FOCS Notes
2/11/2020

Lecture 9: Sums and Asymptotics

- Maximum Substring Sum Problem
- Computing Sums
- Asymptotics: Big-Theta, Big-Oh, and Big-Omega
- Integration Method

Contents

1 Maximum Substring Sum Problem

1 -1 -1 2 3 4 -1 -1 2 3 -4 1 2
Find largest continuous subsequence sum.

$T_1(n) = 2 + \sum_{i=1}^{n}(2 + \sum_{j=i}^{n}(5 + \sum_{k=i}^{j} 2))$ (3 for loops)

$T_2 = 2 + \sum_{i=1}^{n}(3 + \sum_{j=i}^{n} 6)$ (2 for loops)

$$T_3(n) = \begin{cases} 3 & n = 1 \\ 2T_3(\frac{n}{2}) + 6n + 9 & n > 1 \textit{ and even} \\ T(\frac{n+1}{2}) + T(\frac{n-1}{2}) + 6n + 9 & n > 1 \textit{ and odd} \end{cases}$$

$T_4 = 5 + \sum_{i=1}^{n} 10$ (1 for loop)

Q: Which solution is best?

n	1	2	3	4	5
$T_1(n)$	11	29	58	100	157
$T_2(n)$	11	26	47	74	107
$T_3(n)$	3	27	57	87	123
$T_4(n)$	15	25	35	45	55

We need:

- simple formulas for T_1, T_2, T_3, T_4
- A way to compare runtimes that capttures the essence of the algorithm

2 Computing Sums

$S_1 = \sum_{i=1}^{10} 3 = 3 + 3 + 3 + \cdots + 3 = 3 \sum_{i=1}^{10} 1 = 3 * 10 = 30$
$S_2 = \sum_{i=1}^{10} j = j \sum_{i=1}^{10} 1 = 10j$
$S_3 = \sum_{i=1}^{10} i = \frac{10*11}{2} = 55$

Computing Sums:
Rule 1: Pull out constants
Rule 2: Addition Rule
$\sum_{i=1}^{n} (a(i) + b(i) + c(i)) = \sum a(i) + \sum b(i) + \sum c(i)$
Rule 3: Common Sums

- $\sum_{i=k}^{n} 1 = n - k + 1$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
- $\sum_{i=1}^{n} f(x) = nf(x)$
- $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$
  $x = \sum_{i=0}^{n} \frac{1}{2^i} \to 2^n x = \sum_{i=0}^{n} 2^{n-i} = (2^n + 2^{n-1} + \cdots + 2^0) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \Rightarrow x = 2 - \frac{1}{2^n}$
- $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}, r \neq 1$
- $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$
- $\sum_{i=1}^{n} ln(i) = ln(n!)$
  $\sum_{i=1}^{n} ln(i)$
  $exp(x) = e^{ln1} * e^{ln2} * \cdots * e^{ln(x)}$

Ex:

$$\sum_{i=1}^{n} (1 + 2i + 2^{i+2})$$
$$= \sum 1 + \sum 2i + \sum 2^{i+2} \longleftarrow \text{ Addition Rule}$$
$$= n + 2\sum i + 4\sum 2^i \longleftarrow \text{ common sum, constants}$$
$$= n + \frac{2n(n+1)}{2} + 4(\sum 2^{n+1} - 1)$$
$$= n + n(n+1) + 4(2^{n+1} - 2)$$

Nested Sum Rule

$S_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} 1$

$S_2 = \sum_{i=1}^{3}\sum_{j=1}^{i} 1$

Start with the innermost sum and proceed outwards

$S_1 = \sum_{i=1}^{3} 3 = 3\sum_{i=1}^{3} 1 = 9$

$S_2 = \sum_{i=1}^{3} i = \frac{3*4}{2} = 6$

2.1 Computing T_1, T_2, T_4

$T_1(n) = 2 + \sum_{i=1}^{n}[2 + \sum_{j=i}^{n}(5 + \sum_{k=i}^{j} 2)]$

$= 2 + \sum_{i=1}^{n} 2 + \sum_{i=1}^{n}\sum_{j=i}^{n}(5 + \sum_{k=i}^{j} 2)$ (addition)

$= 2 + \sum_{i=1}^{n} 2 + \sum_{i=1}^{n}\sum_{j=i}^{n} 5 + \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 2$ (addition)

