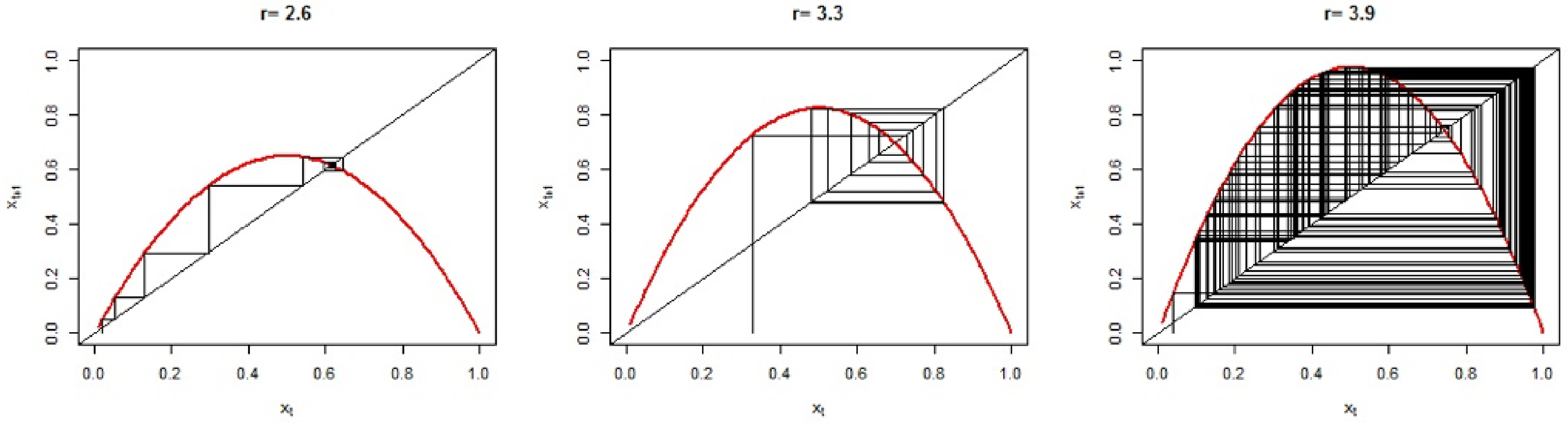
Fundamentals of Dynamical Systems / Discrete-Time Models



Dr. Dylan McNamara
people.uncw.edu/mcnamarad

Dynamical systems theory

- Considers how systems autonomously change along time
  - Ranges from Newtonian mechanics to modern nonlinear dynamics theories
  - Probes underlying dynamical mechanisms, not just static properties of observations
  - Provides a suite of tools useful for studying complex systems

What is a dynamical system?

- A system whose state is uniquely specified by a finite set of variables and whose behavior is uniquely determined by predetermined "rules"
  - Simple population growth
  - Simple pendulum swinging
  - Motion of celestial bodies
  - Behavior of two "rational" agents in a negotiation game

Mathematical formulations of dynamical systems

- **Discrete-time model:** (difference/recurrence equations; iterative maps)

  $$x_t = F(x_{t-1}, t)$$

- **Continuous-time model:** (differential equations)

  $$dx/dt = F(x, t)$$

x_t: State variable(s) of the system at time t

F: Some function that determines the rule that the system's behavior will obey

Discrete-Time Models

Discrete-time model

- Easy to understand, develop and simulate
  - Doesn't require an expression for the rate of change (derivative)
  - Can model abrupt changes and/or chaotic dynamics using fewer variables
  - Directly translatable to simulation in a computer
  - Experimentally, we often have samples of system states at specific points of time

Difference equation and time series

- Difference equation

$$x_t = F(x_{t-1}, t)$$

produces series of values of variable x starting with initial condition x_0:

$$\{ x_0, x_1, x_2, x_3, \dots \} \quad \text{“time series”}$$

  - A prediction made by the above model (to be compared to experimental data)

Linear vs. nonlinear

- **Linear:**
  - Right hand side is just a first-order polynomial of variables

$$x_t = \underline{a\ x_{t-1} + b\ x_{t-2} + c\ x_{t-3}}\ \ldots$$

- **Nonlinear:**
  - Anything else

$$x_t = a\ x_{t-1} + b\ \underline{x_{t-2}^{2}} + c\ \underline{\sqrt{x_{t-1}\ x_{t-3}}}\ \ldots$$

1st-order vs. higher-order

- 1st-order:
  - Right hand side refers only to the immediate past

$$x_t = a \underline{x_{t-1}} (1 - \underline{x_{t-1}})$$

- Higher-order:
  - Anything else

$$x_t = a \underline{x_{t-1}} + b \underline{x_{t-2}} + c \underline{x_{t-3}} \ldots$$

(Note: this is different from the order of terms in polynomials)

Autonomous vs. non-autonomous

- Autonomous:
  - Right hand side includes only state variables (x) and not t itself

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2$$

- Non-autonomous:
  - Right hand side includes terms that explicitly depend on the *value* of t

$$x_t = a\, x_{t-1}\, x_{t-2} + b\, x_{t-3}^2 + \underline{\sin(t)}$$

Things that you should know (1)

- Non-autonomous, higher-order equations can always be converted into autonomous, 1st-order equations
  - $x_{t-2} \rightarrow y_{t-1}$, $y_t = x_{t-1}$
  - $t \rightarrow y_t$, $y_t = y_{t-1} + 1$, $y_0 = 0$
- Autonomous 1st-order equations can cover dynamics of any non-autonomous higher-order equations too!

