3SAT

Approaches and Implementation
A Lightning Talk

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Why 3SAT?

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- Fast heuristic-based solvers exist
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- “Canonical” NP-complete problem
- Fast heuristic-based solvers exist
  - Many other NPC problems are reduced to 3SAT in practice
- Interesting topic at the intersection of:
  - Boolean logic
  - Complexity theory
  - Algorithms

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Lightning talk! All this in under 5 minutes! Let’s get started.

Follow along!

Project is open source on Github @dylhunn:

```
git clone https://github.com/dylhunn/257-max-3sat.git
cd 257-max-3sat/src
make test
```

Solve an arbitrary 3SAT problem:

```
make
./threesat ../test_data/test0.txt 1
```

Boolean satisfiability problem

Boolean formulae composed of clauses in conjunctive normal form

$(A \lor \neg A \lor A) \land (\neg B \lor \neg A \lor C)$

Is this formula satisfiable?

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Is this formula satisfiable?

Yes. One of several possible assignments:

`A: True`

`B: False`

`C: True`

Some terms

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MAX-3SAT: what assignment satisfies the largest number of clauses?

Naïve algorithm

- Enumerate the variables
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- Enumerate the variables
- Recursively generate and check every assignment
- Algorithm:

```
bool naïve_solve(formula f, solution s, int index) {
    if (index >= f.num_vars) {
        if (solution_is_valid(f, s)) return true;
        else return false;
    }
    for (int i = 0; i < 2; i++) {
        s[index] = i; // variable # index
        if (naïve_solve(f, s, index + 1)) return true;
    }
    return false;
}
```

Naïve algorithm

- Enumerate the variables
- Recursively generate and check every assignment
- Try it:

```
make
./threesat ../test_data/test0.txt 0
```

- It works:

```
[dylhunn@dylhunn0 src]$ make
[dylhunn@dylhunn0 src]$ ./threesat ../test_data/test0.txt 0
Solvable. Solution: [var1=false, var2=false, var3=false, var4=false, var5=true]
[dylhunn@dylhunn0 src]$ 
```

Naïve algorithm

- Enumerate the variables
- Recursively generate and check every assignment
- Try it:

```
make
./threesat ../test_data/test2.txt 0
```

- But...

Naïve algorithm

- Enumerate the variables
- Recursively generate and check every assignment
- Try it:

```
make
./threesat ../test_data/test2.txt 0
```

- It’s way too slow on large inputs (*obviously*):

```
[dylhunn@dylhunn0 src]$ ./threesat ../test_data/test2.txt 0
Solvable. Solution: [var1=false, var2=false, var3=true, var4=false, var5=false, var6=false, var7=false, v
ar8=false, var9=false, var10=false, var11=true, var12=false, var13=false, var14=true, var15=false, var16=
false, var17=false, var18=true, var19=false, var20=false, var21=false, var22=false, var23=false, var24=fa
lse, var25=true, var26=false, var27=false, var28=true, var29=false]
[dylhunn@dylhunn0 src]$
```

Runtime: ~10 s for 29 variables in 16 clauses

Bron-Kerbosch algorithm

Consider this formula:

$$(X \lor X \lor Y) \land (\neg X \lor \neg Y \lor \neg Y) \land (\neg X \lor Y \lor Y)$$

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  - nodes that share a clause
  - Incompatible nodes (e.g. $A \land \neg A$)

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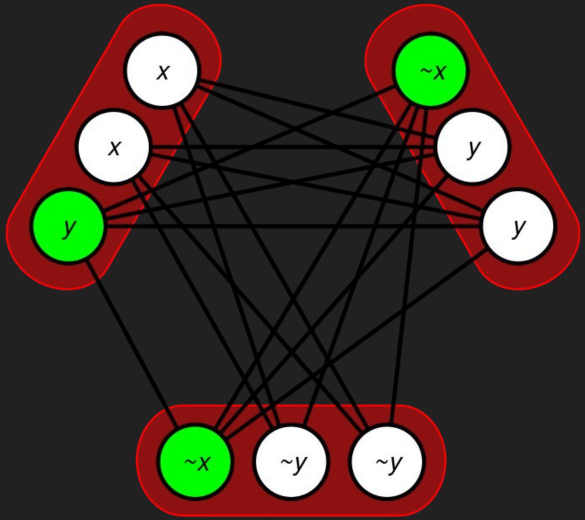

(Example illustration: Thore Husfeldt)

