Continuum Modeling in the Physical Sciences

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A mathematical model in terms of an ordinary differential equation (ODE) is not yet complete if the initial values are left unspecified. This introduces two extra parameters into the system: initial position $u_0 \equiv u(t_0)$ and initial velocity $v_0 \equiv \dot{u}(t_0)$. The solution $u(t)$ thus depends on seven parameters, and we could write it as

$$u = u(t; m, c, k, F_0, \omega, u_0, v_0).$$

For such a simple system this is a huge number to handle, since in an experiment all these parameters could in principle be varied. In the following we show that such a system can essentially be described with fewer parameters, since it does not make sense to vary them all independently. □

The fact that the variables and parameters have physical dimensions can be fruitfully exploited. The techniques of *nondimensionalizing* and *scaling* are extremely powerful tools in analyzing the models. Their importance is fully appreciated only through examples, which account for the largest part of this chapter. The basic idea is to apply a transformation to the variables and parameters such that simplified equations result. It is often amazing how much structure is revealed simply by nondimensionalizing, without solving the model explicitly. Thanks to these techniques it is often known beforehand that the system depends not on all parameters separately but only on certain combinations. In an experimental situation it is of great importance to know how the system depends on the parameters, so this insight may save much time, cost, and energy.

In practice two methods are applied, *dimensional analysis* and *scaling*, each having its own merits. They are dealt with in the subsections below, respectively. Dimensional analysis fully exploits the information contained in the physical dimensions of the variables and parameters. Scaling has a more restricted scope and aims at a reduction of the number of parameters.

1.3 Dimensional analysis

Nondimensionalizing a mathematical model is a constructive way to formulate the model in terms of dimensionless quantities only. A big achievement is that dimensional analysis yields insight in the scaling relations of the system without using knowledge of any governing equation. An advantageous corollary is that the total number of variables and/or parameters is minimal. Reduction of the number of parameters is also the purpose of *scaling*, a technique to be dealt with in the next section. However, dimensional analysis is more general than scaling in that it is based on a transformation of both variables and parameters on the same footing, whereas in scaling only the variables are transformed. Another difference is that scaling starts from the governing equations, whereas dimensional analysis starts much more basically, namely, from the dimensions involved in the system, and it may even predict from them some quantitative features of the model without knowledge of the model equations. The basic idea of dimensional analysis is easily explained. Consider a system with scalar variables $x_1, \ldots, x_k$ and scalar parameters $p_1, \ldots, p_\ell$. So, the total number of quantities involved is $N = k + \ell$. Note that in the model, vectors, matrices, etc., may figure, but for this analysis all their components have to be treated separately. We now form the products

$$x_1^{r_1} \ldots x_k^{r_k} \, p_1^{r_{k+1}} \ldots p_\ell^{r_N}$$

and ask for which choices of the r_i these products are dimensionless. The answer follows from replacing each x_i and p_i with its fundamental dimensions. If, say, m dimensions $d_1, \ldots, d_m$ are involved, the replacement gives rise to another type of product,

$$d_1^{s_1} \ldots d_m^{s_m},$$

with the numbers s_i, $i = 1, \ldots, m$, being linear functions of the r_j, $j = 1, \ldots, N$. The procedure is illustrated several times in the examples below. By requiring

$$s_i = 0, \quad i = 1, \ldots, m,$$

we obtain a set of m linear equations for the N unknowns $r_1, \ldots, r_N$. Note that the numbers r_j, $j = 1, \ldots, N$, are rational, since they are solutions of linear equations with rational coefficients. The rationality of these coefficients stems from the fact that in nature all measurable quantities turn out to have dimensions that are products of integer powers of the fundamental dimensions, as shown in the tables in §1.2. From linear algebra it follows that there are (at most) $N - m$ linearly independent solutions, corresponding to $N - m$ dimensionless quantities q_i, $i = 1, \ldots, (N - m)$. Buckingham formalized this in the following theorem.

Theorem (Buckingham).