Things that you should know (2)

- Linear equations
  - are analytically solvable
  - show either equilibrium, exponential growth/decay, periodic oscillation (with >1 variables), or their combination
- Nonlinear equations
  - may show more complex behaviors
  - do not have analytical solutions in general

Simulating Discrete-Time Models

Simulating discrete-time models

- Simulation of a discrete-time model can be implemented by iterating updating of the system's states
  - Every iteration represents one discrete time step

Exercise

- Implement simulators of the following models and produce time series for t = 1~10

$$x_t = 2\,x_{t-1} + 1,\ x_0 = 1$$

$$x_t = x_{t-1}^2 + 1,\ x_0 = 1$$

Exercise

- Simulate the following set of equations and see what happens if the coefficients are varied

$$x_t = 0.5\, x_{t-1} + 1\, y_{t-1}$$
$$y_t = -0.5\, x_{t-1} + 1\, y_{t-1}$$
$$x_0 = 1,\ y_0 = 1$$

Building Your Own Model Equation

Mathematical modeling tips

- Grab an existing model and tweak it
- Implement each assumption one by one
- Find where to change, replace it by a function, and design the function
- Adopt the simplest form
- Check the model with extreme values

Example: Saturation of growth

- Simple exponential growth model:

$$x_t = a\, x_{t-1}$$

- Problem: How can one implement the saturation of growth in this model?

- Think about a new nonlinear model:

$$x_t = f(x_{t-1})\, x_{t-1}$$

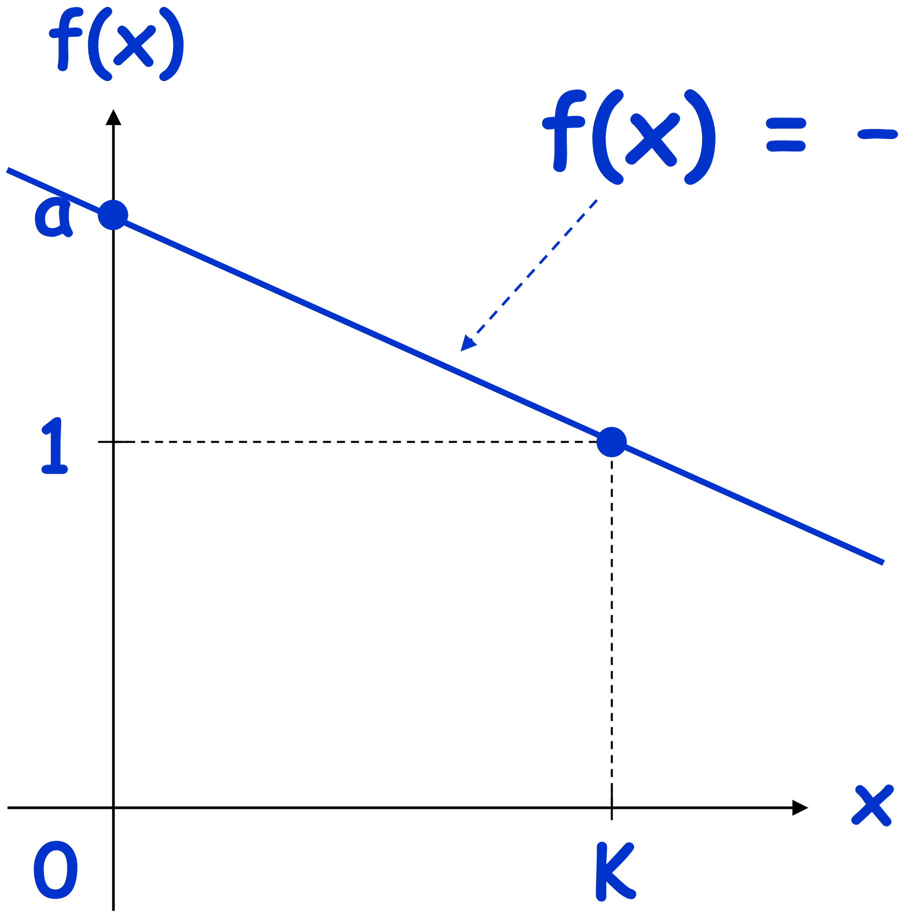
  – Coefficient replaced by a function of x

Modeling saturation of growth

$$x_t = f(x_{t-1})\, x_{t-1}$$

- $f(x)$ should approach 1 (no net growth) when x goes to a carrying capacity of the environment, say K
- $f(x)$ should approach the original growth rate a when x is very small (i.e., with no saturation effect)

What should f(x) be?



$$f(x) = -\frac{a-1}{K}x + a$$

A new model of growth

$$x_t = f(x_{t-1})\, x_{t-1}$$

$$= (-(a-1)\, x_{t-1} / K + a)\, x_{t-1}$$

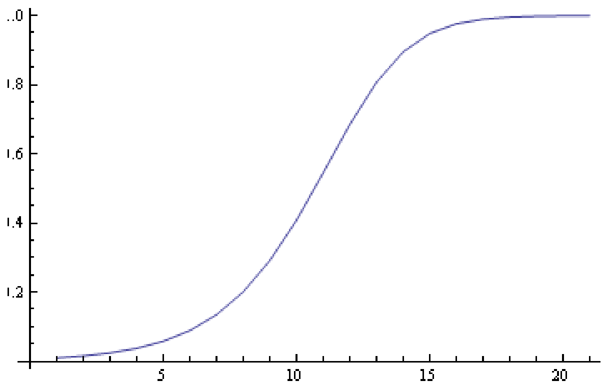
- Using $r = a - 1$:

$$x_t = (-r\, x_{t-1} / K + r + 1)\, x_{t-1}$$

$$= x_{t-1} + \underbrace{r\, x_{t-1} (1 - x_{t-1} / K)}_{\text{Net growth}}$$

Example: Logistic growth model

- N: Population
- r: Population growth rate
- K: Carrying capacity



- Discrete-time version:

$$N_t = N_{t-1} + r\, N_{t-1} \left(1 - N_{t-1}/K \right)$$

- Continuous-time version:

$$dN/dt = r\, N \left(1 - N/K \right)$$

Nonlinear terms

Modeling with multiple variables

- Problem: Develop a nonlinear model of a simple ecosystem made of predator and prey populations