Bron-Kerbosch algorithm

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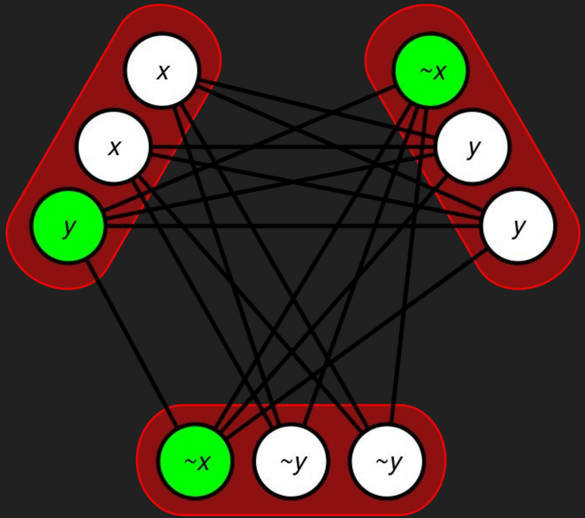
  - (since there are 3 clauses)



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- Connect all nodes except:
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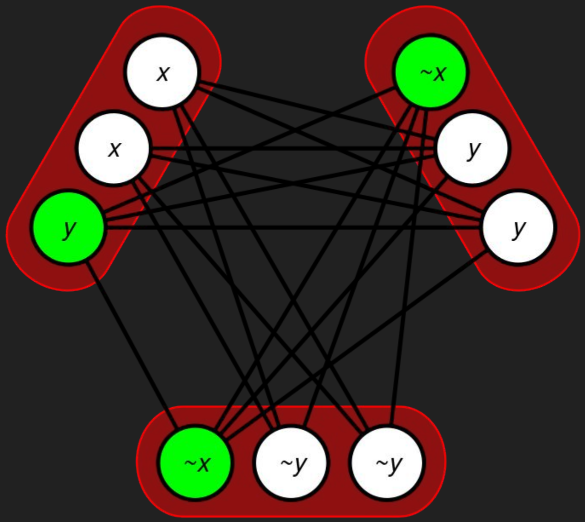
- Any 3-clique is a satisfying assignment
  - (since there are 3 clauses)
- Such an assignment is also a max-clique!



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```
bron_kerbosch(set r, set p, set x) {
     if (p.empty() && x.empty())
          maxcliques_add(r);
     for (vertex v : p) {
          bron_kerbosch(r + v, insersect(p,
     neighbors(v)),
               intersect(x, neighbors(v));
          x = p - v;
          x = x + v;
     }
}
```

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A Hashset!

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- Let’s use these variable numbers [1, n] as indices
- The graph can be represented efficiently as an adjacency matrix
- Bron-Kerbosch requires set operations
- But the set keys are **uniformly distributed integers**!
- Performance boost: implement our own hashset
    - No generic types
    - **Identity hash function**, since the keys are uniformly distributed!

Performant results

- Worst case: no n-vertex graph has more than $3^{(n/3)}$ max-cliques
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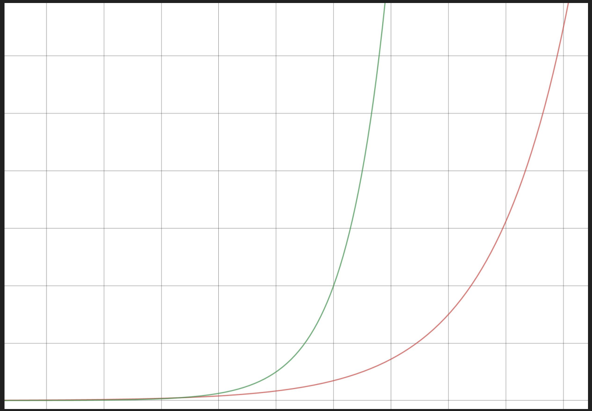

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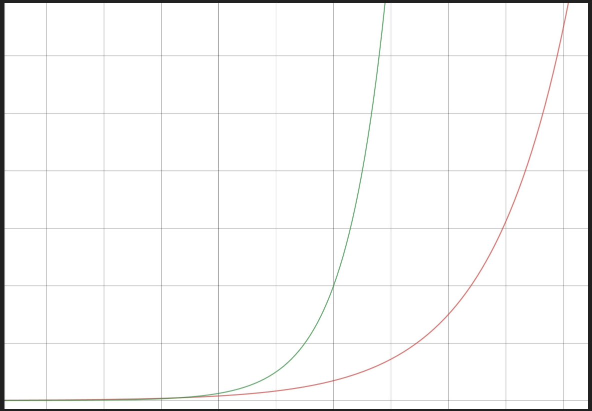

Barring a polynomial solution with P=NP, this much better.

Performant results

- Let’s try the large case again:

```
make
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```

- Much better performance!

```
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- In the **3SAT decision problem**, this can be used as a heuristic!
- In high ratio cases, the naive solver often performs quite well
  - Might terminate after only exploring one assignment!
  - No graph construction overhead
- Thus, empirical verification of the Big-O graph is challenging
  - Must find the 50/50 Satisfiable / Unsatisfiable ratio

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- But it is still the MAX-3SAT solution!
- This is an obvious next step for my implementation

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Questions?