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Question 1

HMM is defined by <T, O, θ>, where,

T = {START, X, Y, Z, STOP}
O = {a, b, c}

θ consists of **transmission parameter** $a_{u,v}$ and **emission parameter** $b_u(o)$

$a_{u,v} = \frac{count(u,v)}{count(u)}$**:**

u\v	X	Y	Z	STOP
START	$\frac{2}{4} = 0.5$	0.0	$\frac{2}{4} = 0.5$	0.0
X	0.0	$\frac{2}{5} = 0.4$	$\frac{2}{5} = 0.4$	$\frac{1}{5} = 0.2$
Y	$\frac{1}{5} = 0.2$	0.0	$\frac{1}{5} = 0.2$	$\frac{3}{5} = 0.6$
Z	$\frac{2}{5} = 0.4$	$\frac{3}{5} = 0.6$	0.0	0.0

$b_u(o) = \frac{count(u \to o)}{count(u)}$**:**

u\o	a	b	c
X	$\frac{2}{5} = 0.4$	$\frac{3}{5} = 0.6$	0.0
Y	$\frac{1}{5} = 0.2$	0.0	$\frac{4}{5} = 0.8$
Z	$\frac{1}{5} = 0.2$	$\frac{3}{5} = 0.6$	$\frac{1}{5} = 0.2$

Question 2

j=0	j=1 [b]	j=2 [c]	j=3
	X	X	
START	Y	Y	STOP
	Z	Z	

Using the Viterbi algorithm, where n=2,

Initialization step:
$\pi(0, \text{START}) = 1$

Forward Step:
$\pi(1, X)$
$= \pi(0, \text{START}) \times b_X(b) \times a_{START,X}$
$= 1 \times 0.6 \times 0.5$
$= 0.3$

$\pi(1, Y)$
$= \pi(0, \text{START}) \times b_Y(b) \times a_{START,Y}$
$= 1 \times 0.0 \times 0.0$
$= 0.0$

$\pi(1, Z)$
$= \pi(0, \text{START}) \times b_Z(b) \times a_{START,Z}$
$= 1 \times 0.6 \times 0.5$
$= 0.3$

$\pi(2, X)$
$= \max_v \{ \pi(1,X) \times b_X(c) \times a_{X,X},$
$\pi(1,Y) \times b_X(c) \times a_{Y,X},$
$\pi(1,Z) \times b_X(c) \times a_{Z,X} \}$
$= \max_v \{ 0.3 \times 0.0 \times 0.0, 0.0, 0.3 \times 0.0 \times 0.4\}$
$= 0.0$

$\pi(2, Y)$
$= \max_v \{ \pi(1,X) \times b_Y(c) \times a_{X,Y},$
$\pi(1,Y) \times b_Y(c) \times a_{Y,Y},$
$\pi(1,Z) \times b_Y(c) \times a_{Z,Y} \}$
$= \max_v \{ 0.3 \times 0.8 \times 0.4, 0.0, 0.3 \times 0.8 \times 0.6\}$
$= 0.144$

$\pi(2, Z)$
$= \max_v \{ \pi(1,X) \times b_Z(c) \times a_{X,Z} ,$
$\pi(1,Y) \times b_Z(c) \times a_{Y,Z} ,$
$\pi(1,Z) \times b_Z(c) \times a_{Z,Z} \}$
$= \max_v \{ 0.3 \times 0.2 \times 0.4, 0.0, 0.3 \times 0.2 \times 0.0\}$
$\underline{= 0.024}$

$\pi(3, STOP)$
$= \max_v \{ \pi(2,X) \times a_{X,STOP} ,$
$\pi(2,Y) \times a_{Y,STOP} ,$
$\pi(2,Z) \times a_{Z,STOP} \}$
$= \max_v \{ 0.0, 0.144 \times 0.6, 0.024 \times 0.0\}$
$\underline{= 0.0864}$

Backtrack step to find path taken:

y^*_2
$= \arg\max_u \{\pi(2,X) \times a_{X, STOP}$
$\pi(2,Y) \times a_{Y, STOP}$
$\pi(2,Z) \times a_{Z, STOP}\}$
$= \arg\max_u \{ 0.0, 0.144 \times 0.6, 0.024\ 0.0\}$
$= Y$

y^*_1
$= \arg\max_u \{\pi(1,X) \times a_{X, Y}$
$\pi(1,Y) \times a_{Y, Y}$
$\pi(1,Z) \times a_{Z, Y}\}$
$= \arg\max_u \{ 0.3 \times 0.4, 0.0, 0.3 \times 0.6\}$
$= \arg\max_u \{ 0.12 , 0.0, 0.18\}$
$= Z$

Therefore, the most probable state sequence by the Viterbi algorithm is **(Z,Y)**.

