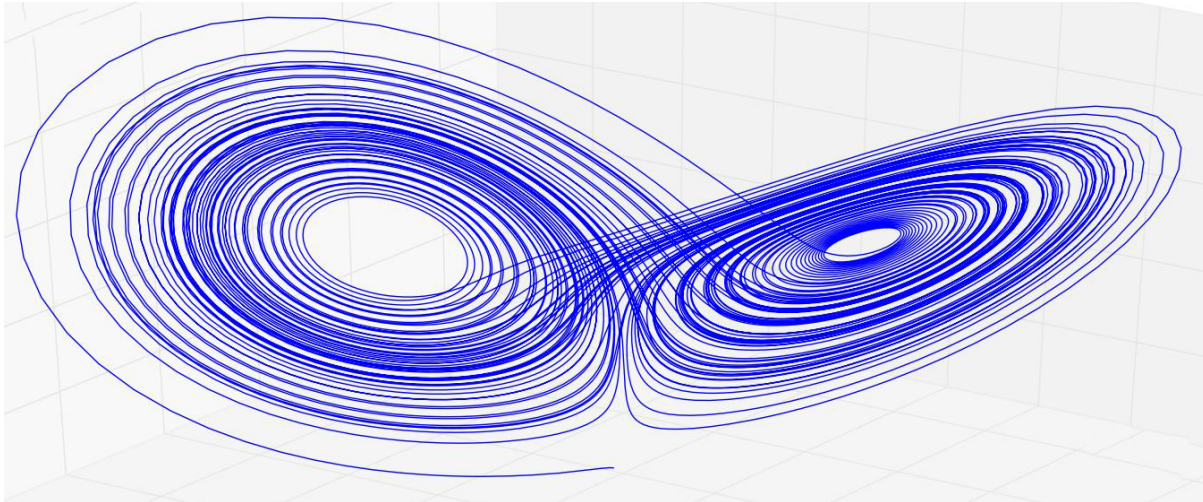


Chaos



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Discovery of chaos

- Discovered in early 1960's by Edward N. Lorenz (in a 3-D continuous-time model)
- Popularized in 1976 by Sir Robert M. May as an example of **complex dynamics caused by simple rules** (he used a 1-D discrete-time logistic map)



Chaos in dynamical systems

- A long-term behavior of a dynamical system that **never falls in any static or periodic trajectories**
- Looks like a **random fluctuation**, but still occurs in **completely deterministic simple systems**
- Exhibits **sensitivity to initial conditions**

