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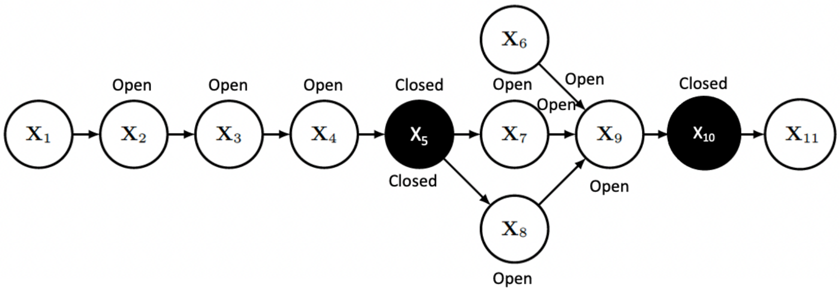
Question 1

Without knowing the actual values of any nodes, X_1 and X_6 are **independent** of each other.



If we are given the values of nodes X_5 and X_{10}, using Bayes' Ball algorithm, since X_5 is closed, X_1 and X_6 are still **independent** of each other.

Question 2

Nodes X_1 and X_6 are at the ends and thus, they only have 2-1 = 1 effective parameter.
Nodes X_9 depend on X_6, X_7 and X_8, therefore 2 x 2 x 2 x (2-1) = 8 effective parameters.
The rest of the nodes have 2 x (2-1) = 2 effective parameters.

Node	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
No. of effective parameters	1	2	2	2	2	1	2	2	8	2	2

The total number of effective parameters needed for the Bayesian Network is:
1 + 2 + 2 + 2 + 2 + 1 + 2 + 2 + 8 + 2 + 2 = **26 effective parameters.**

Given that node X_3, X_8 and X_9 can take 5 different values and all other nodes can only take 4 different values:
Nodes X_1 and X_6 are at the ends and thus, they only have 4-1 = 3 effective parameters.
Nodes X_3 and X_8 have 4 x (5-1) = 16 effective parameters.
Nodes X_4 and X_{10} have 5 x (4-1) = 15 effective parameters.
Nodes X_9 depend on X_6, X_7 and X_8, therefore 4 x 4 x 5 x (5-1) = 320 effective parameters.
The rest of the nodes have 4 x (4-1) = 12 effective parameters.

Node	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
No. of effective parameters	3	12	16	15	12	3	12	16	320	15	12

The total number of effective parameters needed for the Bayesian Network is:
3 + 12 + 16 + 15 + 12 + 3 + 12 + 16 + 320 + 15 + 12 = **436 effective parameters.**

Question 3

	$X_1 = 1$	$X_1 = 2$
	0.5	0.5

X_1	$X_2 = 1$	$X_2 = 2$
1	0.2	0.8
2	0.3	0.7

X_2	$X_3 = 1$	$X_3 = 2$
1	0.3	0.7
2	0.3	0.7

X_3	$X_4 = 1$	$X_4 = 2$
1	0.1	0.9
2	0.5	0.5

X_4	$X_5 = 1$	$X_5 = 2$
1	0.5	0.5
2	0.6	0.4

	$X_6 = 1$	$X_6 = 2$
	0.6	0.4

X_6	X_7	X_8	$X_9 = 1$	$X_9 = 2$
1	1	1	0.8	0.2
1	1	2	0.1	0.9
1	2	1	0.9	0.1
1	2	2	0.7	0.3
2	1	1	0.3	0.7
2	1	2	0.2	0.8
2	2	1	0.2	0.8
2	2	2	0.9	0.1

X_5	$X_7 = 1$	$X_7 = 2$
1	0.2	0.8
2	0.3	0.7

X_5	$X_8 = 1$	$X_8 = 2$
1	0.8	0.2
2	0.7	0.3

X_9	$X_{10} = 1$	$X_{10} = 2$
1	0.8	0.2
2	0.8	0.2

X_{10}	$X_{11} = 1$	$X_{11} = 2$
1	0.7	0.3
2	0.8	0.2

(a)