$= 2 + \sum_{i=1}^{n} 2 + 5\sum_{i=1}^{n}\sum_{j=i}^{n} 1 + 2\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1$ (constant)

$= 2 + 2n + 5\sum_{i=1}^{n}(n-i+1) + 2\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1$ (common sum)

NOTE $\sum_{i=1}^{n}(n-i+1) = \sum_{i=1}^{n} i$

$= 2 + 2n + \frac{5}{2}n(n+1) + 2\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1$ (common sum)

$= 2 + 2n + \frac{5}{2}n(n+1) + 2\sum_{i=1}^{n}\sum_{j=i}^{n}(j-i+1)$ (common sum)

$= 2 + 2n + \frac{5}{2}n(n+1) + 2\sum_{i=1}^{n}(\sum_{j=i}^{n} j - \sum_{j=i}^{n} i + \sum_{j=i}^{n} 1)$ (addition)

$= 2 + 2n + \frac{5}{2}n(n+1) + 2\sum_{i=1}^{n}(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j - i\sum_{j=i}^{n} 1 + (n-i+1))$

$= 2 + 2n + \frac{5}{2}n(n+1) + 2\sum_{i=1}^{n}(\frac{n(n+1)}{2} - \frac{(i-1)i}{2} - i(n-i+1) + (n-i+1))$

Ex

$\sum_{i=1}^{n}\sum_{j=1}^{i} ij = \sum_{i=1}^{n} i\sum_{j=1}^{i} j = \sum_{i=1}^{n} i(\frac{i(i+1)}{2})$

$= \sum_{i=1}^{n}\frac{i^3+i^2}{2} = \frac{1}{2}(\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2)$

$= \frac{1}{2}(\frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1))$

$T_1(n) = 2 + \frac{31}{6}n + \frac{7}{2}n^2 + \frac{1}{3}n^3$

$T_2(n) = 2 + 6n + 3n^2$

$3n(log_2 n + 1) - 9 \leq T_3(n) \leq 12n(log_2 n + 3) - 9$

$T_4(n) = 5 + 10n$

3 Asymptotically Linear Functions (Big-Theta of n)

$f \in \Theta(n) \iff cn \leq \frac{f(n)}{n} \leq Cn$ for some constants c, C
Claim: $T_4(n) = 5 + 10n \in \Theta(n)$
$c = 10$ because $10n < T_4(n)$
$C = 15$ because $T_4(n) = 5 + 10n \leq 5n + 10n = 15n$

Other examples of asymptotically linear:
$2n + 7$, $2n + 15\sqrt{n}$, $10^9 n + 3$, $3n + log(n)$
Not $\Theta(n)$:
$10^{-9}n^2$, $10^9\sqrt{n} + 15$, $n^{1+\epsilon}$, $\frac{n}{log(n)}$, $nlog(n)$

4 General Asymptotics $\Theta(f)$, big-Theta of f

$$\frac{T(n)}{f(n)} \rightarrow_{n \rightarrow \inf} \begin{cases} \text{inf} & t \in \omega(f), \text{, "T¿f"} \\ \text{constant} & T \in \Theta(f), \text{ "T=f"} \\ 0 & T \in o(f), \text{ "T¡f"} \end{cases}$$

5 Integration Method

Given: $\sum\limits_{i=m}^{n} f(i) \longleftarrow$ monotonic increasing

Idea: think of this sum as a Riemannian sum to get a relationship to $\int\limits_{m-1}^{n} f(x)dx$ and $\int\limits_{m}^{n+1} f(x)dx$

$\sum\limits_{i=m}^{n} f(i) \geq \int\limits_{m-1}^{n} f(x)dx$
$\sum\limits_{i=m}^{n} f(i) \leq \int\limits_{m}^{n+1} f(x)dx$

Ex: (Bad version of Stirling's approximation)
$ln(n!) = \sum\limits_{i=1}^{n} ln(i) \leq \int\limits_{1}^{n+1} ln(x)dx = (n+1)ln(n+1) - (n+1) - (1 * ln(1) - 1) = (n+1)ln(n+1) - n$
$n! \leq e^{(n+1)ln(n+1)}e^{-n} = (n+1)^{n+1}e^{-n}$