Glossary: Higher Linear Algebra: Definitions and THeorems

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Definition. Groups.

A group G is a non-empty set with a binary operation defined on it. That is

1. Closure: for all a, $b \in G$ a composition $a * b$ is defined and in G.
2. Associativity: for all $a, b \in G$, $(a * b) * c = a * (b * c)$.
3. Identity: There is an element $e \in G$ such that $a * ee * a = a$.
4. Inverse: for each $a \in G$ there exists an $a' \in G$ such that $a * a' = a' * a = e *$.

If G is a finite set then the order of G is $|G|$, the number of elements in G. Technically, a group is the pair $(G, *)$; said as "the group G under the operation $*$".

*It captures the idea of a collection of objects, and came from symmetry; think about actions on a structure, and composing these actions.

Definition. Abelian Group.

A group G is abelian if the operation satisfies the commutative law

$$a * b = b * a \qquad \text{for all} \qquad a, b \in G.$$

Notes on Groups

- A composition is a function $* : G \times G \to G$.
- The operation $*$ is not restricted to but it s commonly addition (for abelian groups), mulitiplication (often written as juxtaposition) or composition of functions.
- We use power notation for repeated compositions: $a * a * a * \ldots * a = a^n$ and $a^{-n} = (a^{-1})^n$.

Examples

- $(\mathbb{Z}, +)$ is an abelian group;
- $(\mathbb{Z}, -)$ is not a group because the inverse of 2 (or any $n \in \mathbb{Z}, n \neq \pm 1$) is not an integer.
- For any integer m, the set $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}$ of remainders modul m is a group under addition modulo m.
- If p is prime, $\mathbb{Z}_p^* = \mathbb{Z}_p \ \{0\}$ is a group under multiplication modulo p. This only works for primes because non-primes will lead to a zero result (e.g. $p = 6$, $2 * 3 = 6 \mod 6 = 0$.)
- For any set S, the set F of bijective functions $f : S \to S$ is a group under composition, but is not in general abelian.

Lemma. Properties of a Group.

Let $(G, *)$ be a group.

- There is only one identity element in G.
- Each element of G has only one inverse.
- For each $a \in G$, $(a^{-1})^{-1} = a$.
- For every $a, b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$.
- Let $a, b, c \in G$. Then if $a * b = a * c$, $b = c$.Similarly, if $b * a = c * a$ then $b = c$.

Example. Let m be a positive integer and consider the set

$$C_m = \{e, a, a^2, a^3, \ldots, a^{m-1}\},$$

where a is some (undefined) symbol.

Define an operation on C_m by specifying that $a^0 = e$ (soon; $a^m = e$) and

$$a^k * a^l = \begin{cases} a^{k+l} & \text{if } k + l < m \\ a^{k+l-n} & \text{otherwise} \end{cases}.$$

This is an abelian group known as a cyclic group of order m and we often write C_m as $< a : a^m = e >$ and say it is generated by a.

Definition. Permutation Groups.

Let $\Omega_n = \{1, 2, \ldots, n\}$. As an ordered set, Ω_n has $n!$ arrangements or permutations. If we think of these as functions, then each permutation is a bijection on Ω_n (mapping indices).

Then the set $\mathcal{S}_n$ of all permutations of n objects forms a group under composition of order $n!$. The proof follows from the bijective function on a set example.

Definition. Small Finite Groups.

A Cayley table shows the compositions associated with a group, usually for small, finite groups.

Each row must be a permutation of the elements of the group, because:

- If we had a repitition in a row (or column), so that $x * a = x * b$, then the cancellation rule will give $a = b$.
- If $a^2 = a$ then composing with a^{-1} gives $a = e$, so the identity is the only element that can be fixed.

Example.

Example 1.5 *Let G be the set of matrices*

$$\left\{ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Show that G is an abelian group under matrix multiplication and write out its multiplication table.

SOLUTION: Multiplying we get:

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

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Definition. Fields.

