The eqn. ⊛ $y' = -20(y - \sin t) + \cos t$, $y(0) = 1$

For this, we need to use $h = \Delta t < 0.1$

to make the numerical result $\boxed{\text{stable}}$

Stiffness of ODE IVP

Def: An ODE IVP in some intervals $[a, b]$ is called stiff if for typical explicit method requires a much smaller time step to be stable than is needed to represent the solution accurately

Q: how to analyze a method to see the stability constraint?

A: stability analysis on the test eqn $\leftarrow$

$$y'(t) = \lambda y(t), \quad y(0) = y_0$$

where λ is a (complex) number, so

the solution is just $\boxed{y(t) = y_0 e^{\lambda t}}$ $(\lambda = a + bi)$

note $y(t) = y_0 e^{at}(\cos(bt) + i\sin(bt))$

- $Re(\lambda) = a < 0$,

  while $t \to \infty$, $e^{at} \to 0$

  so $y(t) \to 0$ w/ $t \to \infty$

$$e^{\alpha + \beta i} = \boxed{e^{\alpha}} e^{\beta i}$$

$$= \boxed{e^{\alpha}}(\cos\beta + i\sin\beta)$$

$\lambda = a + bi$

- $Re(\lambda) = a > 0$, then $y(t) \nearrow \infty$ $\qquad Re(\lambda) = a$, $Im(\lambda) = b$

Note: a good numerical method ϕ_n for the Test eqn should have the property, when $Re(\lambda) < 0$, $\phi_n \to 0$ as $n \to \infty$

Now let's look at Euler's method: $\phi_{n+1} = \phi_n + h f(t_n, \phi_n)$ on the test eqn $y' = \lambda y$, $y(0) = y_0$ follows

$$\phi_{n+1} = \phi_n + h\lambda\phi_n = (1 + h\lambda)\phi_n$$, then it leads to

$$\begin{cases} \phi_1 = (1+h\lambda)\phi_0 \\ \phi_2 = (1+h\lambda)\phi_1 \\ \phi_{n-1} \\ \phi_n = (1+h\lambda)\phi_{n-1} = (1+h\lambda)^n \phi_0 \end{cases}$$

$\phi_0 \leftarrow y_0$

So now $(1+h\lambda)^n \to 0$ as $n \to \infty$ $\qquad$ condition $(Re(\lambda) < 0)$

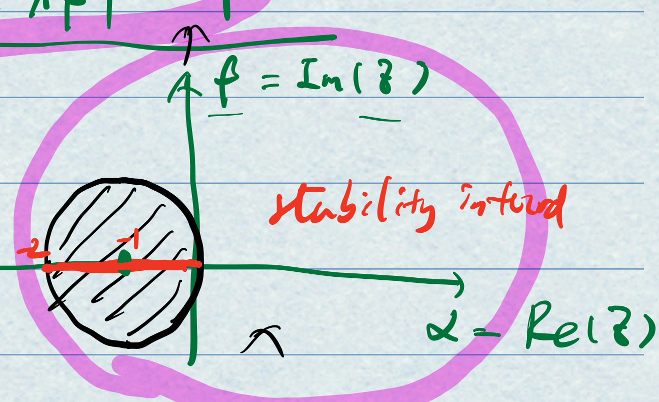
then we need $|1 + h\lambda| < 1$, and this defines a region in the complex plane

let $z = h\lambda = \alpha + i\beta$ complex number, then

$$|1 + h\lambda| = |1 + z| = |1 + \alpha + i\beta| < 1$$

$$|1 + \alpha + \beta i| = 1 \iff \sqrt{(1+\alpha)^2 + \beta^2} = 1$$

$(1+\alpha)^2 + \beta^2 \le 1$ disk of radius 1 centered at $(-1, 0)$



So we need to have $z = \boxed{h\lambda}$ & we need to

check ___ of the region $|h\lambda| < 2 \Rightarrow h \le \frac{2}{|\lambda|}$

Also if $h > \frac{2}{|\lambda|}$, then ϕ_n will grow exponentially

⊙ For Euler method, the stability region is

$R = \{ z \text{ complex} \mid |1+z| < 1 \}$, which is the disk

centered at $(-1, 0)$ and radius 1

⊙ the interval of the region R intersect w/ the

real axis is called the stability interval

For the backward Euler method to test eqn $y' = \lambda y$

$$\phi_{n+1} = \phi_n + h f(t_{n+1}, y_{n+1})$$

condition $Re(\lambda) < 0$

$$\phi_{n+1} = \phi_n + h\lambda\phi_{n+1} \Rightarrow \phi_{n+1} = \frac{\phi_n}{1-h\lambda}$$

