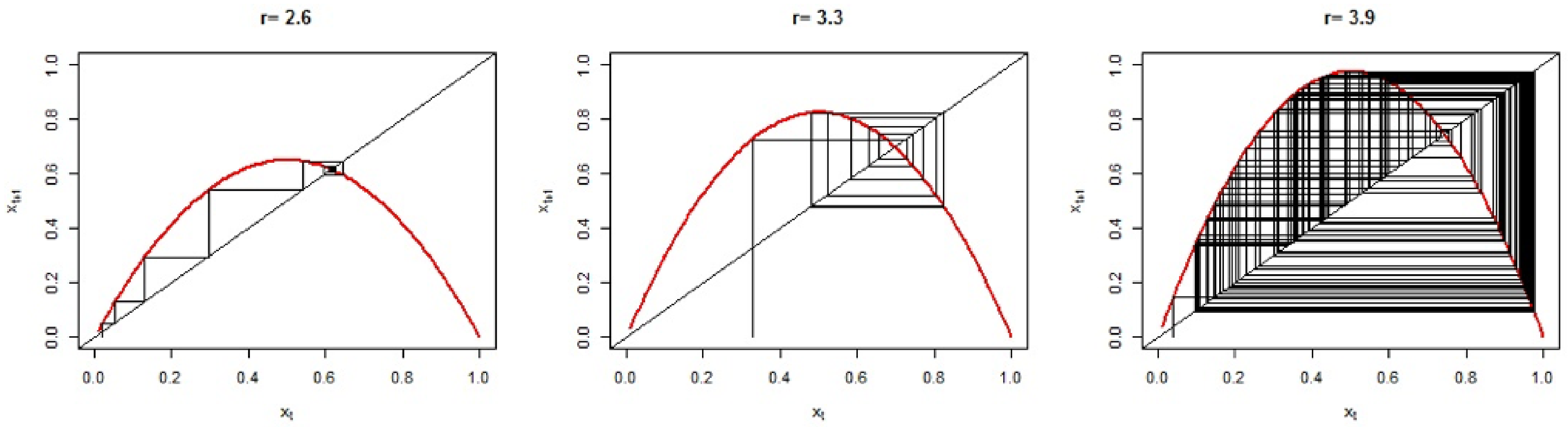
Discrete-Time Models



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Dynamical systems theory

- Considers how systems change along time
  - Ranges from Newtonian mechanics to modern nonlinear dynamics theories
  - Probes underlying dynamical mechanisms, not just static properties of observations
  - Provides a suite of useful tools

What is a dynamical system?

- A system whose state is uniquely specified by a finite set of variables and whose behavior is uniquely determined by predetermined "rules"
  - Simple population growth
  - Simple pendulum swinging
  - Motion of celestial bodies

Mathematical formulations of dynamical systems

- **Discrete-time model:** (difference/recurrence equations; iterative maps)

  $$x_t = F(x_{t-1}, t)$$

- **Continuous-time model:** (differential equations)

  $$dx/dt = F(x, t)$$

x_t: State variable(s) of the system at time t

F: Some function that determines the rule that the system's behavior will obey

Discrete-Time Models

Discrete-time model

- Easy to understand, develop and simulate
  - Doesn't require an expression for the rate of change (derivative)
  - Can model abrupt changes and/or chaotic dynamics using fewer variables
  - Directly translatable to simulation in a computer
  - Experimentally, we often have samples of system states at specific points of time

Difference equation and time series

- Difference equation

  $$x_t = F(x_{t-1}, t)$$

  produces series of values of variable x starting with initial condition x_0:

  $\{ x_0, x_1, x_2, x_3, \ldots \}$ "time series"

  - A prediction made by the above model (to be compared to experimental data)

Types of Discrete-Time Models

- **Linear:**
  - Right hand side is just a first-order polynomial of variables

$$x_t = \underline{a\, x_{t-1} + b\, x_{t-2} + c\, x_{t-3}} \ \ldots$$

- **Nonlinear:**
  - Anything else

$$x_t = a\, x_{t-1} + b\, \underline{x_{t-2}^{\,2}} + c \underline{\sqrt{x_{t-1}\, x_{t-3}}} \ \ldots$$

Types of Discrete-Time Models

- **1st-order:**
  - Right hand side refers only to the immediate past

$$x_t = a\, \underline{x_{t-1}} \,(1 - \underline{x_{t-1}})$$

- **Higher-order:**
  - Anything else

$$x_t = a\, \underline{x_{t-1}} + b\, \underline{x_{t-2}} + c\, \underline{x_{t-3}} \ldots$$

(Note: this is different from the order of terms in polynomials)

Types of Discrete-Time Models

- **Autonomous:**
  - Right hand side includes only state variables (x) and not t itself

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2$$

- **Non-autonomous:**
  - Right hand side includes terms that explicitly depend on the *value* of t

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2 + \underline{\sin(t)}$$

Things that you should know

- Non-autonomous, higher-order equations can always be converted into autonomous, 1st-order equations

Things that you should know

- Linear equations
  - are analytically solvable
  - show either equilibrium, exponential growth/decay, periodic oscillation (with >1 variables), or their combination
- Nonlinear equations
  - may show more complex behaviors
  - do not have analytical solutions in general

Simulating Discrete-Time Models

Simulating discrete-time models

- Simulation of a discrete-time model can be implemented by iterating updating of the system's states
  - Every iteration represents one discrete time step - use a loop!

Try in class Exercise

Building Your Own Model Equation

Mathematical modeling tips

- Grab an existing model and tweak it
- Implement each assumption one by one
- Find where to change, replace it by a function, and design the function
- Adopt the simplest form
- Check the model with extreme values

Example: Saturation of growth

- Simple exponential growth model:

$$x_t = a\ x_{t-1}$$

- Problem: How can one implement the saturation of growth in this model?

