Language Modeling and Probability

2021-9-8

Probability

- The **probability** of an **event** e has a number of epistemological interpretations
- Assuming we have **data**, we can count the number of times e occurs in the dataset to estimate the probability of e, $P(e)$.

$$P(e) = \frac{\text{count}(e)}{\text{count}(\text{all events})}.$$

- If we put all events in a bag, shake it up, and choose one at random (called **sampling**), how likely are we to get e?

Probability

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- What is the probability of heads, $P(e = H)$?
- We have "all" of two possibilities, $e \in \{H, T\}$.
- $P(e = H) = \frac{count(H)}{count(H)+count(T)}$

Probability

- Suppose we have a fair 6-sided die.

$$\frac{count(s)}{count(1) + count(2) + count(3) + \cdots + count(6)}$$
$$= \frac{1}{1+1+1+1+1+1} = \frac{1}{6}$$

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- What's the probability of getting "1"?

$$P(e = 1) = \frac{count(1)}{count(1) + count(2) + count(3)}$$
$$= \frac{2}{2+2+2} = \frac{1}{3}.$$

Probability

- The set of all probabilities for an event e is called a **probability distribution**
- Each die roll is an independent event (Bernoulli trial).

Probability

- Which is greater, $P(HHHHH)$ or $P(HHTHH)$?

Probability

- Which is greater, $P(HHHHH)$ or $P(HHTHH)$?
- Since the events are independent, they're equal

Probability Axioms

1. Probabilities of events must be no less than 0. $P(e) \geq 0$ for all e.
2. The sum of all probabilities in a distribution must sum to 1. That is, $P(e_1) + P(e_2) + \ldots + P(e_n) = 1 $. Or, more succinctly,