Consider a system with variables $x_1, \ldots, x_k$ and parameters $p_1, \ldots, p_\ell$, in which m fundamental dimensions are involved. Then, $k + \ell - m$ dimensionless quantities q_i can be defined, which are products and quotients of the original variables and parameters. Each (scalar) model equation

$$f(x_1, \ldots, x_k, p_1, \ldots, p_\ell) = 0$$

between the x_i and p_i of a mathematical model can be replaced with a corresponding relation between the q_i:

$$f^*(q_1, \ldots, q_{k+\ell-m}) = 0 \,.$$

Since Buckingham [6] denoted the dimensionless quantities by π_i, this theorem is often referred to as the π-theorem of Buckingham. We shall not follow his notation since it is no longer common in the literature. As follows from the construction of the q_i as solutions of an underdetermined set of linear equations, they are not uniquely defined by the procedure. If the procedure yields a set of q_i, we can apply a transformation, e.g., by taking algebraic or even functional combinations of them, obtaining another set of dimensionless quantities of the system. It is a matter of expertise, and partly of taste, to determine a convenient set of q_i for the system under consideration. If the number of variables and parameters is not small, the freedom of choice must be especially exploited with care.

We shall work out the nondimensionalizing procedure for a considerable number of examples, pointing out both the practical aspects of the technique and the insight it may yield about the behavior of the system without solving the equations explicitly.

Example 1.3a. Catapulting.
Let us start with an example in which the mathematics is very basic but the ideas behind dimensional analysis are clearly illustrated. A projectile with mass m is launched vertically. See Fig. **1.2**. At launching it has velocity v_0. Its trajectory, i.e., its vertical position z as a function of time t is assumed to be completely determined by the influence of gravity. The effect of friction due to the air is ignored here (but dealt with in Example 1.3e). The projectile will decelerate because of gravity until it reaches its highest position $z_{\max}$ at time $t_{\max}$. After that it falls back with increasing velocity and arrives on the earth at time t_{final}. Since we take v_0 such that $z_{\max}$ remains small compared to the Earth's radius, we may take the gravity field uniform with gravity constant g.

In this system the variables are z and t and the parameters are m, v_0, and g. The relevant physical dimensions are M, L, and T. So, $k = 2$, $\ell = 3$, and $m = 3$, and the theorem of Buckingham states that the system has two dimensionless quantities. All properties of the system can be expressed in only these two quantities. In this simple case the dimensionless quantities can be easily found from inspection of the dimensions: $[z] = L$, $[t] = T$, $[m] = M$, $[v_0] = L/T$, and $[g] = L/T^2$. An evident choice is

