1 Advection equation and wave equation

Given the constant a, the advection equation is given by

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

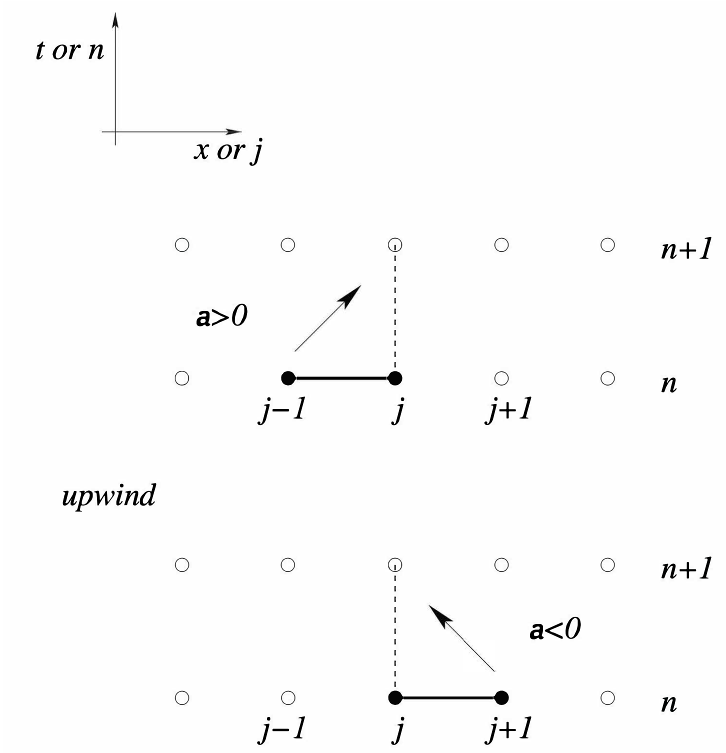
1.1 Upwind scheme

- The upwind scheme is a first-order accurate scheme.
- The upwind scheme is given by

$$U_j^{n+1} = U_j^n - a\frac{\Delta t}{\Delta x}\left(U_j^n - U_{j-1}^n\right) \quad \text{for } a \quad 0$$

$$U_j^{n+1} = U_j^n - a\frac{\Delta t}{\Delta x}\left(U_j^{j+1} - U_j^n\right) \quad \text{for } a \quad 0$$

- Should choose a method based on the direction of wave propagation.
- There is an asymmetry in the advection.
  - If $a > 0$, solution moves to the right.
  - If $a < 0$, solution moves to the left.
- We can analyze the stability using von Neumann stability analysis: let $U_j^n = z^n e^{ij\xi}$, where z is the amplification factor.



Now we can see: the upwind scheme is stable for

1.2 Beam-Warming scheme

- One-sided second order accurate method, and it is derived by following the derivation of the Lax-Wendroff method.

- The beam-warming scheme is given by (for $a > 0$)

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) + \frac{a^2\Delta t^2}{\Delta x^2}\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right)$$

- The method is stable for $|a\Delta t/\Delta x| \leq 2$.

- The beam-warming scheme for $a < 0$ is given by

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{\Delta x}\left(-3U_j^n + 4U_{j+1}^n - U_{j+2}^n\right) + \frac{a^2\Delta t^2}{\Delta x^2}\left(U_j^n - 2U_{j+1}^n + U_{j+2}^n\right)$$

1.3 Lax-Wendroff scheme

We note the advection equation is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \Longrightarrow \frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \tag{1}$$

So we can get

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right) = a^2\frac{\partial^2 u}{\partial x^2} \tag{2}$$

Now we consider using Taylor series expansion at (x_j, t_{n+1})

$$u(x_j, t_{n+1}) = u(x_j, t_n) + \Delta t\frac{\partial u}{\partial t}|_{x_j,t_n} + \frac{1}{2}\Delta t^2\frac{\partial^2 u}{\partial t^2}|_{x_j,t_n} + O(\Delta t^3).$$

