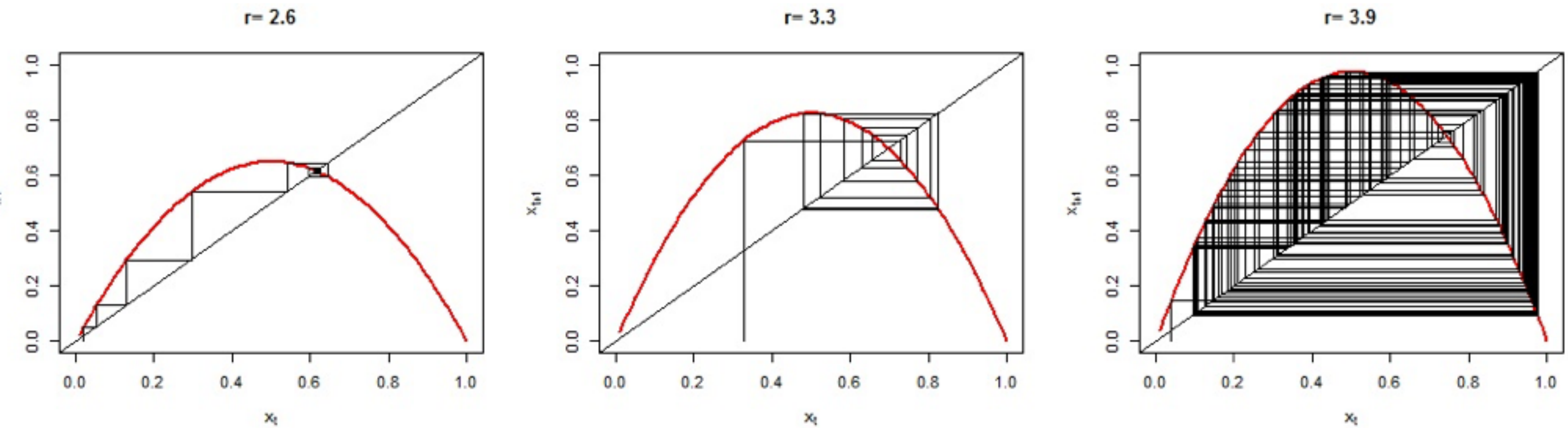
Dynamical Systems (intro) / Discrete-Time Models



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Introduction to Dynamical Systems

Dynamical Systems Theory

- Considers how systems autonomously change along time
  - Ranges from simple Newtonian mechanics to modern nonlinear dynamics theories
  - Probes underlying dynamical mechanisms, not just static properties
  - Provides suite of tools useful for studying complex systems
- A dynamical system is a system with a state that is uniquely specified by a finite set of variables (noise aside) and whose behavior is uniquely determined by “rules”

Mathematical formulations of dynamical systems

- Discrete-time model:

$$x_t = F(x_{t-1}, t)$$

{difference/recurrence equations; iterative maps}

- Continuous-time model:

$$dx/dt = F(x, t)$$

{differential equations}

x_t: State variable(s) of the system at time t

F: Some function that determines the "rules"

Discrete-Time Models

Discrete-Time Model

- **Easy to understand, develop, and simulate**
  - **Doesn't require expression for the rate of change (derivative)**
  - **Can model abrupt changes and/or chaos with few variables**
  - **Directly translatable to computer simulation**
  - **Easier to mesh with experimental data**

Mathematical formulations of dynamical systems

- **Linear: Right hand side is just a first-order polynomial of variables**

$$x_t = \underline{a\, x_{t-1} + b\, x_{t-2} + c\, x_{t-3}} \ldots$$

- **Nonlinear: Pretty much everything else**

$$x_t = a\, x_{t-1} + b\, \underline{x_{t-2}^{\,2}} + c\, \underline{\sqrt{x_{t-1}\, x_{t-3}}} \ldots$$

Mathematical formulations of dynamical systems

- 1st-order: Right hand side refers only to last time step

$$x_t = a\,\underline{x_{t-1}}\,(1 - \underline{x_{t-1}})$$

- Higher-order: Pretty much everything else

$$x_t = a\,\underline{x_{t-1}} + b\,\underline{x_{t-2}} + c\,\underline{x_{t-3}} \ldots$$

Mathematical formulations of dynamical systems

- **Autonomous: Right hand side only includes state variables (x) and not t itself**

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2$$

- **Non-autonomous: Right hand side includes terms that explicitly depend on t**

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2 + \underline{\sin(t)}$$

Mathematical formulations of dynamical systems

- **Non-autonomous, higher order equations can always be converted into autonomous, 1st-order equations**

$$x_{t-2} \rightarrow y_{t-1},\ y_t = x_{t-1}$$

$$t \rightarrow y_t,\ y_t = y_{t-1} + 1,\ y_0 = 0$$

- **Autonomous 1st-order equations can cover dynamics of any non-autonomous higher-order equations!!**

Mathematical formulations of dynamical systems

- **Linear equations:**
  - **are analytically solvable**
  - **show either equilibrium, exponential growth/ decay, periodic oscillation, or combos**
- **Nonlinear equations:**
  - **may show more complex behaviors**
  - **in general, are not analytically solvable**

Simulating Discrete-Time Models

Exercise

- **Simulate the following dynamical model to produce a time series for t = 1,…,10**

$$x_t = 2\,x_{t-1} + 1,\ x_0 = 1$$

$$x_t = x_{t-1}^2 + 1,\ x_0 = 1$$

Exercise

- **Simulate the following set of equations and see what happens if the coefficients are varied**

$$x_t = 0.5\, x_{t-1} + 1\, y_{t-1}$$

$$y_t = -0.5\, x_{t-1} + 1\, y_{t-1}$$

$$x_0 = 1,\ y_0 = 1$$

Build Your Own Model Equation

Modeling tips

- Tweak an existing model
- Implement each assumption one by one
- Find where to change, replace with a newly designed function
- Use simplest form possible
- Check the extreme values

Example: Saturate growth in population model

- Simple exponential growth model:

$$x_t = a\, x_{t-1}$$

- Problem: The model grows/dies only - can one implement saturation of growth in this model?

