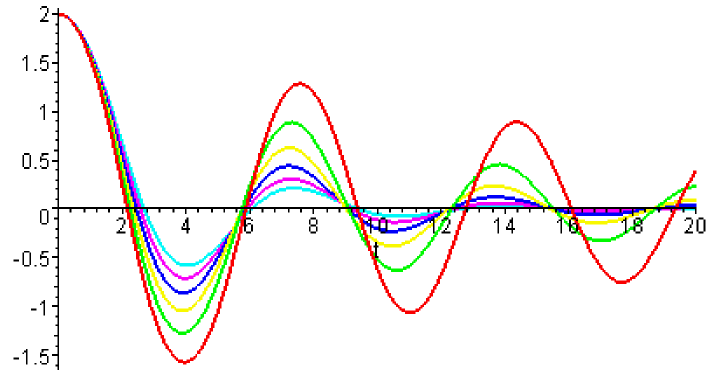
Continuous-Time Models



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Continuous-Time Models with Differential Equations

Mathematical formulations of dynamical systems

- **Discrete-time model:** (difference/recurrence equations; iterative maps)

  $x_t = F(x_{t-1}, t)$

- **Continuous-time model:** (differential equations)

  $dx/dt = F(x, t)$

x_t: State variable(s) of the system at time t

F: Some function that determines the rule that the system's behavior will obey

Including x in F( ) means "feedback loops"

A general form (first-order, autonomous)

$dx_1/dt = F_1\,(x_1,\ x_2,\ x_3,\ \ldots)$
$dx_2/dt = F_2\,(x_1,\ x_2,\ x_3,\ \ldots)$
$dx_3/dt = F_3\,(x_1,\ x_2,\ x_3,\ \ldots)\quad \ldots$

or

$dx/dt = F(x)$

where x is a state vector of the system ($x = \{x_1,\ x_2,\ x_3,\ \ldots\}$)

Higher-order/non-autonomous systems

- **Higher-order systems:**

  Differential equations that include second-order (or higher) derivatives

- **Non-autonomous systems:**

  Differential equations that are time-dependent (i.e., explicitly include t in them)

The following argument also holds for differential equations

- Non-autonomous, higher-order equations can always be converted into autonomous, 1st-order equations
  - $d^2x/dt^2 \rightarrow dy/dt$, $y = dx/dt$
  - $t \rightarrow y$, $dy/dt = 1$, $y_0 = 0$
- Autonomous 1st-order equations can cover dynamics of any non-autonomous higher-order equations too!

Connecting continuous-time models with discrete-time models

$$x_t = F(x_{t-1}) \qquad dx/dt = G(x)$$

- $F(x) \Leftrightarrow x + G(x)\,\Delta t$
- $G(x) \Leftrightarrow (F(x) - x) / \Delta t$

- If $F(x) = Ax$, $G(x) = Bx$:

  $A \Leftrightarrow I + B\,\Delta t$

  $B \Leftrightarrow (A - I) / \Delta t$

How to study differential equations

- Some of them can be analytically solvable
  - Linear systems
  - Simple nonlinear systems

- Analytical solutions are generally not available for nonlinear differential equations

Numerical simulation

- Simplest way: Euler forward method

  $$dx/dt = F(x)$$

  $$\rightarrow x_{t+dt} = x_t + F(x)\,\Delta t$$

- Approximate dynamics using small discrete time steps ($\Delta t << 1$)
- Simulate the model like difference equations

Exercise

- Simulate the following continuous-time logistic growth model in Python, with $r=0.2$, $K=1$, $\Delta t=0.01$:

$$dN/dt = r\,N\,(1 - N/K)$$

Analysis of Continuous-Time Models

Equilibrium point

- A state of the system at which state will not change over time
  - A.k.a. fixed point, steady state
- Can be calculated by solving

$$dx/dt = 0$$

Example

- A simple second-order equation:

$$d^2x/dt^2 = x$$

- Convert this into a first-order form
- Calculate its equilibrium points

Exercise

- A simple pendulum:

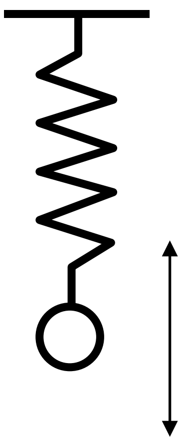
$$d^2\theta/dt^2 = - g/L \sin \theta$$

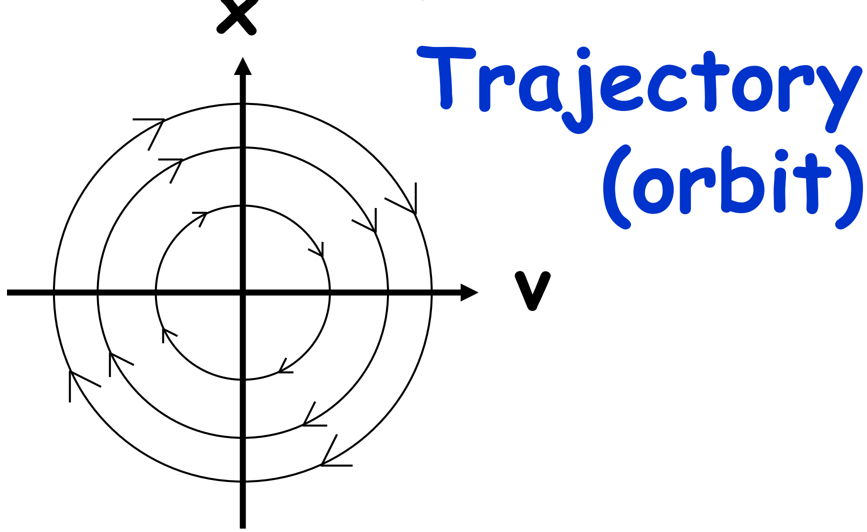
- Convert this into a first-order form
- Calculate its equilibrium points

Visualizing Phase Space of Continuous Models

Phase space of continuous models

- E.g. a simple vertical spring oscillator
- State can be specified by two real variables (location x, velocity v)





Dynamics of continuous models can be depicted as "flow" in a continuous phase space

Visualizing phase space of continuous models

- Vector field
  - Uses many small arrows to show how local derivatives (or direction of trajectories) change from place to place in phase space
- Phase portrait (stream plot)
  - Shows several typical trajectories to illustrate how phase space is globally structured

Exercise

- Write your own code to visualize the phase space of the simple pendulum model

$$d^2\theta/dt^2 = -\, g/L \sin\theta$$

- Include some "damping" effect in the model above and see how it changes the phase space

Visualizing phase space of continuous models manually

- Find "nullclines"
  - Points in the phase space where one of the derivatives is zero (i.e., trajectories are in parallel to one of the axes)
  - Plot where the nullclines are
  - Find how the sign of the derivative changes across the nullclines
  - Find values of other non-zero derivatives
- Draw a "flow" between those nullclines with curves that don't intersect with each other

Exercise

- Draw an outline of the phase space of the following system by studying the distribution of its nullclines:

$$dx/dt = ax - bx\ y$$
$$dy/dt = -cy + dx\ y$$

$$(x >= 0,\ y >= 0)$$

Rescaling Variables

Rescaling variables

- Dynamics of a system won't change qualitatively by linear rescaling of variables (e.g., $x \to \alpha x'$)
- You can set arbitrary rescaling factors for variables to simplify the model equations
- If you have k variables (including t), you may eliminate k parameters

Exercise

- Simplify the logistic growth model by rescaling $N \rightarrow \alpha N'$ and $t \rightarrow \beta t'$

$$dN/dt = r N (1 - N/K)$$

Asymptotic Behavior of Linear Systems

Linear systems

- Linear systems are the simplest cases where states of nodes are continuous-valued and their dynamics are described by a time-invariant matrix
- Continuous-time: $dx/dt = A\,x$
  - A is called a "coefficient" matrix
  - We don't consider constants (as they can be easily converted to the above forms)

