Glossary: Theorems and Definitions From Higher Theory and Application of Differential Equations

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Definition. First-order ODE.

A differential equation of the form $y'(t) = f(t, y(t))$.

*It is called a 'first-order' ODE because the degree of the derivative is only 1.

Definition. Linear first order ODE.

An ODE of the form

$$a(t)y'(t) + b(t)y(t) = f(t)$$

with $a(t) \neq 0$ on some interval $I \subseteq \mathrm{R}$.

*It is 'linear' because the 'variables' (i.e. y, y', y'', etc.) are not raised to a power.

Definition. Separable first order ODE.

An ODE of the form

$$y'(t) = f(t)g(y).$$

The solution is (implicitly) found by writing

$$\int \frac{1}{g(y)}\ \mathrm{d}y = \int f(t)\ \mathrm{d}t.$$

*These equations are called separable differential equations because we can separate everything involving t from everything involving y.

Definition. Integrating factor method.

For a linear first order ODE (i.e. $y' + a(t)y = f(t)$), we generally cannot separate the variables. But suppose there is a function $\mu(t)$, called an **integrating factor** such that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)\ \mathrm{d}t + \frac{C}{\mu(t)}.$$

We find that $\mu(t) = e^{\int a(t)\ \mathrm{d}t}$.

Definition. Homogeneous Differential Equations.

Homogeneous differential equation are differential equations of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

where $y = y(x)$.

They can be solved using the substitution $y = vx$, where v is a function of x. We will always get a separable differential equation. Observe.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}x + v \\
\frac{\mathrm{d}v}{\mathrm{d}x}x + v &= f\left(\frac{vx}{x}\right) = f(v) \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{f(v) - v}{x}.
\end{aligned}$$

Definition. Second Order Linear Differential Equations.

A second order linear differential equation is a differential equation of the form

$$p(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + q(x)\frac{\mathrm{d}y}{x} + r(x)y = G(x),$$

where if $G(x) = 0$ we have a homogenous equation.

Note that the derivative of the exponential function at any point is itself. From this, the generic solution takes the form $y = e^{rx}$.

For the homogenous equation first:

Assume the solution is of the form $y = e^{rx}$ for some constant r. Then

$$y'' = r^2e^{rx},$$

$$y' = re^{rx},$$

$$y = e^{rx}.$$

So substituting into the differential equation gives

$$
\begin{aligned}
e^{rx}\left(p(x)r^2+q(x)r+r(x)\right)&=0,\\
p(x)r^2+q(x)r+r(x)&=0.
\end{aligned}
$$

Solve this quadratic for r, with three cases.

1. $\Delta > 0$: $y = c_1e^{r_1x} + c_2e^{r_2x}$, for some c_1 and c_2. These constants represent that any linear combination of e^{r_1x} and e^{r_2x} is a solution (since $= 0$).
2. $\Delta = 0$: $y = ce^{rx}$.
3. $\Delta < 0 \implies r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$: $y = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$.

Now for the non-homogenous case. We call the solution to this the particular solution. We make a guess from the form of the non-homogeneity.

Non-Homogeneity	Particular Solution Form
Constant term ($f(x) = a$)	$y_p(x) = A$
Polynomial term ($f(x) = P_n(x)$)	$y_p(x) = Q_n(x) \cdot x^m$, where $Q_n(x)$ is a polynomial of degree n
Exponential term ($f(x) = e^{rx}$)	$y_p(x) = A \cdot e^{rx}$, where r is not a root of the characteristic equation
Sine or cosine term ($f(x) = \sin(kx)$ or $f(x) = \cos(kx)$)	$y_p(x) = A \cdot \sin(kx) + B \cdot \cos(kx)$
Exponential times a polynomial ($f(x) = e^{rx} \cdot P_n(x)$)	$y_p(x) = Q_n(x) \cdot e^{rx}$, where $Q_n(x)$ is a polynomial of degree n
Combination of the above forms	The particular solution is the sum of the particular solutions corresponding to each component

Table 1: If any of the coefficients match that of the characteristic polynomial, you must add an extra x.

Then the general solution is $y = y_h + y_p$.

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