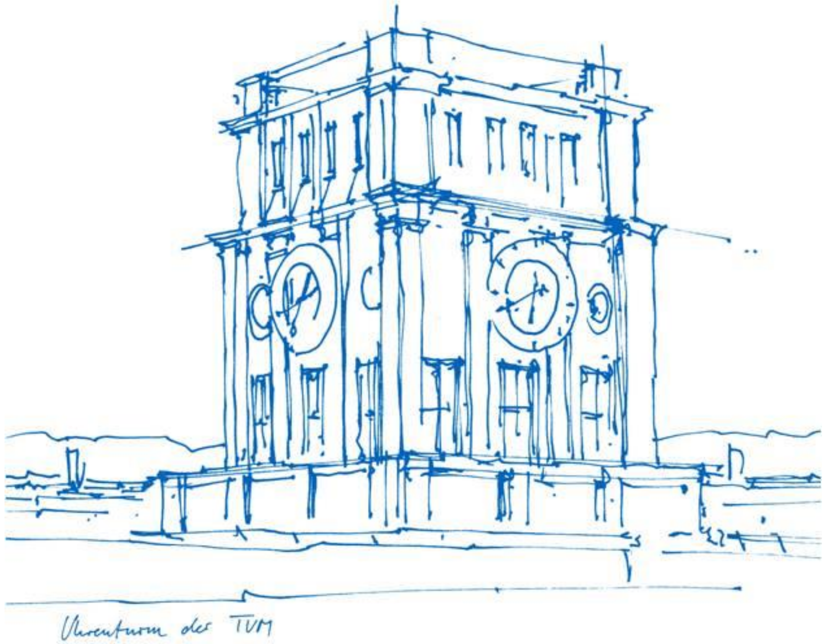
Quantum Algorithm for Solving LSE

Rahul Manavalan

Student of Computational Science and Engineering

Technische Universität München



Desiderata

- Problem Definition
- Classical Solutions
- Problem Re-Definition
- Quantum Solution
  - Insight
  - Algorithm
  - Subtleties
  - Exceptions to the Rule
- Classical Simulation
  - Old Faithful
  - Results

Problem Definition

$$A\,x = b$$

$$A \in R^{N\times N} \qquad x \in R^{N} \qquad b \in R^{N}$$

- Most ubiquitous equation
- Does it make sense to talk about this ? – Yes
  - Big Data in Science – LHC
  - Climate Modelling
  - Deep Learning

Classical Solutions

- Direct Solvers
  - Dense Solvers and var.
  - Sparse Solvers and var.
- Iterative Solvers
  - Stationary Point Solvers
  - Multigrid

Krylov Methods → Steepest Gradient Desc. , CG

Quantum Solution

- HHL Algorithm – 2009
  - Harrow Hassidim Lloyd.

$$\tilde{O}\left(\log(N)s^2\kappa^2/\epsilon\right)$$

- Exponential Speedups
  - Quantum Simulation – RPF
  - Shor‘s Algorithm
  - Grover‘s Search
  - HHL

HHL : Insight

Any Matrix A and its inverse has reciprocating eigenvalues.

$$b := \Sigma b_i u_i$$

$$Ab := \lambda \Sigma b_i u_i$$

$$A^{-1}b := \frac{1}{\lambda} \Sigma b_i u_i$$

If there exists a quantum algorithm to compute λ and in turn 1/λ, we could compute x.

Problem Re-definition

1) How to determine eigenvalues of an arbitrary matrix A ?

2) Given a register of bitwise representations of a decimal number, how an we compute its reciprocal ?

3) Given a bitwise representation of a given normalized angle, how can we compute the sine of the angle ?

1) Eigenvalues of an arbitrary matrix A

Phase Estimation

Assuming that |v〉 is an eigenvector of U and φ the corresponding phase angle.

$$U|v\rangle = exp^{2i\pi\phi}|v\rangle$$

In the real world, Unitary Matrices (Orthogonal Matrices) are hard to come by.

- A is rarely unitary. => ?

Enter : Quantum Simulation … aka Hamiltonian Simulation.

1) Eigenvalues of an arbitrary matrix A

Hamiltonian Simulation

John Preskill

The moral we draw is that "information is physical."

Idea :

1) Convert your matrix A to a valid Hamiltonian H.
2) Allow a closed / open quantum system to evolve regulated by H.

Realization :

