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Supplementary Information: Stability of schooling patterns of a fish pair swimming against a flow

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S1. Expressions for fluid flow model

The flow velocity induced by each of the two fish (point dipoles) in the model is given by

$$\begin{aligned}
\boldsymbol{u}_{f\,1} &= r_0^2 v_0 \frac{(x-x_1+y-y_1)(x-x_1-y+y_1)\cos\theta_1 + 2(x-x_1)(y-y_1)\sin\theta_1}{((x-x_1)^2+(y-y_1)^2)^2}\hat{\boldsymbol{i}} \\
&+ r_0^2 v_0 \frac{2(x-x_1)(y-y_1)\cos\theta_1 - (x-x_1+y-y_1)(x-x_1-y+y_1)\sin\theta_1}{((x-x_1)^2+(y-y_1)^2)^2}\hat{\boldsymbol{j}}, \qquad (1)\\
\boldsymbol{u}_{f\,2} &= r_0^2 v_0 \frac{(x-x_2+y-y_2)(x-x_2-y+y_2)\cos\theta_2 + 2(x-x_2)(y-y_2)\sin\theta_2}{((x-x_2)^2+(y-y_2)^2)^2}\hat{\boldsymbol{i}} \\
&+ r_0^2 v_0 \frac{2(x-x_2)(y-y_2)\cos\theta_2 - (x-x_2+y-y_2)(x-x_2-y+y_2)\sin\theta_2}{((x-x_2)^2+(y-y_2)^2)^2}\hat{\boldsymbol{j}} \qquad (2)
\end{aligned}$$

The flow velocity due to the presence of the walls

$$\begin{aligned}
\boldsymbol{u}_w &= r_0^2 v_0 \Bigg[\left(\frac{\cos\theta_1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{4} f_w(x_1,y_1,\theta_1) - \frac{2(x-x_1)((x-x_1)\cos\theta_1 + (y-y_1)\sin\theta_1)}{((x-x_1)^2+(y-y_1)^2)^2}\right) \\
&+ \left(\frac{\cos\theta_2}{(x-x_2)^2+(y-y_2)^2} - \frac{1}{4} f_w(x_2,y_2,\theta_2) - \frac{2(x-x_2)((x-x_2)\cos\theta_2 + (y-y_2)\sin\theta_2)}{((x-x_2)^2+(y-y_2)^2)^2}\right)\Bigg]\hat{\boldsymbol{i}} \\
&+ r_0^2 v_0 \Bigg[\left(\frac{\sin\theta_1}{(x-x_1)^2+(y-y_1)^2} - \frac{1}{4} f_{\tilde{w}}(x_1,y_1,\theta_1) - \frac{2(y-y_1)((x-x_1)\cos\theta_1 + (y-y_1)\sin\theta_1)}{((x-x_1)^2+(y-y_1)^2)^2}\right) \\
&+ \left(\frac{\sin\theta_2}{(x-x_2)^2+(y-y_2)^2} - \frac{1}{4} f_{\tilde{w}}(x_2,y_2,\theta_2) - \frac{2(y-y_2)((x-x_2)\cos\theta_2 + (y-y_2)\sin\theta_2)}{((x-x_2)^2+(y-y_2)^2)^2}\right)\Bigg]\hat{\boldsymbol{j}}, \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
f_w(x_k,y_k,\theta_k) &= -\frac{\pi^2 e^{-\mathrm{i}\theta_k}}{2h^2}\Bigg(e^{2\mathrm{i}\theta_k}\operatorname{csch}^2\left(\frac{\pi(x-x_k+\mathrm{i}(y-y_k))}{2h}\right) + \operatorname{csch}^2\left(\frac{\pi(x-x_k-\mathrm{i}(y-y_k))}{2h}\right) \\
&+ e^{2\mathrm{i}\theta_k}\operatorname{csch}^2\left(\frac{\pi(x-x_k-\mathrm{i}(y+y_k))}{2h}\right) + \operatorname{csch}^2\left(\frac{\pi(x-x_k+\mathrm{i}(y+y_k))}{2h}\right)\Bigg), \qquad (4a)\\
f_{\tilde{w}}(x_k,y_k,\theta_k) &= \frac{\pi^2 \mathrm{i} e^{-\mathrm{i}\theta_k}}{2h^2}\Bigg(-e^{2\mathrm{i}\theta_k}\operatorname{csch}^2\left(\frac{\pi(x-x_k+\mathrm{i}(y-y_k))}{2h}\right) + \operatorname{csch}^2\left(\frac{\pi(x-x_k-\mathrm{i}(y-y_k))}{2h}\right) \\
&+ e^{2\mathrm{i}\theta_k}\operatorname{csch}^2\left(\frac{\pi(x-x_k-\mathrm{i}(y+y_k))}{2h}\right) - \operatorname{csch}^2\left(\frac{\pi(x-x_k+\mathrm{i}(y+y_k))}{2h}\right)\Bigg). \qquad (4b)
\end{aligned}$$

