weak form derivation:

$$\int_0^1 v(x) u_t \, \partial x - \int_0^1 v(x) u_{xx} \partial x = \int_0^1 v(x) f(x,t) \, \partial x$$

To get rid of u_{xx}, we use integration by parts:

$$\int_0^1 v(x) u_t \partial x - \left[\left[v(x) \, u_x \right]_0^1 - \int_0^1 u_x \frac{\partial v(x)}{\partial x} \, \partial x \right] = \int_0^1 v(x) f(x,t) \partial x$$

$$\int_0^1 v(x) u_t \partial x - v(1) \, u_x(1) + v(0) u_x(0) + \int_0^1 u_x \frac{\partial v(x)}{\partial x} \partial x = \int_0^1 v(x) f(x,t) \partial x$$

because $u(0,t) = u(1,t) = 0$, we have:

$$\int_0^1 v(x) u_t \partial x - 0 + 0 + \int_0^1 u_x \frac{\partial v(x)}{\partial x} \, \partial x = \int_0^1 v(x) f(x,t) \partial x$$

because $v(0) = v(1) = 0$ as well

so: $$\int_0^1 v(x) u_t \, \partial x + \int_0^1 u_x \, v_x(x) \, \partial x = \int_0^1 v(x) f(x,t) \partial x$$