Chaos in Discrete-Time Models

Simple example: Logistic map

- A simple difference equation used by Robert M. May in his paper in 1976

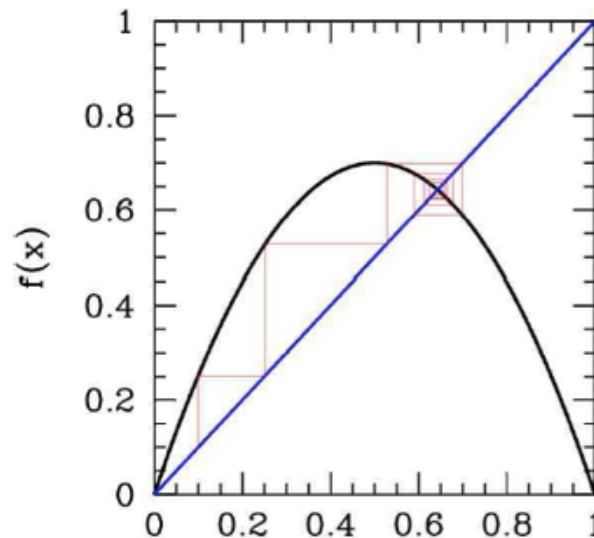
$$x_t = a x_{t-1} (1 - x_{t-1})$$

- Shows quite complex dynamics as control parameter a is varied

Simple example: Logistic map

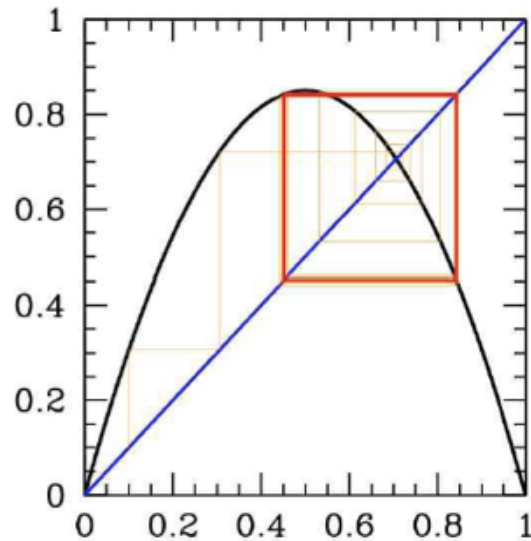
$$x_t = a x_{t-1} (1 - x_{t-1})$$

$$a = 2.8$$



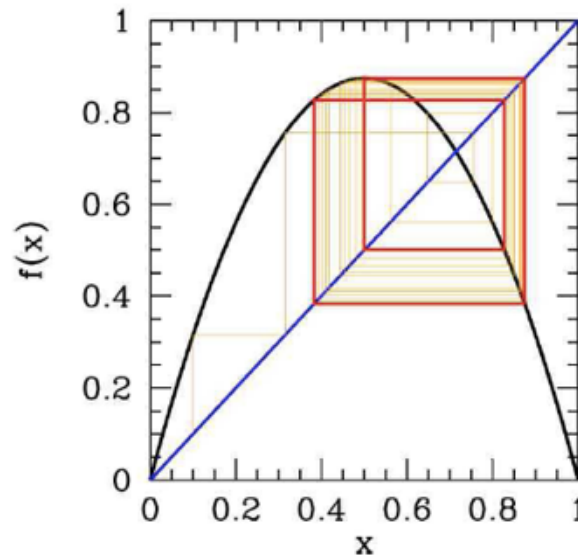
Simple example: Logistic map

$$x_t = a x_{t-1} (1 - x_{t-1})$$
$$a = 3.4$$