A field $(\mathbb{F}, +, \times)$ is a set $\mathbb{F}$ with two binary operations on it, addition $(+)$ and multiplication $(\times)$, where

1. $(\mathbb{F}, +)$ is an abelian gruop;
2. $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group under multiplication, with 0 being the additive identity;
3. The distributive laws $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$ hold.

Note this satisfies the $12 = 5 + 5 + 2$ number laws.

*It is called a 'field' to resemble fields in physics; that is, a premise upon which there is rich structure and interplay.

Lemma. Interplay Between $+$ and $\times$.

Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$. Then

- $a0 = 0$ (additive identity);
- $a(-b) = -(ab)$ (associativity);
- a(b-c) = ab - ac (distributive law);
- if $ab = 0$ then either $a = 0$ or $b = 0$ (proof: suppose $ab = 0$ and $a \neq 0$, then multiply both sides by a^{-1} to give $0 = a^{-1}0 = a^{-1}(ab) = (a^{-1}a)b = 1b = b$).

Definition. Subgroups.

Let $(G, *)$ be a group and H a non-empty subset of G.

Then H is a subgroup of G if H is a group under the restriction of $*$ to H.

We write this as $H \leq G$ and say H inherits the group structure from G.

*The idea is that H is just a smaller bunch of objects within G, the whole collection, and also has its structure and properties.

Lemma. The Subgroup Lemma.

Let $(G, *)$ be a group and H and non-empty subset of G.

Then H is a subgroup of G if and only if

- for all $a, b \in H$, $a * b \in H$;
- for all $a \in H$, $a^{-1} \in H$;

These are all proved from closure and inverse of the group; associativity is inherited from G and an identity exists via the inverse.

Note that any subgroup of an abelian group is also an abelian group.

Examples

1. Every non-trivial group G has at least two subgroups: $\{e\}$ and G.
2. For any integer m let $m\mathbb{Z}$ be the set of all multiples of m. Then $(m\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$. The converse is also true; any subgroup of $(\mathbb{Z}, +)$ is $(m\mathbb{Z}, +)$ for some m.
3. Consider $C_m =< a : a^m = e >$ the cyclic group of order m.
   Picking any integer k with $1 < k < m$ we could generate a subgroup $H_k =< a^k >$ of C_m by looking at all powers of a_k. Clearly, the order of H_k cannot be larger than m; it is strictly less than m if and only if k and m have a common divisor $d > 1$.
4. Let $n \geq 1$ be any integer. The set of invertible $n \times n$ matrices over a field $\mathbb{F}$ s a group under matrix multiplication. It is non-abelian if $n > 1$. It is called the general linear group $\mathrm{GL}(n, \mathbb{F})$.
   They have important subgroups, such as the special linear groups $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ of matrices with determinant 1, and the group of orthogonal matrices $O(n) \leq \mathrm{GL}(n, \mathbb{R})$.

Definition. Subfields.

If $(\mathbb{F}, +, \times)$ is a field and $\mathbb{E} \subseteq \mathbb{F}$ is also a field under the same operations, then $(\mathbb{E}, +, \times)$ is a subfield of $(\mathbb{F}, +, \times)$, usually written $\mathbb{E} \leq \mathbb{F}$.

Lemma. The Subfield Lemma.

Let $\mathbb{E} \neq \{0\}$ be a non-empty subset of field $\mathbb{F}$.

Then $\mathbb{E}$ is a subfield of $\mathbb{F}$ if and only if for all $a, b \in \mathbb{E}$

$$a + b \in \mathbb{E}, \quad -b \in \mathbb{E}, \quad a \times b \in \mathbb{E}, \quad b^{-1} \in \mathbb{E} \text{ if } b \neq 0.$$

Proof: The distributive laws are inherited from $\mathbb{F}$, and the rest of the proof comes from applying the subgroup lemma to $(\mathbb{E}, +)$ and $(\mathbb{E}, \times)$.

Definition. Morphisms.