Think about how variables behave in isolation

Naturally grows to carrying capacity if isolated

Naturally decays if isolated

Initial assumptions

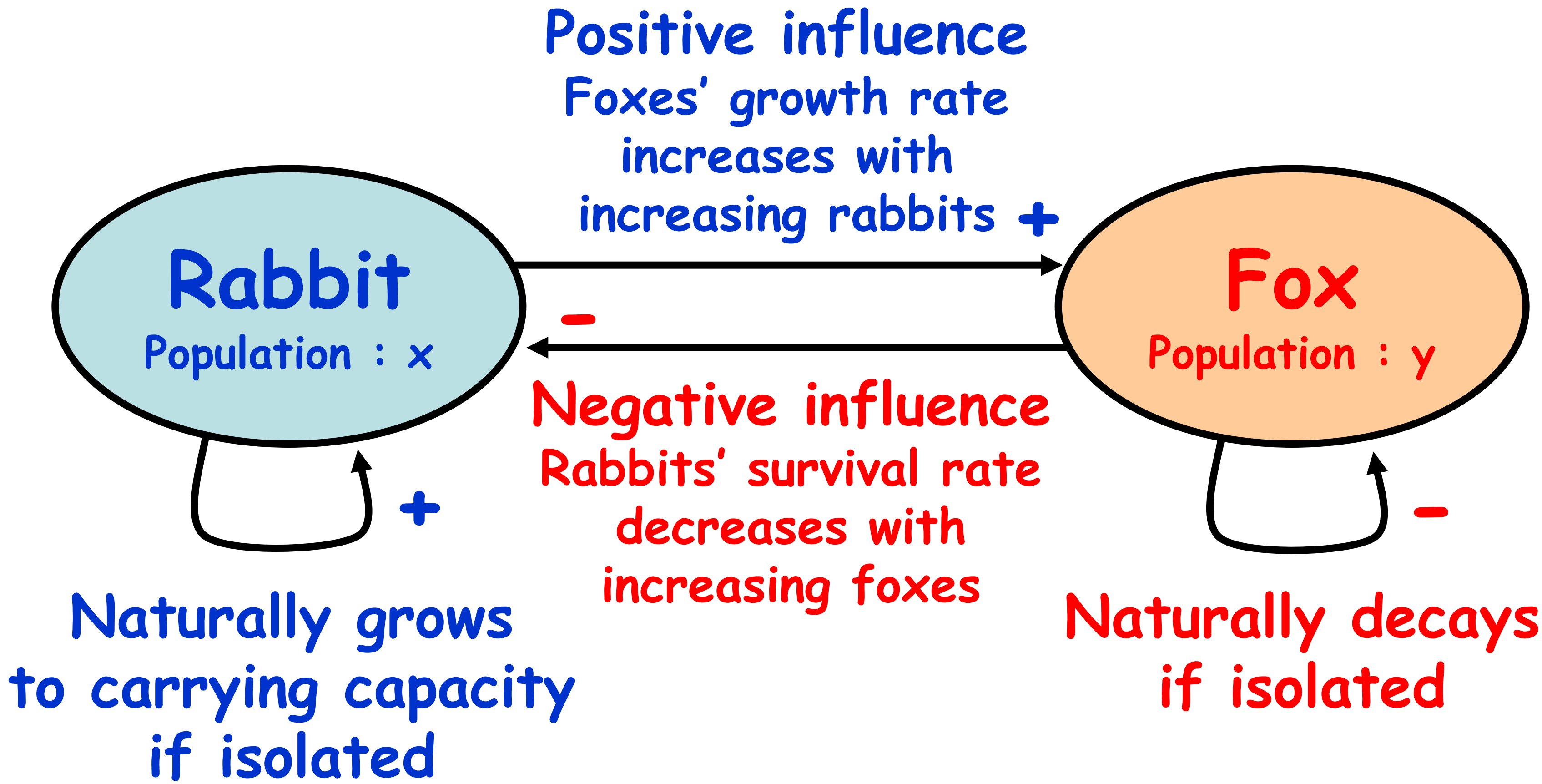
- Rabbits will grow based on the logistic growth model, with carrying capacity = 1 for simplicity
- Foxes will decay exponentially

Rabbit: $x_t = x_{t-1} + a\, x_{t-1} (1 - x_{t-1})$

Fox: $y_t = b\, y_{t-1}$

$(0<a,\ 0<b<1)$

Think about how variables interact with each other



Revised model

- Introduced coefficient $(1 - c\, y_{t-1})$ $(0<c)$ to the first term of x
  - Negative influence of foxes on rabbits' survival rate
- Replaced b with $(b + d\, x_{t-1})$ $(0<d)$
  - Positive influence of rabbits on foxes' growth rate

Rabbit: $x_t = (1 - c\, y_{t-1})\, x_{t-1}$
$\qquad + a\, x_{t-1}\,(1 - x_{t-1})$

Fox: $y_t = (b + d\, x_{t-1})\, y_{t-1}$

$(0<a,\ 0<b<1,\ 0<c,\ 0<d)$

FYI: Lotka-Volterra model

- This model can be rewritten as:

$$x_t - x_{t-1} = \alpha\, x_{t-1}\,(1 - x_{t-1}) - \beta\, x_{t-1}\, y_{t-1}$$

$$y_t - y_{t-1} = -\,\gamma\, y_{t-1} + \delta\, x_{t-1}\, y_{t-1}$$

  - Known as the “Lotka-Volterra” equations (of discrete-time version with carrying capacity)
  - Models predator-prey dynamics in a general form
  - One of the most famous nonlinear systems with multiple variables