- Think about a new nonlinear model:

$$x_t = f(x_{t-1})\ x_{t-1}$$

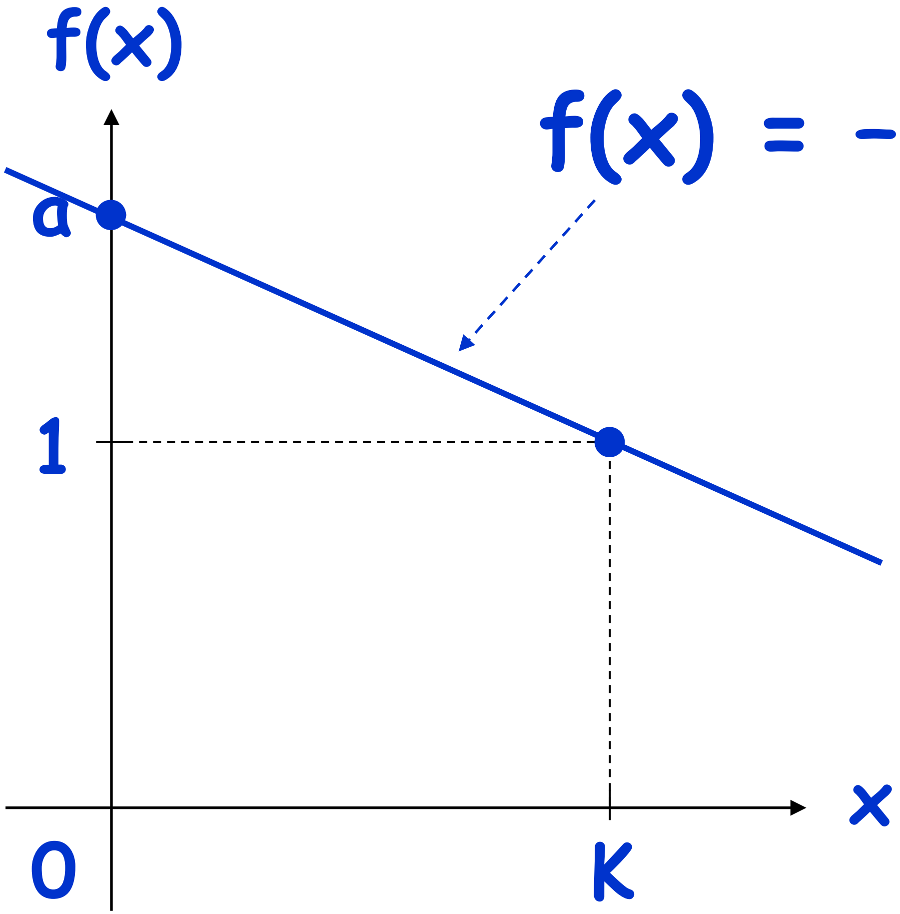
  – Coefficient replaced by a function of x

Modeling saturation of growth

$$x_t = f(x_{t-1})\, x_{t-1}$$

- $f(x)$ should approach 1 (no net growth) when x goes to a carrying capacity of the environment, say K
- $f(x)$ should approach the original growth rate a when x is very small (i.e., with no saturation effect)

What should f(x) be?



$$f(x) = -\frac{a-1}{K}x + a$$

A new model of growth

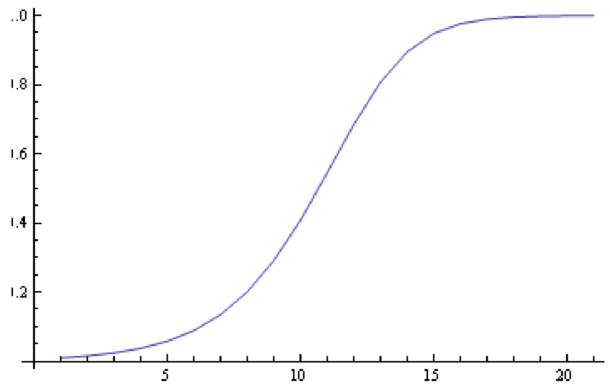
$$x_t = f(x_{t-1})\, x_{t-1}$$
$$= (\,-(a-1)\, x_{t-1} / K + a\,)\, x_{t-1}$$

- Using $r = a - 1$:

$$x_t = (\,-r\, x_{t-1} / K + r + 1\,)\, x_{t-1}$$
$$= x_{t-1} + \underbrace{r\, x_{t-1} (\,1 - x_{t-1} / K\,)}_{\text{Net growth}}$$

Example: Logistic growth model

- N: Population
- r: Population growth rate
- K: Carrying capacity



- Discrete-time version:

$$N_t = N_{t-1} + r\, N_{t-1} \left(1 - N_{t-1}/K \right)$$

Nonlinear terms

Modeling with multiple variables

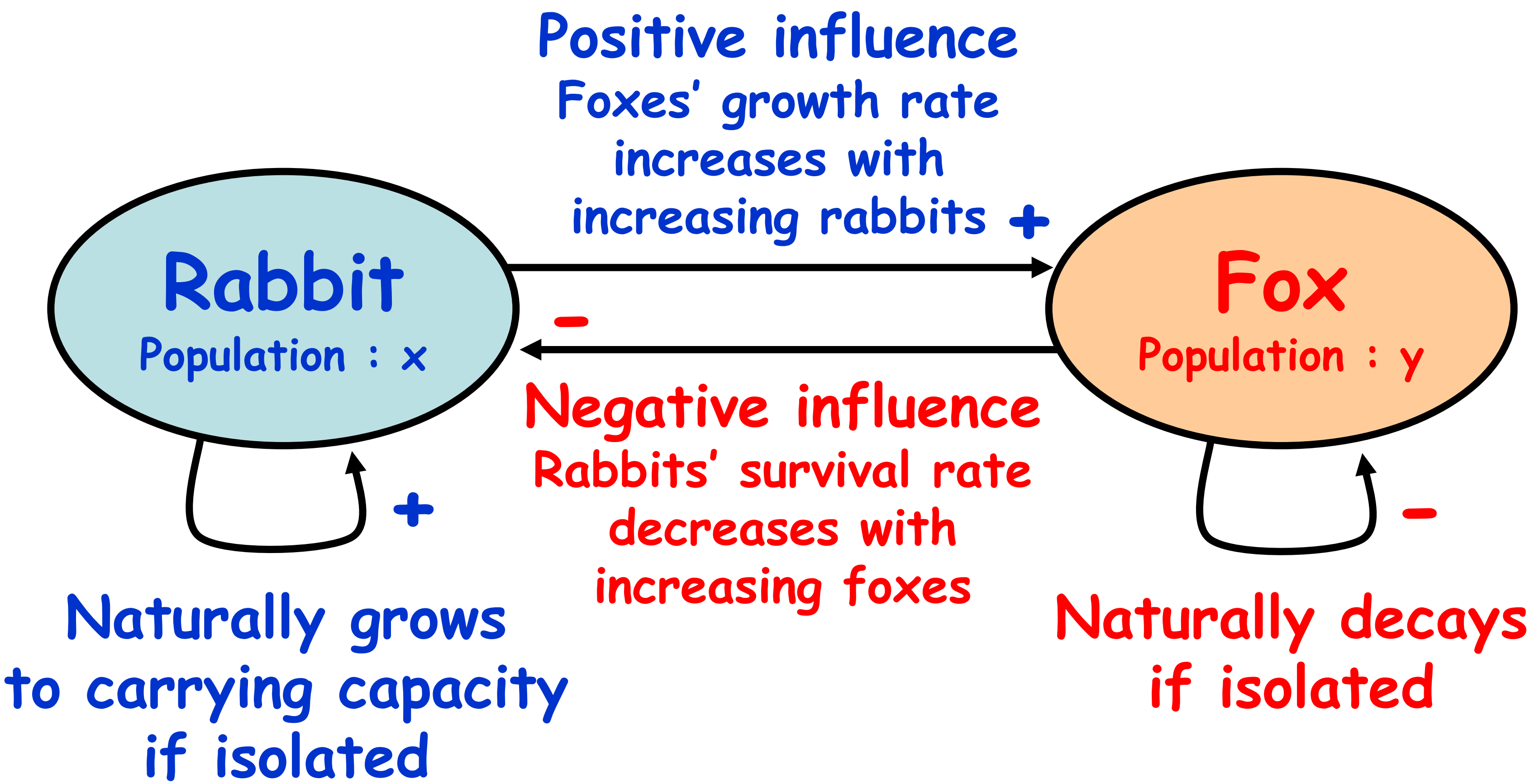
- Problem: Develop a nonlinear model of a simple ecosystem made of predator and prey populations

Think about how variables behave in isolation

Naturally grows
to carrying capacity
if isolated

Naturally decays
if isolated

Think about how variables interact with each other



Lotka-Volterra model

- The model derived in class can be rewritten

$$x_t - x_{t-1} = \alpha\, x_{t-1}\,(1 - x_{t-1}) - \beta\, x_{t-1}\, y_{t-1}$$

$$y_t - y_{t-1} = -\,\gamma\, y_{t-1} + \delta\, x_{t-1}\, y_{t-1}$$

  - Known as the "Lotka-Volterra" equations (of discrete-time version with carrying capacity)
  - Models predator-prey dynamics in a general form
  - One of the most famous nonlinear systems with multiple variables

Analysis of Discrete-Time Models

Equilibrium/Fixed point

- A state of the system at which state will not change over time
  - A.k.a. steady state