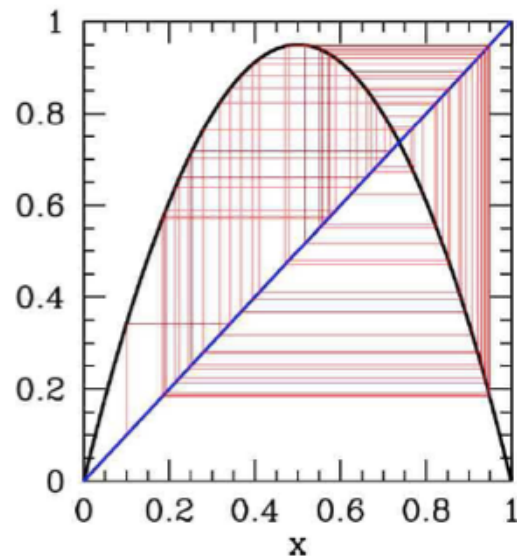
Simple example: Logistic map

$$x_t = a x_{t-1} (1 - x_{t-1})$$
$$a = 3.5$$

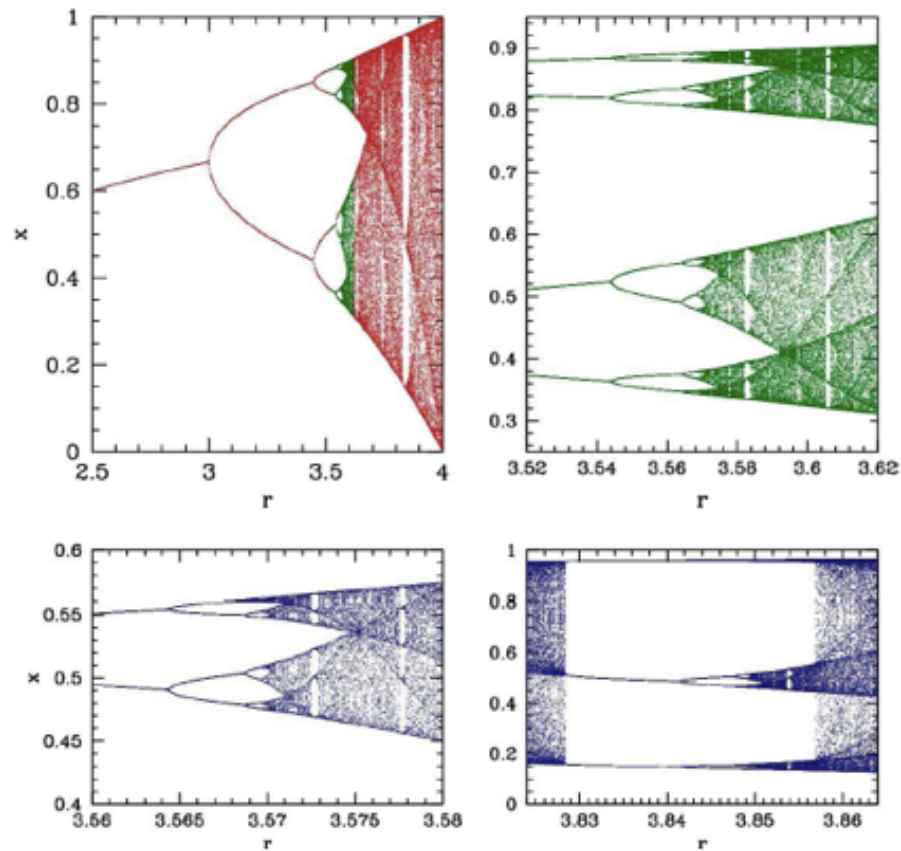


Simple example: Logistic map

$$x_t = a x_{t-1} (1 - x_{t-1})$$
$$a = 3.8$$



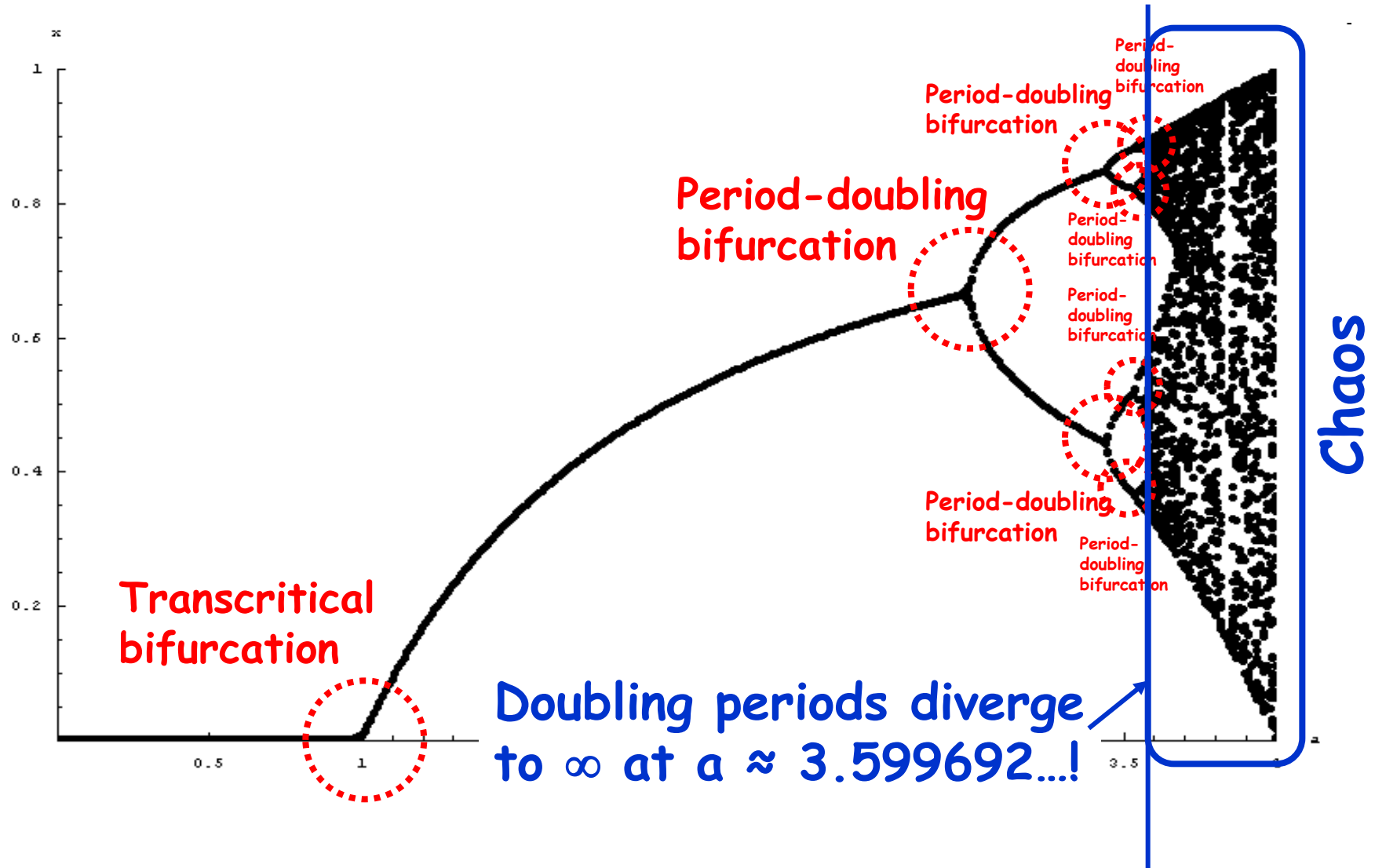
Simple example: Logistic map



Drawing bifurcation diagrams using numerical results

- For systems with periodic (or chaotic) long-term behavior, it is useful to draw a bifurcation diagram using numerical simulation results instead of analytical results