- Think about a new nonlinear model:

$$x_t = f(x_{t-1})\, x_{t-1}$$

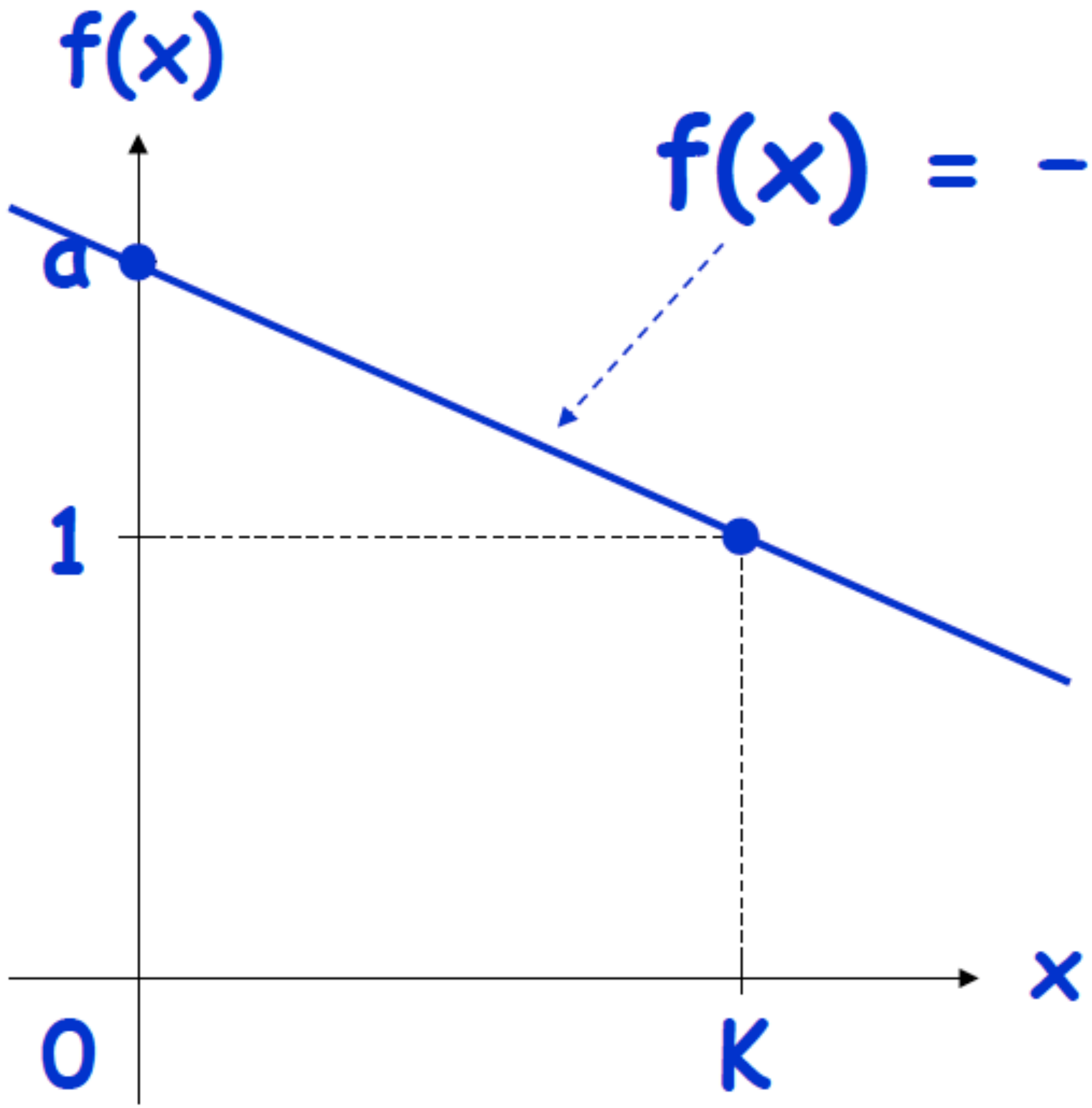
Example: Saturate growth in population model

$$x_t = f(x_{t-1})\, x_{t-1}$$

- $f(x)$ should approach 1 (no growth) when x goes to a "carrying capacity", call it K
- $f(x)$ should approach the original growth rate a when x is very small

Example: Saturate growth in population model

$$f(x) = -\frac{a-1}{K}x + a$$



Example: Saturate growth in population model

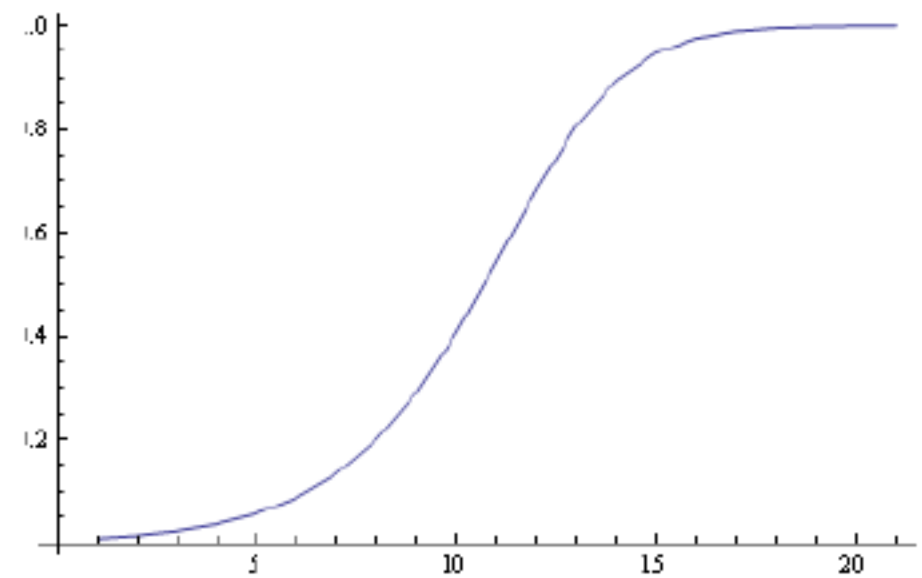
$$x_t = f(x_{t-1})\, x_{t-1}$$
$$= (\,-\,(a - 1)\, x_{t-1} / K + a\,)\, x_{t-1}$$