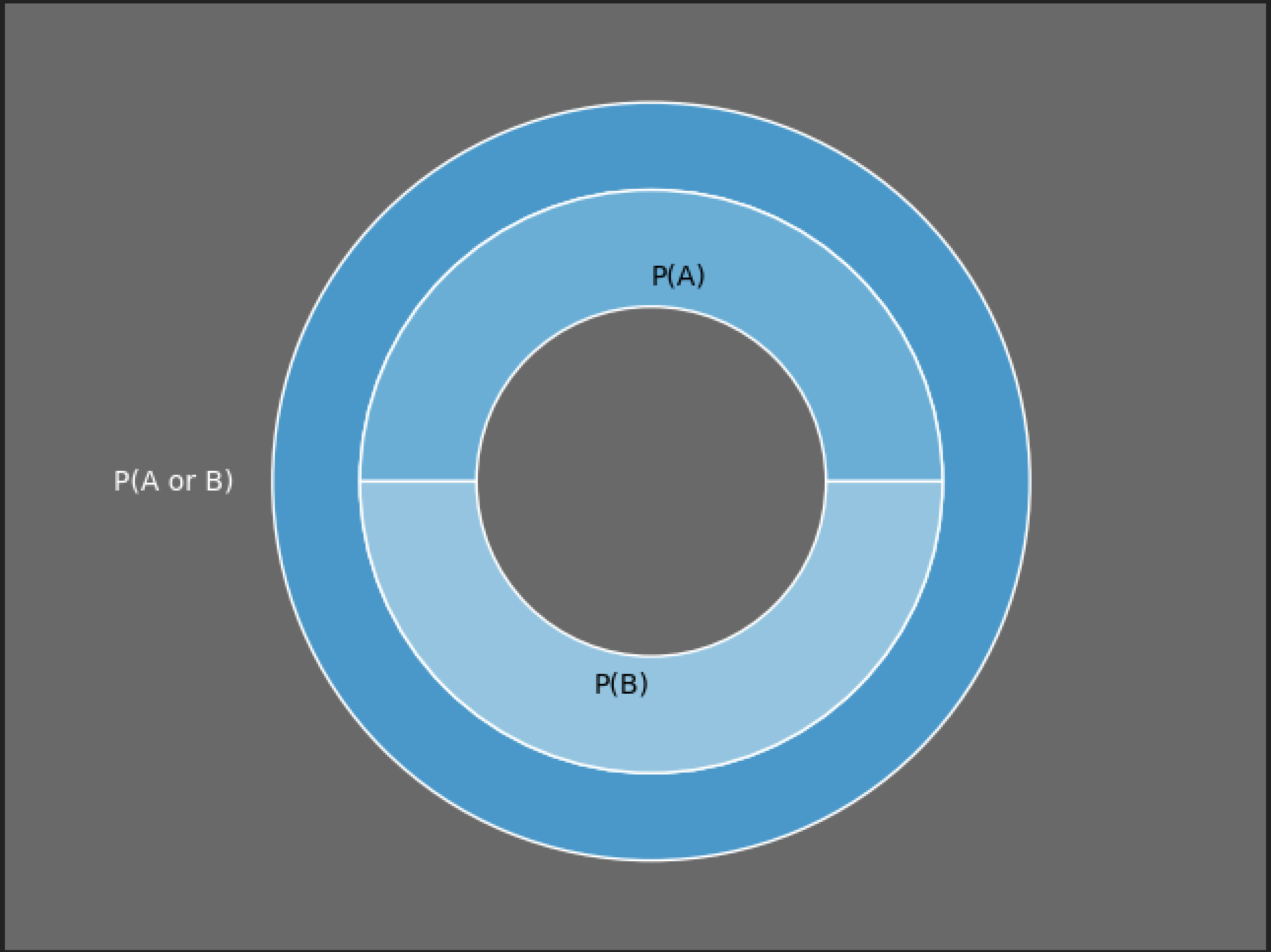
$$\sum_{e \in E} P(e) = 1.$$

3. The probability that one or both of two independent events e_1 and e_2 will occur is the sum of their respective probabilities.

$$P(e_1 \text{ or } e_2) = P(e_1 \cup e_2) = P(e_1) + P(e_2) \text{ when } e_1 \cap e_2 = \emptyset$$