Analysis of Discrete-Time Models

Equilibrium point

- A state of the system at which state will not change over time
  - A.k.a. fixed point, steady state
- Can be calculated by solving

$$x_t = x_{t-1}$$

Exercise

- Calculate equilibrium points in the following models

$$N_t = N_{t-1} + r\,N_{t-1}\,(\,1 - N_{t-1}/K\,)$$

$$x_t = 2x_{t-1} - x_{t-1}^{\,2}$$

$$x_t = x_{t-1} - x_{t-2}^{\,2} + 1$$

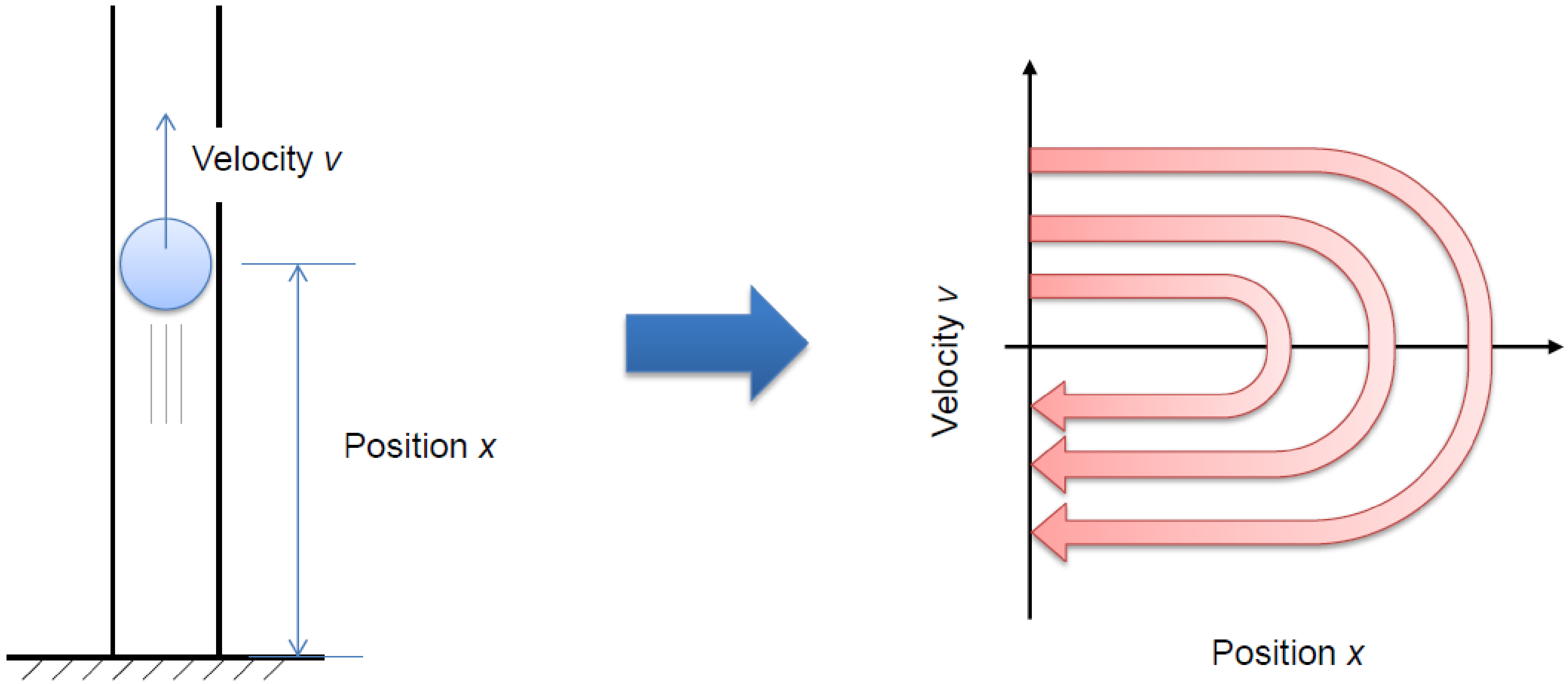
Phase Space Visualization

Geometrical approach

- Developed in the late 19C by J. Henri Poincare
- Visualizes the behavior of dynamical systems as trajectories in a phase space
- Produces a lot of intuitive insights on geometrical structure of dynamics that would be hard to infer using purely algebraic methods

Phase space (state space)

- A theoretical space in which every state of a dynamical system is mapped to a spatial location



Phase space (state space)

- Created by “orthogonalizing” state variables of the system
- Its dimensionality equals # of variables needed to specify the system state (a.k.a. degrees of freedom)
- Temporal change of the system states can be drawn in it as a trajectory

Attractor and basin of attraction

- **Attractor:**

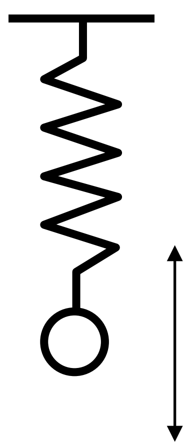
  A state (or a set of states) from which no outgoing edges or flows running in phase space
  - Static attractors (equilibrium points)
  - Dynamic attractors (*e.g.* limit cycles)

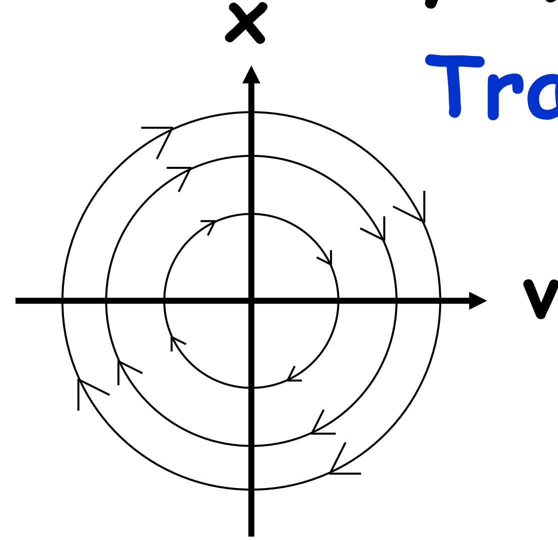
- **Basin of attraction:**

  A set of states which will eventually end up in a given attractor

Phase space of continuous-state models

- E.g. a simple vertical spring oscillator
- State can be specified by two real variables (location x, velocity v)





