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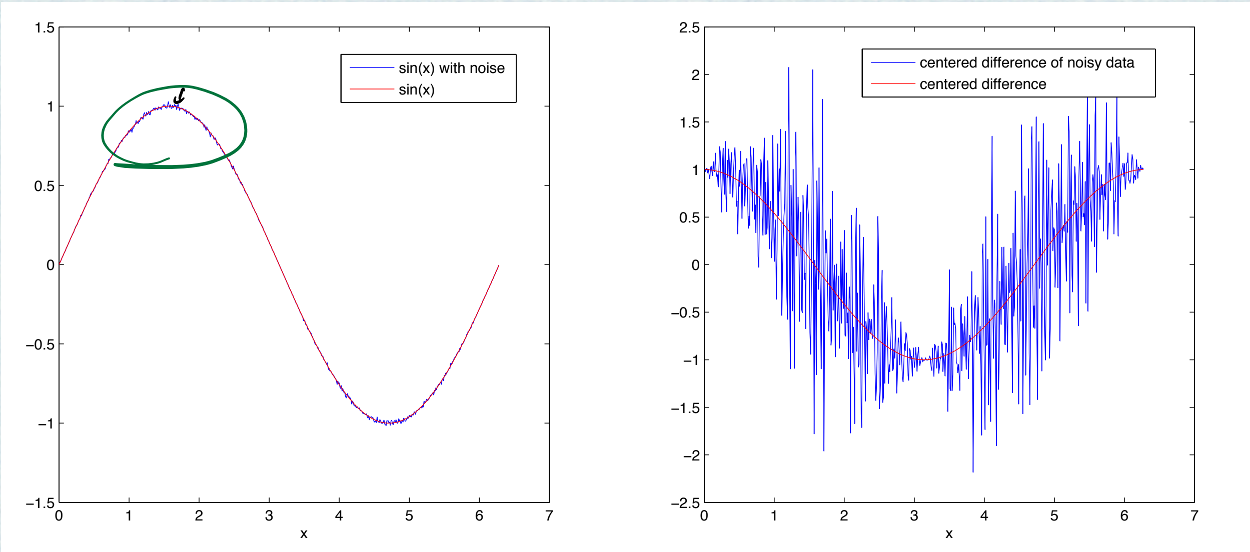
% Example 14.6 -- Figure 14.3 : differentiating noisy data

x = 0:.01:2*pi;
l = length(x);
sinx = sin(x);
sinp = (1+.01*randn(1,l)).*sinx;

cosx = (sinx(3:l)-sinx(1:l-2))/.02;
cosp = (sinp(3:l)-sinp(1:l-2))/.02;
err_f = max(abs(sinx-sinp))
err_fp = max(abs(cosx-cosp))

subplot(1,2,1)
plot(x,sinp,x,sinx,'r')
legend('sin(x) with noise', 'sin(x)');
xlabel('x')
%title('sin (x) with 1% noise')