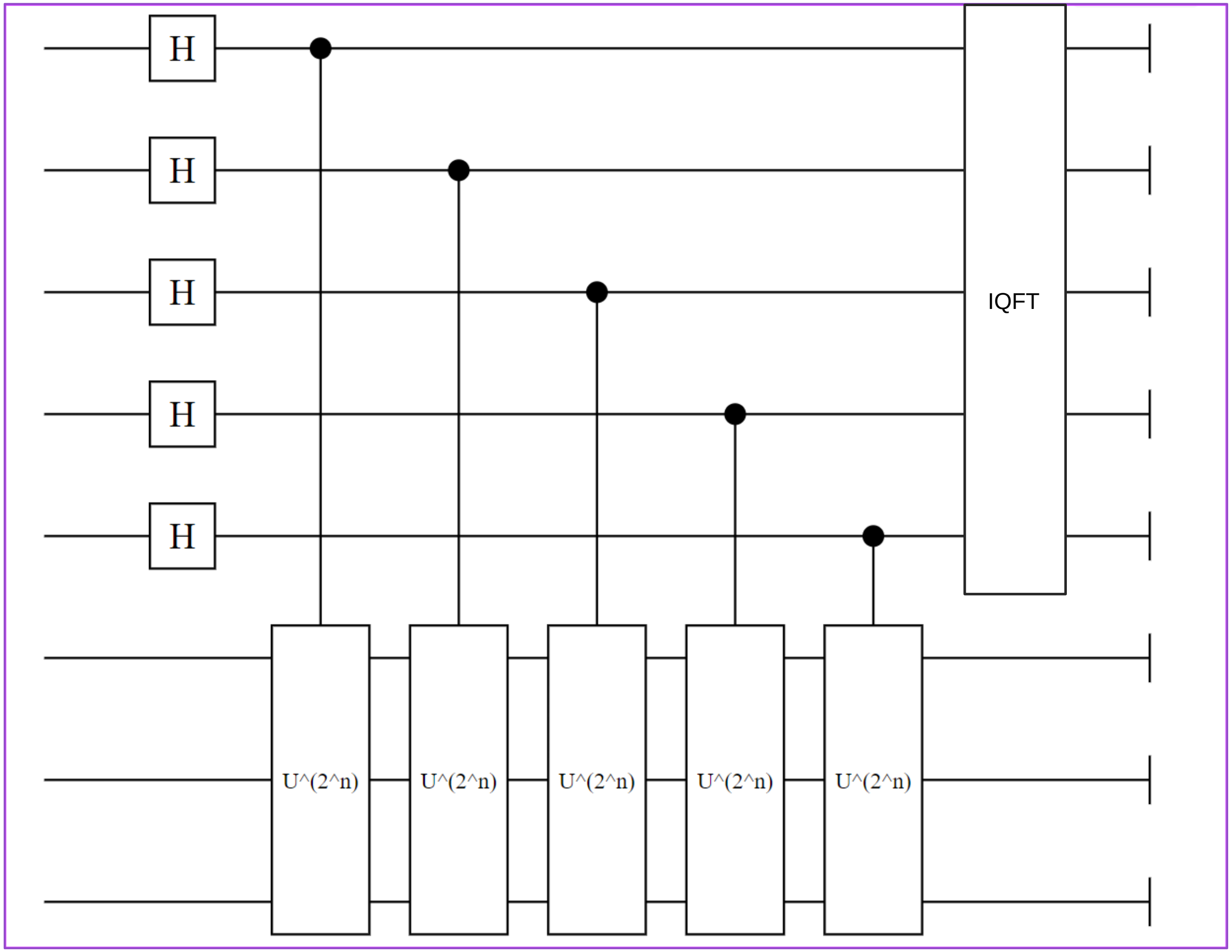
$$\tilde{O}(\log(N)s^2t)$$

1) HARD ! : Sparse and Dense Hamiltonians -> Quantum Simulations.
2) Smaller ε and Faster τ are focal points.

Lie Trotter decomp / recomb ; Galaxy decomp ; Forrest decomp …..

1) Eigenvalues of an arbitrary matrix A

Phase Estimation : Successive Controlled Hamiltonian Simulations



Phase Estimation – Crash Course

$$|0\rangle^{\otimes t} \otimes |u\rangle$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}^{\otimes t} \otimes |u\rangle$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}^{\otimes t} \otimes U|u\rangle$$

$$\frac{|0\rangle + exp^{2i\pi\phi}|1\rangle}{\sqrt{2}} \otimes U|u\rangle$$

$$\bigotimes_{j=0}^{t-1}\left[\frac{|0\rangle + exp^{2i\pi\phi * 2^j}|1\rangle}{\sqrt{2}} \otimes U^{2^j}|u\rangle\right]$$

$$\Sigma_{k=0}^{2^t-1} exp^{2ij\pi\phi}|k\rangle$$

Eigenvalues of U take the form $exp(2i\pi\phi)$.

$U = exp(2\pi iA)$.

Eigenvalues of U are now $exp(2i\pi\lambda)$ where λ is the eigenvalue of A
At the end of PE, the register contains ϕ for the normal case.
In the QS case PE produces λ in the register.

3) Sine of a bitstring – Controlled Rotation

$$|\theta\rangle|0\rangle - > |\theta\rangle(cos\theta|0\rangle + sin\theta|1\rangle)$$

$$(|\theta\rangle\theta| \otimes R_y)(|\theta\rangle \otimes |0\rangle) = |\theta\rangle \otimes R_y|0\rangle$$

$$R_y = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$R_y|0\rangle = cos\theta|0\rangle + sin\theta|1\rangle$$

Let $\theta = \theta.\theta_1\theta_2\theta_3 \ldots\ldots.\theta_f$ be decimal representation

To transform $|\theta\rangle \rightarrow \sin|\theta|$, we use the following idea.

If

$$\theta = \sum_{i=1}^{f} \theta_i * 2^{-i}$$

Then

$$e^{\theta} = \prod_{i=1}^{f} e^{\theta_i * 2^{-i}}$$

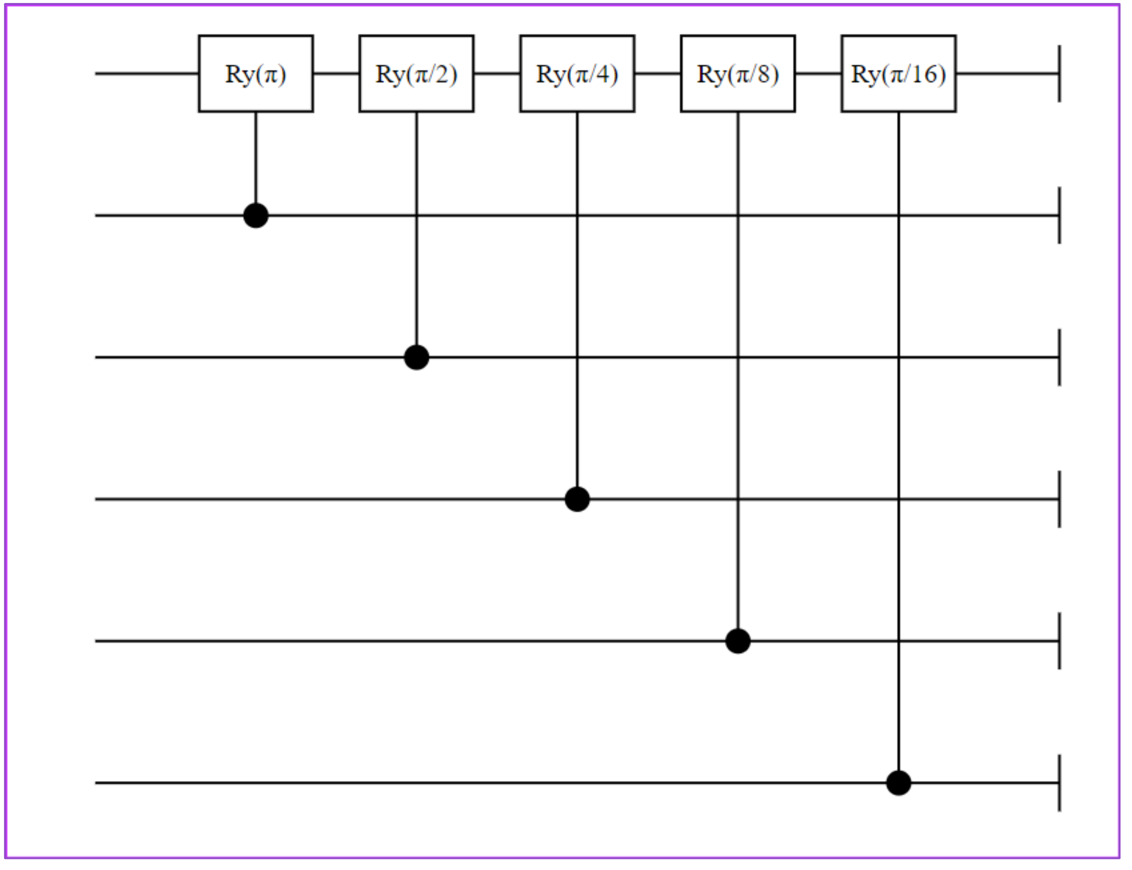
WKT $|sin\theta| < 1$.

