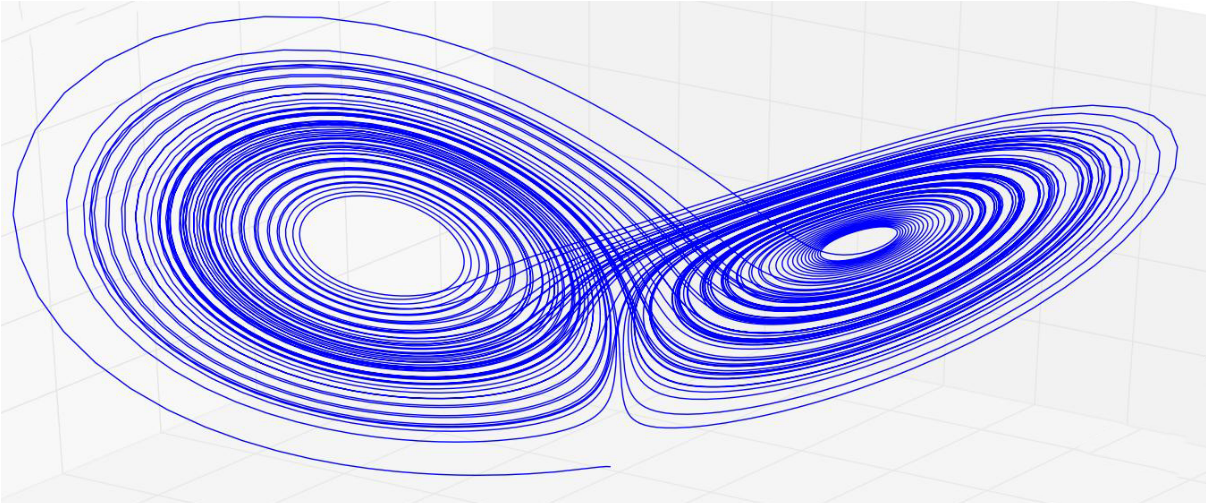
Chaos



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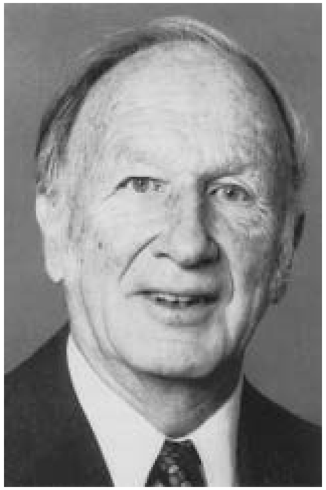
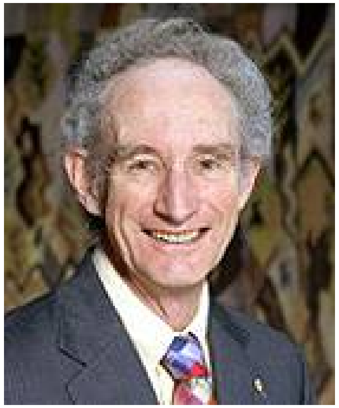
Discovery of chaos

- Discovered in early 1960's by Edward N. Lorenz (in a 3-D continuous-time model)



- Popularized in 1976 by Sir Robert M. May as an example of complex dynamics caused by simple rules (he used a 1-D discrete-time logistic map)



Chaos in dynamical systems

- A long-term behavior of a dynamical system that never falls in any static or periodic trajectories
- Looks like a random fluctuation, but still occurs in completely deterministic simple systems
- Exhibits sensitivity to initial conditions
- Can be found everywhere

