# Prediction and Logistic Regression

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# Prediction

- Fundamental in modern machine learning approaches to computational linguistics
- Recall that we use our **training data** to set parameters in a model (train the model) and test on **test data**
- Given some data, e.g., a sentence, we want to predict something about it.
- For example, SPAM vs NOT SPAM or whether a product review is positive or negative

# Prediction

- Assume we have a corpus of reviews in which each review is labeled **classes** of positive (P) or negative (N) sentiment.
- Binary classification problem (2 classes/labels)
- We want to guess P or N (the classes) based only on the review text
- We need a set of **features** from the review text to make this prediction
- We **train** a model to predict a class label on its own.

# Splitting the Data

- Suppose we have 1,000 labeled reviews
- We can Wet aside ~10% as our **development set** and ~10% as our **test set**.
- We use the remaining 80% as our **training set** to train the model
- Researchers shouldn't look at test data (and preferably training data) if possible
- Sets should usually be randomized to prevent biased model
- Training data ideally balanced by classes, but not always possible

# Splitting Data

- Alternative called **k-fold cross-validation**.
  - Run several training and testing sessions over random splits of the data
- Related: leave one out (LOO) evaluation, where you train on all but one test example over the hole dataset
  - Useful when data are small

# Training

- Process of making iterative changes to model's *parameters* to increase its performance
- One common measure of performance is **accuracy.**

$$\text{accuracy} = \frac{\text{correct predictions}}{\text{all predictions}}.$$

# Training

- In **supervised learning** we used these labeled examples to train our model.
- The correct labels are called **gold standard** labels.
- By looking at these labels, we want to **fit** the model to the data, making predictions on *unseen* test data

# Review: The Perceptron

- We know that $y = mx + b$ represents a line in two-dimensional real number space -- what mathematicians call $\mathbb{R}^2$
- The $m$ represents the slope of the line, while $b$ is the $y$-intercept -- how much the line is shifted up or down. We'll use $w$ instead of $m$.
- In linear algebra and statistics parlance, the $w$ coefficient is known as a **weight**.

# The Perceptron

We can use more terms

$$y = w_1x_1 + w_2x_2 + b$$

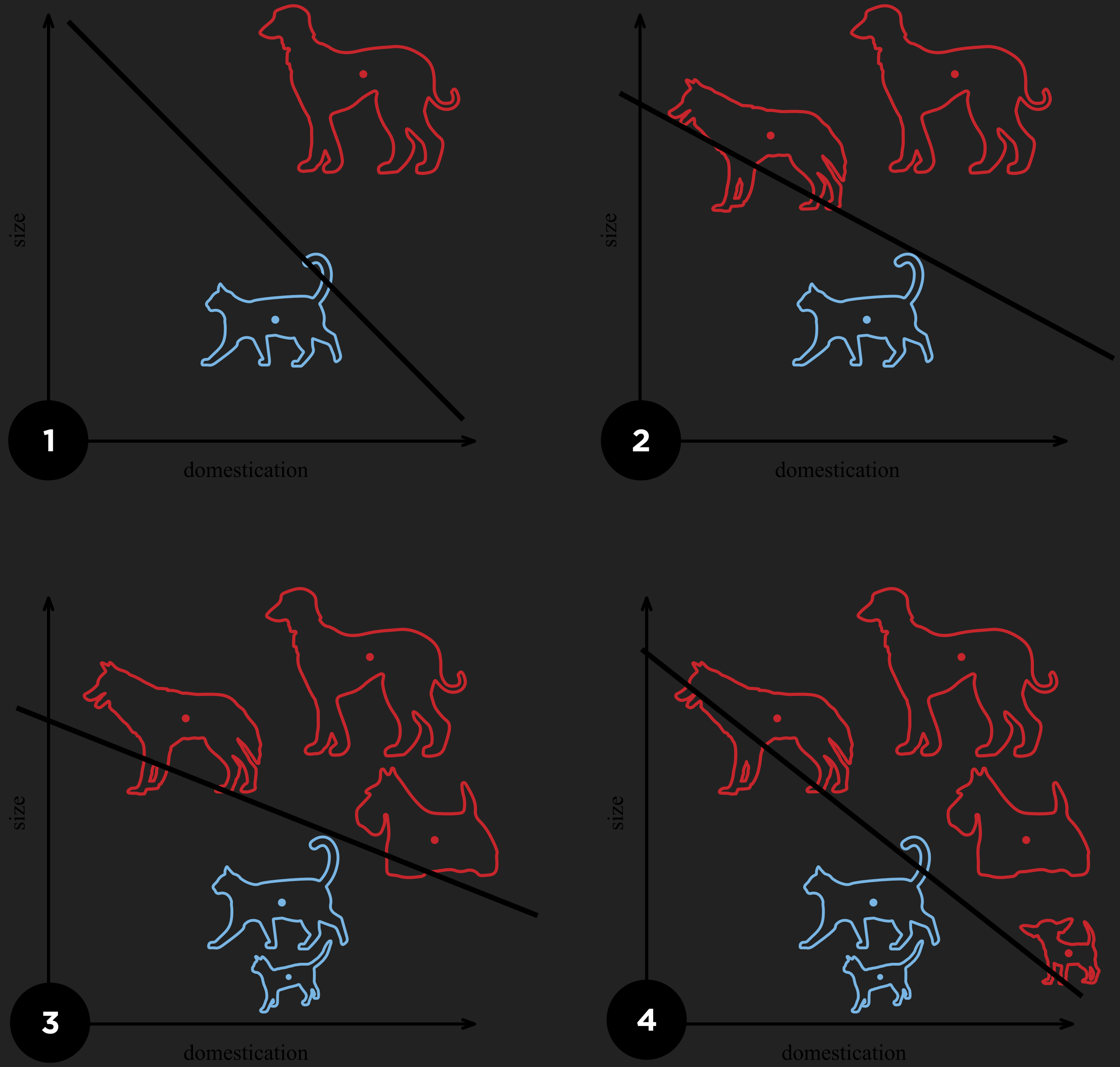
- The **weights** tell us how much influence a given term will have.
- Every $x$ has a corresponding $y$

