Basic Probability and Statistics

Alvin Grissom II, Haverford College

Probability

- The **probability** of an **event** e has a number of epistemological interpretations
- Assuming we have **data**, we can count the number of times e occurs in the dataset to estimate the probability of e, $P(e)$.

$$P(e) = \frac{\text{count}(e)}{\text{count}(\text{all events})}.$$

- If we put all events in a bag, shake it up, and choose one at random (called **sampling**), how likely are we to get e?

Probability

- Suppose we flip a fair coin
- What is the probability of heads, $P(e = H)$?

Probability

- Suppose we flip a fair coin
- What is the probability of heads, $P(e = H)$?
- We have "all" of two possibilities, $e \in \{H, T\}$.

Probability

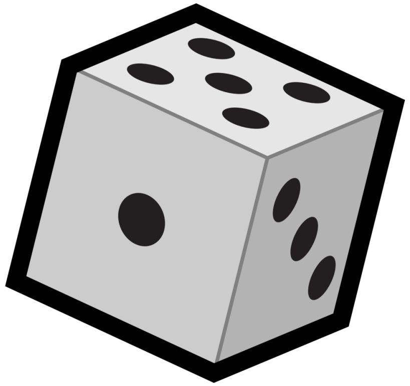
- Suppose we flip a fair coin
- What is the probability of heads, $P(e = H)$?
- We have "all" of two possibilities, $e \in \{H, T\}$.
- $P(e = H) = \frac{count(H)}{count(H)+count(T)}$

Probability

- Suppose we have a fair 6-sided die.

$$\frac{count(s)}{count(1)+count(2)+count(3)+\cdots+count(6)} = \frac{1}{1+1+1+1+1+1} = \frac{1}{6}$$

Probability



- What about a die with on ly three numbers $\{1, 2, 3\}$, each of which appears twice?
- What's the probability of getting "1"?

Probability

- What about a die with only three numbers $\{1, 2, 3\}$, each of which appears twice?
- What's the probability of getting "1"?

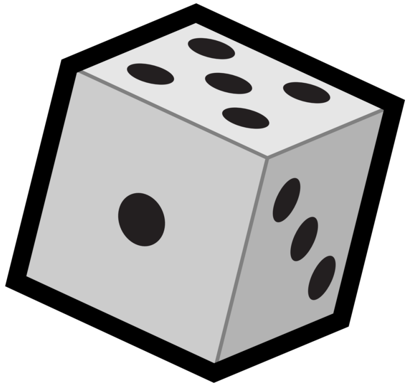
$$P(e = 1) = \frac{count(1)}{count(1) + count(2) + count(3)}$$

Probability

- What about a die with on ly three numbers $\{1, 2, 3\}$, each of which appears twice?
- What's the probability of getting "1"?

$$P(e=1) = \frac{count(1)}{count(1) + count(2) + count(3)} = \frac{2}{2+2+2} = \frac{1}{3}.$$

Probability



- The set of all probabilities for an event e is called a **probability distribution**
- Each die roll is an independent event (Bernoulli trial).

Probability

- Which is greater, $P(HHHHH)$ or $P(HHTHH)$?