S2. Expressions for fish dynamics model

The advection velocities of the two fish are given by the following expressions:

$$\begin{aligned}
\boldsymbol{U}_1 = & \Bigg[U_0\left(1-\epsilon+\frac{4\epsilon y_1}{h}\left(1-\frac{y_1}{h}\right)\right)-\frac{\pi^2 r_0^2 v_0}{24h^2}\Bigg\{e^{-\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_1}+3e^{-\mathrm{i}\theta_1}\csc^2\left(\frac{\pi y_1}{h}\right)+3e^{\mathrm{i}\theta_1}\csc^2\left(\frac{\pi y_1}{h}\right) \\
& - 3e^{\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2+\mathrm{i}(y_1-y_2))}{2h}\right)-3e^{-\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2-\mathrm{i}(y_1-y_2))}{2h}\right) \\
& - 3e^{\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2-\mathrm{i}(y_1+y_2))}{2h}\right)-3e^{-\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2+\mathrm{i}(y_1+y_2))}{2h}\right)\Bigg\}\Bigg]\hat{i} \\
& + \Bigg[\frac{\pi^2 r_0^2 v_0}{24h^2}e^{-\mathrm{i}(\theta_1+\theta_2)}\Bigg\{e^{\mathrm{i}\theta_2}-e^{\mathrm{i}(2\theta_1+\theta_2)}-3e^{\mathrm{i}\theta_2}\csc^2\left(\frac{\pi y_1}{h}\right)+3e^{\mathrm{i}(2\theta_1+\theta_2)}\csc^2\left(\frac{\pi y_1}{h}\right) \\
& + 3e^{\mathrm{i}(\theta_1+2\theta_2)}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2+\mathrm{i}(y_1-y_2))}{2h}\right)-3e^{\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2-\mathrm{i}(y_1-y_2))}{2h}\right) \\
& - 3e^{\mathrm{i}(\theta_1+2\theta_2)}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2-\mathrm{i}(y_1+y_2))}{2h}\right)+3e^{\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_1-x_2+\mathrm{i}(y_1+y_2))}{2h}\right)\Bigg\}\Bigg]\hat{j},
\end{aligned} \tag{5}$$