# The Perceptron

We can have as weights and variables as we like, allowing us to generalize

$$
\begin{aligned}
a = h_{\mathbf{w}}(\mathbf{x}) &= \mathbf{w} \cdot \mathbf{x} + b \\
&= w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + b \\
&= \sum_{i=1}^{n} w_i x_i + b
\end{aligned}
$$

- Equation represents $n - 1$ dimensional hyperplane in $\mathbb{R}^n$ to separate data
- Each weight $w_1$ coresponds to a **feature** $x_i$.
- Number of features = number of dimensions



1
size
domestication
2
size
domestication
3
size
domestication
4
size
domestication

# The Perceptron

Prediction algorithm

```
function predict(x)
    return sign(dot(w,x) + b)
```

$$\text{predicted class} = sign(\mathbf{w} \cdot \mathbf{x} + b)$$

# Perceptron Learning Algorithm

```
1.   Initialize weights vector w to random numbers in [0, 1].
2.   for each example (x, y) in D:
         prediction = predict(x)
         if not sign(y) == sign(prediction):
             for each w in weights:
                 w = w + y * x
                 b = b + y * x
```

- Error-based learning
- If algorithm predicts wrong class, update weights

# Perceptron Learning Algorithm

```
1.  Initialize weights vector w to random numbers in [0, 1].
2.  for each example (x, y) in D:
        prediction = predict(x)
        if not sign(y) == sign(prediction):
            for each w in weights:
                w = w + y * x
            b = b + y * x
```

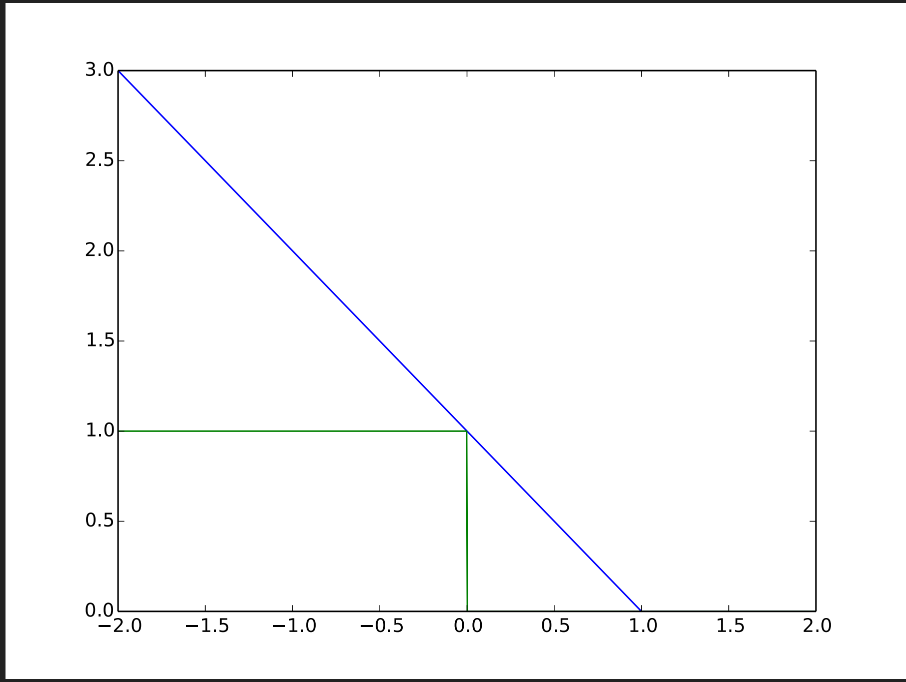
## Update Rule for Weights

$$w_i := w_i + \eta y x_i.$$

# Perceptron Loss Function

$$\mathcal{L}(h_{\mathbf{w}}(\mathbf{x}), y) = \mathcal{L}(p, y) = \begin{cases} y - p & \text{if } yp > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= \max(0, 1 - yp)$$

