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CS 498 HW 9.1

10.1 $f: \mathbb{R}^d \rightarrow \mathbb{R}$ by $f(\vec{x}) = \vec{a}^T \vec{x}$

$\{\vec{x}\} = \{\vec{x}_1, \vec{x}_2, \dots \vec{x}_N\}$
$\{f\} = \{f(\vec{x}_1), f(\vec{x}_2), \dots f(\vec{x}_N)\}$

a.) $$\text{mean}(\{f\}) = \frac{1}{N} \sum_i f(\vec{x}_i)$$

$$= \frac{1}{N} \sum_i \vec{a}^T \vec{x}_i = \vec{a}^T \underbrace{\left(\frac{1}{N} \sum \vec{x}_i\right)}$$

$$= a^T \text{mean}(\{\vec{x}\})$$

b.) $$\text{Covmat}(\{\vec{x}\}) = \frac{1}{N} \sum_i (\vec{x}_i - \text{mean}(\{\vec{x}\})) \cdot (\vec{x}_i - \text{mean}(\{x\}))^T$$
$$\in \mathbb{R}^{d \times d}$$

$$\text{var}(\{f\}) = \text{var}(\{f(\vec{x}_1), \dots f(\vec{x}_N)\})$$

$$= \frac{1}{N} \sum_i (\vec{a}^T \vec{x}_i - \vec{a}\,\text{mean}(\{\vec{x}\}))^2$$

$$= \frac{1}{N} \sum_i (a^T(\vec{x}_i - \text{mean}(\{x\})))^2$$

$$= \frac{1}{N} \sum_i \vec{a}^T(\vec{x}_i - \text{mean}(\{\vec{x}\}))(\vec{a}^T(\vec{x}_i - \text{mean}(\{\vec{x}\})))^T$$

$$= \frac{1}{N} \sum_i \vec{a}^T(\vec{x}_i - \text{mean}(\{\vec{x}\}))(\vec{x}_i - \text{mean}(\{\vec{x}\}))^T \vec{a}$$

$$= \vec{a}^T \text{Covmat}(\{\vec{x}\}) \vec{a}$$

10.1 c.) Let $\vec{a}^T \text{Covmat}(\{\vec{x}\})\vec{a} = \emptyset$
which means $\text{var}(\{\vec{x}\}) = \emptyset$

So,
$$\frac{1}{N}\sum_i \left(\vec{a}^T \vec{x}_i - \vec{a}^T \text{mean}(\{\vec{x}\})\right)^2 = \emptyset$$

Which implies $\vec{a}^T \vec{x}_i - a^T \text{mean}(\{\vec{x}\}) = \emptyset \;\forall i \in \mathbb{R}$

Hence, each $\vec{x}_i$ lies on a hyperplane with normal $\vec{a}$ and intercept $-\vec{a}^T \text{mean}(\{\vec{x}\})$

10.2 (Drawn of figure 10.31 below)