1 Elliptic PDEs

1.1 Laplace's equation and Poisson's equation

Now we look at the steady state of the 2D heat equation. The steady state is the solution to the PDE such that the time derivative is zero. The steady state will lead to the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

and the Poisson equation.

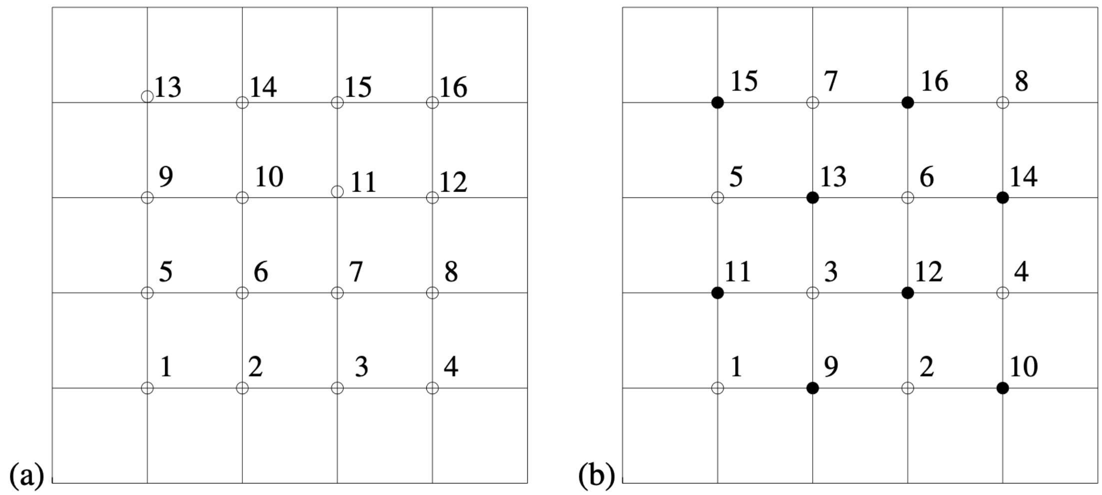
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{2}$$

which are both elliptic PDEs.

1.2 Boundary conditions

- Dirichlet boundary conditions: $u(x, y) = g(x, y)$ on $\partial\Omega$
- Neumann boundary conditions: $\frac{\partial u}{\partial n} = h(x, y)$ on $\partial\Omega$
- Mixed boundary conditions: $u(x, y) = g(x, y)$ on $\partial\Omega_1$ and $\frac{\partial u}{\partial n} = h(x, y)$ on $\partial\Omega_2$

1.3 Finite difference method



The general procedure for the finite difference method is as follows:

- Discretize the domain into a grid
- Approximate the derivatives using finite differences
- Solve the resulting system of equations

Recall for the 5-point stencil, we have:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} = f_{i,j} \tag{3}$$

If $h = \Delta x = \Delta y$, we can write this as:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = f_{i,j} \tag{4}$$

or

$$u_{i,j+1} + u_{i+1,j} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j} \tag{5}$$

Also recall the matrix form of the coefficient in the finite difference can be written as: And we can write it in the block form:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
-4 & 1 & & & 1 & & & & & & & & & & & \\
1 & -4 & 1 & & & 1 & & & & & & & & & & \\
& 1 & -4 & 1 & & & 1 & & & & & & & & & \\
& & 1 & -4 & & & & 1 & & & & & & & & \\
\hline
1 & & & & -4 & 1 & & & 1 & & & & & & & \\
& 1 & & & 1 & -4 & 1 & & & 1 & & & & & & \\
& & 1 & & & 1 & -4 & 1 & & & 1 & & & & & \\
& & & 1 & & & 1 & -4 & & & & 1 & & & & \\
\hline
& & & & 1 & & & & -4 & 1 & & & 1 & & & \\
& & & & & 1 & & & 1 & -4 & 1 & & & 1 & & \\
& & & & & & 1 & & & 1 & -4 & 1 & & & 1 & \\
& & & & & & & 1 & & & 1 & -4 & & & & 1 \\
\hline
& & & & & & & & 1 & & & & -4 & 1 & & \\
& & & & & & & & & 1 & & & 1 & -4 & 1 & \\
& & & & & & & & & & 1 & & & 1 & -4 & 1 \\
& & & & & & & & & & & 1 & & & 1 & -4
\end{array}\right]$$

$$D = \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} D & I & 0 & 0 \\ I & D & I & 0 \\ 0 & I & D & I \\ 0 & 0 & I & D \end{pmatrix} \tag{6}$$

We note the matrix A is symmetric, and is negative definite and almost diagonally dominant, which implies that the eigenvalues are all real with ≤ 0.