 - Plot actual system states sampled after a long period of time has passed
 - Can capture period-doubling bifurcation by a “set” of points

Cascade of period-doubling bifurcations leading to chaos



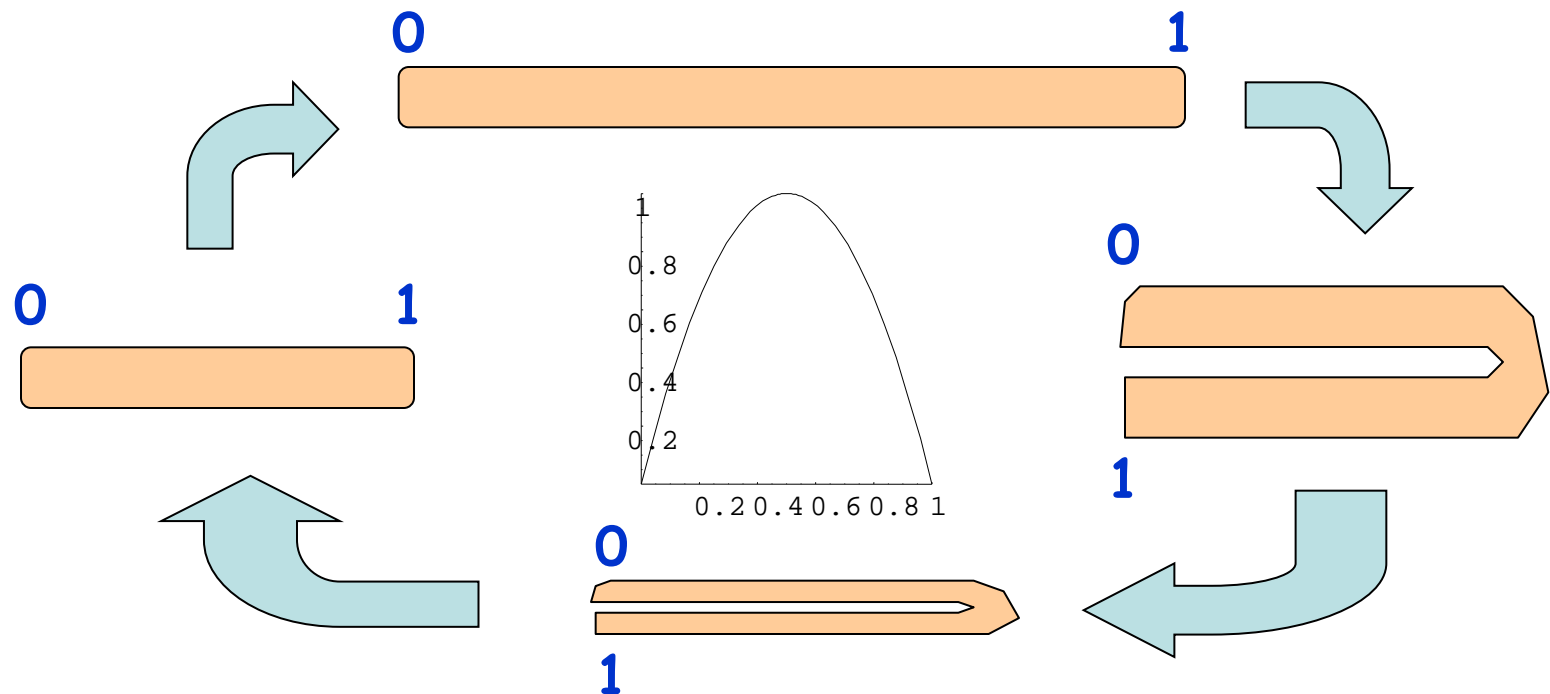
Reinterpreting chaos

- Where a period diverges to infinity
 - Periodic behavior with infinitely long periods means “aperiodic” behavior
- Where (almost) no periodic trajectories are stable
 - No fixed points or periodic trajectories you can sit in (you have to always deviate from your past course!!)