$$t^* = \frac{gt}{v_0}, \quad z^* = \frac{gz}{v_0^2}.$$

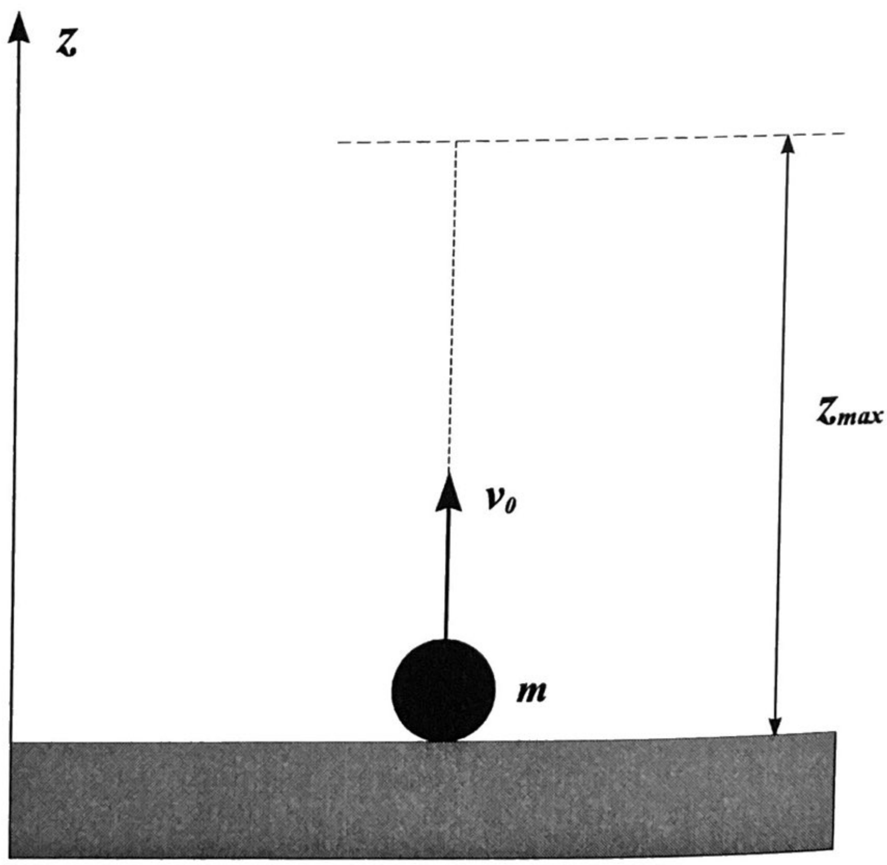

Figure 1.2. The main scaling characteristics of a mass m, launched with initial speed v_0, are easily predicted by dimensional analysis.

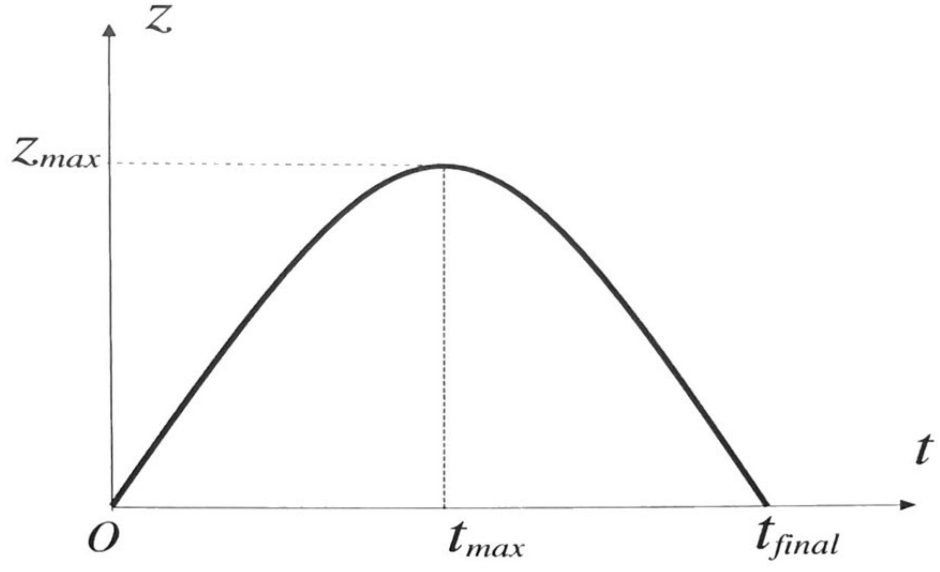

Figure 1.3. The height z of a mass, launched at speed v_0, as a function of time t. It will reach a maximum height z_{max} at time t_{max} and reach the earth's surface again at time t_{final}.

Note that the mass m is not present in t^* and z^*, since the physical dimension M is not present in one of the other variables and parameters. The Buckingham theorem yields that its motion is described by a relation between z^* and t^*. This immediately leads to the conclusion that the motion of the projectile is independent of its mass. From experimental evidence we know that the relation between z and t is more or less as sketched in Fig. **1.3**. The function $z(t)$ reaches a maximum z_{max} at t_{max} and vanishes at t_{final}. Since z^* and t^* are just scaled versions of z and t, z^* apparently can be written as an explicit function of t^*:

$$z^* = f^*(t^*). \tag{1.2}$$

The theorem does not specify any information about f^* but only ensures its existence and the insight that the form of f^* does not depend on any of the parameters m, v_0, and g separately. The latter property thus also holds for the dimensionless quantities z^*_{max}, t^*_{max}, and t^*_{final}. These are just fixed numbers, as shown in Exercise 1.3a. Using the relations between dimensional and dimensionless quantities, we have that

$$z_{\text{max}} = \frac{v_0^2}{g} z^*_{\text{max}}\,, \quad t_{\text{max}} = \frac{v_0}{g} t^*_{\text{max}}\,, \quad t_{\text{final}} = \frac{v_0}{g} t^*_{\text{final}}\,.$$

This yields the insight that z_{max} scales with v_0^2 and both t_{max} and t_{final} with v_0 for fixed value of g. We denote this as

$$z_{\text{max}} \sim v_0^2\,, \quad t_{\text{max}} \sim v_0\,, \quad t_{\text{final}} \sim v_0\,.$$