$$\begin{aligned}
\boldsymbol{U}_2 = & \Bigg[U_0\left(1-\epsilon+\frac{4\epsilon y_2}{h}\left(1-\frac{y_2}{h}\right)\right)-\frac{\pi^2 r_0^2 v_0}{24h^2}\Bigg\{e^{-\mathrm{i}\theta_2}+e^{\mathrm{i}\theta_2}+3e^{-\mathrm{i}\theta_2}\csc^2\left(\frac{\pi y_2}{h}\right)+3e^{\mathrm{i}\theta_2}\csc^2\left(\frac{\pi y_2}{h}\right) \\
& - 3e^{\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2-y_1))}{2h}\right)-3e^{-\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2-y_1))}{2h}\right) \\
& - 3e^{\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_1+y_2))}{2h}\right)-3e^{-\mathrm{i}\theta_1}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_1+y_2))}{2h}\right)\Bigg\}\Bigg]\hat{i} \\
& + \Bigg[\frac{\pi^2 r_0^2 v_0}{24h^2}e^{-\mathrm{i}(\theta_1+\theta_2)}\Bigg\{e^{\mathrm{i}\theta_1}-e^{\mathrm{i}(\theta_1+2\theta_2)}-3e^{\mathrm{i}\theta_1}\csc^2\left(\frac{\pi y_2}{h}\right)+3e^{\mathrm{i}(\theta_1+2\theta_2)}\csc^2\left(\frac{\pi y_2}{h}\right) \\
& + 3e^{\mathrm{i}(2\theta_1+\theta_2)}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2-y_1))}{2h}\right)-3e^{\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2-y_1))}{2h}\right) \\
& - 3e^{\mathrm{i}(2\theta_1+\theta_2)}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_1+y_2))}{2h}\right)+3e^{\mathrm{i}\theta_2}\operatorname{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_1+y_2))}{2h}\right)\Bigg\}\Bigg]\hat{j}.
\end{aligned} \tag{6}$$

The flow-induced angular velocity of each of the two fish are given by:

$$
\begin{aligned}
\Omega_1 = {} & \frac{1}{16h^3} e^{-\mathrm{i}(2\theta_1+\theta_2)} \Bigg[32U_0 y_1 h \epsilon e^{\mathrm{i}\theta_2} \left(1 + e^{2\mathrm{i}\theta_1}\right)^2 - 64 U_0 h^2 \epsilon e^{\mathrm{i}(2\theta_1+\theta_2)} \cos^2(\theta_1) \\
& + \pi^3 {r_0}^2 v_0 \Bigg\{ e^{\mathrm{i}(5\theta_1+\theta_2)} \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) - e^{\mathrm{i}(\theta_1+\theta_2)} \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) \\
& - 2e^{\mathrm{i}(\theta_1+\theta_2)} \cos^2(\theta_1) \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) - 2e^{\mathrm{i}(3\theta_1+\theta_2)} \cos^2(\theta_1) \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) \\
& - 2\mathrm{i} \coth\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 - y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 - y_2))}{2h}\right) \\
& + 2\mathrm{i} e^{2\mathrm{i}(2\theta_1+\theta_2)} \coth\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 - y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 - y_2))}{2h}\right) \\
& - 2\mathrm{i} e^{2\mathrm{i}(\theta_1+\theta_2)} \cos^2(\theta_1) \coth\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \\
& + \mathrm{i}\left(-1 + e^{4\mathrm{i}\theta_1}\right) \coth\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \\
& + 2\mathrm{i} e^{2\mathrm{i}\theta_1} \cos^2(\theta_1) \coth\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \\
& - 4\mathrm{i} e^{\mathrm{i}(\theta_1+\theta_2)} \cos(\theta_1) \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) \sin(\theta_1) \\