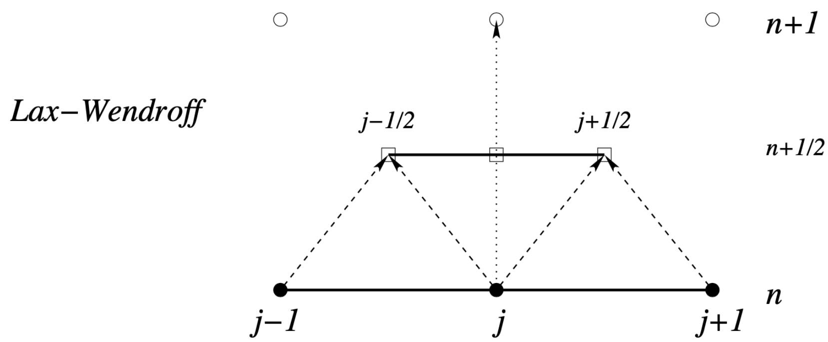
We can use (1) and (2) to replace the first and second derivatives with respect to time to derive the Lax-Wendroff scheme, as follows:

$$u(x_j, t_{n+1}) = u(x_j, t_n) - a\Delta t\frac{\partial u}{\partial x}|_{x_j,t_n} + \frac{1}{2}a^2\Delta t^2\frac{\partial^2 u}{\partial x^2}|_{x_j,t_n} + O(\Delta t^3)$$

Using central difference for both derivatives, we can write the Lax-Wendroff scheme in a more compact form as

$$U_j^{n+1} = U_j^n - \frac{a\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

- We can derive the Lax-Wendroff scheme using the combination of the Lax-Friedrichs scheme and the Leap-Frog scheme too.



1. The Lax-Friedrichs scheme of half time step is given by

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{j+1}^n + u_j^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

$$u_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_j^n + u_{j-1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

2. The Leap-Frog scheme (now over 2 half steps: $n, n+\frac{1}{2}, n+1$) is then given by

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)$$

Combining the two gives us

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

- We can use stability analysis on the Lax-Wendroff scheme: assuming $U_j^n = z^n e^{ij\xi}$, we can get the amplification factor

$$\begin{aligned} z =& 1 - \frac{a\Delta t}{2\Delta x}\left(e^{i\xi} - e^{-i\xi}\right) + \frac{a^2\Delta t^2}{2\Delta x^2}\left(e^{i\xi} - 2 + e^{-i\xi}\right) \\ =& 1 - \frac{a\Delta t}{\Delta x} i \sin(\xi) + \frac{a^2\Delta t^2}{\Delta x^2}\left(\cos(\xi) - 1\right) \end{aligned}$$

And the amplification factor's norm is given by

$$|z|^2 = \left(1 + \frac{a^2\Delta t^2}{\Delta x^2}\left(\cos(\xi) - 1\right)\right)^2 + \left(\frac{a\Delta t}{\Delta x}\sin(\xi)\right)^2$$

Let $\lambda = \frac{a\Delta t}{\Delta x}$, we can get

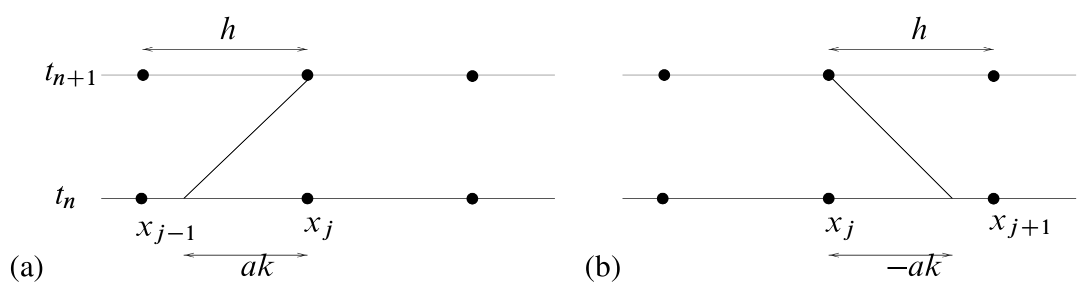
$$|z|^2 = \left(1 + \lambda^2\left(\cos(\xi) - 1\right)\right)^2 + \left(\lambda\sin(\xi)\right)^2 = 1 - \lambda^2(1-\lambda^2)(1-\cos(\xi))^2$$

- The Lax-Wendroff scheme is stable for $|a\Delta t/\Delta x| \leq 1$.

