eg. $\boxed{y'(t) = 3y(t)}$ sol: $y(t) = k e^{3t}$ general sol $t=0$

ODEs: Initial Value Problems

assume I.C. $y(0) = 2 \Rightarrow k = 2$

$t = 0$

1 Review of ODEs

Now we focus on first-order differential equations, in particular to initial value problems (IVPs) of the form

D.E.

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a, b], \\ y(a) = y_0. \end{cases} \tag{1}$$

$\frac{dy(t)}{dt} = f(t, y(t))$

I.C.

Examples of first-order ODEs include

I.F. $= e^{\int p(t)dt}$

- Linear ODEs: $y' + p(t)y = g(t)$, $\quad y' + 2y = \sin(t)$
- Separable ODEs: $y' = g(t)h(y)$, $\quad y' = t^2 y^3$
- Exact ODEs: $M(t, y) + N(t, y)y' = 0$. $\quad M(t,y)dt + N(t,y)dy = 0$ $\quad y' = \frac{dy}{dt}$
- Bernoulli ODEs: $y' + p(t)y = q(t)y^n$. $\quad y' + y = 2y^3$

Example 1: For the equation $y'(t) = \cos(t)$, we have general solution $y(t) = \sin(t) + C$.

Example 2: Solve the IVP $y' = 2ty^2$, $y(0) = y_0$. We can use the method of separation of variables to obtain

I.C.

$\frac{dy}{dt} = 2ty^2$

$$\int \frac{dy}{y^2} = \int 2tdt \Rightarrow -\frac{1}{y} = t^2 + C \Rightarrow y(t) = \frac{1}{-t^2 - C}.$$

sol. to D.E.

$-y^{-1} = t^2 + C$

I.C.

assume $y(0) = \frac{1}{-0^2 - C} = \boxed{y_0}$

$\frac{1}{-C} = y_0 \Rightarrow \boxed{C = -\frac{1}{y_0}}$

I.V.P

(a) $\begin{cases} y' = f(t, y) \\ y(t) = y_0 \end{cases}$ ← eg.

general sol

$t=0$

With the initial condition $y(0) = y_0$, we have $y(t) = \frac{1}{-t^2+1/y_0}$. $= \boxed{\dfrac{y_0}{-y_0 t^2+1}}$

general sol for I.V.P

$\begin{cases} \frac{dy}{dt} = 2t\,y^2 \\ y(0) = y_0 \end{cases}$

Existence and Uniqueness The existence and uniqueness of solutions to the IVP (1) are guaranteed under certain conditions, and we need to check these conditions before solving the IVP, either analytically or numerically.

Theorem 1. *Let* $\boxed{f(t, y)}$ *and* $\boxed{f_y(t, y)}$ *be continuous functions of* t *and* y *at all points* (t, y) *in some neighborhood* S *of the point* (t_0, y_0)*. Then, there is a unique function* $y = \phi(t)$ *defined on some interval* $t \in [t_0 - \delta, t_0 + \delta]$ *satisfying* (1).

We can check the example above to see if the conditions are satisfied.

eg: → $\begin{cases} y' = 2t\,y^2 = f(t, y) \\ y(0) = y_0 \end{cases}$

$f(t, y) = 2t\,y^2$ continuous

$\frac{\partial f}{\partial y} = 4ty$

$y(t) = \dfrac{y_0}{-y_0 t^2+1}$

(b)

$y(t) = \dfrac{y_0}{-y_0 t^2 + 1}$ I.C. $y(0) = y_0$

↓

continuous function

$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ $t \in [t_0, T]$

initial time

2 Numerical Methods for ODEs

$y_0 = y(t_0)$ $y_1 = y(t_1)$ $y_{n-1} = y(t_{n-1})$ Goal: find $y_i = y(t_i)$

2.1 Notations:

t_0 t_1 t_2 ... t_{n-1} $T = t_n$

$t_i, i = 0, 1, \ldots, n$

Suppose we want to approximate the solution $y(t)$ of (1) in some interval $t \in [t_0, T]$. Obviously, we can not solve for all t in this interval, so we consider discrete *time* points $t_0 < t_1 < \ldots < t_{n-1} < t_n \leq T$. Our goal is to obtain $y_1 = y(t_1), y_2 = y(t_2), \ldots, y_n = y(t_n)$ as the exact solution (true value) $y(t)$, and we use $\tilde{y}_i, i = 1, \ldots, n$ (or ϕ_i in some cases) to approximate values of the true solution y_i at these $t_1, t_2, \ldots, t_n$ time points. For simplicity, we can assume the times points t_i are equally spaced, so the timestep $t_{i+1} - t_i = h$ for all i indices. We note, the methods we will discuss may not require equally space timesteps. This even spaced time points simplification lead to simpler discussions of analysis in convergence and stability.