- Using $r = a\text{-}1$:

$$x_t = (\,-\,r\, x_{t-1} / K + r + 1\,)\, x_{t-1}$$
$$= x_{t-1} + \underbrace{r\, x_{t-1} (\,1 - x_{t-1} / K\,)}_{\text{Net growth}}$$

Example: Saturate growth in population model

- N: Population
- r: Population growth rate
- K: Carrying capacity



- Discrete-time version:

$$N_t = N_{t-1} + r\, N_{t-1} \left(1 - N_{t-1}/K \right)$$

- Continuous-time version:

$$dN/dt = r\, N \left(1 - N/K \right)$$

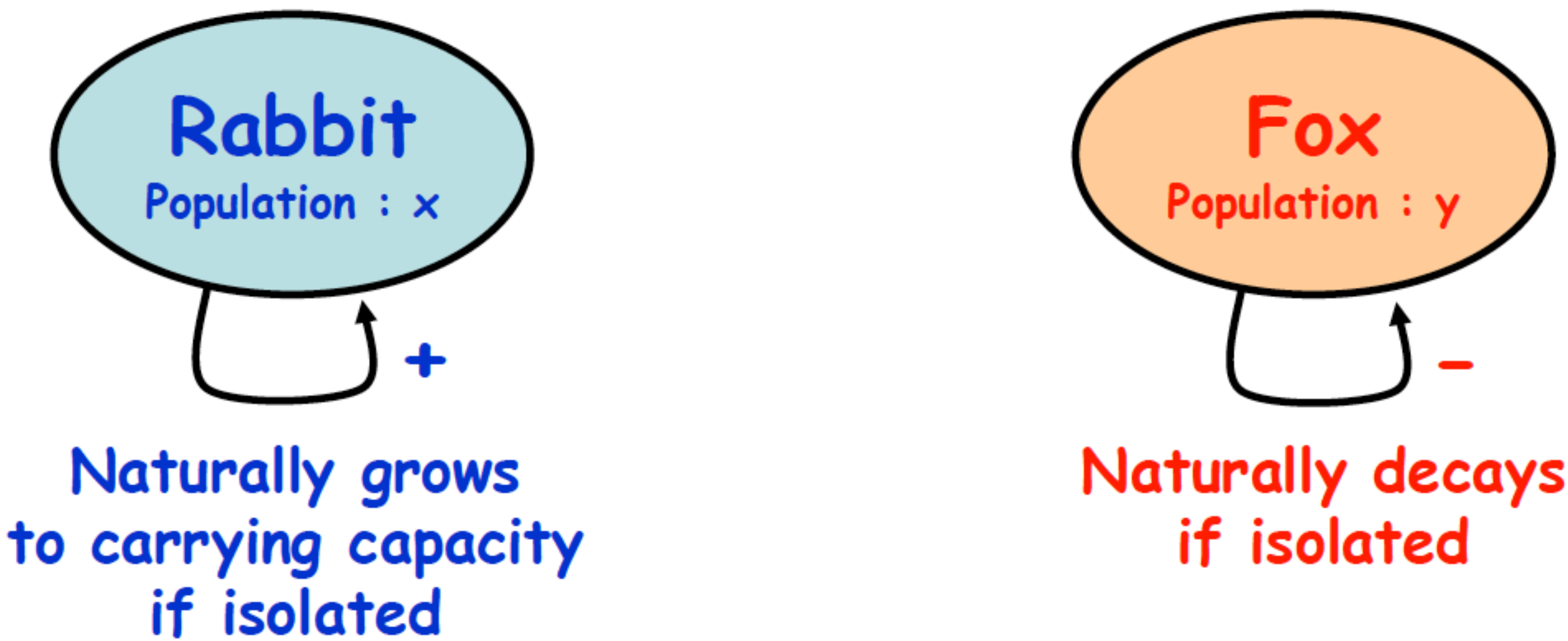
Nonlinear terms

Example: Predator and prey model

- **A nonlinear model of a simple ecosystem comprised of foxes (predators) and rabbits (prey)**

Example: Predator and prey model

- A nonlinear model of a simple ecosystem comprised of foxes (predators) and rabbits



Example: Predator and prey model

- **A nonlinear model of a simple ecosystem comprised of foxes (predators) and rabbits (prey)**

