FOCS Notes
1/31/2020

Stronger forms of induction

- "Leaping" induction (multiple base cases)
- Strong induction

Contents

1 Recap

Given $P(n)$, goal is to show $P(n) \to \forall n \geq b, P(b) \to P(b+1) \to P(b+2) \to \dots$ via principle of induction.
$P(b)$ is the base case, $P(b) \to P(b+1)$ justified by induction step.

Template for Inductive Proofs

- We prove by induction $P(n), n \geq b)$
- Base case: Show that $P(b)$ is true
- Inductive step: (to prove $P(n) \to P(n+1), \forall n \geq b)$
  We state our method of proof
  Use the fact that $P(n)$ holds to get that $P(n+1)$ holds
- Conclude by the principle of induction, that $P(n), \forall n \geq b$

2 Leaping Induction

$P(1) \to P(3)$, $P(2) \to P(4)$, $\dots$
Multiple base cases ($P(1)$ and $P(2)$ above)
$P(n) \to P(n+2)$ induction step above

3 Strong Induction

$P(1) \to P(2)$, $(P(1) \land P(2)) \to P(3)$, $\dots$
$P(1)$ is the base case
$P(1) \land P(2) \land \dots \land P(n) \to P(n+1)$ inductive step.

4 Examples

Outline

- Example of weak induction using a lemma
- Weak induction with a strengthened hypothesis
  - $n^2 < 2^n$ when $n \geq 4$
  - tiling dyadic floors with triominoes
- Two more flavors of induction
  - Leaping: $n^3 < 2^n$ for $n \geq 10$
  - Strong induction:
    - Fundamental Theorem of Arithmetic
    - Nim game

4.1 $\forall n \geq 1 : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

$P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
Base case $P(1) : 1 \leq 2$ true
Inductive step show $P(n) \to P(n+1)$
Assume $P(n)$, so $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$ and show $P(n+1) : \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq \underline{2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}}$
Is the underlined section true? Show $P(n+1) : \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$. I can use the assumption that $P(n)$ holds to conclude that $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$.

Lemma: If $n \geq 1$ then $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$.

Proof. Direct proof. $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \iff 2\sqrt{n(n+1)} + 1 \leq 2(n+1) \iff 2\sqrt{n(n+1)} \leq 2n+1 \iff 4n(n+1) \leq (2n+1)^2 \iff 4n^2 + 4n \leq 4n^2 + 4n + 1$ True $\square$

4.2 Proving claims by strengthening our inductive hypothesis

Ex: $n^2 \leq 2^n$ for $n \geq 4$
Base case $P(4) : 16 \leq 16$ True
Inductive step $P(n) \to P(n+1)$ via direct proof
Assume $P(n)$, namely $n^2 \leq 2^n$. We want to show $P(n+1) : (n+1)^2 \leq 2^{n+1}$
Calculate: $(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1$. Want to prove that is $\leq 2^{n+1} = 2^n + 2^n$
so true if $2n + 1 \leq 2^n$

We wanted to prove $P(n) : n^2 \leq 2^n$ for $n \geq 4$
We see it's useful to prove the stronger claim
$Q(n)$:

(i) $n^2 \leq 2^n$

(ii) $2n + 1 \leq 2^n$

for $n \geq 4$

Proof. We prove by induction that $Q(n), \forall n \geq 4$.
Base step $Q(4) : (i) 4^2 \leq 2^4$ True, $(ii) 2 * 4 + 1 \leq 2^4$ True
Inductive step We use direct proof (for both parts of $Q(n+1)$)
Assume $Q(n) : (i) n^2 \leq 2^n \ (ii) 2n + 1 \leq 2^n$. We need to show $Q(n+1) : (i)(n+1)^2 \leq 2^{n+1} \ (ii) 2(n+1) + 1 \leq 2^{n+1}$
(i)
$(n+1)^2 = n^2 + (2n+1) \leq 2^n + (2n+1)$ Using $Q(n)$ part (i) $\leq 2^n + 2^n$ using $Q(n)$ part (ii) $= 2^{n+1}$
(ii)
$2(n+1) + 1 = (2n+1) + 2 \leq 2^n + 2$ using $Q(n)$ part (ii) $\leq 2^n + 2^n$ because $2 < 2^n$ when $n \geq 4 = 2^{n+1}$ □

