FOCS Notes
Lecture
1/28/2020

Agenda:

- What is induction?
- What do we need it for
- The principle of induction. ”Toppling the dominos”
- The template for proof by Induction
- Examples
- Induction and the well-ordering principle, with an example of the smallest counter-example

# Contents



# 1 Introduction

## 1.1 Dispensing postage using 5 and 7 cent stamps

19 cents: 5, 7, 7
20 cents: 5 5 5 5
21 cents: 7 7 7
22 cents: 5 5 5 7
23 cents: ?

Claim: 23 cents cannot be achieved using 5 and 7 cent stamps $\iff \forall (x, y) \in \mathbb{N}_0 : 23 \neq 5x + 7y$

*Proof.* Direct proof, by enumeration. Clearly $x$ must be $\leq 5$ if it solves the equation, so consider $x \in \{0, 1, \ldots, 5\}$ and show that the corresponding y is non-integral. □

From tinkering, we convince ourselves that any amount of postage greater than 23 cents can be written using 5 and 7 cent stamps
$\forall n \in \mathbb{N}, n > 23 : (\exists (x, y) \in \mathbb{N}_0 : n = 5x + 7y)$

# 2 Statements we may prove by Induction

| Predicate | Proposition |
|---|---|
| (i) $P(n)$: "5 and 7 cent stamps can make postage $n$" | $\forall n \geq 24 : P(n)$ |
| (ii) $P(n)$: "$n^2 + n + 41$ is a prime number" | $\forall n \geq 1 : P(n)$ |
| (iii) $P(n)$: "$4^n - 1$ is divisible by 3" | $\forall n \geq 1 : P(n)$ |

## 2.1 Is $4^n - 1$ divisible by 3 if $n \geq 1$?

Last class we proved: If $4^n - 1$ is divisible by 3 then $4^{n+1} - 1$ is divisible by 3
We proved, therefore $\forall n \in \mathbb{N} : P(n) \to P(n+1)$ (*)
We also see that $P(1)$ is true
This together with (*) allows us to see $P(2)$ is true, then $P(3)$ is true, ...

To make the leap that $P(1)$ true and $\forall n : P(n) \to P(n+1)$ implies $\forall n : P(n)$, we use the <u>principle of induction</u>

<u>Principle of Induction</u>

(i) If $P(n)$ is a predicate on $\mathbb{N}$, and

(ii) $\forall n \geq b : P(n) \to P(n+1)$, and $\longleftarrow$ induction step

(iii) $P(b)$ is true $\longleftarrow$ base step

then we conclude that $\forall n \geq b : P(n)$.
Using the principle of induction, we conclude that $4^n - 1$ is divisible by 3 for all $n \geq 1$

# 3 Template for Proof by Induction

$\forall n \geq 1 : P(n)$

<u>Proof</u>

(1) We use proof by induction to prove $\forall n \geq 1 : P(n)$

(2) Establish that $P(1)$ is true

(3) Show that $P(n) \to P(n+1)$. Prove this implication directly of by contraposition
<u>Direct</u>

- Assume $P(n)$ is true
- Via mathematically valid derivations conclude $P(n+1)$ is true

<u>Contrapostion</u>

- Assume $P(n+1)$ is false
- Via valid derivations show $P(n)$ is false as well

(4) State that by the principle of induction, $\forall n \geq: P(n)$

# 4 Examples

## 4.1 $\sum_{i=1}^{n} i$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\begin{array}{cccccc} & 1 & 2 & 3 & \dots & \text{n} \\ + & n & n-1 & n-2 & \dots & 1 \\ \hline & n+1 & n+1 & n+1 & \dots & n+1 \end{array}$$

*Proof.* We prove this via induction. First, we establish out base case. Let $n = 1, P(1) : \sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$ so our base case is true. We now establish our inductive step, then $P(n) \rightarrow P(n+1)$.
We will use a direct proof. Assume $P(n)$ is true, that is, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We now have that $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n^2+n) + (n+1) = \frac{1}{2}(n^2+n+2n+2) = \frac{1}{2}(n+1)(n+2)$. Therefore, $P(n+1)$ is true.
By the principle of induction, $\forall n \geq 1 : P(n)$ $\square$

## 4.2 $S(n) = 1^2 + 2^2 + \cdots + n^2$

$$S(n) = 1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$$

Q: what is $S(n)$ equal to?