Let $(G, *)$ and $(H, \circ)$ be two groups. A (group) homomorphism from G to H is a map $\phi : G \to H$ that respects the two operations, that is where

$$\phi(a * b) = \phi(a) \circ \phi(b) \quad \text{for all } a, b \in G.$$

A bijective homomorphism $\phi : G \to H$ is called an isomorphism; the groups are then said to be isomorphic, and are considered the same group. isomorphism is an equivalence relation.

Example. Let $m \geq 2$ be any integer. Define $\phi : (\mathbb{Z}, +) \to (m\mathbb{Z}, +)$ by $\phi(a) = ma$. Show that ϕ is an isomorphism of groups.

Solution. Firstly

$$\phi(a + b) = m(a + b) = ma + mb = \phi(a) + \phi(b).$$

The easiest way to show ϕ is a bijection is to find the inverse. But if $g \in m\mathbb{Z}$ then $g = ma$ for some $a \in \mathbb{Z}$ and clearly $\phi(a) = g$, so $\phi^{-1}(g) = a$. Thus ϕ is an isomorphism.

Lemma. Homomorphism Lemma.

Let $(G, *)$ and $(H, \circ)$ be two groups and ϕ a homomorphism between them. Then

- ϕ maps the identity of G to the identity of H.
- ϕ maps inverses to inverses, i.e. $\phi(a^{-1}) = (phi(a))^{-1}$ for all $a \in G$.
- if ϕ is an isomorphism frmo G to H then ϕ^{-1} is an isomorphism from H to G.

Definition. Kernel and Image.

Let $\phi : G \to H$ be a group homomorphism, with e' the identity of H.

The kernel of ϕ is the set

$$\ker(\phi) = \{g \in G : \phi(g) = e'\}.$$

It is the information lost in the map.

The image of ϕ is the set

$$\text{im}(\phi) = \{h \in H : h = \phi(g)\}, \text{ for some } g \in G$$

.It is the approximation given by the map.

Lemma. Kernel and Image Lemma.

For $\phi : G \to H$ a group homomorphism, $\ker(\phi) \leq G$ and $\text{im}(\phi) \leq H$.

Lemma. Isomorphism Lemma.

A homomorphism ϕ is one-to-one if and only if $\ker(\phi) = \{e\}$, with e the identity of G. If ϕ is one-to-one then $textim(\phi)$ is isomorphic to G.

Definition. Linear Representation of G on $\mathbb{F}^n$.

If there exists a homomorphism $\phi : G \to \text{GL}(n, \mathbb{F})$ for some n and some field $\mathbb{F}$, then the group $\text{im}(\phi)$ is called a linear representation of G on $\mathbb{F}^n$.

If ϕ is one-to-one (so every element maps to a distinct matrix), we call the representation faithful.

Definition. Permutation Representation.

From above, in the case where G is finite and H is $\mathcal{S}_n$ for some n, then we get a permutation representation of the group G as a subgroup of S_n.

Definition. Vector Space.

Let $\mathbb{F}$ be a field. A vector space over the field $\mathbb{F}$ consists of an abelian group $(V, +)$ and a function from $\mathbb{F} \times V$ to V called scalar multiplication and written $\alpha\mathbf{v}$ where

1. $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{v} \in V$.
2. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.
3. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all $\alpha \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$.
4. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ for all $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u} \in V$.

There are ten axioms here; 5 from the abelian group, closure of scalar multiplication, and four explicit ones. The $+$ in $(V, +)$ may be distinguished as vector addition.

Lemma. 2.1.

Let V be a vector space over field $\mathbb{F}$. For all $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$ then

- $0\mathbf{v} = \mathbf{0}$ and $\lambda\mathbf{0} = \mathbf{0}$.
- $(-1)\mathbf{v} = -\mathbf{v}$.
- $\lambda\mathbf{v} = \mathbf{0}$ implies either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.
- If $\lambda\mathbf{v} = \lambda\mathbf{w}$ and $\lambda \neq 0$ then $\mathbf{v} = \mathbf{w}$.