So, launching with a twice-as-large velocity leads to a four-times-larger maximal height of the projectile. In the same way we conclude that

$$z_{\text{max}} \sim \frac{1}{g}\,, \quad t_{\text{max}} \sim \frac{1}{g}\,, \quad t_{\text{final}} \sim \frac{1}{g}$$

for a fixed value of v_0. So, catapulting on the moon, where g is (approximately six times) smaller than on the earth, enhances z_{max}, t_{max}, and t_{final} all by the same factor. □

Exercise 1.3a.

Check these conclusions on catapulting by explicitly solving the equation of motion

$$m\frac{d^2z}{dt^2} = -mg.$$

Show that f^* *in* (1.2) *has the explicit form as given in Fig.* **1.4**. *Calculate explicitly the values of* z^*_{max}, t^*_{max}, *and* ${}^*_{\text{final}}$. *Note that this function cannot be found from dimensional analysis only.*

Example 1.3b. Swinging pendulum.
Consider the motion of a mathematical swing: this pendulum has mass m concentrated in a point at the end of a rigid rod of length ℓ. The motion is restricted to a vertical plane. See Fig. **1.5**. The position of the swinging pendulum is completely specified by the angle φ with the vertical. This is the independent variable, and time t is the dependent variable. Parameters are mass m, rod length ℓ, gravitational acceleration g, and the initial position $\varphi_0 = \varphi(0)$. For convenience we take the initial velocity vanishing. So, $k + \ell = 6$, and since the three fundamental dimensions M, L, and T are involved, the system has three dimensionless quantities. Since φ and φ_0 are already dimensionless, they form an obvious choice. To find the third, we form the products

$$t^{r_1}\ell^{r_2}m^{r_3}g^{r_4}.$$

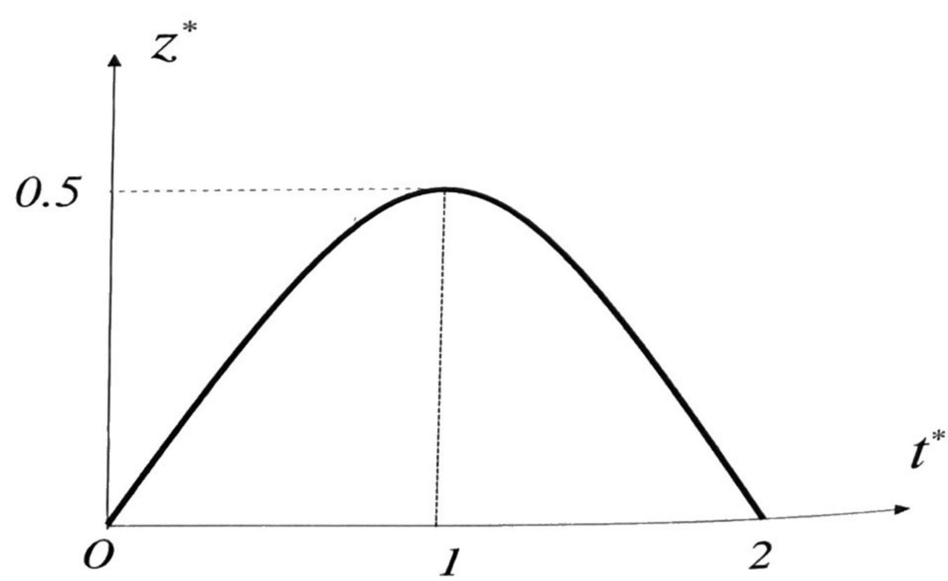

Figure 1.4. Explicit form of the dimensionless function f^* in (1.2). Note that this function is independent of the parameters m, g, and v_0 of the system. The dimensionless height z^* reaches a maximum value $z^*_{max} = 1/2$ at time $t^*_{max} = 1$ and hits the earth's surface again at time $t^*_{final} = 2$.