Where will the system go eventually?

$$dx/dt = A\,x$$

These equations give the following exact solution:

$$x_t = e^{At}\,x_0$$

$$= \Sigma_{k=0\sim\infty}\,(At)^k/k!\;\;x_0$$

FYI: Exponential operator for matrices

- $x_t = e^{At} x_0$

  $= \Sigma_{k=0\sim\infty} (At)^k / k! \;\; x_0$

  - Similar to the Taylor series expansion of the exponential function:

    $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

  - e^M converges for any square matrix M
  - If M's eigenvalues are $\{\lambda_i\}$, then e^M's eigenvalues are $\{ e^{\lambda_i} \}$, with all eigenvectors unchanged (you can prove this)

Where will the system go eventually?

$$dx/dt = A\,x$$

- What happens if the system starts from non-equilibrium initial states and goes on for a long period of time?
- Let's think about their asymptotic behavior $\lim_{t \to \infty} x(t)$

Considering asymptotic behavior (1)

- Let $\{ v_i \}$ be n linearly independent eigenvectors of the coefficient matrix

  (They might be fewer than n, but here we ignore such cases for simplicity)

- Write the initial condition using eigenvectors, i.e.

  $x_0 = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$

Considering asymptotic behavior (2)

- Then:

$$x_t = e^{At}\, x_0$$
$$= e^{\lambda_1 t}\, b_1 v_1 + e^{\lambda_2 t}\, b_2 v_2 + \ldots + e^{\lambda_n t}\, b_n v_n$$

Dominant eigenvector

- If $Re(\lambda_1) > Re(\lambda_2), Re(\lambda_3), \ldots,$

$$x_t = e^{\lambda_1 t}\{ b_1 v_1 \cancel{+ e^{(\lambda_2-\lambda_1)t}} b_2 v_2 + \ldots \cancel{+ e^{(\lambda_n-\lambda_1)t}} b_n v_n \}$$

$$\lim_{t \to \infty} x_t \sim e^{\lambda_1 t} b_1 v_1$$

If the system has just one such **dominant eigenvector** v_1, its state will be eventually along that vector **regardless of where it starts**

What eigenvalues and eigenvectors can tell us

- An eigenvalue tells whether a particular "state" of the system (specified by its corresponding eigenvectors) grows or shrinks by interactions between parts
  - $|\lambda| > 1$ -> growing
  - $|\lambda| < 1$ -> shrinking

  for discrete-time cases

  - $\mathrm{Re}(\lambda) > 0$ -> growing
  - $\mathrm{Re}(\lambda) < 0$ -> shrinking

  for continuous-time cases

Linear Stability Analysis of Nonlinear Systems

Linearizing continuous-time models

- For continuous-time models:

$dx/dt = F(x)$

$\text{Left} = d(x_e + \Delta x)/dt = d\Delta x/dt$

$\text{Right} = F(x_e + \Delta x)$

$\sim F(x_e) + F'(x_e)\,\Delta x$

$= F'(x_e)\,\Delta x$

Therefore,

$$d\Delta x/dt = F'(x_e)\,\Delta x$$

Review: First-order derivative of vector functions

- **Continuous-time:** $d\Delta x/dt = F'(x_e)\,\Delta x$

  These can hold even if x is a vector

  What corresponds to the first-order derivative in such a case:

$$F'(x_e) = dF/dx_{(x=x_e)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}_{(x=x_e)}$$

Jacobian matrix at $x=x_e$

Eigenvalues of Jacobian matrix

- A Jacobian matrix is a linear approximation around the equilibrium point, telling you the local dynamics: "how a small perturbation will grow, shrink or rotate around that point"
  - The equilibrium point serves as a local origin
  - The Δx serves as a local coordinate
  - Eigenvalue analysis applies