We normalize theta using a C such that

$sin(C*\theta) \approx C*\theta$

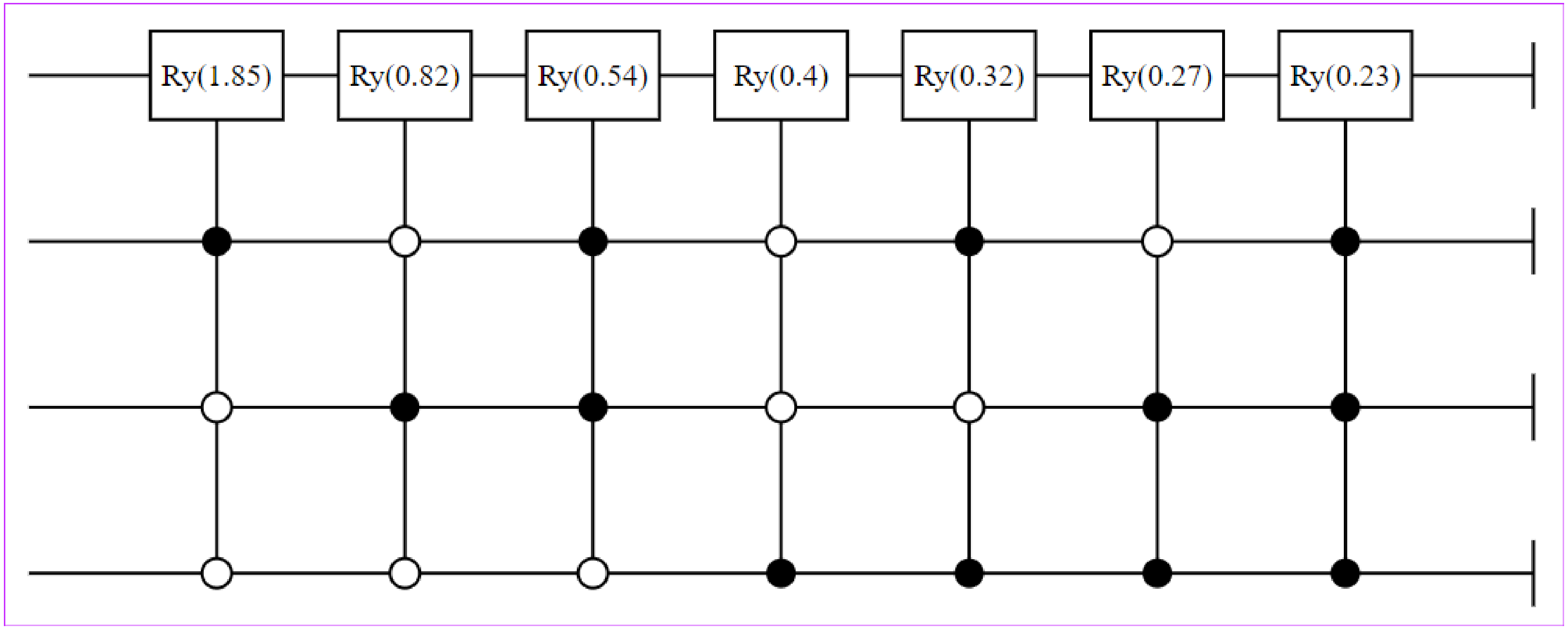
Sine of a Bitstring

Controlled Rotation : Eigenvalues a priori.

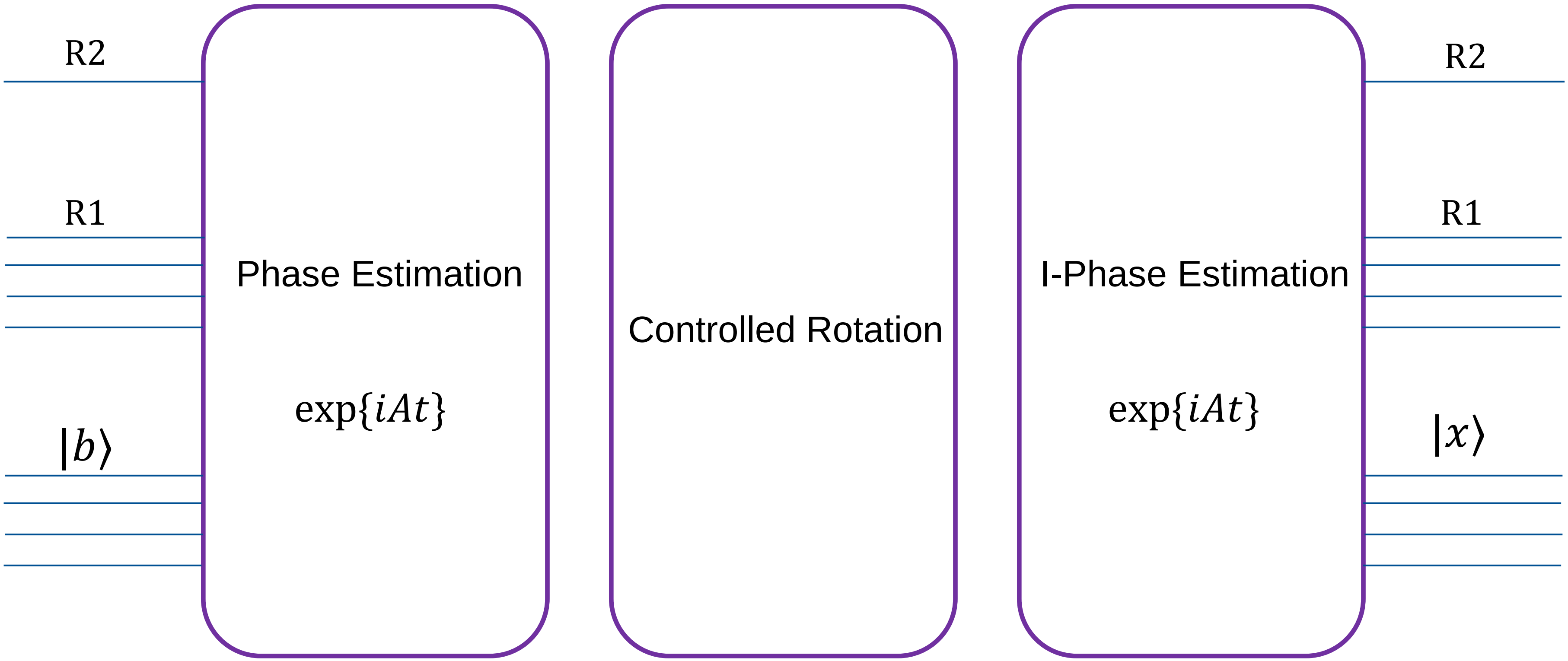


Sine of a Bitstring

Controlled Rotation : More robust



Algorithm – Components



A – Hermitian , sparse , well conditioned.

Brass Tacks – (i)

Encode $|b\rangle$ using Amplitude encoding.

