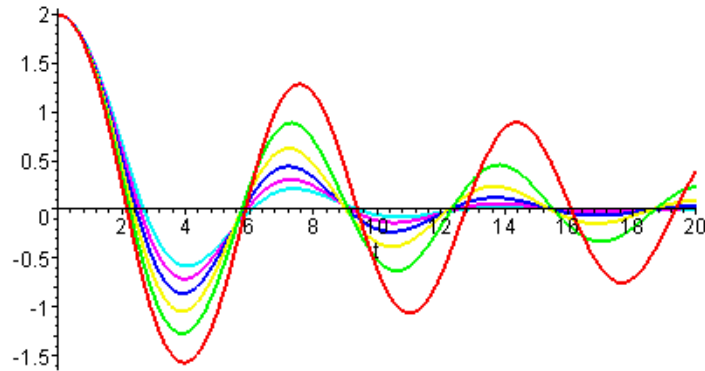# Continuous-Time Models

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Continuous-Time Models with Differential Equations

Mathematical formulations of dynamical systems

- **Discrete-time model:** (difference/recurrence equations; iterative maps)

$$x_t = F(x_{t-1}, t)$$

- **Continuous-time model:** (differential equations)

$$dx/dt = F(x, t)$$

x_t: State variable(s) of the system at time t

F: Some function that determines the rule that the system's behavior will obey

A general form (first-order, autonomous)

$$dx_1/dt = F_1 (x_1, x_2, x_3, ...)$$
$$dx_2/dt = F_2 (x_1, x_2, x_3, ...)$$
$$dx_3/dt = F_3 (x_1, x_2, x_3, ...) \ ...$$

or

$$dx/dt = F(x)$$

where x is a **state vector** of the system ($x = \{x_1, x_2, x_3, ...\}$)

Higher-order/non-autonomous systems

- **Higher-order systems:**
  Differential equations that include second-order (or higher) derivatives

- **Non-autonomous systems:**
  Differential equations that are time-dependent (i.e., explicitly include t in them)

The following argument also holds for differential equations

- **Non-autonomous, higher-order equations can always be converted into autonomous, 1st-order equations**

    - $d^2x/dt^2 \to dy/dt$, $y = dx/dt$

    - $t \to y$, $dy/dt = 1$, $y_0 = 0$

- **Autonomous 1st-order equations can cover dynamics of any non-autonomous higher-order equations too!**

Connecting continuous-time models with discrete-time models

$$x_t = F(x_{t-1}) \qquad dx/dt = G(x)$$

- $F(x) \Leftrightarrow x + G(x) \, \Delta t$
- $G(x) \Leftrightarrow (F(x) - x) / \Delta t$

- If $F(x) = Ax$, $G(x) = Bx$:

  $A \Leftrightarrow I + B \, \Delta t$

  $B \Leftrightarrow (A - I) / \Delta t$

How to study differential equations

- **Some of them can be analytically solvable**
  - **Linear systems**
  - **Simple nonlinear systems**

- **Analytical solutions are generally not available for nonlinear differential equations**

Numerical simulation

- **Simplest way: Euler forward method**

$$dx/dt = F(x)$$

$$\rightarrow x_{t+dt} = x_t + F(x)\, \Delta t$$

- **Approximate dynamics using small discrete time steps ($\Delta t \ll 1$)**

- **Simulate the model like difference equations**

Exercise

- **Simulate the following continuous-time logistic growth model in Python, with r=0.2, K=1, Δt=0.01:**

$$dN/dt = r N (1 - N/K)$$

Analysis of Continuous-Time Models

Equilibrium point

- **A state of the system at which state will not change over time**
  - A.k.a. fixed point, steady state
- **Can be calculated by solving**

