Computations of the dynamics through an electron gun in DYNAC V6.0

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1. Introduction:

One focuses on a numerical step-by-step method based on the 5 points Bode's method $[2]$. One considers steps of size h in the azimuthally direction z and one divides the length h in 4 elements of equivalent lengths. This gives way to the azimuthally positions z_0, z_1, z_2, z_3, z_4 such that:

$$z_1 - z_0 = h/4 \, , \, z_2 - z_0 = h/2 \, , \, z_3 - z_0 = 3h/4, \, z_4 - z_0 = h \qquad (1)$$

One assumes that in the region of interest the electric field is not affected by the distance from the axis of the gun (paraxial approximation). In addition, the electric field $E(z)$ does not depend on time, i.e. $\frac{\partial E}{\partial t} = 0$.

2. Computation of the dynamics in the longitudinal direction.

When crossing the step of size h, the energy $W = m_0 c^2 \gamma$ is changed by the amount $\Delta\gamma$:

$$\Delta\gamma = \frac{q}{m_0 c^2} \int_{z_1}^{z_4} E_z \, dz \qquad (2)$$

Rewriting eq.2 with the 5 points Bode's rule, one obtains:

$$\Delta\gamma = \frac{q}{m_0 c^2} \frac{h}{90} \left[7E(z_0) + 32E(z_1) + 12E(z_2) + 32E(z_3) + 7E(z_4) \right] \qquad (3)$$

Let γ_0 be the relativistic γ at the entrance of the step of length h. At the end of this step one will have:

$$\gamma_4 = \gamma_0 + \Delta\gamma \qquad (4)$$

The time t is changed by the amount Δt such that $[1]$:

$$\Delta t = \frac{q}{m_0 c^3} \int_{z_0}^{z_4} \frac{E(z)}{\beta^3 \gamma^3} (z - z_0) dz \qquad (5)$$

Provided that the step size h has been taken sufficiently small, at the end of the step one can write:

$$t_4 = t_0 + \Delta t + \frac{h}{\beta_0 c} \qquad (6)$$

The quantities β_0 and t_0 represent the velocity and the time at the entrance of the step.

Rewriting eq.5 with the 5 points Bode's rule, one obtains:

$$\Delta t = \frac{q}{m_0 c^3} \frac{h^2}{90} \left[\frac{8}{\beta_1^3 \gamma_1^3} E(z_1) + \frac{6}{\beta_2^3 \gamma_2^3} E(z_2) + \frac{24}{\beta_3^3 \gamma_3^3} E(z_3) + \frac{7}{\beta_4^3 \gamma_4^3} E(z_4) \right] \qquad (7)$$

Eq.7 requires knowing the values γ_1, γ_2 and γ_3 at positions z_1, z_2, z_3, respectively. For this, the step size h being assumed small all over, one considers the series expansion of γ_4 with respect to γ_0:

$$\gamma_4 = \gamma_0 + h \left(\frac{\partial \gamma}{\partial z} \right)_{z_0} + \frac{h^2}{2} \left(\frac{\partial^2 \gamma}{\partial z^2} \right)_{z_0} \qquad (8)$$

With:

$$\left(\frac{\partial \gamma}{\partial z}\right)_{z_0} = \frac{q}{m_0 c^2} E(z_0) \tag{9}$$

Since $\gamma_4 = \gamma_0 + \Delta\gamma$, from eq.8 one obtains:

$$\left(\frac{\partial^2 \gamma}{\partial z^2}\right)_{z_0} = \frac{2\Delta\gamma}{h^2} - 2\left(\frac{\partial \gamma}{\partial z}\right)_{z_0} \frac{1}{h} \tag{10}$$

Hence, eq.9 and eq.10 allow computing γ_1, γ_2 and γ_3 in the form:

$$\gamma_1 = \gamma_0 + \frac{h}{4}\left(\frac{\partial \gamma}{\partial z}\right)_{z_0} + \frac{h^2}{32}\left(\frac{\partial^2 \gamma}{\partial z^2}\right)_{z_0} \tag{11}$$

$$\gamma_2 = \gamma_0 + \frac{h}{2}\left(\frac{\partial \gamma}{\partial z}\right)_{z_0} + \frac{h^2}{8}\left(\frac{\partial^2 \gamma}{\partial z^2}\right)_{z_0} \tag{12}$$

$$\gamma_3 = \gamma_0 + \frac{3h}{4}\left(\frac{\partial \gamma}{\partial z}\right)_{z_0} + \frac{9h^2}{32}\left(\frac{\partial^2 \gamma}{\partial z^2}\right)_{z_0} \tag{13}$$

3. Computation of the dynamics in the transverse directions.

The transverse motion can be derived from an integration of the equation of the type:

$$\frac{d(mv_r)}{dt} = q(E_r - v_z B_\theta)$$

After integration over the step size h, the transverse momentum is changed by the amount $\Delta(mv_r)$ and the variation in slope r' becomes:

$$\Delta r' = \frac{\Delta(mv_r)}{mv_z} - \frac{mv_r}{mv_z}\Delta(mv_z) + ...$$

The extra-terms are due to the fact that r and r' are not canonically conjugate, the conjugate of r is mv_r. As a consequence, computations are complicated and developing second order corrections to improve the accuracy of the transverse motion is hardly possible. This problem can be resolved by using the following Picht transformations:

$$R = r\sqrt{\beta\gamma} \ , \ R' = dR/dz \tag{14}$$

The quantities R, R' are referred to as the 'reduced coordinates'. The advantage of Picht transformations result from the fact that R, R' are canonically conjugate.