Probability

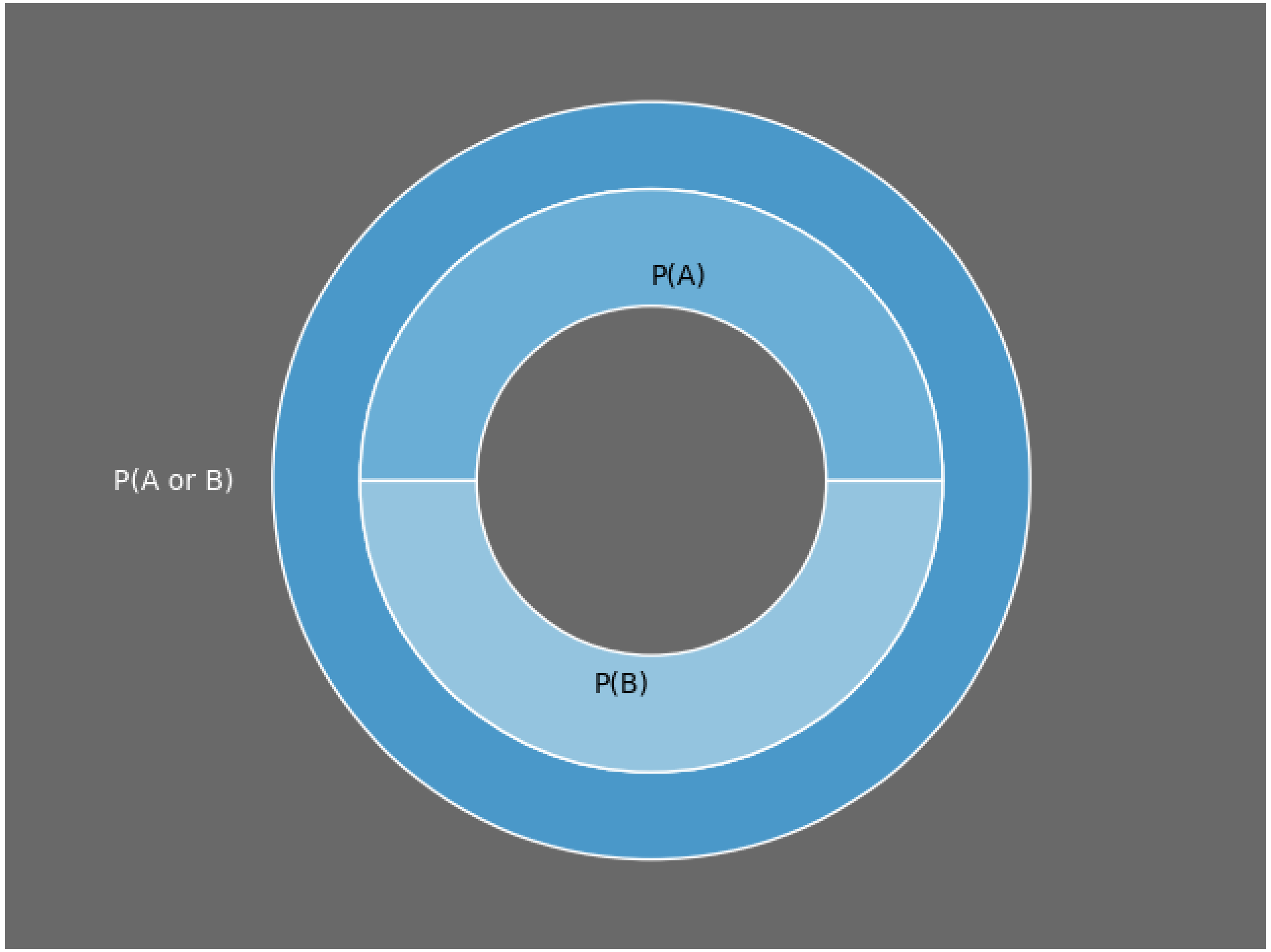
- Which is greater, $P(HHHHH)$ or $P(HHTHH)$?
- Since the events are independent, they're equal

Probability Axioms

1. Probabilities of events must be no less than 0. $P(e) \geq 0$ for all e.
2. The sum of all probabilities in a distribution must sum to 1. That is, $P(e_1) + P(e_2) + \ldots + P(e_n) = 1$. Or, more succinctly,
$$\sum_{e \in E} P(e) = 1.$$
3. The probability that one or both of two independent events e_1 and e_2 will occur is the sum of their respective probabilities.
$$P(e_1 \text{ or } e_2) = P(e_1 \cup e_2) = P(e_1) + P(e_2) \text{ when } e_1 \cap e_2 = \emptyset$$

