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CS 498 HW 9.2

10.3 a) Let Σ represent $\text{covmat}(\{\vec{x}\})$
$\Rightarrow \vec{v} = \vec{x}_i - \text{mean}(\{\vec{x}\})$
And let $U = [\vec{v}_1 \dots \vec{v}_n]$ (from pg. 305)

So, $\Sigma = \frac{1}{N} U U^T$

Let $\mathcal{U}$ be the eigenvalue of Σ

$$\mathcal{U} = [\vec{E}_1 \dots \vec{E}_p]$$

And $P = \mathcal{U}^T U$ (projected data onto eigenvectors)
So,
$$\text{covmat}(\{\vec{p}\}) = \frac{1}{N} P P^T = \Lambda \quad (\vec{p}_i = \vec{u}_i \cdot \vec{v}_i)$$

As Σ has one eigenvalue, $\Lambda \in \mathbb{R}^{1 \times 1} \Rightarrow \Lambda = \lambda_1$
We also know $\vec{p}_i \in \mathbb{R}^{1 \times 1}$, so $\vec{p}_i = p_i$
Thus, $p_i = p_1 + t_i (p_2 - p_1)$ for some t

Since $p_i = u^T \vec{v}_i$, and $\vec{v}_i = \vec{x}_i - \text{mean}(\{\vec{x}_i\})$,

$$\vec{u}^T \vec{v}_i = \vec{u}^T \vec{v}_1 + t_i (\vec{u}^T \vec{v}_2 - \vec{u}^T \vec{v}_1)$$
$$\vec{u}^T \vec{v}_i = \vec{u}^T (\vec{v}_1 + t_i (\vec{v}_2 - \vec{v}_1))$$
$$\vec{v}_i = \vec{v}_1 + t_i (\vec{v}_2 - \vec{v}_1)$$

And as $\vec{x}_i - \text{mean} = (\vec{x}_1 - \text{mean}) + t_i (\vec{x}_2 - \text{mean})$

So, $x_i = x_1 + t_i (x_2 - x_1)$

$\square$

(continued)

10.3 b) We know that $t_i = \dfrac{p_i - p_1}{p_2 - p_1}$ (from part a)

We want $\text{std}(t_i) = \text{std}\left(\dfrac{p_i - p_1}{p_2 - p_1}\right)$

$\text{std}(t_i) \Rightarrow \dfrac{1}{p_2 - p_1} \text{std}(p_i)$

$$\text{Var}(t_i) = \frac{1}{(p_2 - p_1)^2} \text{Var}(p_i)$$

We also know (from def.):

$$\text{covmat}(\{p_i\}) = \frac{1}{N} p \cdot p^T = \Lambda \in \mathbb{R}^{p \times p}$$

And since we know that there is 1 eigenvalue,

$$\text{covmat}(\{p_i\}) = \lambda_1$$

So,

$$\text{Var}(t_i) = \frac{1}{(p_2 - p_1)^2} (\lambda_1)$$

and,

$$\text{std}(t_i) = \frac{1}{|p_2 - p_1|} \cdot \sqrt{\lambda_1}$$

$\square$