$$\phi_n = \boxed{\frac{1}{(1-h\lambda)^n}} \phi_0$$, so we need to have

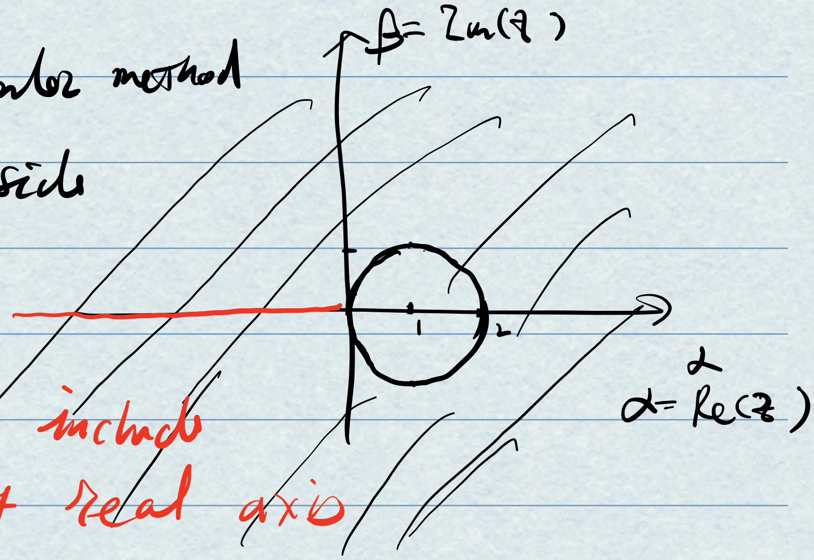
$\Leftarrow |1-h\lambda| > 1$ let $z = h\lambda = \alpha + i\beta$

$|1-\alpha-\beta i| > 1 \Rightarrow \sqrt{(1-\alpha)^2+\beta^2} \ge 1$

let's check $(1-\alpha)^2+\beta^2 = 1$ circle w/ center @

(1, 0) and radius 1

So R of the backward Euler method on test eqn is the out side of the circle, if $Re(\lambda)<0$



and the stability interval include the whole negative part of real axis

So you can choose any h

Def: A method for which the stability region R contains all of $\{Re(z)<0\}$ is called A-stable and there is no stability conditions required for a stiff eqn for those A-stable methods

- Not all implicit methods are A-stable

- all explicit methods have finite intervals of absolute stability, so we need to choose carefully the h

general case $f_i = f(t_n + \boxed{c_i h}, \phi_n + h\boxed{\sum_{j=1}^{s} a_{ij} f_j})$

$i = 1, 2, \cdots, s,$

1 More topics on ODE IVPs methods

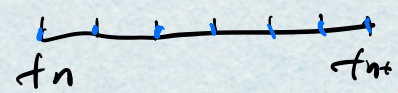
1.1 Review of the Runge-Kutta methods

The Runge-Kutta methods are a class of numerical methods for solving ordinary differential equations. The general form of the Runge-Kutta method is

s-stage

$y' = f(t, y)$

$$\boxed{\phi_{n+1} = \phi_n + h \sum_{i=1}^{s} b_i f_i,}$$



where h is the step size, y_n is the numerical solution at time t_n, y_{n+1} is the numerical solution at time $t_{n+1} = t_n + h$, b_i are the weights, and f_i are the slopes. The slopes f_i are defined as

explicit RK

$$f_1 = f(t_n, \phi_n), \; f_2 = f(t_n + c_2 h, \phi_n + h(a_{21} f_1)), \; \ldots, \; f_s = f(t_n + c_s h, \phi_n + h \sum_{j=1}^{s-1} a_{sj} f_j),$$

where $f(t, y)$ is the right-hand side of the ODE, a_{ij} are the coefficients of the method, and c_i are the nodes of the method. We have the constraint $\sum_{i=1}^{s} b_i = 1$ for the weights, and $\sum_{j=1}^{s} a_{ij} = c_i$ for the coefficients. If $a_{ij} = 0, i \leq j$, the RK method will be explicit. And if $a_{ii} \neq 0, a_{ij} = 0, i < j$, we have diagonally implicit RK method (DIRK). We can use the Butcher