Probability Disjunction

Probability space of two independent events, A and B



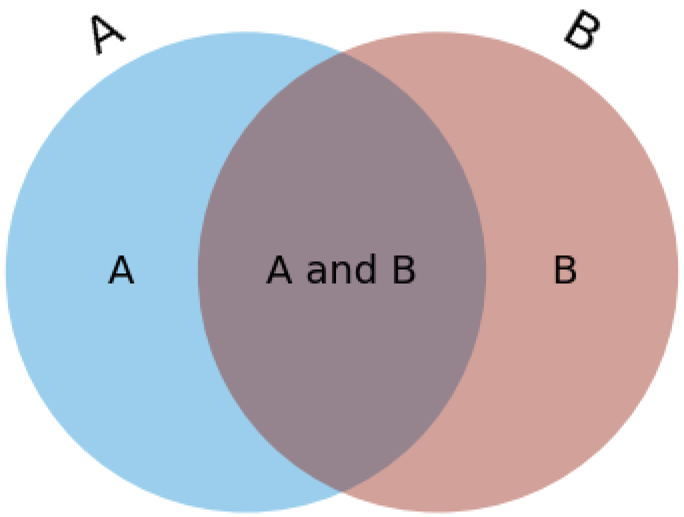
Joint Probability

The probability that two independent events e_1 and e_2 *both* occur is given by their product.

$$P(e_1 \wedge e_2) = P(e_1 \cap e_2) = P(e_1)P(e_2) \textbf{ when } e_1 \cap e_2 = \emptyset$$

- Intuitively, think of every probability as a *scaling factor*.
- You can think of a probability as the fraction of the probability space occupied by an event e_1.
  - $P(e_1 \wedge e_2)$ is the fraction of of e_1's probability space wherein e_2 also occurs.
  - So, if $P(e_1) = \frac{1}{2}$ and $P(e_2) = \frac{1}{3}$, then $P(e_2, e_2)$ is a third of a half of the probability space or $\frac{1}{3} \times \frac{1}{2}$.

Joint Probability



Conditional Probability

- A **conditional probability** is the probability that one event occurs given that we take another for granted.
- The probability of e_2 given e_1 is $P(e_2 \mid e_1)$.
- This is the probability that e_2 will occur given that we take for granted that e_1 occurs.

Conditional Probability

If e_1 and e_2 are independent, then

$$P(e_1)(e_2|e_1) = P(e_2, e_1) = P(e_2 \cap e_1) = P(e_1)P(e_2) = P(e_1 \cap e_2) = P(e_1)P(e_2).$$

Conditional Probability

If e_1 and e_2 are independent, then

$$P(e_1)P(e_2|e_1) = P(e_2, e_1) = P(e_2 \cap e_1) = P(e_1)P(e_2) = P(e_1 \cap e_2) = P(e_1)P(e_2).$$

- But what if they're not independent?

Conditional Probability

In general,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

when $P(A) \neq 0$.

Conditional Probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Suppose we have some probabilities of properties of toys:
$P(\text{round}) = 0.3$ and $P(\text{blue}, \text{round}) = 0.2$.
Then, $P(\text{blue}|\text{round})$ is the fraction of the round toys that are also blue.

$$P(\text{blue}|\text{round}) = \frac{P(\text{blue}, \text{round})}{P(\text{round})} = \frac{0.2}{0.3} = 0.667$$

Answers question: If we know that the toy is round, how likely is it to be blue?
Interpretation: Denominator is the probability space of round toys; numerator giving us the fraction of that space containing blue toys.

