Math Models, Spring 2026: HW 3

Note that some problems or parts of problems may not be graded.

1. Let's consider our constant-gravity model of the projectile problem, but now let's include the force that air resistance exerts on the projectile. To be as simple as possible, we assume that air resistance exerts a force of the form $-k\dfrac{dx}{dt}$, where $k > 0$ is a constant.

(a) Apply Newton's Second Law to conclude that the IVP for the projectile is

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} = -mg, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = V. \tag{1}$$

(b) Note that the differential equation in (1) is linear, 2nd-order, constant coefficient, and inhomogeneous. Solve the IVP using techniques you have learned for solving 2nd-order, constant-coefficient equations.

(c) Note that the differential equation in (1) is also a 1st-order linear equation for $\dfrac{dx}{dt}$. Solve the IVP for $\dfrac{dx}{dt}$ using techniques you have learned for solving 1st-order linear equations. Then integrate your solution to recover x.

(d) Rescale (1) so that you end up with a single dimensionless parameter $\varepsilon = kV/mg$ that appears only next to the $dy/d\tau$ term in the rescaled differential equation (use y and τ as your rescaled variables).

(e) Give a physical interpretation of ε. How do you expect the projectile to behave when ε is small? (Hint: What is the 'limiting case' corresponding to $\varepsilon = 0$?)

(f) Find the solution to the rescaled version of (1). (You can use either the approach from (b) or the approach from (c).) 'Unscale' this solution by converting y and τ back to x and t. Verify that this unscaled solution is the same one you got in parts (b) and (c).

2. Consider the ODE

$$\dot{x} = rx + x^3, \tag{2}$$

where x is the dependent variable and r is a parameter.

(a) By imitating what we did in class for similar scalar ODEs, construct the bifurcation diagram for (2).

(b) What is the bifurcation point? Describe in words what happens to the number and stability of branches as r is increased through the bifurcation point. (This example illustrates what is called a subcritical pitchfork bifurcation. The pitchfork bifurcation we did in class is called a supercritical pitchfork bifurcation.)

(c) Again by imitating what we did in class, sketch the bifurcation diagram again and then add 3 phaselines, one at a typical value of r below the bifurcation point, one at the bifurcation point, and one at a typical value of r above the bifurcation point.

If you are taking the class at the 500 level, do the next problem.

3. It is know that a physical phenomena is described by the quantities ρ, t, m, l, and P. These represent density, time, mass, length, and pressure, respectively. It is know that there is a physical law

$$g(\rho, t, m, l, P) = 0$$

relating these quantities. Show that there is an equivalent physical law of the form

$$G(l^3\rho/m, t^6P^3/m^2\rho) = 0.$$

(Note that $[P] = \frac{\text{force}}{\text{area}}$.)

Additional Instructions

To facilitate the HW grading, please do the following.

- Write your HW solutions double spaced.
- Write your HW solutions in the same order in which the problems are assigned above.
- Clearly label each problem. Draw a dark box or circle around the label—the point is that I should be able to quickly flip through your HW solution and find a given problem.