- Amplitude embedding : *roof(log(2,length(b))*

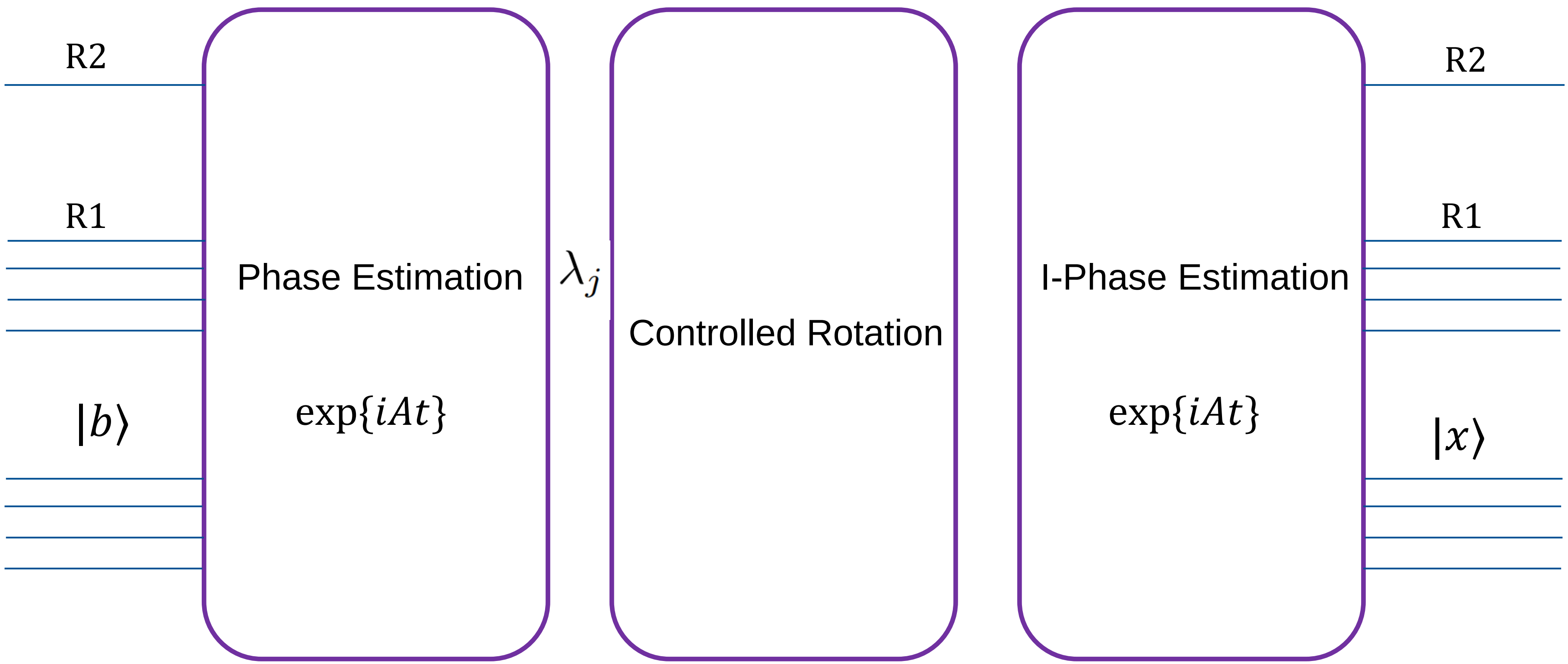
$$\frac{|b\rangle}{|||b\rangle||}$$

- Caveat : Solution is also amplitude embedded

`

$$\frac{|x\rangle}{|||x\rangle||}$$

Algorithm – Components



A – Hermitian , sparse , well conditioned.

Brass Tacks – (v)

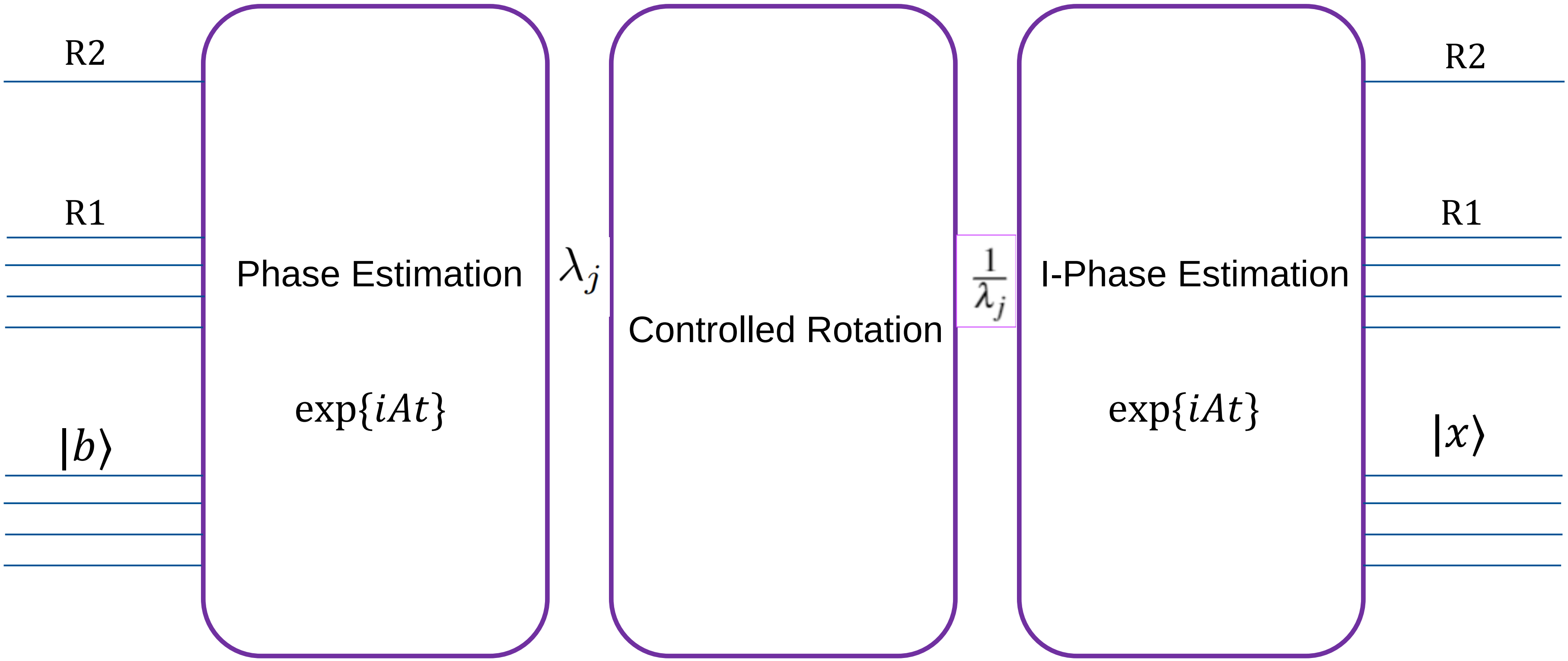
Controlled Rotation

- $|\lambda\rangle$ is housed in Reg. A in a bitwise manner.
- Add a ancilla qubit R2.
- Prior to controlled rotation we reciprocate. i.e , transform $|\lambda\rangle \rightarrow \left|\frac{1}{\lambda}\right\rangle$
- If $|\lambda\rangle$ is not normalized, choose C such that $C < \min \lambda_j$.
- State of R2 after CRot :

$$\sqrt{1 - \frac{C^2}{\lambda_j^2}}\,|0\rangle + \frac{C}{\lambda_j}\,|1\rangle$$

Algorithm – Components

$$\sqrt{1-\frac{C^2}{\lambda_j^2}}\,|0\rangle + \frac{C}{\lambda_j}\,|1\rangle$$



A – Hermitian , sparse , well conditioned.

