Q1) Let f have 5 bounded derivatives on $[x_0-2h, x_0+2h]$

~ Deffine g & $\hat{g}$:

→ $g = \frac{1}{h^2}(f(x_0-h) - 2f(x_0) + f(x_0+h))$

→ $\hat{g} = \frac{1}{4h^2}(f(x_0-2h) - 2f(x_0) + f(x_0+2h)$

(i) show that $\hat{c} = \frac{g-\hat{g}}{3}$, estimates the error ~ $f''(x_0) - g$

(ii) Show that g aproximates $f''(x_0)$ w/ error order 2

→ Centered second dif. numr.:

$f(x_0-h) - 2f(x_0) + f(x_0+h) =$

$(\cancel{f_0} - \cancel{hf_1} + \frac{h^2}{2}f_2 - \cancel{\frac{h^3}{6}f_3} + \frac{h^4}{24}f_4 - \cancel{\frac{h^5}{120}f_5} + R_{-h})$

$- \cancel{2f_0} +$

$(\cancel{f_0} + \cancel{hf_1} + hf_1 + \frac{h^2}{2}f_2 + \cancel{\frac{h^3}{6}f_3} + \frac{h^4}{24}f_4 + \cancel{\frac{h^5}{120}f_5} + R_{+h})$

$\Leftrightarrow f(x_0-h) - 2f(x_0) + f(x_0+h) =$

$h^2 f_2 + \frac{h^4}{12}f_4 + (R_{-h} + R_{+h})$

$\div h^2$ → → $\boxed{g = f''(x_0) + \frac{h^2}{12}f^{(4)}(x_0) + \frac{R_{-h} + R_{+h}}{h^2}}$

~ Since $R_{\pm h} = O(h^5)$, $(R_{-h} + R_{+h})/h^2 = O(h^3)$

⋆ $\Rightarrow \boxed{g = f''(x_0) + \frac{h^2}{12}f^{(4)}(x_0) + O(h^3)}$

Expanding $\hat{g}$:

→ $f(x_0-2h) - 2f(x_0) + f(x_0+2h) =$

$(2h)^2 f_2 + \frac{(2h)^4}{12}f_4 + (R_{-2h} + R_{+2h})$

⋆ $\Leftrightarrow 4h^2 f_2 + \frac{16h^4}{12}f_4 + O(h^5)$

$\div 4h^2$ $\Rightarrow \boxed{\hat{g} = f''(x_0) + \frac{h^3}{3}f^{(4)}(x_0) + O(h^3)}$

Showing g is 2nd order accurate:

→ $g - f''(x_0) = \frac{h^2}{12}f^{(4)}(x_0) + O(h^3)$

~ leading term is proportional to h^2, so $\boxed{g = f''(x_0) + O(h^2)}$

Taylor Expansion reasoning

Truncation Error Analysis

Order of error Analysis

Richardson Expansion Logic

→ Derivation using Taylor Series about x_0

→ Indentification of

→ Why g approximates $f''(x_0)$

→ The leading error term of g

→ Why $\hat{g}$ has a different leading error

→ Argoment showing: $\hat{c} = \frac{g-\hat{g}}{3} \approx f''(x_0) - g$

→ $f^{(k)}(x_0) = f_k$ ~ For $\pm h$

$f(x_0 \pm h) = f_0 \pm hf_1 + \frac{h^2}{2}f_2 + \frac{h^3}{6}f_3 + \frac{h^4}{24}f_4 \pm \frac{h^5}{120}f_5 + R_{\pm h}$

→ w/ $|R_{\pm h}| \le Ch^5$

→ For $\pm 2h$

$f(x_0 \pm 2h) = f_0 \pm 2hf_1 + \frac{(2h)^2}{2}f_2 + \frac{(2h)^3}{6}f_3 + \frac{(2h)^4}{24}f_4 \pm \frac{(2h)^5}{120}f_5 + R_{\pm 2h}$