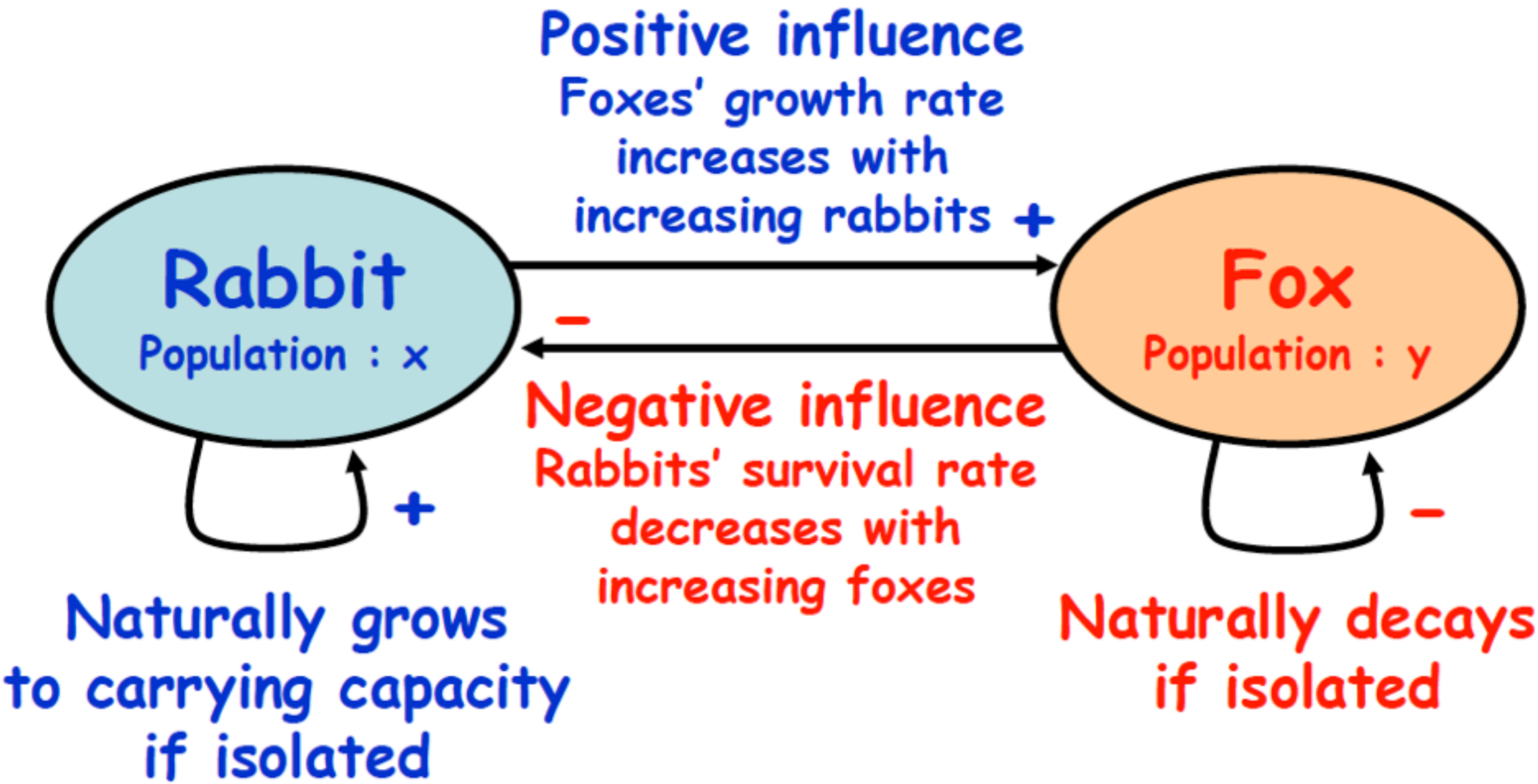
Rabbit: $x_t = x_{t-1} + a\, x_{t-1}\,(1 - x_{t-1})$

Fox: $y_t = b\, y_{t-1}$

$(0<a,\ 0<b<1)$

Example: Predator and prey model

- A nonlinear model of a simple ecosystem comprised of foxes (predators) and rabbits



Example: Predator and prey model

- Introduced coefficient to the first term of x that captures the negative influence of foxes on rabbits survival rate

- Replaced b with a coefficient that captures the positive influence of rabbits on foxes' growth

Rabbit: $x_t = (1 - c\, y_{t-1})\, x_{t-1} + a\, x_{t-1}\, (1 - x_{t-1})$

Fox: $y_t = (b + d\, x_{t-1})\, y_{t-1}$

$(0 < a,\ 0 < b < 1,\ 0 < c,\ 0 < d)$

Lotka-Volterra Model

- This model can be rewritten as:

$$x_t - x_{t-1} = \alpha\, x_{t-1}\, (1 - x_{t-1}) - \beta\, x_{t-1}\, y_{t-1}$$
$$y_t - y_{t-1} = -\gamma\, y_{t-1} + \delta\, x_{t-1}\, y_{t-1}$$

- Known as the "Lotka-Volterra" equations

- One of the most famous early nonlinear systems with more than one variable

Analysis of Discrete-Time Models

Fixed points

- Fixed points represent a state of the system that will not change in time (a.k.a steady state)
- Can be found by setting all X values to X*
- Examples