Characteristics of Chaos

Common mechanism of chaos: Stretch and folding in phase space

- In chaotic systems, it is generally seen that a phase space is stretched and folded, like dough kneading



Sensitivity to initial conditions

- Stretching and folding mechanisms always dig out microscale details hidden in a system's state and expand them to macroscale visible levels
- This causes chaotic systems very sensitive to initial conditions
- Think about where a grain in a dough will eventually move during kneading

Lyapunov exponent (1)

- A quantitative measure of a system's sensitivity to small differences in initial condition
- Characterizes how the distance between initially close two points will grow over time

$$| F^t(x_0 + \varepsilon) - F^t(x_0) | \sim \varepsilon e^{t\lambda} \quad (\text{for large } t)$$

Lyapunov exponent (2)

$$| F^t(x_0 + \varepsilon) - F^t(x_0) | \sim \varepsilon e^{t\lambda} \text{ (for large } t)$$

$$\lambda = \lim_{\substack{t \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{1}{t} \log \left| \frac{F^t(x_0 + \varepsilon) - F^t(x_0)}{\varepsilon} \right|$$

$$= \lim_{t \rightarrow \infty} \frac{1}{t} \log \left| \frac{d}{dx} F^t(x_0) \right|$$

$$= \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=0}^{t-1} \log |F'(x_i)|$$

Chaos in Continuous-Time Models

Chaos in continuous-time models

- Requires three or more dimensions
 - Because in 1-D or 2-D phase space, every trajectory works as a “wall” and thus confines where you can go in the future; stretching and folding are not possible in such an environment

Edward N. Lorenz's model

- A model of fluid convection:

$$dx/dt = s (y - x)$$

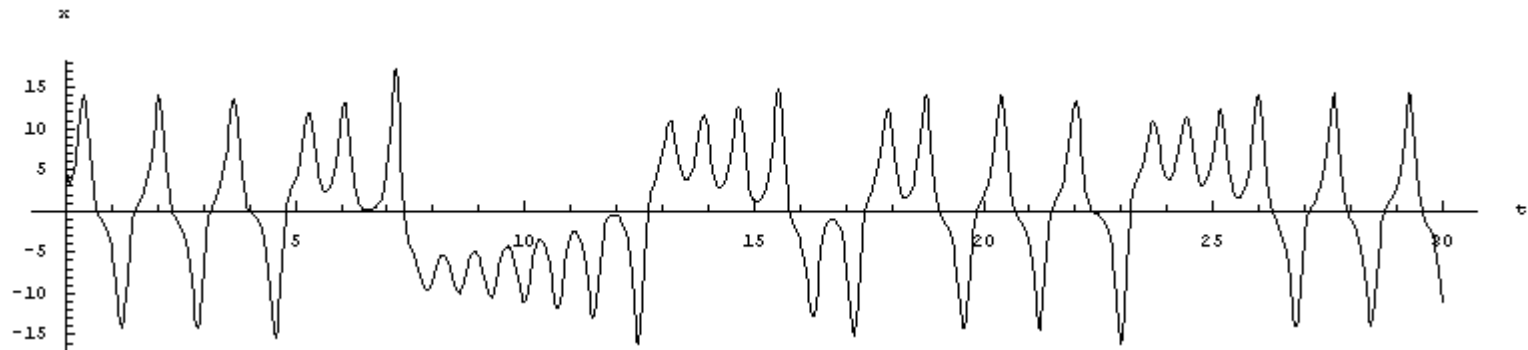
$$dy/dt = r x - y - x z$$

$$dz/dt = x y - b z$$

(s, r and b are positive parameters)



- Typical behavior (x plotted over time):



Strange attractor

- A bounded region in a phase space that attracts nearby trajectories but also exhibits **sensitive dependence on initial conditions** inside it (i.e., no convergence to fixed points or periodic trajectories)
 - A.k.a.: chaotic attractor, fractal attractor

Other Routes to Chaos

- Intermittency

- characterized by dynamics with bursts of chaotic and periodic behavior. as control parameter is changed the chaotic bursts become longer until full chaos occurs

- Quasiperiodicity

- develops when two orbits/modes are nonlinearly coupled