→ w/ $|R_{\pm 2h}| \le C(2h)^5 = C'h^5$

A numerical aprox. $g(h)$ to a quality Q is second order accurate if $|Q - g(h)| \approx Ch^2$, for small h

Show $\hat{c} = (g - \hat{g})/3$, estimates $f''(x_0) - g$:

→ $g - \hat{g} = (f'' + \frac{h^2}{12}f^{(4)} + O(h^5)) - (f'' + \frac{h^3}{3}f_3 + O(h^3))$

$\Leftrightarrow (\frac{1}{12} - \frac{1}{3})h^2 f^{(4)}(x_0) + O(h^3) = -\frac{h^2}{4}f^{(4)}(x_0) + O(h^3)$

$[\div 3]$ ~ $\boxed{\hat{c} = \frac{g-\hat{g}}{3} = -\frac{h^2}{12}f^{(4)}(x_0) + O(h^3)}$

⇒ since $f''(x_0) - g = \frac{h^2}{12}f^{(4)}(x_0) + O(h^3)$

⇒ $\hat{c}$ is an estimate for $f''(x_0) - g$

Q2) Aprox. $f'(x)$, for $f(x) = Ln(x^2+1)$ at $x = 1.3$, using;
→ forward difference: $D_f(h) = \frac{f(x+h) - f(x)}{h}$
→ central difference: $D_c(h) = \frac{f(x+h) - f(x-h)}{2h}$
, for $h = 0.01$ & $h = 0.001$

Implementation of finite differences formulas
Truncation vs roundoff error

$f(x) = \ln(x^2+1) = f'(x) = \frac{1}{x^2+1}(2x)$
$= \frac{2x}{x^2+1}$

→ At $x = 1.3 \sim f'(1.3) = \frac{2(1.3)}{(1.3)^2+1} = \frac{2.6}{2.69}$
$\approx 0.96654275092936 79$

See Script

⇒ $f'(1.3) \approx 0.966542750929$
→ For $h = 0.01$:
Forward: $0.9655834256 16$
→ error $\approx 9.593253 \times 10^{-4}$
Centered: $0.9665426 92603$
→ error $\approx 5.832671 \times 10^{-8}$

Q3) Solve the IVP, $x^3 y'(x) + 20x^2 y(x) = x$
$x \in [2, 10]$, $y(2) = 0$, w/ Forward & Backward Euler for step sizes $h = 10^{-2}$, $h = 10^{-3}$, $h = 10^{-4}$, Compare

Divide the ODE by x^3

$\Rightarrow y'(x) + \frac{20}{x} y(x) = \frac{1}{x^2}$

$\sim y'(x) = f(x,y) = \frac{1}{x^2} - \frac{20}{x} y$

Let $x_n = 2 + nh$, $n = 0, 1, \dots N$

$Nh = 8$, and the numerical aproximation by $y_n \approx y(x_n)$ w/ $y_0 = 0$

→ Forward Euler:

$y_{n+1} = y_n + (h)(f(x_n, y_n))$,

$\rightarrow$ w/ $f(x_n, y_n) = \frac{1}{x^2} - \frac{20}{x} y$

$\Leftrightarrow y_{n+1} = y_n + h\left(\frac{1}{x_n^2} - \left(\frac{20}{x_n}\right)(y_n)\right)$

$= \left(1 - \frac{20h}{x_n}\right) y_n + \frac{h}{x_n^2}$

→ Backward Euler

$y_{n+1} = y_n + (h)(f(x_{n+1}, y_{n+1}))$

$\rightarrow y_{n+1} = y_n + h\left(\frac{1}{x_{n+1}^2} - \left(\frac{20}{x_{n+1}}\right) y_{n+1}\right)$

$\Leftrightarrow y_{n+1} + \left(\frac{20h}{x_{n+1}}\right) y_{n+1} = y_n + \frac{h}{x^2_{n+1}}$

$\Rightarrow y_{n+1} = \left[\left(y_n + \frac{h}{x^2_{n+1}}\right) / \left(1 + 20h / x_{n+1}\right)\right]$

Time stepping methods
Stability behaviours
Error Convergence