Probability Distribution Functions

- Ultimately, a probability function is a *mathematical function* called a **probability distribution**.
- Input a value or values and get a probability

$$P(x) = p$$

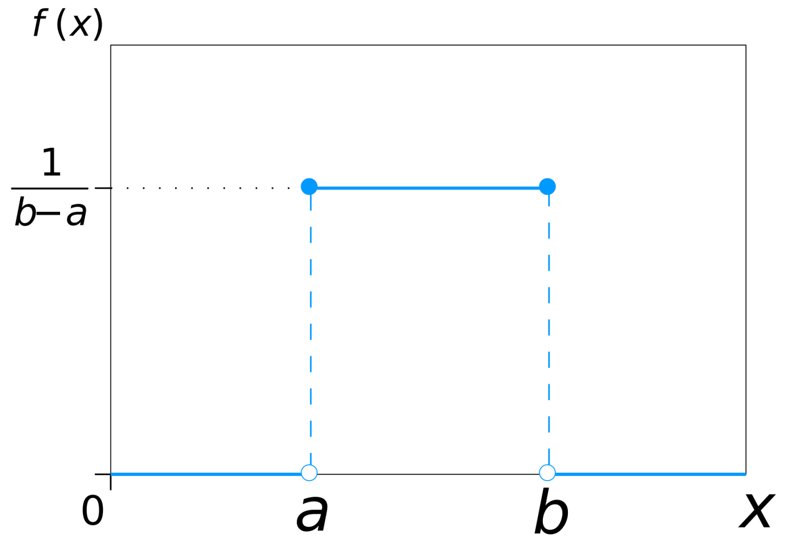
- Some are discrete; others are continuous.
- All possible inputs must sum to 1.

Probability Distribution Functions

- Like any other function, probability functions can be graphed.
- There are several common distributions.