- Can be calculated by solving

$$x_t = x_{t-1}$$

Exercise

- Calculate equilibrium points in the following model

$$N_t = N_{t-1} + r\, N_{t-1}\,(1 - N_{t-1}/K)$$

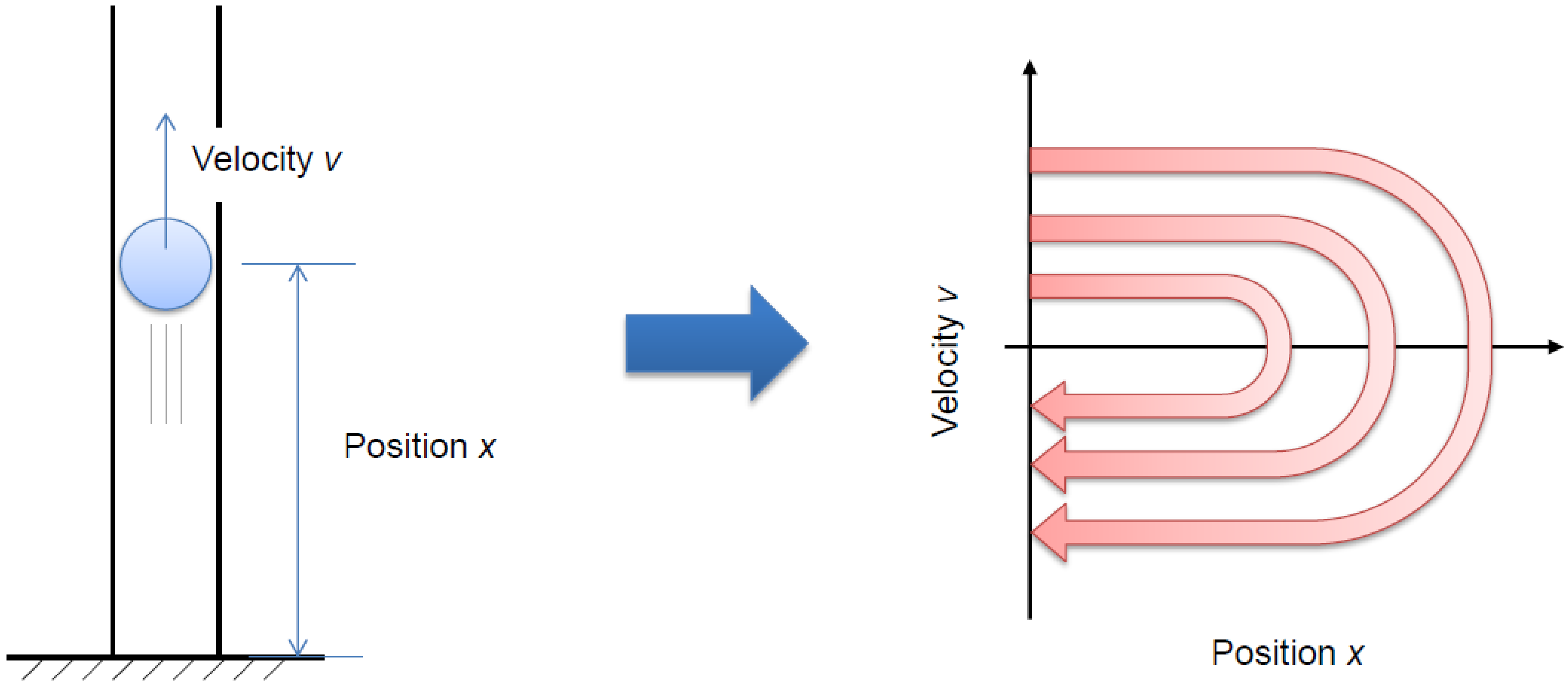
Phase Space Visualization

Geometrical approach

- Developed in the late 19C by J. Henri Poincare
- Visualizes the behavior of dynamical systems as trajectories in a phase space
- Produces a lot of intuitive insights on geometrical structure of dynamics that would be hard to infer using purely algebraic methods

Phase space (state space)

- A theoretical space in which every state of a dynamical system is mapped to a spatial location



Phase space (state space)

- Created by "orthogonalizing" state variables of the system
- Its dimensionality equals # of variables needed to specify the system state (a.k.a. degrees of freedom)
- Temporal change of the system states can be drawn in it as a trajectory

Attractor and basin of attraction

- **Attractor:**

  A state (or a set of states) from which no outgoing edges or flows running in phase space

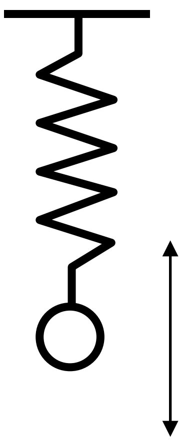
  - Static attractors (equilibrium points)
  - Dynamic attractors (e.g. limit cycles)

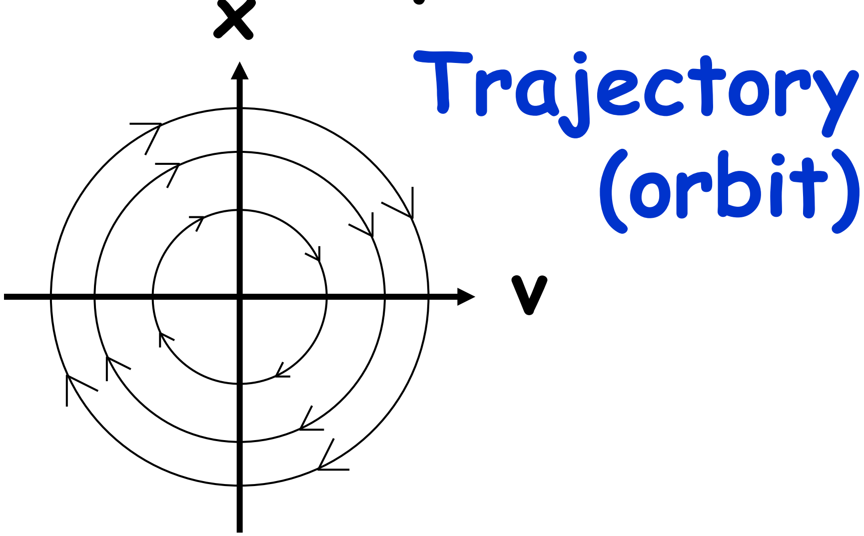
- **Basin of attraction:**

  A set of states which will eventually end up in a given attractor

Phase space of continuous-state models

- E.g. a simple vertical spring oscillator
- State can be specified by two real variables (location x, velocity v)



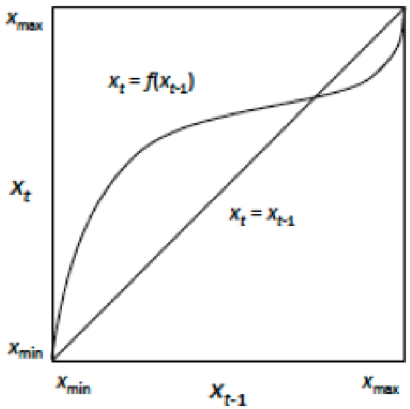


Dynamics of continuous models can be depicted as "flow" in a continuous phase space