subplot(1,2,2)
plot(x(2:l-1),cosp,x(2:l-1),cosx,'r')
xlabel('x')
legend('centered difference of noisy data', ...
    'centered difference ');
%title('cos (x) by noisy numerical differentiation')
```



using finite difference in noisy data will lead to "large" error !!

Method of polynomial approximation

Example: Find a one-sided approximation to $f'(x_0)$ based on $f(x_0)$, $f(x_0+h)$ and $f(x_0+2h)$.

1. Approximate $f(x)$ by a degree-2 polynomial $p(x)$ with the given 3 points. Use Newton polynomial.

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - (x_0 + h)) \tag{11}$$

$$p(x_0) \;=\; c_0 = f(x_0) \tag{12}$$

$$p(x_0 + h) \;=\; c_0 + c_1 h = f(x_0 + h) \tag{13}$$

$$p(x_0 + 2h) \;=\; c_0 + c_1(2h) + c_2(2h)(h) = f(x_0 + 2h) \tag{14}$$

Solve the system with substitution or divided differences

$$c_0 = f(x_0)$$

$$c_1 = \frac{f(x_0 + h) - f(x_0)}{h}$$

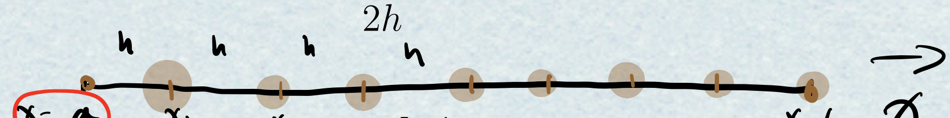
$$c_2 = \frac{\frac{f(x_0+2h)-f(x_0+h)}{h} - \frac{(f(x_0+h)-f(x_0))}{h}}{2h} = \frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{2h^2}$$

Put together

$$p(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) = f(x_0) + \frac{f(x_0+h) - f(x_0)}{h}(x - x_0)$$
$$+ \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{2h^2}(x - x_0)(x - x_0 - h)$$

2. Approximate $f'(x_0)$ by

$$\begin{aligned} p'(x_0) &= c_1 + c_2(x_0 - x_1) + c_2(x_0 - x_2) = c_1 + c_2(-h) \\ &= \frac{f(x_0+h) - f(x_0)}{h} - \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{2h} \\ &= \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} \end{aligned}$$



Finite difference differentiation matrices

- Let $x_i \in [a, b]$ such that $x_0 = a$, $x_1 = a + h$, $x_i = a + ih$, $\dots$, $x_n = b$.
- $h = (b - a)/n$.
- $f(x_i) = f_i$ is the given data.
- If we want to approximate the derivative $f'(x_i)$ using centered differences, interior points x_i : $1 \le i \le n-1$

$$f_1' = \frac{f_2 - f_0}{2h}, \quad \boxed{f_i' = \frac{f_{i+1} - f_{i-1}}{2h}}, \quad f_{n-1}' = \frac{f_n - f_{n-2}}{2h}.$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$1 \le i \le n-1$

$\frac{1}{2h}(f_2 - f_0) = f_1'$

$\frac{1}{2h}(f_3 - f_1) = f_2'$

$\begin{pmatrix} 0 & \frac{1}{2h} & \cdots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} \quad \begin{pmatrix} f_1' + \frac{f_0}{2h} \\ f_2' \end{pmatrix}$

$\frac{1}{2h}(f_{i+1} - f_{i-1}) = f_i'$,

$\frac{1}{2h}(f_n - f_{n-2}) = f_{n-1}'$

$i \to \begin{pmatrix} 0 & -\frac{1}{2h} & 0 & \frac{1}{2h} & 0 \\ & & & -\frac{1}{2h} & 0 \end{pmatrix} \begin{pmatrix} f_i \\ \vdots \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} f_i' \\ \vdots \\ f_{n-1}' \end{pmatrix}$

$i-1 \quad i \quad i+1 \qquad (n-1)(n-1) \qquad -\frac{f_n}{2h}$

- Then we can write the operator in matrix-vector form $\mathbf{f}' = D\mathbf{f}$, where

$$\mathbf{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad D = \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \end{bmatrix}.$$

- Dimensions of vectors and matrices?

$$D = \frac{1}{2h} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & & \\ 0 & -1 & 0 & 1 & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \quad (n-1)(n-1)$$

- $\mathbf{f}'$ is uniquely determined from $\mathbf{f}$.

- Using backward differences and the condition $f(x_0) = 0$,

$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} = D^- f(x_0)$

$f_i' = \frac{f_i - f_{i-1}}{h}$

$$\begin{bmatrix} f'(x_1) + \frac{f(x_0)}{h} \\ f'(x_2) \\ \vdots \\ f'(x_n) \end{bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

$n \times n \qquad n$

- If we are given $\mathbf{f}'$ and the initial condition, then we can solve the linear system for $\mathbf{f}$.