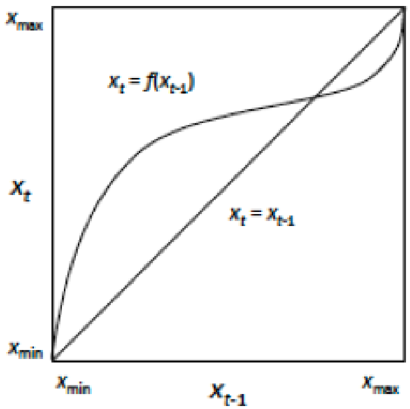
Trajectory (orbit)

Dynamics of continuous models can be depicted as "flow" in a continuous phase space

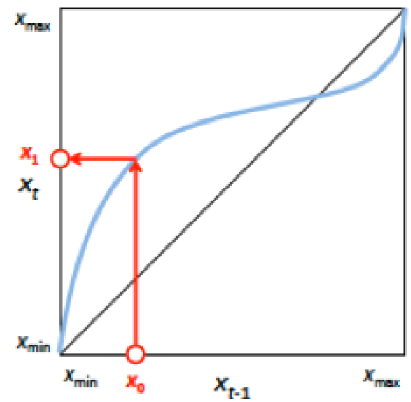
Cobweb plot

- A visual tool to study the behavior of 1-D iterative maps
- Take x_{t-1} and x_t for two axes
- Draw the map of interest ($x_t=F(x_{t-1})$) and the "$x_t=x_{t-1}$" reference line
  - They will intersect at "equilibrium points"
- Trace how time series develop from an initial value by jumping between these two curves

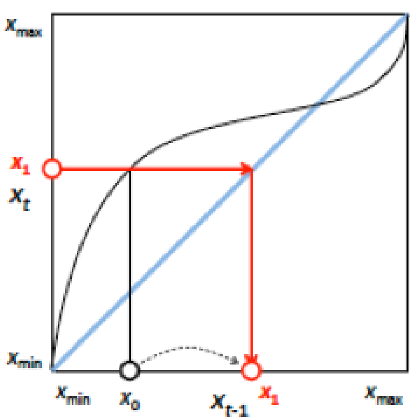
Cobweb Plot



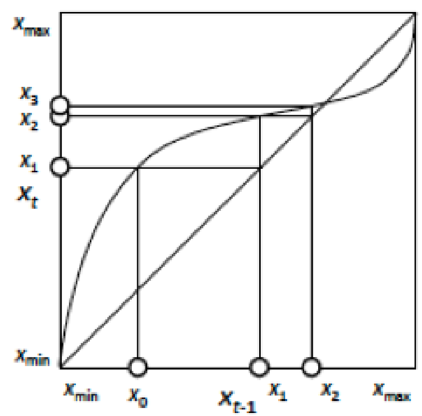
x_{max}
$x_t = f(x_{t-1})$
x_t
$x_t = x_{t-1}$
x_{min}
x_{min}
x_{t-1}
x_{max}



x_{max}
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_{max}

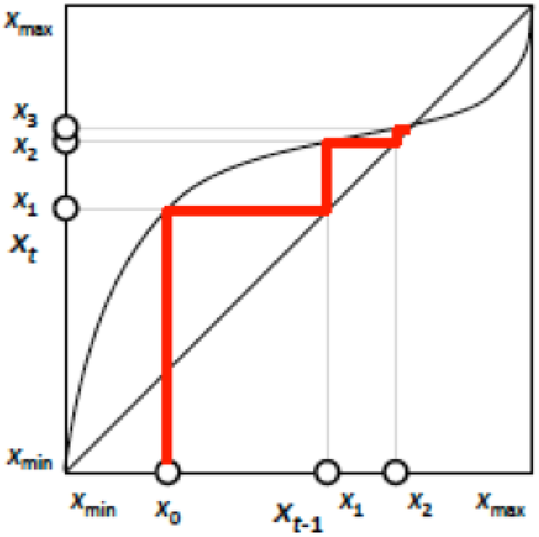


x_{max}
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_1
x_{max}



x_{max}
x_3
x_2
x_1
x_t
x_{min}
x_{min}
x_0
x_{t-1}
x_1
x_2
x_{max}

Cobweb Plot



Rescaling Variables

Rescaling variables

- Dynamics of a system won't change qualitatively by linear rescaling of variables (e.g., $x \rightarrow \alpha\, x'$)
- You can set arbitrary rescaling factors for variables to simplify the model equations

- If you have k variables, you may eliminate k parameters

Exercise

- Simplify the logistic growth model by rescaling $x \rightarrow \alpha\, x'$

$$x_t = x_{t-1} + r\, x_{t-1}\, (1 - x_{t-1}/K)$$

Linear Systems

Dynamics of linear systems

- Some systems can be modeled as linear systems
  - Their dynamics is described by a product of matrix and state vector
  - Either in continuous or discrete time

- Dynamics of such linear systems can be studied analytically

Linear systems

- Linear systems are the simplest cases where states of nodes are continuous-valued and their dynamics are described by a time-invariant matrix
- Discrete-time: $x_t = A\, x_{t-1}$
  - A is called a "coefficient" matrix
  - We don't consider constants (as they can be easily converted to the above forms)

Asymptotic Behavior of Linear Systems

Where will the system go eventually?

$$x_t = A\, x_{t-1}$$

This equation gives the following exact solution:

$$x_t = A^t\, x_0$$

Where will the system go eventually?

$$x_t = A\, x_{t-1}$$

- What happens if the system starts from non-equilibrium initial states and goes on for a long period of time?
- Let's think about their asymptotic behavior $\lim_{t \to \infty} x_t$

Considering asymptotic behavior (1)

- Let { v_i } be n linearly independent eigenvectors of the coefficient matrix

  (They might be fewer than n, but here we ignore such cases for simplicity)

- Write the initial condition using eigenvectors, i.e.