  $$dx/dt = 0$$

Example

- **A simple second-order equation:**

$$d^2x/dt^2 = x$$

- **Convert this into a first-order form**
- **Calculate its equilibrium points**
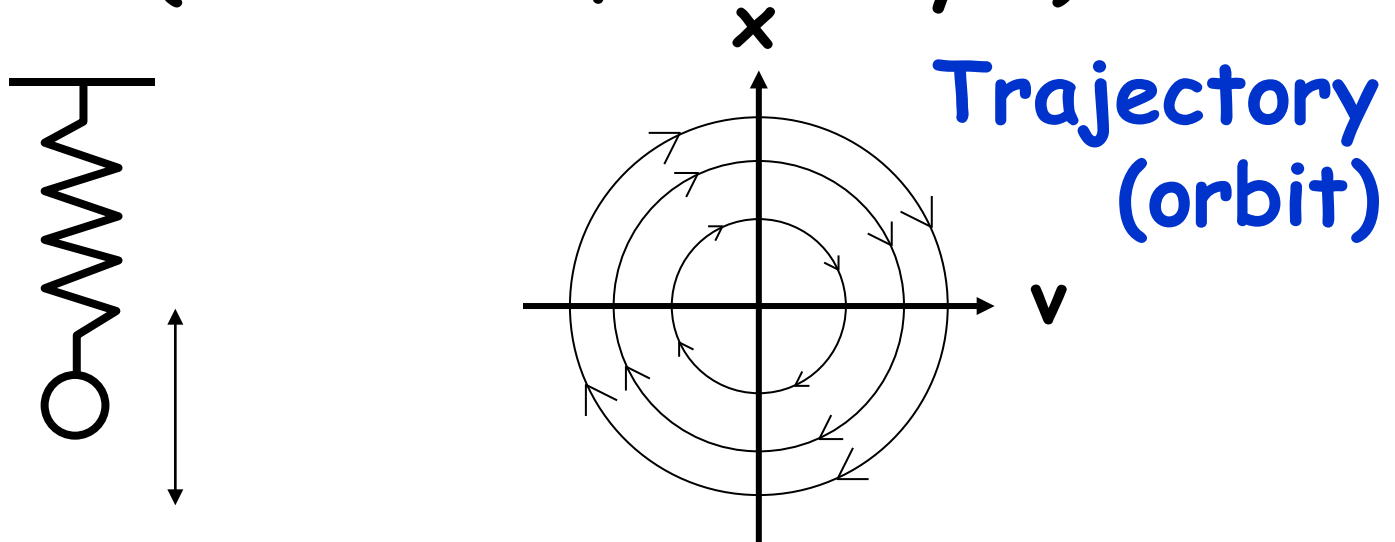
Exercise

- **A simple pendulum:**

$$d^2\theta/dt^2 = -g/L \sin \theta$$

- **Convert this into a first-order form**
- **Calculate its equilibrium points**

Visualizing Phase Space of Continuous Models

Phase space of continuous models

- E.g. a simple vertical spring oscillator
- State can be specified by two real variables (location x, velocity v)

Trajectory (orbit)

Dynamics of continuous models can be depicted as "flow" in a continuous phase space

Visualizing phase space of continuous models

- ## Vector field

  - Uses many small arrows to show how local derivatives (or direction of trajectories) change from place to place in phase space

- ## Phase portrait (stream plot)

  - Shows several typical trajectories to illustrate how phase space is globally structured

Exercise

- **Write your own code to visualize the phase space of the simple pendulum model**

$$d^2\theta/dt^2 = - g/L \sin \theta$$

- **Include some "damping" effect in the model above and see how it changes the phase space**

Visualizing phase space of continuous models manually

- Find "nullclines"
  - Points in the phase space where one of the derivatives is zero (i.e., trajectories are in parallel to one of the axes)
  - Plot where the nullclines are
  - Find how the sign of the derivative changes across the nullclines
  - Find values of other non-zero derivatives
- Draw a "flow" between those nullclines with curves that don't intersect with each other

Exercise

- Draw an outline of the phase space of the following system by studying the distribution of its nullclines:

$$dx/dt \ = \ ax \ - \ bx \ y$$
$$dy/dt \ = \ -cy \ + \ dx \ y$$

$$(x \ >= \ 0, \ y \ >= \ 0)$$

Rescaling Variables

Rescaling variables

- **Dynamics of a system won't change qualitatively by linear rescaling of variables (e.g., $x \rightarrow \alpha \, x'$)**

- **You can set arbitrary rescaling factors for variables to simplify the model equations**

- **If you have k variables (including t), you may eliminate k parameters**

Exercise

- **Simplify the logistic growth model by rescaling** $N \rightarrow \alpha\, N'$ **and** $t \rightarrow \beta\, t'$

$$dN/dt = r\, N\, (\, 1 - N/K\,)$$

Asymptotic Behavior of Linear Systems

Linear systems

- Linear systems are the simplest cases where states of nodes are continuous-valued and their dynamics are described by a time-invariant matrix

