11.3) $f(R) = \begin{cases} \frac{2}{\theta^2}(\theta - R) & 0 < R < \theta \\ 0 & \text{otherwise} \end{cases}$ find c such that $R < \theta < cR$ is a $(1-\alpha)100\%$ C.I. for θ

$$1-\alpha = P(R < \theta < cR) = P\left(1 < \frac{\theta}{R} < c\right) = P\left(\frac{1}{c} < \frac{R}{\theta} < 1\right) = P\left(\frac{\theta}{c} < R < \theta\right)$$

So $$1-\alpha = \int_{\theta/c}^{\theta} \frac{2}{\theta^2}(\theta - R)\, dR = \frac{2}{\theta^2}\left(\theta R - \frac{R^2}{2}\right)\Big|_{\theta/c}^{\theta} = \frac{2}{\theta^2}\left[\left(\theta^2 - \frac{\theta^2}{2}\right) - \left(\frac{\theta^2}{c} - \frac{\theta^2}{2c^2}\right)\right]$$

$$\Rightarrow \quad 1-\alpha = 1 - \frac{2}{c} + \frac{1}{c^2} \quad \Rightarrow \quad c^2(1-\alpha) = c^2 - 2c + 1 \quad \Rightarrow \quad \alpha c^2 - 2c + 1 = 0$$

By quadratic formula:

$$c = \frac{2 \pm \sqrt{4 - (4)(\alpha)(1)}}{2\alpha} = \frac{2 \pm 2\sqrt{1-\alpha}}{2\alpha} = \frac{1 \pm \sqrt{1-\alpha}}{\alpha}$$

11.9) $$\frac{(n_1+n_2-2)\, S_p^2}{\sigma^2} \sim \chi^2_{(n_1+n_2-2)}$$

So $$E\left[\frac{(n_1+n_2-2)S_p^2}{\sigma^2}\right] = n_1+n_2-2$$

$$\Rightarrow \quad \frac{(n_1+n_2-2)}{\sigma^2} E(S_p^2) = n_1+n_2-2$$

$$\Rightarrow \quad E(S_p^2) = \sigma^2 \quad \Rightarrow \quad S_p^2 \text{ is an unbiased estimator of } \sigma^2$$

And

$$Var\left[\frac{(n_1+n_2-2)S_p^2}{\sigma^2}\right] = 2(n_1+n_2-2)$$

*under conditions of Thm 11.5

$$\Rightarrow \quad \frac{(n_1+n_2-2)^2}{\sigma^4} Var(S_p^2) = 2(n_1+n_2-2)$$

$$\Rightarrow \quad Var(S_p^2) = \frac{2\sigma^4}{n_1+n_2-2}$$

11.22)

$n = 120$ $\sigma = 10.5\,mm$ probability of 0.99 $\Rightarrow$ $\alpha = .01$

$\Rightarrow z_{\alpha/2} = z_{.005} = 2.575$

So max Error $= z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = (2.575)\left(\frac{10.5}{\sqrt{120}}\right) = 2.4682\,mm$

11.23)

$n = 120$ $\sigma = 10.5\,mm$ $\bar{x} = 141.8\,mm$ $\alpha = 1 - .98 = .02$

$z_{\alpha/2} = z_{.01} = 2.33$

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$141.8 \pm (2.33)\left(\frac{10.5}{\sqrt{120}}\right)$$

$\Rightarrow 141.8 \pm 2.2333$

$\Rightarrow (139.5667\,mm,\ 144.0333\,mm)$

We are 98% confident that the true mean blood pressure of women in their fifties is between 139.5667 mm and 144.0333 mm of mercury.

11.28)

$\sigma = 12.2\,s$ $e = 2.5\,s$ $\alpha = 1 - .95 = .05$

$\Rightarrow z_{\alpha/2} = z_{.025} = 1.96$

From Ex #6: $n = \left[\frac{z_{\alpha/2}\,\sigma}{e}\right]^2$

So

$n = \left[\frac{1.96(12.2)}{2.5}\right]^2 = 91.485$ (round up for sample size)

$n = 92$

11.29)

