FOCS Notes
2/7/2020

Lecture 8: Structural induction for proving properties of recursively defined sets

1. Two types of questions about recursive sets
2. Ex: Matched Parentheses
3. Structural Induction:
   - $\mathbb{N}$
   - Palindromes
   - Arithmetic Expressions
4. Properties of RBTs:
   - are trees
   - $\text{size}(T) \leq 2^{\text{height}+1} - 1$

Contents

1 Questions about recursive sets

Two types of questions:

(i) Is there some property that every element in A has?
Ex: Is every RBT a tree?

(ii) Is everything with a particular property in this set?
Ex: Is every tree a RBT?

Recursively defined set A $\quad A = \{0, 4, 8, 12, \dots\}$

1. $0 \in A$ [base]
2. $x \in A \to x + 4 \in A$

Q: if $x \in A$, is x divisible by 4?
Q: is every even number in this set?

2 Structural Induction

Recursion is useful for definition
Induction is useful for proving properties
For recursively defined sets, use structural induction

Structural Induction

- the base case(s) has a certain property
- if the ancestors of this structure has a property then the structure itself has this property
- conclude by structural induction that every structure in this set has this property

Ex: Adam and Eve were humans
Adam and Eve had two kidneys
It's the case that if their parents have two kidneys, the child will have two kidneys

2.1 Matched Parentheses

Ex: (()), (()())((()))()

1. $\epsilon \in M$
2. if $x, y \in M$ then $(x) \cdot y \in M$

Show ()() in M

Proof. Direct. $x = y = \epsilon \in M \Rightarrow (\epsilon) \cdot \epsilon = () \in M$
$x = \epsilon, y = () \Rightarrow (x) \cdot y = ()() \in M$ □

Q: $()(\notin M$

Strings in M are balanced (number of opening and closing parentheses are equal)

Proof. Structural induction.
Base ϵ trivially is balanced
Structural Induction Step if x and y have the same number of opening and closing parentheses, then so does $(x) \cdot y$
By structural induction every string in M is balanced □

Q: $)(\notin M$ left as an exercise. $\Longleftarrow$ to answer show that if $x \in M$ then every prefix of x has at least as many opening parentheses as closing

Structural Induction
Induction with recursively defined sets is called strucural induction Let S be recursively defined set, meaning

1. Base cases $s_1, s_2, \ldots, s_n$ are in S
2. Constructor rules that use elements in S to construct new elements in S.

Let $P(S)$ be a property defined for any element $s \in S$. To show $\forall s : P(s)$, you must show

1. $P(s_1), \ldots, P(s_n)$ are true
2. [Inductive Step] For every constructor rule:
   If $P(s)$ is true for all parents, then $P(s)$ is true for their child.
3. By structural induction, $\forall s : P(s)$

2.2 Palindromes P

$P = \{0, 1, 00, 11, 010, 101, \dots\}$

1. $0 \in P$
2. $1 \in P$
3. $x \in P \rightarrow 0 \cdot x \cdot 0 \in P$
4. $x \in P \rightarrow 1 \cdot x \cdot 1 \in P$
5. $x \in P \rightarrow x \cdot x \in P$

Q: 1101011
$1 \in P \Rightarrow 0 \cdot 1 \cdot 0 \in P \Rightarrow 1 \cdot 010 \cdot 1 \in P \Rightarrow 1101011 \in P$

Claim: Every palindrome is in P

Proof. Induction (on the length of the string). Let $x \in P$
Base Cases
Length(x)=2: x=00 or x=11, so $x \in P$
Length(x)=1: x=0 or x=1, so $x \in P$
Inductive Step
By leaping induction. Let length(x)=n. Assume all palindromes of length $n-2$ are in P. Note that either x=0$\cdot y\cdot$0 where y is a palindrome of length $n-2$ or x=1$\cdot y\cdot$1 where y is a palindrome of length $n-2$.
By the inductive hypothesis, $y \in P$. One of the constructor rules gives x from y $\square$

3 RBTs

3.1 RBTs are trees

Claim: RBT with $n \geq 1$ vertices have $n-1$ edges

Proof. $P(t)$: If t has $n \geq 1$ vertices, then t has $n-1$ edges
Base Case $P(\epsilon)$ true vacuously/trivially because ϵ has no vertices
Induction step Consider our constructor rule that takes RBTs T_1 and T_2, where T_1 has n_1 vertices and l_1 edges and T_2 has n_2 vertices and l_2 edges, and constructs a RBT T, with n vertices and l edges. Direct proof.
Case: $T_1 = \epsilon$ and $T_2 = \epsilon$. T has $n = 1$ and $l = 0 \Rightarrow P(T)$ is true
Case: $T_1 = \epsilon; T_2 \neq \epsilon \ \Rightarrow \ n = n_2 + 1; l = l_2 + 1$ and by inductive hypothesis, since $n_2 \geq 1$, $l_2 = n_2 - 1$, so $l = n_2 - 1 + 1 = n_2 = n - 1$ so $P(T)$ is true
Case: $T_1 \neq \epsilon; T_2 = \epsilon$ argument same as above (could use without loss of generality)
Case: $T_1 \neq \epsilon; T_2 \neq \epsilon \ \Rightarrow \ n = n_1 + n_2 + 1; l = l_1 + l_2 + 2$. By the inductive hypothesis, since $n_1, n_2 \geq 1$, we have $l_1 = n_1 - 1$ and $l_2 = n_2 - 1$. Plugging in, we see that $l = (n_1 - 1) + (n_2 - 1) + 2 = n_1 + n_2 = n - 1$ so $P(T)$ $\square$

3.2 size(T)$\leq 2^{\textbf{height}+1} - 1$

Given a RBT we define two functions height and size recursively
Size

1. size(ϵ)=0

2. If T_1 and T_2 were used to construct T, then size(T)=size(T_1)+size(T_2)+1

Height

1. height(ϵ)=-1

2. If T_1 and T_2 used to construc T, then height(T)=1+max(height(T_1), height(T_2))

   NOTE: a single node has size 1 and height 0

<u>Claim</u> size(T)$\leq 2^{\text{height}(T)+1} - 1$

Proof. By structural induction. $P(T)$: size(T)$\leq 2^{\text{height}(T)+1} - 1$ <u>Base case</u> size(ϵ), height(ϵ)=-1 so $0 \leq 2^{-1+1} - 1 = 0$ True <u>Inductive Step</u> Direct proof. Assume $P(T_1)$ and $P(T_2)$ hold. We want to show size(T)$\leq 2^{\text{height}(T)+1} - 1$. To do so, note that size(T)=size(T_1)+size(T_2)+1$\leq 2^{\text{height}(T_1)+1} - 1 + 2^{\text{height}(T_2)+1} - 1 + 1$. That is size($T$)$\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1} - 1$. Now recall height($T$)=1+max(height($T_1$),height($T_2$)), so size($T$)$\leq 2^{\text{height}(T)} + 2^{\text{height}(T)} - 1 = 2 * 2^{\text{height}(T)} - 1 = 2^{\text{height}(T)+1} - 1$ so $P(T)$ is true.
By structural induction, $\forall T : P(T)$ □