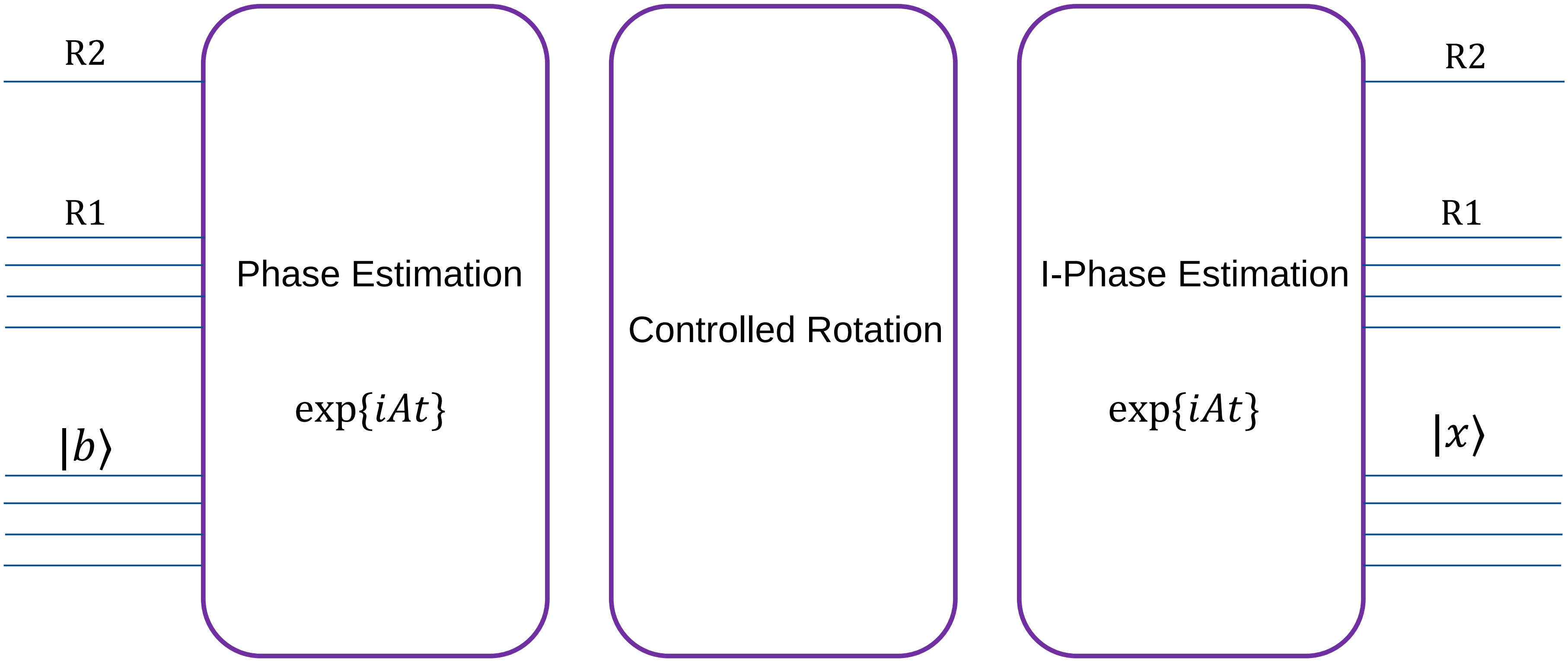
Brass Tacks – (vi)

Un-compute PE

- Throw away garbage in R1
- Use Inverse Phase estimation
- Runtime Significance
- Error Significance

Algorithm – Components

$$\sqrt{1-\frac{C^2}{\lambda_j^2}}\,|0\rangle + \frac{C}{\lambda_j}\,|1\rangle$$



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If there exists a quantum algorithm to compute λ and in turn 1/λ, we could compute x.

Brass Tacks – (vii)