- Hinge loss




Plot: hinge vs. 0-1 loss

# Logistic Regression

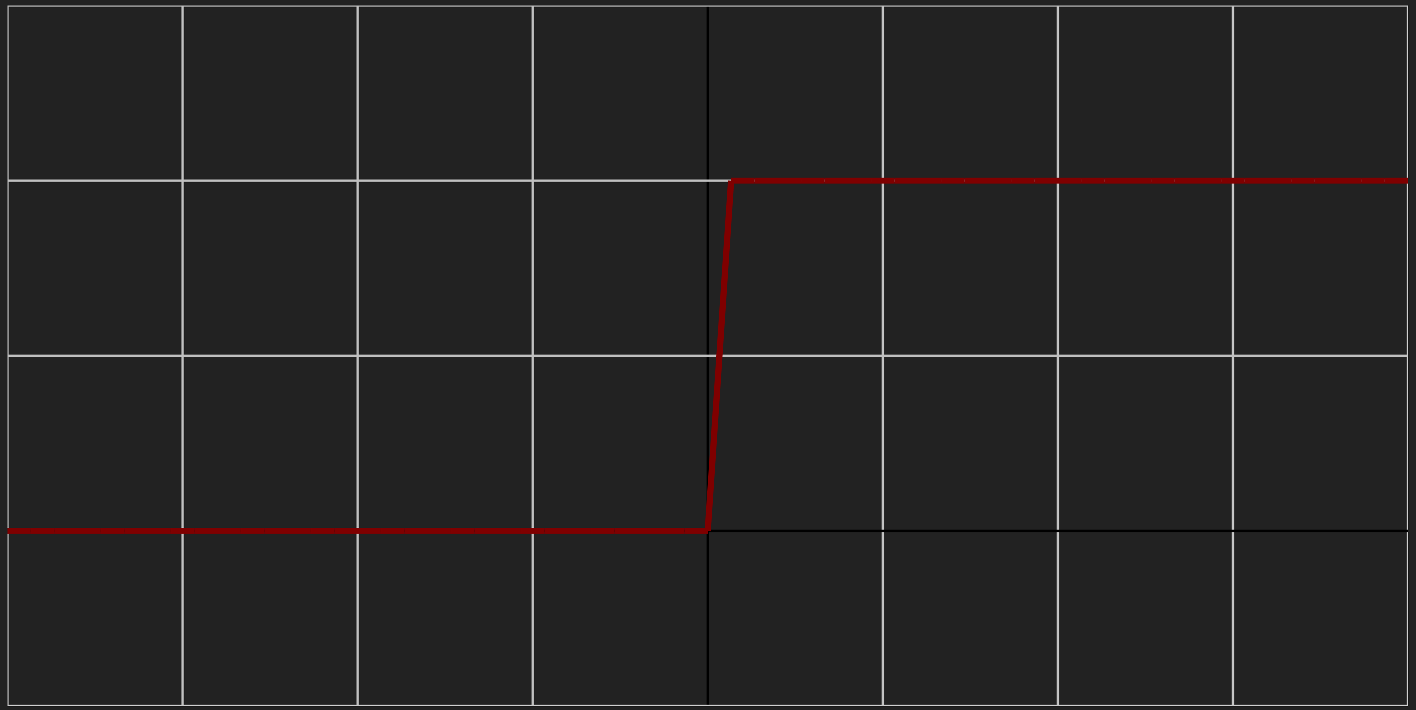
# Logistic Regression

- Like perceptron, but better
- Perceptron tends to **overfit** training data
  - Activation $a$ result of pure linear combination
  - In principle, unbounded

# Logistic Regression

- Using an **activation funtion** bounds the output.
  For example, the binary step activation function:
- Send $a = \mathbf{w} \cdot \mathbf{x} + b$ through $f(t)$.

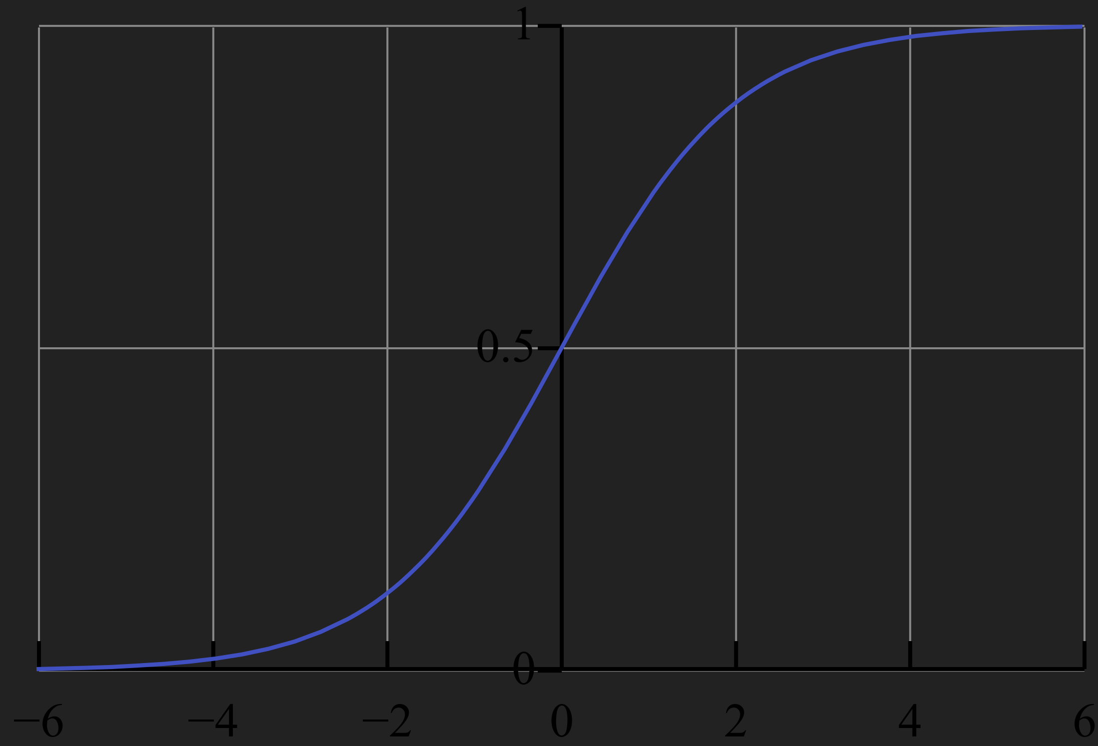
$$f(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise} f \end{cases}$$