From the graph, given X_4, nodes X_1, X_2, X_3 and X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} are conditionally independent of each other, thus:

$$P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_1, X_2, X_3, X_4) = P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_4)$$

This means that:

$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11})$$

$$= P(X_1, X_2, X_3, X_4)\, P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_1, X_2, X_3, X_4)$$

$$= P(X_1, X_2, X_3, X_4)\, P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_4)$$

This also means that:

$$P(X_3, X_4)$$

$$= \sum_{X_1, X_2, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}} P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11})$$

$$= \sum_{X_1, X_2, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}} P(X_1, X_2, X_3, X_4)\, P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_4)$$

$$= \sum_{X_1, X_2} P(X_1, X_2, X_3, X_4) \sum_{X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}} P(X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11} \mid X_4)$$

$$= \sum_{X_1, X_2} P(X_1, X_2, X_3, X_4)$$

$$= \sum_{X_1, X_2} P(X_1)\, P(X_2 \mid X_1)\, P(X_3 \mid X_2)\, P(X_4 \mid X_3)$$

$$= \sum_{X_2} P(X_3 \mid X_2)\, P(X_4 \mid X_3)$$

Therefore,

P(X_3 = 1| X_4 = 2)

$$= \frac{P(X_3=1, X_4=2)}{\sum_{i=\{1,2\}} P(X_3=i, X_4=2)}$$

$$= \frac{\sum_{X_2} P(X_3=1 \mid X_2) P(X_4=2 \mid X_3=1)}{\sum_{i=\{1,2\}} P(X_3=i, X_4=2)}$$

$$= \frac{(0.3+0.3)\times 0.9}{(0.3+0.3)\times 0.9 + (0.7+0.7)\times 0.5}$$

= 0.435

Therefore, the conditional probability of P(X_3 = 1| X_4 = 2) is **0.435.**

(b)

If we look closer at the probability tables of X_3 and X_{10}, their values are the same regardless of the given values. Therefore, we can infer that X_3 is independent of X_2 and X_{10} is independent of X_9. Since this is so, value of X_5 will not depend on the given values of X_1, X_2 and X_{11} such that:

$P(X_5=2 \mid X_2=1, X_{11}=2, X_1=1)$
$= P(X_5=2)$
$= P(X_5=2 \mid X_4=1)\ P(X_4=1) + P(X_5=2 \mid X_4=2)\ P(X_4=2)$

$P(X_3=1) = 0.3$
$P(X_3=2) = 0.7$

$P(X_4=1)$
$= P(X_4=1 \mid X_3=1)\ P(X_3=1) + P(X_4=1 \mid X_3=2)\ P(X_3=2)$
$= 0.1 \times 0.3 + 0.5 \times 0.7$
$= 0.38$

$P(X_4=2)$
$= P(X_4=2 \mid X_3=1)\ P(X_3=1) + P(X_4=2 \mid X_3=2)\ P(X_3=2)$
$= 0.9 \times 0.3 + 0.5 \times 0.7$
$= 0.62$

Therefore,

$P(X_5=2 \mid X_4=1)\ P(X_4=1) + P(X_5=2 \mid X_4=2)\ P(X_4=2)$
$= 0.5 \times 0.38 + 0.4 \times 0.62$
$= 0.438$

Therefore, the conditional probability of $P(X_5=2 \mid X_2=1, X_{11}=2, X_1=1)$ is **0.438.**

Question 4

Using Maximum Likelihood estimation to estimate the probability tables for the nodes,

	X_7	
X_5	1	2
1	1/4	3/4
2	3/4	1/4

			X_9	
X_6	X_7	X_8	1	2
1	1	1	1/3	2/3
1	1	2	1	0
1	2	1	1	0
1	2	2	1/2	1/2
2	1	1	1	0
2	1	2	0	1
2	2	1	1	0
2	2	2	0	1