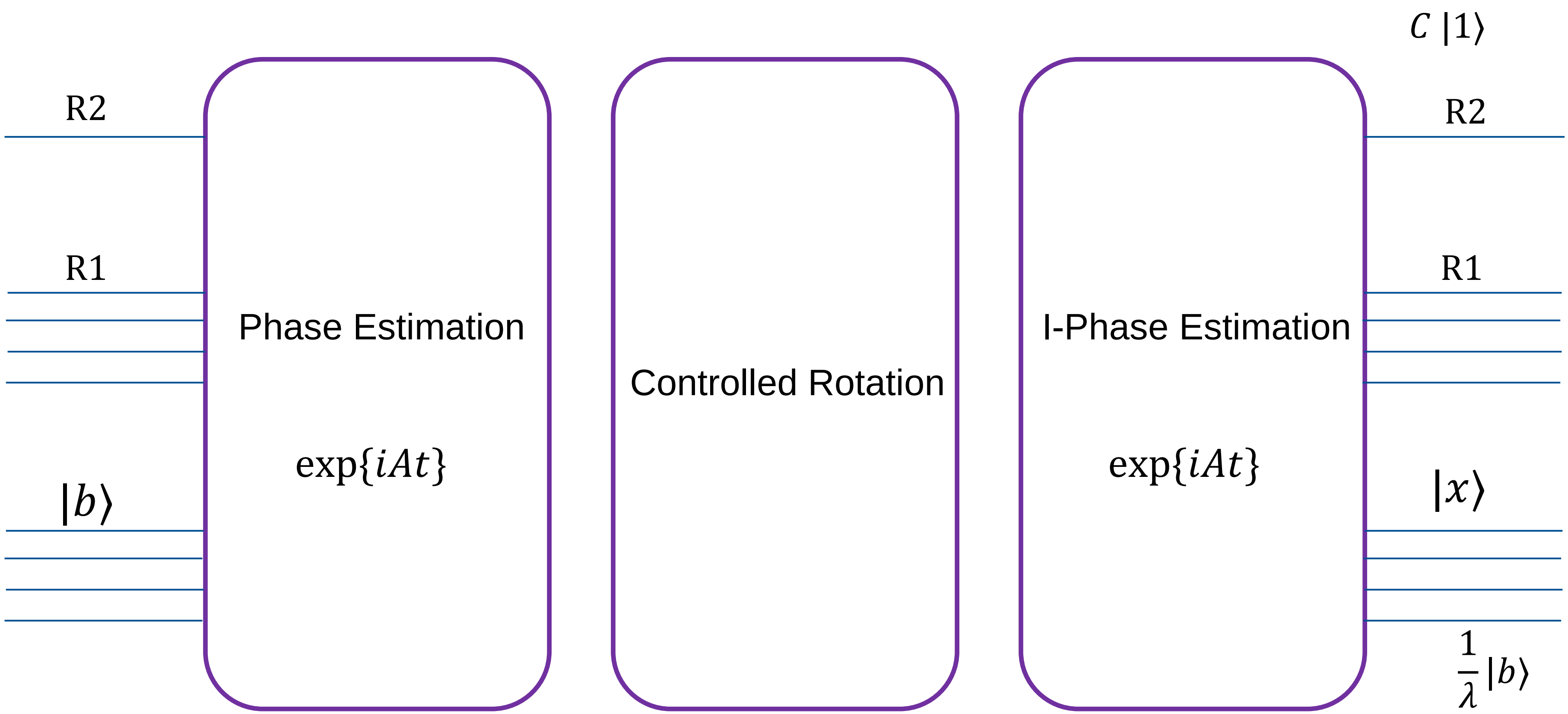
Measure ancilla qubit

- Most boring yet crucial part of the algorithm
- Measure ancilla
- If we get 1 => success, solution at the base register
- Else Back to square one

What could we measure with respect to?

- Projectors
- Compute Expectation

Algorithm – Components



A – Hermitian , sparse , well conditioned.

Caveats

- norm in b -> $|b\rangle$
- t in Quantum Simulation
- n_A - number of qubits in Register A.
- C in Controlled Rotation

Difficulties in Actual Q.Computation

- Preparation of $|b\rangle$ efficiently
- Availability of qubits for a practical problem.
  - $If\ \lambda \approx \epsilon\ , we\ require\ as\ many\ as\ 50\ qubits\ for\ Register\ A\ alone\ for\ A\ \epsilon\ R^{32x32}$.
- The final measurement is a hit and miss. Need to perform several experiments.
- Not realizable ! - However (see ref.)

1) Eigenvalues of an arbitrary matrix A

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Exceptions to the Rule

- Non Hermitian Matrices
- Non Square Matrices
- Dense Matrices
- Non Invertible Matrices

Classical Simulation Frameworks

- Yao.jl
- Quaintum.jl
- Qiskit Terra

Old Faithful – Poisson Equation

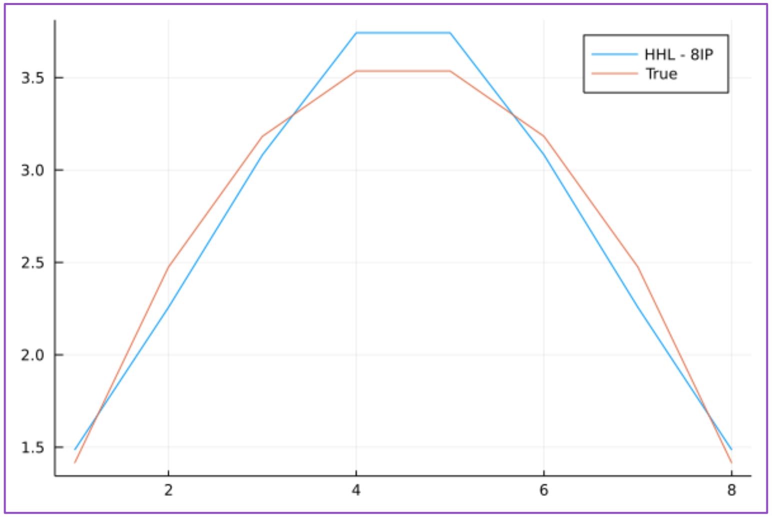
- $\Delta\phi = f \quad ; \mathrm{x} \in [a, b]$
- $\delta\Omega : \phi(a) = 0 \ ; \phi(b) = 0$
- $f = ones(N)$

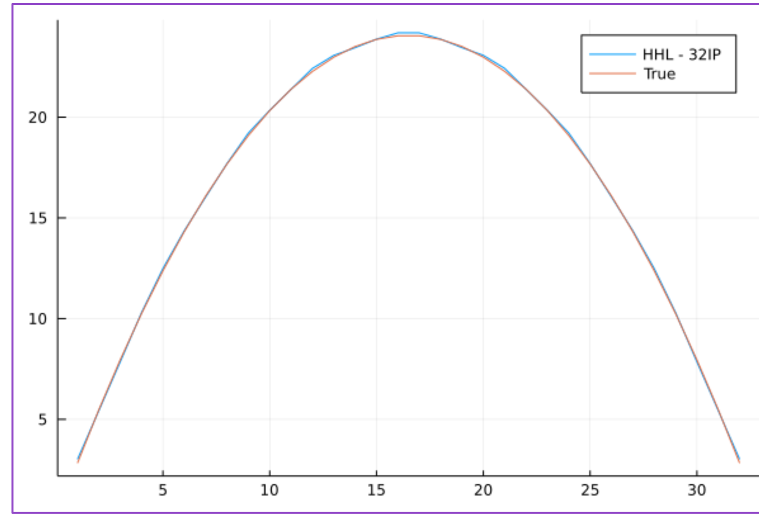
- Finite Difference Method. HHL vs Julia „\“

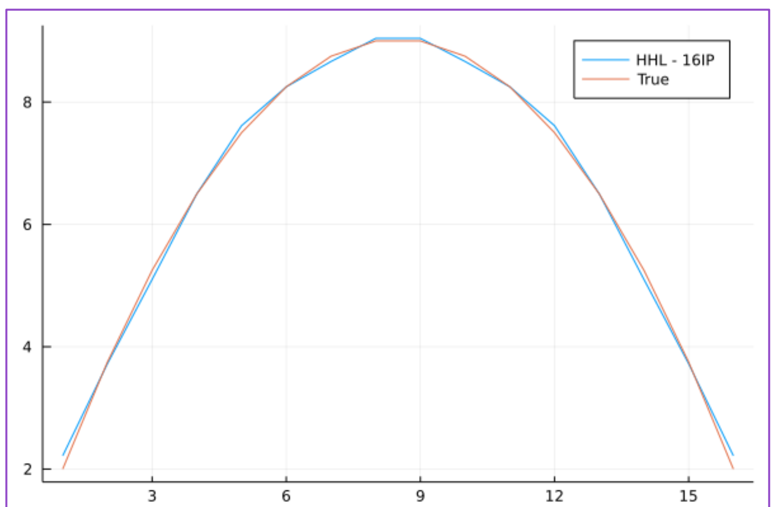
$$\underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}}_{:=A} \times \underbrace{\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \vdots \\ \Phi_{N-1} \\ \Phi_N \end{pmatrix}}_{:=\boldsymbol{\Phi}} = \underbrace{\begin{pmatrix} h^2 f_1 - \Phi_a \\ h^2 f_2 \\ h^2 f_3 \\ h^2 f_4 \\ h^2 f_5 \\ \vdots \\ h^2 f_{N-1} \\ h^2 f_N - \Phi_b \end{pmatrix}}_{:=\boldsymbol{F}}$$