# Logistic Regression

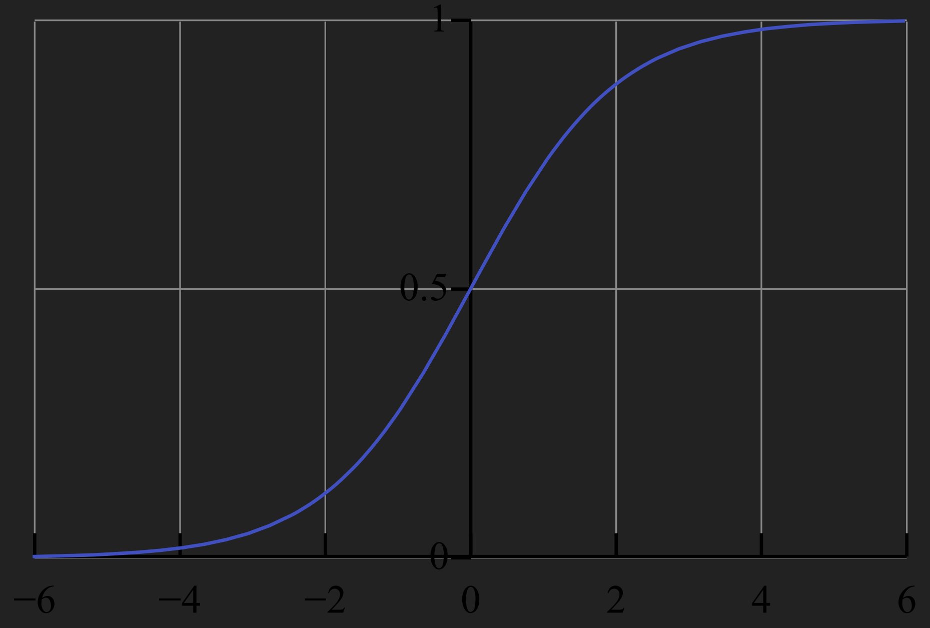
- Binary step function forfeits all ifnormation about confidence in prediction
- We can use a smooth version, a **sigmoid** (S-shaped) function called the **logistic function**.

$$\sigma(t) = \frac{e^t}{1+e^t} = \frac{1}{1+e^{-t}}.$$



- This function gives ua a **probability** of the positive class.

$$
\begin{aligned}
p &= \sigma(\mathbf{w} \cdot \mathbf{x} + b) \\
&= \frac{1}{1 + \exp[-(\mathbf{w} \cdot \mathbf{x} + b)]} \\
&= \frac{1}{1 + \exp[-(\sum_{i=1}^{n} w_i x_i + b)]} \\
&= P(y = 1 | \mathbf{w}; \mathbf{x})
\end{aligned}
$$



# Logistic Regression Prediction Algorithm

- Our prediction and algorithms are slightly changed from those for the Perceptron.
- Classes are now 0 or 1 instead of -1 or 1

```
function classify(x):
    p = predict(x)
    if p < 0.5:
        return 0
    else:
        return 1

function predict(x):
    return logistic(dot(w, x) + b))
```

# Logistic Regression

- Note: we use **log probabilities** instead of pure probabilities
- Prevents underflow
  - Monotonicity maintained.

$$log(ab) = log(a) + log(b)$$

So,

$$\log[p(x)p(y)] = \log p(x) + \log p(y)$$

# Logistic Regression Learning Algorithm

- (Very slightly) different update rule, but same principle

$$w_i := w_i + \eta(y - p)x_i.$$

```
function train(x, y, learning_rate):
1.   Initialize weights vector w to 0 in [0, 1].
2.   for each example (x, y) in D:
         p = predict(x)
         for each w in weights:
             w = w + learning_rate * (y - p) * x
         b = b + learning_rate * (y - p) * x
```

- In perceptron, update is entirely determined by learning rate $\eta$.
- In logistic regression, update is determined by learning rate $\eta$ *and how wrong you were*, $y - p$

# Logistic Regression Loss Function

- Various names: logistic loss, cross-entropy loss, negative log likelihood
- Entropy is a term from information theory that we'll discuss later
    - Measurement of information content
- Logistic regreission is a **discriminative classifier**
    - Doesn't have a prior, unlike a generative classifier
    - Directly optimizes $P(y|x)$ over training data

# Logistic Regression Loss Function

- Learns weights that predict

$$\operatorname*{argmax}_y P(y|\mathbf{x};\mathbf{w})$$

- I.e., maximize the probability that it chooses the correct class, $y$.
- Only two possible classes, 1 or 0, so $P(y) = 1 - P(\neg y)$

# Logistic Regression Loss Function

- The **odds** or **likelihood** that an event $e$ occurs is the probability that it will occur divided by the probability that it won't

$$\frac{P(y=1)}{1-P(y=1)} = \frac{P(y=1)}{P(y=0)}$$

- Monotonic transformation of probabilities
- Maps probabilities from $[0,1]$ to $[0,\infty]$
  - Reverse of what logistic function does
- Ex: if $P(y=1) = .75$, then $P(y=0) = .25$.
  - Odds of class 1 is .75/.25, or 3 to 1.

# Logistic Regression Loss Function

- Assume $P(y=1) = p = \sigma(\mathbf{w} \cdot \mathbf{x} + b)$
  Then,
  $$\begin{aligned} P(y=0) &= 1 - p \\ &= 1 - \sigma(\mathbf{w} \cdot \mathbf{x} + b) \end{aligned}$$
- Or, more compactly,
  $$p^y (1-p)^{1-y} = y\sigma(\mathbf{w} \cdot \mathbf{x} + b)^y (1 - \sigma(\mathbf{w} \cdot \mathbf{x} + b))^{1-y}.$$
- Taking the log, we have:
  $$\begin{gathered} y \log p + (1-y) \log(1-p) \\ = y \log \sigma(\mathbf{w} \cdot \mathbf{x} + b) + (1-y)(1 - \log \sigma(\mathbf{w} \cdot \mathbf{x} + b)), \end{gathered}$$
  which is the **log likelihood** of a correct answer.