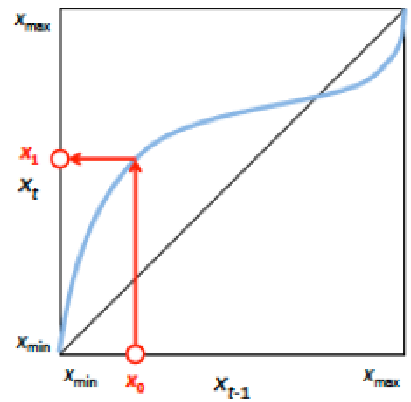
Cobweb plot

- A visual tool to study the behavior of 1-D iterative maps
- Take x_{t-1} and x_t for two axes
- Draw the map of interest ($x_t=F(x_{t-1})$) and the "$x_t=x_{t-1}$" reference line
  - They will intersect at "equilibrium points"
- Trace how time series develop from an initial value by jumping between these two curves

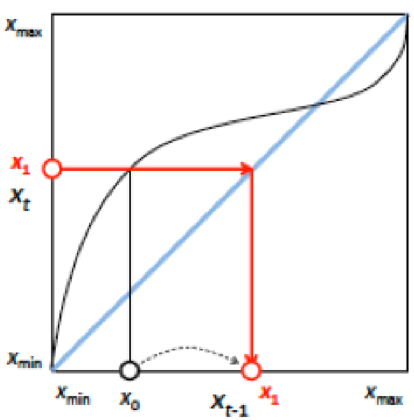
Cobweb Plot



x_{max}
$x_t = f(x_{t-1})$
x_t
$x_t = x_{t-1}$
x_{min}
x_{min}
x_{t-1}
x_{max}



x_{max}
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_{max}



x_{max}
x_1
x_t
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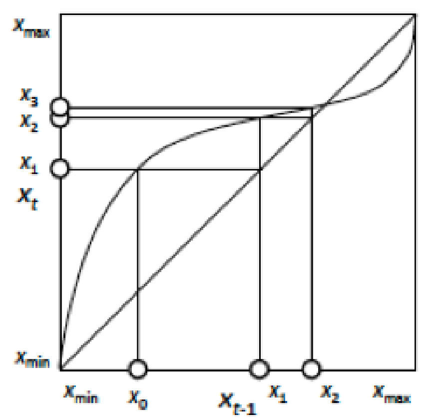
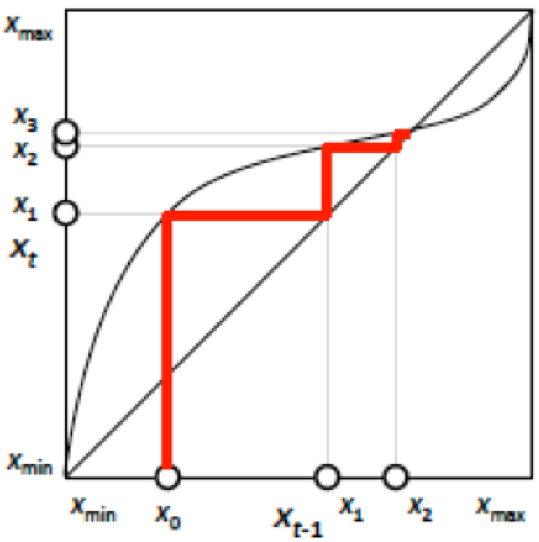
x_{min}
x_0
x_{t-1}
x_1
x_{max}



x_{max}
x_3
x_2
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_1
x_2
x_{max}

Cobweb Plot



Rescaling Variables

Rescaling variables

- Dynamics of a system won't change qualitatively by linear rescaling of variables (e.g., $x \rightarrow \alpha x'$)
- You can set arbitrary rescaling factors for variables to simplify the model equations
- If you have k variables, you may eliminate k parameters

Exercise

- Simplify the logistic growth model by rescaling $x \rightarrow \alpha\, x'$

$$x_t = x_{t-1} + r\, x_{t-1}\, (1 - x_{t-1}/K)$$

Linear Systems

Linear systems

- Some systems can be modeled as linear systems
  - Their dynamics is described by a product of matrix and state vector
  - Either in continuous or discrete time

- Dynamics of such linear systems can be studied analytically

Linear systems

- Linear systems are the simplest cases where states of nodes are continuous-valued and their dynamics are described by a time-invariant matrix
- Discrete-time: $x_t = A x_{t-1}$
  - A is called a "coefficient" matrix
  - We don't consider constants (as they can be easily converted to the above forms)

Asymptotic Behavior of Linear Systems

Where will the system go eventually?

$$x_t = A\, x_{t-1}$$

This equation gives the following exact solution:

$$x_t = A^t\, x_0$$

Where will the system go eventually?

$$x_t = A\, x_{t-1}$$

- What happens if the system starts from non-equilibrium initial states and goes on for a long period of time?
- Let's think about their asymptotic behavior $\lim_{t \to \infty} x_t$

Considering asymptotic behavior

- Let $\{ v_i \}$ be n linearly independent eigenvectors of the coefficient matrix (They might be fewer than n, but here we ignore such cases for simplicity)
- Write the initial condition using eigenvectors, i.e.

$$x_0 = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$$

Considering asymptotic behavior

- Then:

$$x_t = A^t x_0$$
$$= \lambda_1^t b_1 v_1 + \lambda_2^t b_2 v_2 + \ldots + \lambda_n^t b_n v_n$$

Dominant eigenvector

- If $|\lambda_1| > |\lambda_2|, |\lambda_3|, \ldots,$

  $x_t = \lambda_1^t \{ b_1 v_1 + \cancel{(\lambda_2/\lambda_1)^t b_2 v_2} + \ldots + \cancel{(\lambda_n/\lambda_1)^t b_n v_n} \}$

  $\lim_{t \to \infty} x_t \sim \lambda_1^t b_1 v_1$

If the system has just one such **dominant eigenvector** v_1, its state will be eventually along that vector **regardless of where it starts**