Question 3

s_i*

= arg $\max_{si}$ P(s_i | o_1 = b, o_2 = c)

$$= \arg\max_{si} \frac{\alpha_u(i)\ \beta_u(i)}{\sum_v \alpha_v(j)\ \beta_v(j)}$$

= arg $\max_u \alpha_u(i)\ \beta_u(i)$ with i ∈ {1,2}

where,

$\alpha_u(i)$ =P(x_1, x_2, …, x_{i-1}, y_i = u; θ)

$\beta_u(i)$ =P(x_i, x_{i+1} …, x_n, | y_i = u; θ)

Forward Step:

(Base case)

$\alpha_u(1) = \alpha_{START,u}$ where u ∈ {X, Y, Z}

$\alpha_X(1) = \alpha_{START,X} = 0.5$

$\alpha_Y(1) = \alpha_{START,Y} = 0.0$

$\alpha_Z(1) = \alpha_{START,Z} = 0.5$

(Recursive case)

For i=2,

$\alpha_u(i + 1) = \sum_v \alpha_v(i)\ a_{v,u}\ b_v(x_i)$ where u ∈ {X, Y, Z}

$\alpha_X(2)$

$= \sum_v \alpha_v(1)\ a_{v,X}\ b_v(x_1)$

$= \alpha_X(1)\ a_{X,X}\ b_X(b) + \alpha_Y(1)\ a_{Y,X}\ b_Y(b) + \alpha_Z(1)\ a_{Z,X}\ b_Z(b)$

$= 0.5 \times 0.0 + 0.0 + 0.5 \times 0.4 \times 0.6$

= 0.12

$\alpha_Y(2)$

$= \sum_v \alpha_v(1)\ a_{v,Y}\ b_v(x_1)$

$= \alpha_X(1)\, a_{X,Y}\, b_X(b) \;+\; \alpha_Y(1)\, a_{Y,Y}\, b_Y(b) \;+ \alpha_Z(1)\, a_{Z,Y}\, b_Z(b)$

$= 0.5 \times 0.4 \times 0.6 + 0.0 + 0.5 \times 0.6 \times 0.6$

<u>= 0.3</u>

$\alpha_Z(2)$

$= \sum_v \alpha_v(1)\, a_{v,Z}\, b_v(x_1)$

$= \alpha_X(1)\, a_{X,Z}\, b_X(b) \;+\; \alpha_Y(1)\, a_{Y,Z}\, b_Y(b) \;+ \alpha_Z(1)\, a_{Z,Z}\, b_Z(b)$

$= 0.5 \times 0.4 \times 0.6 + 0.0 + 0.0$

<u>= 0.12</u>

Backward Step:

(Base case)

$\beta_u(n) = \alpha_{u,\,STOP}\, b_u(x_n)$ where u ∈ {X, Y, Z}

$\beta_X(2)$

$= a_{X,\,STOP}\, b_X(c)$

= 0.2 x 0.0

<u>= 0.0</u>

$\beta_Y(2)$

$= a_{Y,\,STOP}\, b_Y(c)$

= 0.6 x 0.8

<u>= 0.48</u>

$\beta_Z(2)$

$= a_{Z,\,STOP}\, b_Z(c)$

<u>= 0.0</u>

(Recursive case)

For i=1, calculate:

$\beta_u(i) = \sum_v a_{u,v}\, b_u(x_i)\, \beta_v(i+1)$

$\beta_X(1)$

$= \sum_v a_{X,v}\, b_X(b)\, \beta_v(2)$

$= a_{X,X}\, b_X(b)\, \beta_X(2) + a_{X,Y}\, b_X(b)\, \beta_Y(2) + a_{X,Z}\, b_X(b)\, \beta_Z(2)$