# Logistic Regression Loss Function

- Our loss function is log likelihood defined over all of our training data.
  - So, we sum over the individual examples' losses in all of our training data for our loss function

$$\begin{aligned}\mathscr{L}(\mathbf{x}) &= \sum_{(\mathbf{x},y)\in D} [y \log p + (1-y)\log(1-p)] \\ &= \sum_{(\mathbf{x},y)\in D} [y \log \sigma(\mathbf{w}\cdot\mathbf{x}) + (1-y)(1-\log\sigma(\mathbf{w}\cdot\mathbf{x}+b))],\end{aligned}$$

# Side note:

$$
\begin{aligned}
\text{Assume } \ln \frac{p}{1-p} &= \mathbf{w} \cdot \mathbf{x} + b. \\
\frac{p}{1-p} &= e^{\mathbf{w}\cdot\mathbf{x}+b} \\
p &= (1-p)e^{\mathbf{w}\cdot\mathbf{x}+b} \\
p &= e^{\mathbf{w}\cdot\mathbf{x}+b} - pe^{\mathbf{w}\cdot\mathbf{x}+b} \\
p + pe^{\mathbf{w}\cdot\mathbf{x}+b} &= e^{\mathbf{w}\cdot\mathbf{x}+b} \\
p(1 + e^{\mathbf{w}\cdot\mathbf{x}+b}) &= e^{\mathbf{w}\cdot\mathbf{x}+b} \\
p &= \frac{e^{\mathbf{w}\cdot\mathbf{x}+b}}{1 + e^{\mathbf{w}\cdot\mathbf{x}+b}} \\
p &= \frac{1}{1 + e^{-(\mathbf{w}\cdot\mathbf{x}+b)}}
\end{aligned}
$$

🤯

# Optimization

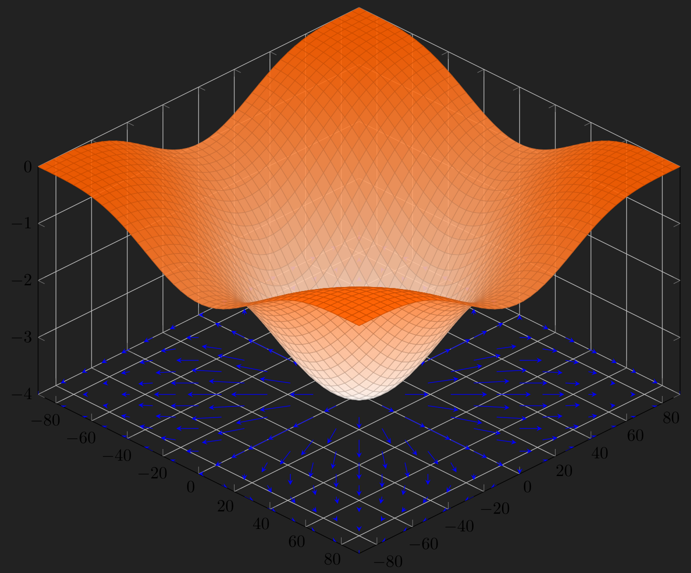
## Review: Derivatives

- Rate of change or slope of a function
- Given a line, $y = mx + b$,
  if $m = \frac{\Delta y}{\Delta x} = 3/4$, this is the slope.
- Since lines have the same slpoe everywhere, we need only find it once
- For curves, the derivative function gives us the slope.
- Simple rule: If $f(x) = x^2$, then $\frac{dy}{dx} = f'(x) = 2x$ at an arbitrary point

# Optimization

## Partial Derivatives

- Given functions of more than one variable, rate of change along one dimension, assuming other ones held constant
    - For $f(x, y)$, when differentiating along $x$ axis, treat $y$ as a constant
    - Written as $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, etc.

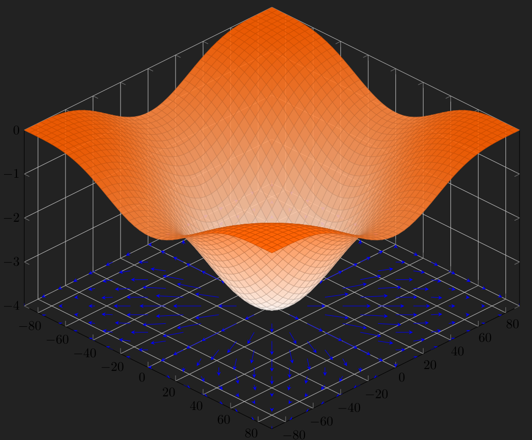


# Optimization

## The Gradient

- The **gradient** is a vector of partial derivatives
    - One for each dimension
- Given a function of $n$ variables, the gradient has an partial derivative for very variable.
- Given a function $f(x, y, z)$, the graident of $f$ is

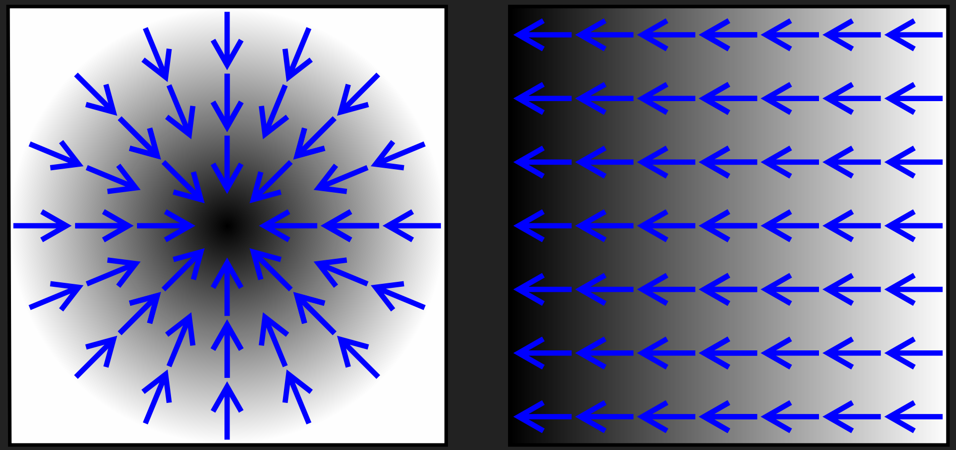
$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right].$$