Chaos in Discrete-Time Models

Simple example: Logistic map

- A simple difference equation used by Robert M. May in his paper in 1976

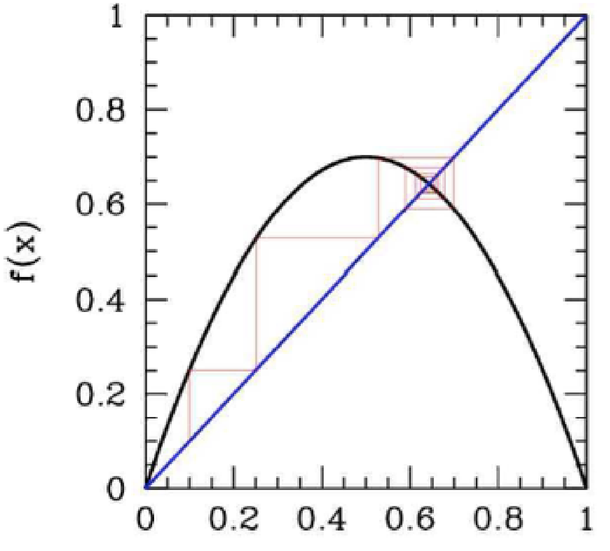
$$x_t = a\, x_{t-1} (1 - x_{t-1})$$

  - Similar to (but not quite the same as) the discrete-time logistic growth model (missing first x_t on the right hand side)
  - Shows quite complex dynamics as control parameter a is varied