  $x_0 = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$

Considering asymptotic behavior (2)

- Then:

$$x_t = A^t x_0$$

$$= \lambda_1^t b_1 v_1 + \lambda_2^t b_2 v_2 + \dots + \lambda_n^t b_n v_n$$

Dominant eigenvector

- If $|\lambda_1| > |\lambda_2|, |\lambda_3|, \ldots,$

  $x_t = \lambda_1^t \{ b_1 v_1 + \cancel{(\lambda_2/\lambda_1)^t b_2 v_2} + \ldots + \cancel{(\lambda_n/\lambda_1)^t b_n v_n} \}$

  $\lim_{t \to \infty} x_t \sim \lambda_1^t b_1 v_1$

If the system has just one such dominant eigenvector v_1, its state will be eventually along that vector regardless of where it starts

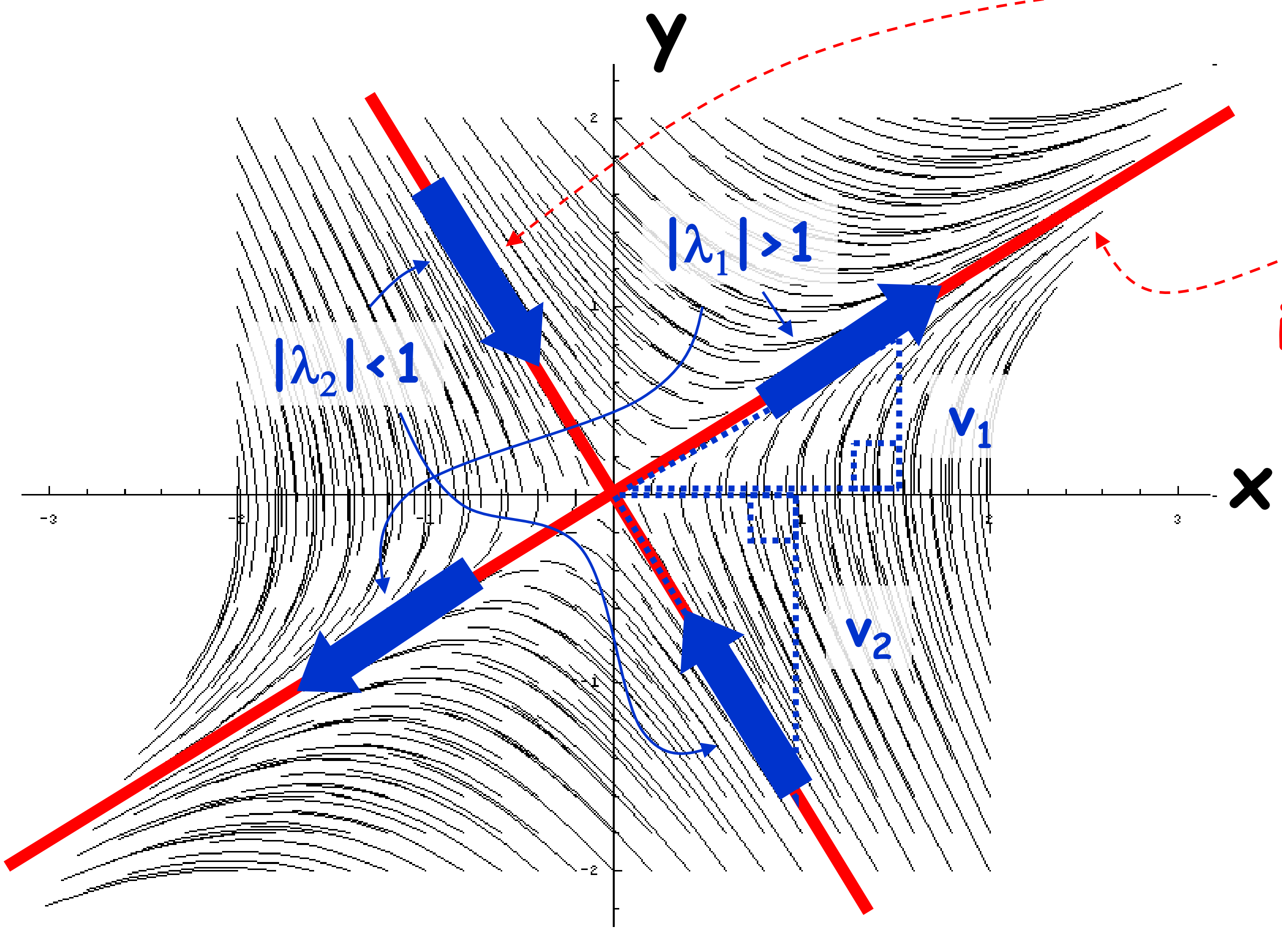
What eigenvalues and eigenvectors can tell us

- An eigenvalue tells whether a particular "state" of the system (specified by its corresponding eigenvectors) grows or shrinks by interactions between parts