In terms of Cartesian coordinates $r \equiv (x, y)$ and $r' \equiv (x', y')$, the 'reduced coordinates' $R \equiv (X, Y)$ and $R' \equiv (X', Y')$ are given by:

$$R = r(\gamma^2 - 1)^{0.25} \ , \ R' = r'(\gamma^2 - 1)^{0.25} + 0.5R\gamma(\gamma^2 - 1)^{-1} \tag{15}$$

Since one has $\frac{\partial E}{\partial t} = 0$, the transverse motion is controlled by the following relation (see eq.10, ref.[1] p. 218):

$$\frac{d^2 R}{dz^2} + R(z)\, G(\gamma)\, E^2(z) = 0 \tag{16}$$

With:

$$G(\gamma) = \left(\frac{q}{2m_0 c^2}\right)^2 \frac{\gamma^2 + 2}{(\gamma^2 - 1)^2} \tag{17}$$

The coordinate $R(z)$ can be expanded on the general form as:

$$R(z) = a + b(z - z_0) + c(z - z_0)^2 + ... \quad \text{, with} \quad z_0 \le z \le z_4$$

The step size h being taken sufficiently small, this development may be limited as:

$$R(z) = R_0 + R_0^{'}(z - z_0) \tag{18}$$

in which R_0 , $R_0^{'}$ represent the 'reduced coordinates' at the entrance of the step.

From eq.16 and eq.18, the slope $R^{'}$ is changed by the amount $\Delta R^{'}$ such that:

$$\Delta R^{'} = \int_{z_0}^{z_4} \frac{d^2R}{du^2} du = -\int_{z_0}^{z_4} G(\gamma)E^2(z)\left(R_0 + R_0^{'}(z - z_0)\right)dz \tag{19}$$

One separates the two integrals in eq.19. For the first one, one obtains with the 5 points Bode's rule:

$$\int_{z_0}^{z_4} G(\gamma)E^2(z)\,dz = \tag{20}$$

$$\frac{h}{90}\left[7G(\gamma_0)E^2(z_0) + 32G(\gamma_1)E^2(z_1) + 12G(\gamma_2)E^2(z_2) + 32G(\gamma_3)E^2(z_3) + 7G(\gamma_4)E^2(z_4)\right]$$

For the second one, one will have:

$$\int_{z_0}^{z_4} G(\gamma)E^2(z)\,(z - z_0)dz = \tag{21}$$

$$= \frac{h^2}{90}\left[8G(\gamma_1)E^2(z_1) + 6G(\gamma_2)E^2(z_2) + 24G(\gamma_3)E^2(z_3) + 7G(\gamma_4)E^2(z_4)\right]$$

At the exit of the step h the slope $R_4^{'}$ is given by:

$$R_4^{'} = R_0^{'} + \Delta R^{'} \tag{22}$$

Now let ΔR be the jump of the reduced coordinate R over the step h ; at the end of this step one has:

$$R_4 = R_0 + \Delta R + hR_0^{'} \tag{23}$$

with:

$$\Delta R = \int_{z_0}^{z_4} \Delta R^{'}(z)dz \tag{24}$$

From eq.19, one can rewrite eq.24 as:

$$\Delta R = -R_1\int_{z_0}^{z_4} dz\int_{z_0}^{z} G(\gamma)E^2(u)du - R_1^{'}\int_{z_0}^{z_4} dz\int_{z_0}^{z} G(\gamma)E^2(u)(u - z_0)du \tag{25}$$

Consider in this relation the first integral:

$$I = \int_{z_0}^{z_4} dz\int_{z_0}^{z} G(\gamma)E^2(u)du \tag{26}$$

The variable z varies between z_0 and z_4 and the variable u varies between z_0 and z . One changes the order of integration between u and z (in this case, the variable u varies between z_0 and z_4 and the variable z varies between u and z_4) . It follows that one can rewrite eq.26 on the form:

$$I = \int_{z_0}^{z_4} G(\gamma)E^2(u)du\int_{u}^{z_4} dz = \int_{z_0}^{z_4} G(\gamma)E^2(u)(z_4 - u)du \tag{27}$$

The 5 points Bode's rule gives way to:

$$I = \frac{h^2}{90}\left[7G(\gamma_0)E^2(z_0) + 24G(\gamma_1)E^2(z_1) + 6G(\gamma_2)E^2(z_2) + 8G(\gamma_3)E^2(z_3)\right] \tag{28}$$

Consider the second integral in the right hand side of eq.25:

$$J = \int_{z_0}^{z_4} dz \int_{z_0}^{z} G(\gamma)E^2(u)(u - z_0)du \tag{29}$$

Changing the order of integration between u and z, it can be rewritten as:

$$J = \int_{z_0}^{z_4} G(\gamma)E^2(u)(u - z_0)du \int_{z_0}^{u} dz = \int_{z_0}^{z_4} G(\gamma)E^2(u)(u - z_0)^2 du \tag{30}$$

The 5 points Bode's rule allows writing:

$$\int_{z_0}^{z_4} G(\gamma)E_z^2(z - z_0)^2 dz = \frac{h^3}{90}\left[2G(\gamma_1)E^2(z_1) + 3G(\gamma_2)E^2(z_2) + 18G(\gamma_3)E^2(z_3) + 7G(\gamma_4)E^2(z_4)\right] \tag{31}$$

Eq.28 and eq.31 permit computing ΔR and, thus, R_4 from eq.23. Given R_4 , $R_4^{'}$, it is a simple matter to return to the Cartesian coordinates r_4 , $r_4^{'}$ from eq.15.

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