$\Rightarrow y(t) = t + C$

$\Rightarrow y(t) = y_n + t$

For $y'(t) = 1, \; y(t_n) = \boxed{y_n}$ $\qquad y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} y'(t)\,dt$

exact sol : $y(t_n + z) = y_n + z$

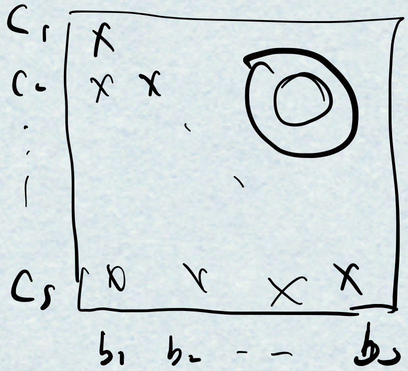
For this ODE $y' = f \equiv 1$, then apply R.K.

we have $h \sum_{j=1}^{s} a_{ij} f_j = h \sum_{j=1}^{s} a_{ij} = c_i h$

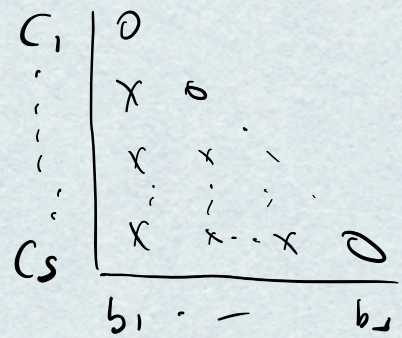
So $\boxed{\sum_{j=1}^{s} a_{ij} = c_i}$ this ensure the internal stages are at the right time

For $a_{ii} \neq 0$ $a_{ij} = 0$ 对 $i<j$

tableau to describe the coefficients of the Runge-Kutta method:



$$\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s-1} & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s-1} & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss-1} & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_{s-1} & b_s
\end{array}$$



DIRK

ERK

We also note, the RK methods can be written as

$$\frac{\phi_{n+1} - \phi_n}{h} = \sum_{i=1}^{s} b_i f_i = \sum_{i=1}^{s} b_i f(t_n + c_i h, \phi_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, \phi_n)).$$

So this is the approximation of the equation:

$$\frac{y_{n+1} - y_n}{h} = b_1 f_1 + b_2 f_2 + \ldots + b_s f_s + \mathcal{O}(h^p),$$

For the order of a s stage Runge-Kutta method, we have 1) for $s \leq 4$, the best possible order is s, and 2) for $s \geq 5$, the best possible order is $s - 1$. The order of the method is determined by the number of stages s and the coefficients of the method.

- You can get the RAS for RK method on test eqn $y' = \lambda y$
- minimal requirement of stability is to maintain stable for $y' = 0$

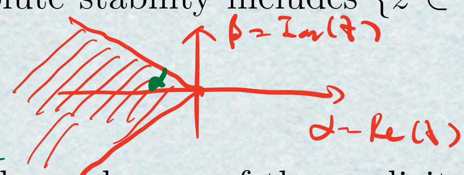
1.2 Review of stability

A stiff problem usually means there are multiple scales in the solutions to an ODE IVP.

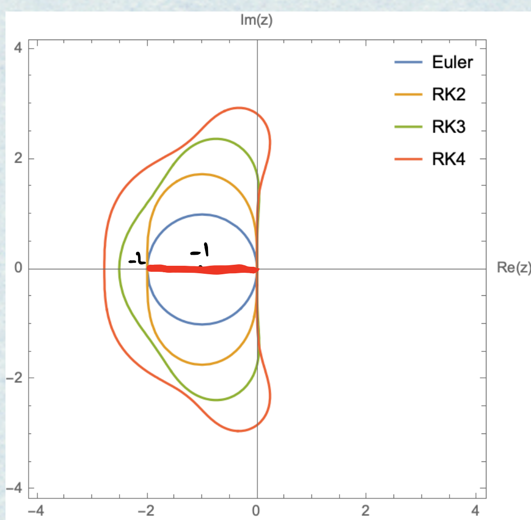
Region of absolute stability: The region of absolute stability of a method is the set of points $z = h\lambda$ in the complex plane for which the numerical solution of the test equation $y' = \lambda y$ (h is the step size of the numerical method and λ is a complex number) $\phi_n, n = 0, 1, \ldots$ satisfies $\phi_n \to 0$ as $n \to \infty$.

A-Stability: A method is A-stable if its region of absolute stability contains the entire left half-plane, i.e., the region of absolute stability includes $\{z \in \mathbb{C} : \Re(z) \leq 0\}$.

$A(\alpha)$- stability



We note that backward Euler method is A-stable, and none of the explicit Runge-Kutta methods are A-stable.



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