& - 4e^{2\mathrm{i}(\theta_1+\theta_2)} \cos(\theta_1) \coth\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \sin(\theta_1) \\
& + 2e^{\mathrm{i}(\theta_1+\theta_2)} \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) \sin^2(\theta_1) + 2e^{\mathrm{i}(3\theta_1+\theta_2)} \cot\left(\frac{\pi y_1}{h}\right) \csc^2\left(\frac{\pi y_1}{h}\right) \sin^2(\theta_1) \\
& + 2\mathrm{i} e^{2\mathrm{i}(\theta_1+\theta_2)} \coth\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 - \mathrm{i}(y_1 + y_2))}{2h}\right) \sin^2(\theta_1) \\
& - 2\mathrm{i} e^{2i\theta_1} \coth\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \operatorname{csch}^2\left(\frac{\pi(x_1 - x_2 + \mathrm{i}(y_1 + y_2))}{2h}\right) \sin^2(\theta_1) \Bigg\} \Bigg],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\Omega_2 = & \frac{1}{16h^3} e^{-\mathrm{i}(2\theta_2+\theta_1)} \Bigg[32U_0 y_2 h\epsilon e^{\mathrm{i}\theta_1} \left(1+e^{2i\theta_2}\right)^2 - 64U_0 h^2 \epsilon e^{\mathrm{i}(2\theta_2+\theta_1)} \cos^2(\theta_2) \\
& + \pi^3 {r_0}^2 v_0 \Bigg\{ e^{\mathrm{i}(5\theta_2+\theta_1)} \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) - e^{\mathrm{i}(\theta_2+\theta_1)} \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) \\
& - 2e^{\mathrm{i}(\theta_2+\theta_1)} \cos^2(\theta_2) \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) - 2e^{\mathrm{i}(3\theta_2+\theta_1)} \cos^2(\theta_2) \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) \\
& - 2\mathrm{i} \coth\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2-y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2-y_1))}{2h}\right) \\
& + 2\mathrm{i}e^{2\mathrm{i}(2\theta_2+\theta_1)} \coth\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2-y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2-y_1))}{2h}\right) \\
& - 2\mathrm{i}e^{2\mathrm{i}(\theta_2+\theta_1)} \cos^2(\theta_2) \coth\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \\
& + \mathrm{i}\left(-1+e^{4\mathrm{i}\theta_2}\right) \coth\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \\
& + 2\mathrm{i}e^{2\mathrm{i}\theta_2} \cos^2(\theta_2) \coth\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \\
& - 4\mathrm{i}e^{\mathrm{i}(\theta_2+\theta_1)} \cos(\theta_2) \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) \sin(\theta_2) \\
& - 4e^{2\mathrm{i}(\theta_2+\theta_1)} \cos(\theta_2) \coth\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \sin(\theta_2) \\
& + 2e^{\mathrm{i}(\theta_2+\theta_1)} \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) \sin^2(\theta_2) + 2e^{\mathrm{i}(3\theta_2+\theta_1)} \cot\left(\frac{\pi y_2}{h}\right) \csc^2\left(\frac{\pi y_2}{h}\right) \sin^2(\theta_2) \\
& + 2\mathrm{i}e^{2\mathrm{i}(\theta_2+\theta_1)} \coth\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1-\mathrm{i}(y_2+y_1))}{2h}\right) \sin^2(\theta_2) \\
& - 2\mathrm{i}e^{2\mathrm{i}\theta_2} \coth\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \mathrm{csch}^2\left(\frac{\pi(x_2-x_1+\mathrm{i}(y_2+y_1))}{2h}\right) \sin^2(\theta_2) \Bigg\} \Bigg].
\end{aligned} \tag{8}
$$