Results $n\_\{\lambda\} = 12$

1D

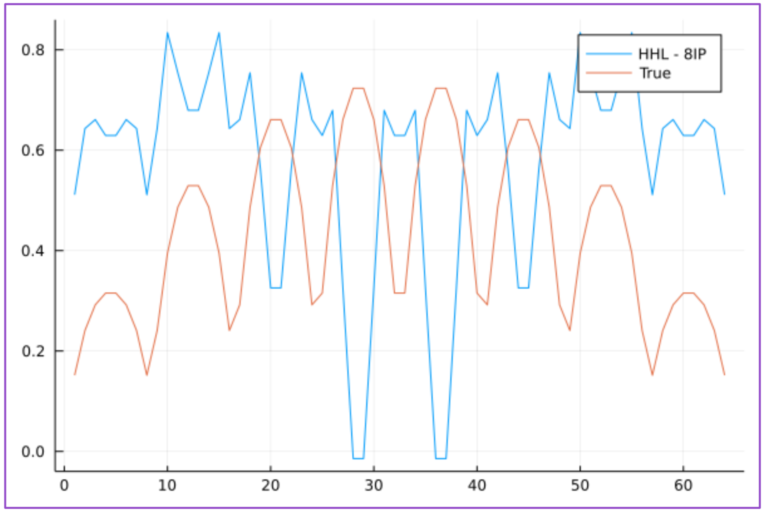


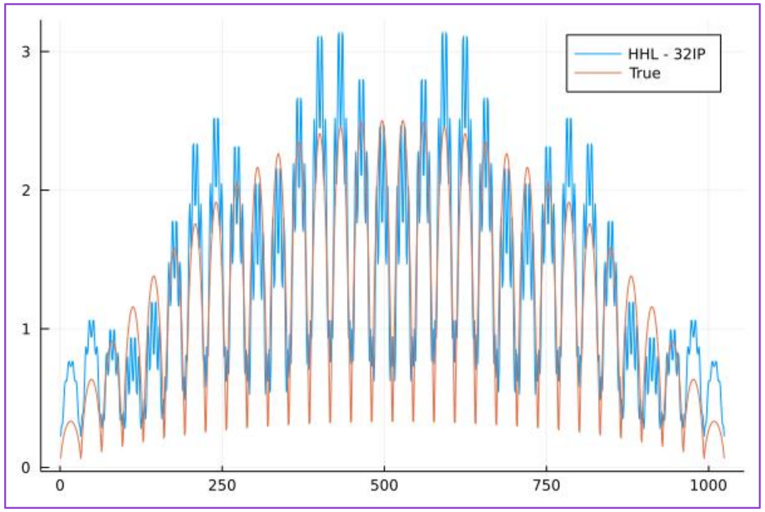


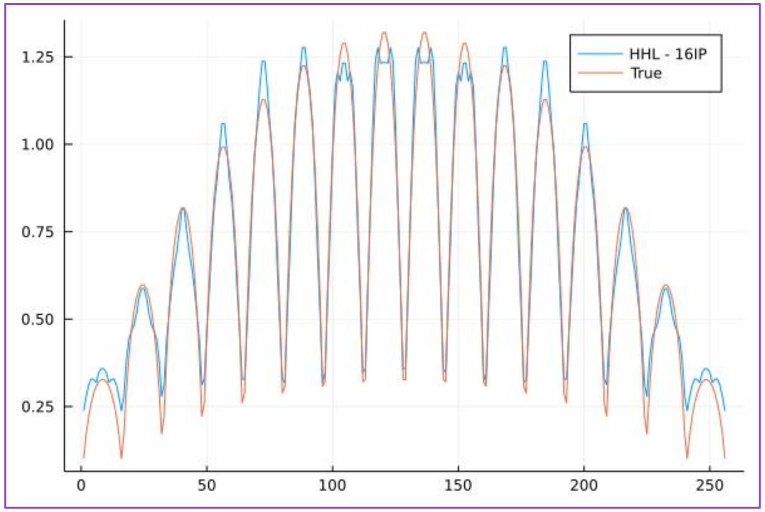


Results $n_{\lambda} = 12$

2D

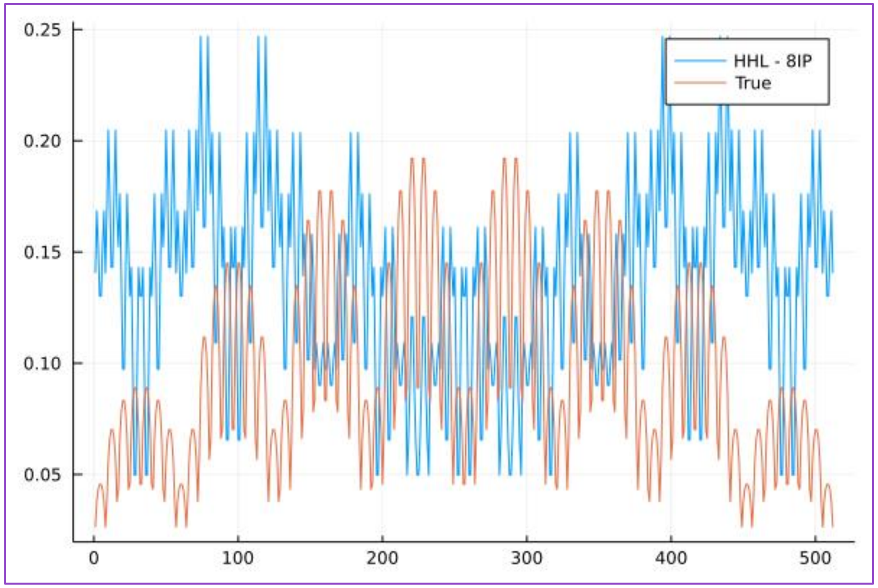






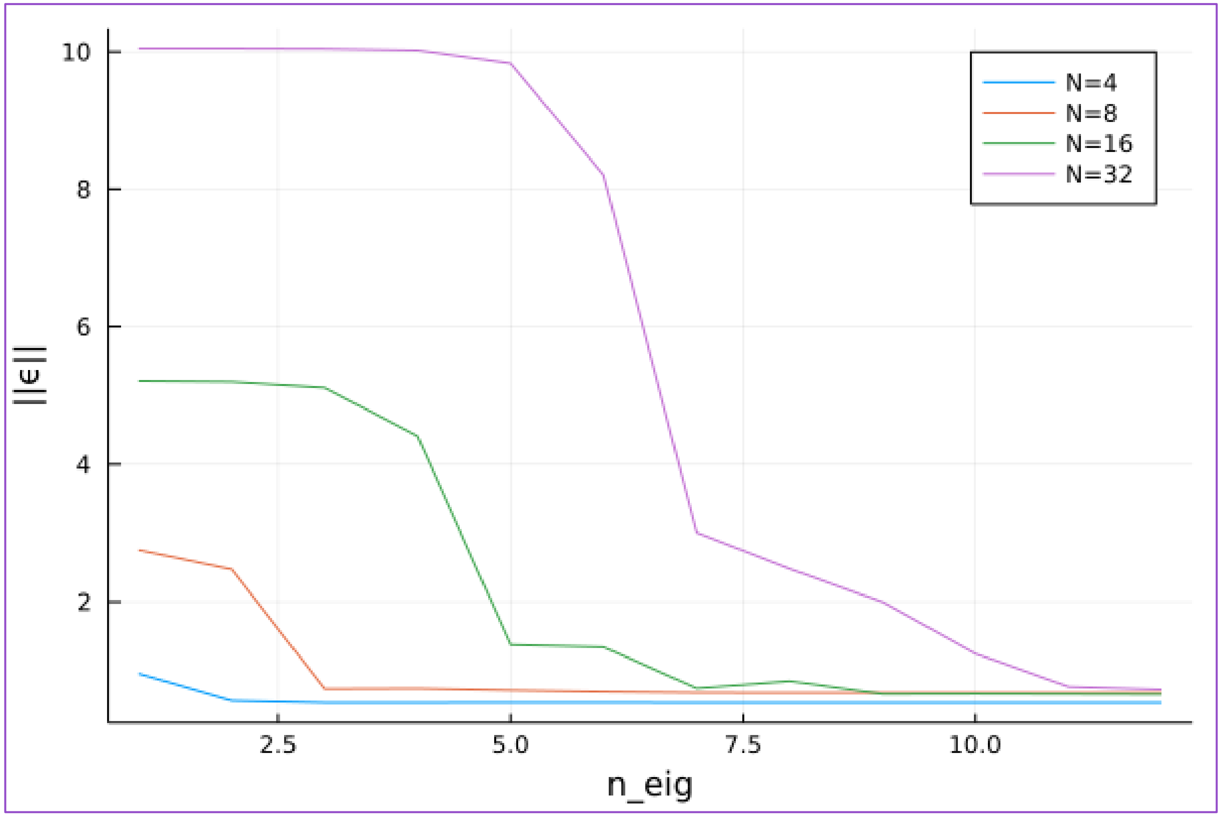
Results $n\_\{\lambda\} = 12$

3D



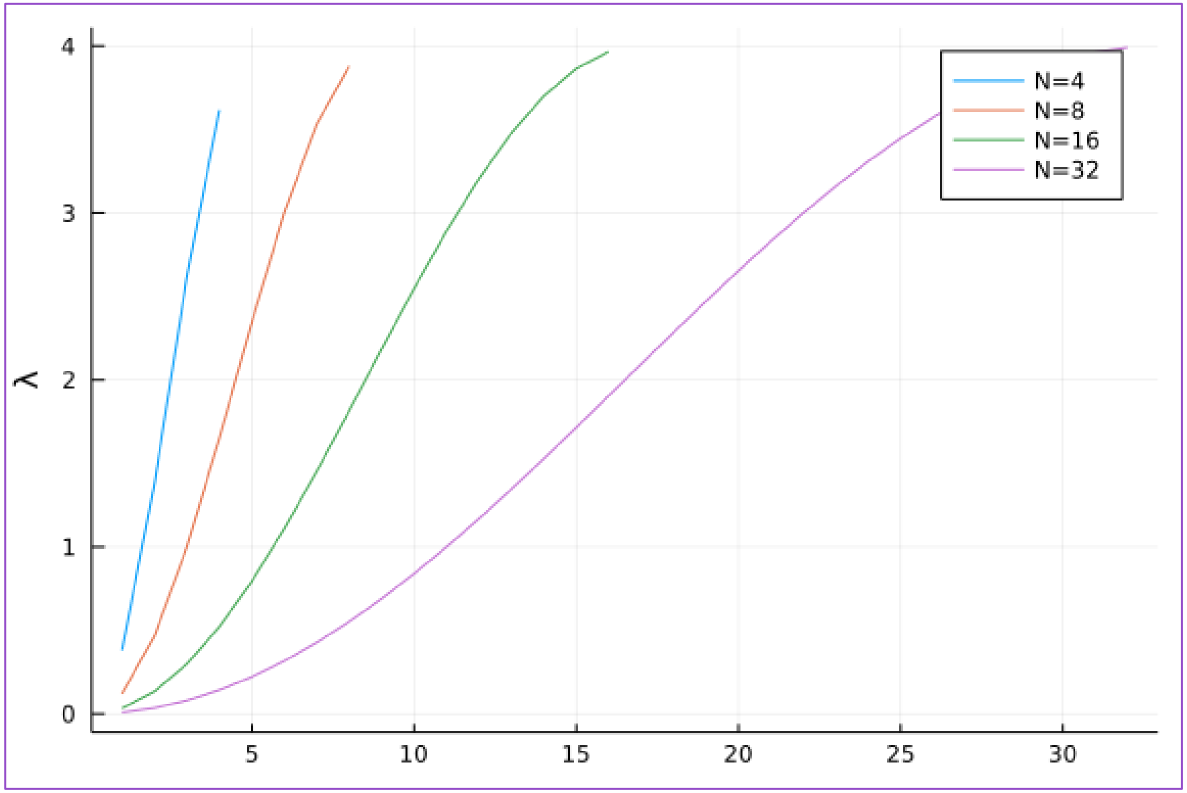
Results

Error Analysis – 1D



Results

Error Analysis – 1D – Inference : Relation to eigenvalues



More on HHL

- Baseline Implementation
- Rahul Manavalan / Solving the Heat Equation with the HHL Algorithm · GitLab (lrz.de)

- Elaborations into the claims made in the HHL paper.
  - The Why of HHL.
  - https://dynamic-queries.medium.com/chapter-1-1-solutions-to-linear-systems-and-all-that-69dc73e02739

References

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- 1301.2340.pdf (arxiv.org) : Box to Box Solution
- 1302.1210.pdf (arxiv.org) : Experimental Implementation albeit for a small system.
- qiskit-terra/exact_reciprocal.py at a9289c085036002fa864b30edcd0b8ce066ea858 · Qiskit/qiskit-terra (github.com) : Exact reciprocal.
- 1802.08227.pdf (arxiv.org) : Primer
- arXiv:0811.3171v3 [quant-ph] 30 Sep 2009 : Base Paper : Hard Read
- 1110.2232.pdf (arxiv.org) : Saved hours , quite comprehensive