$\begin{cases} y' = f(t, y(t)) & (1) \\ y(t_0) = y_0 \end{cases}$

In general, if we integrate the equation in (1) from t_n to t_{n+1}, we have

$y' = f(t, y(t))$

$$\int_{t_n}^{t_{n+1}} y'(t)dt = \int_{t_n}^{t_{n+1}} f(t, y(t))dt \Longrightarrow y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{2}$$

$$y(t_{n+1}) - y(t_n) = y(t)\Big|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

$y(t_{n+1}) - y(t_n)$

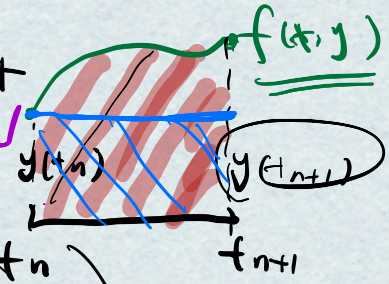
note $y_n = y(t_n)$

key task:

Construct approx. to $\int_{t_n}^{t_{n+1}} f(t, y(t))dt$

given $\to y_n \approx y(t_n)$ is known

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$



2.2 (Forward) Euler's Method

The simplest method to the integral in (2) is to use a constant for the $f(t, y(t))$, for example we let the constant to be $f(t_n, \underset{\text{know}}{\overset{y_n}{y(t_n)}})$.

we know $\boxed{y_n = y(t_n)}$

$$y(t_{n+1}) \approx y(t_n) + (t_{n+1} - t_n) f(t_n, y(t_n)).$$

And this is the (forward or explicit) Euler's method.

If we use ϕ_n to denote the approximate solution of the IVP at t_n, then the Euler's method has formula note $\boxed{\phi_{n+1} \approx y(t_{n+1})}$

Forward

$$\phi_{n+1} = \phi_n + (t_{n+1} - t_n) f(t_n, \phi_n). \tag{3}$$

explicit method

We note this is one step method, and it is *explicit in time* method.

Example: Solve the IVP $y' = (-2t + 1)y$, $y(0) = 1$ using forward Euler's method with $h = 0.2$. (fixed h means equally spaced time: $t_{n+1} - t_n = h$)

$\Rightarrow f(t, y) = (-2t + 1)y$

as $f(t, y) = (1 - 2t)y$

Answer: We have $f(t_n, \phi_n) = (1 - 2t_n)\phi_n$, so

plug $t_n \to t$, $\phi_n \to y$ we get

$$\phi_{n+1} = \phi_n + h(1 - 2t_n)\phi_n = \phi_n(1 + h(1 - 2t_n)).$$

$t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, and $t_3 = 0.6$ if $h = 0.2$

$f(t_n, \phi_n) = (1 - 2t_n)\phi_n$.

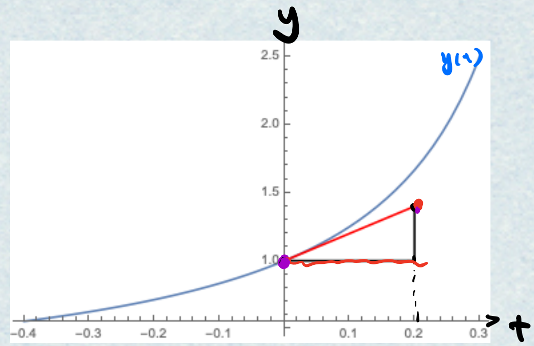
$\phi_{n+1} = \phi_n + (t_{n+1} - t_n) f(t_n, \phi_n)$

$t_{n+1} - t_n = h$

S.

$\phi_{n+1} = \phi_n + h\,\underbrace{(1 - 2t_n)\phi_n}_{f(t_n, \phi_n) = y'(t_n)}$

I.C. $y(0) = 1$



Euler 'method

n	t_n	$f(t_n, \phi_n) = (1 - 2t_n)\phi_n$	$\phi_{n+1} = \phi_n + (t_{n+1} - t_n) f(t_n, \phi_n)$
0	0	1	–
1	0.2	$(1 - 2 \cdot 0.2) \cdot 1.2 = 0.72$	$1 + 0.2(1) = 1.2$
2	0.4	$(1 - 2 \cdot 0.4) \cdot 1.344 = 0.2688$	$1.2 + 0.2(0.72) = 1.344$

We should avoid using Euler's method for stiff problems, where the solution changes rapidly.

forward

2.3 Backward Euler's Method

Recall the integral in (2), $\boxed{y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt}$, we can approximate the integral by using the value of $f(t_{n+1}, y(t_{n+1}))$ instead of $f(t_n, y(t_n))$, and this leads to the (backward or implicit) Euler's method,

$$y(t_{n+1}) \approx y(t_n) + (t_{n+1} - t_n) f(t_{n+1}, y(t_{n+1})). \tag{4}$$

If ϕ_n is approx to $y(t_n)$