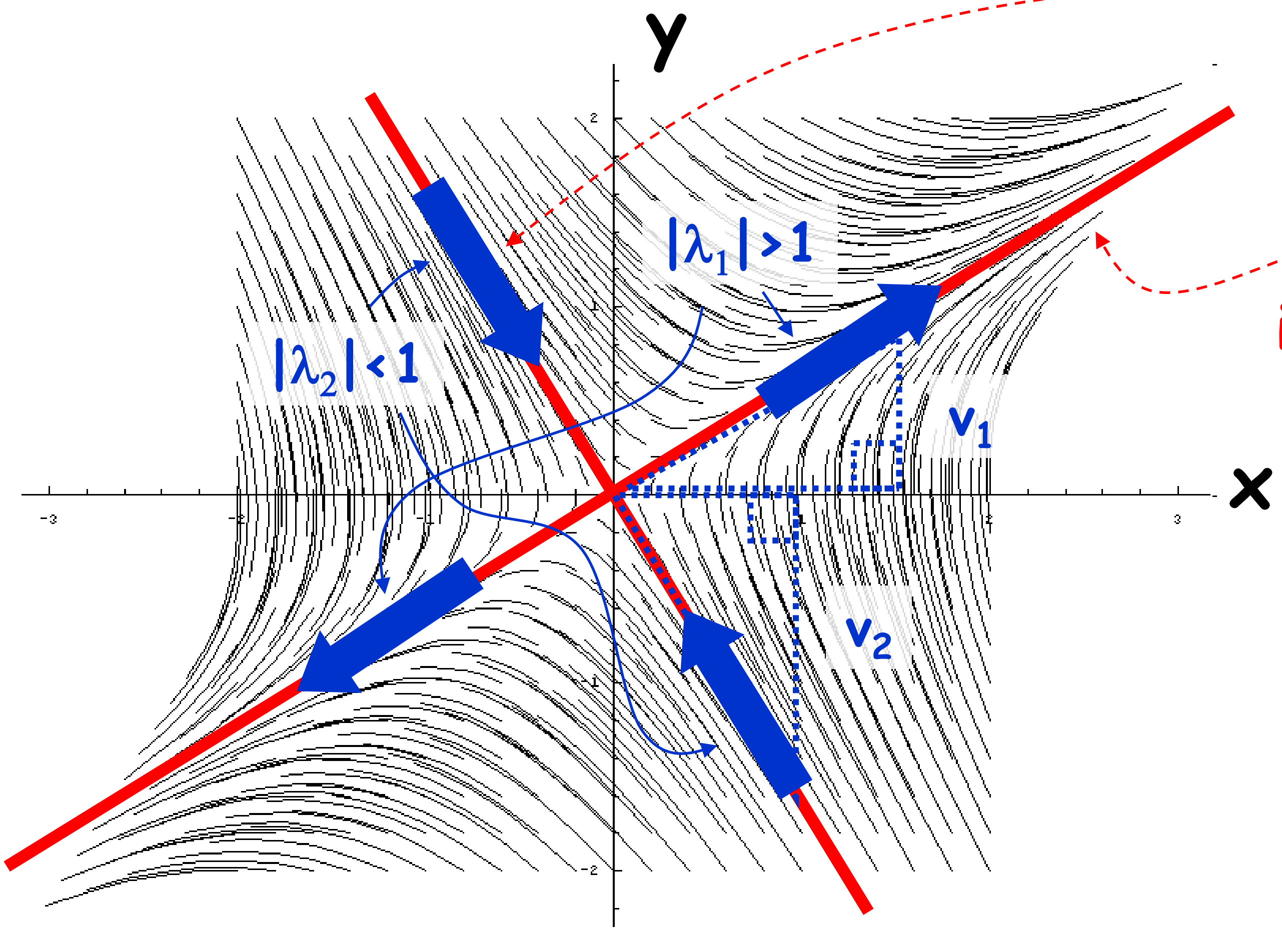
What eigenvalues and eigenvectors can tell us

- An eigenvalue tells whether a particular "state" of the system (specified by its corresponding eigenvectors) grows or shrinks by interactions between parts

  - $|\lambda| > 1$ -> growing
  - $|\lambda| < 1$ -> shrinking

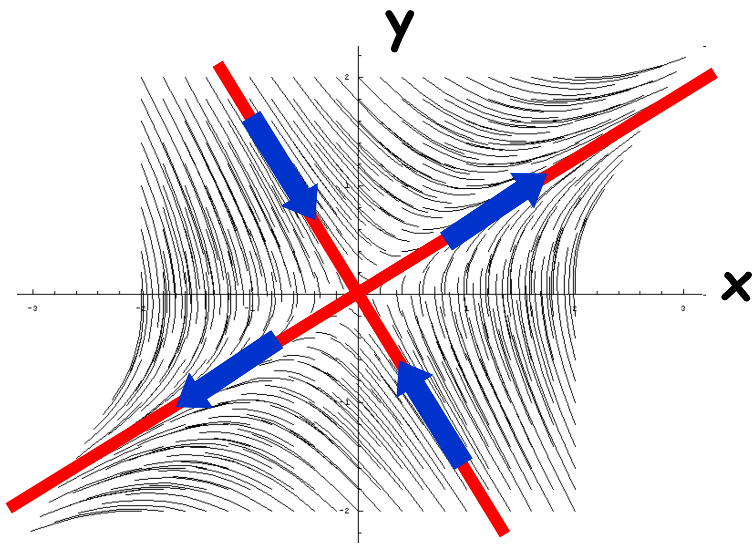
Example

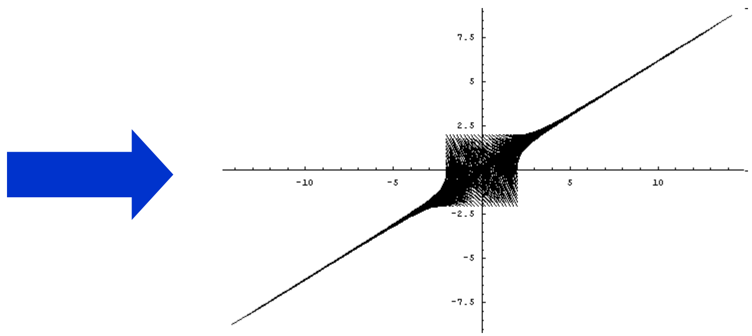
- Phase space of a two-variable linear difference equation with (a, b, c, d) = (1, 0.1, 0.1, 0.9)