Probability Disjunction

Probability space of two events, A and B



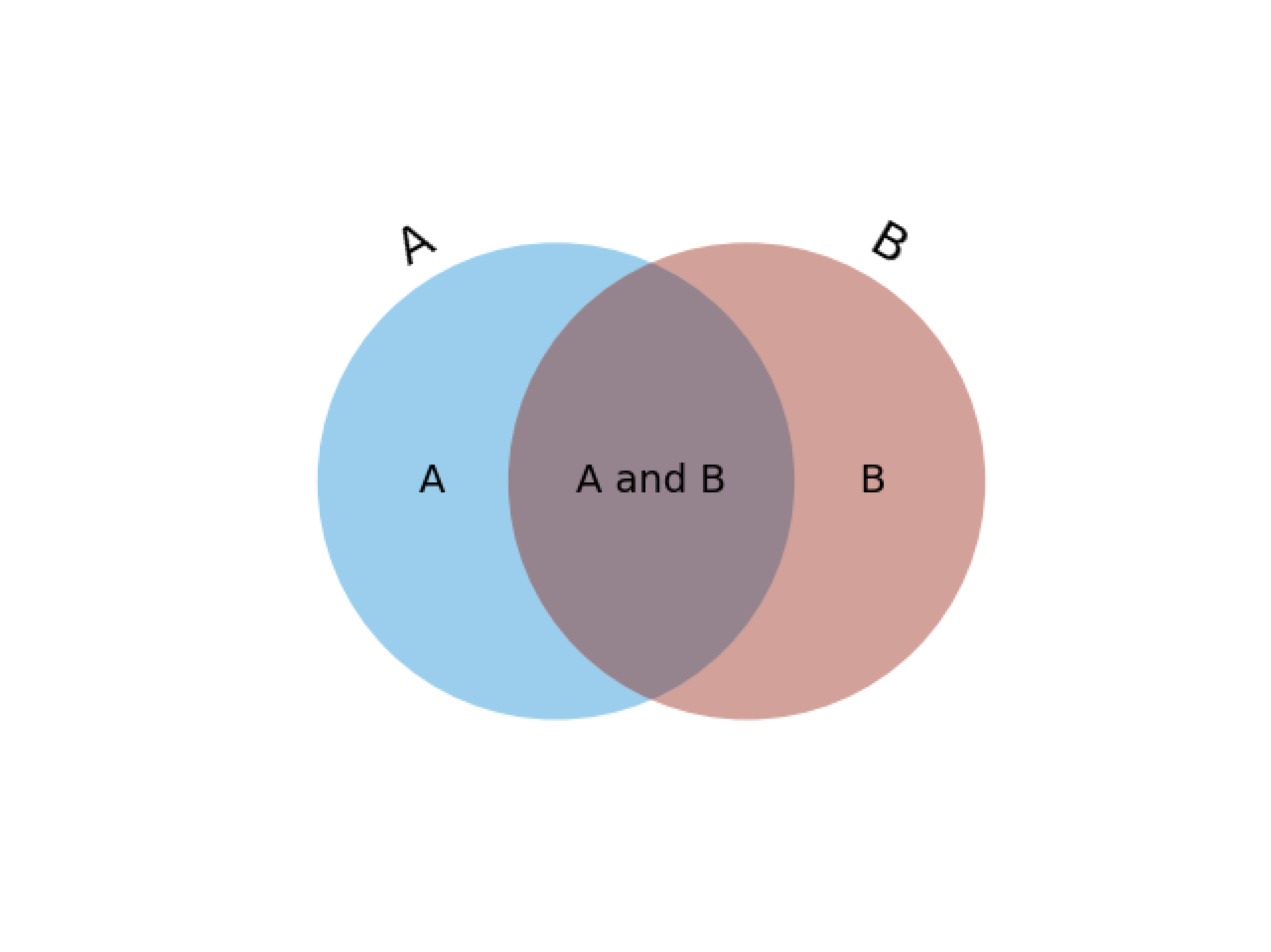
Joint Probability

The probability that two independent events e_1 and e_2 *both* occur is given by their product.

$$P(e_1 \wedge e_2) = P(e_1 \cap e_2) = P(e_1)P(e_2) \text{ when } e_1 \cap e_2 = \emptyset$$

- Intuitively, think of every probability as a *scaling factor*.
- You can think of a probability as the fraction of the probability space occupied by an event e_1.
    - $P(e_1 \wedge e_2)$ is the fraction of of e_1's probability space wherein e_2 also occurs.
    - So, if $P(e_1) = \frac{1}{2}$ and $P(e_2) = 1/3)$, then $P(e_2, e_2)$ is a third of a half of the probability space or $\frac{1}{3} \times \frac{1}{2}$.

Joint Probability



Probability of a Word

- Given a **corpus** of text, we can estimate the probability of the word "cat" occurring by counting. If the corpus has 100 word tokens and "cat" appears 10 times, then

$$P(\text{cat}) = \frac{count(\text{cat})}{count(\text{all words})} = \frac{10}{100} = 0.1$$

Probability of Words

- Suppose that "cat" appears ten times and "dog" appears five times. What is $P(\text{“cat dog”})$, i.e., the probability that we pick "cat" the first time and the next word is "dog"?

Probability of Words

- Suppose that "cat" appears ten times and "dog" appears five times. What is $P(\text{``cat dog''})$, i.e., the probability that we pick "cat" the first time and the next word is "dog"?
- Assmuing independence,

$$P(\text{``cat dog''}) = P(\text{cat}, \text{dog}) = P(\text{cat})P(\text{dog}) = \frac{10}{100} \times \frac{5}{100} = \frac{50}{10000} = .005$$

Probability of Words

- Is this reasonable?

Probability of Words

- Is this reasonable?
- Some words are more likely to come after others
- When we make independence assumptions, we call it a **bag of words** model
- We can do better

n-grams

- An n-gram is a sequence of tokens
- The n representes the number of tokens
  - n=1, 2, 3 are called unigrams, bigrams, and trigrams, respectively
  - After that, just say the number, e.g., 4-grams.
- Ex: In the sentence, "the quick brown fox jumped over the

Examples of n-gram

Given sentence: "the quick brown fox jumped over the lazy dog"

- unigrams: {the, quick, brown, fox, jumped, ...}
- bigrams: {the quick, quick brown, brown fox, fox jumped, jumped over, ...}
- Trigrams?
- 4-grams?