Simple example: Logistic map

$$x_t = a\, x_{t-1} (1 - x_{t-1})$$

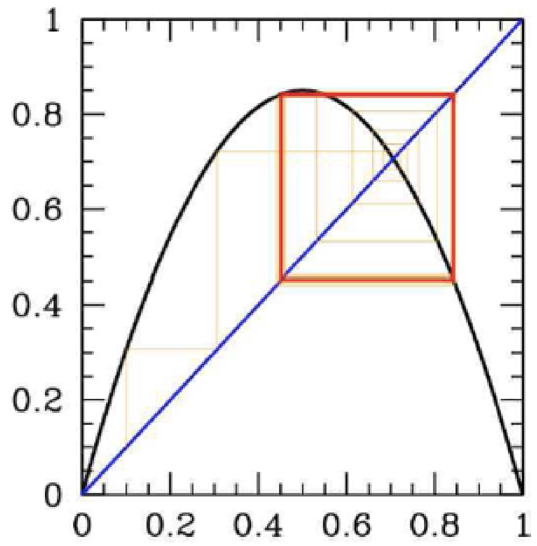
$$a = 2.8$$



Simple example: Logistic map

$$x_t = a\, x_{t-1} (1 - x_{t-1})$$

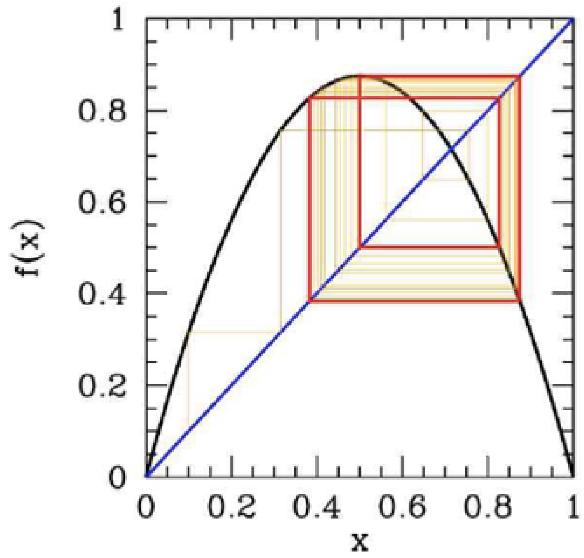
$$a = 3.4$$



Simple example: Logistic map

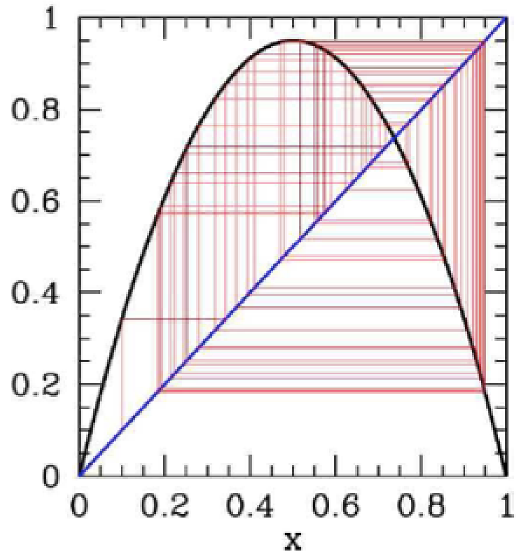
$$x_t = a\, x_{t-1}\,(1 - x_{t-1})$$

$$a = 3.5$$

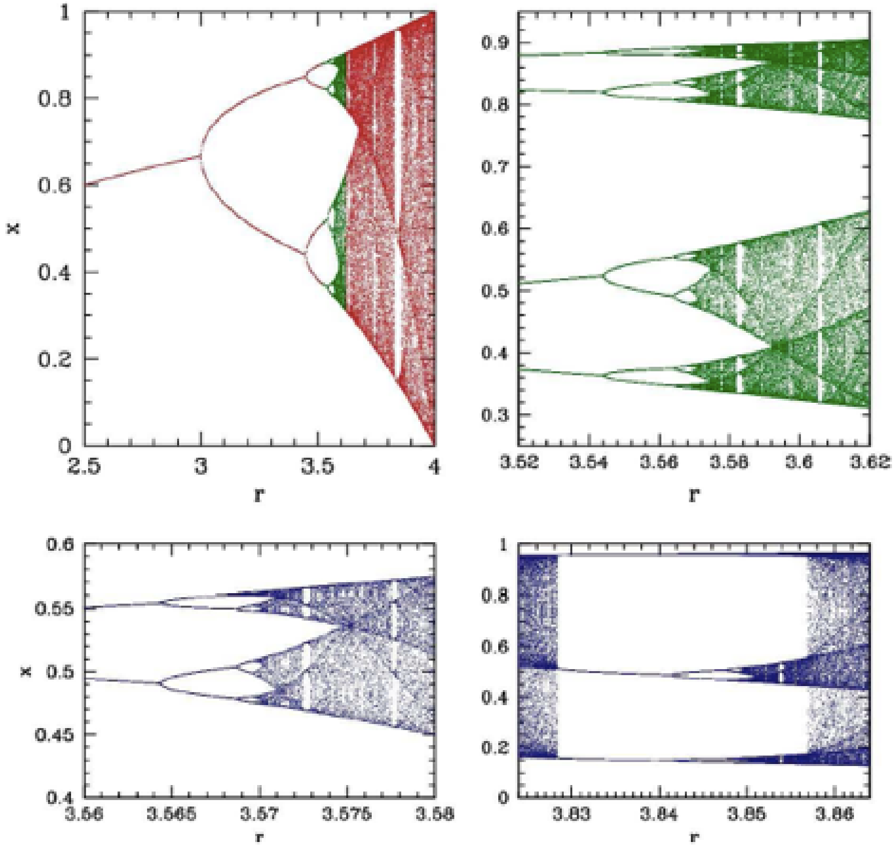


Simple example: Logistic map

$$x_t = a\, x_{t-1} \,(\, 1 - x_{t-1}\,)$$