# Optimization

## The Gradient

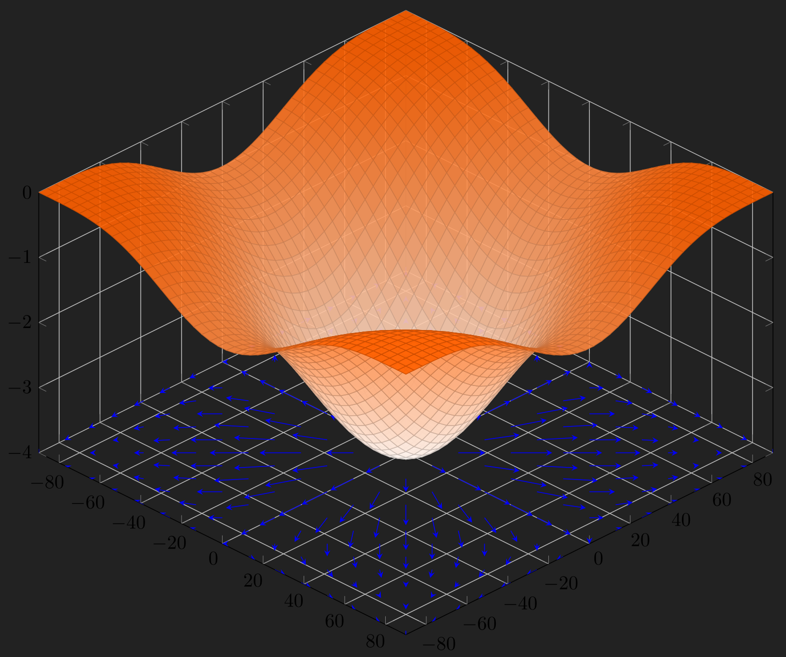
- Gradient describes a **direction** within an $n$-dimensional space
  - Each element just a number (a slope along one dimension)
  - Key: direction of steepest ascent along the surface of $f$



# Optimization

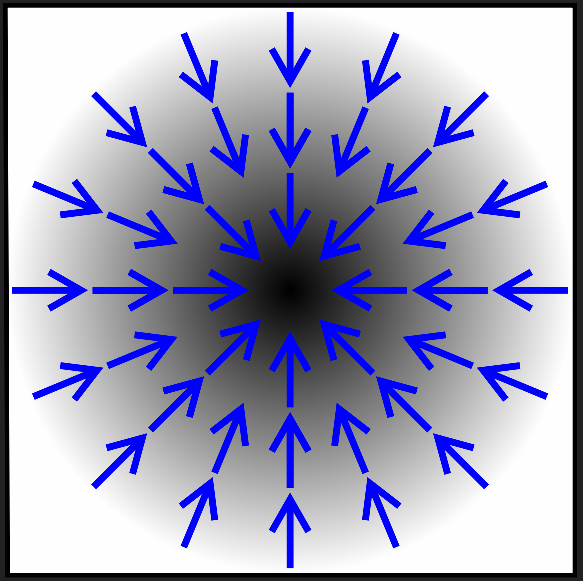
## The Gradient

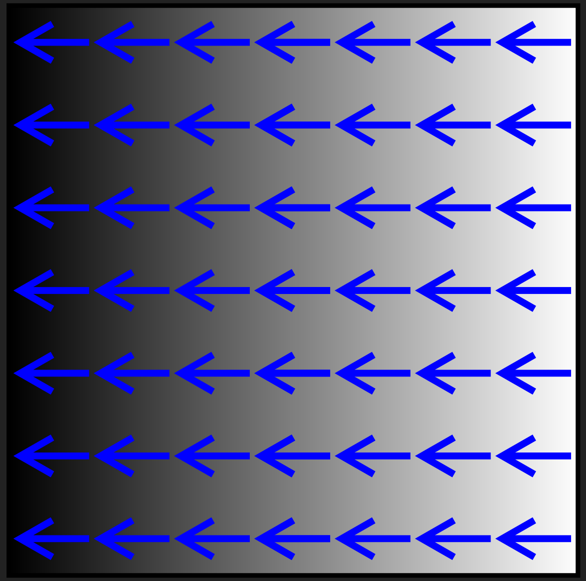
- Intuition:
  - What is the maximum rate at which one can walk up a line?
  - What is the maximum rate at which one can walk up a hill at a given point?
    - What about along a single dimension?



# Optimization

- Intuition:
  - What is the maximum rate at which one can walk up a line?
  - What is the maximum rate at which one can walk up a hill at a given point?
    - What about along a single dimension?
  - The magnitude gradient is just the aggregate of all of the slopes in every direction.