If we use the red-black ordering, we can write the matrix in the following form: And the block form is then

$$
\left[\begin{array}{cccccccc|cccccccc}
-4 & & & & & & & & 1 & & 1 & & & & & \\
 & -4 & & & & & & & 1 & 1 & & 1 & & & & \\
 & & -4 & & & & & & 1 & & 1 & 1 & 1 & & & \\
 & & & -4 & & & & & & 1 & & 1 & & 1 & & \\
 & & & & -4 & & & & & & 1 & & 1 & & 1 & \\
 & & & & & -4 & & & & & & 1 & 1 & 1 & & 1 \\
 & & & & & & -4 & & & & & & 1 & & 1 & 1 \\
 & & & & & & & -4 & & & & & & 1 & & 1 \\
\hline
1 & 1 & 1 & & & & & & -4 & & & & & & & \\
 & 1 & & 1 & & & & & & -4 & & & & & & \\
1 & & 1 & & 1 & & & & & & -4 & & & & & \\
 & 1 & 1 & 1 & & 1 & & & & & & -4 & & & & \\
 & & 1 & & 1 & 1 & 1 & & & & & & -4 & & & \\
 & & & 1 & & 1 & & 1 & & & & & & -4 & & \\
 & & & & 1 & & 1 & & & & & & & & -4 & \\
 & & & & & 1 & 1 & 1 & & & & & & & & -4
\end{array}\right]
$$

$$
A = \begin{pmatrix} D & B \\ B^T & D \end{pmatrix} \tag{7}
$$

1.4 Gauss-Seidel and SOR method

Now given the matrix form of lienar equation $\mathtt{A}\vec{u} = \vec{f}$, we can split the matrix

$$
\mathtt{A} = \mathtt{D} - \mathtt{L} - \mathtt{L}^T \tag{8}
$$

where $\mathtt{D}$ is the diagonal part, $\mathtt{L}$ is the lower triangular part, and $\mathtt{L}^T$ is the upper triangular part. We can then write the equation as:

$$
\mathtt{D}\vec{u} - \mathtt{L}\vec{u} - \mathtt{L}^T\vec{u} = \vec{f} \Longrightarrow (\mathtt{D} - \mathtt{L})\vec{u} = \mathtt{L}^T\vec{u} + \vec{f} \tag{9}
$$

We can then rearrange the equation to get:

$$(\mathtt{D}-\mathtt{L})\vec{u}^{(n+1)}=\mathtt{L}^T\vec{u}^{(n)}+\vec{f}\Longrightarrow\vec{u}^{(n+1)}=(\mathtt{D}-\mathtt{L})^{-1}(\mathtt{L}^T\vec{u}^{(n)}+\vec{f}) \tag{10}$$

Now consider the ith row of the matrix equation:

$$(\mathtt{D}-\mathtt{L})\vec{u}^{(n+1)}=\mathtt{L}^T\vec{u}^{(n)}+\vec{f} \tag{11}$$

which is given by ($\mathtt{U}=\mathtt{L}^T$):

$$\mathtt{D}_{i,i}u_i^{(k+1)}+\sum_{j=1}^{N}\mathtt{L}_{i,j}u_j^{(k+1)}+\sum_{j=1}^{N}\mathtt{U}_{i,j}u_j^{(k+1)}=f_i \tag{12}$$

where f_i is the ith component of the vector $\vec{f}$, and $u_i^{(k+1)}$ is the ith component of the vector $\vec{u}^{(k+1)}$. We can then write the equation in the following form:

$$\mathtt{D}_{i,i}u_i^{(k+1)}=\sum_{j=1}^{i-1}\mathtt{L}_{i,j}u_j^{(k)}+\sum_{j=i+1}^{N}\mathtt{L}_{i,j}u_j^{(k)}+f_i \tag{13}$$

This is the Gauss-Seidel method. The Gaus-Seidel method is an iterative method that is used to solve linear equations.

We can then use the SOR method to solve the equation. The SOR method is an iterative method that is used to solve linear equations. The SOR method starts with ($\mathtt{U} = \mathtt{L}^T$):

$$\mathtt{D}\vec{u} - \mathtt{L}\vec{u} - \mathtt{U}\vec{u} = \vec{f} \tag{14}$$

$$\mathtt{D}\vec{u} + \omega(\mathtt{D}\vec{u} - \mathtt{L}\vec{u} - \mathtt{U}\vec{u}) = \mathtt{D}\vec{u} + \omega\vec{f} \tag{15}$$

where ω is the relaxation parameter. And it is indeed

$$(\mathtt{D} - \omega\mathtt{L})\vec{u} = -\omega(\mathtt{D}\vec{u} - \mathtt{U}\vec{u}) + \mathtt{D}\vec{u} + \omega\vec{f} \tag{16}$$

i.e.

$$(\mathtt{D} - \omega\mathtt{L})\vec{u} = \omega\vec{f} - [(\omega - 1)\mathtt{D} - \omega\mathtt{U}]\,\vec{u} \tag{17}$$

So we can write the equation in the following form:

$$(\mathtt{D} - \omega\mathtt{L})\vec{u}^{(n+1)} = \omega\vec{f} - [(\omega - 1)\mathtt{D} - \omega\mathtt{U}]\,\vec{u}^{(n)} \tag{18}$$

which has the component form:

$$u_i^{(n+1)} = (1 - \omega)u_i^{(n)} + \frac{\omega}{D_{ii}}\left(\sum_{j=1}^{i-1}\mathtt{L}_{i,j}u_j^{(n)} + \sum_{j=i+1}^{N}\mathtt{U}_{i,j}u_j^{(n)} + f_i\right) \tag{19}$$