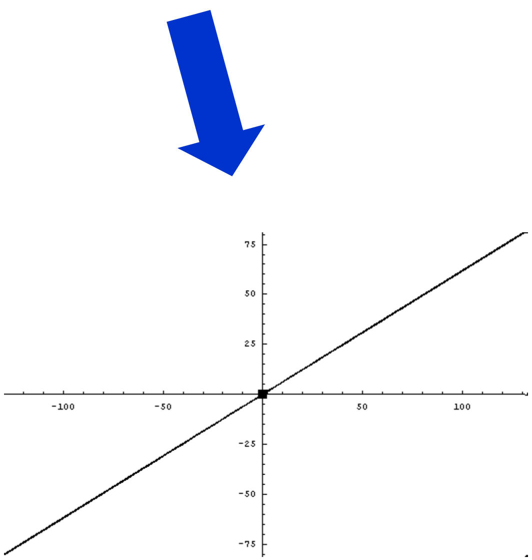


Along these lines (called invariant lines), the dynamics can be understood as simple exponential growth/decay

Example







This could be regarded as a very simple form of self-organization (though completely predictable);

Order spontaneously emerges in the system as time goes on

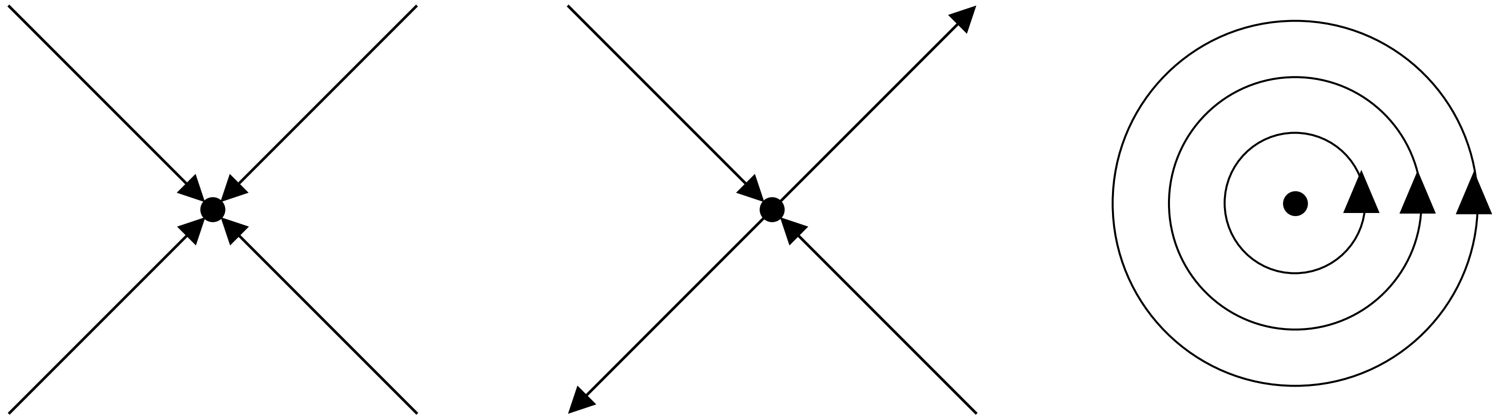
Linear Stability Analysis of Nonlinear Systems

Stability of equilibrium points

- If a system at its equilibrium point is slightly perturbed, what happens?
- The equilibrium point is called:
  - Stable (or asymptotically stable) if the system eventually falls back to the equilibrium point
  - Lyapunov stable if the system doesn't go far away from the equilibrium point
  - Unstable otherwise

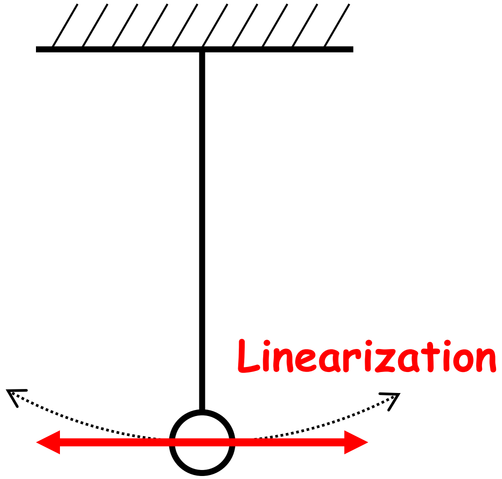
Question

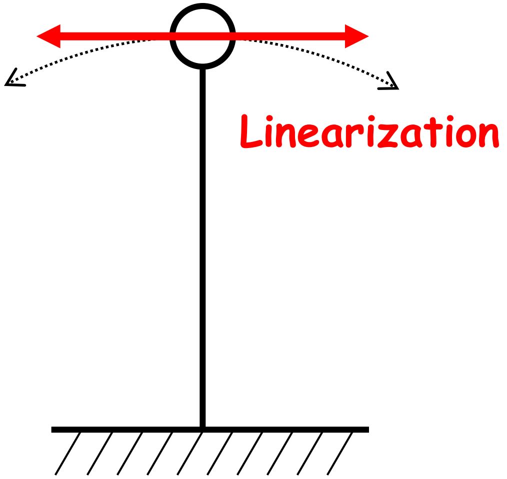
- What is the stability of each of the following equilibrium points?