= 0.0 + 0.4 x 0.6 x 0.48 + 0.0
<u>= 0.1152</u>

$\beta_Y(1)$

$= \sum_v a_{Y,v}\, b_Y(b)\, \beta_v(2)$
$= a_{Y,X}\, b_Y(b)\, \beta_X(2) + a_{Y,Y}\, b_Y(b)\, \beta_Y(2) + a_{Y,Z}\, b_Y(b)\, \beta_Z(2)$
<u>= 0.0</u>

$\beta_Z(1)$

$= \sum_v a_{Z,v}\, b_Z(b)\, \beta_v(2)$
$= a_{Z,X}\, b_Z(b)\, \beta_X(2) + a_{Z,Y}\, b_Z(b)\, \beta_Y(2) + a_{Z,Z}\, b_Z(b)\, \beta_Z(2)$
= 0.0 + 0.6 x 0.6 x 0.48 + 0.0
<u>= 0.1728</u>

s_1^*
$= \arg\max_u \alpha_u(1)\, \beta_u(1)$
$= \arg\max_u \{\alpha_X(1)\, \beta_X(1),\ \alpha_Y(1)\, \beta_Y(1),\ \alpha_Z(1)\, \beta_Z(1)\}$
$= \arg\max_u \{0.5 \times 0.1152,\ 0.0,\ 0.5 \times 0.1728\}$
$= \arg\max_u \{0.0576,\ 0.0,\ 0.0864\}$
= Z

s_2^*
$= \arg\max_u \alpha_u(2)\, \beta_u(2)$
$= \arg\max_u \{\alpha_X(2)\, \beta_X(2),\ \alpha_Y(2)\, \beta_Y(2),\ \alpha_Z(2)\, \beta_Z(2)\}$
$= \arg\max_u \{0.12 \times 0.0,\ 0.3 \times 0.48,\ 0.12 \times 0.0\}$
$= \arg\max_u \{0.0,\ 0.144,\ 0.0\}$
= Y

Therefore, the state sequence is (Z,Y) and the marginal distribution for s_1 is 0.0864 while the marginal distribution for s_2 is 0.144.

Question 4

Given that states are from 0 to N, where 0 = START and N = STOP, using the EM algorithm,

$P(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_{i-1}, z_i = u, x_i, x_{i+1} \ldots, x_n, y_i, y_{i+1} \ldots, y_n; \theta)$

$= P(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_{i-1}, z_i = u; \theta) \times$
$P(x_i, x_{i+1} \ldots, x_n, y_i, y_{i+1} \ldots, y_n \mid z_i = u; \theta)$

$= \alpha_u(i) \times \beta_u(i)$

where,
$\alpha_u(i) = P(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_{i-1}, z_i = u; \theta)$
$\beta_u(i) = P(x_i, x_{i+1} \ldots, x_n, y_i, y_{i+1} \ldots, y_n \mid z_i = u; \theta)$

Forward Step:

$\alpha_u(1) = \alpha_{START,u}$ where $u \in \{1, \ldots, N-1\}$

For i=2 to i=n, calculate:
$\alpha_u(i + 1)$

$= \sum_v \alpha_v(i)\, a_{v,u}\, b_v(x_i)\, c_{x_i}(y_i)$

where,
$c_{x_i}(y_i) = P(y \mid x)$

Backward Step:

$\beta_u(n) = a_{u,STOP}\, b_u(x_n)\, c_{x_n}(y_n)$ where $u \in \{1, \ldots, N-1\}$

For i=n-1 to i=1, calculate:

$\beta_u(i) = \sum_v a_{u,v}\, b_u(x_i)\, c_{x_i}(y_i)\, \beta_v(i + 1)$

where,
$c_{x_i}(y_i) = P(y \mid x)$

In each step, it takes O(N) operations to calculate a single term. Given that there are N terms to calculate, it will take $O(N^2)$ for each step. The total time complexity is therefore **$O(nN^2)$** since there are n steps.