FOCS Notes
2/25/2020

Q: review on graphs
Let G be a connected graph whose degree sequence consists of all 2s (ever vertex has degree 2). Argue that G has at least one cycle.

1st proof

Proof. Contradiction. Assume G is acyclic. Then G is a tree because it is connected and acyclic, so it has $n-1$ edges. By the handshaking theorem, $\sum_{i=1}^{n} i = 2|E| \Rightarrow 2n = 2(n-1)$. This contradiction shows G has at least one cycle. $\square$

2nd proof

Proof. Direct. Construct a largest length acyclic path in G, $v_1 v_2 \dots v_l$. Consider v_1. Deg(v_1)=2, so v_1 is connected to another vertex u.
Case 1 u is not in the path p. Then $p' = uv_1 \dots v_l$ is acyclic. This contradicts the maximality of p. This case does not happen
Case 2 u is in the path p. In which case $uv_1, v_2 \dots u$ is a cycle. $\square$

Lecture 12 Graph matching and Graph coloring.
A set of edges is independent if they do not share any vertices (they don't touch)
A matching in a graph is a maximal set of independent edges

Q: When, in a bipartite graph, does there exist a matching that covers all of the left vertices?

Hall's matching theorem
There exists a matching on a bipartite graph with $V = (L, R)$ that covers $L \iff$ the "matching" or "marriage" condition: $\forall x \subseteq L : |\mathcal{N}(x)| \geq |x|$ is satisfied. Here, $\mathcal{N}(x) = \{u \in R : (u, v) \in E \text{ for some } v \in X\}$
Q: Why is there no matching that covers L if the matching condition is violated?
violation of the matching condition $\Rightarrow \exists x \subseteq L : |\mathcal{N}(x)| < |x|$. If $|\mathcal{N}(x)| < |x|$ then there cannot be a set of independent edges that covers X, hence there is no set of independent edges that covers G
Proof of Hall's theorem

Proof. We just showed that existence of a covering matching $\Rightarrow$ marriage condition is satisfied (by contrapositive).
Marriage condition is satisfied $\Rightarrow$ exists a covering matching.
By induction on the number of left vertices n.
Base case $n = 1$. The marriage condition says the left nodde has at least one neighbor. Thus this vertex can be covered with a matching.
Induction
Assume for $n-1$ marriage condition implies a matching exists.
Given a graph with n left vertices, consider the two possible cases.
Case 1 $\forall n \subseteq L : |\mathcal{N}(x)| \geq |x| + 1$
Pick an arbitrary vertex $a \in L$ and $b \in \mathcal{N}(\{a\})$
Consider $S \subseteq L - \{a\}$. Consider $\overline{\mathcal{N}}(S) = \{u : (u, v) \in E \text{ for some } v \in S \text{ and } v \neq b\}$
Note $|\overline{\mathcal{N}}(S)| \geq |\mathcal{N}(S)| - 1$
So for every $S \subseteq L - \{a\}$, $|\overline{\mathcal{N}}(S)| \geq |\mathcal{N}(S)| - 1 \geq |S| + 1 - 1 = |S|$ so the residual graph satisfies the matching condition. This graph has $n-1$ left vertices, so by the inductive hypothesis we can match all of its left vertices. This matching together with (a, b) gives a matching for all n left vertices of our original graph.
Case 2 $\exists x \subseteq L : |\mathcal{N}(x)| = |x|$. See book for proof. $\square$

Ex Claim: if min(left degrees)$\geq$max(right degrees) in a bipartite graph, then a matching covering of the left vertices exists.
Equivalently: min(left degrees)$\geq$max(right degrees) implies $\forall s \subseteq L : |\mathcal{N}(S)| \geq |S|$
Given an S, the number of edges from S to the R

$$\begin{aligned}
&= |S| * (\text{average degree of vertices in } S) \\
&= |\mathcal{N}(S)| * (\text{average degree of vertices in } \mathcal{N}(S)) \\
|S| * (\text{average degrees of vertices in } S) &\geq |S| * \min(\text{left degrees}) \\
|\mathcal{N}(S)| * (\text{average degrees of vertices in } \mathcal{N}(S)) &\leq |\mathcal{N}(S) * \max(\text{right degrees}) \\
|S| * \min(\text{left degrees}) &\leq |\mathcal{N}(S)| * \max(\text{right degreees}) \\
\frac{\min(\text{left degrees})}{\max(\text{right degrees})} &\leq \frac{|\mathcal{N}(S)|}{|S|} \\
&\Rightarrow |S| \leq |\mathcal{N}(S)|
\end{aligned}$$

Graph coloring

Graph coloring: a k-coloring of a graph G is an assignment of colors $\{1, \ldots, k\}$ to the vertices of G so that no two adjacent vertices have the same color.
Example applications (conflict graphs)
Ex Radio frequency assignment. Radio stations cannot share a frequency with any stations that they overlap
Ex Scheduling course exams. Conflict if there is a student in two classes with exam at the same time

Greedy algorithm for coloring a graph

1. Colors $\{1, 2, 3, \ldots\}$
2. Order the vertices in the graph $v_1, \ldots, v_n$
3. Assume vertices v_1, v_i have been colored. Color v_{i+1} with the smallest color so that it does not conflict with any previously colored vertex

Using the greedy coloring algorithm, color(v_i)$\leq S_i + 1$
$\Rightarrow$
$\mathcal{X}(G) \leq \max_{i=1,\ldots,n} S_i + 1$ where $\mathcal{X}(G) = \min\{k : G \text{ is } k\text{-colorable}\}$