Standard Vector Space Examples.

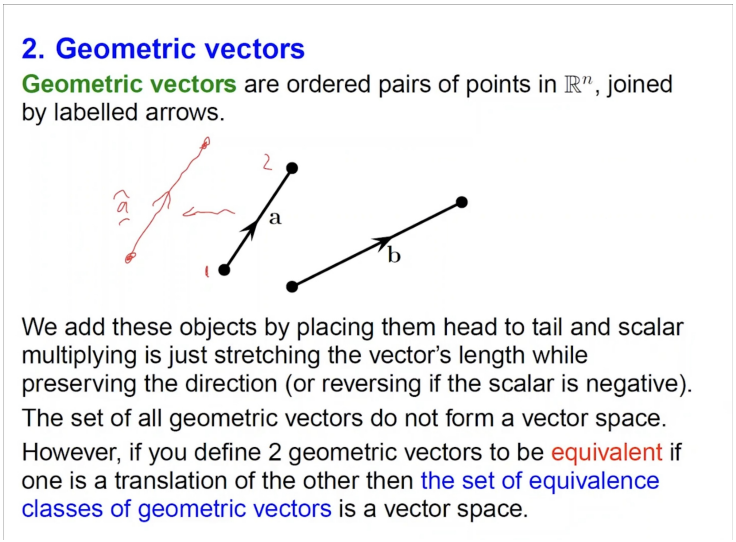
1. n-tuples.

The set $\mathbb{F}^n$ consists of all n-tuples of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \alpha_i \in \mathbb{F} \right\}.$$

If $\mathbf{x} = (\alpha_i)_{1 \leq i \leq n}$, $\mathbf{y} = (\beta_i)_{1 \leq i \leq n}$ are elements of $\mathbb{F}^n$, then vector addition on $\mathbb{F}^n$ is defined as $\mathbf{x} + \mathbf{y} = (\alpha_i + \beta_i)_{1 \leq i \leq n}$.
Scalar multiplication on $\mathbb{F}^n$ is $\lambda\mathbf{x} = (\lambda\alpha_i)_{1 \leq i \leq n}$.
With these operations, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.
You met $\mathbb{R}^n$ in first year – mainly $\mathbb{R}^2$ and $\mathbb{R}^3$ of course – and you should be familar with those vector spaces.

2. Geometric Vectors.



3. Matrices.

   For any positive integers p and q the set $M_{p,q}(\mathbb{F})$ is the set of $p \times q$ matrices with elements from $\mathbb{F}$. Then $M_{p,q}(\mathbb{F})$ is a vector spsace over $\mathbb{F}$ with vector addition with the usual addition of matrices and scalar multiplication multiplying each element of the matrix.

4. Polynomials.

   The set of all Polynomials with coefficients in $\mathbb{F}, \mathcal{P}(\mathbb{F})$, is a vector space over $\mathbb{F}$ with

$$(f+g)(x) = f(x) + g(x) \quad \text{for all} \quad x \; in \mathbb{F} (\lambda f)(x) = \lambda f(x) \quad \text{for all} \quad \lambda, x \in \mathbb{F}.$$

   Similarly, $\mathcal{P}(F)$ (polynomials of degree n or less) is a vector space over $\mathbb{F}$. Note this example shows 'vector spaces' can appear abstract from actual vectors! We just need vector $+$ and $\times$.

5. Function Spaces.

   Let X be a non-empty set and $\mathbb{F}$ be a field. Then define

$$\mathcal{F}[X] = \{f : X \to \mathbb{F}\}.$$

   The set $\mathcal{F}[X]$ is a vector space over $\mathbb{F}$ if we define

   - the zero in $\mathcal{F}[X]$ to be the zero function: $x \to 0$ for all $x \in X$.
   - $(f+g)(x) = f(x) + g(x)$ for all $x \; in X$.
   - $(\lambda f)(x) = \lambda(f(x))$ for all $x \in X$.