4.3 Tiling Dyadic Floor with Triominoes

Given floor with $2^n \times 2^n$ tiles, remove 1 of the 4 center tiles.
Question: Can you tile the floor L-shaped with triominoes (3-block shape similar looking to tetris)
Answer: yes
when n=1, yes. Use an L such that the bend is opposite the missing tile.

General Case
$P(n)$: can tile a $2^n \times 2^n$ floor with a missing center square using triominoes
Base case $P(1)$ true (above)
Inductive step $P(n) \to P(n+1)$?
Adding an initial triomino making a 2×2 square with the missing tile effectively removes a tile from corner of each quadrant.

- reduces to the question: can we tile a $2^n \times 2^n$ floor with a missing corner square using triominoes?

So, strengthen our inductive hypothesis:

$$\begin{aligned} Q(n) : &(i) \text{ we can tile with a missing center square} \\ &(ii) \text{ we can tile with a missing corner square} \end{aligned}$$

Proof that $Q(n) \to Q(n+1)$
(i) can tile a $2^{n+1} \times 2^{n+1}$ with a missing center square

- use one triomino to reduce to tiling 4 floors of $2^n \times 2^n$ with missing corner squares
- by $Q(n)$ part (ii) we can tile each of these 4 quadrants using triominoes
- Hence we get a tiling for our $2^{n+1} \times 2^{n+1}$ floor

(ii) can tile a $2^{n+1} \times 2^{n+1}$ with a missing corner square

- use one triomino to reduce to tiling 4 floors with missing corner squares
- ...

4.4 Leaping Induction

Ex: $n^3 < 2^n$ for $n \geq 10$
Can be accomplished by strengthening inductive hypothesis to

$$\begin{aligned} Q(n) : &(i) n^3 < 2^n \\ &(ii) 6n^2 + 1 \leq 2^n \text{ for } n \geq 10 \\ &(iii) 2n + 1 \leq n^2 \end{aligned}$$

Easier way: observe that $(n+2)^3 = n^3 + 6n^2 + 12n + 8$. Recall that

$$\begin{aligned} n > 10 \Rightarrow n &> 6 \\ n^2 &> 12 \\ n^3 &> 8 \end{aligned}$$

$= 4n^3 < 4 * 2^n$ by inductive hypothesis $n^3 < 2^n = 2^{n+2}$
So we showed that if $P(n)$, then $P(n+2)$ when $n \geq 10$. Need to prove $P(10)$ and $P(11)$ to get $\forall n \geq 10$
Two base cases:

- $P(10) : 1000 < 1024$ True
- $P(11) : 1331 < 2048$ True

4.5 Strong Induction

$P(1) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$ is the inductive step.
Ex: Fundamental Theorem of Arithmetic
Suppose $n \geq 2$. Then

(i) n can be written as a product of factors each of which is prime

(ii) This representation is unique up to ordering

$2020 = 2 * 1010 = 2 * 2 * 505 = 2 * 2 * 5 * 101$
$2021 = 43 * 47$
$P(n) \rightarrow P(n+1)$ looks untrue/unclear

Proof. Base Case $P(2)$: 2 has a unique prime factorization. True
Inductive Step $P(1) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$
Direct proof. Cases:

(i) $n+1$ is prime. Clearly $P(n+1)$

(ii) $n+1$ is composite, that is $n+1 = k * l$ where $k < n+1, l < n+1$ are in $\mathbb{N}$. Each of k and l have unique prime factorizations because $P(k) \wedge P(l)$ is true by the inductive hypothesis. Implies $n+1$ has a unique prime factorization

$\square$