1.4 Characteristic method

Recall the solution to the advection equation $u_t + au_x = 0, u(x,0) = \eta(x)$ is $u(x,t) = \eta(x - at) = u_0(x - at)$.

- Over a single time step $u(x_j, t_{n+1}) = u(x_j - a\Delta t, t_n)$.
- If $a > 0$, $x_j - a\Delta t$ lies between x_{j-1} and x_j.
- If $a\Delta t/\Delta x = 1$, $x_j - a\Delta t = x_{j-1}$ and $u(x_j - a\Delta t, t_n) = u(x_{j-1}, t_n)$.
  - Solution shifts one grid cell to the right in each time step.
  - The numerical exact solution is $U_j^{n+1} = U_{j-1}^n$.
  - If $a\Delta t/\Delta x < 1$, $x_j - a\Delta t$ does not lie exactly at a grid point.



- Can develop a numerical method based on interpolating U_j^n at nearby grid points.
- The linear interpolant at U_j^n and U_{j-1}^n is

$$p(x) = U_j^n + (x - x_j)\left(\frac{U_j^n - U_{j-1}^n}{\Delta x}\right)$$

- Evaluate at $U_j^{n+1} = p(x_j - a\Delta t)$:

- Note that we are interpolating if $x = x_j - a\Delta t$ lies between x_{j-1} and x_j. Otherwise, we are extrapolating.

- The Lax-Wendroff method can be derived from computing a quadratic polynomial using U_j^n, U_{j-1}^n, and U_{j+1}^n.

The Courant-Friedrichs-Lewy condition

A necessary condition for any method to solve the advection:

If U_j^{n+1} is computed based on values U_{j+p}^n, U_{j+p+1}^n, $\ldots, U_{j+q}^n$ with $p \leq q$, we must have $x_{j+p} \leq x_j - a\Delta t \leq x_{j+q}$.

- Otherwise the method will not converge.

- Since $x_j = jh$, $-q \leq \dfrac{a\Delta t}{\Delta x} \leq -p$.

- Called the CFL condition.

- The Courant number: $\nu = a\Delta t/\Delta x$.

Domain of dependence: for the advection equation, the solution $u(X,T)$ at a fixed point (X,T) depends on the initial data η at a fixed point $X - aT$.

- Then we define the domain of dependence to be $\mathcal{D}(X,T) = \{X - aT\}$.
- The data at any other point will have no effect on the solution at this point.

For the diffusion equation $u_t = u_{xx}$, the solution at any point (X,T) depends on the data everywhere.

- The domain of dependence is $\mathcal{D}(X,T) = (-\infty, \infty)$.
- Infinite propagation speed.

A finite-difference method also has a domain of dependence.

- The domain of dependence of a grid point (x_j, t_n) is the set of grid points x_i at the initial time $t = 0$ with the property that the data U_i^0 at x_i has an effect on the solution U_j^n.

Consider Lax-Wendroff:

$$U_j^{n+1} = U_j^n - a\frac{\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right) + a^2\frac{\Delta t^2}{2\Delta x^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

- The solution at U_j^{n+1} on depends U_j^n, U_{j+1}^n, and U_{j-1}^n.

- Go back another time level. Then U_j^n depends on U_j^{n-1}, U_{j+1}^{n-1}, and U_{j-1}^{n-1}, and U_{j-1}^n depends on U_j^{n-1}, U_{j-1}^{n-1}, and U_{j-2}^{n-1}, U_{j+1}^n depends on U_j^{n-1}, U_{j+1}^{n-1}, and U_{j+2}^{n-1}.

- Can trace back to U^0.

- Let $\Delta t/\Delta x = r$, and then refine the grid.

- The numerical method will fill the interval $[X - T/r, X + T/r]$. We want $X - T/r \leq X - aT \leq X + T/r$ to match the equation's domain of dependence.

  - Need $|a| \leq 1/r$ or $|a\Delta t/\Delta x| \leq 1$.

Definition. The CFL condition: A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as Δt and Δx go to zero.

- If it is violated, then the method cannot be convergent.

- If it is satisfied, then the method might be convergent.

- Still need to do proper stability analysis to determine the proper stability restriction on Δt, Δx.