  - $|\lambda| > 1$ -> growing
  - $|\lambda| < 1$ -> shrinking

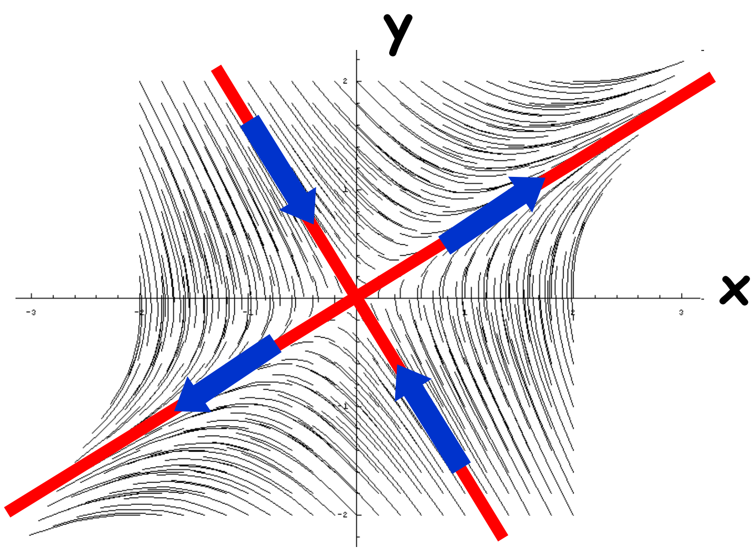
Example

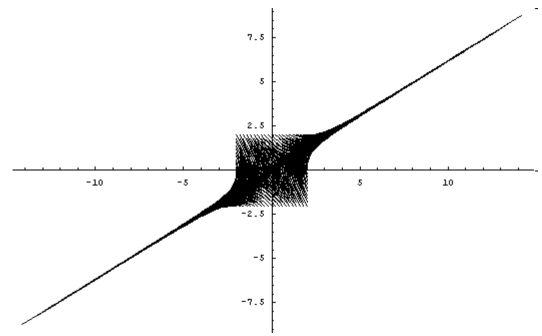
- Phase space of a two-variable linear difference equation with (a, b, c, d) = (1, 0.1, 0.1, 0.9)

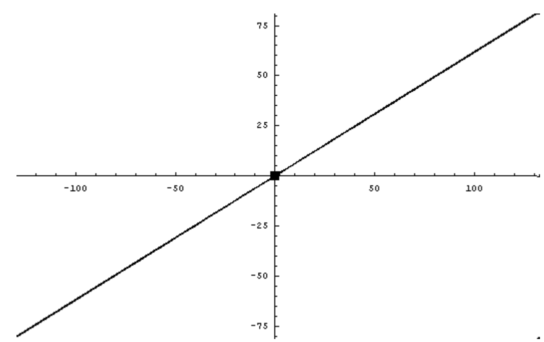


Along these lines (called invariant lines), the dynamics can be understood as simple exponential growth/decay

Example







This could be regarded as a very simple form of self-organization (though completely predictable);

Order spontaneously emerges in the system as time goes on

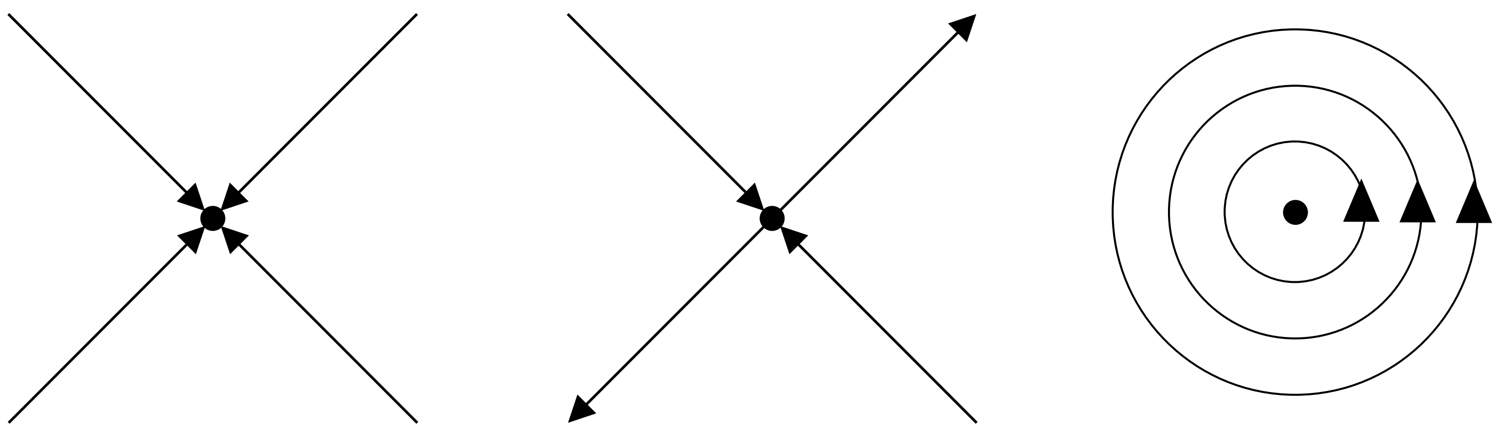
Linear Stability Analysis of Nonlinear Systems

Stability of equilibrium points

- If a system at its equilibrium point is slightly perturbed, what happens?
- The equilibrium point is called:
  - Stable (or asymptotically stable) if the system eventually falls back to the equilibrium point
  - Lyapunov stable if the system doesn't go far away from the equilibrium point
  - Unstable otherwise

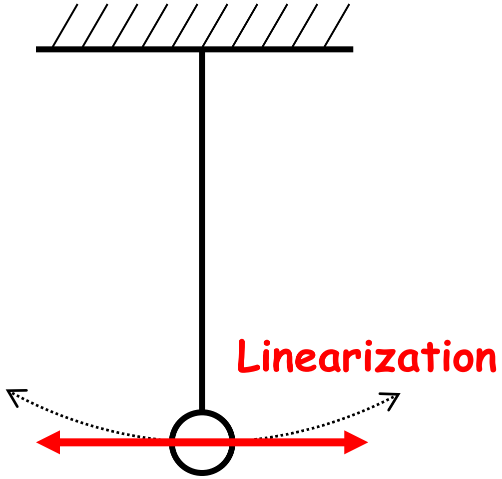
Question

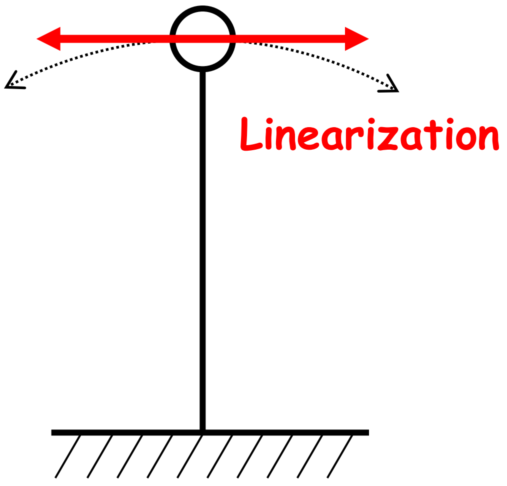
- What is the stability of each of the following equilibrium points?