$$N_t = N_{t-1} + r\,N_{t-1}\,(1 - N_{t-1}/K)$$

$$x_t = 2x_{t-1} - x_{t-1}^2$$

$$x_t = x_{t-1} - x_{t-2}^2 + 1$$

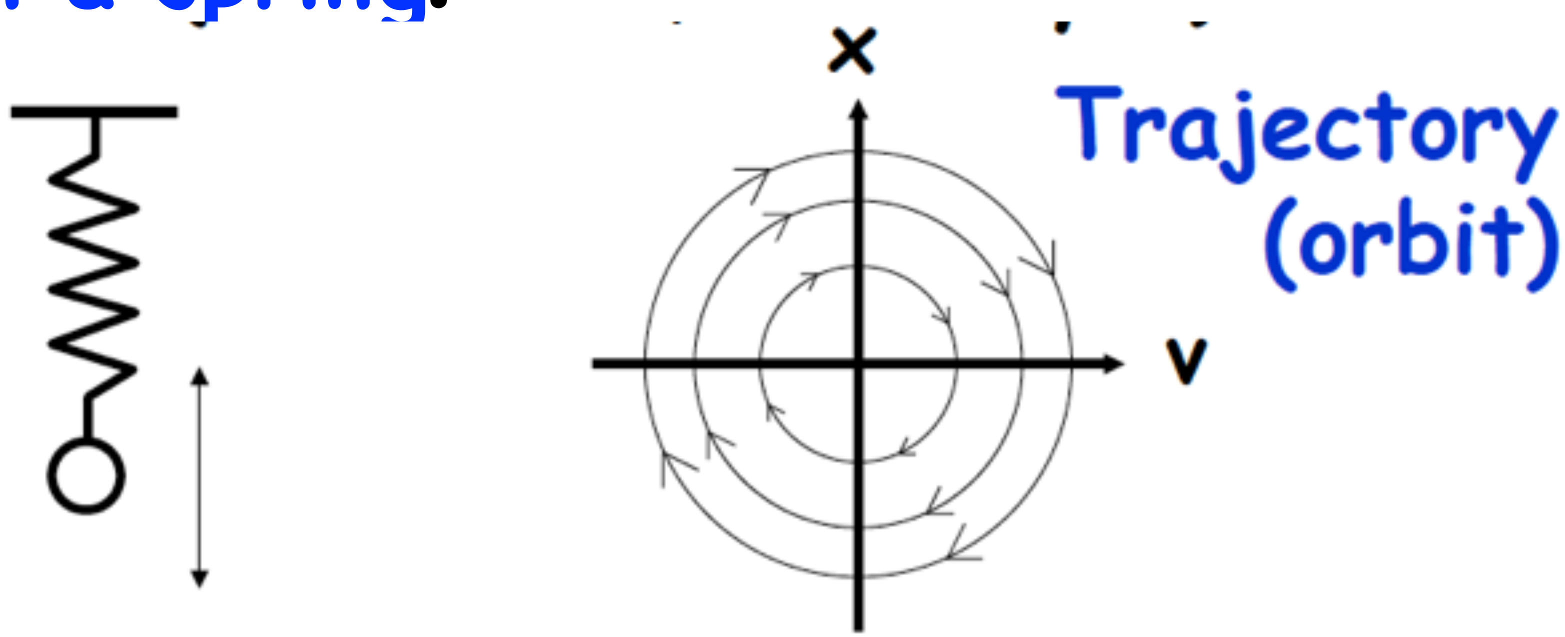
Phase Space Visualization

Phase space

- Developed in late 19th century by J. Henri Poincare
- Way to visualize the behavior of dynamical systems as trajectories in a phase space - dynamics become a "flow" in the space
- Every state of the system corresponds to a point in the phase space - the axis thus represent the degrees of freedom (variables)
- Produces insight from geometrical structure and global behavior in the space that can't be gleaned from algebraic methods

Phase space

- **Attractor**: A subset of the space that nearby trajectories flow into and then do not leave (as long as the "forcing" doesn't change)
- **Basin of attraction**: A set of states which will eventually end up in a given attractor
- **Mass on a spring**:



Exercise

- Draw the phase space and some sample trajectories for the following model:

$$x_t = 0.5\, x_{t-1} + 1\, y_{t-1}$$
$$y_t = -0.5\, x_{t-1} + 1\, y_{t-1}$$
$$x_0 = 1,\ y_0 = 1$$

Exercise

- **Draw the phase space and some sample trajectories for the following model:**

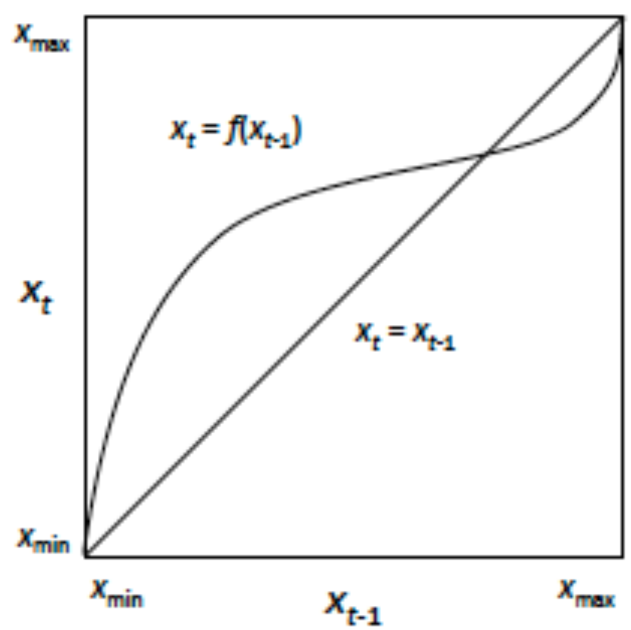
$$x_t = -0.5\, x_{t-1} - 0.7\, y_{t-1}$$
$$y_t = x_{t-1} - 0.5\, y_{t-1}$$

Cobweb Plot

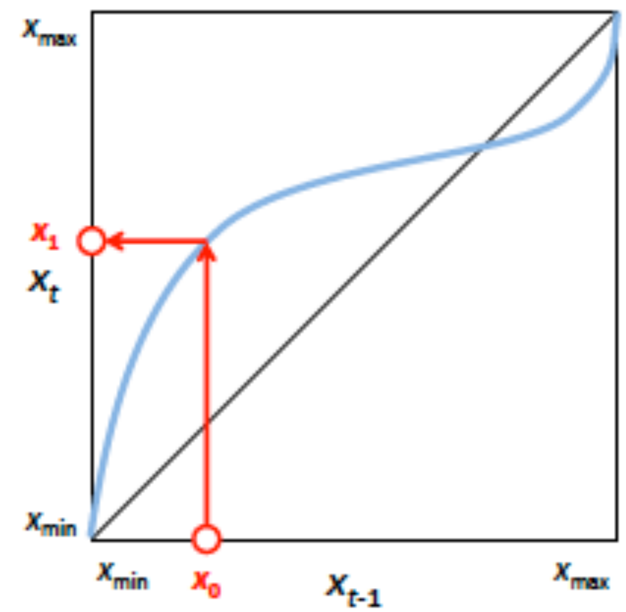
Cobweb Plot

- A visual tool to study the behavior of 1-D iterative maps
- Take x_{t-1} and x_t as the two axes
- Draw the map of interest ($x_t=F(x_{t-1})$) and the $x_t=x_{t-1}$ reference line - these will intersect at "fixed points"
- Trace the development of the system by jumping between these two lines