To build an intuition, consider $\sum_{i=1}^{n} i$ geometrically

The sum of the natural numbers can be represented as the area of a triangle. Duplicating and reflecting that triangle forms a rectangle with area $n(n+1)$, therefore the area of the two triangles is $\frac{n(n+1)}{2}$. Higher dimensions work similarly.
Suggest that $S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3$.

How to guess the coefficients? Plug in values of $n$ to get constraints and solve for the $a_i$
$S(1) = 1 = a_0 + a_1 + a_2 + a_3$
$S(2) = 5 = a_0 + 2a_1 + 4a_2 + 8a_3$
$S(3) = 14 = a_0 + 3a_1 + 9a_2 + 27a_3$
$S(4) = 30 = a_0 + 4a_1 + 16a_2 + 64a_3$
solve $\longrightarrow$ $a_0 = 0, a_1 = \frac{1}{6}, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}$

Conjecture $\forall n \geq 1 : S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$

*Proof.* We prove that $\forall n : S(n) = \frac{1}{6}n(n+1)(2n+1)$ by induction. First we establish the base case, that is, $P(1)$. $P(1)$ is the claim that $S(1) = \frac{1}{6}1(1+1)(2*1+1) = 1$. This is trivially true.
Next we establish our inductive implication, $P(n) \rightarrow P(n+1)$, using direct proof. Assume $P(n)$ is true, that is, $S(n) = \frac{1}{6}n(n+1)(2n+1)$. We want to show $P(n+1)$ is also true, that is, $S(n+1) = \frac{1}{6}(n+1)(n+2)(2n+3)$. To show this, note that $S(n+1) = \sum_{i=1}^{n+1} i^2 = S(n) + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$ [algebra not shown]. Therefore $P(n+1)$ is true.
By the principle of induction, $\forall n \geq 1 : P(n)$, that is, $\forall n \geq 1 : S(n) = \frac{1}{6}n(n+1)(2n+1)$. $\square$

## 4.3 Generalized Example

Fact:
Can use the principle of induction to prove $\forall n \geq b : P(n)$

Let $Q(n) = P(n + (b - 1))$
Then use principle of induction to establish $\forall n \geq 1 : Q(n)$

# 5 Induction and Axioms

Recall the Well-Ordering Principle
Any non-empty set of natural numbers has a smallest element

Face
The Principle of Induction is equivalent to the Well-Ordering Princple (WOP). Full proof online

Well-ordering Principle $\rightarrow$ The Principle of Induction

*Proof.* We prove it by contradiction. We assume that the Well-Ordering Principle holds and the principle of induction does not. That is, there is a predicate $P$ on the natural numbers for which $P(1)$is true, $\forall n : P(n) \rightarrow P(n+1)$, but there exists an $m$ such that $P(m)$ is not true.
Let $S = \{k : P(k) \textit{ is not true}\} \subseteq \mathbb{N}$. Since $m \in S$, $S$ is non-empty and by WOP, there is $n_* \in S$ such that $n_*$ is the smallest element in $s$. That is, $n_*$ is the smallest number such that $P(n_*)$ is false. Since $P(n_* - 1) \rightarrow P(n_*)$ is false, it is the case that $P(n_* - 1)$ is also false. This means $n_* - 1 \in S$. Since $n_* - 1 < n_*$ this contradicts the minimality of $n_*$. These derivations were all mathematically sound, so one of the assumptions must be false. We conclude that if WOPhods, then the principle of induction holds. $\square$