Linear stability analysis

- Studies whether a nonlinear system is stable or not at its equilibrium point by locally linearizing its dynamics around that point





Local linearization (1)

- Let Δx be a small difference between the system's current state x and its equilibrium point x_e, i.e. $x = x_e + \Delta x$
- Plug $x = x_e + \Delta x$ into differential equations and ignore quadratic or higher-order terms of Δx (hence the name "linearization")

Local linearization (2)

- This operation does the trick to convert the dynamics of Δx into a product of a matrix and Δx
- By analyzing eigenvalues of the matrix, one can predict whether x_e is stable or not
  - I.e. whether a small perturbation (Δx) grows or shrinks over time

Mathematically speaking…

- This operation is similar to "linear approximation" in calculus

  Taylor series expansion:

  $$F(x) = \Sigma_{n=0\sim\infty} F^{(n)}(a)/n!\ (x-a)^n$$

  Let $x \to x_e+\Delta x$ and $a \to x_e$, then

  $$F(x_e+\Delta x) = F(x_e) + F'(x_e)\,\Delta x \cancel{+ O(\Delta x^2)}$$

  Ignore

Linearizing discrete-time models

- For discrete-time models:

$$x_t = F(x_{t-1})$$

$$\text{Left} = x_e + \Delta x_t$$

$$\text{Right} = F(x_e + \Delta x_{t-1})$$

$$\sim F(x_e) + F'(x_e)\,\Delta x_{t-1}$$

$$= x_e + F'(x_e)\,\Delta x_{t-1}$$

Therefore,

$$\Delta x_t = F'(x_e)\,\Delta x_{t-1}$$

First-order derivative of vector functions

- Discrete-time: $\Delta x_t = F'(x_e)\,\Delta x_{t-1}$

  This can hold even if x is a vector

  What corresponds to the first-order derivative in such a case:

$$F'(x_e) = dF/dx_{(x=x_e)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}_{(x=x_e)}$$

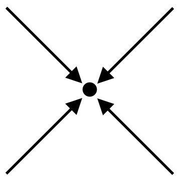
Jacobian matrix at $x=x_e$

Eigenvalues of Jacobian matrix

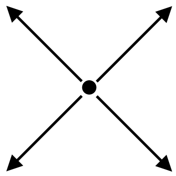
- A Jacobian matrix is a linear approximation around the equilibrium point, telling you the local dynamics: "how a small perturbation will grow, shrink or rotate around that point"
  - The equilibrium point serves as a local origin
  - The Δx serves as a local coordinate
  - Eigenvalue analysis applies

With real eigenvalues

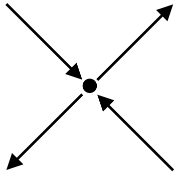
- If all the eigenvalues indicate that Δx will shrink over time
  -> stable point



- If all the eigenvalues indicate that Δx will grow over time
  -> unstable point



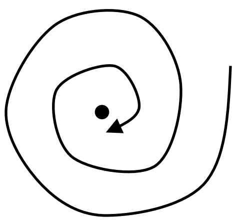
- If some eigenvalues indicate shrink and others indicate grow of Δx over time
  -> saddle point (this is also unstable)



With two complex conjugate eigenvalues (for 2-D systems)

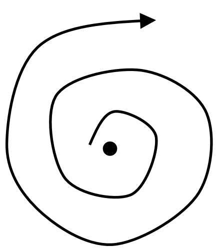
- If both eigenvalues indicate that Δx will shrink over time

  -> stable spiral focus



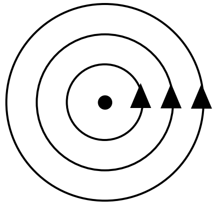
- If both eigenvalues indicate that Δx will grow over time

  -> unstable spiral focus



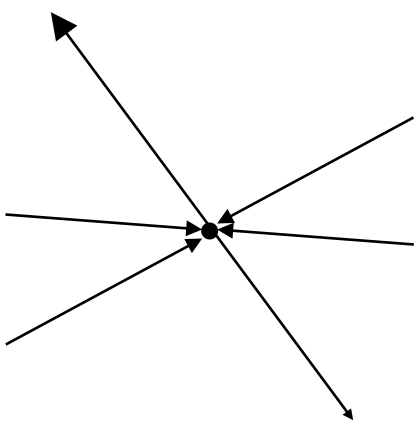
- If both eigenvalues indicate neither shrink nor growth of Δx

  -> neutral center (but this may or may not be true for nonlinear models; further analysis is needed to check if nearby trajectories are truly cycles or not)

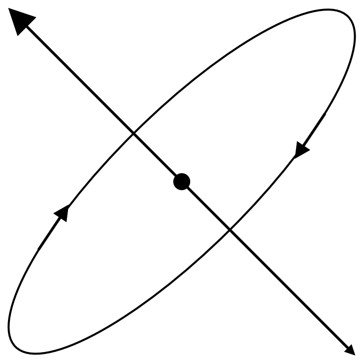


With real and complex eigenvalues mixed (for higher-dimm. systems)

- Each eigenvalue (or a pair of complex conjugate eigenvalues) tell you distinct dynamics simultaneously seen at the equilibrium point:



All real eigenvalues (1 indicates growth; other 2 indicates shrink)



1 real eigenvalue indicates growth; other 2 indicates rotation (complex conjugates with no growth or shrink)

Exercise

- Find all equilibrium points of the following model, and study their stability

$$x_t = x_{t-1}\, y_{t-1}$$
$$y_t = y_{t-1}\,(x_{t-1} - 1)$$