FOCS Notes
2/21/2020

Contents

1 Modular Arithmetic Recap and Examples

$a \equiv b \ mod \ d$ if $d|a-b$
From quotient remainder theorem $a \equiv b \ modd$ for exactly one integer $b \in [0, d-1\}$
$17 \equiv 2 \ mod \ 5$ because $17 = 5*3+2$, $5|17-2$

Q: it's currently 7 past the hour. What minute past thehour will it be 6231 minutes from now?
$7 + 6231 \equiv ? \ mod \ 60$
$6238 \equiv ? \ mod \ 60 \ \longrightarrow \ 6238 \equiv 58 \ mod \ 60$

Addition
$(x+y) \equiv [(x \ mod \ d) + (y \ mod \ d)] \ mod \ d$
$7 + 6231 \equiv [(7 \ mod \ 60) + (6231 \ mod \ 60)] \ mod \ 60 \equiv [7+51] \ mod \ 60 \equiv 58 \ mod \ 60$

Proof. $x = md + r$ by Quotient-Remainder Theorem (QRT)
$y = nd + s$
$x + y = (m+n)d + (r+s) \Rightarrow x + y \equiv r + s \mod d$ □

$10^{10} + 101237 \ mod \ 10 = 0 + 7 \ mod \ 10 \equiv 7 \ mod \ 10$

Multiplication
$x * y \equiv (x \ mod \ d) * (y \ mod \ d) \ mod \ d$

Proof. $x = md + r$, $y = nd + s$
$x * y = (md+r)(nd+s) = (\ldots)d + r * s$
$x * y \equiv r * s \ mod \ d$ □

Ex: What is the last digit of 29^{29}?
$29^{29} \ mod \ 10 = 9^{29} \ mod \ 10$
$9^{29} = 9^{2*14+1} = 81^{14} * 9$
$9^{29} \ mod \ 10 = (81^{14} \ mod 10) * (9 \ mod \ 10) \ mod \ 10 = 1 * 9 \ mod \ 10 = 9$

2 Graphs

$g = (V, E)$ is a graph, where V = set of vertices and E = set of edges
$V = \{a, b, c, d, e, f, g\}$
$E = \{(a,b), (a,c), (b,c), (b,d), (b,e), (c,d), (d,e), (f,g)\}$

Given a graph G:

- a path from v_1 to v_2 is a sequence of vertices (starts with v_1 and ends with v_2) such that every two adjacent vertices in this path are connected by an edge
  e.g. $p = acde$
- If there is a path that connects v_1 and v_2 we say v_1 and v_2 are connected

- The length of a path p is the number of edges crossed in that path
- Cycles are paths that start and end at a vertexwithout repeating any vertices or edges
- A graph G is connected if for any two vertices v_1 and v_2 there is a path connected v_1 and v_2

Adjancency list: lists vertices that are connected to specific other vertices.
Adjancency matrices: for n nodes, and $n \times n$ matrix where location i, j has a 1 if nodes i and j are adjacent.

Degree Sequences
degree $S_i = \#$ of neighbors of $v_i = \sum_{j=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ji}$
Complete, K_5 is a fully-connected graph with 5 vertices. $S = [4, 4, 4, 4, 4]$
Cycle, C_5 is 5-vertex graph with all vertices connected to 2 others in ring. $S = [2, 2, 2, 2, 2]$
Biparttite, $K_{3,2}$ is a graph with 2 columns, 3 in left and 2 in right. All left nodes and connected to all right nodes and vice versa. $S = [3, 3, 2, 2, 2]$
Line, L_5 is a 5-vertex line. $S = [2, 2, 2, 1, 1]$
Star, S_6 is a star with 1 hub vertex. $S = [5, 1, 1, 1, 1, 1]$
Ex:
$S = [2, 2, 2, 2, 2, 2]$ could be a C_6 benzene ring or 2 triangles.
Takeaway:

- Graphs unique determine their degree sequence
- Realizable degree sequences do not in general uniquely determine the corresponding graphs
- Not all degree sequences are realizable

Handshaking Theorem
$\sum_{i=1}^{n} S_i = 2|E|$ if $G = (V, E)$ is a graph with n vertices.
Idea: every edge contributes a 2 to the sum $\sum_{i=1}^{n} S_i$
Missed an example

Proof. Construct a graph G where $V = \{\text{partygoers}\}$ and $E = \{(u, v) : u \text{ and } v \text{ shook hands}\}$.
A partygoer v_i is odd *Rightarrow* S_i is odd
Consider $\sum_{i=1}^{n} S_i = \sum_{i:S_i \text{ is even}} S_i + \sum_{i:S_i \text{ is odd}} = 2|E|$ By handshaking theorem.
Therefore, there must be an even number of odd people in order for the sum to be even. □

Trees
Acyclic (no cycles exist) connected graph.
Claim A tree with n vertices has $n - 1$ edges.

Planar Graph
A graph is planar if you can draw it (in a plane) without edge crossings
Euler's Invariant: for planar, connected graphs: $F + V - E = 2$
pyramid $4 + 4 - 6 = 2$
cube $6 + 8 - 12 = 2$