure(figsize=LARGE_FIGSIZE)
ax = fig.add_subplot(111, projection='3d')
ax.plot(asol_long[0:35:,0],asol_long[0:35,1],asol_long[0:35,2],'b');
ax.set_xlabel('X');
ax.set_ylabel('Y');
ax.set_zlabel('Z');
```

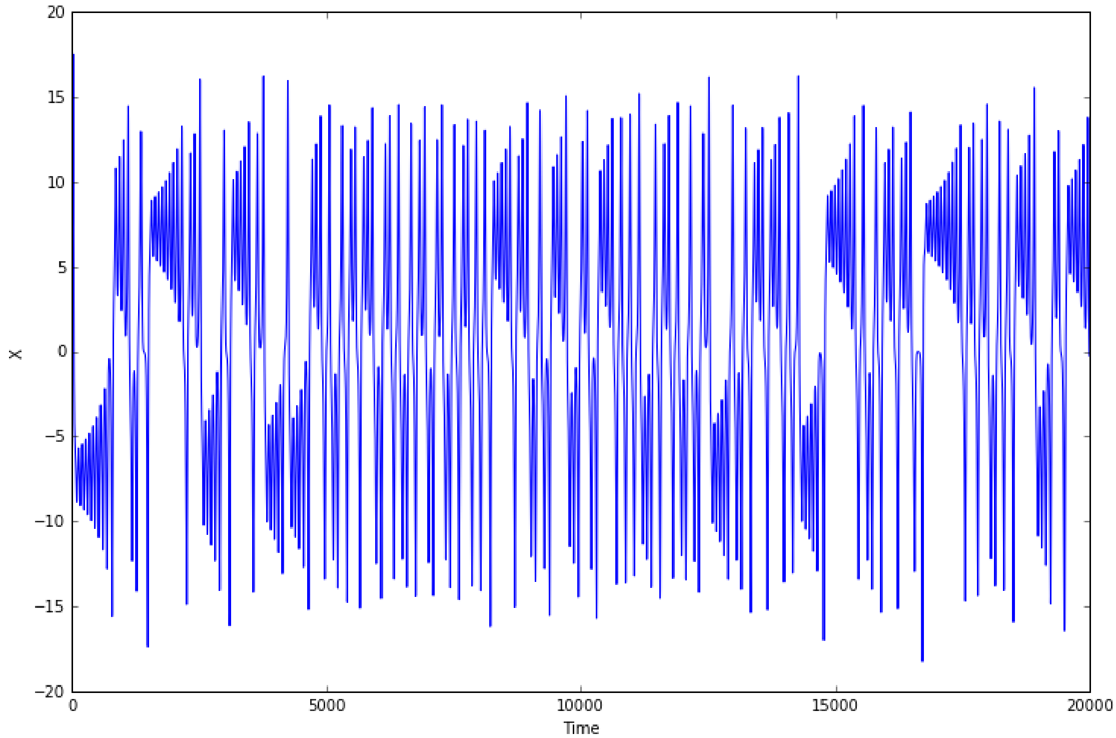


In [39]:
```
p=28
sigma=10
beta=8/3
a_t_long = np.arange(0, 200, 0.01)
asol_long = integrate.odeint(solvr_long, [5.5, 5.5, 5.5], a_t_long)
fig = plt.figure(figsize=LARGE_FIGSIZE)
ax = fig.add_subplot(111, projection='3d')
ax.plot(asol_long[:,0],asol_long[:,1],asol_long[:,2],'b')
ax.set_xlabel('X');
ax.set_ylabel('Y');
ax.set_zlabel('Z');
```



Look at just X

```
In [41]: fig, ax = plt.subplots(figsize=LARGE_FIGSIZE)
         ax.plot(asol_long[:,0])
         ax.set_xlabel('Time');
         ax.set_ylabel('X');
```




```
In [ ]:
```