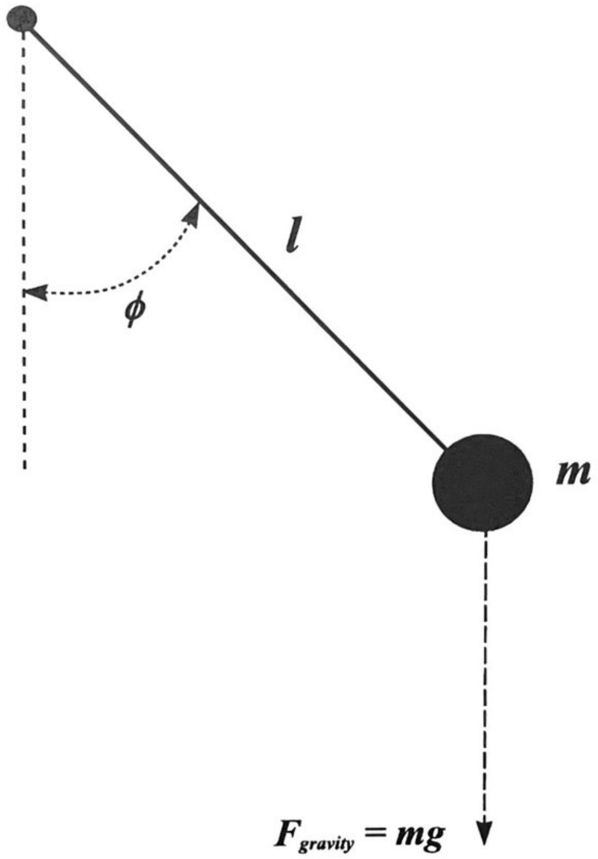

Figure 1.5. Swinging pendulum of mass m and length l. The motion is confined to a vertical plane, and the position of the pendulum can be indicated by the angle ϕ with the vertical.

The condition that this product must be dimensionless leads to the linear equations

$$\begin{aligned} r_1 - 2r_4 &= 0, \\ r_2 + r_4 &= 0, \\ r_3 &= 0. \end{aligned}$$

$r_4 = \frac{1}{2}$

The choice $\cancel{(r_1, r_2) = (1, 0)}$ then yields

$$t^* = t\sqrt{\frac{g}{\ell}}.$$

Note that the mass m is not present in any of the dimensionless quantities φ, φ_0, and t^*. This implies that pendulum motion is independent from m. The movement of the pendulum is given by some relation between φ, φ_0, and t^*. With φ_0 constant and t^* monotonously increasing, we may write φ as an explicit function of t^*:

$$\varphi = f^*(t^*, \varphi_0).$$

This allows for a conclusion about the period of the system. One should realize that dimensional analysis as such does not reveal that φ is a periodic function of time. However, if we take this for granted in view of the observations, we have that $f^*(t^* + \tau^*) = f^*(t^*)$, with τ^* the dimensionless period. Since $\tau^* = \tau\sqrt{g/\ell}$ and τ^* does not depend on any of the parameters, we find that τ scales with $\sqrt{\ell/g}$, so

$$\tau \sim \sqrt{\frac{\ell}{g}}.$$

□

Exercise 1.3b.

a. *Give the dimensionless form of the exact pendulum equation*

$$m\ell\ddot{\varphi} + mg\sin\varphi = 0.$$

b. *If* $|\varphi| \ll 1$, *the linearized pendulum equation*

$$m\ell\ddot{\varphi} + mg\varphi = 0$$

is a good approximation. Give its dimensionless form.

c. *Write the solution of the equation under* b *and check that the period indeed scales with* $\sqrt{\ell/g}$ *as derived in Example* 1.3b. *Determine how the period is influenced if the length is doubled and also when the pendulum is placed on the moon.*

Example 1.3c. Harmonic oscillator.
Here, we revisit the harmonic oscillator introduced in Example 1.2a. Setting the initial values at zero for convenience, the model equation

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \omega t$$

has the two variables u and t and the five parameters m, c, k, F_0, and ω. So, $N = 7$ in this case. The fundamental dimensions involved are mass M, length L, and time T. Forming the products

$$u^{r_1} t^{r_2} m^{r_3} c^{r_4} k^{r_5} F_0^{r_6} \omega^{r_7}$$

and substituting the dimensions, we arrive at the products

$$L^{r_1} T^{r_2} M^{r_3} \left(\frac{M}{T}\right)^{r_4} \left(\frac{M}{T^2}\right)^{r_5} \left(\frac{ML}{T^2}\right)^{r_6} \left(\frac{1}{T}\right)^{r_7}.$$