$\sigma = 3.2$ hours ← *must be same → $\alpha = 1 - .95 = .05$

$e = 20$ minutes $= \frac{1}{3}$ hours $\Rightarrow z_{\alpha/2} = z_{.025} = 1.96$

$$n = \left[\frac{z_{\alpha/2}\,\sigma}{e}\right]^2 = \left[\frac{(1.96)(3.2)}{1/3}\right]^2 = 354.0419 \quad \text{(round up)}$$

$n = 355$

11.30)

$n = 10$

$\bar{x} = 5.68$ cm

$s = .29$ cm

$\alpha = 1 - .95 = .05$

$\Rightarrow t_{\alpha/2,\,n-1} = t_{.025,\,9} = 2.262$

$\bar{x} \pm t_{\alpha/2,\,n-1}\, s/\sqrt{n}$

$5.68 \pm (2.262)\left(\frac{.29}{\sqrt{10}}\right)$

$\Rightarrow 5.68 \pm 0.2074$

$\Rightarrow (5.4726 \text{ cm},\ 5.8874 \text{ cm})$

We are 95% confident that the true mean length of the skulls of this species of bird is between 5.4726 cm and 5.8874 cm.

11.32)

$n = 12$ $\quad x_i$'s: 2.3, 1.9, 2.1, 2.8, 2.3, 3.6, 1.4, 1.8, 2.1, 3.2, 2.0, 1.9

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{12}\sum_{i=1}^{12} x_i = 2.2833$$

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{1}{11}\sum_{i=1}^{12}(x_i - 2.2833)^2} \approx .6250$$

$\alpha = 1 - .95 = .05$

$\Rightarrow t_{\alpha/2,\,n-1} = t_{.025,\,11} = 2.201$

$$\text{maxError} = t_{\alpha/2,\,n-1}\, s/\sqrt{n} = (2.201)\left(\frac{.6250}{\sqrt{12}}\right) = .3971$$

11.34)

$\bar{x}_1 = 80.7$ min $\qquad \bar{x}_2 = 88.1$ min

$s_1 = 19.4$ min $\qquad s_2 = 18.8$ min

$n_1 = 61 \qquad n_2 = 61$

$\alpha = 1 - .99 = .01$

$\Rightarrow z_{\alpha/2} = z_{.005} = 2.575$

Since n_1 & n_2 are large ($n_1, n_2 \geq 30$) we use

$$\bar{x}_1 - \bar{x}_2 \pm z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$\Rightarrow (80.7 - 88.1) \pm 2.575 \sqrt{\frac{(19.4)^2}{61} + \frac{(18.8)^2}{61}}$$

$$\Rightarrow -7.4 \pm 8.9066$$

$$\Rightarrow (-16.3066 \text{ min}, 1.5066 \text{ min})$$

We are 99% confident that the true difference between the mean amounts of time it takes to repair failures of the two kinds of photocopying equipment lies between -16.3066 min and 1.5066 min.

11.36)

X_{1j}'s (Mine A): 8500, 8330, 8480, 7960, 8030

X_{2j}'s (Mine B): 7710, 7890, 7920, 8270, 7860

Assuming data comes from normal populations of equal variance

$n_1 = 5$

$$\bar{X}_1 = \frac{1}{n}\sum_{j=1}^{n} X_{1j} = \frac{1}{5}\sum_{j=1}^{5} X_{1j} = 8260$$

$$s_1 = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \bar{X}_1)^2} = \sqrt{\frac{1}{4}\sum_{j=1}^{5}(X_{1j} - 8260)^2} = 251.8928$$

$n_2 = 5$

$$\bar{X}_2 = \frac{1}{n}\sum_{j=1}^{n} X_{2j} = \frac{1}{5}\sum_{j=1}^{5} X_{2j} = 7930$$

$$s_2 = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(X_{2j} - \bar{X}_2)^2} = \sqrt{\frac{1}{4}\sum_{j=1}^{5}(X_{2j} - 7930)^2} = 206.5188$$

$$S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-1}} = \sqrt{\frac{4(257.8928)^2 + 4(206.5788)^2}{5+5-2}}$$

$$= 230.3259$$

$\alpha = 1 - .99 = .01$

$$\Rightarrow t_{\alpha/2,\, n_1+n_2-2} = t_{.005,\,8} = 3.355$$

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2,\, n_1+n_2-2} \cdot S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\Rightarrow 8260 - 7930 \pm 3.355\,(230.3259)\sqrt{\frac{1}{5} + \frac{1}{5}}$$

$$\Rightarrow 330 \pm 488.7258$$

$\Rightarrow (-158.7258,\ 818.7258)$ millions of calories per ton

We are 99% confident that the true difference between the mean heat-producing capacities of coal from the two mines is between -158.7258 and 818.7258 millions of calories per ton.