- Continuous-time: $dx/dt = A\,x$

  - A is called a "coefficient" matrix

  - We don't consider constants (as they can be easily converted to the above forms)

Where will the system go eventually?

$$dx/dt = A x$$

These equations give the following exact solution:

$$x_t = e^{At} x_0$$

$$= \Sigma_{k=0\sim\infty} (At)^k/k! \; x_0$$

FYI: Exponential operator for matrices

- $x_t = e^{At} x_0$

  $= \Sigma_{k=0\sim\infty} (At)^k / k! \ x_0$

  – **Similar to the Taylor series expansion of the exponential function:**

  $e^x = 1 + x + x^2/2! + x^3/3! + \dots$

  – e^M **converges for any square matrix M**

  – **If M's eigenvalues are $\{\lambda_i\}$, then e^M's eigenvalues are $\{ e^{\lambda_i} \}$, with all eigenvectors unchanged** (you can prove this)

Where will the system go eventually?

$$dx/dt = A\,x$$

- **What happens if the system starts from non-equilibrium initial states and goes on for a long period of time?**

- Let's think about their asymptotic behavior $\lim_{t \to \infty} x(t)$

Considering asymptotic behavior (1)

- **Let { v_i } be n linearly independent eigenvectors of the coefficient matrix**
  (They might be fewer than n, but here we ignore such cases for simplicity)

- **Write the initial condition using eigenvectors, i.e.**
  $x_0 = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$

Considering asymptotic behavior (2)

- **Then:**

$$x_t = e^{At} x_0$$
$$= e^{\lambda_1 t} b_1 v_1 + e^{\lambda_2 t} b_2 v_2 + \ldots + e^{\lambda_n t} b_n v_n$$

Dominant eigenvector

- If $\text{Re}(\lambda_1) > \text{Re}(\lambda_2), \text{Re}(\lambda_3), \ldots,$

$x_t = e^{\lambda_1 t} \{ b_1 v_1 + e^{(\lambda_2 - \lambda_1)t} b_2 v_2 + \ldots + e^{(\lambda_n - \lambda_1)t} b_n v_n \}$

$\lim_{t \to \infty} x_t \sim e^{\lambda_1 t} b_1 v_1$

If the system has just one such dominant eigenvector v_1, its state will be eventually along that vector regardless of where it starts

What eigenvalues and eigenvectors can tell us

- An eigenvalue tells whether a particular "state" of the system (specified by its corresponding eigenvectors) grows or shrinks by interactions between parts
  - $|\lambda| > 1$ -> growing
  - $|\lambda| < 1$ -> shrinking
    } for discrete-time cases
  - $Re(\lambda) > 0$ -> growing
  - $Re(\lambda) < 0$ -> shrinking
    } for continuous-time cases

Linear Stability Analysis of Nonlinear Systems

Linearizing continuous-time models

- **For continuous-time models:**

$dx/dt = F(x)$

$\text{Left} = d(x_e + \Delta x)/dt = d\Delta x/dt$

$\text{Right} = F(x_e + \Delta x)$

$\sim F(x_e) + F'(x_e) \, \Delta x$

$= F'(x_e) \, \Delta x$

Therefore,

$$d\Delta x/dt = F'(x_e) \, \Delta x$$

Review: First-order derivative of vector functions

- **Continuous-time:** $d\Delta x/dt = F'(x_e)\,\Delta x$

These can hold even if x is a vector

What corresponds to the first-order derivative in such a case:

$$F'(x_e) = dF/dx_{(x=x_e)} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\[2mm] \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_n} \\[2mm] \vdots & \vdots & & \vdots \\[2mm] \dfrac{\partial F_n}{\partial x_1} & \dfrac{\partial F_n}{\partial x_2} & \cdots & \dfrac{\partial F_n}{\partial x_n} \end{pmatrix}_{(x=x_e)}$$

Jacobian matrix at $x=x_e$

Eigenvalues of Jacobian matrix

- **A Jacobian matrix is a linear approximation around the equilibrium point, telling you the local dynamics:** **"how a small perturbation will grow, shrink or rotate around that point"**

  - The equilibrium point serves as a local origin
  - The Δx serves as a local coordinate
  - Eigenvalue analysis applies