Collecting powers of M, L, and T, we obtain the following three linear equations for the r_i:

$$\begin{aligned} &r_1 + r_6 = 0, \\ &r_2 - r_4 - 2r_5 - 2r_6 - r_7 = 0, \\ &r_3 + r_4 + r_5 + r_6 = 0. \end{aligned}$$

Here, we meet with three equations for seven unknowns, so four unknowns can be treated as free parameters. For example, we could take $r_1, \ldots, r_4$. The choices $(r_1, r_2, r_3, r_4) = (1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$, respectively, yield the dimensionless quantities

$$u^* = \frac{uk}{F_0}, \; t^* = \omega t, \; m^* = \frac{m\omega^2}{k}, \; c^* = \frac{c\,\omega}{k}.$$

The dimensionless spring equation then reads as

$$m^* \ddot{u}^* + c^* \dot{u}^* + u^* = \sin t^*,$$

where the time derivative is taken with respect to t^*. □

Exercise 1.3c.

The approach used above for the driven spring system is based on the assumption $F_0 \neq 0$. Apply dimensional analysis to the case $F_0 = 0$ but with the initial position u_0 and initial velocity v_0 both nonvanishing.

Example 1.3d. Estimating the power of explosions.

Details of the strength of the first atomic bomb in 1945 were classified until the 1960s. However, the British physicist G.I. Taylor was able to give a very accurate estimate of the strength from dimensional analysis by using available film of the expansion of the mushroom shape of the explosion. His arguments proceed as follows (see, e.g., [31] and [3, 4]).

The basic appearance of the explosion is an expanding spherical fireball whose edge corresponds to a powerful shock wave, as sketched in Fig. **1.6**. Let R be the radius of the shock wave. It will depend on E, the energy released by the explosion; t, the time elapsed since the explosion; ρ, the initial and ambient air density, and p, the initial and ambient

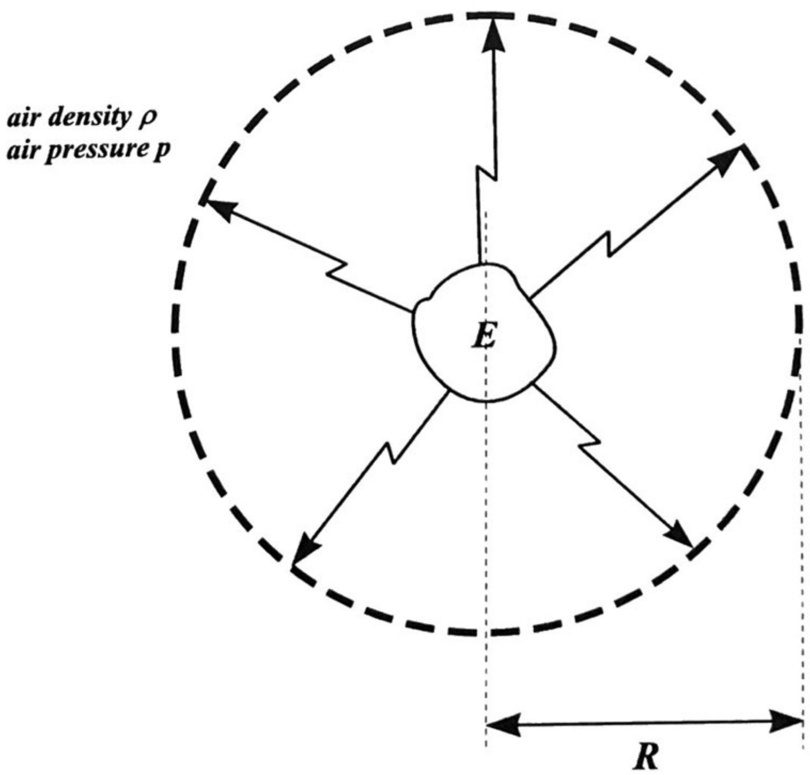

Figure 1.6. Sketch of a shock wave propagating from an explosion with energy E. Dimensional analysis shows that the energy E can be estimated from the propagation velocity of the front.