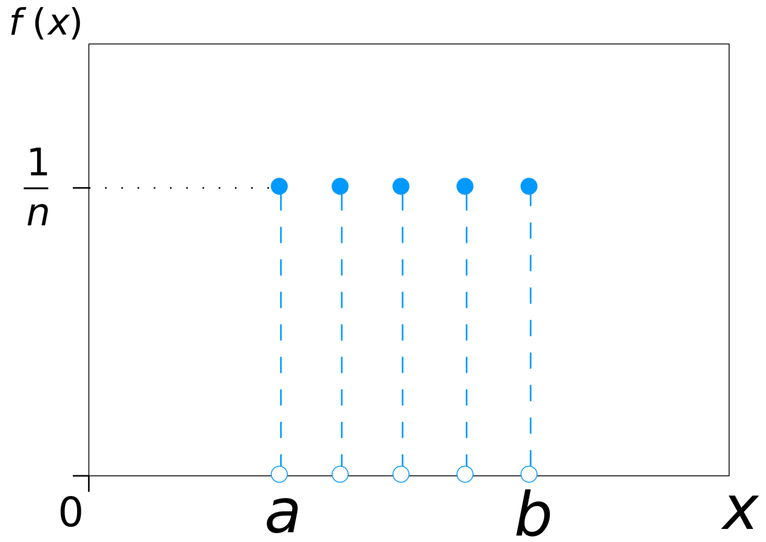
Probability Distribution Functions

Continuous Uniform Distribution



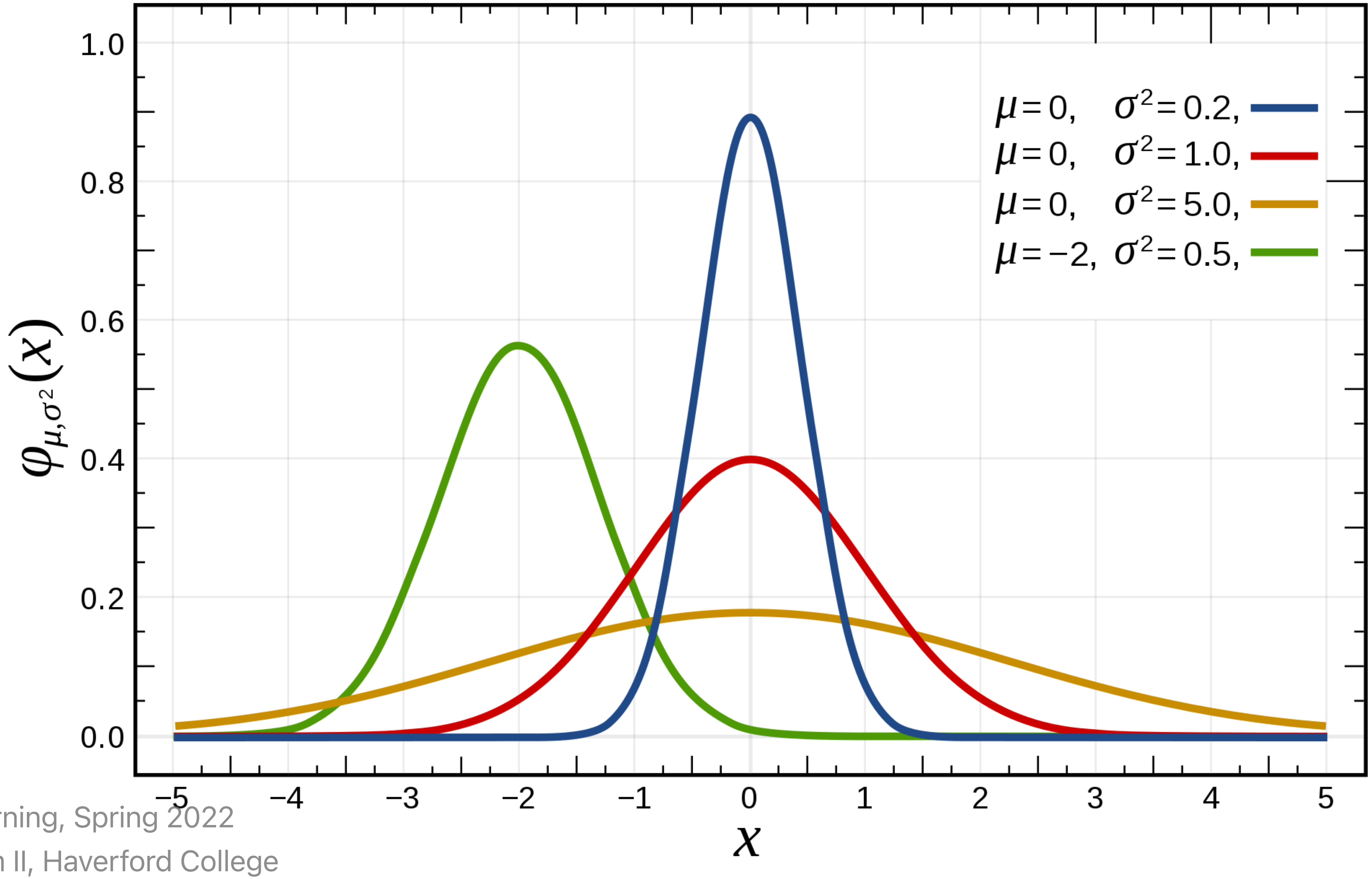
Probability Distribution Functions

Discrete Uniform Distribution



Probability Distribution Functions

- Normal/Gaussian Distribution



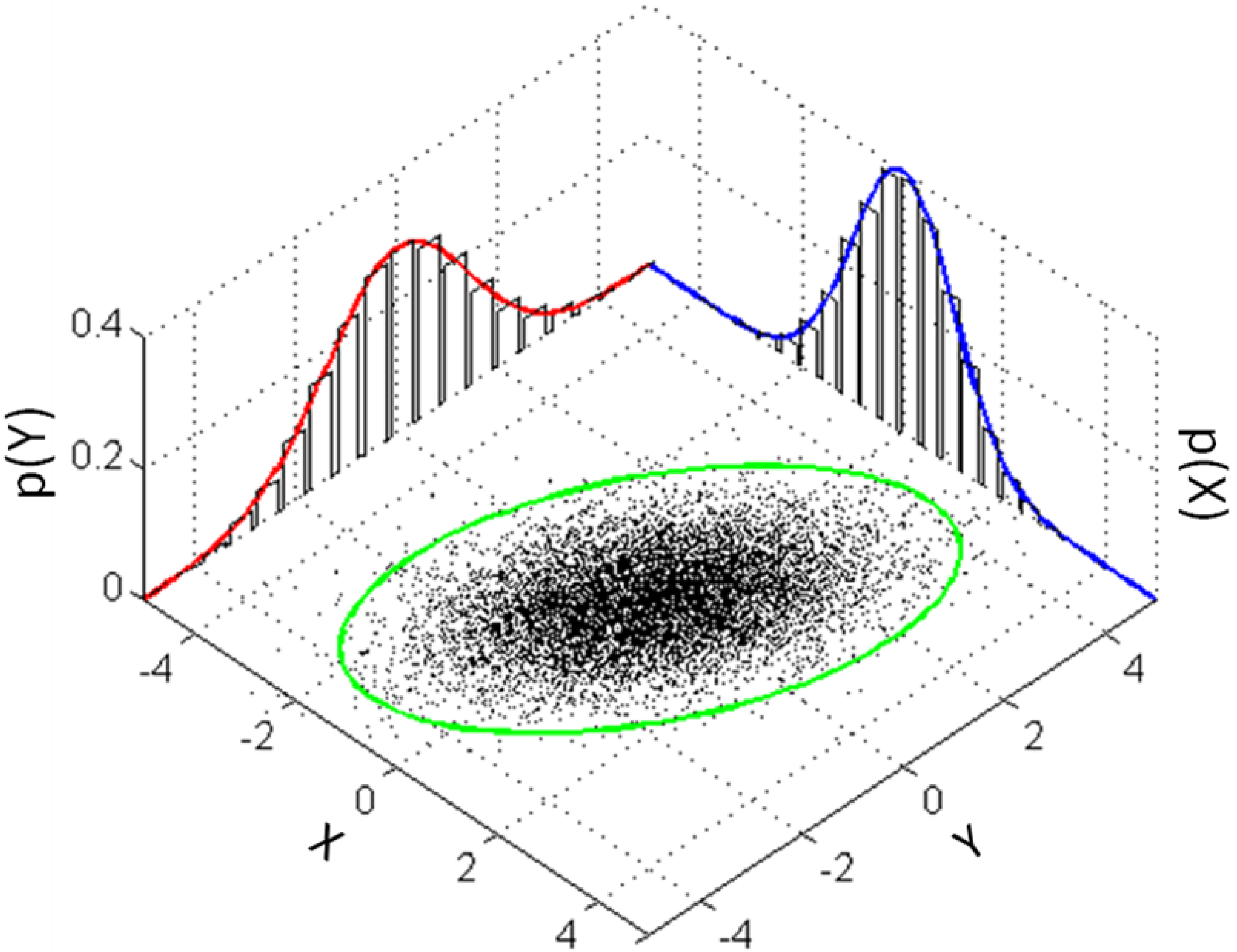
Conditional Probability Distributions

$$P(X|Y)$$

- Still a probability distribution, as always.
- All inputs must still sum to 1.
- How do we find cumulative probability for continuous vs. discrete distributions?

Joint/Multvariate Probability Distributions

$$P(X, Y)$$



Marginal Probability Distributions

- Given a discrete joint probability distribution function

$$P(X, Y),$$

how would we find

$$P(X)?$$

Marginal Probability Distributions

Given a discrete joint probability distribution function $P(X, Y)$, how would we find $P(X)$?

- "Marginalize out" the Y.
  - Sum up all y's.
- Discrete Case: $p(x) = \sum_{y \in Y} P(x, y)$

Marginal Probability Distributions

Given a discrete joint probability distribution function $P(X, Y)$, how would we find $P(X)$?

- "Marginalize out" the Y (sum over all all $y\in Y).
- Fix the X.
- Discrete Case: $p(x) = \sum_{y \in Y} P(x, y)$
- Continuous Case: $p(x) = \int p(x, y) dy$

Marginal Probability Distributions

Given a discrete joint probability distribution function $P(X, Y)$, how would we find $P(X)$?

- "Marginalize out" (fix) the Y.