$$a = 3.8$$



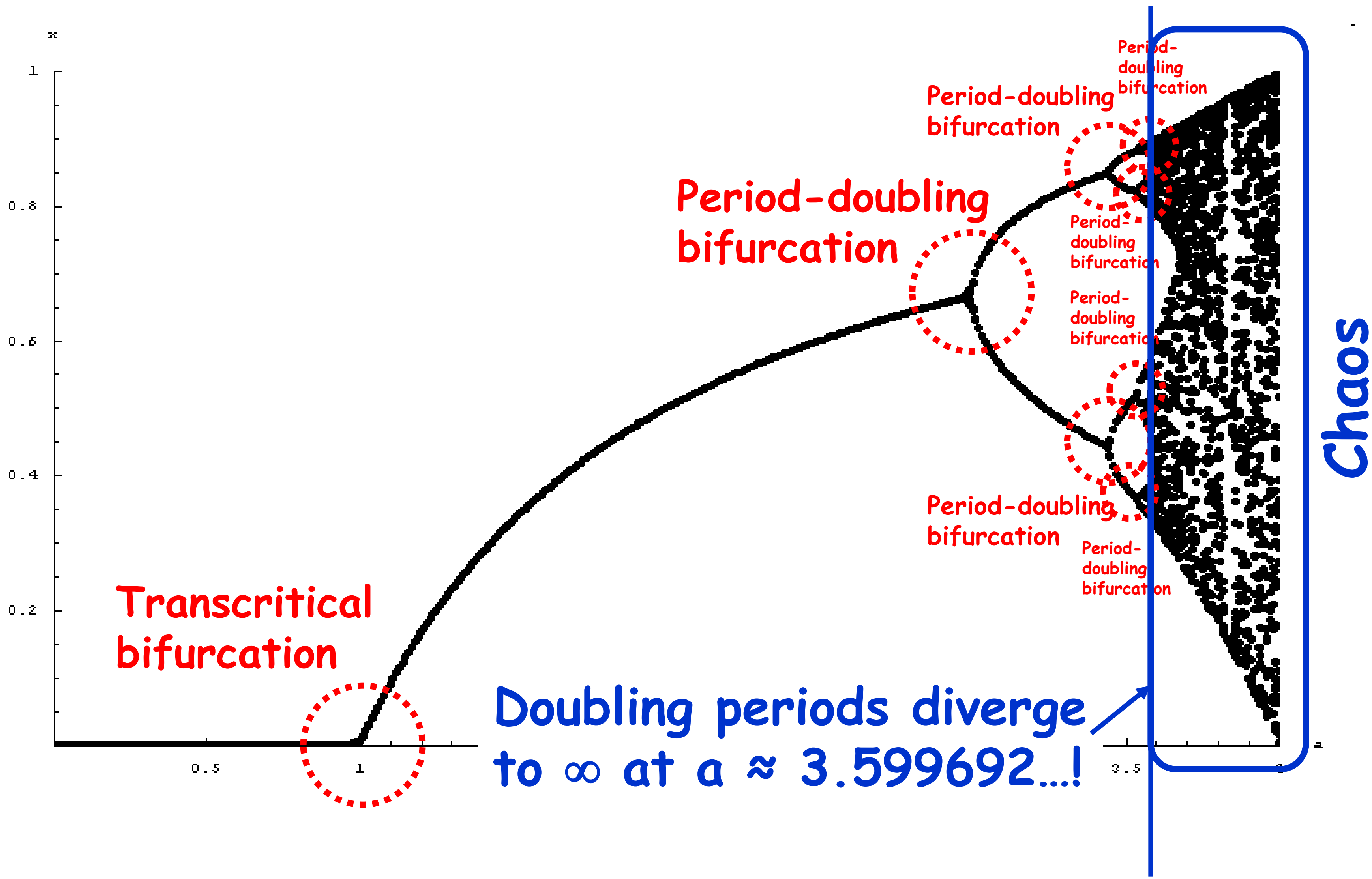
Simple example: Logistic map



Drawing bifurcation diagrams using numerical results

- For systems with periodic (or chaotic) long-term behavior, it is useful to draw a bifurcation diagram using numerical simulation results instead of analytical results
  - Plot actual system states sampled after a long period of time has passed
  - Can capture period-doubling bifurcation by a "set" of points