Conditional Probability

- A **conditional probability** is the probability that one event occurs given that we take another for granted.
- The probability of e_2 given e_1 is $P(e_2 \mid e_1)$.
- This is the probability that e_2 will occur given that we take for granted that e_1 occurs.

Conditional Probability

If e_1 and e_2 are independent, then

$$\begin{aligned} P(e_2|e_1) &= P(e_2, e_1) \\ &= P(e_2 \cap e_1) \\ &= P(e_1)P(e_2) \\ &= P(e_1 \cap e_2) \\ &= P(e_1)P(e_2). \end{aligned}$$

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- But what if they're not independent?

Conditional Probability

- What if seeing word w_i affects the probability of word w_{i+1}?
- Knowing the previous word gives us *more information* with which we can make a *more informative estimate* of the probability

n-gram Probability

- Suppose we've seen the word "computer." How would we calculate the probability that the next word is "science" given the **context** "computer"?

n-gram Probability

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$$P(\text{science}|\text{computer}) = \frac{count(\text{computer science})}{count(\text{computer})}.$$

- This is the fraction of occurrences of "computer science" to the occurrences of just "computer."
- This is answering the question, Of the instances of "computer," how many of them are followed by "science?"

n-gram Probability

- Another way of looking at it

$$P(\text{science}|\text{computer}) = \frac{P(\text{computer science})}{P(\text{computer})}$$

$$= \frac{\frac{C(\text{computer science})}{C(\text{all bigrams})}}{\frac{C(\text{computer }*)}{C(\text{all bigrams})}}$$

$$= \frac{C(\text{computer science})}{C(\text{computer})}$$

where $$*$$ refers to any word.

n-gram Probability

More generally, given words w_1, w_2,

$$P(w_2|w_1) = \frac{C(w_1, w_2)}{C(w_1)} = \frac{P(w_1, w_2)}{P(w_1)}$$

n-gram Probability

- Be sure to distinguish between the probability of "computer science" and the probability of "science" given "computer." This is called a **bigram** probability, because we're using a sequence of two words in our calculation.
  - Often, these are both called "bigram probability," but the first is the probability of a bigram while the second is bigram-based conditional probability.

n-gram Probability

- Trigram probability

$$\frac{P(\text{the computer science})}{P(\text{the computer})}$$

- We can keep going to 4-grams, etc.

n-gram Probability

- Trigram probability

$$\frac{P(\text{the computer science})}{P(\text{the computer})}$$

- We can keep going to 4-grams, etc.
- Why not always use huge number of n-grams?

n-gram Probability

- The longer a sequence, the less likely it is to occur
- If $C(\text{the computer science})$ is 0---or, worse, $C(\text{the computer})$ is 0---our calculations aren't useful.
- Data sparsity

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- Extremely unlikely; possibly 0.

Probability of a Sentence

- Instead, assume that each word depends only on the previous word (Markov property).
  - Bigram model
- Pad the sentence with special characters `<s>` and `</s>`, so that we know the probability of the first word and the end of the sentence

Probability of a Sentence

With a bigram model,

$$P(\text{<s> colorless kumquat ideas sleep furiously</s>})$$

is estimated by

$$P(\text{colorless}|\text{<s>})P(\text{kumquat}|\text{colorless})P(\text{ideas}|\text{kumquat})P(\text{sleep}|\text{ideas})P(\text{furiously}|\text{sleep})P(\text{</s>}|\text{sleep}).$$

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- What would be the *bag of words* probability of this sentence?

Smoothing

- Zeroes are a problem!
- Also called **pseudocounting**

Smoothing

- Suppose "inexorably" never appears.
- Then $P(\text{inexorably})) = 0$
- How can we deal with this?

Smoothing

$$\frac{C(w)}{C(\text{all words})},$$

becomes

$$\frac{C(w)+1}{C(\text{all words})+|V|}.$$

Smoothing

In general,

$$\frac{C(w) + \alpha}{C(\text{all words}) + \alpha|V|}$$

where $\alpha < 1$