S3. Expressions for dynamical system

The final system of equations of the dynamical system governing the position and orientation of both the fish swimming in an infinite channel is given by Eq. (**??**). The expressions of all the required functions are given below:

$$
\begin{aligned}
f_\xi(\xi_1,\xi_2,\theta_1,\theta_2,\Lambda) = & \frac{\pi^2}{24} \Bigg[-3\mathrm{i}e^{-\mathrm{i}\theta_2} \Big\{ \mathrm{csch}^2(\psi_1) + \mathrm{sech}^2(\psi_2) \Big\} + 3\mathrm{i}e^{\mathrm{i}\theta_2} \Big\{ \mathrm{csch}^2(\psi_3) + \mathrm{sech}^2(\psi_2) \Big\} \\
& + (\cos(2\pi\xi_1) - 5) \sec^2(\pi\xi_1) \sin\theta_1 \Bigg],
\end{aligned} \tag{9a}
$$

$$
\begin{aligned}
f_\theta(\xi_1,\xi_2,\theta_1,\theta_2,\Lambda) \;=\; & \frac{\pi^3}{8}(\sin(\theta_2)+\mathrm{i}\,\cos(\theta_2))\Bigg[-\cos(2\theta_1)\coth\left(\psi_2-\mathrm{i}\frac{\pi}{2}\right)\mathrm{csch}^2\left(\psi_2-\mathrm{i}\frac{\pi}{2}\right)\\
&+\;\cos(2\theta_1)\cos(\theta_2)^2\coth\left(\psi_4+\mathrm{i}\frac{\pi}{2}\right)\mathrm{csch}^2\left(\psi_4+\mathrm{i}\frac{\pi}{2}\right)\\
&+\;\coth(\psi_1)\,\mathrm{csch}^2(\psi_1)\,(\cos(2\theta_1)-\mathrm{i}\,\sin(2\theta_1))\\
&-\;\mathrm{i}\,\coth\left(\psi_2-\mathrm{i}\frac{\pi}{2}\right)\mathrm{csch}^2\left(\psi_2-\mathrm{i}\frac{\pi}{2}\right)\sin(2\theta_1)\\
&-\;\coth\left(\psi_4+\mathrm{i}\frac{\pi}{2}\right)\mathrm{csch}^2\left(\psi_4+\mathrm{i}\frac{\pi}{2}\right)\Big\{\mathrm{i}\,\cos^2(\theta_2)\sin(2\theta_1)+\cos(2\theta_1)\sin^2(\theta_2)\\
&-\;\mathrm{i}\,\sin(2\theta_1)\sin^2(\theta_2)-\mathrm{i}\,\cos(2\theta_1)\sin(2\theta_2)-\sin(2\theta_1)\sin(2\theta_2)\Big\}\\
&-\;(\cos(2\theta_1+2\theta_2)+\mathrm{i}\,\sin(2\theta_1+2\theta_2))\coth(\psi_3)\,\mathrm{csch}(\psi_3)\\
&-\;2\mathrm{i}\,\cos(\theta_1)\cos(\theta_2)\sec^2(\pi\xi_1)\tan(\pi\xi_1)+2\cos(\theta_1)\sec^2(\pi\xi_1)\sin(\theta_2)\tan(\pi\xi_1)\Bigg],
\end{aligned}
\tag{9b}
$$

$$
g_\theta(\xi_1,\theta_1) \;=\; 8\xi_1\cos^2\theta_1, \tag{9c}
$$

$$
\begin{aligned}
f_\Lambda(\xi_1,\xi_2,\theta_1,\theta_2,\Lambda) \;=\; & \frac{\pi^2}{24}\Bigg[\cos\theta_1(7+\cos(2\pi\xi_1))\sec^2(\pi\xi_1)-\cos\theta_2(7+\cos(2\pi\xi_2))\sec^2(\pi\xi_2)\\
&+\;3(\cos\theta_1-\mathrm{i}\sin\theta_1)\Big\{\mathrm{csch}^2(\psi_1)-\cos(2\theta_1)\mathrm{sech}^2(\psi_2)-\mathrm{sech}^2(\psi_4)\\
&+\;\mathrm{csch}^2(\psi_1)(\cos(2\theta_1)+\mathrm{i}\sin(2\theta_1))-\mathrm{i}\,\mathrm{sech}^2(\psi_2)\sin(2\theta_1)\Big\}\\
&-\;3(\cos\theta_2-\mathrm{i}\sin\theta_2)\Big\{\mathrm{csch}^2(\psi_1)-\mathrm{sech}^2(\psi_2)-\cos(2\theta_2)\mathrm{sech}^2(\psi_4)\\
&+\;\mathrm{csch}^2(\psi_3)(\cos(2\theta_2)+\mathrm{i}\sin(2\theta_2))-\mathrm{i}\,\mathrm{sech}^2(\psi_4)\sin(2\theta_2)\Big\}\Bigg],
\end{aligned}
\tag{9d}
$$

where

$$
\begin{aligned}
\psi_1 &= \frac{\pi}{2}(\Lambda+\mathrm{i}(\xi_1-\xi_2)), \quad \psi_2=\frac{\pi}{2}(\Lambda-\mathrm{i}(\xi_1+\xi_2)),\\
\psi_3 &= \frac{\pi}{2}(\Lambda-\mathrm{i}(\xi_1-\xi_2)), \quad \psi_4=\frac{\pi}{2}(\Lambda+\mathrm{i}(\xi_1+\xi_2)).
\end{aligned}
\tag{10}
$$

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