Linear stability analysis

- Studies whether a nonlinear system is stable or not at its equilibrium point by locally linearizing its dynamics around that point





Local linearization

- Let Δx be a small difference between the system's current state x and its equilibrium point x_e, i.e. $x = x_e + \Delta x$
- Plug $x = x_e + \Delta x$ into differential equations and ignore quadratic or higher-order terms of Δx (hence the name "linearization")

Local linearization

- This operation does the trick to convert the dynamics of Δx into a product of a matrix and Δx
- By analyzing eigenvalues of the matrix, one can predict whether x_e is stable or not
  - I.e. whether a small perturbation (Δx) grows or shrinks over time

Mathematically speaking…

- This operation is similar to "linear approximation" in calculus

  Taylor series expansion:

  $$F(x) = \Sigma_{n=0\sim\infty} F^{(n)}(a)/n!\ (x-a)^n$$

  Let $x \to x_e+\Delta x$ and $a \to x_e$, then

  $$F(x_e+\Delta x) = F(x_e) + F'(x_e)\,\Delta x \cancel{+ O(\Delta x^2)}$$

  Ignore

Linearizing discrete-time models

- For discrete-time models:

$x_t = F(x_{t-1})$

$\text{Left} = x_e + \Delta x_t$

$\text{Right} = F(x_e + \Delta x_{t-1})$

$\sim F(x_e) + F'(x_e)\,\Delta x_{t-1}$

$= x_e + F'(x_e)\,\Delta x_{t-1}$

Therefore,

$$\Delta x_t = F'(x_e)\,\Delta x_{t-1}$$

First-order derivative of vector functions

- Discrete-time: $\Delta x_t = F'(x_e)\,\Delta x_{t-1}$

This can hold even if x is a vector

What corresponds to the first-order derivative in such a case:

$$F'(x_e) = dF/dx_{(x=x_e)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}_{(x=x_e)}$$

Jacobian matrix at $x=x_e$

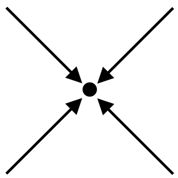
Eigenvalues of Jacobian matrix

- A Jacobian matrix is a linear approximation around the equilibrium point, telling you the local dynamics: "how a small perturbation will grow, shrink or rotate around that point"
  - The equilibrium point serves as a local origin
  - The Δx serves as a local coordinate
  - Eigenvalue analysis applies

With real eigenvalues

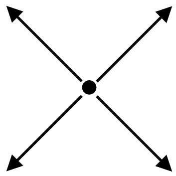
- If all the eigenvalues indicate that Δx will shrink over time

  -> stable point



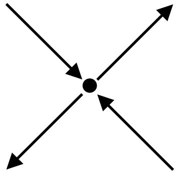
- If all the eigenvalues indicate that Δx will grow over time

  -> unstable point



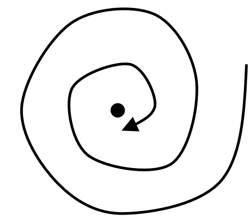
- If some eigenvalues indicate shrink and others indicate grow of Δx over time

  -> saddle point (this is also unstable)

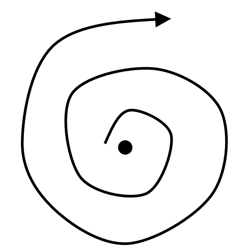


With two complex conjugate eigenvalues (for 2-D systems)

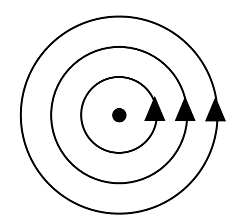
- If both eigenvalues indicate that Δx will shrink over time
  -> stable spiral focus



- If both eigenvalues indicate that Δx will grow over time
  -> unstable spiral focus

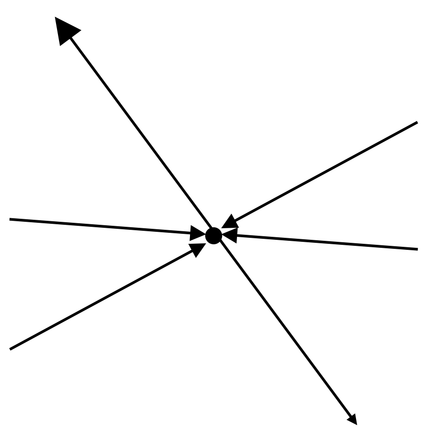


- If both eigenvalues indicate neither shrink nor growth of Δx
  -> neutral center (but this may or may not be true for nonlinear models; further analysis is needed to check if nearby trajectories are truly cycles or not)

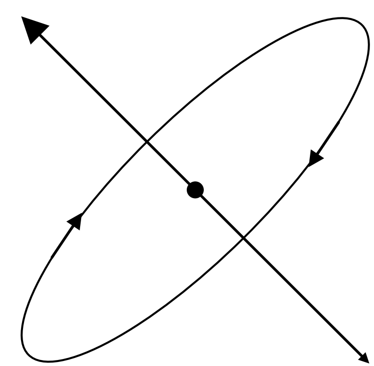


With real and complex eigenvalues mixed (for higher-dimm. systems)

- Each eigenvalue (or a pair of complex conjugate eigenvalues) tell you distinct dynamics simultaneously seen at the equilibrium point:



All real eigenvalues (1 indicates growth; other 2 indicates shrink)



1 real eigenvalue indicates growth; other 2 indicates rotation (complex conjugates with no growth or shrink)