11.42) $n = 100$ $\quad x = 18$ $\quad \alpha = 1 - .99 = .01$

$\hat{\theta} = \frac{x}{n} = .18$ $\quad \Rightarrow z_{\alpha/2} = z_{.005} = 2.575$

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$$

$$\Rightarrow .18 \pm (2.575)\sqrt{\frac{.18(1-.18)}{100}}$$

$$\Rightarrow .18 \pm 0.0989$$

$\Rightarrow (0.0811,\ 0.2789)$

We are 99% confident that the true proportion of inedible fish as a result of chemical pollution is between 0.0811 and 0.2789.

11.50) θ_1 be proportion for population 1.
θ_2 be proportion for population 2

$$\hat{\theta}_1 = \frac{x_1}{n_1} = \frac{48}{500} = 0.096 \qquad \alpha = 1 - .99 = .01$$

$$\hat{\theta}_2 = \frac{x_2}{n_2} = \frac{68}{400} = 0.170 \qquad \Rightarrow z_{\alpha/2} = z_{.005} = 2.575$$

$$\hat{\theta}_1 - \hat{\theta}_2 \pm z_{\alpha/2} \sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1} + \frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}$$

$$\Rightarrow .096 - .170 \pm (2.575) \sqrt{\frac{.096(1-.096)}{500} + \frac{.170(1-.170)}{400}}$$

$$\Rightarrow -0.074 \pm 0.0591$$

$$\Rightarrow (-0.1331, -0.0149)$$

We are 99% confident that the true difference between the corresponding proportions of marriage license applications in which the woman was at least one year older than the man is between -0.1331 and -0.0149.

11.52) $\alpha = 1 - .95 = .05$ $\qquad e = .05$

$\Rightarrow z_{\alpha/2} = z_{.025} = 1.96$

From Ex 11.16: $n = \dfrac{z^2_{\alpha/2}}{2e}$

$$n = \frac{(1.96)^2}{2(.05)} = 768.32 \quad \text{(round up)}$$

$n = 769$

11.54) From Ex 11.32, $s = 0.6250$!

$\alpha = 1 - .90 = .10$

$n = 12$

$$\chi^2_{\alpha/2, n-1} = \chi^2_{.05, 11} = 19.675$$

$$\chi^2_{1-\alpha/2, n-1} = \chi^2_{.95, 11} = 4.575$$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}}$$

$$\Rightarrow \quad \frac{(12-1)(.625)^2}{19.675} < \sigma^2 < \frac{(12-1)(.625)^2}{4.575}$$

$$\Rightarrow \quad 0.2184 < \sigma^2 < 0.9392$$

$$\Rightarrow \quad 0.4673 < \sigma < 0.9691$$

We are 90% confident that the true population standard deviation (percentage of impurities in the given brand of peanut butter) is between 0.4673 and 0.9691.

11.57) From Ex 11.34:

$n_1 = 61$ $\quad n_2 = 61$ $\quad \alpha = 1 - .98 = .02$

$s_1 = 19.4$ $\quad s_2 = 18.8$

$$f_{\alpha/2, n_1-1, n_2-1} = f_{.01, 60, 60} = 1.84$$

$$f_{\alpha/2, n_2-1, n_1-1} = f_{.01, 60, 60} = 1.84$$

$$\frac{s_1^2}{s_2^2} \cdot \frac{1}{f_{\alpha/2, n_1-1, n_2-1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} \cdot f_{\alpha/2, n_2-1, n_1-1}$$

$$\Rightarrow \frac{(19.4)^2}{(18.8)^2} \cdot \frac{1}{1.84} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(19.4)^2}{(18.8)^2} \cdot 1.84$$

$$\Rightarrow 0.5787 < \frac{\sigma_1^2}{\sigma_2^2} < 1.9593$$

We are 98% confident that the ratio of the variances of the two populations lies between 0.5787 and 1.9593.