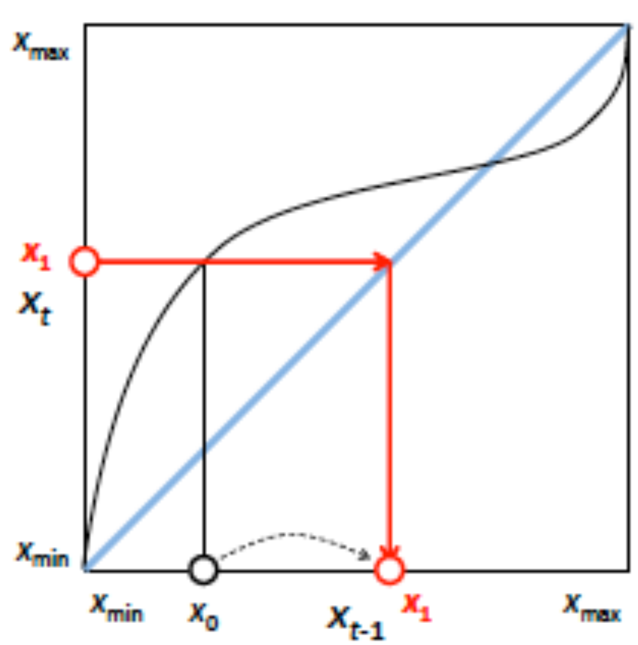
Cobweb Plot



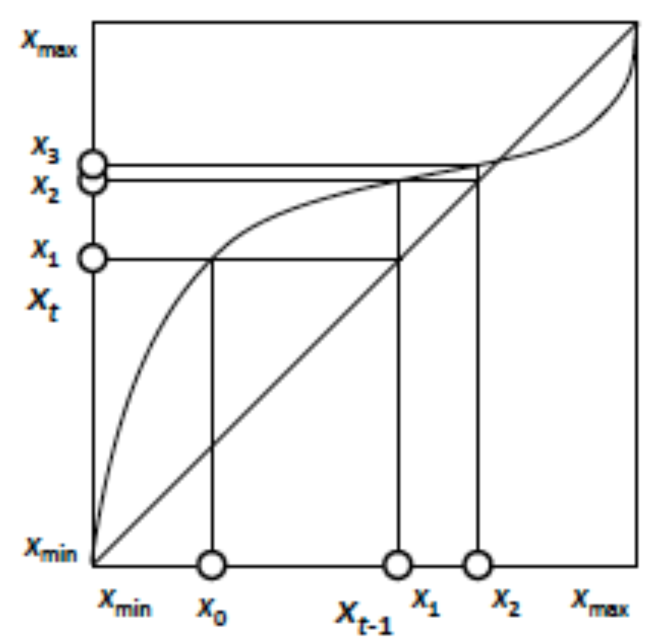
x_{max}
$x_t = f(x_{t-1})$
x_t
$x_t = x_{t-1}$
x_{min}
x_{min}
x_{t-1}
x_{max}



x_{max}
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_{max}

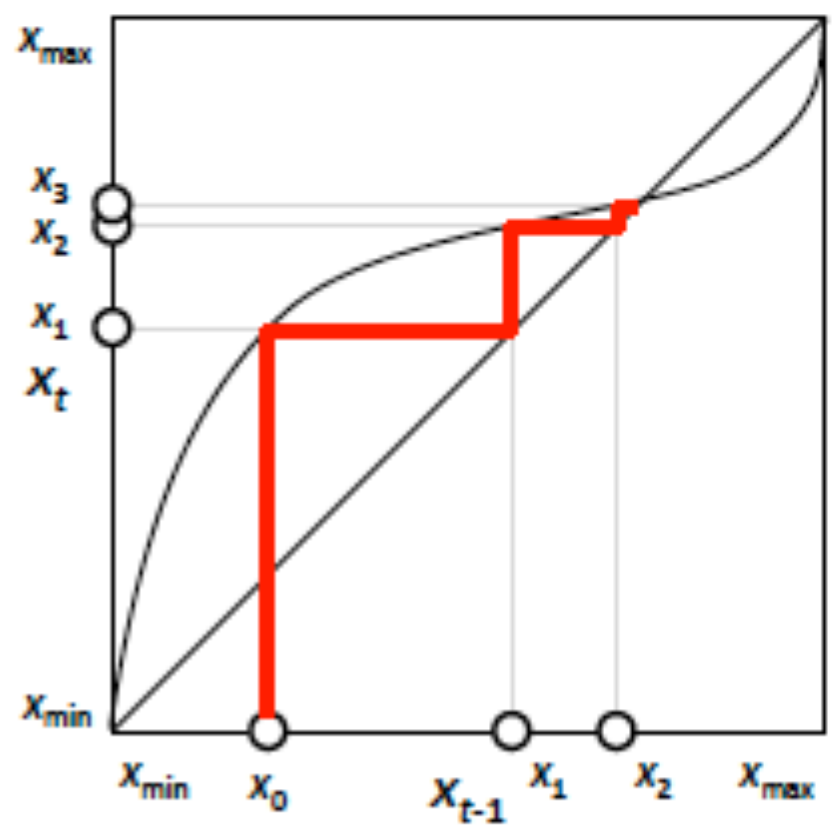


x_{max}
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_1
x_{max}



x_{max}
x_3
x_2
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_1
x_2
x_{max}

Cobweb Plot



Rescaling Variables

Rescaling Variables

- Dynamics of a system should not depend on unit size - so should be able to choose any linear rescaled version of variables and keep same behavior

- Simplify the logistic growth model by rescaling $x \rightarrow \alpha x'$

$$x_t = x_{t-1} + r\, x_{t-1} (1 - x_{t-1}/K)$$

Analysis of Linear Systems

Linear systems - asymptotic behavior

- Many systems can be considered as being linear in certain regions of phase space. Dynamics of linear systems can be studied analytically.

- A linear system is: $x_t = A\, x_{t-1}$ where A is a "coefficient" matrix

- A good question to ask of a linear system is where will the system go eventually?

$$x_t = A^t\, x_0$$

Linear systems - asymptotic behavior

- Let $\{v_i\}$ be n linearly independent eigenvectors of the coefficient matrix

- Write the initial condition using eigenvectors,

  $x_0 = b_1v_1 + b_2v_2 + \ldots + b_nv_n$

Linear systems - asymptotic behavior

- **Then:**

$$x_t = A^t x_0$$

$$= \lambda_1^t b_1 v_1 + \lambda_2^t b_2 v_2 + \ldots + \lambda_n^t b_n v_n$$

Linear systems - asymptotic behavior

- **Dominant eigenvector:**

If $|\lambda_1| > |\lambda_2|, |\lambda_3|, \ldots,$

$$x_t = \lambda_1^t \{ b_1 v_1 + \cancel{(\lambda_2/\lambda_1)^t b_2 v_2} + \ldots + \cancel{(\lambda_n/\lambda_1)^t b_n v_n} \}$$

$$\lim_{t \to \infty} x_t \sim \lambda_1^t b_1 v_1$$

- **If the system has just one such dominant eigenvector, it's state will be eventually along that vector regardless of where it starts**