Cascade of period-doubling bifurcations leading to chaos



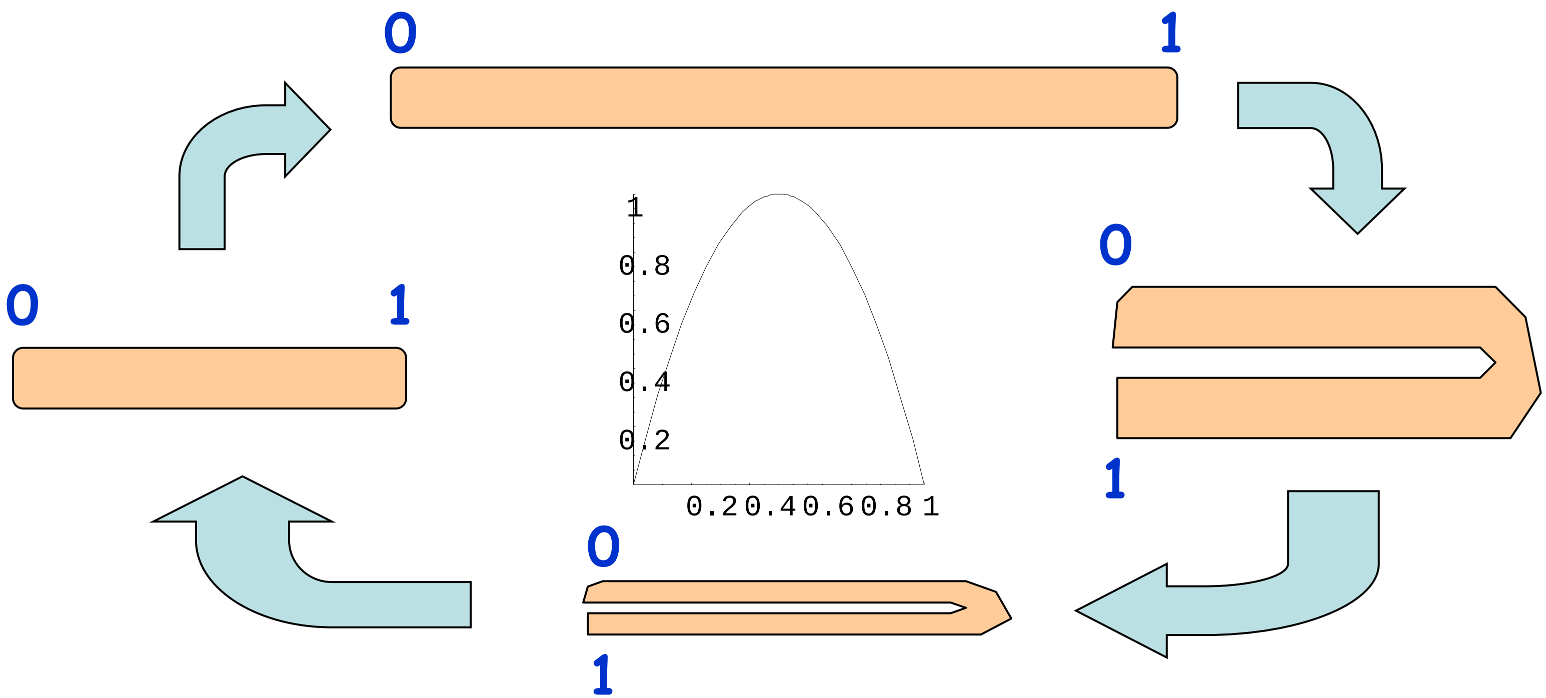
Reinterpreting chaos

- Where a period diverges to infinity
  - Periodic behavior with infinitely long periods means "aperiodic" behavior
- Where (almost) no periodic trajectories are stable
  - No fixed points or periodic trajectories you can sit in (you have to always deviate from your past course!!)

Characteristics of Chaos

Common mechanism of chaos: Stretch and folding in phase space

- In chaotic systems, it is generally seen that a phase space is stretched and folded, like dough kneading



Sensitivity to initial conditions

- Stretching and folding mechanisms always dig out microscale details hidden in a system's state and expand them to macroscale visible levels
- This causes chaotic systems very sensitive to initial conditions

- Think about where a grain in a dough will eventually move during kneading

Lyapunov exponent (1)

- A quantitative measure of a system's sensitivity to small differences in initial condition
- Characterizes how the distance between initially close two points will grow over time

$$| F^t(x_0+\varepsilon) - F^t(x_0) | \sim \varepsilon\, e^{t\underline{\lambda}} \quad \text{(for large } t\text{)}$$