# Optimization

## Intuition: Alternate View of the Gradient

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\
&= \frac{\partial f}{\partial x}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{\partial f}{\partial z}\begin{bmatrix}0\\0\\1\end{bmatrix} \\
&= \begin{bmatrix}\frac{\partial f}{\partial x}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\\frac{\partial f}{\partial y}\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{\partial f}{\partial z}\end{bmatrix} \\
&= \begin{bmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial x}\end{bmatrix}
\end{aligned}
$$

# Optimization

## Gradient

- The magnitude of the gradient $|\nabla f|$ is the **Euclidean distance**, **Euclidean norm** or $\ell^2$ norm
- Same distance formula we learned in high school or earlier

$$|\nabla f(x,y)| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

$$|\nabla f(x,y,z)| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2},$$

# Optimization

## Gradient Descent

- Goal: Minimize loss/maximize log likelihood or correct answers
- Loss function is **convex**
- If we can walk down the loss function, we can minimize the error on the training data
  - This is what traning is
- Use the negative gradient $-|\nabla f|$ to minimize loss
  - Same as taking the negative partial derivative of every element in gradient vector

# Optimization

## Gradient Descent

- Recall our loss function is

$$\begin{aligned} \mathscr{L}(\mathbf{x}) &= -[y \log p + (1-y)\log(1-p)] \\ &= -[y \log \sigma(\mathbf{w} \cdot \mathbf{x}) + (1-y)(1 - \log \sigma(\mathbf{w} \cdot \mathbf{x} + b)] \end{aligned}$$

# Optimization

## Gradient Descent

- Recall our loss function is

$$\begin{aligned}\mathscr{L}(\mathbf{x}) &= -[y\log p + (1-y)\log(1-p)] \\ &= -[y\log\sigma(\mathbf{w}\cdot\mathbf{x}) + (1-y)(1-\log\sigma(\mathbf{w}\cdot\mathbf{x}+b)]\end{aligned}$$

Its derivative is:

$$\begin{aligned}\mathscr{L}\prime(\mathbf{x}) &= \sum_{(x,y)\in D}(y-p)x_i \\ &= \sum_{(\mathbf{x},y)\in D}(y-\sigma(\mathbf{w}\cdot\mathbf{x}+\mathbf{b}))x_i\end{aligned}$$

# Optimization

## Gradient Descent

$$\mathscr{L}\prime(\mathbf{x}) = \sum_{(x,y)\in D} (y-p)x_i$$
$$= \sum_{(\mathbf{x},y)\in D} (y-\sigma(\mathbf{w}\cdot\mathbf{x}+\mathbf{b}))x_i$$

- If we remove the summation, we have most of the update rule!

$$(y-p)x_i$$

# Optimization

## Gradient Descent

$$\mathscr{L}\prime(\mathbf{x}) = \sum_{(x,y)\in D} (y - p)x_i$$
$$= \sum_{(\mathbf{x},y)\in D} (y - \sigma(\mathbf{w} \cdot \mathbf{x} + \mathbf{b}))x_i$$

- If we remove the summation, we have most of the update rule!

$$(y - p)x_i$$

- Just add the learning rate:

$$w_i := w_i + \eta(y - p)x_i$$
$$w_i := w_i + \eta\nabla\mathscr{L}$$

# Optimization

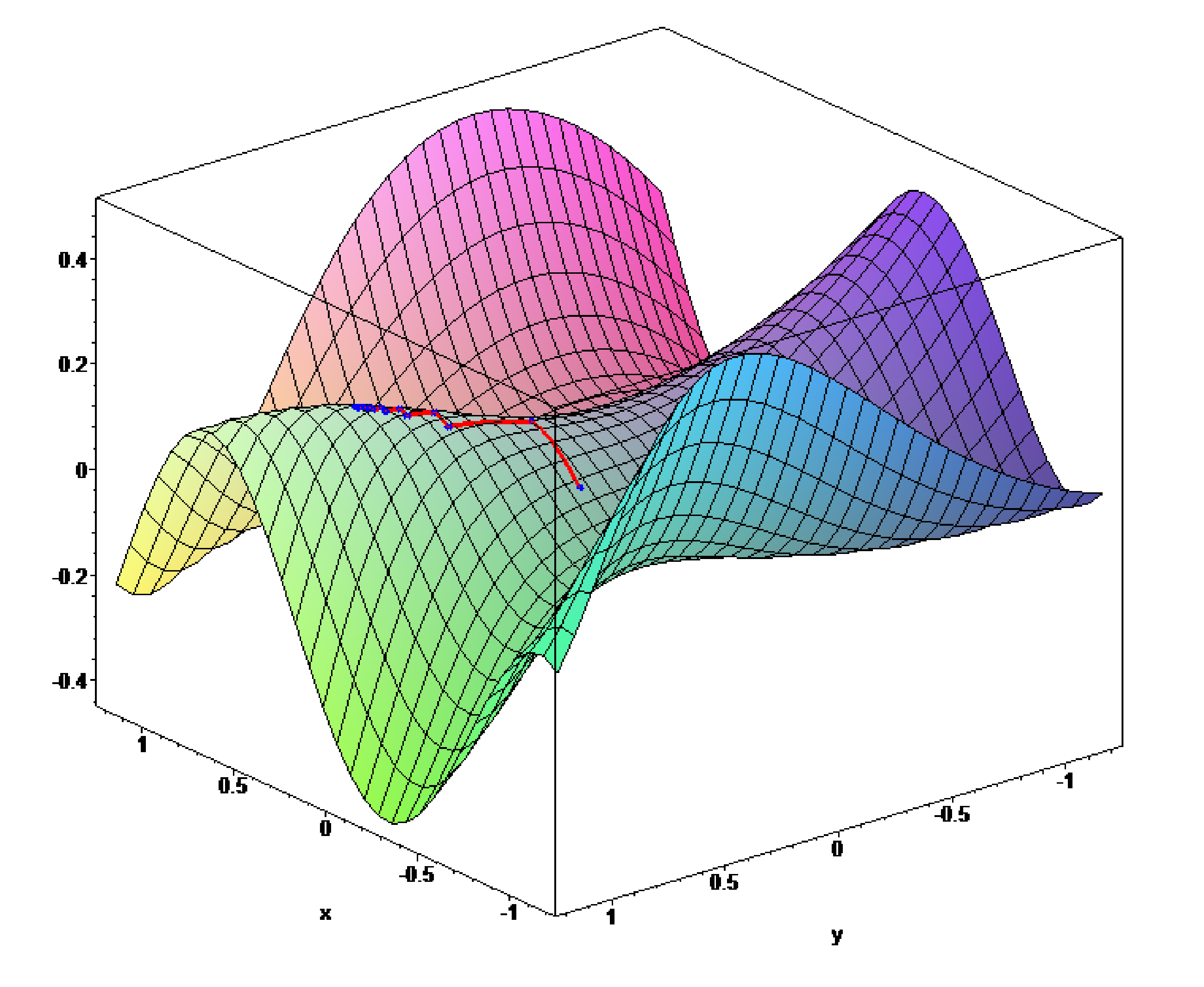
## Gradient Descent

$$w_i := w_i + \eta(y - p)x_i$$
$$w_i := w_i + \eta\nabla\mathscr{L}$$

- The loss function's derivative is our update rule, changes weight $w_i$ by the gradient given by the example!
- The learning rate $\eta$ is a **hyperparameter** that determines how big of a step we take in the direction of given by the gradient

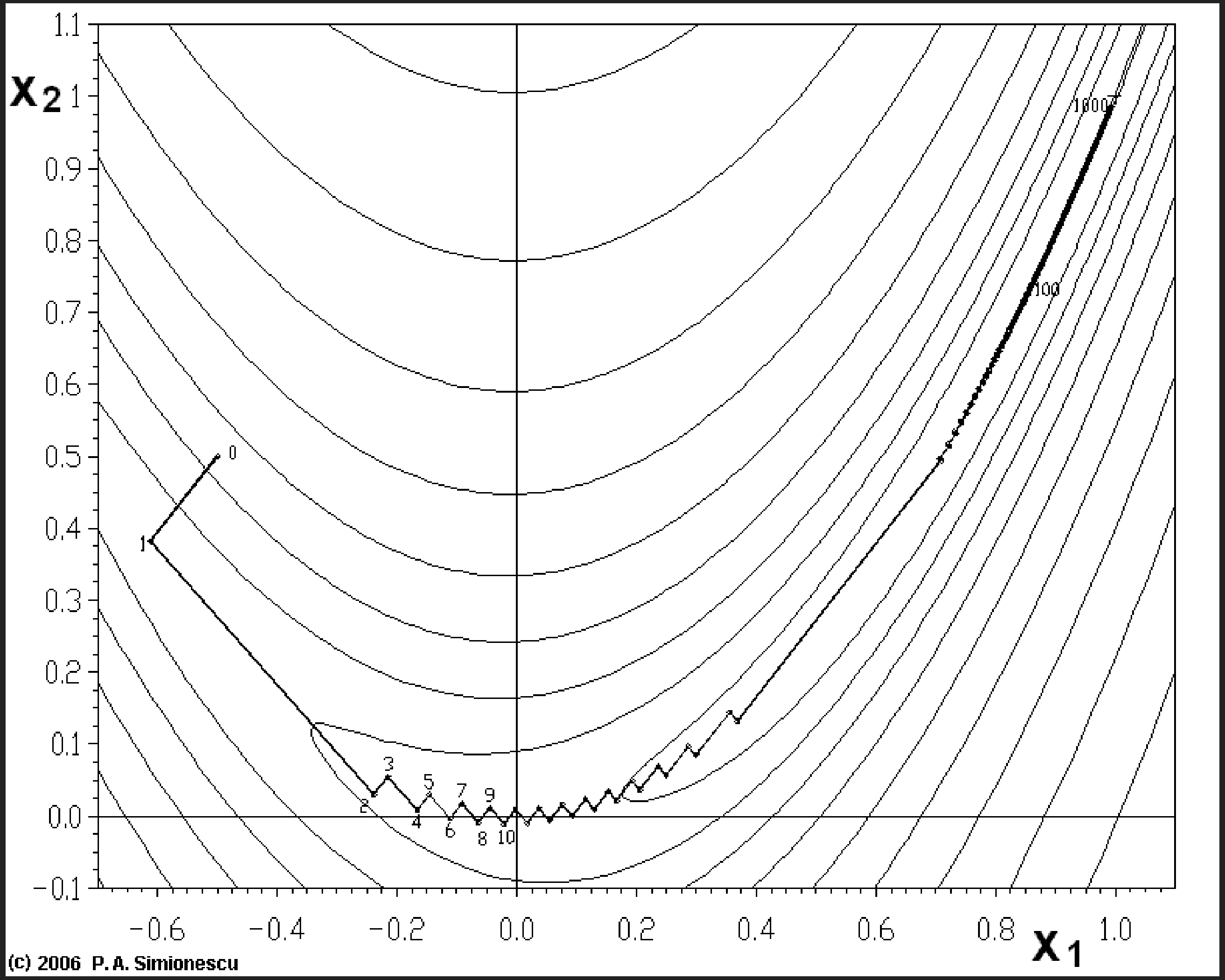
# Optimization

## Gradient Descent



# Optimization

## Gradient Descent



# Optimization

## Stochastic Gradient Descent

- Full gradient descent requires going over all training data for one update
- Instead, we use **stochastic** gradient descent (SGD), which uses a single example from training data to update all parameters on one iteration
- Understanding logistic regression and SGD is crucial for understanding more complex algorithms