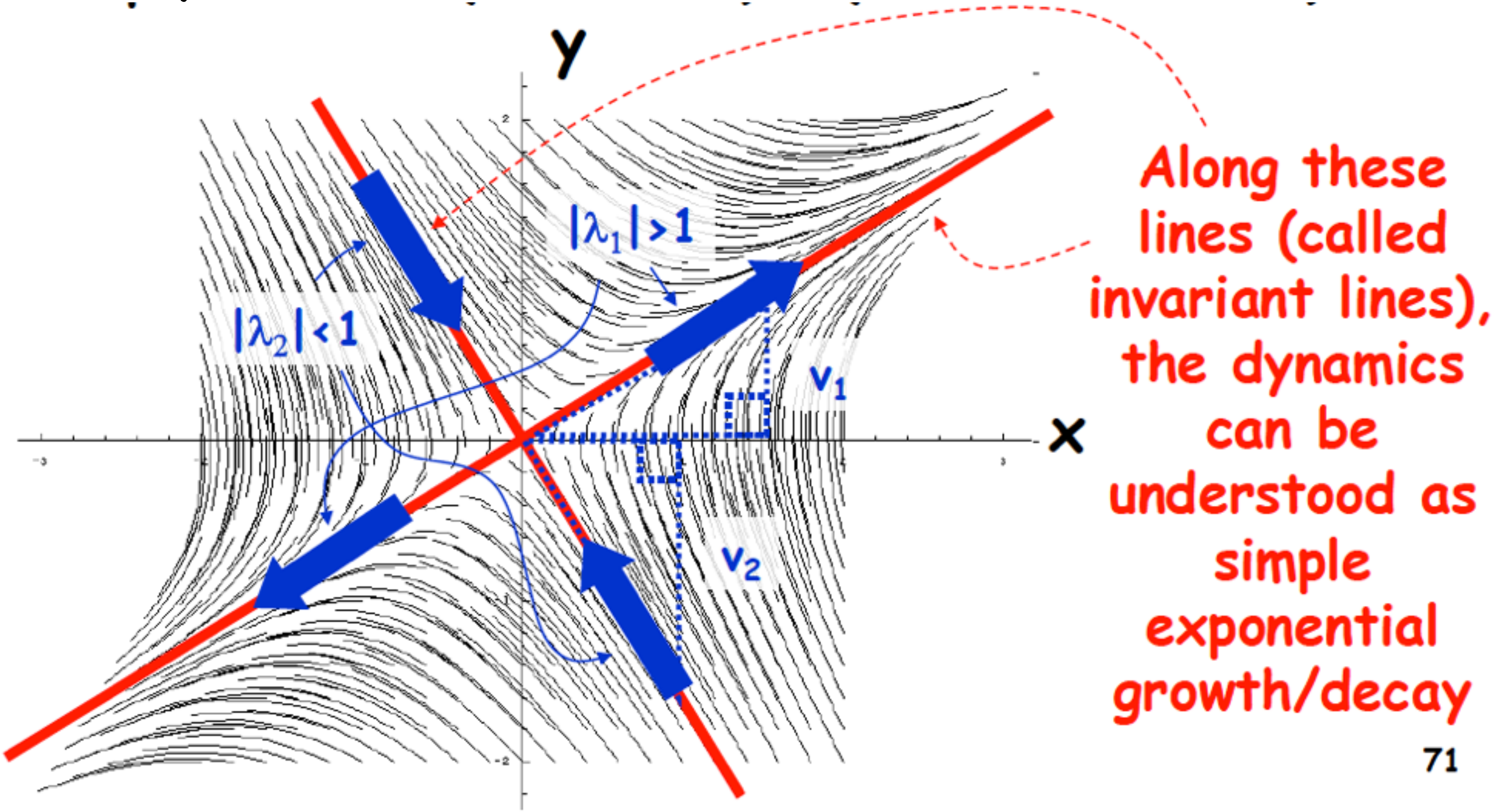
Linear systems - asymptotic behavior

- **An eigenvalue tells whether a particular "state" of the system (specified by its corresponding eigenvectors) grows or shrinks**

$$|\lambda| > 1 \rightarrow \text{growing}$$
$$|\lambda| < 1 \rightarrow \text{shrinking}$$

Linear systems - asymptotic behavior

- **Phase space of a two variable linear difference equation**



Linear stability analysis

- When a system is at it's fixed point, what happens when it is slightly perturbed?

- Fixed point is called:

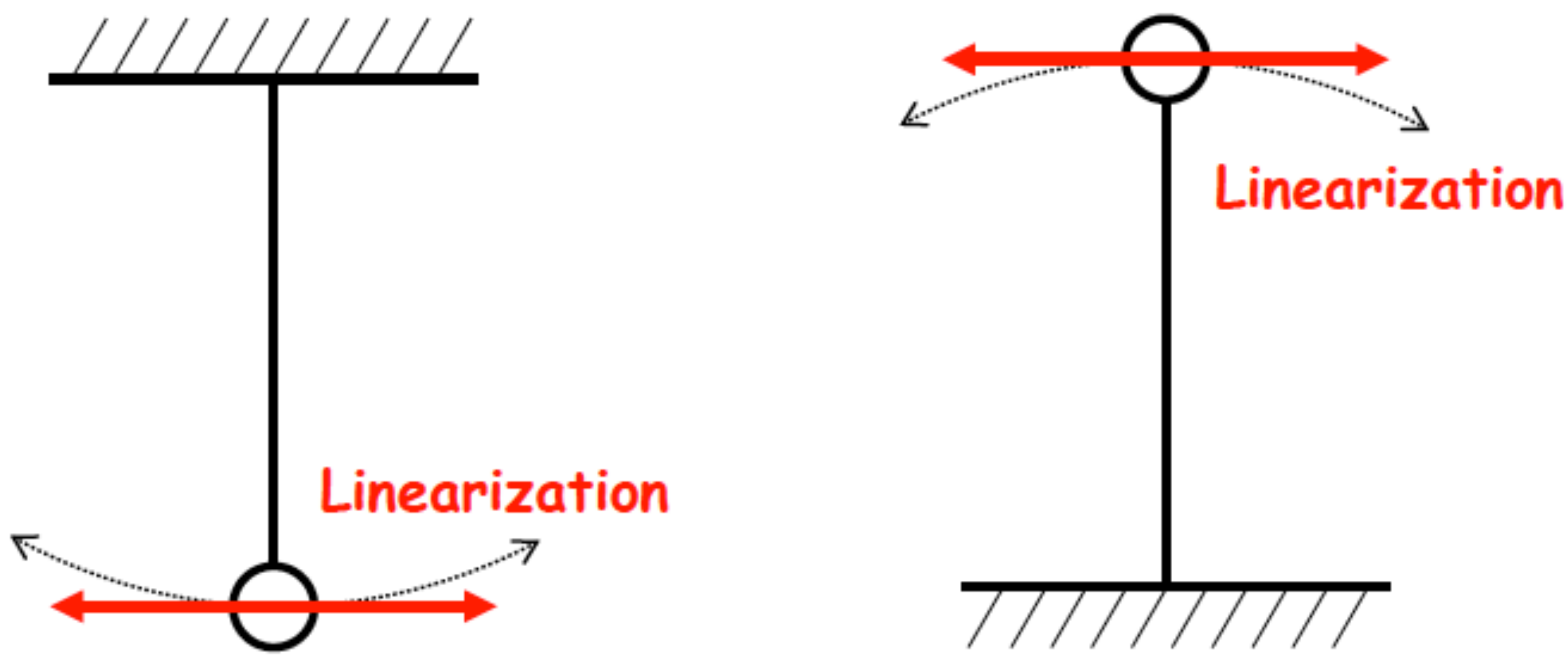
  - Stable if the system eventually falls back to the fixed point

  - Lyapunov stable if the system doesn't go far form the fixe point

  - Unstable otherwise

Linear stability analysis

- Linear stability analysis studies whether a nonlinear system is stable or not at its fixed point by locally linearizing the dynamics near that point



Linear stability analysis

- Let Δx be a small difference between the system's current state x and it's fixed point x_e i.e.
- Plug $x = x_e + \Delta x$ into differential equations and ignore quadratic or higher order terms of Δx
- This operation does the trick of converting the dynamics of Δx into a product of a matrix and Δx
- The eigenvalues of the matrix reveal whether x_e is stable or not.

Linear stability analysis

- **Mathematically speaking…**

Taylor series expansion:

$$F(x) = \Sigma_{n=0\sim\infty} F^{(n)}(a)/n!\ (x-a)^n$$

Let $x \rightarrow x_e+\Delta x$ and $a \rightarrow x_e$, then

$$F(x_e+\Delta x) = F(x_e) + F'(x_e)\,\Delta x \cancel{+ O(\Delta x^2)}$$

Ignore

Linear stability analysis

- For discrete-time models…

$$x_t = F(x_{t-1})$$

$$\text{Left} = x_e + \Delta x_t$$

$$\begin{aligned}\text{Right} &= F(x_e + \Delta x_{t-1}) \\ &\sim F(x_e) + F'(x_e)\,\Delta x_{t-1} \\ &= x_e + F'(x_e)\,\Delta x_{t-1}\end{aligned}$$

Therefore,

$$\Delta x_t = F'(x_e)\,\Delta x_{t-1}$$

Linear stability analysis

- **For discrete-time models…**

This can hold even if x is a vector

What corresponds to the first-order derivative in such a case:

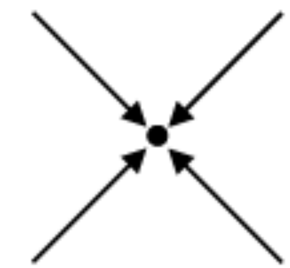
$$F'(x_e) = dF/dx_{(x=x_e)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}_{(x=x_e)}$$

Jacobian matrix at $x=x_e$

Linear stability analysis

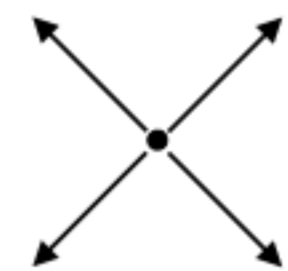
- If all the eigenvalues indicate that Δx will shrink over time

  -> stable point



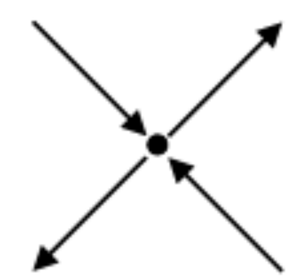
- If all the eigenvalues indicate that Δx will grow over time

  -> unstable point



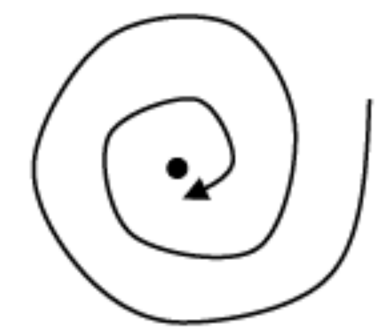
- If some eigenvalues indicate shrink and others indicate grow of Δx over time

  -> saddle point (this is also unstable)

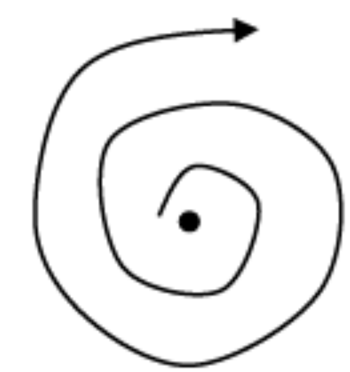


Linear stability analysis (2-D systems)

- If both eigenvalues indicate that Δx will shrink over time
  -> stable spiral focus



- If both eigenvalues indicate that Δx will grow over time
  -> unstable spiral focus



- If both eigenvalues indicate neither shrink nor growth of Δx
  -> neutral center (but this may or may not be true for nonlinear models; further analysis is needed to check if nearby trajectories are truly cycles or not)