Lyapunov exponent (2)

$$| F^t(x_0+\varepsilon) - F^t(x_0) | \sim \varepsilon\, e^{t\underline{\lambda}} \quad \text{(for large t)}$$

$$\lambda = \lim_{\substack{t\to\infty \\ \varepsilon\to 0}} \frac{1}{t} \log \left| \frac{F^t(x_0+\varepsilon) - F^t(x_0)}{\varepsilon} \right|$$

$$= \lim_{t\to\infty} \frac{1}{t} \log \left| \frac{d}{dx} F^t(x_0) \right|$$

$$= \lim_{t\to\infty} \frac{1}{t} \Sigma_{i=0\sim t-1} \log |F'(x_i)|$$

Chaos in Continuous-Time Models

Chaos in continuous-time models

- Requires three or more dimensions
  - Because in 1-D or 2-D phase space, every trajectory works as a "wall" and thus confines where you can go in the future; stretching and folding are not possible in such an environment

Edward N. Lorenz's model

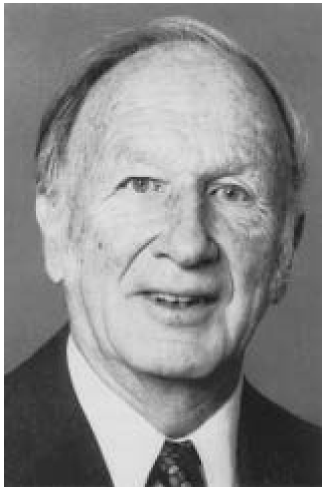
- A model of fluid convection:

  $dx/dt = s\,(y - x)$

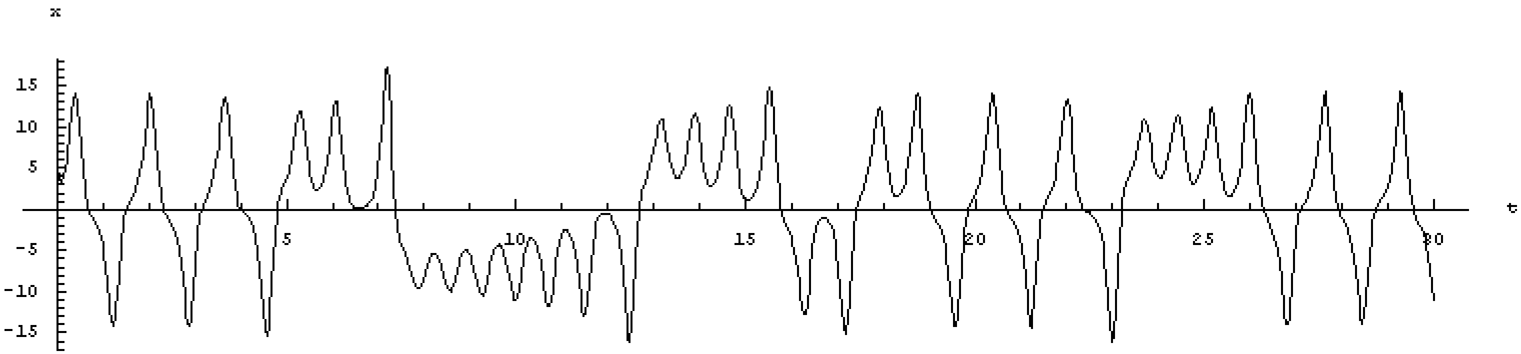
  $dy/dt = r\,x - y - x\,z$

  $dz/dt = x\,y - b\,z$

  (s, r and b are positive parameters)



- Typical behavior (x plotted over time):



Strange attractor

- A bounded region in a phase space that attracts nearby trajectories but also exhibits sensitive dependence on initial conditions inside it (i.e., no convergence to fixed points or periodic trajectories)
  - A.k.a.: chaotic attractor, fractal attractor

Other Routes to Chaos

- Intermittency
  - characterized by dynamics with bursts of chaotic and periodic behavior. as control parameter is changed the chaotic bursts become longer until full chaos occurs

- Quasiperiodicity
  - develops when two orbits/modes are nonlinearly coupled